MATH 213 – FALL 2015 – SUPPLEMENTARY PROBLEMS

1. Problems

(1) Do the problems in Chapter 2 of Kuttler (the text-e-book) dealing with transpose, inverse and elementary matrices.

(2) (a) Let m, n_1, n_2 and p be positive integers. Let S^1, S^2, T_1 and T_2 have sizes $m \times n_1$, $m \times n_2$, $n_1 \times p$ and $n_2 \times p$, respectively. Prove that

$$\begin{pmatrix} S^1 & S^2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = S^1 T_1 + S^2 T_2.$$

You might proceed as follows: Let s^k_{ij} (respectively, t^k_{ij}) be the (i,j)-entry of S^k (respectively, T_k) for $1 \leq i \leq m$ and $1 \leq j \leq n_k$ (respectively, $1 \leq i \leq n_k$ and $1 \leq j \leq p$). Let $S = \begin{pmatrix} S^1 & S^2 \end{pmatrix}$, let $T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ and let s_{ij} (respectively, t_{ij}) be the (i,j)-entry of S (respectively, T). Prove that

$$s_{ij} = \begin{cases} s^1_{ij} & \text{if } 1 \leq j \leq n_1, \\ s^2_{i,?} & \text{if } n_1 + 1 \leq j \leq n_1 + n_2. \end{cases}$$

(Replace the ? with the correct column index.) Prove a similar statement expressing the entries of T in terms of those of T_1 and T_2. Now, use the definition of matrix multiplication to evaluate the (i,j)-entry of ST, using the above to express it in terms of the entries of S^1, S^2, T_1 and T_2. Finally, use the definitions of matrix multiplication and addition to evaluate the (i,j)-entry of $S^1T_1 + S^2T_2$. This should match the previous calculation.

(b) Let m_1, m_2, n, p_1 and p_2 be positive integers. Let U_1, U_2, V^1 and V^2 have sizes $m_1 \times n$, $m_2 \times n$, $n \times p_1$ and $n \times p_2$, respectively. Prove that

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} V^1 & V^2 \end{pmatrix} = \begin{pmatrix} U_1V^1 & U_1V^2 \\ U_2V^1 & U_2V^2 \end{pmatrix}.$$

First establish the following facts: For any matrix U with n columns and any matrix W with n rows, we have

$$U \begin{pmatrix} V^1 & V^2 \end{pmatrix} = \begin{pmatrix} UV^1 & UV^2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} W = \begin{pmatrix} U_1W \\ U_2W \end{pmatrix}.$$

(Actually, I already gave you this exercise on a previous worksheet.)

(c) Let m_1, m_2, n_1, n_2, p_1, p_2 be positive integers. For $i = 1,2$ and $j = 1,2$, let A_{ij} be an $m_i \times n_j$ matrix and let B_{ij} be an $n_i \times p_j$ matrix. Prove that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}.$$

You might proceed as follows: Let $A^j = \begin{pmatrix} A_{1j} \\ A_{2j} \end{pmatrix}$ and $B_i = \begin{pmatrix} B_{i1} & B_{i2} \end{pmatrix}$ and apply (a) to deduce that

$$AB = A^1B_1 + A^2B_2.$$

Now use (b) to evaluate the right hand side.

(3) Let A be an $m \times m$ matrix and B be an $n \times n$ matrix. Show that the *block diagonal matrix*

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$$

is invertible if and only if A and B are. (Note that the two $\mathbf{0}$s in the above matrix have differnt sizes.)

(4) Let m_1, m_2, n_1 and n_2 be positive integers. For $i = 1, 2$ and $j = 1, 2$, let A_{ij} be an $m_i \times n_j$ matrix. Prove that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix}.$$

What conditions need to be imposed on the A_{ij} to guarantee that A is symmetric? skew-symmetric?

(5) Let A be an $n \times n$ matrix and let m be a positive integer.
  (a) Suppose that A is invertible. Show that A^m is invertible.
  (b) Suppose that A^m is invertible. Show that A is invertible.

(6) Let $u_{ij}(a)$ be the 3×3 elementary matrix corresponding to the elementary row operation "add a times row i to row j". Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find a matrix B such that

$$B^{-1} = u_{12}(3)Au_{23}(5).$$

(7) Let a, b and c be numbers. Find

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1}.$$

(8) Let A and B be 2×2 matrices. Is it always true that

$$(A + B)^2 = A^2 + 2AB + B^2?$$

Provide a proof or a counterexample.

(9) Let

$$A = \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & e \end{pmatrix}.$$

Prove that A satisfies the equation

$$X^3 - (a + b + c)X^2 + (ab + ac + bc)X - abcI = \mathbf{0}$$

and that B satisfies the equation

$$X^3 - (a + d + e)X^2 + (ad + ae + de - bc)X - e(ad - bc)I = \mathbf{0},$$

where $\mathbf{0}$ is the 3×3 zero matrix and I is the 3×3 identity matrix.

(10) Let A be an $m \times n$ matrix, let B be an $n \times m$ matrix, and suppose that AB is invertible. Prove that A has a right inverse and that B has a left inverse. (A right inverse of an $m \times n$ matrix A is an $n \times m$ matrix U such that $AU = I$. The notion of left inverse is defined similarly.) (Hint: Since AB is invertible, there is a matrix C such that $(AB)C = I$. Stare at this equation until you see the right inverse of A.)

(11) We say that an $m \times n$ matrix B is *left invertible* (respectively, right invertible) if there is an $n \times m$ matrix U (respectively, an $n \times m$ matrix V) such that $UB = I_n$ (respectively, $BV = I_m$), in which case we say that U (respectively, V) is the left (respectively, right) inverse of A. Let A be an $n \times n$ matrix and suppose that A is left invertible with left inverse U. What is the reduced row echelon form of A? Justify your answer.

(12) Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and let V be the vector space of functions $f : \mathbb{Z} \to \mathbb{R}$. For $f \in V$, define the *support of* f, written $\operatorname{supp} f$, by $\operatorname{supp} f = \{n \in \mathbb{Z} : f(n) \neq 0\}$. Let $W = \{f \in V : \operatorname{supp} f \text{ is a finite set}\}$.
(a) Show that W is a subspace of V.
(b) For $n \in \mathbb{Z}$, define $f_n \in W$ by

$$f_n(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Show that $\operatorname{span}\{f_n : n \in \mathbb{Z}\} = W$.

(13) Show that

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 11 \\ -12 \\ 21 \end{pmatrix}, \begin{pmatrix} 16 \\ -12 \\ 12 \end{pmatrix} \right\}$$

(14) Do the polynomials

$$p(x) = x^2 + x + 1, \quad q(x) = x^2 + 3x - 3 \quad \text{and} \quad r(x) = x^2 + 9x + 9$$

span the space P_2 of polynomials in x of degree ≤ 2?

(15) Let

$$f_1(x) = 2x+1, \quad f_2(x) = -5x+3, \quad g_1(x) = -x, \quad g_2(x) = 1-3x, \quad h_1(x) = -7x+4 \quad \text{and} \quad h_2(x) = 5x-8.$$

(a) Write f_1 and f_2 as linear combinations of g_1 and g_2.
(b) Write g_1 and g_2 as linear combinations of h_1 and h_2.
(c) Write f_1 and f_2 as linear combinations of h_1 and h_2.

(16) Write $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ as a linear combination of the matrices
(a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
(b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(17) Find a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} \right\} \quad \text{and} \quad \begin{pmatrix} -3 & 4 & x \\ 2 & 2 & y \\ -1 & 5 & z \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(18) Let P_n be the space of polynomials in x of degree $\leq n$.
(a) Show that, for any number a, the polynomials 1, $x - a$ and $\frac{1}{2}(x-a)^2$ span P_2.
(b) Show that, for any number a, the polynomials 1, $x - a$ and $\frac{1}{2}(x-a)^2$, $\frac{1}{6}(x-a)^3$ span P_3.
(c) Show that, for any number a, the polynomials 1 and

$$\frac{1}{2 \cdot 3 \cdot 4 \cdot \cdots \cdot (k-1) \cdot k}(x-a)^k, \qquad 1 \leq k \leq n,$$

span P_n.

(19) Suppose that A is a 3×3 matrix satisfying the following properties:
 (i) The leftmost two columns of A are $\mathbf{a}_1 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$ and $\mathbf{a}_2 = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}^T$.
 (ii) The vector $\mathbf{x} = \begin{pmatrix} 1 & 1 & 9 \end{pmatrix}^T$ belongs to the nullspace of A.
Prove that $C(A) = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2\}$.

(20) Let S and S' be linearly independent subsets of V. Suppose that
$$\operatorname{span} S \cap \operatorname{span} S' = \{0\}.$$
Prove that $S \cup S'$ is linearly independent.

(21) Consider the following subspaces of the vector space $\mathcal{F}$ of functions $f : \mathbb{R} \to \mathbb{R}$:
$$W = \operatorname{span}\{\cos(nt) : n = 0, 1, 2, \ldots\} \quad \text{and} \quad V = \operatorname{span}\{\sin(nt) : n = 1, 2, 3, \ldots\}.$$
 (a) Show that $\cos(mt)\cos(nt)$ and $\sin(mt)\sin(nt)$ belong to W.
 (b) Show that if $f(t), g(t) \in W$ and $h(t), k(t) \in V$ then $f(t)g(t)$ and $h(t)k(t)$ both belong to W.
 (c) Show that $\cos(mt)\sin(nt)$ belongs to V.
 (d) Show that if $f(t) \in W$ and $g(t) \in V$ then $f(t)g(t) \in V$.
 (e) Show that $W \cap V = \{0\}$. (Functions in W are even and functions in V are odd so functions in $W \cap V$ are both even and odd.)
 (f) Show that the set $\{\cos(nt) : n = 0, 1, 2, \ldots\}$ is linearly independent. You might proceed as follows:
  (i) Prove that
$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = \begin{cases} \pi & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$
  (ii) Let
$$x_1 \cos(n_1 t) + x_2 \cos(n_2 t) + \cdots + x_k \cos(n_k t) = 0$$
  be a linear dependence relation with the n_j pairwise distinct. Multiply both sides by $\cos(n_j t)$ and then integrate both sides from 0 to 2π. The identity you derive shows that $x_j = 0$.
 (g) Prove that the set $\{\sin(nt) : n = 1, 2, \ldots\}$ is linearly independent. (Hint: Adapt the procedure from (f).)
 (h) Show that the set
$$X = \{\cos(nt) : n = 0, 1, 2, \ldots\} \cup \{\sin(nt) : n = 1, 2, 3, \ldots\}$$
 is linearly independent. (Hint: Use (e), (f), (g) and (12).)
 (i) Let $f(t) \in \operatorname{span} X$:
$$f(t) = a_0 + \sum_{n=1}^{N} a_n \cos(nt) + b_n \sin(nt).$$
 Show that
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)dt \qquad \text{and} \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)dt.$$
 (Hint: Show that
$$\int_{-\pi}^{\pi} \cos(mt)\sin(nt) = 0$$
 for all $m = 0, 1, 2, \ldots$ and all $n = 1, 2, 3, \ldots$ and adapt the proofs of (f) and (g).)

2. Solutions

(1) (a) Let A, B, C and D have sizes $1 \times m$, $1 \times n$, $m \times 1$ and $n \times 1$, respectively. We claim that

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = AC + BD.$$

To see this, write

$$A = \begin{pmatrix} a_1 & a_2 \cdots & a_m \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \cdots & b_n \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} &= \begin{pmatrix} a_1 & \cdots a_m & b_1 & \cdots b_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \\ d_1 \\ \vdots \\ d_n \end{pmatrix} \\ &= a_1c_1 + \cdots + a_mc_m + b_1d_1 + \cdots b_nd_n \\ &= (a_1c_1 + \cdots + a_mc_m) + (b_1d_1 + \cdots b_nd_n) \\ &= AC + BD. \end{aligned}$$

(b) Let E, F, G and H have sizes $m \times n$, $m \times p$, $n \times q$ and $p \times q$, respectively. Then $\begin{pmatrix} E & F \end{pmatrix}$ has size $m \times (n+p)$ and $\begin{pmatrix} G \\ H \end{pmatrix}$ has size $(n+p)q$. Write E_i and F_i for the i-th rows of E and F and write G_j and H_j for the j-th rows of G and H. Then the i-th row of $\begin{pmatrix} E & F \end{pmatrix}$ is $\begin{pmatrix} E_i & F_i \end{pmatrix}$ and the j-th column of $\begin{pmatrix} G \\ H \end{pmatrix}$ is $\begin{pmatrix} G_j \\ H_j \end{pmatrix}$.

$$\begin{aligned} (i,j)\text{-entry of } \begin{pmatrix} E & F \end{pmatrix} \begin{pmatrix} G \\ H \end{pmatrix} &= \left(i\text{-th row of } \begin{pmatrix} E & F \end{pmatrix}\right)\left(j\text{-th column of } \begin{pmatrix} G \\ H \end{pmatrix}\right) \\ &= \begin{pmatrix} E_i & F_i \end{pmatrix} \begin{pmatrix} G_j \\ H_j \end{pmatrix} \\ &= E_iG_j + F_iH_j \\ &= \big((i,j)\text{-entry of } EG\big) + \big((i,j) \text{ entry of } FH\big) \\ &= (i,j)\text{-entry of } EG + FH, \end{aligned}$$

where the third equality follows from (a). Thus,

$$\begin{pmatrix} E & F \end{pmatrix} \begin{pmatrix} G \\ H \end{pmatrix} = EG + FH.$$

(c) Let U, V and w have sizes $m \times n$, $n \times p$ and $n \times q$, respectively. Let V_j and W_j be the j-th columns of V and W, respectively. Observe that

$$j\text{-th column of } \begin{pmatrix} V & W \end{pmatrix} = \begin{cases} V_j & \text{if } 1 \le j \le p, \\ W_{j-p} & \text{if } p+1 \le j \le p+q. \end{cases}$$

Then

$$j\text{-th column of } U \begin{pmatrix} V & W \end{pmatrix} = \begin{cases} UV_j = j\text{-th column of } UV & \text{if } 1 \le j \le p, \\ UW_{j-p} = (j-p)\text{-th column of } UW & \text{if } p+1 \le j \le p+q. \end{cases}$$

Note that as j runs from $p+1$ through $p+q$, W_{j-p} runs through the q columns of W from left to right and UW_{j-p} runs through the q columns of UW from left to right. Thus,

$$U\begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} UV & UW \end{pmatrix}.$$

(d) Let X, Y and Z have sizes have sizes $m \times p$, $n \times p$ and $p \times q$, respectively. We claim that

$$\begin{pmatrix} X \\ Y \end{pmatrix} Z = \begin{pmatrix} XZ \\ YZ \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} Z = \left(Z^T \begin{pmatrix} X^T & Y^T \end{pmatrix}\right)^T$$

(Subexercise: Prove that $\begin{pmatrix} X \\ Y \end{pmatrix}^T = \begin{pmatrix} X^T & Y^T \end{pmatrix}$.)

$$\begin{aligned}
&= \begin{pmatrix} Z^T X^T & Z^T Y^T \end{pmatrix}^T \\
&= \begin{pmatrix} (XZ)^T & (YZ)^T \end{pmatrix}^T \\
&= \begin{pmatrix} XZ \\ YZ \end{pmatrix}.
\end{aligned}$$

(e) For $i, j = 1, 2$, let A_{ij} and B_{ij} have sizes $m_i \times n_j$ and $n_i \times p_j$, respectively. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$A\begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} AB_1 & AB_2 \end{pmatrix} \qquad \text{(by (c))}$$

$$\begin{aligned}
AB_1 &= \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} \\
&= A_1 B_{11} + A_2 B_{21} && \text{(by (b))} \\
&= \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} B_{11} + \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} B_{12} \\
&= \begin{pmatrix} A_{11} B_{11} \\ A_{21} B_{11} \end{pmatrix} + \begin{pmatrix} A_{12} B_{12} \\ A_{22} B_{21} \end{pmatrix} && \text{(by (d))} \\
&= \begin{pmatrix} A_{11} B_{11} + A_{12} B_{21} \\ A_{21} B_{11} + A_{22} B_{21} \end{pmatrix}
\end{aligned}$$

Similarly,

$$AB_2 = \begin{pmatrix} A_{11} B_{12} + A_{12} B_{22} \\ A_{21} B_{12} + A_{22} B_{22} \end{pmatrix}.$$

Therefore,

$$AB = A\begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} A_{11} B_{11} + A_{12} B_{21} & A_{11} B_{12} + A_{12} B_{22} \\ A_{21} B_{11} + A_{22} B_{21} & A_{21} B_{12} + A_{22} B_{22} \end{pmatrix},$$

as was to be shown.

(2) Suppose A and B are invertible $m \times m$ matrices; let A^{-1} and B^{-1} be their inverses. Then by (1e),

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \begin{pmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{pmatrix} = \begin{pmatrix} AA^{-1} + \mathbf{00} & A\mathbf{0} + \mathbf{0}B^{-1} \\ \mathbf{0}A^{-1} + B\mathbf{0} & \mathbf{00} + BB^{-1} \end{pmatrix} = \begin{pmatrix} AA^{-1} & \mathbf{0} \\ \mathbf{0} & BB^{-1} \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} = I^{2m \times 2m}.$$

Similarly,

$$\begin{pmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{pmatrix} \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} = I^{2m \times 2m}.$$

Therefore, $\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{pmatrix}$.

(3) (a) Let A a $m \times n$ matrix and let B be an $m \times p$ matrix. We want to show that

$$\begin{pmatrix} A & B \end{pmatrix}^T = \begin{pmatrix} A^T \\ B^T \end{pmatrix}.$$

We just write it out. Let a_{ij} and b_{ij} be the (i, j)-entries of A and B, respectively.

$$\begin{pmatrix} A & B \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} & \cdots b_{1p} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & b_{22} & \cdots b_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} & b_{m2} & \cdots b_{mp} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \\ b_{11} & b_{21} & \cdots & b_{m1} \\ b_{12} & b_{22} & \cdots & b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1p} & b_{2p} & \cdots & b_{mp} \end{pmatrix} = \begin{pmatrix} A^T \\ B^T \end{pmatrix}.$$

(b) Setting $A = C^T$ and $B = D^T$ in (a), we get

$$\begin{pmatrix} C^T & D^T \end{pmatrix}^T = \begin{pmatrix} C \\ D \end{pmatrix}.$$

Transposing both sides gives

$$\begin{pmatrix} C^T & D^T \end{pmatrix} = \begin{pmatrix} C \\ D \end{pmatrix}^T.$$

(c) For $j = 1, 2$, let $A_j = \begin{pmatrix} A_{1j} \\ A_{2j} \end{pmatrix}$. We compute:

$$\begin{aligned} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T &= \begin{pmatrix} A_1 & A_2 \end{pmatrix}^T \\ &= \begin{pmatrix} A_1^T \\ A_2^T \end{pmatrix} && \text{(by (a)).} \end{aligned}$$

By (b), $A_1^T = \begin{pmatrix} A_{11}^T & A_{21}^T \end{pmatrix}$. Similarly, $A_2^T = \begin{pmatrix} A_{12}^T & A_{22}^T \end{pmatrix}$. Therefore,

$$\begin{pmatrix} A_1^T \\ A_2^T \end{pmatrix} = \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix}.$$

(4) (a) Let A^{-1} be the inverse of A. We claim that $(A^{-1})^m$ is the inverse of A^m.

$$
\begin{aligned}
A^m(A^{-1})^m &= A^{m-1}(AA^{-1})(A^{-1})^{m-1} && \text{(by the associativity of matrix multiplication)}\\
&= A^{m-1}I(A^{-1})^{m-1} && \text{(as } AA^{-1} = I\\
&= A^{m-1}(A^{-1})^{m-1} &&\\
&= A^{m-2}(AA^{-1})(A^{-1})^{m-2} && \text{(by the associativity of matrix multiplication)}\\
&= A^{m-2}I(A^{-1})^{m-2} && \text{(as } AA^{-1} = I)\\
&= A^{m-2}(A^{-1})^{m-2} &&\\
&\quad\vdots &&\\
&= AA^{-1} &&\\
&= I. &&
\end{aligned}
$$

(b) Suppose A^m is invertible, with inverse $(A^m)^{-1}$. Then

$$A(A^{m-1}(A^m)^{-1}) = A^m(A^m)^{-1} = I.$$

Therefore, A is invertible with inverse $A^{m-1}A^{-m}$.

(5)

$$B = (B^{-1})^{-1} = (u_{12}(3)Au_{23}(5))^{-1} = u_{23}(5)^{-1}A^{-1}u_{12}(3)^{-1} = u_{23}(-5)A^{-1}u_{12}(-3).$$