MATH 213 – TEST 3 REVIEW

(1) Which of $\begin{bmatrix} -23 \\ 5 \\ -3 \\ 10 \end{bmatrix}$, $\begin{bmatrix} -47 \\ 4 \\ -34 \\ -18 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\}$?

(2) Are the vectors

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -6 \\ -4 \\ 5 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}$$

linearly independent? If not, find a nontrivial linear dependence relation among them.

(3) Find bases for the row and column spaces of

$$A = \begin{bmatrix} 0 & 4 & 8 & -20 \\ 1 & -1 & -6 & 9 \\ -5 & 6 & 32 & -50 \end{bmatrix}.$$

(4) Extend the set $S = \left\{ \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 9 \\ 3 \\ -11 \end{bmatrix} \right\}$ to a basis of $\mathbf{R}^3$.

(5) Let

$$A = \begin{pmatrix} 2 & 3 & -4 & 6 \\ 7 & -1 & 9 & -25 \\ -4 & 5 & -14 & 32 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad C = \frac{1}{31} \begin{pmatrix} 0 & 5 & 1 \\ 0 & 4 & 7 \\ 31 & -22 & -23 \end{pmatrix}.$$

Given that $CA = B$, find bases of $C(A)$, $R(A)$, $N(A)$ and $N(A^T)$.

(6) Let A be a 3×4 matrix and suppose that

$$N(A) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Let

$$B = \begin{pmatrix} A & A \end{pmatrix} \quad \text{and let} \quad C = \begin{pmatrix} A \\ A \end{pmatrix}.$$

Compute the dimensions of $C(X)$, $R(X)$, $N(X)$ and $N(X^T)$ for $X = A$, B, or C.

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(7) (a) Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbf{R}^{n\times 1}$. What is the size of the matrix $\mathbf{u}\mathbf{v}^T$? What is the rank of the matrix $\mathbf{u}\mathbf{v}^T$?

(b) Find vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^{3\times 1}$ such that

$$\mathbf{u}\mathbf{v}^T = \begin{pmatrix} 1 & 3 & 7 \\ -2 & -6 & -14 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) Show that if A is an $n \times n$ matrix of rank 1 then there is are vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^{n\times 1}$ such that $A = \mathbf{u}\mathbf{v}^T$.

(1) We put the vectors in a matrix and reduce it to RREF:

$$\begin{bmatrix} 1 & 5 & -2 & -23 & -47 \\ 0 & 0 & 1 & 5 & 4 \\ 1 & 5 & 2 & -3 & -34 \\ 1 & 4 & 4 & 10 & -18 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If follows that

$$\begin{bmatrix} 5 \\ 0 \\ 5 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ 0 \\ 5 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} -2 \\ 1 \\ 2 \\ 4 \end{bmatrix} \in W, \qquad \begin{bmatrix} -47 \\ 4 \\ -34 \\ -18 \end{bmatrix} \notin W.$$

(2) Let $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ and reduces it to RREF:

$$\begin{bmatrix} 2 & -6 & 1 \\ 1 & -4 & -3 \\ -1 & 5 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the columns of $\mathrm{RREF}(A)$ are linearly independent, so are those of A.

(3) We reduce A to RREF:

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nonzero rows of $\mathrm{RREF}(A)$ span $R(A)$:

$$R(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 & -4 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & -5 \end{bmatrix} \right\}.$$

Since columns 1 and 2 of $\mathrm{RREF}(A)$ have leading 1s, columns 1 and 2 of A form a basis of the column space:

$$C(A) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 9 \\ -50 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix} \right\}.$$

(4) Unless we're very unlucky, any random vector we add to S will yield a basis. For instance, let's try adding in the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We need to verify that $\left\{ \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 9 \\ 3 \\ -11 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly independent. To do this, we put these vectors into the columns of a matrix and reduce it to RREF:

$$\begin{bmatrix} 4 & 9 & 1 \\ 1 & 3 & 0 \\ -4 & -11 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the RREF has linearly independent columns, so does the original matrix.

(5) Note that B is the reduced row echelon form of A. Since B_1 and B_2 form a basis of $C(B)$, A_1 and A_2 form a basis of $C(A)$. As A and B are row equivalent, they have the same row space. Thus,

$$R(A) = R(B) = \operatorname{span}\{\begin{pmatrix} 1 & 0 & 1 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 & 4 \end{pmatrix}\}.$$

To find basis vectors for $N(A)$, we compute a general solution of $A\mathbf{x} = \mathbf{0}$. Letting $x_3 = s$ and $x_4 = t$, we have $x_2 = 2s - 4t$ and $x_1 = -s + 3t$, so

$$\mathbf{x} = s \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -4 \\ 0 \\ 1 \end{pmatrix}.$$

The two vectors in this expression form a basis of $N(A)$. Finally, since $AC = B$,

$$C^3 \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} = B^3 = \mathbf{0},$$

where B^i and C^i indicate the i-th row of B and C. Thus,

$$C^3 = \frac{1}{31} \begin{pmatrix} 31 & -22 & -23 \end{pmatrix} \in N(A^T).$$