1. Linear combinations and spans

Let V be a vector space.

Definition 1.

- A sum of the form
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$
where $x_1, x_2, \ldots, x_n$ are scalars and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$, is called a *linear combination* of the vectors $v_1, v_2, \ldots, v_n$.
- The *span of* S, written $\operatorname{span} S$ is the set of vectors $\mathbf{v} \in V$ expressible as a linear combination of elements of S:
$$\mathbf{v} \in \operatorname{span} S \iff \begin{array}{c} \exists \textbf{ finitely many} \text{ vectors } \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in S \\ \text{and scalars } x_1, x_2, \ldots x_n \text{ such that} \\ \mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n. \end{array}$$

Theorem 2. *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. *Then* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a subspace of* V.

Proof. Let $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and let $\mathbf{x}, \mathbf{y} \in W$ and let x be a scalar. Then there are scalars $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ such that
$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \qquad \text{and} \qquad \mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n$$
and
$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n) + (y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n) \\ &= (x_1 + y_1)\mathbf{v}_1 + (x_2 + y_2)\mathbf{v}_2 + \cdots + (x_n + y_n)\mathbf{v}_n.\end{aligned}$$
Thus, $\mathbf{x} + \mathbf{y}$ is a linear combination of the $\mathbf{v}_i$ and $\mathbf{x} + \mathbf{y} \in W$. We have established that W is closed under addition. We have
$$x\mathbf{x} = x(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n) = (xx_1)\mathbf{v}_1 + (xx_2)\mathbf{v}_2 + \cdots + (xx_n)\mathbf{v}_n \in W,$$
so W is closed under scalar multiplication as well. $\square$

Example 3. Consider the vectors following vectors in $\mathbf{R}^3$:
$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$
Then $\mathbf{a} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as
$$\mathbf{a} = \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3.$$

Example 4. The zero vector $\mathbf{0}$ belongs to the span of any set of vectors $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ because
$$0\mathbf{v}_1 + \mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

Example 5. Let $\mathbf{e}_i \in \mathbf{R}^n$ be the column vector whose only nonzero entry is a 1 in row i. If $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T \in \mathbf{R}^n$, then $\mathbf{x} \in \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ as
$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Theorem 6 (Linear combinations are matrix-vector products)**.** *Let* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbf{R}^m$ *and let* A *be the matrix with column vectors* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$*:*

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}.$$

Then the linear combinations of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *are precisely the matrix-vector products* $A\mathbf{x}$ *for* $\mathbf{x} \in \mathbf{R}^n$*.*

Proof. The identity

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

holds. $\square$

Corollary 7. *Let* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *and* A *be as in Theorem 6. The following are equivalent for a vector* $\mathbf{b} \in \mathbf{R}^m$*.*

(1) $\mathbf{b}$ *can be expressed as a linear combination of the vectors* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$*.*
(2) $\mathbf{b} \in \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.
(3) $A\mathbf{x} = \mathbf{b}$ *has a solution.*
(4) $\mathbf{b}$ *belongs to* $C(A)$*.*

Remark 8. The column space is called the column space because it's the space spanned by the columns.

Example 9. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ be as in Example 3. Since the matrix

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is invertible, the solution $\mathbf{x} = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}^T$ of the equation $A\mathbf{x} = \mathbf{b}$ is unique. Therefore, $\mathbf{b}$ does not belong to any of $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_3\}$ or $\operatorname{span}\{\mathbf{e}_2, \mathbf{e}_3\}$.

Example 10. Let P be the set of polynomials in the variable x with coefficients in $\mathbf{R}$. Then

$$\operatorname{span}\{1, x, x^2, x^3, \ldots\} = P$$

because every polynomial is a **finite** sum of monomials[1] and a finite sum of monomials is a linear combinations of elements of $\{1, x, x^2, x^3, \ldots\}$.

1.1. **The pivot columns span the column space.**

Theorem 11. *Let* $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ *be an* $m \times n$ *matrix in reduced row echelon form with leading* 1*s in positions* (i, j_i)*,* $1 \le i \le r$*. Then*

$$C(A) = \operatorname{span}\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}\}.$$

[1]A *monomial* is a polynomial of the form ax^n, where $a \in \mathbf{R}$ and $n \ge 0$.

Proof. Since rows $r+1$ through m of A have no leading 1s and A is in row echelon form, they are zero rows. Therefore, $a_{ij} = 0$ for $r+1 \le i \le m$ and $1 \le j \le n$. Since A is in reduced row echelon form, the 1 in row i of the column vector $\mathbf{a}_{j_i}$ is the only nonzero entry in $\mathbf{a}_{j_i}$. Combining these facts, we see that

$$\mathbf{a}_j = a_{1j}\mathbf{a}_{j_1} + a_{2j}\mathbf{a}_{j_2} + \cdots + a_{rj}\mathbf{a}_{j_r} \in \operatorname{span}\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}\}$$

for $1 \le j \le n$, proving that all the columns of A belong to $\operatorname{span}\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}\}$. □

Exercise 12. Suppose $1 \le i < j \le n$. Let $\mathbf{v}^{ij} \in \mathbf{R}^n$ be the vector with 1s in positions i and j and 0s in all other positions and let

$$S_n = \{\mathbf{v}^{ij} : 1 \le i < j \le n\}.$$

For $n = 2, 3, 4$, answer the following:

(1) Is it true that $\operatorname{span} S_n = \mathbf{R}^n$?
(2) Find all ways of writing the zero vector $\mathbf{0} \in \mathbf{R}^n$ as a linear combination of elements of S_n.
(3) Find all ways of writing the "ones vector" $\mathbf{1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbf{R}^n$ as a linear combination of elements of S_n.

Exercise 13. Let $\mathbf{e}^{ij} \in M_2(\mathbf{R})$ be the matrix whose only nonzero entry is a 1 in position (i, j). Let

$$R = \{\mathbf{e}^{11}, \mathbf{e}^{12}, \mathbf{e}^{21}, \mathbf{e}^{22}\}, \qquad S = \{\mathbf{e}^{11}, \mathbf{e}^{12} + \mathbf{e}^{21}, \mathbf{e}^{22}\}, \qquad \text{and} \qquad T = \{\mathbf{e}^{12} - \mathbf{e}^{21}\}.$$

Show that:

(1) $\operatorname{span} R = M_2(\mathbf{R})$.
(2) $\operatorname{span} S$ is the set of symmetric matrices in $M_2(\mathbf{R})$. (A matrix A is *symmetric* if $A^T = A$.)
(3) $\operatorname{span} T$ is the set of skew-symmetric matrices in $M_2(\mathbf{R})$. (A matrix A is *skew-symmetric* if $A^T = A$.)

Exercise 14. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ be list of vectors in $\mathbf{R}^n$. Prove that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subset \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ if and only if $\mathbf{v}_i \in \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ for all i, $1 \le i \le m$. (How might you apply the result of this exercise to prove equality of two spans rather than containment?)

Exercise 15. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and let x and y be scalars with $x \ne 0$. Prove:

(1) $\operatorname{span}\{x\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
(2) $\operatorname{span}\{\mathbf{v}_1 + y\mathbf{v}_2, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

1.2. A spanning subset of $\mathbf{R}^m$ can't be too small.

Theorem 16. *Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be vectors in $\mathbf{R}^m$ and suppose that $n < m$. Then the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ do not span $\mathbf{R}^m$.*

Proof. Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$. Let R be any row echelon form of A and let U be an invertible $m \times m$ matrix such that $R = UA$. Since R has n columns, A has at most n pivot rows. Since $m > n$, R must have at least one zero row. In particular, the m-th row of R is zero. Therefore, $A\mathbf{x} = \mathbf{e}_m$ has no solution, where $\mathbf{e}_m \in \mathbf{R}^m$ is the column vector whose only nonzero entry is a 1 in row m. Setting $\mathbf{b} = U^{-1}\mathbf{e}_m$, it follows that $A\mathbf{x} = \mathbf{b}$ has no solution. Therefore, by Corollary 7, $\mathbf{b} \notin \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$. □

1.3. **Uniqueness of reduced row echelon form.** (Under construction; do not read!)

Lemma 17. *Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots \mathbf{a}_n \end{pmatrix}$ be an $m \times n$ matrix reduced row echelon form with pivot columns $j_1 < j_2 < \ldots < j_r$. The following are equivalent:*

(1) $\mathbf{a}_j$ *is a pivot column of* A.
(2) $\mathbf{a}_j \notin \operatorname{span}\{\mathbf{a}_{j_i} : j_i < j\}$.
(3) $\mathbf{a}_j \notin \operatorname{span}\{\mathbf{a}_k : k < j\}$.

Proof. Let $A_j = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_j \end{pmatrix}$. Then A_j is in reduced row echelon form as A is (reference an exercise?) and $\{\mathbf{a}_{j_i} : j_i < j\}$ is a basis for $C(A_j)$ by Theorem 11. In particular, if $1 \leq p \leq r$ then $\{\mathbf{a}_{j_i} : 1 \leq i \leq p\}$ is a basis of $C(A_{j_p})$. Since bases are linearly independent,

$$\mathbf{a}_{j_p} \notin \operatorname{span}\{\mathbf{a}_{j_i} : 1 \leq i < p\},$$

proving that (1) implies (2).

Conversely, suppose $\mathbf{a}_j$ is not a pivot column of A. Now the column space of A_j is spanned by its pivot columns, namely, the $\mathbf{a}_{j_i}$ with $j_i < j$, so $\mathbf{a}_j \in \operatorname{span}\{\mathbf{a}_{j_i} : j_i < j\}$. This proves that (2) implies (1).

To see that (2) and (3) are equivalent, observe that $\operatorname{span}\{\mathbf{a}_k : k < j\}$ is by definition, the column space of A_j while $\operatorname{span}\{\mathbf{a}_{j_i} : j_i < j\}$ is the span of the pivot columns of A_j. But these spans are equal by Theorem 11. $\square$

Exercise 18. Let A be an $m \times n$ matrix in row echelon form whose leading 1s lie in positions (i, j_i), $1 \leq i \leq r$. Suppose $j_i < j < j_{i+1}$. Prove that $a_{pj} = 0$ for $i < p \leq m$.

Theorem 19. *Let A and B be row equivalent matrices in reduced row echelon form. Then $A = B$.*

Proof. Suppose not. Then there is a pair of $m \times n$ matrices (A, B) witnessing the failure of the theorem such that n minimal among all such witnesses. In other words:

(i) A and B are distinct, row equivalent matrices in reduced row echelon form.
(ii) If A' and B' is another such pair then A has fewer columns than A'.

Since A and B are row equivalent and in reduced row echelon form, so are A_{n-1} and B_{n-1}. (Why?) By the minimality of n, the theorem holds for A_{n-1} and B_{n-1}. Therefore, $A_{n-1} = B_{n-1}$. Thus, to prove the theorem, it suffices to show that $\mathbf{a}_n = \mathbf{b}_n$.

Suppose, first, that $\mathbf{a}_n$ is a pivot column of A. Then $\mathbf{a}_n \notin C(A_{n-1})$ by Lemma 17. Since A and B are row equivalent, $\mathbf{a}_n \in C(A_{n-1})$ if and only if $\mathbf{b}_n \in C(B_{n-1})$. (Row equivalence preserves linear relations between columns.) Therefore, $\mathbf{b}_n \notin C(B_{n-1})$ and, by Lemma 17 again, $\mathbf{b}_n$ is a pivot column of B. Let $r = \operatorname{rank} A_{n-1} = \operatorname{rank} B_{n-1}$. Then $\mathbf{a}_n$ and $\mathbf{b}_n$ must both equal $\mathbf{e}_{r+1}$ because A and B are in reduced row echelon form. Thus, $\mathbf{a}_n = \mathbf{b}_n$.

Now suppose that $\mathbf{a}_n$ is not a pivot column of A. Then $\mathbf{a}_n \in C(A_{n-1})$ by Lemma 17, so there is a vector $\mathbf{x} \in \mathbf{R}^{n-1}$ such that $\mathbf{a}_n = A_{n-1}\mathbf{x}$. Since A and B are row equivalent, $\mathbf{b}_n = B_{n-1}\mathbf{x}$. But $A_{n-1} = B_{n-1}$, so it follows once again that $\mathbf{a}_n = \mathbf{b}_n$. $\square$

Exercise 20. Let W and W' be subspaces of W. Let S and S' be spanning sets of W and W', respectively. Show that $S \cup S'$ is a spanning set of $W \cup W'$.

Exercise 21. Let $A \in \mathbf{R}^{m \times n}$ and let $\mathbf{v} \in R(A)$. Show that if $\mathbf{v}^T \in N(A)$ then $\mathbf{v} = \mathbf{0}^{1 \times m}$. (Hint: Since $\mathbf{v} \in R(A)$, there is a row vector $\mathbf{x} \in \mathbf{R}^{1 \times m}$ such that $\mathbf{v} = \mathbf{x}A$. If $A\mathbf{v}^T = \mathbf{0}^{m \times 1}$ then

$$\mathbf{v}\mathbf{v}^T = \mathbf{x}A\mathbf{v}^T = \mathbf{x}\mathbf{0}^{m \times 1} = \mathbf{0}^{m \times 1}.$$

Explain why $\mathbf{v}\mathbf{v}^T = 0$ implies $\mathbf{v} = \mathbf{0}^{1\times m}$.) Conclude that $R(A^T) \cap N(A) = \{0\}$.