MATH 213 – LAB 2

1. Introduction

The Gaussian elimination algorithm we've developed for transforming a matrix into reduced row echelon form allows us to solve any system of m linear equations in n variables. An important issue that we haven't touched on is the *complexity* of this algorithm: How much computational effort does Gaussian elimination require? How large a system can a computer solve in a reasonable amount of time? The issue of complexity is not one that we are going to discuss in any detail. It is, however, crucially important since it determines the practical utility of the algorithm. We will content ourselves with some naïve operation counts.

For concreteness, here is an explicit description of the Gaussian elimination algorithm for transforming a matrix into reduced row echelon via a sequence of elementary row operations:

Gaussian elimination algorithm:

Set $r = 0$.

for j from 1 to n:

- Case 1: Suppose $a_{ij} = 0$ for all $i > r$.
  - Do nothing.
- Case 2: At least one of a_{ij} for $i > r$, is nonzero.
  - Increment r by 1.
  - If $a_{r,j} = 0$:
    * Choose the smallest index $p \geq r$ such that $a_{pj} \neq 0$.
    * Swap rows r and p.
  - If $a_{r,j} \neq 1$, multiply row r by $a_{r,j}^{-1}$.
  - For $i \neq r$ such that $a_{ij} \neq 0$: Subtract a_{ij} times row r from row i.

The final value of r is the rank of the matrix.

2. Terminology

- An $n \times n$ matrix is called a *permuation matrix* if its entries are all either 0 or 1 and it has a single 1 in every row and every column.
- Elementary row operations:
  I. Swap two rows.
  II. Multiply a row by a nonzero number.
  III. Add a multiple of one row to a different row.

  Multiplying a row by 1 (respectively, adding 0 times one row to a different row) is called the <u>trivial</u> elementary row operation of Type II (respectively, Type III).

3. Exercises

(1) Use the Gaussian elimination algorithm **as described above** to transform the following matrices into reduced row echelon form.

(a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 1 & 1 & 7 & 0 \end{pmatrix}$

(f) $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$,

(g) $A_3 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$

(h) $A_4 = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 6 \\ 1 & 2 & 4 & 7 \\ 1 & 2 & 4 & 8 \end{pmatrix}$

(2) Find an $n \times n$ matrix that the Gaussian elimination algorithm transforms it into reduced row echelon form by applying exactly n Type II elementary row operations, all nontrivial, and no operations of any other type.

(3) Find an $n \times n$ matrix such that Gaussian elimination transforms it into reduced row echelon form by applying exactly $n - 1$ Type I elementary row operations and no operations of any other type. You should start by playing with examples for some small values of n. Hint: Consider permutation matrices (see Terminology).

(4) Give a upper bound on the number of Type I elementary row operations needed in applying the Gaussian elimination algorithm to an $n \times n$ matrix A. Your answer should depend on n but not on any of the entries of A and is, thus, a "worst case" upper bound. Do the same for elementary row operations of Types II and III. In the next section, we'll construct some "worst case" matrices for Type III operations.

4. Challenge

Definition: We say that an $m \times n$ matrix <u>has property $(\star)$</u> if the Gaussian elimination algorithm as described above transforms A into reduced row echelon form using

$$(m-1)\min\{m,n\}$$

elementary row operations of Type III, all nontrivial, and no elementary row operations of any other type.

(1) $(**)$ Let $m \geq n$. Find an $m \times n$ matrix with property $(\star)$.

This problem is hard. Below are some subproblems to steer you in a useful direction.

(a) Let $A_1 = \begin{pmatrix} 1 \end{pmatrix}$ and let A_2, A_3 and A_4 be as in (1f), (1g), and (1h) of the previous section. Verify that A_n has property $(\star)$.

(b) Find 2×1, 3×2, 4×3 and 5×4 matrices that have property $(\star)$. Hint: Add rows onto A_1, A_2, A_3 and A_4.

(c) Let A_n be an $n \times n$ matrix with property $(\star)$. Show that

$$B_{n+1} = \begin{pmatrix} A_n \\ \mathbf{a}_n \end{pmatrix}$$

has property $(\star)$, where $\mathbf{a}_n$ is the n-th row of A_n.

(d) Let A_n and $\mathbf{a}_n$ be as above. Let $\mathbf{b}_n$ be an n-component column vector, let $\mathbf{1}$ be the n-component column vector consisting entirely of 1s, and let x be a number. Suppose that the $j = 1$ through $j = n$ stages Gaussian elimination transforms

$$\begin{pmatrix} A_n & \mathbf{b}_n \\ \mathbf{a}_n & x \end{pmatrix} \qquad \text{into} \qquad \begin{pmatrix} I_n & \mathbf{1} \\ \mathbf{0} & 1 \end{pmatrix}$$

to I_n has the effect of reducing $\begin{pmatrix} A_n & \mathbf{b}_n \end{pmatrix}$ to $\begin{pmatrix} I_n & \mathbf{1} \end{pmatrix}$. For $n = 1, 2, 3, 4$, find $\mathbf{b}_n$ and x. Find a 5×5 matrix A_5 with property $(\star)$.

(e) Can you describe an $n \times n$ matrix A_n with property $(\star)$ for arbitrary n? Look for patterns in the matrices you've already computed.