1. Linear (in)dependence

Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a finite sequence of elements of V.

Definition 1.

- Let $x_1, x_2, \ldots, x_n$ be scalars. An identity of the form

$$\mathbf{0} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_n\mathbf{v}_n \tag{1}$$

  is called a *linear dependence relation* among $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Such a relation is called *nontrivial* of not all the x_i are zero.
- We say that the finite sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is *linearly dependent* if there is a nontrivial linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and *linearly independent* otherwise.
- Let S be a subset of V. We say that S is linearly dependent if there is a finite sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of distinct elements of S that is linearly dependent and linearly independent otherwise.

Remark 2. The linear dependence of the sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ does not depend on the order in which the elements of the sequence are listed. (Why?) Therefore, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a finite subset of V then S is linearly dependent if and only if the list $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is. Defining linear dependence for sequences as well as for sets is important because sequences allow for repetition of elements; this added flexibility is useful.

Exercise 3. Prove that S is linearly independent if and only if every finite subset of S is linearly independent.

Theorem 4. *Let S be a subset of V. Then S is linearly independent if and only if every element of* $\operatorname{span} S$ *can be written* uniquely *as a linear combination of elements of S.*

Proof. Suppose every element of $\operatorname{span} S$ can be written uniquely as a linearly combination of elements of S. Let $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ be distinct elements of S and let

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0} \tag{2}$$

be a linear dependence relation among the $\mathbf{v}_j$. Since $\mathbf{0} \in \operatorname{span} S$ ($\mathbf{0}$ belongs to *every* subspace of V), $\mathbf{0}$ can be written uniquely as a linear combination of elements of S. Since $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}$, we must have $x_j = 0$ for all j, making the linear dependence relation 2 trivial. Thus, S is linearly independent.

Conversely, suppose that S is linearly independent. Let $\mathbf{w} \in V$ and suppose that

$$\mathbf{w} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_n\mathbf{v}_n \qquad \text{and} \qquad \mathbf{w} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots y_n\mathbf{v}_n,$$

are two representations of $\mathbf{w}$ as linear combinations of elements of S, where the x_j and the y_j are scalars. Subtracting these two identities, we get

$$\begin{aligned}\mathbf{0} &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_n\mathbf{v}_n) - (y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots y_n\mathbf{v}_n) \\ &= (x_1 - y_1)\mathbf{v}_1 + (x_2 - y_2)\mathbf{v}_2 + \cdots + (x_n - y_n)\mathbf{v}_n.\end{aligned}$$

But since S is linearly independent, this linear dependence relation must be trivial, i.e., we must have $x_1 - y_1 = 0$, $x_2 - y_2 = 0$, $\ldots$, $x_n - y_n = 0$. Thus, $x_j = y_j$ for all j and the representation of $\mathbf{v}$ as a linear combination of the elements of S is unique. $\square$

1.1. **Linear (in)dependence in $\mathbf{R}^m$.** The following result is an immediate consequence of the "linear combinations are matrix-vector products" theorem.

Theorem 5. *The linear dependence relations among column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbf{R}^n$ are in one-to-one correspondence with $N(A)$, where $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$:*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \qquad \textit{if and only if} \qquad A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}.$$

In particular the vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly independent if and only if A has nonzero nullspace.

Example 6. Let

$$\mathbf{a}_1 = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 7 \\ -7 \\ 7 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{a}_3 = \begin{pmatrix} 9 \\ -13 \\ -11 \end{pmatrix}.$$

Then the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly dependent as the matrix equation

$$\begin{pmatrix} -1 & 7 & 9 \\ 3 & -7 & -13 \\ 9 & 7 & -11 \end{pmatrix}\begin{pmatrix} x_1 \\ x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has nontrivial solutions: Its general solution is $\mathbf{x} = t\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}^T$. The linear dependence relation corresponding to this basic solution is $2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$.

Corollary 7. *Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots \mathbf{a}_n \end{pmatrix}$ and $B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots \mathbf{b}_n \end{pmatrix}$ be row equivalent matrices. Then*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \quad \textit{if and only if} \quad x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n = \mathbf{0}.$$

In words, row equivalence preserves linear relations between columns.

Proof. Use Theorem 5 and the fact that row equivalent matrices have the same nullspace. (Prove this fact as an exercise.) □

1.2. **Linearly independent subsets of $\mathbf{R}^m$ can't be too big.**

Theorem 8. *Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and A be as in Theorem 5. If $m < n$ then the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly dependent.*

Proof. This follows from Theorem 5 and the fact that if A is an $m \times n$ matrix with $m < n$ then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. □

Example 9. Let

$$\mathbf{a}_1 = \begin{pmatrix} -7 \\ -2 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 17 \\ 5 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 6 \\ -4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{a}_4 = \begin{pmatrix} -4 \\ -7 \end{pmatrix}.$$

Since the matrix

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{pmatrix} = \begin{pmatrix} -7 & 17 & 6 & -4 \\ -2 & 5 & -4 & -7 \end{pmatrix}$$

has more rows than columns, its columns must satisfy nontrivial linear dependence relations. To find some, we reduce A to its reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & -98 & -99 \\ 0 & 1 & -40 & -41 \end{pmatrix}$$

We deduce that

$$98\mathbf{a}_1 + 40\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0} \qquad \text{and} \qquad 99\mathbf{a}_1 + 41\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}.$$

1.3. **Linear independence of pivot columns.**

Theorem 10. *Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots \mathbf{a}_n \end{pmatrix}$ be an $m \times n$ matrix with reduced row echelon form R. Suppose R has leading 1s in columns $j_1 < j_2 < \ldots < j_r$. Then the column vectors $\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}$ of A are linearly independent.*

Proof. Since R is in reduced row echelon form and has a leading 1 in position (i, j_i) its j_i-th column must be $\mathbf{e}_i$. Being a subset of the linearly independent set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$, the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r\}$ is linearly independent. Therefore, by Corollary 7, the vectors $\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}$ are linearly independent, too. ☐

1.4. **Shrinking linearly dependent sets; growing linearly independent sets.**

Exercise 11. Let S and T be linearly independent subsets of V.

(1) Let $\mathbf{v}$ be a nonzero element of $\operatorname{span} S$. Prove that $S \cup \{\mathbf{v}\}$ is linearly dependent.
(2) Suppose $S \subset T$. Prove that if $\operatorname{span} S = \operatorname{span} T$ then $S = T$. (Suggestion: Prove the contrapositive implication "if $S \subsetneq T$ then $\operatorname{span} S \neq \operatorname{span} T$".)

Theorem 12. *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$.*

(1) *Suppose S is linearly dependent. Then there is an index j, $1 \leq j \leq n$, such that* $\operatorname{span}(S - \{\mathbf{v}_j\}) = \operatorname{span} S$.
(2) *Suppose that S is linearly independent and that $\mathbf{w} \notin \operatorname{span} S$. Then $S \cup \{\mathbf{w}\}$ is linearly independent.*

Proof. We begin with (1). Since S is linearly dependent, there is a nontrivial linear dependence among its elements S:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}. \tag{3}$$

Being nontrivial, there is an index j such that $x_j \neq 0$, allowing us to solve 3 for $\mathbf{v}_j$:

$$\mathbf{v}_j = -\left(\frac{x_1}{x_j}\mathbf{v}_1 + \frac{x_2}{x_j}\mathbf{v}_2 + \cdots + \frac{x_n}{x_j}\mathbf{v}_n\right). \tag{4}$$

We claim that $\operatorname{span}(S - \{v_j\}) = \operatorname{span} S$. Since $S - \{v_j\} \subset S$, we have $\operatorname{span}(S - \{v_j\}) \subset \operatorname{span} S$. To prove the reverse containment, let $\mathbf{w}$ be an arbitrary element of $\operatorname{span} S$. Then there are scalars $y_1, y_2, \ldots, y_n$ such that

$$\mathbf{w} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n.$$

Substituting for $\mathbf{v}_j$ from (4), we get

$$\mathbf{w} = y_1\mathbf{v}_1 + \cdots + y_{j-1}\mathbf{v}_{j-1} - \left(\frac{x_1}{x_j}\mathbf{v}_1 + \frac{x_2}{x_j}\mathbf{v}_2 + \cdots + \frac{x_n}{x_j}\mathbf{v}_n\right) + y_{j+1}v_{j+1} + \cdots + y_n\mathbf{v}_n$$

$$= \left(y_1 - \frac{x_1}{x_j}\right)\mathbf{v}_1 + \cdots + \left(y_{j-1} - \frac{x_{j-1}}{x_j}\right)\mathbf{v}_{j-1} + \left(y_{j+1} - \frac{x_{j+1}}{x_j}\right)\mathbf{v}_{j+1} + \cdots + \left(y_n - \frac{x_n}{x_j}\right)\mathbf{v}_n.$$

Therefore, $\mathbf{w} \in \operatorname{span}(S - \{\mathbf{v}_j\})$ and $\operatorname{span} S \subset \operatorname{span}(S - \{v_j\})$. We conclude that $\operatorname{span} S = \operatorname{span}(S - \{v_j\})$, as desired.

Moving on to (2), let

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n + y\mathbf{w} = \mathbf{0} \tag{5}$$

be a linear dependence among the elements of $S \cup \{\mathbf{w}\}$. If y were nonzero, then we could solve for $\mathbf{w}$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

$$\mathbf{w} = -\frac{x_1}{y}\mathbf{v}_1 - \frac{x_2}{y}\mathbf{v}_2 - \cdots - \frac{x_n}{y}\mathbf{v}_n,$$

implying $\mathbf{w} \in \operatorname{span} S$ contrary to our hypothesis. Therefore, $y = 0$ and (5) becomes

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}.$$

But this is a linear dependence among the elements of S, a linearly independent set by hypothesis. Therefore, $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_n = 0$. We have shown that the linear dependence (5) is trivial, proving that $S \cup \{\mathbf{w}\}$ is linearly dependent. $\square$

Corollary 13. *Let S be a finite subset of V such that $V = \operatorname{span} S$. Then there is a subset $B \subset S$ such that the following conditions hold:*

(1) $\operatorname{span} B = V$,
(2) *B is linearly independent.*

Proof. Let

$$\mathcal{F} = \{T \subset S : \operatorname{span} T = V\}.$$

Let B be an element of $\mathcal{F}$ with the smallest possible number of elements. Since $B \in \mathcal{F}$, we have $\operatorname{span} B = V$. We claim that B is linearly independent. For suppose not. Then by Theorem 12 (2), there is an element $\mathbf{v} \in B$ such that $\operatorname{span}(B - \{\mathbf{v}\}) = \operatorname{span} B$. But then $\operatorname{span}(B - \{\mathbf{v}\}) = V$ and $B - \{\mathbf{v}\} \in \mathcal{F}$, contradicting the minimality of B. Therefore, B must be linearly independent. $\square$

Definition 14. Let $\mathcal{F}$ be a set of sets. We say that an element $S \in \mathcal{F}$ is

(1) *maximal* if $T \in \mathcal{F}$ and $S \subset T$ imply $S = T$, i.e., S is not properly contained in any element of $\mathcal{F}$.
(2) *minimal* if $T \in \mathcal{F}$ and $T \subset S$ imply $S = T$, i.e., S does not properly contain any element of $\mathcal{F}$.

Exercise 15. Let X be a finite set and let $\mathcal{F}$ be a set of subsets of X. Prove that $\mathcal{F}$ has a maximal element and a minimal element. Does this remain true if we drop the finiteness assumption on X?

Exercise 16. Let S be a subset of V. (You shouldn't need to assume that S is finite.)

(1) Prove that the following statements are equivalent:

(a) S is a minimal element of the set $\mathcal{F}$ of spanning subsets of V.
(b) S linearly independent.

(2) Prove that the following statements are equivalent:
(a) S is a maximal element of the set $\mathcal{F}$ of linearly independent subsets of V.
(b) S spans V.

(For (1), use the proof of Corollary 13 for inspiration.)

Exercise 17. Let S be a subset of V. Prove that the following are equivalent:

(1) S is linearly independent.
(2) For every partition of S into disjoint subsets S_1 and S_2, the spaces span S_1 and span S_2 are linearly disjoint.
(3) For all $\mathbf{v} \in S$, the spaces $\operatorname{span}\{\mathbf{v}\}$ and $\operatorname{span}(S - \{\mathbf{v}\})$ are linearly disjoint.