1. Subspaces

Definition 1. Let V be a vector space and let W be a subset of V. We say that W is

- *closed under addition* if $v_1 + v_2 \in W$ for all $v_1, v_2 \in W$.
- *closed under scalar multiplication* if $xv \in W$ for all $x \in \mathbf{R}$ and all $v \in W$.
- a *(vector) subspace* of V if it is closed under both addition and scalar multiplication.

Example 2. Let $V = \mathbf{R}^n$ and let

$$W = \left\{ \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \in \mathbf{R}^n : v_1 + v_2 + \cdots + v_n = 0 \right\}.$$

We claim that W is a subspace of V. To check closure under addition, we let

$$v = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} v_1' & v_2' & \cdots & v_n' \end{pmatrix}$$

be arbitrary elements of W. Then

$$v + v' = \begin{pmatrix} v_1 + v_1' & v_2 + v_2' & \cdots & v_n + v_n' \end{pmatrix}.$$

To show that $v + v' \in W$, we must show that the sum of its entries is zero:

$$\begin{aligned} (v_1 + v_1) + (v_2 + v_2') + \cdots + (v_n + v_n') &= (v_1 + v_2 + \cdots v_n) + (v_1' + v_2' + \cdots + v_n') \\ &= 0 + 0 \qquad (\text{as } v \in W \text{ and } v' \in W) \\ &= 0. \end{aligned}$$

Thus, W is closed under addition. To show that W is closed under scalar multiplication, we must show that the sum of the entries of

$$xv = \begin{pmatrix} xv_1 & xv_2 & \cdots & xv_n \end{pmatrix}$$

is zero:

$$xv_1 + xv_2 + \cdots + xv_n = x(v_1 + v_2 + \cdots + v_n) \stackrel{*}{=} x0 = 0,$$

where the equality marked with $*$ uses the fact that $v \in W$. Thus, W is closed under scalar multiplication. Being closed under addition and scalar multiplication, we conclude that W is a subspace of V.

Exercise 3. Let V and W be as in Definition 1. We say that W is

- *closed under additive inverse* if $-v \in W$ for all $v \in W$.
- *closed under subtraction* if $v_1 - v_2 \in W$ for all $v_1, v_2 \in W$.

Show that if W is a subspace of V, then the neutral element 0 of V belongs to W and that V is closed under additive inverse and subtraction.

Exercise 4. Let V be a vector space and let W be a subset of V. Show that W is a subspace of V if and only if

$$xv_1 - v_2 \in W$$

for all $x \in \mathbf{R}$ and all $v_1, v_2 \in W$.

1.1. The nullspace of a matrix.

Definition 5. Let A be an $m \times n$ matrix. The *nullspace of* A, written $N(A)$, is the set of vectors $\mathbf{x} \in \mathbf{R}^n$ such that $A\mathbf{x} = \mathbf{0}$. The nullspace of A is also referred to as the *kernel* of A and written $\ker A$.

The nullspace is not a new object for us, it is just new terminology:

Example 6. The solution set of a homogeneous system of m equations in n variables is the nullspace of its coefficient matrix.

Theorem 7. *The nullspace $N(A)$ is a subspace of $\mathbf{R}^n$.*

Proof. To prove closure under addition, let $\mathbf{x}, \mathbf{y} \in N(A)$. We show that $\mathbf{x} + \mathbf{y} \in N(A)$:

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} \overset{*}{=} \mathbf{0} + \mathbf{0} = \mathbf{0},$$

where the equality marked with $*$ uses our hypothesis that $\mathbf{x}, \mathbf{y} \in N(A)$. Thus, $\mathbf{x}+\mathbf{y} \in N(A)$. The proof of scalar multiplication is similar. Let $\mathbf{x} \in N(A)$ and let x be a scalar. Then

$$A(x\mathbf{x}) = xA\mathbf{x} = x\mathbf{0} = \mathbf{0}.$$

(Where did we use the hypothesis that $\mathbf{x} \in N(A)$?) Therefore, $x\mathbf{x} \in N(A)$. □

Exercise 8. Show that

$$N(A^T) = \{\mathbf{x}^T : \mathbf{x} \in \mathbf{R}^{1\times m} \text{ and } \mathbf{x}A = \mathbf{0}\}.$$

The set $N(A^T)$ is sometimes called the *left nullspace* or *left kernel* of A.

1.2. Column space and row space.

Let A be an $m \times n$ matrix.

Definition 9. The *column space* of A, written $C(A)$, is the following subset of $\mathbf{R}^{m\times 1}$:

$$C(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbf{R}^{n\times 1}\}.$$

The *row space* of A, written $R(A)$, is the following subset of $\mathbf{R}^{1\times n}$:

$$R(A) = \{\mathbf{w}A : \mathbf{w} \in \mathbf{R}^{1\times m}\}.$$

Theorem 10. *$C(A)$ is a subspace of $\mathbf{R}^m$ and $R(A)$ is a subspace of $\mathbf{R}^{1\times n}$*

Proof. We prove the result for $C(A)$ and leave the corresponding proof for $R(A)$ as an exercise for the reader. Let $\mathbf{y}, \mathbf{y}' \in C(A)$ and let α be a scalar. By definition of $C(A)$, there are vectors $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^n$ such that $\mathbf{y} = A\mathbf{x}$ and $\mathbf{y}' = A\mathbf{x}'$. Then

$$\mathbf{y} + \mathbf{y}' = A\mathbf{x} + A\mathbf{x}' = A(\mathbf{x} + \mathbf{x}')$$

by the distributive law of matrix arithmetic. Therefore, $\mathbf{y} + \mathbf{y}' \in C(A)$. Also,

$$\alpha\mathbf{y} = \alpha A\mathbf{x} = A(\alpha\mathbf{x}),$$

so $\alpha\mathbf{y} \in C(A)$. Having established closure of $\operatorname{im} A$ under addition and scalar multiplication, we conclude that $\operatorname{im} A$ is a subspace of $\mathbf{R}^m$. □

Exercise 11. Let A be an $m \times n$ matrix.

(1) Let U be an invertible $m \times m$ matrix and let A be an $m \times n$ matrix. Show that $R(UA) = R(A)$. (Suggestion: Prove this first when U is an elementary matrix.)
(2) Let V be an invertible $n \times n$ matrix. Prove that $C(AV) = C(A)$. (Suggestion: Use (1) and the fact that $C(B) = R(B^T)^T$ for any matrix B.)

1.3. **Intersection and sum.** Let W and W' be subspace of the vector space V.

Definition 12. The *intersection of* W *and* W', written $W \cap W'$, is the subset of V defined by

$$W \cap W' = \{\mathbf{v} \in V : \mathbf{v} \in W \text{ and } \mathbf{v} \in W'\}.$$

Exercise 13. Prove that $W \cap W'$ is a subspace of V.

Definition 14. The *sum of* W *and* W', written $W + W'$, is the subset of V defined by

$$W + W' = \{\mathbf{w} + \mathbf{w}' : \mathbf{w} \in W,\, \mathbf{w}' \in W'\}.$$

Exercise 15. Prove that $W + W'$ is a subspace of V.

Exercise 16. The *union of* W *and* W', written $W \cup W'$, is the subset of V defined by

$$W \cap W' = \{\mathbf{v} \in V : \mathbf{v} \in W \text{ or } \mathbf{v} \in W'\}.$$

Prove that $W \cup W'$ is a subspace of V if and only if $W \subset W'$ or $W' \subset W$.

Exercise 17. Let A be an $m \times n$ matrix and let A' be an $m' \times n$ matrix. What is the relationship between

$$N(A), \qquad N(A') \qquad \text{and} \qquad N\begin{pmatrix} A \\ A' \end{pmatrix}?$$

Exercise 18. Let B be an $m \times n$ matrix and let B' be an $m \times n'$ matrix. What is the relationship between

$$\operatorname{im} B, \qquad \operatorname{im} B' \qquad \text{and} \qquad \operatorname{im}\begin{pmatrix} B & B' \end{pmatrix}?$$

Exercise 19. Let $W_1, W_2, \ldots, W_k$ be subspaces of V. Give definitions for k-fold intersections and sums $W_1 \cap W_2 \cap \cdots \cap W_k$ and $W_1 + W_2 + \cdots W_k$ and prove that they are subspaces of V.

2. Additional exercises

$$\begin{aligned} \mathbf{R}^n &= n\text{-dimensional row or column vectors with entries in } \mathbf{R} \\ \mathbf{R}^{m\times n} &= m \times n \text{ matrices with entries in } \mathbf{R} \\ M_n(\mathbf{R}) &= n \times n \text{ matrices with entries in } \mathbf{R} \\ P_n &= \text{polynomials of degree} \le n \text{ with coefficients in } \mathbf{R} \\ U^X &= \text{functions from the set } X \text{ into the vector space } U \end{aligned}$$

(1) Is W a subspace of V?

(a) $V = \mathbf{R}^n$, $W = \{\mathbf{x} \in \mathbf{R}^n : x_1 \le x_i \text{ for } i = 1, \dots, n\}$
(b) $V = \mathbf{R}^n$, $W = \{\mathbf{x} \in \mathbf{R}^n : |x_1| \le x_i \text{ for } i = 1, \dots, n\}$
(c) $V = \mathbf{R}^n$, $W = \{\mathbf{x} \in \mathbf{R}^n : |x_i| \le 1 \text{ for } i = 1, \dots, n\}$
(d) $V = M_n(\mathbf{R})$, $W = \{A \in V : A \text{ is upper triangular}\}$
(e) $V = M_n(\mathbf{R})$, $W = \{A \in V : A \text{ is invertible}\}$
(f) $V = M_n(\mathbf{R})$, $W = \{A \in V : \operatorname{rank} A < n\}$
(g) $V = M_n(\mathbf{R})$, $W = \{A \in V : A^2 = \mathbf{0}\}$
(h) $V = \mathbf{R}^{m\times n}$, $W = \{A \in V : A \text{ is in row echelon form}\}$
(i) $V = \mathbf{R}^{m\times n}$, $W = \{A \in V : A^T = -A\}$
(j) $V = P_n$, $W = \{f \in P_n : f(0) = 0\}$
(k) $V = P_n$, $W = \{f \in P_n : f(0) = 1\}$
(l) $V = P_n$, $W = \{f \in P_n : f(1) = 0\}$
(m) $V = P_n$, $W = \{f \in P_n : f(-x) = f(x)\}$
(n) $V = P_n$, $W = \{f \in P_n : f(-x) = -f(x)\}$
(o) $V = \mathbf{R}^{\mathbf{R}}$, $W = \{f \in V : f \text{ is bounded}\}$ (A function $f : \mathbf{R} \to \mathbf{R}$ is bounded if there is a number $M > 0$ such that $|f(x)| \le M$ for all $x \in \mathbf{R}$.)
(p) $V = \mathbf{R}^{\mathbf{R}}$, $W = \{f \in V : f \text{ is continuous}\}$.
(q) $V = \mathbf{R}^{\mathbf{R}}$, $W = \{f \in V : f \text{ is continuous and } f(0) = 0\}$.
(r) $V = \mathbf{R}^{\mathbf{R}}$, $W = \{f \in V : f \text{ is differentiable}\}$.
(s) $V = \mathbf{R}^{\mathbf{R}}$, $W = \{f \in V : f \text{ is differentiable and } f'(0) = 0\}$
(t) $V = \mathbf{R}^{\mathbf{R}}$, $W = \{f \in V : f \text{ is and } f'(0) = 0\}$
(u) $V = \mathbf{R}^{\mathbf{R}}$, $W = \{f \in V : f \text{ is } n\text{-times differentiable and } f^{(k)}(0) = 0 \text{ for all } k \le n\}$
(v) $V = \mathbf{R}^{\mathbf{R}}$, $W = \{f \in V : f(x) = 0 \text{ for } |x| > 1\}$
(w) $V = [0,1]^{\mathbf{R}}$, $W = C[0,1]$
(x) $V = C[0,1]$, $W = \left\{ f \in C[0,1] : \int_0^1 f(x)dx = 0 \right\}$
(y) $V = C(-\infty,\infty)$, $W = \{f \in V : \int_{-\infty}^{\infty} |f(x)|dx < \infty\}$
(z) $V = C(-\infty,\infty)$, $W = \{f \in V : \int_{-\infty}^{\infty} f(x)^2 dx < \infty\}$