1. Vector spaces

1.1. Addition. Let V be a set.

Definition 1. An *addition law* $+$ on V is a rule that, to every pair (v_1, v_2) of elements of V, assigns a *sum element* $v_1 + v_2 \in V$ to every pair (v_1, v_2) where $v_1, v_2 \in V$ such that:

(1) (Associative law) $(v_1 + v_2) + v_3 = v_1 \oplus (v_2 + v_3)$ for all $v_1, v_2, v_3 \in V$.
(2) (Commutative law) $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$.
(3) (Neutral element) There is a *neutral element* $0 \in V$ such that $0 + v = v$ for all $v \in V$.
(4) (Additive inverse) For every $v \in V$ there is an *additive inverse* $-v \in V$ such that $v + (-v) = 0$.

Exercise 2. Here are four standard examples of addition laws. Prove that they are, in fact, addition laws by identifying the neutral elements and additive inverses and verifying the required properties.

(1) The usual addition law on $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$ is an addition law in the sense of Definition 1.
(2) Let $V = \mathbf{R}^n$ be the set of n-dimensional (row or column) vectors with entries in $\mathbf{R}$. Then addition of vectors is an addition law in the sense of Definition 1.
(3) More generally, let $V = \mathbf{R}^{m\times n}$ be the set of $m \times n$ matrices with entries in $\mathbf{R}$. Then addition of matrices is an addition law in the sense of Definition 1.
(4) Let V be the set of polynomials of degree $\le n$ with variable x and coefficients in $\mathbf{R}$:

$$V = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 : a_i \in \mathbf{R}\}.$$

Then addition of polynomials is an additon law in the sense of Definition 1.

Exercise 3. Let $v, v_1, v_2 \in V$. Prove that: (1) $v + 0 = v$, (2) $(-v) + v = 0$, (3) $-(-v) = v$, (4) $-(v_1 + v_2) = (-v_1) + (-v_2)$.

Exercise 4. Let V be a set equipped with an addition law $+$ in the sense of Definition 1. Let X be a set and let W be the set of functions $f : X \to V$. For $f, g \in W$, define $f + g \in W$ by

$$f + g : X \longrightarrow V \qquad \text{by} \qquad (f + g)(x) = f(x) + g(x).$$

(The sum $f(x) + g(x)$ is computed in V.) Define

$$0 : X \longrightarrow V \qquad \text{by} \qquad 0(x) = 0.$$

(The 0 on the right is the neutral element of V.) If $f \in W$, we define

$$-f : X \to V \qquad \text{by} \qquad (-f)(x) = -f(x).$$

(The minus on the right indicates additive inverse in V.) Show that, with $f + g$, 0 and $-f$ defined as above, we have defined an addition law on W in the sense of Definition 1.

Let V be a set equipped with an eddition law $+$ in the sense of Definition 1.

Definition 5. For $v_1, v_2 \in V$, define the *subtraction* $v_1 - v_2 \in V$ by

$$v_1 - v_2 = v_1 + (-v_2).$$

Exercise 6. Let $v, v_1, v_2 \in V$. Show that (1) $v - v = 0$, (2) $-v = 0 - v$; (3) $-(v_1 + v_2) = -v_1 - v_2$. (Which $-$ signs in these equations indicate additive inverse? Which indicate subtraction?)

1.2. **Scalar multiplcation.** Let V be a set equipped with an addition law $+$.

Definition 7. A *scalar multiplicaiton law* on V is a rule that, to each pair (x, v) with $x \in \mathbf{R}$ and $v \in V$, assigns a *product element* $xv \in V$ such that

1. $(xy)v = x(yv)$ for all $x, y \in \mathbf{R}$ and all $v \in V$.
2. $(x + y)v = xv + yv$ for all $x, y \in \mathbf{R}$ and all $v \in V$.
3. $x(v_1 + v_2) = xv_1 + xv_2$ for all $x \in \mathbf{R}$ and all $v_1, v_2 \in V$.
4. $1v = v$ for all $v \in V$.

Definition 8. A *vector space* is a set V equipped with addition and scalar multiplication laws as in Definitions 1 and 7, respectively.

Exercise 9. Define scalar multiplication laws on the sets V of Example 2 equipped with their respective addition laws. Prove that these scalar multiplcation laws satisfy the properties of Definition 7.

Exercise 10. Let X, V and W be as in Example 4. If $f \in W$ and $x \in W$, we define

$$xf : X \longrightarrow V \qquad \text{by} \qquad (xf)(x) = xf(x).$$

(The product on the right is scalar multiplication in V.) Show that this defines a scalar multiplication law on W.

Lemma 11. *Let V be a vector space.*

1. $0f = 0$ *for all* $v \in V$.
2. $(-1)v = -v$ *for all* $v \in V$.

Remark 12. The 0 on the right of (1) is the scalar $0 \in \mathbf{R}$; the 0 on the right is the neutral element $0 \in V$. The minus sign on the right hand side of V indicates additive inverse in V.

Proof.

1. We have
$$0f = (0 + 0)f = 0f + 0f.$$
Subtracting $0f$ from both sides, we get $0f = 0$.
2. We have
$$0 = 0v = (1 + (-1))v = 1v + (-1)v = v + (-1)v.$$
Subtracting v from both sides, we get $(-1)v = v$.

□

Exercise 13. Flesh out the proof of Lemma 11 by justifying the equalities and concluding sentences in each part.