MATH 213 – LINEAR ALGEBRA – EXERCISES

1. Linear equations

Terminology:

- A linear equation is called *degenerate* if all of its coefficients are equal to 0. (Its constant term need not be 0.)
- If k is a nonzero number, the linear equations
$$a_1x_1 + \cdots a_nx_n = b \qquad \text{and} \qquad k(a_1x_1 + \cdots a_nx_n) = kb$$
said to be *proportional*.

(1) Consider the linear equation $x - 9y = 5$.
  (a) Find the general solution corresponding to the choice of parameter $x = s$.
  (b) Find the general solution corresponding to the choice of parameter $y = t$.

(2) Find general solutions of the following linear equations:

  (a) $x - y = 0$
  (b) $\pi y - x = \pi^2$
  (c) $2x = -3y$
  (d) $x + 0y = 1$
  (e) $0x + 2y = 5$
  (f) $x - y + z = 3$
  (g) $x + y = y - z$
  (h) $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$.

(3) Find a linear equation with the given general solution:

  (a) $(x, y) = (s, 2s)$
  (b) $(x, y) = (3 - 2t, t)$
  (c) $(x, y) = (2 - t, 2 + t)$
  (d) $(x, y) = (4s - 5, 5s - 4)$
  (e) $(x, y) = (1, t)$
  (f) $(x, y, z) = (s, t, s + t)$
  (g) $(x, y, z) = (1 - s, 2 - 3t, 4t - 5s - 1)$
  (h) $(x, y, z) = (2s - t, \pi, s + t + 1)$

(4) Both $(x, y) = (2s+1, s-1)$ and $(x, y) = (17+2t, 7+t)$ are general solutions of the equation $x-2y = 3$.
  (a) What values of s and t give the solution $(x, y) = (7, 2)$?
  (b) Let s_0 be a fixed value of the parameter s. Find the value of t corresponding to the solution $(x, y) = (2s_0 + 1, s_0 - 1)$.
  (c) Let t_0 be a fixed value of the parameter t. Find the value of s corresponding to the solution $(x, y) = (17 + 2t_0, 7 + t_0)$.

(5) Prove the statement or provide a counterexample.
  (a) If two linear equations in n variables are proportional then they have the same solutions.
  (b) $(*)$ If two linear equations in n variables have the same solutions then they are proportional. (This might be tricky without using some theory that we'll develop.)

(6) (a) Find a nondegenerate linear equation in x, y whose solutions include $(2, 1)$ and $(-1, 4)$.
  (b) Find infinitely many linear equations in x, y whose solutions include $(2, 1)$ such that no two of these equations are proportional.

(7) (a) Is there a nondegenerate linear equation in x, y whose solutions include $(0, 0)$, $(1, 0)$ and $(0, 1)$?

(b) Can a degenerate linear equation have a unique solution?
(c) What are the possible graphs of degenerate linear equations in x, y?

(8) ($*$) (This one might be tricky without some techniques we'll develop.)
(a) Find a nondegenerate linear equation in x, y, z whose solutions include $(1, 2, 3)$ and $(4, 5, 6)$.
(b) Find infinitely many linear equations in x, y, z whose solutions include $(1, 2, 3)$ and $(4, 5, 6)$ no two of which are proportional.

2. Systems of linear equations

(1) Find the reduced row echelon form of the matrix:

(a) $\begin{pmatrix} 9 & 9 & 7 & 3 \\ 5 & 5 & 4 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 3 & 6 \\ -1 & -2 & -1 \\ 0 & 6 & 13 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & -16 & 1 \\ 5 & -20 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -4 \\ 0 & -1 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & 5 & -2 \\ 1 & 2 & 0 \\ 4 & 10 & -3 \end{pmatrix}$

(2) Solve the system:

(a) $\begin{array}{rrcr} 15x & -2x & = & -41 \\ -7x & +y & = & 19 \end{array}$

(b) $\begin{array}{rrrcr} x_1 & -6x_2 & -12x_3 & = & 13 \\ & x_2 & +2x_3 & = & -2 \end{array}$

(c) $\begin{array}{rrrcr} 67x_1 & -134x_2 & -201x_3 & = & 26 \\ -18x_1 & +36x_2 & +54x_3 & = & -7 \end{array}$

(d) $\begin{array}{rrrcr} -7x & +10y & -20z & = & 0 \\ x & -2y & +4z & = & 1 \\ 5x & -7y & +14z & = & 0 \end{array}$

(e) $\begin{array}{rrrcr} -5x_1 & -3x_2 & -9x_3 & = & 10 \\ 4x_1 & +2x_2 & +5x_3 & = & -9 \\ 2x_1 & +x_2 & +2x_3 & = & -5 \end{array}$

(f) $\begin{array}{rrrrrrcr} 4x_1 & -x_2 & & -2x_4 & +7x_5 & & = & 6 \\ -3x_1 & +x_2 & & +2x_4 & -5x_5 & & = & -4 \\ -6x_1 & +x_2 & +x_3 & +x_4 & -13x_5 & -3x_6 & = & -12 \end{array}$

(g) $\begin{array}{rrrrrrcr} 6x_1 & & +6x_3 & +6x_4 & +18x_5 & +2x_6 & = & 26 \\ 3x_1 & +x_2 & +6x_3 & +6x_4 & +11x_5 & & = & 9 \\ 27x_1 & & +27x_3 & +27x_4 & +81x_5 & +9x_6 & = & 117 \\ -11x_1 & -4x_2 & -23x_3 & -23x_4 & -41x_5 & & = & -33 \end{array}$

(3) For what values of c does the system

$$\begin{array}{rrcl} x & +2y & = & 5 \\ 3x & +6y & = & c \end{array}$$

have a unique solution? No solution? Infinitely many solutions?

(4) For what values of a and b does

$$\begin{array}{ccccccc} ax & & & + & bz & = & 2 \\ ax & + & 2y & + & az & = & b \\ bx & + & 2y & + & az & = & a \end{array}$$

have a unique solution? Infinitely many solutions? No solutions?

(5) Suppose that a system of 5 equations in 3 variables has a unique solution. What is the reduced row echelon form of its coefficient matrix?

(6) Write down all possible forms of $m \times n$ matrices in reduced row echelon form for each choice of m, n with $1 \leq m, n \leq 3$. Sort your list by rank. Use $*$ to indicate an arbitrary number. For example, a 2×2 matrix in reduced row echelon form must look like one of the following:

$$\begin{array}{ll} \text{rank 0:} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \text{rank 1:} & \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \text{rank 2:} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

(7) Consider the following augmented matrix in which $*$ denotes any number and $\square$ denotes any *nonzero* number:

$$\begin{pmatrix} \square & * & * & * & * & * \\ 0 & \square & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \square & 0 \\ 0 & 0 & 0 & 0 & * & \square \end{pmatrix}$$

Is the corresponding system of equations consistent?

(8) (a) Under what conditions on a and b does the homogeneous system

$$\begin{array}{ccccc} x & + & ay & = & 0 \\ x & + & by & = & 0 \end{array}$$

have a unique solution?

(b) Under what conditions on a, b and c does the homogeneous system

$$\begin{array}{ccccccc} x & + & ay & + & a^2z & = & 0 \\ x & + & by & + & b^2z & = & 0 \\ x & + & cy & + & c^2z & = & 0 \end{array}$$

have a unique solution?

(c) ($*$) Under what conditions on $a_1, \ldots, a_{n-1}$ does the $n \times n$ homogeneous system

$$x_0 + a_i x_1 + a_i^2 x_2 + \cdots + a_i^{n-1} x_{n-1} = 0 \qquad (1 \leq i \leq n)$$

have a unique solution?

(9) Let A be an $m \times n$ matrix in (reduced) row echelon form. Let p be an integer with $1 \leq p \leq m-1$ and let q be an integer with $1 \leq q \leq n-1$. Prove the statement or provide a counterexample:

(a) Let A' be the matrix consisting of the top p rows of A. Then A' is in (reduced) row echelon form.

(b) Let A' be the matrix consisting of the bottom p rows of A. Then A' is in (reduced) row echelon form.

(c) Let A' be the matrix consisting of the leftmost q columns of A. Then A' is in (reduced) row echelon form.

(d) Let A' be the matrix consisting of the rightmost q columns of A. Then A' is in (reduced) row echelon form.

(10) Let A and $(A \mid \mathbf{b})$, respectively, be the coefficient and augmented matrix of a system of linear equations. Suppose that

$$\operatorname{rank}(A \mid \mathbf{b}) > \operatorname{rank} A.$$

Explain why the system is inconsistent.

(11) Let B be an $m \times q$ matrix such that the submatrix of B consisting of its first $q-1$ columns is in reduced row echelon form. Show that there is a sequence of elementary row operations transforming B into reduced row echelon form.

(12) Write a computer program (in your favorite language) to reduce a matrix to (reduced) row echelon form.

3. Matrix arithmetic

(1) Express $\mathbf{w}$ as a linear combination of the $\mathbf{v}_i$, or show that there is no such expression.

(a) $\mathbf{w} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -20 \\ 24 \end{pmatrix}$

(b) $\mathbf{w} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 \begin{pmatrix} 7 \\ 6 \end{pmatrix}$

(c) $\mathbf{w} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

(d) $\mathbf{w} = \begin{pmatrix} 13 \\ -21 \\ 34 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ 5 \\ -8 \end{pmatrix}$

(e) $\mathbf{w} = \begin{pmatrix} 13 \\ -21 \\ 34 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(2) If A is invertible, find A^{-1} and write A as a product of elementary matrices. If A is not invertible, find an invertible matrix B such that BA is in reduced row echelon form and write B as a product of elementary matrices.

(a) $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & -4 \\ 0 & 13 \end{pmatrix}$

(d) $\begin{pmatrix} -5 & 15 & 4 \\ 29 & -87 & -23 \end{pmatrix}$

(e) $\begin{pmatrix} -1 & 2 & 0 \\ -8 & 16 & 1 \\ 2 & -4 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 2 & -2 \\ 2 & -2 & 3 \\ -4 & 5 & -6 \end{pmatrix}$

(g) $\begin{pmatrix} 13 & -4 & -28 \\ -6 & 2 & 13 \\ -3 & 1 & 6 \end{pmatrix}$

(h) $\begin{pmatrix} -6 & 9 & 28 \\ 1 & 0 & -5 \\ 5 & -4 & -24 \end{pmatrix}$

(i) $\begin{pmatrix} -7 & -7 & -1 & 42 \\ -2 & -2 & 1 & 4 \\ -1 & -1 & 1 & -1 \end{pmatrix}$

(3) For $1 \le i, j \le 2$, let $\mathbf{e}^{ij} = (e^{ij}_{k\ell})$ be the 2×2 matrix such that

$$e^{ij}_{k\ell} = \begin{cases} 1 & \text{if } i = k \text{ and } j = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write down the matrices $\mathbf{e}^{ij}$ for all i, j. (That is, parse and understand the definition of $\mathbf{e}^{ij}$.)
(b) Compute $(\mathbf{e}^{ij})^2$ for all i, j.
(c) A "number" x is called *idempotent* if $x^2 = x$ and *nilpotent* if $x^n = 0$ for some n. What are the nonidentity ($\neq 1$) idempotent elements of $\mathbf{R}$? What are the nonzero nilpotents elements of $\mathbf{R}$?

(d) Observe that the set of 2×2 matrices contains both nonidentity idempotents and nonzero nilpotents.
(e) Can you find a 3×3 matrix such that $A^3 = 0$ but $A^2 \neq 0$?
(f) (*) Can you find a 2×2 matrix A such that $A^3 = 0$ but $A^2 \neq 0$?

(4) (a) Let A, A', B and B' be $m \times n$, $m' \times n$, $n \times p$ and $n \times p'$ matrices, respectively. Show that

$$\text{(i)} \quad A\begin{pmatrix} B & B' \end{pmatrix} = \begin{pmatrix} AB & AB' \end{pmatrix} \qquad \text{and} \qquad \text{(ii)} \quad \begin{pmatrix} A \\ A' \end{pmatrix} B = \begin{pmatrix} AB \\ A'B \end{pmatrix}.$$

Solution skeleton:

Let's prove (i).

- The goal is to prove equality of two matrices. To do this, we show that their (k, ℓ)-entries are equal for an arbitrary pair (k, ℓ).
- Mini-exercise: Show that $A\begin{pmatrix} B & B' \end{pmatrix}$ and $\begin{pmatrix} AB & AB' \end{pmatrix}$ both have size $m \times (p + p')$.
- Let (k, ℓ) be such that $1 \leq k \leq m$ and $1 \leq \ell \leq p + p'$. We need to show that the (k, ℓ)-entry of $A\begin{pmatrix} B & B' \end{pmatrix}$ equals the (k, ℓ)-entry of $\begin{pmatrix} AB & AB' \end{pmatrix}$.
- Introduce notation: Let a_{ij}, b_{ij} and b'_{ij} be the (i, j)-entries of A, B and B', respectively.
- *Evaluate the (k, ℓ)-entry of $A\begin{pmatrix} B & B' \end{pmatrix}$*: The (k, ℓ)-entry of $A\begin{pmatrix} B & B' \end{pmatrix}$ is the k-th row of A times the ℓ-th column of $\begin{pmatrix} B & B' \end{pmatrix}$. The k-th row of A is $\begin{pmatrix} a_{k1} & \cdots a_{kn} \end{pmatrix}$. The ℓ-th column of $\begin{pmatrix} B & B' \end{pmatrix}$ is

$$\begin{cases} \text{the } \underline{\qquad\quad}\text{-th column of } B & \text{if } 1 \leq \ell \leq \underline{\qquad\quad}, \\ \text{the } \underline{\qquad\quad}\text{-th column of } B' & \text{if } \underline{\qquad\quad} < \ell \leq p + p'. \end{cases}$$

Therefore, by the definition of matrix multiplication, the (k, ℓ)-entry of $A\begin{pmatrix} B & B' \end{pmatrix}$ is

$$\begin{cases} a_{\_\_}b_{\_\_} + a_{\_\_}b_{\_\_} + \cdots + a_{\_\_}b_{\_\_} & \text{if } 1 \leq \ell \leq \underline{\qquad\quad}, \\ a_{\_\_}b'_{\_\_} + a_{\_\_}b'_{\_\_} + \cdots + a_{\_\_}b'_{\_\_} & \text{if } \underline{\qquad\quad} < \ell \leq p + p'. \end{cases}$$

- *Evaluate the (k, ℓ)-entry of $\begin{pmatrix} AB & AB' \end{pmatrix}$*: The (k, ℓ)-entry of $\begin{pmatrix} AB & AB' \end{pmatrix}$ is the

$$\begin{cases} \text{the } (\underline{\qquad\quad}, \underline{\qquad\quad})\text{-entry of } AB & \text{if } 1 \leq \ell \leq \underline{\qquad\quad}, \\ \text{the } (\underline{\qquad\quad}, \underline{\qquad\quad})\text{-entry of } AB' & \text{if } \underline{\qquad\quad} < \ell \leq p + p'. \end{cases}$$

Therefore, by the definition of matrix multiplication, the (k, ℓ)-entry of $\begin{pmatrix} AB & AB' \end{pmatrix}$ is

$$\begin{cases} a_{\_\_}b_{\_\_} + a_{\_\_}b_{\_\_} + \cdots + a_{\_\_}b_{\_\_} & \text{if } 1 \leq \ell \leq \underline{\qquad\quad}, \\ a_{\_\_}b'_{\_\_} + a_{\_\_}b'_{\_\_} + \cdots + a_{\_\_}b'_{\_\_} & \text{if } \underline{\qquad\quad} < \ell \leq p + p'. \end{cases}$$

- Put it all together.

(b) Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be the rows of A and let $\mathbf{b}_1, \ldots, \mathbf{b}_p$ be the columns of B. Show that

$$AB = \begin{pmatrix} \mathbf{a}_1 B \\ \vdots \\ \mathbf{a}_m B \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 \\ \vdots \\ A\mathbf{b}_p \end{pmatrix}.$$

(c) Let A_i be an $m_i \times n$ matrix, $1 \leq i \leq r$ and let B_j be an $n \times p_j$ matrix, $1 \leq j \leq s$. Show that for any $m \times n$ matrix A and any $n \times p$ matrix B,

$$A\begin{pmatrix} B_1 & \cdots & B_s \end{pmatrix} = \begin{pmatrix} AB_1 & \cdots & AB_s \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} A_1 \\ \vdots \\ A_r \end{pmatrix} B = \begin{pmatrix} A_1 B \\ \vdots \\ A_r B \end{pmatrix}.$$

(5) Let $\mathbf{e}_i$ be the *i-th standard basis vector* of dimension m (i.e., with m columns):

$$\mathbf{e}_i = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \qquad \text{(the 1 is in column } i\text{).}$$

Let A be an $m \times n$ matrix. What is $\mathbf{e}_i A$?

Solution skeleton: This problem has two parts: (1) find a candidate for $\mathbf{e}_i A$ and (2) prove that $\mathbf{e}_i A$ equals your candidate. To find a candidate, make up and calculate some small examples to see what's going on:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{pmatrix} = \begin{pmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \end{pmatrix} = \begin{pmatrix} \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{pmatrix} = \begin{pmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{pmatrix}$$

You should conclude that the $\mathbf{e}_i A$ is the i-th row $\mathbf{a}_i$ of A.

Of course, a couple of motivating examples aren't proof. You need to prove, in general, that $\mathbf{e}_i A = \mathbf{a}_i$. A strategy for proving that two matrices (like $\mathbf{e}_i A$ and $\mathbf{a}_i$) are equal is to let (i, j) be an arbitrary pair with $1 \le i \le m$ and $1 \le j \le n$ and argue that the (i, j)-entry of one matrix equals the (i, j)-entry of the other.

We introduce some notation. Write a_{ij} for the (i, j)-entry of the matrix A. Write e_{ij} for the entry of the row vector $\mathbf{e}_i$ in the j-th column. Then

(†)
$$e_{ij} = \begin{cases} \underline{\qquad} & \text{if } i \ne j, \\ \underline{\qquad} & \text{if } i = j. \end{cases}$$

By definition of matrix multiplication, $\mathbf{e}_i A$ is an _______-dimensional _______ (row/column) vector whose j-th entry is

$$e_{i\_}a_{\_\_} + e_{i\_}a_{\_\_} + \cdots + e_{i\_}a_{\_\_}.$$

By (†),

$$e_{i\_}a_{\_\_} + e_{i\_}a_{\_\_} + \cdots + e_{i\_}a_{\_\_} = \underline{\qquad}.$$

Summarizing the above calculation, we have shown that the j-th entry of the _______ (row/column) vector $\mathbf{e}_i A$ is _______:

$$\mathbf{e}_i A = \begin{pmatrix} a_{\_\_} & a_{\_\_} & \cdots & a_{\_\_} \end{pmatrix}$$

Thus, $\mathbf{e}_i A$ is simply the _______-th _______ (row/column) of A.

(6) Let A be an $m \times n$ matrix. Let O_k be an elementary row operation of type k, $k \in \{1, 2, 3\}$. Let A' be the matrix you get by applying O_k to A. Show that

$$A_k = E_k A.$$

Here are some suggestions:

- Follow the proof we did in class for $k = 1$.
- Let O_1 be a type 1 row operation, interchanging rows p and q, $p \ne q$, say. Let O_2 be a type 2 row operation, multiplying row p by $c \ne 0$, say. Let O_3 be a type 3 row operation, say adding c times row p to row q, $p \ne q$.

- Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be the rows of A. Let $\mathbf{a}'_1, \ldots, \mathbf{a}'_m$ be the rows of A'. For each k, express $\mathbf{a}'_i$ in terms of the $\mathbf{a}_j$. Let $\mathbf{v}_i$ be the i-th row of E_k.
- Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the rows of E_k. For each k, express $\mathbf{v}_i$ in terms of the $\mathbf{e}_j$.
- For each k, describe the i-th row of $E_k A$; (4b) might be useful here.

(7) Show that every elementary matrix is invertible, and that its inverse is an elementary matrix.

(8) Let

$$\mathbf{b}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

For $i = 1, 2, 3$, solve the matrix equation $A\mathbf{x} = \mathbf{b}_i$. (Suggestion: Find A^{-1} use that to solve for $\mathbf{x}$. Don't do a separate Gaussian elimination procedure for each i.)

(9) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & k \end{pmatrix}.$$

(a) Find all k for which A is invertible. For these k:
   (i) Find A^{-1}.
   (ii) Write A as a product of elementary matrices.

(b) For k for which A is not invertible, write A in the form UR where U is invertible and R is in reduced row echelon form. Express U as a product of elementary matrices.

(10) Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & a & a^2 \end{pmatrix}$$

For all a, find an invertible matrix U such that UA is in reduced row echelon form.