MATH 213 – FALL 2015 – TEST 1

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Solve all five problems. Write neatly in the booklet provided; write your name and student number on it. Point values are indicated. You have 50 minutes. Good luck!

(1) (a) 1 pt: Define what it means for a matrix A to be in *reduced row echelon form.*

Solution:

1. All the zero rows of A are at the bottom.
2. Each nonzero row of A has a leading 1.
3. A leading 1 is to the right of the leading ones in all the rows above it.
4. A leading 1 is the only nonzero entry in its column.

(b) 2 pts: Write down all 4×3 matrices $A = (a_{ij})$ in reduced row echelon form such that $a_{12} = 4$. Use "$*$" to indicate "arbitrary number". Be sure to explain why your list is complete.

Solution: Since $a_{12} = 4$, the topmost row a_1 of A is not a zero row. Therefore, by criterion 2., the a_1 must contain a leading 1. Since it's *leading* (the leftmost nonzero entry in its row) the leading 1 be in position $(1, 1)$. Thus, $a_{11} = 1$ and a_1 has the form $\begin{pmatrix} 1 & 4 & * & * \end{pmatrix}$. If A has rank 1 then A has only one nonzero row because it's in RREF. Thus, it must look like

$$A = \begin{pmatrix} 1 & 4 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{$\dagger$}$$

Suppose now that A has rank ≥ 2. Then the second row a_2 of A must have a leading 1. Since it must be to the right of the leading 1 in a_1, the leading 1 in a_2 must be in column 2 or 3. Were in column 2 then A would look like

$$\begin{pmatrix} 1 & 4 & * \\ 0 & 1 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}.$$

But this matrix is not in RREF since the leading 1 in position $(2, 2)$ is not the only nonzero entry in its column. Thus, the leading 1 in a_2 must be in column 3 and A must take the form

$$\begin{pmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{$\ddagger$}$$

A cannot have rank ≥ 3 because then the leading 1 in the third row a_3 of A would have to lie to the right of the leading 1 in a_2, which is clearly impossible. Thus, ($\dagger$) and ($\ddagger$) are the only possibilities.

(2) Consider the following system of equations:

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}$$

(a) 2 pts: Write down the augmented matrix associated to this system and find its reduced row echelon form. (The reduced row echelon form of the matrix has integer entries. If yours doesn't, you've made a mistake.)

Solution: The augmented matrix of the system is

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

We reduce it to RREF:

- Multiply row 3 by $\frac{1}{3}$:

$$\longrightarrow \begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{pmatrix}$$

- Swap rows 1 and 2:

$$\longrightarrow \begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

- Subtract 3 times row 1 from row 2:

$$\longrightarrow \begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

- Multiply row 2 by $\frac{1}{2}$:

$$\longrightarrow \begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

- Add 3 times row 2 to row 1; add -3 times row 2 to row 3:

$$\longrightarrow \begin{pmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

- Add -3 times row 2 to row 1; add -3 times row 2 to row 3:

$$\longrightarrow \begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

This last matrix is in RREF.

(b) 2 pts: Find a general solution of the system.

Solution: There are no leading 1s in columns 3 and 4. Thus, we assign parameters to the corresponding variables:

$$x_3 = s, \qquad x_4 = t.$$

Solving for the leading variables, we get the general solution

$$x_1 = -24 + 2s - 3t, \quad x_2 = -7 + 2s - 2t, \quad x_3 = s, \quad x_4 = t, \quad x_5 = 4.$$

(c) 1 pt: Check your answer.

Solution: If we set $s = t = 0$, we get the solution

$$(x_1, x_2, x_3, x_4, x_5) = (-24, -7, 0, 0, 4).$$

Plug this into the left hand side of the third equation of the original system and evaluate:

$$3(-24) - 9(-7) + 12(0) - 9(0) + 6(4) = 15.$$

This is gives us confidence in our solution.

(3) (a) 1 pt: Define what is meant by the *rank* of a matrix A.

Solution: The *rank* of a matrix is the number of leading 1s (equivalently, of nonzero rows) in its reduced row echelon form.

(b) 4 pts: Let

$$A = \begin{pmatrix} a & a & 0 \\ a & b & b \\ -a & -b & a \end{pmatrix}.$$

For all pairs (a, b), determine $\operatorname{rank}(A)$.

Solution: We reduce A to echelon form. This form will, of course, depend on (a, b). In attempting to create a leading 1 in row 1, we are immediately faced with two possibilities: $a = 0$ (Case 1) or $a \neq 0$ (Case 2).

- *Case 1:* In this case,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & b \\ 0 & -b & 0 \end{pmatrix}.$$

Again, we are faced with two possibilities: $b = 0$ (Case 1a) or $b \neq 0$ (Case 1b).

  – *Case 1a:* A is the zero matrix and, thus, has rank 0.

  – *Case 1b:* In this case, we can divide rows 2 and 3 by b and, subsequently,add row 2 to row 3. This reduces A to the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

From here, it is easy to see that $\operatorname{rank}(A) = 2$.

- *Case 2:* Apply the following row operations: Multiply row 1 by $\frac{1}{a}$; subtract a times row 1 to row 2; add a times row 1 to row 3. This reduces A to the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & b-a & b \\ 0 & a-b & a \end{pmatrix}$$

Whether we can create a leading 1 in position $(2, 2)$ or not depends on whether $a = b$ (Case 2a) or $a \neq b$ (Case 2b).

  – *Case 2a:* In this case,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & b-a & b \\ 0 & a-b & a \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & a \end{pmatrix}.$$

Since we are in Case 2 where $a \neq 0$, we can multiply rows 2 and 3 by $\frac{1}{a}$ and then subtract row 2 from row 3. This gives the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is in reduced row echelon form and has rank 2.

– *Case 2b:* In this case, we perform the row operation "multiply row 2 by $\frac{1}{b-a}$" followed by "subtract $a-b$ times row 2 from row 3:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & b-a & b \\ 0 & a-b & a \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{b}{b-a} \\ 0 & a-b & a \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{b}{b-a} \\ 0 & 0 & a+b \end{pmatrix}$$

Whether or not we can construct a leading 1 in position $(3,3)$ depends on whether $a = -b$ (Case 2a-i or $a \neq -b$ (Case 2a-ii).

∗ *Case 2a-i:* In this case,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{b}{b-a} \\ 0 & 0 & a+b \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix has rank 2.

∗ *Case 2a-ii:* Multiply row 3 by $\frac{1}{a+b}$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{b}{b-a} \\ 0 & 0 & a+b \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{b}{b-a} \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has rank 3.

We summarize:

- A has rank 0 when $a = b = 0$.
- A never has rank 1.
- A has rank 2 when $a = 0$ and $b \neq 0$ and $0 \neq a = \pm b$.
- A has rank 3 when $a \neq 0$ and $b \neq \pm a$.

(4) Let $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) 1 pt: Show that all matrices of the form $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ commute with B, i.e., satisfy $AB = BA$.

Solution:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & a+b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(b) 2 pts: Show that any matrix A that commutes with B has the above form.

Solution: Suppose the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commutes with B. Then

$$\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

Equating entries, we see that $a + c = c$, implying $c = 0$, and $a + b = b + d$, implying $a = d$. Thus,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

as desired.

(5) (a) 1 pt: Define what it means for a system of linear equations to be *homogeneous*.

Solution: A system of equations is *homogeneous* if the constant term of each of its constituent equations is 0.

(b) 2 pts: Is it possible for a homogeneous system to have no solutions? Explain.

Solution: No. The solution $(x_1, \ldots, x_n) = (0, \ldots, 0)$ is a solution of any homogeneous system. (This solution is called the *trivial solution*.)

(c) 1 pt: Let A be an $m \times n$ matrix with rank r. Let p be the number of free variables (parameters) in a general solution of the corresponding homogeneous system. Write down an equation expressing p in terms of m, n and r. You do not need to offer an explanation for this part.

Solution: $p = n - r$. (Number of free variables $=$ number of variables $-$ number of leading variables.)