MATH 213 – REVIEW PROBLEMS

(1) Find the reduced row echelon form of A. Solve the equation $A\mathbf{x} = \mathbf{0}$.

(a) $A = \begin{pmatrix} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{pmatrix}$

Answer: $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{pmatrix}$

Answer: $\begin{pmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(2) Find an invertible matrix U such that UA is in reduced row echelon form.

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

Answer: $U = A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ $(bc \neq 0)$

Answer: $U = A^{-1} = -\dfrac{1}{bc}\begin{pmatrix} d & -b \\ -c & 0 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & b & 2 \\ 0 & d & 3 \end{pmatrix}$ $(d \neq 0)$

Answer: $U = A^{-1} = \begin{pmatrix} 1 & -bd^{-1} \\ 0 & d^{-1} \end{pmatrix}$

(d) $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

Answer: $U = \begin{pmatrix} 0 & 1 & 0 & 1 \\ \frac{1}{2} & -1 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & \frac{1}{2} \\ -1 & -1 & 1 & 0 \end{pmatrix}$

(e) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{pmatrix}$

Answer: $U = \begin{pmatrix} \frac{5}{4} & -\frac{3}{4} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & 0 \\ -2 & -1 & -2 & 1 \end{pmatrix}$

(3) For the matrices A in (1) and (2), determine the dimensions of $C(A)$, $R(A)$, $N(A)$, and $N(A^T)$. Find bases for these spaces.

(4) Let D, and P are 2×2 matrices with D diagonal and P invertible. Set $A = PDP^{-1}$. Show that A has two linearly independent eigenvectors.

Answer: Suppose $D = \operatorname{diag}(\lambda_1, \lambda_2)$ and let P_j be the i-th column of P. Then

$$AP_j = PDP^{-1}P_j = PD\mathbf{e}_j = \lambda P\mathbf{e}_j = \lambda_j P_j.$$

Therefore, P_1 and P_2 are eigenvectors of A. Since P is invertible, its columns, namely, P_1 and P_2, are linearly independent.

(5) Find all eigenvectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Is A diagonalizable?

Answer: The eigenvalues of A are the solutions of

$$(\lambda - 1)^2 = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 0.$$

Therefore, the only eigenvalue of A is $\lambda = 1$. The eigenvectors of A are the elements of

$$N(\lambda I - A) = N\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}.$$

In particular, A does not have two linearly independent eigenvectors. Therefore, by (4), A is not diagonalizable.

(6) Show that $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \cdots & 0 \end{pmatrix}^T \in \mathbf{R}^{n\times 1}$ is an eigenvector of every upper-triangular matrix $U \in \mathbf{R}^{n\times n}$. Show that $\mathbf{e}_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^T \in \mathbf{R}^{n\times 1}$ is an eigenvector of every lower-triangular matrix $L \in \mathbf{R}^{n\times n}$.

Answer: Let U_1 be the first column of U. Since U is upper-triangular, $U_1 = U_{11}\mathbf{e}_1$. Therefore,

$$U\mathbf{e}_1 = U_1 = U_{11}\mathbf{e}_1,$$

and $\mathbf{e}_1$ is an eigenvector of U with eigenvalue U_{11}.

(7) Let $A \in \mathbf{R}^{n\times n}$ and let $\lambda \in \mathbf{R}$. Show that

$$V_\lambda = \{\mathbf{x} \in \mathbf{R}^{n\times 1} : A\mathbf{x} = \lambda\mathbf{x}\}$$

is a subspace of $\mathbf{R}^{n\times 1}$.

Answer: We have $V_\lambda = N(\lambda I - A)$ and the nullspace of a matrix in $\mathbf{R}^{n\times n}$ is a subspace of $\mathbf{R}^{n\times 1}$.

You could also argue from the definition of subspace: If $\mathbf{x}, \mathbf{x}' \in V_\lambda$ and $t \in \mathbf{R}$, then

$$A(\mathbf{x}+\mathbf{x}') = A\mathbf{x} + A\mathbf{x}' = \mathbf{0}+\mathbf{0} = \mathbf{0} \quad \text{and} \quad A(t\mathbf{x}) = t(A\mathbf{x}) = t\mathbf{0} = \mathbf{0},$$

so $\mathbf{x}+\mathbf{x}', t\mathbf{x} \in V_\lambda$ and V_λ is closed under addition and scalar multiplication. Therefore, V_λ is a subspace of $\mathbf{R}^{n\times 1}$.

(8) Is W a subspace of $\mathbf{R}^{1\times 3}$?

(a) $W = \{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \in \mathbf{R}^{1\times 3} : 3x_1 - 2x_3 = 0\}$

Answer: Yes. The slick way to dispatch this one is to observe that

$$W = N(A)^T, \quad \text{where} \quad A = \begin{pmatrix} 3 & 0 & -2 \end{pmatrix},$$

and use the fact that nullspaces are subspaces.

Alternatively, you can argue directly from the definition. Let $x, x' \in W$ and let $t \in \mathbf{R}$. Then

$$3(x_1 + x_1') - 2(x_3 + x_3') = (3x_1 - 2x_3) + (3x_1' - 2x_3') = 0 + 0 = 0$$
$$\text{and} \quad 3(tx_1) - 2(tx_3) = t(3x_1 - 2x_3) = t(0) = 0.$$

Therefore,

$$x + x' = \begin{pmatrix} x_1 + x_1 = & x_2 + x_2' & x_3 + x_3' \end{pmatrix} \in W \quad \text{and} \quad tx = \begin{pmatrix} tx_1 & tx_2 & tx_3 \end{pmatrix} \in W.$$

(b) $W = \{\begin{pmatrix} s & s+t & s+2t \end{pmatrix} : s, t \in \mathbf{R}\}$

Answer: Yes. The slick way to dispatch this one is to observe that

$$W = R(A), \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix},$$

and use the fact that row spaces are subspaces.

Alternatively, you can argue directly from the definition. Let

$$x = \begin{pmatrix} s & s+t & s+2t \end{pmatrix}, \, x' = \begin{pmatrix} s' & s'+t' & s'+2t' \end{pmatrix} \in W$$

and let $a \in \mathbf{R}$. Then

$$\begin{aligned} x + x' &= \begin{pmatrix} s & s+t & s+2t \end{pmatrix} + \begin{pmatrix} s' & s'+t' & s'+2t' \end{pmatrix} \\ &= \begin{pmatrix} s+s' & (s+s')+(t+t') & (s+s')+2(t+t') \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} ax &= \begin{pmatrix} as & a(s+t) & a(s+2t) \end{pmatrix} \\ &= \begin{pmatrix} as & as+at & as+2at \end{pmatrix}. \end{aligned}$$

Thus, $s+s'$ and $t+t'$ witness $x+x' \in W$ and as and at witness $ax \in W$.

(c) $W = \{\begin{pmatrix} s-t & st & s+t \end{pmatrix} : s,t \in \mathbf{R}\}$

Answer: No. Let

$$x = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-0 & 1 \cdot 0 & 1+0 \end{pmatrix} \in W$$

and

$$x' = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0-1 & 0 \cdot 1 & 0+1 \end{pmatrix} \in W.$$

Then

$$x + x' = \begin{pmatrix} 0 & 0 & 2 \end{pmatrix}.$$

I claim that $x+x' \notin W$. We have $x+x' \in W$ if and only if the system of (nonlinear!) equations

$$(*) \qquad s+t=0, \quad st=0, \quad \text{and} \quad s-t=2$$

has a solution. The system of linear equations $s+t=0$, $s-t=2$ has unique solution $s=1$, $t=-1$. But then $st = 1(-1) = -1 \neq 0$. Thus, $(*)$ has no solution and $x+x' \notin W$. As W is not closed under addition, it is not a subspace of $\mathbf{R}^{1\times 3}$.

(d) $W = \{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} : x_1 \le x_2 \le x_3\}$

Answer: No. We have $x = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \in W$ but $-x = \begin{pmatrix} -1 & -2 & -3 \end{pmatrix} \notin W$. Thus, W is not closed under scalar multiplication and, consequently, is not a subspace of $\mathbf{R}^{1\times 3}$.

(9) Let V be the vector space of all polynomials with coefficients in $\mathbf{R}$ and let W be the subset of V consisting of all polynomials of degree ≥ 5. Is W a subspace of V?

Answer: No. We have $f(x) = x^5 + 1 \in W$ and $g(x) = -x^5 \in W$, but $f(x) + g(x) = 1 \notin W$. Thus, W is not closed under scalar multiplication and, consequently, is not a subspace of V.

(10) Let E be an elementary matrix. What is $\det E$? (The answer depends of the type of row operation encoded by the elementary matrix.)

Answer:

$E \longleftrightarrow$ interchange two different rows, $\quad \det E = -1$

$E \longleftrightarrow$ multiply a row by $k \neq 0$, $\quad \det E = k$

$E \longleftrightarrow$ add a multiple of one row to a different row, $\quad \det E = 1$

(11) Find the coordinate vector of $\mathbf{x}$ with respect to the basis $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ of V.

(a) $\mathbf{x} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$, $V = \mathbf{R}^{2\times 1}$

Answer: $[\mathbf{x}]_B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(b) $\mathbf{x} = \begin{pmatrix} 24 \\ 33 \\ 42 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $V = \mathbf{R}^{3\times 1}$

Answer: $[\mathbf{x}]_B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

(c) $\mathbf{x} = -\begin{pmatrix} \frac{1}{2} \\ 3 \\ \frac{3}{2} \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $V = \mathbf{R}^{3\times 1}$

Answer: $[\mathbf{x}]_B = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}$

(d) $\mathbf{x} = -\begin{pmatrix} 1 & 3 \\ -5 & 0 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,
$V = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$

Answer: $[\mathbf{x}]_B = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$

(12) Consider the basis $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ of $\mathbf{R}^{1\times 3}$ given by

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Find a matrices A and B such that $A[\mathbf{x}]_{\mathbf{a}} = \mathbf{x}$ and $B\mathbf{x} = [\mathbf{x}]_{\mathbf{a}}$, for all $\mathbf{x} \in \mathbf{R}^{3\times 1}$.

Answer: $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$, $\quad B = A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

(13) Consider the bases

$$B = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$C = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

of $\mathbf{R}^{2\times 2}$. Find a matrices X and Y such that $X[A]_C = [A]_B$ and $Y[A]_B = [A]_C$ for all matrices $A \in \mathbf{R}^{2\times 2}$.

Answer: $X = [C]_B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Y = [B]_C = [C]_B^{-1} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(14) Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{pmatrix}$.

(a) Show that

$$W = \{\mathbf{b} \in \mathbf{R}^{4\times 1} : A\mathbf{x} = \mathbf{b} \text{ has a solution}\}.$$

is a subspace of $\mathbf{R}^{4\times 1}$.

Answer: W is the column space $C(A)$ of A (why?); we know that the column space is a subspace of an $m \times n$ matrix is a subspace of $\mathbf{R}^{m\times 1}$.

(b) Find a basis of W.

Answer: We have

$$\mathrm{RREF}(A) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the leftmost three columns of A form a basis of W.

(15) Let A be the $n \times n$ defined by

$$A_{ij} = \begin{cases} 1 & \text{if } j \geq i, \\ 0 & \text{otherwise.} \end{cases}$$

Let I_j be the j-th column of the $n \times n$ identity matrix. Evaluate

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}}_{\text{all entries} = 1} AI_j.$$

(Your answer will depend on j.)

(16) Let $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{n\times p}$.

(a) Show that $C(AB) \subseteq C(A)$ and $R(AB) \subseteq R(B)$.

(b) Show that $\operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$.

(c) Give an examples of matrices A and B such that $\operatorname{rank} AB < \min\{\operatorname{rank} A, \operatorname{rank} B\}$.

Answer:

(a) Writing B_j for the j-th column of B, we have

$$AB = \begin{pmatrix} AB_1 & AB_2 & \cdots AB_p \end{pmatrix}.$$

Now AB_j is a linear combinations of the columns of A (why?), so $AB_j \in C(A)$. Therefore,

$$C(AB) = \operatorname{span}\{AB_1, AB_2, \ldots, AB_p\} \subset C(A).$$

Writing A^i for the i-th row of A, we have

$$AB = \begin{pmatrix} A^1 B \\ A^2 B \\ \vdots \\ A^m B \end{pmatrix}.$$

Now $A^i B$ is a linear combinations of the rows of B (why?), so $A^i B \in R(B)$. Therefore,

$$R(AB) = \operatorname{span}\{A^1 B, A^1 B, \ldots, A^m B\} \subset R(B).$$

(b) Since $C(AB) \subset C(A)$,

$$\operatorname{rank} AB = \dim C(AB) \leq \dim C(A) = \operatorname{rank} A.$$

Since $R(AB) \subset R(B)$,

$$\operatorname{rank} AB = \dim R(AB) \leq \dim R(B) = \operatorname{rank} B.$$

So $\operatorname{rank} AB \leq \operatorname{rank} A$ and $\operatorname{rank} AB \leq \operatorname{rank} B$. These two inequalities are equivalent to the single inequality $\operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$.

(c) Take $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$0 = \operatorname{rank} AB < 1 = \min\{1, 1\} = \min\{\operatorname{rank} A, \operatorname{rank} B\}.$$

(17) (a) Define what is meant by an *inverse* of an $n \times n$ matrix A.

(b) Define what it means for A to be *invertible*.

(c) Show that A can have at most one inverse.

(d) Show that A is the inverse of A^{-1}.

(e) Show that if A^{-1} and B^{-1} exist, then $(AB)^{-1}$ exists and equals $B^{-1}A^{-1}$.

Answer:

(a) An inverse of $A \in \mathbf{R}^{n \times n}$ is a matrix $B \in \mathbf{R}^{n \times n}$ such that $AB = BA = I$, where $I \in \mathbf{R}^{n \times n}$ is the identity matrix.

(b) A matrix is invertible if it has an inverse.

(c) If B and C are both inverses of A, then

$$C = IC = (BA)C = B(AC) = BI = B.$$

(d) Observe that the A and B play symmetric roles in the definition — see (a) — of the inverse of A.

(e) We compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

(18) (a) Let E be the elementary matrix corresponds to the elementary row operation "interchange rows p and q." Find an elementary row operation whose corresponding elementary matrix is E^T.

(b) Let E be the elementary matrix corresponds to the elementary row operation "Multiply row p by k." Find an elementary row operation whose corresponding elementary matrix is E^T.

(c) Let E be the elementary matrix corresponds to the elementary row operation "Add k times row p to row q." Find an elementary row operation whose corresponding elementary matrix is E^T.

(d) Conclude that E is an elementary matrix if and only if E^T is.

(e) Let E be an elementary matrix. Show that $\det E^T = \det E$. (Hint: Use (a)-(c).)

(f) Suppose that A is invertible. Using the fact that $\det BC = \det B \det C$ for all $B, C \in \mathbf{R}^{n\times n}$, show that $\det A = \det A^T$. (Hint: Invertible matrices can be written as products of elementary matrices.) Can you show that $\det A = \det A^T$ when A is not invertible?

Answer:

(a) Elementary matrices corresponding to elementary row operations of the type "interchange rows p and q" are symmetric. (Why?) Therefore, $E^T = E$ and E^T corresponds to this same elementary row operation.

(b) Elementary matrices corresponding to elementary row operations of the type "multiply row p by k" are symmetric. (Why?) Therefore, $E^T = E$ and E^T corresponds to this same elementary row operation.

(c) If E corresponds to the elementary row operation "add k times row p to row q" then E^T corresponds to the elementary row operation "add k times row q to row p". (Why?)

(d) We have shown that if E is an elementary matrix then so is E^T. Conversely, if E^T is an elementary matrix then, by the same argument, $(E^T)^T$ is too. But $(E^T)^T = E$.

(19) Find the eigenvalues and eigenvectors of the matrix.

(a) $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$

Answer: To find eigenvalues of A, we solve:

$$\begin{aligned} \begin{vmatrix} \lambda - 4 & 5 \\ -2 & \lambda + 3 \end{vmatrix} &= 0 \\ (\lambda - 4)(\lambda + 3) - 5(-2) &= 0 \\ \lambda^2 - \lambda - 2 &= 0 \\ (\lambda - 2)(\lambda + 1) &= 0 \\ \lambda &= -1,\ 2. \end{aligned}$$

The eigenspace of $\lambda = -1$ is

$$N(-1I - A) = N\begin{pmatrix} -5 & 5 \\ -2 & 2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

The eigenspace of $\lambda = 2$ is

$$N(-1I - A) = N\begin{pmatrix} -2 & 5 \\ -2 & 5 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 5 \\ 2 \end{pmatrix}\right\}.$$

(b) $A = \begin{pmatrix} 7 & -3 \\ -2 & 6 \end{pmatrix}$

Answer: To find eigenvalues of A, we solve:

$$\begin{aligned} \begin{vmatrix} \lambda - 7 & 3 \\ 2 & \lambda - 6 \end{vmatrix} &= 0 \\ (\lambda - 7)(\lambda - 6) - 3 \cdot 2 &= 0 \\ \lambda^2 - 13\lambda + 36 &= 0 \\ (\lambda - 4)(\lambda - 9) &= 0 \\ \lambda &= 4,\ 9. \end{aligned}$$

The eigenspace of $\lambda = 4$ is

$$N(4I - A) = N\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

The eigenspace of $\lambda = 9$ is

$$N(9I - A) = N\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 3 \\ -2 \end{pmatrix}\right\}.$$

(c) $A = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Answer: To find eigenvalues of A, we solve:

$$\begin{vmatrix} \lambda - 3 & -4 & -2 \\ 0 & \lambda - 1 & -2 \\ 0 & 0 & \lambda \end{vmatrix} = 0$$

$$(\lambda - 3)(\lambda - 1)\lambda = 0$$

$$\lambda = 0,\ 1,\ 3.$$

The eigenspace of $\lambda = 4$ is

$$N(0I - A) = N \begin{pmatrix} -3 & -4 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The eigenspace of $\lambda = 1$ is

$$N(1I - A) = N \begin{pmatrix} -2 & -4 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

The eigenspace of $\lambda = 3$ is

$$N(3I - A) = N \begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

(20) Let $A, B \in \mathbf{R}^{n\times n}$ and suppose that $AB = BA$. Show that if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ then so is $B\mathbf{x}$.

Answer: Suppose $\mathbf{x}$ is an eigenvector for A with eigenvalue λ, $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides on the left by B and using the fact that scalar multiplication commutes with matrix multiplication, we get

$$BA\lambda\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x}.$$

Using the (given) identity $AB = BA$, we can write the above in the form

$$A(B\mathbf{x}) = \lambda(B\mathbf{x}).$$

Therefore, by the defintions of eigenvalue and eigenvector, $B\mathbf{x}$ is an eigenvector of A with eigenvalue λ.

(21) Suppose the matrix $A \in \mathbf{R}^{2\times 2}$ has eigenvalues 1 and 2. What are the eigenvalues of $5I + A$?

Answer: That 1 and 2 are eigenvalues of A means that $I - A$ and $2I - A$ are not invertible. But

$$I - A = 6I - (5I + A) \quad \text{and} \quad 2I - A = 7I - (5I + A).$$

Therefore, 6 and 7 are eigenvalues of $5I + A$.

(22) Let $\mathbf{u} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$, and set $A = \mathbf{u}^T\mathbf{v}$. Find the eigenvalues and eigenvectors of A.

Answer: Observe that

$$A\mathbf{u}^T = (\mathbf{u}^T\mathbf{v})\mathbf{u}^T = \mathbf{u}^T(\mathbf{v}\mathbf{u}^T) = (4\cdot 1 + 5\cdot 2 + 6\cdot 3)\mathbf{u}^T = 32\mathbf{u}^T.$$

Therefore, $\mathbf{u}^T$ is an eigenvector of A with eigenvalue 32.

The matrix A has rank 1. (Why?) Therefore, $\dim N(A) = 2$. But $N(A)$ is the space of eigenvectors of A with eigenvalue 0. Computing this nullspace, we see that the eigenvalues for A with eigenvalue 0 are the vectors of the form

$$s\begin{pmatrix} -\frac{5}{4} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}.$$

(23) Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^{n\times 1}$ and set $A = \mathbf{u}\mathbf{v}^T$. Show that $\mathbf{u}$ is an eigenvector of A. What is the corresponding eigenvalue?

Answer: We have:

$$A\mathbf{u} = (\mathbf{u}\mathbf{v}^T)\mathbf{u} = \mathbf{u}(\mathbf{v}^T\mathbf{u}) = (\mathbf{v}\cdot\mathbf{u})\mathbf{u}.$$

Therefore, $\mathbf{u}$ is an eigenvector of A with eigenvalue $\mathbf{v}\cdot\mathbf{u}$.

(24) Let $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

(a) Diagonalize A, i.e., find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Answer: To find eigenvalues of A, we solve:

$$\begin{aligned}
\begin{vmatrix} \lambda - 4 & 3 \\ 1 & \lambda - 2 \end{vmatrix} &= 0 \\
(\lambda-4)(\lambda-2) - 3\cdot 1 &= 0 \\
\lambda^2 - 6\lambda + 5 &= 0 \\
(\lambda-1)(\lambda-5) &= 0 \\
\lambda &= 1,\ 5.
\end{aligned}$$

The eigenspace of $\lambda = 1$ is

$$N(1I - A) = N\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} = \operatorname{span}\{\mathbf{v}_1\}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The eigenspace of $\lambda = 5$ is

$$N(5I - A) = N\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = \operatorname{span}\{\mathbf{v}_5\}, \quad \mathbf{v}_5 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Therefore,

$$A\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_5 \end{pmatrix} = \begin{pmatrix} 1\mathbf{v}_1 & 5\mathbf{v}_5 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Solving for A, we get

$$A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_5 \end{pmatrix}^{-1}.$$

(b) Compute A^{100}.

Answer:

$$\begin{aligned} A^{100} &= \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1^{100} & 0 \\ 0 & 5^{100} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}^{-1} \\ &= -\frac{1}{4} \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5^{100} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 1 + 3 \cdot 5^{100} & 1 - 5^{100} \\ 3 - 3 \cdot 5^{100} & 3 + 5^{100} \end{pmatrix} \end{aligned}$$

(25) Show that $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \dfrac{1}{2} \begin{pmatrix} 3^n + 1 & 3^n - 1 \\ 3^n - 1 & 3^n 1 \end{pmatrix}$.

(26) (a) Find a formula for A^n, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Explain why the second column of A^n equals $A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) Let F_n be the Fibonacci sequence:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

Show that

$$A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}, \quad n \geq 1.$$

(d) Using the previous parts, show that

$$F_n = \frac{\rho^n - \bar{\rho}^n}{\rho - \bar{\rho}}, \quad \text{where} \quad \rho = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \bar{\rho} = \frac{1 - \sqrt{5}}{2}.$$

(27) Find the matrix $[T]$ of the linear transformation T satisfying

$$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 17 \end{pmatrix}.$$

Answer: We solve the matrix equation

$$A \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 12 & 17 \end{pmatrix},$$

giving

$$A = \begin{pmatrix} 5 & 7 \\ 12 & 17 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}.$$

(28) Consider the basis $B = (\mathbf{b}_1, \mathbf{b}_2)$ of $\mathbf{R}^{2\times 1}$ given by

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Let T be the linear transformation defined by $T(\mathbf{x}) = [\mathbf{x}]_B$, where $[\mathbf{x}]_B$ means the coordinate vector of $\mathbf{x}$ with respect to the basis B. Find the matrix $A \in \mathbf{R}^{2\times 2}$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $x \in \mathbf{R}^{2\times 1}$.

Answer: By definition of coordinate vector,

$$\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix} [x]_B = \mathbf{x}.$$

Therefore,

$$\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix}^{-1} \mathbf{x} = [\mathbf{x}]_B.$$

Thus, the answer is

$$A = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

(29) Let P_3 be the space of polynomials of degree ≤ 3. Is T a linear transformation, where $T : P_3 \to P_3$ is given by:

(a) $T(f(x)) = f(x) + x$

Answer: No: $T_1(2x^3) = 2x^3 + x \neq 2x^3 + 2x = 2(x^3 + x) = 2T_1(x^3)$.

(b) $T_2(f(x)) = xf(x)$

Answer: Yes:

$$T_2(f(x) + g(x)) = x(f(x) + g(x)) = xf(x) + xg(x) = T_2(f(x)) + T_2(g(x))$$
$$T_2(af(x)) = x(af(x)) = a(xf(x)) = aT_2(f(x)).$$

(c) $T_3(f(x)) = f'(x)$ (derivative)

Answer: Yes – see your calculus notes:

$$T_3(f(x) + g(x)) = (f(x) + g(x))' = f'(x) + g'(x) = T_3(f(x)) + T_3(g(x))$$
$$T_3(af(x)) = (af(x))' = af'(x) = aT_3(f(x)).$$

(d) $T_4(f(x)) = xf'(x)$

Answer: Yes: $T_4(f(x)) = T_2(T_3(x))$ and compositions of linear transformations are linear.