MATH 213 – FALL 2015 – HOMEWORK 3

DUE: FRIDAY, 13.11.2016

(1) Let n_1 and n_2 be positive integers. For $i = 1, 2$ and $j = 1, 2$, let A_{ij} be an $n_i \times n_j$ matrix. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

(a) Find matrices X and Y such that

$$\begin{pmatrix} I & \mathbf{0} \\ X & I \end{pmatrix} A \begin{pmatrix} I & Y \\ \mathbf{0} & I \end{pmatrix} = \begin{pmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$

(b) Find $\begin{pmatrix} I & \mathbf{0} \\ X & I \end{pmatrix}^{-1}$ and $\begin{pmatrix} I & Y \\ \mathbf{0} & I \end{pmatrix}^{-1}$. (Your inverses should be block matrices.)

(c) Let $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$. Show that A is invertible if and only if A_{11} and S are both invertible, in which case A^{-1} can be expressed in terms of A_{11}^{-1}, S^{-1} and the A_{ij}. (The matrix S is called the *Schur complement* of A_{11} in A.)

(d) Use (c) to find A^{-1}, where

$$A = \begin{pmatrix} 4 & 3 & 1 & 2 \\ 5 & 4 & 0 & 1 \\ 1 & 0 & 7 & 10 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

(Subdivide A into 2×2 blocks.)

(e) Let

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}, \quad A_{21} = \begin{pmatrix} a_{31} & a_{32} \end{pmatrix} \quad \text{and} \quad A_{22} = \begin{pmatrix} a_{33} \end{pmatrix}.$$

Assuming that A_{11} is invertible, find S. Suggestion: Use the fact that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible if and only if $ad - bc \neq 0$, in which case

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Under what condition(s) is S invertible?

(f) Find a polynomial in the nine variables a_{ij} such that A is invertible if and only if this polynomial is nonzero. (Hint: Two numbers are nonzero if and only if their product is nonzero.) Express your polynomial as a sum of monomials, i.e., as a sum of terms where each individual term is a products of variables.

(g) Stare at the polynomial you constructed in (f) and notice some patterns in it.

(2) Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and let V be the vector space of functions $f : \mathbb{Z} \to \mathbb{R}$. For $f \in V$, define the *support of* f, written $\operatorname{supp} f$, by $\operatorname{supp} f = \{n \in \mathbb{Z} : f(n) \neq 0\}$. Let $W = \{f \in V : \operatorname{supp} f \text{ is a finite set}\}$.

(a) Show that W is a subspace of V.

(b) For $n \in \mathbb{Z}$, define $f_n \in W$ by

$$f_n(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Show that $\{f_n : n \in \mathbb{Z}\}$ is a basis of W.

(3) Let

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 & 19 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

(a) Find a basis for the column space of A.

(b) Find a basis for the nullspace of A.

(c) Write the rightmost column of A as a linear combination of the leftmost four columns.

(4) Let $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ be an n-dimensional row vector and let B be an $n \times p$ matrix. Prove that the n-dimensional row vector $\mathbf{a}B$ belongs to the row space of B.

(5) Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix.

(a) Prove that $R(AB) \subset R(B)$. (Hint: Use (4).)

(b) Prove that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$. (Hint: Use (a) and the fact that, for any matrix X, $\dim R(X) = \operatorname{rank} X$.)

(c) Prove that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. (Hint: Prove results analogous to (4) and (5a) for the column space and use the fact that, for any matrix X, $\dim C(X) = \operatorname{rank} X$.)