MATH 213 – LAB 9

- A *linear transformation* is a function $T : V \to W$ such that
  (i) $T(v + v') = T(v) + T(v')$, and
  (ii) $T(xv) = xT(v)$
  for all $v, v' \in V$ and all $x \in \mathbf{R}$.

- Let $A \in \mathbf{R}^{n\times n}$. A nonzero vector $v \in \mathbf{R}^{n\times 1}$ is an eigenvector of A belonging to the eigenvalue λ if $Av = \lambda v$. A number λ is an eigenvalue of A if and only if $N(\lambda I - A) \neq \{0\}$, in which case the eigenvectors of A belonging to λ are the elements of $N(\lambda I - A)$.

(1) Define
$$T : \mathbf{R}^{2\times 1} \to \mathbf{R}^{2\times 1} \quad \text{by} \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ 2x - 7y \end{pmatrix}.$$
(a) Show that T is a linear transformation. To begin, take
$$v = \begin{pmatrix} x \\ y \end{pmatrix}, \; v' = \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbf{R}^{2\times 1}.$$
(b) Let
$$A = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \in \mathbf{R}^{2\times 2}.$$
Show that $T(v) = Av$ for all $v \in \mathbf{R}^{2\times 1}$. Start by writing
$$v = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(2) Define
$$T : \mathbf{R} \to \mathbf{R} \quad \text{by} \quad T(x) = x + 1.$$
Show that T is not a linear transformation. One possible starting point is to compute $T(2)$ and $T(1) + T(1)$.

(3) Define
$$T : \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2} \quad \text{by} \quad T(A) = A^T.$$
Show that T is a linear transformation. Can you generalize your argument to show that the transposition transformation $T : \mathbf{R}^{m\times n} \to \mathbf{R}^{n\times m}$?

(4) Let $M, N \in \mathbf{R}^{2\times 2}$. Define:
$$\begin{aligned} T &: \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2} \quad \text{by} \quad T(A) = MA, \\ U &: \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2} \quad \text{by} \quad U(A) = AN, \\ V &: \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2} \quad \text{by} \quad U(A) = MAN. \end{aligned}$$

(a) Show that T, U and V are linear transformations.

(b) ($*$) Let $e = (e_{ij} : 1 \leq i, j \leq 2)$, be the standard basis of $\mathbf{R}^{2\times 2}$. Find matrices $\mathcal{T}, \mathcal{U}, \mathcal{V} \in \mathbf{R}^{4\times 4}$ such that

$$T(A) = \mathcal{T}[A]_e, \quad U(A) = \mathcal{U}[A]_e \quad \text{and} \quad V(A) = \mathcal{V}[A]_e.$$

(5) Let

$$A = \begin{pmatrix} 2 & 3 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Compute the polynomial $\det(xI - A)$, where

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

(b) Find the eigenvalues of A by solving the equation $\det(xI - A) = 0$.

(c) For each root λ of the equation $\det(xI - A) = 0$, find corresponding eigenvector(s), i.e., a basis of $N(\lambda I - A)$.

(d) If v_j is an eigenvector belonging to the ieigenvector λ_j, show that

$$Av = v \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}, \quad \text{where} \quad v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}.$$

(e) Find v^{-1}. Compute

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} v \begin{pmatrix} \lambda_1^n & \\ & \lambda_2^n \end{pmatrix} v^{-1} = v \left(\lim_{n\to\infty} \begin{pmatrix} \lambda_1^n & \\ & \lambda_2^n \end{pmatrix} \right) v^{-1}.$$

(6) Cities A and B have initial populations of 1 million and 1.5 million, respectively. Every year, 3% of the population of city A moves to city B and 2.5% of the population of city B moves to city A. What happens to the populations of the cities in the long run?