MATH 213 ASSIGNMENT 2

DUE: FRIDAY 23/10/2015

(1) For $1 \leq i, j \leq 2$, let $\mathbf{e}^{ij} = (e^{ij}_{k\ell})$ be the 2×2 matrix such that
$$e^{ij}_{k\ell} = \begin{cases} 1 & \text{if } i = k \text{ and } j = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write down the matrices $\mathbf{e}^{ij}$ for all i, j. (That is, parse and understand the definition of $\mathbf{e}^{ij}$.)

(b) Compute $(\mathbf{e}^{ij})^2$ for all i, j.

(c) A object x is called *idempotent* if $x^2 = x$ and *nilpotent* if $x^n = 0$ for some n. What are the nonidentity ($\neq 1$) idempotent elements of $\mathbf{R}$? What are the nonzero nilpotents elements of $\mathbf{R}$?

(d) Observe that the set of 2×2 matrices contains both nonidentity idempotents and nonzero nilpotents.

(e) Find a 3×3 matrix such that $A^3 = 0$ but $A^2 \neq 0$?

(2) Let
$$A = \begin{pmatrix} 20 & -9 & 0 \\ 81 & -36 & -1 \\ -70 & 31 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -7 & -6 & -4 & 11 \\ -3 & -5 & -5 & -1 \\ 3 & 2 & 1 & -6 \end{pmatrix}.$$

(a) Find A^{-1}.

(b) Write A as a product of elementary matrices.

(c) Find an invertible matrix U such that UB is in reduced row echelon form.

(3) (Lab 5, 13/10/2015) Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Prove that
$$(AB)^T = B^T A^T.$$

(4) (Lab 6, 20/10/2015) Let $M_n(\mathbf{R})$ be the vector space of $n \times n$ matrices with real entries and let
$$W = \{A \in M_n(\mathbf{R}) : A^T = A\}.$$
Prove that W is a subspace of $M_n(\mathbf{R})$. (W is called the space of *symmetric matrices.*)

1. Solutions

(1) (a) $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $e_{11}^2 = e_{11}$, $e_{12}^2 = \mathbf{0}$, $e_{21}^2 = \mathbf{0}$, $e_{22}^2 = e_{22}$

(c) The only idempotents in $\mathbf{R}$ are 0 and 1. The only nipontent in $\mathbf{R}$ is 0.

(d) The matrices e_{11} and e_{22} are nonidentity, nonzero idempotents in $M_2(\mathbf{R})$; e_{12} and $e21$ are nonzero idempotents in $M_2(\mathbf{R})$.

(e) If x and z are nonzero, then

$$A = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$$

satisfies $A^3 = \mathbf{0}$ and $A^2 \neq \mathbf{0}$.

(2) (a) $A^{-1} = \begin{pmatrix} 5 & -9 & -9 \\ 11 & -20 & -20 \\ 9 & -10 & -9 \end{pmatrix}$

(b) We record the elementary row operations we used to bring A to reduced row echelon form:

(i) Multiply row 1 by $-\frac{1}{20}$.

(ii) Add -81 times row 1 to row 2.

(iii) Add 70 times row 1 to row 3.

(iv) Multiply row 2 by $\frac{20}{9}$.

(v) Add $\frac{9}{20}$ times row 2 to row 1.

(vi) Add $\frac{1}{2}$ times row 2 to row 3.

(vii) Multiply row 3 by -9.

(viii) Add row 3 to row 1.

(ix) Add $\frac{20}{9}$ times row 3 to row 2.

Therefore,

$$A^{-1} = E_{\text{ix}} E_{\text{viii}} E_{\text{vii}} E_{\text{vi}} E_{\text{v}} E_{\text{iv}} E_{\text{iii}} E_{\text{ii}} E_{\text{i}},$$

where $E_?$ is the elementary matrix associated to the elementary row operation performed in step (?). Inverting both sides and applying the 9-term generalization of the rule $(AB)^{-1} = B^{-1}A^{-1}$, we get

$$A^{-1} = E_{\text{i}}^{-1} E_{\text{ii}}^{-1} E_{\text{iii}}^{-1} E_{\text{iv}}^{-1} E_{\text{v}}^{-1} E_{\text{vi}}^{-1} E_{\text{vii}}^{-1} E_{\text{viii}}^{-1} E_{\text{ix}}^{-1}.$$

Noting that $E_?^{-1}$ is elementary matrix associated to the inverse of the elementary row operation performed in step (?), we get

$$E_{\text{i}}^{-1} = \begin{pmatrix} -20 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad E_{\text{ii}}^{-1} = \begin{pmatrix} 1 & & \\ 81 & 1 & \\ & & 1 \end{pmatrix}, \quad E_{\text{iii}}^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ -70 & & 1 \end{pmatrix}, \quad \dots$$

(c) Augment B by the 3×3 identity matrix and bring to reduced row echelon form:

$$\begin{pmatrix} B & I \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -3 & 5 & -2 & 10 \\ 0 & 1 & 0 & 1 & -12 & 5 & -23 \\ 0 & 0 & 1 & 1 & 9 & -4 & 17 \end{pmatrix}$$

We take $U = \begin{pmatrix} 5 & -2 & 10 \\ -12 & 5 & -23 \\ 9 & -4 & 17 \end{pmatrix}$.

(3) Let a_{ij} be the (i,j)-entry of A and let b_{ij} be the (i,j) entry of B. Note that both $(AB)^T$ and $B^T A^T$ are $p \times m$ matrices. So let i and j integers with $1 \leq i \leq p$ and $1 \leq j \leq m$. We need to show that the (i,j)-entry $(AB)^T$ equals the (i,j)-entry of $B^T A^T$.

By definition of transpose, the (i,j)-entry of $(AB)^T$ is the (j,i)-entry of AB. By definition of matrix multiplication, the (j,i) entry of AB is

$$\sum_{k=1}^{n} a_{jk} b_{ki}. \tag{$*$}$$

By definition of matrix multiplication, the (i,j)-entry of $B^T A^T$ is

$$\sum_{k=1}^{n} \Big((i,k)\text{-entry of } B^T\Big)\Big((k,j)\text{-entry of } A^T\Big)$$

By definition of transpose, the (i,k)-entry of B^T is the (k,i)-entry of B, namely, b_{ki}. Similarly, the (k,j)-entry of A^T is a_{jk}. Therefore,

$$\sum_{k=1}^{n} \Big((i,k)\text{-entry of } B^T\Big)\Big((k,j)\text{-entry of } A^T\Big) = \sum_{k=1}^{n} b_{ki} a_{jk}. \tag{$**$}$$

Since the expression on the right hand side of $(**)$ is the same as the expressin in $(*)$, we are done.

(4) (I think I did this in class. Sorry if I spoiled your fun.) Let A and B be elements of W, i.e., A and B are matrices satisfying $A^T = A$. Then

$$\begin{aligned} (A+B)^T &= A^T + B^T && \text{(You proved this in one of the labs.)} \\ &= A + B && \text{(Since } A = A^T \text{ and } B = B^T.\text{)} \end{aligned}$$

Therefore, $A + B \in W$ and W is closed under addition. Similarly, if x is a scalar then

$$\begin{aligned} (xA)^T &= xA^T && \text{(You proved this in one of the labs.)} \\ &= xA && \text{(Since } A = A^T.\text{)} \end{aligned}$$

Thus, W is closed under scalar multiplication. Being closed under addition and scalar multiplication, we conclude that W is a subspace of $M_n(\mathbf{R})$.