MATH 213 LAB 5 – THE TRANSPOSE; TECHNIQUES FOR PROVING THINGS ABOUT MATRICES

Definition: Let $A = (a_{ij})$ be an $m \times n$ matrix. Define the *transpose* A^T of A to be the matrix whose (i, j)-entry is a_{ji}:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \ddots \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots a_{m1} \\ a_{12} & a_{22} & \cdots a_{m2} \\ \vdots & \vdots & \ddots \vdots \\ a_{1n} & a_{2n} & \cdots a_{mn} \end{pmatrix}.$$

In other words, the rows (columns) of A are the columns (rows) of A^T. In particular, A^T as size $n \times m$.

Examples:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 34 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 13 \\ 1 & 5 & 21 \\ 2 & 8 & 34 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 \\ 5 & 7 \\ 11 & 13 \end{pmatrix}^T = \begin{pmatrix} 2 & 5 & 11 \\ 3 & 7 & 13 \end{pmatrix}$$

Exercise 1: Show that if A is an $m \times n$ matrix and c is a scalar then

$$(cA)^T = cA^T.$$

Flesh out the proof skeleton below by filling in the blanks and justifying your equations by quoting the appropriate definitions.

Proof skeleton:

- *Think and articulate*: We need to prove the equality of two matrices, namely, $(cA)^T$ (the transpose of cA) and cA^T (the multiple of A^T by the scalar c). To prove that two matrices are equal, we need to show that their (i, j)-entries are equal for an arbitrary pair (i, j). So we're going to need notation for these entries.
- *Introduce notation*: Write a_{ij} for the (i, j)-entry of A.
- *Calculate the left hand side*: Express the (i, j)-entry of $(cA)^T$ in terms of c and the entries of A:
  - The (i, j) entry of cA is _______. (Here, you use the definition of scalar multiplication.)
  - The (i, j) entry of $(cA)^T$ is _______. (Here, you use the definition of transpose.)
- *Calculate the right hand side*: Express the (i, j)-entry of cA^T in terms of c and the entries of A:
  - The (i, j) entry of A^T is _______. (Here, you use the definition of transpose.)
  - The (i, j) entry of cA^T is _______. (Here, you use the definition of scalar multiplication.)
- *Put it all together.*

Exercise 2: Show that if A and B are $m \times n$ matrices then

$$(A + B)^T = A^T + B^T.$$

Flesh out the proof skeleton below by filling in the blanks and justifying your equations by quoting the appropriate definitions.

Proof skeleton:

- *Think and articulate*: We need to prove the equality of two matrices, namely, _______ and _______. To prove that two matrices are equal, we need to show that ___________________________________.
- *Introduce notation*: Write _______ and _______ for the (i, j)-entries of A and B, respectively.
- *Calculate the left hand side*: Express the (i, j)-entry of $(A+B)^T$ in terms of the entries of A and B:
  - The (i, j) entry of $A + B$ is _______.
  - The (i, j) entry of $(A + B)^T$ is _______.
- *Calculate the right hand side*: Express the (i, j)-entry of $A^T + B^T$ in terms of the entries of A and B:

- The (i,j) entry of A^T is _______. The (i,j)-entry of B^T is _______.
- The (i,j) entry of $A^T + B^T$ is _______.

- *Put it all together.*

Exercise 3: [Recall] Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{ij})$ be an $n \times p$ matrix. Then the (i,j) entry of AB is

$$a_{\_\_}b_{\_\_} + a_{\_\_}b_{\_\_} + \cdots + a_{\_\_}b_{\_\_}.$$

Exercise 4: Show that if A is an $m \times n$ matrix and B is an $n \times p$ matrix then

$$(AB)^T = B^T A^T.$$

Flesh out the proof skeleton below by replacing the ... with full sentences, justifying your equations by quoting the appropriate definitions.

Proof skeleton:

- *Think and articulate*: We need to...
- *Introduce notation*: Write...
- *Calculate the left hand side*: ... (Exercise 3 might be useful here.)
- *Calculate the right hand side*: ... (Here, too.)
- *Put it all together.*

Definition:

- An *inverse* of an $n \times n$ matrix A is an $n \times n$ matrix B with the property that
$$AB = BA = I_n.$$
- An $n \times n$ matrix A is *invertible* if A has an inverse, i.e., there exists a matrix B such that $AB = I_n$ and $BA = I_n$.

Exercise 5: Suppose B and B' are inverses of A. Prove that $B = B'$.

Flesh out the following proof skeleton below by justifying each of the following inequalities by quoting either a definition, or a property of matrix.

$$\begin{aligned} B' &= B' I_n \\ &= B'(AB) \\ &= (B'A)B \\ &= I_n B \\ &= B. \end{aligned}$$

Exercise 5 shows that an inverse of A, if it exists, is *unique*. Therefore, we can refer to it as *the* inverse of A. We write A^{-1} for the inverse of A, a reasonable thing to do since its meaning is unambiguous.

Exercise 6: Show that if A and B are invertible. Show that AB is invertible with inverse $B^{-1}A^{-1}$.

Proof skeleton:

- *Think and articulate*: By definition of inverse, to show that $B^{-1}A^{-1}$ is an inverse of AB, we need to check that the identies ______________ and ______________ hold.
- *Calculate*: Show that these two identities hold, justifying each step by quoting the relevant definition or property of matrix arithmetic.

($*$) **Exercise 7:** Write down definitions for a *left inverse* and a *right inverse* of an $n \times n$ matrix A. Adapt the proof in Exercise 5 to prove that if A has a left inverse B and a right inverse B' then $B = B'$. (Note that we have *not* shown that the existence of a left (right) inverse implies the existence of a right (left) inverse. This implication is true, though.)