ASSIGNMENT 1 – SOLUTIONS

(3g) We want to find a, b and c and d such that $ax + by + cz = d$ has general solution

$$(x, y, z) = (1 - s, 2 - 3t, 4t - 5s - 1).$$

Substituting, we see that the equation

$$a(1 - s) + b(2 - 3t) + c(4t - 5s - 1) = d$$

must be satisfied. Rerranging terms, we get

$$s(a - 5c) + t(-3b + 4c) = d - a - 2b + c.$$

This identity holds if $-a - 5c = 0$, $-3b + 4c = 0$ and $d - a - 2b + c = 0$ since, in this case, both sides evaluate to 0. Thus, we want to solve the following system of 3 equations in 4 variables:

$$\begin{array}{rcrcrcrcr} -a & & & - & 5c & & & = & 0 \\ & - & 3b & + & 4c & & & = & 0 \\ a & + & 2b & - & c & - & d & = & 0 \end{array}$$

The reduced row echelon form of this system is

$$\begin{array}{ccccccccc} a & & & & & - & \frac{3}{2}d & = & 0 \\ & & b & & & + & \frac{2}{5}d & = & 0 \\ & & & & c & + & \frac{3}{10}d & = & 0. \end{array}$$

Thus, d is a free variable, so we can find a solution with $d = 10$ (or any other value):

$$(a, b, c, d) = (15, -4, -3, 10).$$

Thus, $(x, y, z) = (1 - s, 2 - 3t, 4t - 5s - 1)$ is a general solution of

$$15x - 4y - 3z = 10.$$

(4b) Since $(2s + 1, s - 1)$ is a solution of our equation for any s and $(x, y) = (17 + 2t, 7 + t)$ is a general solution, there must be a t (depending on s) such that

$$(2s + 1, s - 1) = (17 + 2t, 7 + t).$$

This is equivalent to the pair of equations $2s + 1 = 17 + 2t$ and $s - 1 = 7 + t$. Both have solution

$$t = s - 8.$$

(6) (a) The line through $(2, 1)$ and $(-1, 4)$ has slope

$$\frac{4 - 1}{-1 - 2} = -1.$$

The line through $(2, 1)$ and (x, y) with slope -1 is given by

$$\frac{y - 1}{x - 2} = -1 \iff y = -x + 3.$$

(b) The line through $(2, 1)$ with slope m is given by

$$\frac{y - 1}{x - 2} = m \iff y = mx - 2m + 1.$$

Since m can take on any value, this gives infinitely many linear equations one of whose solutions is $(2, 1)$.

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(7) (a) The graph of a nondegenerate linear equation is a line in the xy-plane. There is no line that passes through all three of the points $(0,0)$, $(1,0)$ and $(1,1)$. Those points form the form the vertices of a triangle. Therefore, there can be no nondegenerate linear equation whose solution set contains alll three.

(b) A degenerate linear equation has the form $0x + 0y = c$. If $c = 0$ then the solution set of this equation is *all* pairs (x, y). If $c \neq 0$ then the solution set is empty. In neither case is there exactly one element in the solution set.

(c) The graph of $0x + 0y = 0$ is the entire xy-plane. The graph of $0x + 0y = c$ for $c \neq 0$ is empty.

(2f) The augmented matrix of the system has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & -2 & -3 & -2 \end{pmatrix}$$

The leading variables are x_1, x_2 and x_3; the free variables are x_4, x_5 and x_6. We assign parameters to the free variables:

$$x_4 = s, \qquad x_5 = t, \qquad x_6 = u.$$

Solving for the leading variables yields

$$x_1 = 2 - 2t, \qquad x_2 = 2 - 2s - t, \qquad x_3 = -2 + s + 2t + 3u.$$

(4) Consider the augmented matrix of the system:

$$A = \begin{pmatrix} a & 0 & b & 2 \\ a & 2 & a & b \\ b & 2 & a & a \end{pmatrix}$$

Let's dispense with some of the easier cases first. Suppose $a = b = 0$. Then

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

From the first row, we see that the system has no solutions since $0 \neq 2$. Suppose now that $a = 0$ and $b \neq 0$. Then

$$A = \begin{pmatrix} 0 & 0 & b & 2 \\ 0 & 2 & 0 & b \\ b & 2 & 0 & 0 \end{pmatrix}$$

from which it is easy to see that both coefficient and augmented matrices have rank 3. It follows that the system has a unique solution when $a = 0$ and $b \neq 0$.

Assume now that $a \neq 0$ and $b = 0$. Then

$$A = \begin{pmatrix} a & 0 & 0 & 2 \\ a & 2 & a & 0 \\ 0 & 2 & a & a \end{pmatrix},$$

which can be reduced to

$$\begin{pmatrix} 1 & 0 & 0 & 2/a \\ 0 & 1 & a/2 & -1 \\ 0 & 0 & 0 & a+2 \end{pmatrix}$$

If $a = -2$ then we conclude that both the coefficient matrix and the augmented matrix of the system have rank 2, less than the number of variables, implying that the system has infinitely many solutions. If $a \neq -2$, then the coefficient matrix has rank 2 while the augmented matrix has rank 3, making the system inconsistent. We summarize our findings so far:

- If $a = 0$ and $b = 0$ then the system has no solutions.
- If $a = 0$ and $b \neq 0$ then the system has a unique solution.

- If $a \neq 0, -2$ and $b = 0$ then the system has no solutions.
- If $a = -2$ and $b = 0$ then the system has infinitely many solutions.

Now let's look at the generic case $a \neq 0$, in which case A can be reduced to

$$\begin{pmatrix} 1 & 0 & \frac{b}{a} & \frac{2}{a} \\ 0 & 1 & \frac{a-b}{2} & \frac{b-2}{2} \\ 0 & 0 & -\frac{b(b-a)}{a} & \frac{a^2-2b-ab+2a}{a} \end{pmatrix}$$

If $b \neq a$ (in addition to our running assumption that $b \neq 0$) then both the coefficient and augmented matrices have rank 3 and the system has a unique solution. If $b = a$ then the bottom row is zero, and it follows that the system has infinitly many solutions. We summarize our findings in the case $a \neq 0$ and $b \neq 0$:

- If $a \neq 0$ and $b \neq 0, a$ then the system has a unique solution.
- If $a \neq 0$ and $b = a$ then the system has infinitely many solutions.

Parsing the above summaries, we conclude that the system has

- no solutions if $b = 0$ and $a \neq -2$,
- infinitely many solutions if $a \neq 0$ and $b = a$,
- a unique solution otherwise.

(5) A system of equations has a unique solution if and only if its coefficient and augmented matrices both have rank equal to the number of variables, 3 in this case. There is only one 5×3 matrix in reduced row echelon form that has rank 3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(10) Let R and S be the reduced row echelon forms of A and $(A \mid \mathbf{b})$, respectively. Note that R is the matrix you get by deleting the rightmost column from S. Therefore, if $\operatorname{rank}(A \mid \mathbf{b}) > \operatorname{rank} A$, then S must have a leading 1 in its rightmost column. (Otherwise, all the leading 1s of S actually lie in R and the matrices have the same rank.) Because a leading 1 is *leading*, the other entries in its row – all of which lie to the left of this leading 1 – must be 0. In other words, the row of S containing the leading 1 looks like $(0 \cdots 0 \mid 1)$. The equation corresponding to this row is $0 = 1$, which has no solutions. Consequently, the system can have no solutions either.