MATH 213 – ASSIGNMENT 3 – SOLUTIONS

(1) (a) We assume that A_{11} is invertible. To kill the $(2,1)$-entry of A, we perform the "block row operation" subtract $A_{21}A_{11}^{-1}$ times row one from row two from by multiplying A on the left by the "block elementary matrix" $\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix}$:

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix}.$$

To kill the $(1,2)$-entry of this resulting matrix, we apply the "block column operation" subtract $-A_{11}^{-1}A_{12}$ times column one from column two by multiplying it on the right by the "block elementary matrix" $\begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}$:

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix} \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{pmatrix}.$$

Thus, we can take

$$X = -A_{21}A_{11}^{-1} = -\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 1 & -1 \end{pmatrix}$$

and

$$Y = -A_{11}^{-1}A_{12} = -\begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -5 \\ 5 & 6 \end{pmatrix}$$

(b) $\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}$, $\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix}$

(c) By (a) and (b),

$$A = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix}.$$

The matrix $\begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix}$ is invertible if and only if A_{11} and S are, in which case

$$\begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$$

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(d) With $A_{11} = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$,

$$A_{11}^{-1} = \frac{1}{4 \cdot 4 - 3 \cdot 5} \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix}.$$

We have

$$\begin{aligned} S &= A_{22} - A_{21} A_{11}^{-1} A_{12} \\ &= \begin{pmatrix} 7 & 10 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

and

$$S^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} A^{-1} &= \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \\ &= \begin{pmatrix} A_{11}^{-1} + Y S^{-1} X & Y S^{-1} \\ S^{-1} X & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 21 & -17 & -3 & 5 \\ -28 & 23 & 4 & -7 \\ -13 & 11 & 2 & -5 \\ 7 & -6 & -1 & 3 \end{pmatrix} \end{aligned}$$

(e) Assume that A_{11} is invertible, so that $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Then

$$\begin{aligned} S &= A_{22} - A_{21} A_{11}^{-1} A_{12} \\ &= a_{33} - \begin{pmatrix} a_{31} & a_{32} \end{pmatrix} \left[\frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \right] \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} \\ &= \frac{a_{33}(a_{11}a_{22} - a_{12}a_{21}) - \begin{pmatrix} a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}}{a_{11}a_{22} - a_{12}a_{21}} \\ &= \frac{a_{33}(a_{11}a_{22} - a_{12}a_{21}) - (a_{13}a_{22}a_{31} - a_{13}a_{21}a_{32} - a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32})}{a_{11}a_{22} - a_{12}a_{21}} \\ &= \frac{a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}}{a_{11}a_{22} - a_{12}a_{21}} \end{aligned}$$

S is invertible if and only if $S \neq 0$.

(f) A_{11} and S are both invertible if and only if $(a_{11}a_{22} - a_{12}a_{21})S \neq 0$, i.e.,

$$a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} \neq 0.$$

(2) (a) To show that W is a subspace of V, let $f, g \in W$ and let x be a scalar. To show that $f+g \in W$, we must show that $f+g$ has finite support. Note that $\operatorname{supp}(f+g) \subset \operatorname{supp} f \cup \operatorname{supp} g$. (Why?) Since f and g are in W, both $\operatorname{supp} f$ and $\operatorname{supp} g$ are finite. Therefore, $\operatorname{supp} f \cup \operatorname{supp} g$ is finite, as is its subset $\operatorname{supp}(f+g)$. Thus, $f+g \in W$. Similarly, $\operatorname{supp}(xf) \subset \operatorname{supp} f$, so $xf \in W$.

(b) Let $n_1, n_2, \ldots, n_k \in \mathbf{Z}$ be distinct integers and let $x_1, x_2, \ldots, x_k$ be scalars. Suppose

$$\sum_{i=1}^{k} x_k f_{n_k} = 0.$$

Evaluating both sides at n_i gives

$$x_i = x_i f_{n_i}(n_i) = 0.$$

Therefore, $\{f_n : n \in \mathbf{Z}\}$ is linearly independent. To see that this set spans $\mathbf{Z}$, observe that if $\operatorname{supp} f \subset \{n \in \mathbf{Z} : -N \leq n \leq N\}$, then

$$f = \sum_{n=-N}^{N} f(n) f_n.$$

(3) We put A into reduced row echelon form:

$$\begin{pmatrix} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 & 19 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 2 & 14 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 2 & 14 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Since columns 1, 3 and 4 of the reduced row echelon form of A contain leading 1s, columns 1, 3 and 4 of A, i.e.,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

form a basis of $C(A)$.

(b) Since we already found the reduced row echelon form of A, it's easy to solve the homogeneous system $Ax = 0$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3s+2t \\ s \\ -5t \\ -7t \\ t \end{pmatrix} = s \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ -5 \\ -7 \\ 1 \end{pmatrix}.$$

Therefore, the vectors

$$\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 0 \\ -5 \\ -7 \\ 1 \end{pmatrix}$$

form a basis of $N(A)$.

(c) It's easy to write the fifth column of the reduced row echelon form of A as a linear combination of the basis elements of $C(A)$:

$$\begin{pmatrix} -2 \\ 7 \\ 5 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The same linear dependence relation holds between the columns of A itself:

$$\begin{pmatrix} -2 \\ 5 \\ 19 \\ 7 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

(4) $\begin{pmatrix} x_1 & x_2 & \cdots & x_m \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{pmatrix} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots x_m\mathbf{b}_m \in R(B)$.

(5) (a) Write $\mathbf{a}_i$ for the i-th row of A. Then

$$AB = \begin{pmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{pmatrix}.$$

By (4), $\mathbf{a}_i B \in R(B)$ for all i. Since the $\mathbf{a}_i B$ span the row space of AB, $R(AB) \subset R(B)$.

(b) We have $\dim R(B) = \operatorname{rank} B$, $\dim R(AB) = \operatorname{rank} AB$. Since $R(AB) \subset R(B)$, $\dim R(AB) \leq \dim R(B)$. Therefore,

$$\operatorname{rank} AB = \dim R(AB) \leq \dim R(B) = \operatorname{rank} B.$$

(c) We prove a results analogous to (4) and (5a) for the column space. Observe that

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 + \cdots + b_n\mathbf{a}_n \in C(A).$$

Since the j-th column of AB is $A\mathbf{b}_j$ where $\mathbf{b}_j$ is the j-th column of B, it follows from the observation that the columns of AB belong to the column space of A. Since the columns of AB span $C(AB)$, we see that $C(AB) \subset C(A)$. Arguing now as in (b), we have

$$\operatorname{rank} AB = \dim C(AB) \leq \dim C(A) = \operatorname{rank} A.$$