MATH 213 – LAB 3 – TEST 1 REVIEW

(1) For what pairs (a, b) does

$$\begin{array}{ccccccc} x & + & ay & + & a^2z & = & 1 \\ x & + & ay & + & abz & = & a \\ bx & + & a^2y & + & a^2bz & = & a^2b \end{array}$$

have no solution? A unique solution? Infinitely many solutions?

(2) Find a homogeneous linear equation with general (parametric) solution

$$(x, y, z) = (s - 2t, 2s - 10t, s + 4t).$$

(3) Find a nondegenerate linear equation whose solution set contains

$$(1, 1, 1, 1) \qquad \text{and} \qquad (1, 0, 1, 0).$$

(4) Find s, t, u such that

$$s\begin{pmatrix}1\\0\\2\end{pmatrix} + t\begin{pmatrix}1\\1\\2\end{pmatrix} + u\begin{pmatrix}-1\\3\\1\end{pmatrix} = \begin{pmatrix}2\\1\\10\end{pmatrix}.$$

(5) Let A be an $m \times n$ matrix and let p, q and r be distinct numbers between 1 and n. We apply, sequentially, the following elementary row operations:

1. Swap rows q and r.
2. Add a times row q to row p.
3. Swap rows q and r.

Let A''' be the resulting matrix.

$$A \xrightarrow{1.} A' \xrightarrow{2.} A'' \xrightarrow{3.} A'''.$$

There is a *single* elementary row operation that transforms A into A'''. Find it.

(6) Let

$$A = \begin{pmatrix}1 & 2 & 3\\0 & 1 & 6\\0 & 0 & 1\end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix}1 & x & y\\0 & 1 & z\\0 & 0 & 1\end{pmatrix}.$$

(a) Write down a system $\mathcal{S}$ of linear equations in three variables such that (x, y, z) is a solution of $\mathcal{S}$ if and only if $AB = I$.

(b) Find all matrices B such that $AB = I$.

(7) A system of equations has the following augmented matrix:

$$\left(\begin{array}{cccc|c}1 & a & 0 & 0 & 0\\0 & 0 & b & 0 & b\\0 & 0 & b(c-1) & c & c\end{array}\right)$$

For what values of a, b, c does the system have:

(a) no solutions?

(b) a unique solution?

(c) infinitely many solutions?

(8) In this problem, you may freely use any result proved in class. (Results connecting ranks with solution sets of systems of linear equation might be helpful, for example.)

(a) Define what it means for a system of linear equations to be *homogeneous*.

(b) The quadruple $(x_1, x_2, x_3, x_4) = (1, 1, 2, 3)$ is a solution of a homogeneous system of four equations in four variables. Explain why this system must, in fact, have infinitely many solutions.

(c) Explain why any homogeneous system with more variables than equations has infinitely many solutions.

(d) Give an example of a system of linear equations with more variables than equations that has no solutions. (In light of the (c), this system cannot be homogeneous.)

(e) Can a system of linear equations with more variables than equations have a unique solution? Explain.

(9) Write down all 3×3 matrices $A = (a_{ij})$ in row echelon form for which $a_{22} = 5$.

(10) Find all matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A.$$

(11) (a) Show that

$$kI = k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

commutes with every 2×2 matris A. Conversely, show that if B is a 2×2 matrix that commutes with every 2×2 matrix A, then B has the form kI for some k.

(b) Generalize (a) to $n \times n$ matrices.

(12) (a) Let $\mathbf{e}_j$ be the n-dimensional row vector with 1 in the j-th column and 0 in all other columns. Let B be an $n \times p$ matrix. What is $\mathbf{e}_j B$?

(b) Let $\mathbf{e}_i$ be the n-dimensional column vector with 1 in the i-th row and 0 in all other row. Let A be an $m \times n$ matrix. What is $A\mathbf{e}_i$?

(13) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 matrix. Show that

$$A^2 - (a + d)A + (ad - bc)I = 0.$$

(14) Let

$$A = \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix}.$$

Find a polynomial $f(a, b, c, e, f, i)$ such that

$$A^3 - (a + e + i)A^2 + f(a, b, c, e, f, i)A - aeiI = 0.$$