MATH 213 – FALL 2015 – TEST 2

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Solve **two** of problems (1)-(3) and solve **two** of problems (4)-(6). Write neatly in the booklet provided; write your name and student number on it. Point values are indicated. You have 50 minutes. Good luck!

(1) Let

$$V = \{A \in M_2(\mathbb{R}) : a_{11} + a_{22} = 0\} \qquad \text{and} \qquad W = \{A \in M_2(\mathbb{R}) : A \text{ is not invertible}\}.$$

(a) (2 pts) Is V a subspace of $M_2(\mathbb{R})$? Justify your answer.

(b) (2 pts) Is W a subspace of $M_2(\mathbb{R})$? Justify your answer.

(2) Let

$$U = \left\{ \begin{pmatrix} x & x+2y & 3x+5y \end{pmatrix} : x, y \in \mathbb{R} \right\} \subset \mathbb{R}^{1\times 3}.$$

(a) (2 pts) Show that U is a subspace of $\mathbb{R}^{1\times 3}$.

(b) (2 pts) Find an element of $\mathbb{R}^{1\times 3}$ that is <u>not</u> in U. Justify your answer.

(3) (4 pts) Let P_2 be the vector space of polynomials in x of degree ≤ 2:

$$P_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}.$$

Write $x^2 + x + 1 \in P_2$ as a linear combination of $2x^2 + 3x + 1$, $4x^2 + 3x$ and $4x^2$.

(4) (4 pts) Let

$$A = \begin{pmatrix} 3 & -5 & 14 \\ 1 & -2 & 5 \end{pmatrix}$$

Write A in the form

$$E_1 E_2 \cdots E_k R,$$

where the E_i are elementary matrices and R is in reduced row echelon form.

(5) (4 pts) Let A be an invertible 2×2 matrix, let C be an arbitrary 2×2 matrix and let

$$X = \begin{pmatrix} A & \mathbf{0} \\ C & A^{-1} \end{pmatrix},$$

where $\mathbf{0}$ denotes the 2×2 zero matrix. Find X^{-1}. Feel free to use any facts you know about algebra with block matrices.

(6) (4 pts) Let

$$A = \begin{pmatrix} -2 & -5 & 8 \\ 3 & 7 & -11 \\ 9 & 21 & -34 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 7 & -11 \\ 9 & 21 & -34 \end{pmatrix}.$$

(a) (1 pt) Find a single elementary row operation that transforms A into B.

(b) (3 pts) Given that

$$A^{-1} = \begin{pmatrix} 7 & 2 & 1 \\ -3 & 4 & -2 \\ 0 & 3 & -1 \end{pmatrix},$$

find B^{-1}. Please use the given information and (a); <u>don't</u> find the inverse of B directly using, e.g., row-reduction.

Solutions

(1) (a) We claim that V is a subspace of $M_2(\mathbb{R})$. To see the, let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be elements of V. Then

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

To prove that $A + B \in W$, we must show that $(a_{11} + b_{11}) + (a_{22} + b_{22}) = 0$:

$$\begin{aligned} (a_{11} + b_{11}) + (a_{22} + b_{22}) &= (a_{11} + a_{22}) + (b_{11} + b_{22}) \\ &= (a_{11} + b_{11}) + (a_{22} + b_{22}) \\ &= 0 + 0 && \text{as } A \in V \text{ and } B \in V \\ &= 0. \end{aligned}$$

Thus, V is closed under addition.
If c is any scalar, then

$$cA = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

To prove that $cA \in V$, we must show that $ca_{11} + ca_{22} = 0$:

$$\begin{aligned} ca_{11} + ca_{22} &= c(a_{11} + a_{22} \\ &= c(0) && \text{as } A \in V \\ &= 0. \end{aligned}$$

Thus, V is closed under scalar multiplication.
Being closed under both addition and scalar multiplication, we conclude that V is a subspace of $M_2(\mathbb{R})$.

(b) W is not a subspace of $M_2(\mathbb{R})$. To see this, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Neither A nor B is invertible, so $A, B \in V$. However, $A + B = I$ (2×2 identity matrix) and I is invertible. Therefore, $A + B \notin W$ and W is not closed under addition. Since it's not closed under addition, W is not a subspace of V.

(2) (a) Let $\mathbf{u}, \mathbf{u}' \in U$. Then there are numbers $x, x', y, y' \in \mathbb{R}$ such that

$$\mathbf{u} = \begin{pmatrix} x & x + 2y & 3x + 5y \end{pmatrix} \quad \text{and} \quad \mathbf{u}' = \begin{pmatrix} x' & x' + 2y' & 3x' + 5y' \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u} + \mathbf{u}' &= \begin{pmatrix} x & x + 2y & 3x + 5y \end{pmatrix} + \begin{pmatrix} x' & x' + 2y' & 3x' + 5y' \end{pmatrix} \\ &= \begin{pmatrix} x + x' & x + 2y + x' + 2y' & 3x + 5y + 3x' + 5y' \end{pmatrix} \\ &= \begin{pmatrix} x + x' & x + x' + 2(y + y') & 3(x + x') + 5(y + y') \end{pmatrix} \end{aligned}$$

and, setting

$$x'' = x + x' \quad \text{and} \quad y'' = y + y',$$

we have

$$\mathbf{u} + \mathbf{u}' = \begin{pmatrix} x'' & x'' + 2y'' & 3x'' + 5y'' \end{pmatrix}.$$

Thus, $\mathbf{u} + \mathbf{u}' \in U$ and U is closed under addition.

If c is any scalar then

$$\begin{aligned} c\mathbf{u} &= c\begin{pmatrix} x & x+2y & 3x+5y \end{pmatrix} \\ &= \begin{pmatrix} cx & c(x+2y) & c(3x+5y) \end{pmatrix} \\ &= \begin{pmatrix} cx & cx+2cy & 3cx+5cy \end{pmatrix} \end{aligned}$$

and, setting

$$x' = cx \quad \text{and} \quad y' = cy,$$

we have

$$c\mathbf{u} = \begin{pmatrix} x' & x'+2y' & 3x'+5y' \end{pmatrix}.$$

Thus, $c\mathbf{u} \in U$ and U is closed under scalar multiplication. Being closed under addition and scalar multiplication, U is a subspace of $\mathbb{R}^{1\times 3}$.

(b) Let $\mathbf{v} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. If $\mathbf{v}$ was in U then there would be numbers x and y such that

$$1 = x, \quad 0 = x + 2y \quad \text{and} \quad 0 = 3x + 5y.$$

If $x = 1$ and $0 = x + 2y$, then $y = -\frac{1}{2}$. But then $3x + 5y = -\frac{1}{2} \neq 0$. Thus, $\mathbf{v}$ cannot be in U. (Any randomly selected vector would likely work just as well as this $\mathbf{v}$.)

(3) We want to find scalars a, b, c such that

$$x^2 + x + 1 = a(2x^2 + 3x + 1) + b(4x^2 + 3x) + c(4x^2).$$

Rearranging the right hand side, we get

$$x^2 + x + 1 = (2a + 4b + 4c)x^2 + (3a + 3b)x + a.$$

Equating coefficients of like powers of x, we get $a = 1$, $3a + 3b = 1$ and $2a + 4b + 4c = 1$. Solving this yields

$$b = \frac{1}{3}(1 - 3(1)) = -\frac{2}{3} \quad \text{and} \quad c = \frac{1}{4}(1 - 2(1) - 4(-\frac{2}{3})) = \frac{5}{12}.$$

Thus,

$$x^2 + x + 1 = 1(2x^2 + 3x + 1) - \frac{1}{3}(4x^2 + 3x) + \frac{5}{12}(4x^2).$$

(4) We bring A to reduced row echelon form:

$$\begin{pmatrix} 3 & -5 & 14 \\ 1 & -2 & 5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -2 & 5 \\ 3 & -5 & 14 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}.$$

Inverting this sequence of elementary row operations gives

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & -2 & 5 \\ 3 & -5 & 14 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & -5 & 14 \\ 1 & -2 & 5 \end{pmatrix}$$

Associating the corresponding elementary matrix with each of these elementary row operations gives

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}}_{R}.$$

(5) Let's try to solve the equation

$$\begin{pmatrix} A & \mathbf{0} \\ C & A^{-1} \end{pmatrix} \begin{pmatrix} R & S \\ T & U \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$$

for R, S, T, U:

$$\begin{pmatrix} AR & AS \\ CR + A^{-1}T & CS + A^{-1}U \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$$

Equating top left entries gives $AR = I$. But A is invertible, so $R = A^{-1}$. Equating top right entries gives $AS = \mathbf{0}$. But A is invertible, so $S = \mathbf{0}$. Equating bottom right entries gives $CR + A^{-1}T = \mathbf{0}$

or $A^{-1}T = -CA^{-1}$. Now multiply both sides by A **on the right** to get $T = -ACA^{-1}$. Equating bottom right entries gives $CS + A^{-1}U = I$ or $A^{-1}U = I$ or $U = A$. Thus,

$$X^{-1} = \begin{pmatrix} A^{-1} & \mathbf{0} \\ -ACA^{-1} & A \end{pmatrix}.$$

(You can check directly that $X^{-1}X = I$.)

(6) B is obtained from A by adding row 2 to row 1. Thus, $B = EA$ where

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$B^{-1} = (EA)^{-1} = A^{-1}E^{-1} = \begin{pmatrix} 7 & 2 & 1 \\ -3 & 4 & -2 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -5 & 1 \\ -3 & 7 & -2 \\ 0 & 3 & -1 \end{pmatrix}.$$