# MATH 213 – REVIEW PROBLEMS

(1) Find the reduced row echelon form of $A$. Solve the equation $A\mathbf{x} = \mathbf{0}$.

(a) $A = \begin{pmatrix} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{pmatrix}$

(2) Find an invertible matrix $U$ such that $UA$ is in reduced row echelon form.

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ $(bc \neq 0)$

(c) $A = \begin{pmatrix} 1 & b & 2 \\ 0 & d & 3 \end{pmatrix}$ $(d \neq 0)$

(d) $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

(e) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{pmatrix}$

(3) For the matrices $A$ in (1) and (2), determine the dimensions of $C(A)$, $R(A)$, $N(A)$, and $N(A^T)$. Find bases for these spaces.

(4) Let $D$, and $P$ are $2 \times 2$ matrices with $D$ diagonal and $P$ invertible. Set $A = PDP^{-1}$. Show that $A$ has two linearly independent eigenvectors.

(5) Find all eigenvectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Is $A$ diagonalizable?

(6) Show that $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \cdots & 0 \end{pmatrix}^T \in \mathbf{R}^{n\times 1}$ is an eigenvector of every upper-triangular matrix $U \in \mathbf{R}^{n\times n}$. Show that $\mathbf{e}_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^T \in \mathbf{R}^{n\times 1}$ is an eigenvector of every lower-triangular matrix $L \in \mathbf{R}^{n\times n}$.

(7) Let $A \in \mathbf{R}^{n\times n}$ and let $\lambda \in \mathbf{R}$. Show that

$$V_\lambda = \{\mathbf{x} \in \mathbf{R}^{n\times 1} : A\mathbf{x} = \lambda\mathbf{x}\}$$

is a subspace of $\mathbf{R}^{n\times 1}$.

(8) Is $W$ a subspace of $\mathbf{R}^{1\times 3}$?

(a) $W = \{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \in \mathbf{R}^{1\times 3} : 3x_1 - 2x_3 = 0\}$

(b) $W = \{\begin{pmatrix} s & s+t & s+2t \end{pmatrix} : s, t \in \mathbf{R}\}$

(c) $W = \{\begin{pmatrix} s-t & st & s+t \end{pmatrix} : s, t \in \mathbf{R}\}$

(d) $W = \{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} : x_1 \le x_2 \le x_3\}$

(9) Let $V$ be the vector space of all polynomials with coefficients in $\mathbf{R}$ and let $W$ be the subset of $V$ consisting of all polynomials of degree $\ge 5$. Is $W$ a subspace of $V$?

(10) Let $E$ be an elementary matrix. What is $\det E$? (The answer depends of the type of row operation encoded by the elementary matrix.)

(11) Find the coordinate vector of $\mathbf{x}$ with respect to the basis $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ of $V$.

(a) $\mathbf{x} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$, $V = \mathbf{R}^{2\times 1}$

(b) $\mathbf{x} = \begin{pmatrix} 24 \\ 33 \\ 42 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $V = \mathbf{R}^{3\times 1}$

(c) $\mathbf{x} = -\begin{pmatrix} \frac{1}{2} \\ 3 \\ \frac{3}{2} \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $V = \mathbf{R}^{3\times 1}$

(d) $\mathbf{x} = -\begin{pmatrix} 1 & 3 \\ -5 & 0 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,
$V = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$

(12) Consider the basis $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ of $\mathbf{R}^{1\times 3}$ given by

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Find a matrices $A$ and $B$ such that $A[\mathbf{x}]_{\mathbf{a}} = \mathbf{x}$ and $B\mathbf{x} = [\mathbf{x}]_{\mathbf{a}}$, for all $\mathbf{x} \in \mathbf{R}^{3\times 1}$.

(13) Consider the bases

$$B = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$C = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

of $\mathbf{R}^{2\times 2}$. Find a matrices $X$ and $Y$ such that $X[A]_C = [A]_B$ and $Y[A]_B = [A]_C$ for all matrices $A \in \mathbf{R}^{2\times 2}$.

(14) Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{pmatrix}$.

(a) Show that

$$W = \{\mathbf{b} \in \mathbf{R}^{4\times 1} : A\mathbf{x} = \mathbf{b} \text{ has a solution}\}.$$

is a subspace of $\mathbf{R}^{4\times 1}$.

(b) Find a basis of $W$.

(15) Let $A$ be the $n \times n$ defined by

$$A_{ij} = \begin{cases} 1 & \text{if } j \geq i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $I_j$ be the $j$-th column of the $n \times n$ identity matrix. Evaluate

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}}_{\text{all entries} = 1} AI_j.$$

(Your answer will depend on $j$.)

(16) Let $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{n\times p}$.

(a) Show that $C(AB) \subseteq C(A)$ and $R(AB) \subseteq R(B)$.

(b) Show that $\operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$.

(c) Give an examples of matrices $A$ and $B$ such that $\operatorname{rank} AB < \min\{\operatorname{rank} A, \operatorname{rank} B\}$.

(17) (a) Define what is meant by an *inverse* of an $n \times n$ matrix $A$.

(b) Define what it means for $A$ to be *invertible*.

(c) Show that $A$ can have at most one inverse.

(d) Show that $A$ is the inverse of $A^{-1}$.

(e) Show that if $A^{-1}$ and $B^{-1}$ exist, then $(AB)^{-1}$ exists and equals $B^{-1}A^{-1}$.

(18) (a) Let $E$ be the elementary matrix corresponds to the elementary row operation "interchange rows $p$ and $q$." Find an elementary row operation whose corresponding elementary matrix is $E^T$.

(b) Let $E$ be the elementary matrix corresponds to the elementary row operation "Multiply row $p$ by $k$." Find an elementary row operation whose corresponding elementary matrix is $E^T$.

(c) Let $E$ be the elementary matrix corresponds to the elementary row operation "Add $k$ times row $p$ to row $q$." Find an elementary row operation whose corresponding elementary matrix is $E^T$.

(d) Conclude that $E$ is an elementary matrix if and only if $E^T$ is.

(e) Let $E$ be an elementary matrix. Show that $\det E^T = \det E$. (Hint: Use (a)-(c).)

(f) Suppose that $A$ is invertible. Using the fact that $\det BC = \det B \det C$ for all $B, C \in \mathbf{R}^{n\times n}$, show that $\det A = \det A^T$. (Hint: Invertible matrices can be written as products of elementary matrices.) Can you show that $\det A = \det A^T$ when $A$ is not invertible?

(19) Find the eigenvalues and eigenvectors of the matrix.

(a) $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 7 & -3 \\ -2 & 6 \end{pmatrix}$

(c) $A = \begin{pmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(20) Let $A, B \in \mathbf{R}^{n\times n}$ and suppose that $AB = BA$. Show that if $\mathbf{x}$ is an eigenvector of $A$ with eigenvalue $\lambda$ then so is $B\mathbf{x}$.

(21) Suppose the matrix $A \in \mathbf{R}^{2\times 2}$ has eigenvalues 1 and 2. What are the eigenvalues of $5I + A$?

(22) Let $\mathbf{u} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$, and set $A = \mathbf{u}^T\mathbf{v}$. Find the eigenvalues and eigenvectors of $A$.

(23) Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^{n\times 1}$ and set $A = \mathbf{u}\mathbf{v}^T$. Show that $\mathbf{u}$ is an eigenvector of $A$. What is the corresponding eigenvalue?

(24) Let $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

(a) Diagonalize $A$, i.e., find an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.

(b) Compute $A^{100}$.

(25) Show that $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \frac{1}{2}\begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n1 \end{pmatrix}$.

(26) (a) Find a formula for $A^n$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Explain why the second column of $A^n$ equals $A^{n-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) Let $F_n$ be the Fibonacci sequence:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

Show that

$$A^{n-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}, \quad n \geq 1.$$

(d) Using the previous parts, show that

$$F_n = \frac{\rho^n - \bar{\rho}^n}{\rho - \bar{\rho}}, \quad \text{where} \quad \rho = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \bar{\rho} = \frac{1-\sqrt{5}}{2}.$$

(27) Find the matrix $[T]$ of the linear transformation $T$ satisfying

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 17 \end{pmatrix}.$$

(28) Consider the basis $B = (\mathbf{b}_1, \mathbf{b}_2)$ of $\mathbf{R}^{2\times 1}$ given by

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Let $T$ be the linear transformation defined by $T(\mathbf{x}) = [\mathbf{x}]_B$, where $[\mathbf{x}]_B$ means the coordinate vector of $\mathbf{x}$ with respect to the basis $B$. Find the matrix $A \in \mathbf{R}^{2\times 2}$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $x \in \mathbf{R}^{2\times 1}$.

(29) Let $P_3$ be the space of polynomials of degree $\leq 3$. Is $T$ a linear transformation, where $T : P_3 \to P_3$ is given by:

(a) $T(f(x)) = f(x) + x$

(b) $T_2(f(x)) = xf(x)$

(c) $T_3(f(x)) = f'(x)$ (derivative)

(d) $T_4(f(x)) = xf'(x)$