MATH 213 LAB 6 – PRODUCTS OF SETS; DIRECT SUMS OF VECTOR SPACES

1. Products of sets

Let X and Y be sets.

Definition 1. The *(Cartesian) product of* X *and* Y, written $X \times Y$, is the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$:

$$X \times Y = \{(x, y) : x \in X,\ y \in Y\}.$$

We write X^2 as a shorthand for $X \times X$.

Example 2. Let $X = \{1, 2\}$ and let $Y = \{a, b, c\}$. Then

$$X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\} \quad \text{and}$$
$$Y \times X = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 3)\}.$$

Exercise 3. Typically, the set $\mathbf{R}^2$ is represented graphically as the (Cartesian) xy-plane. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. We define the *graph of* f to be the subset of $\mathbf{R}^2$ defined by

$$\operatorname{graph} f = \{(x, f(x)) : x \in \mathbf{R}\}.$$

(1) Is this "set theoretic" definition of graph compatible with the graphical notion of the graph of a function?

(2) Let Z be a subset of $X \times Y$ such that for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in Z$. Show that there is a unique function $g : X \to Y$ such that $\operatorname{graph} g = Z$. (In mathematical contexts, a function $X \to Y$ is often *defined* as a subset Z of $X \times Y$ such that for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in Z$ because it seems more precise than the "wordy" definition we gave — a rule assigning to every element of X a single element of Y.)

Exercise 4. Prove or provide a counterexample:

(1) $(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z)$

(2) $(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z)$

(3) $(W \cap X) \times (Y \cap Z) = (W \times Y) \cap (X \times Z)$

(4) $(W \cup X) \times (Y \cup Z) = (W \times Y) \cup (X \times Z)$

(5) $(W \cup X) \times (Y \cup Z) = (W \times Y) \cup (W \times Z) \cup (X \times Y) \cup (X \times Z)$

2. Direct sums of vector spaces

Let V and W be vector spaces. Let $U = V \times W$. Define a addition operation $+$ on U by the rule

$$(v, w) + (v', w') = (v + v', w + w') \qquad \text{for all } (v, w),\ (v', w') \in U. \tag{1}$$

(On the right hand side, $v + v'$ is computed in V and $w + w'$ is computed in W.)

Exercise 5. Prove that (1) satisfies the properties characterizing an addition operation:

(1) $a + (b + c) = (a + b) + c$ for all $a, b, c \in U$.

(2) $a + b = b + a$ for all $a, b \in W$.

(3) There is an element $0_U \in U$ such that $a + 0_U = a$ for all $a \in U$.

(4) For all $a \in U$, there is an element $-a \in U$ such that $a + (-a) = 0_U$.

Define a scalar multiplication operation on U by the rule

$$x(v, w) = (xv, xw) \qquad \text{for all } (v, w) \in U \text{ and all scalars } x. \tag{2}$$

(On the right hand side, xv is computed in V and xw is computed in W.)

Exercise 6. Prove that (2) satisfies the properties characterizing a scalar multiplication operation:

(1) $x(ya) = (xy)a$ for all scalars x and y and all $a \in U$.

(2) $(x + y)a = xa + ya$ for all scalars x and y and all $a \in U$.

(3) $x(a + b) = xa + xb$ for all scalars $x$and all a, $b \in U$.

(4) $1a = a$ for all $a \in U$.

Definition 7. The vector space obtained by endowing the set $V \times W$ and with the addition and scalar operations defined in (1) and (2), respectively, is called the *direct sum of V and W* and is denoted $V \oplus W$.

3. Subspaces

Exercise 8. Let

$$V = \{A \in M_n(\mathbf{R}) : A^2 = 0\}.$$

Is V a subspace of $M_n(\mathbf{R})$? Explain.

> We recall the definition of the vector space of functions from a set X into a vector space V: If f, $g \in \mathcal{F}(X, V)$ and c is a scalar, then $f + g \in \mathcal{F}(X, V)$ and $cf \in \mathcal{F}(X, V)$ are defined by
>
> $$(f + g)(x) = \underline{\hspace{3cm}} \qquad \text{and} \qquad (cf)(x) = \underline{\hspace{1.5cm}},$$
>
> where the right hand sides are computed in ______.

Exercise 9. Is the given subset of $\mathcal{F}(\mathbf{R}, \mathbf{R})$ a subspace?

(1) $\{f \in \mathcal{F}(\mathbf{R}, \mathbf{R}) : f(3) = 0\}$

(2) $\{f \in \mathcal{F}(\mathbf{R}, \mathbf{R}) : f \text{ is differentiable}\}$

(3) $\{f \in \mathcal{F}(\mathbf{R}, \mathbf{R}) : f \text{ is differentiable and } f(3) = f'(3) = 0\}$

(4) $\{f \in \mathcal{F}(\mathbf{R}, \mathbf{R}) : f \text{ is bounded}\}$ (A function $f : \mathbf{R} \to \mathbf{R}$ is bounded if there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbf{R}$.)