# MATH 213 – TEST 3 REVIEW

(1) Let $A$ be an $m \times n$ matrix and let $B$ be its reduced row echelon form.

(a) If $\operatorname{rank} A = r$, then any $r$ columns of $A$ form a basis of $C(A)$.

(b) If $m > n$ then $\dim R(A) > \dim C(A)$.

(c) If $\operatorname{rank} A = n$ then the rows of $A$ form a basis of $\mathbf{R}^n$.

(d) If the rows of $A$ span $\mathbf{R}^n$ then the columns of $A$ span $\mathbf{R}^m$.

(e) If the columns of $A$ are linearly independent and the rows of $A$ span $\mathbf{R}^n$, then $m = n = \operatorname{rank} A$.

(f) $A$ and $B$ have the same column space.

(g) If $N(A) \neq \{\mathbf{0}\}$ then $C(A) \neq \mathbf{R}^m$.

(h) If the columns of $A$ are linearly independent and the rows of $A$ are linearly independent, then $N(A) = \{0\}$ and $N(A^T) = \{0\}$.

(i) $\operatorname{rank} A = \operatorname{rank} A^T$

(j) The columns of $A$ satisfy the same linear dependence relations as the columns of $B$.

(k) If the columns of $A$ are linearly independent then $A\mathbf{x} = \mathbf{0}$ has a unique solution.

(l) If the columns of $A$ are linearly independent then $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbf{R}^m$.

(m) If $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbf{R}^n$, then $N(A^T) = \{0\}$.

(n) $\operatorname{rank}\begin{pmatrix} A & A \end{pmatrix} = \operatorname{rank} A$.

(2) Explain why there is **no** $3 \times 3$ matrix $A$ such that

$$R(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 & 2 & 1 \end{pmatrix},\ \begin{pmatrix} 2 & 1 & 2 \end{pmatrix} \right\} \quad \text{and} \quad N(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

(Hint: Consider products of basis elements of $R(A)$ with basis elements of $N(A)$.)

(3) Find a basis for the vector space

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : x + 2y + z = 0 \right\}.$$

(Hint: Find a matrix whose nullspace is $W$.)

Find a basis for the vector space

$$W = \left\{ \begin{pmatrix} x+y \\ x \\ x-y \end{pmatrix} : x, y \in \mathbf{R} \right\}$$

(Hint: Find a matrix whose column space is $W$.)

(4) Let $\mathbf{e}_1, \mathbf{e}_2$ be the standard basis of $\mathbf{R}^{2\times 1}$. Show that

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_2 \\ \mathbf{e}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_2 \\ \mathbf{e}_2 \end{pmatrix}$$

is *not* a basis of $\mathbf{R}^{4\times 1}$ and that

$$\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_2 & \mathbf{e}_2 \end{pmatrix}$$

is a basis of $\mathbf{R}^{2\times 2}$.

(5) Let $P_2$ be the space of polynomials of degree $\leq 2$ and consider the transformation

$$T : P_2 \longrightarrow \mathbf{R}^3 \quad \text{given by} \quad T(f) = \begin{pmatrix} f(2) \\ f'(2) \\ f''(2) \end{pmatrix}$$

(The $'$ means derivative.)

(a) Show that $T$ is linear. Feel free to invoke properties of the derivative that you know from calculus.

(b) Let $B = (1, x, x^2)$ be the standard basis of $P_2$. Find a $3 \times 3$ matrix $A$ such that

$$T(f) = A[f]_B \quad \text{for all } f \in P_2.$$

[Linear transformations are *not* on Friday's test.]