MATHEMATICS 271 FALL 2015
ASSIGNMENT 1 SOLUTIONS

1. For each of the following statements, determine whether the statement is true or false. Prove the true statements, and for the false statements, write down their negations and prove them. Note that you are allowed to use $\sqrt{2}$ as an irrational number without clarification. However, for any irrational numbers other than $\sqrt{2}$, you must prove that they are irrational.

(a) For all real numbers x and y, if x and y are irrational then $x+y$ is irrational.
Solution: This statement is false. Its negation is: "There are real numbers x and y so that x is irrational and y is irrational, but $x+y$ is rational."

For example, consider the case $x=\sqrt{2}$ and $y=-\sqrt{2}$. In this case, x and y are irrational, but $x+y=0$ which is rational. We now prove that $-\sqrt{2}$ is irrational by contradiction. Suppose that $-\sqrt{2}$ is rational; that is, $-\sqrt{2}=\dfrac{m}{n}$ for some integers m and n where $n \neq 0$. Then, $\sqrt{2}=-\left(-\sqrt{2}\right)=\dfrac{-m}{n}$ where $-m$ and n are integers, which means $\sqrt{2}$ is rational which contradicts the fact that $\sqrt{2}$ is irrational. Thus, $-\sqrt{2}$ is irrational.

(b) For all real numbers x, there exists a real number y so that $x+y$ is rational.
Solution: This statement is true and here is a proof. Let x be a real number. Put $y=-x$. Then y is a real number and $x+y=0$ which is rational.

(c) For all real numbers x, there exists a real number y so that $x-y$ is irrational.
Solution: This statement is true and here is a proof. Let x be a real number. Put $y=x-\sqrt{2}$. Then y is a real number and $x-y=\sqrt{2}$ which is irrational.

(d) For all real numbers x, there exists a real number y so that $x+y$ is rational and $x-y$ is irrational.
Solution: This statement is false. Its negation is: "There is a real number x so that for all real numbers y, $x+y$ is irrational or $x-y$ is rational." Below is a proof of its negation.

Let x be the real number 0, and suppose that y is a real number. We have two cases.

Case 1: y is irrational. Then $x+y=0+y=y$ which is irrational.

Case 2: y is rational. Then $y=\dfrac{m}{n}$ for some integers m and n where $n \neq 0$, and so $x-y=0-y=-y=\dfrac{-m}{n}$ where $-m$ and n are integer, so $x-y$ is rational.

2. For each of the following statements, determine whether the statement is true or false. Prove the true statements, and for the false statements, write down their negations and prove them.

(a) For all real numbers x and y, if x and y are not integers then $\lfloor xy \rfloor = \lfloor x \rfloor \lceil y \rceil$.
Solution: This statement is false. Its negation is: "There are real numbers x and y so that x and y are not integers and $\lfloor xy \rfloor \neq \lfloor x \rfloor \lceil y \rceil$". For example, let $x=y=1.1$. Then x and y are non-integer real numbers. Also, $\lfloor xy \rfloor = \lfloor 1.21 \rfloor = 1$, and $\lfloor x \rfloor \lceil y \rceil = \lfloor 1.1 \rfloor \lceil 1.1 \rceil = 1 \times 2 = 2$, so $\lfloor xy \rfloor \neq \lfloor x \rfloor \lceil y \rceil$.

(b) For all real numbers x and y, if x and y are not integers then $\lfloor xy \rfloor \neq \lfloor x \rfloor \lceil y \rceil$.
Solution: This statement is false. Its negation is: "There are real numbers x and y so that x and y are not integers and $\lfloor xy \rfloor = \lfloor x \rfloor \lceil y \rceil$". For example, let $x = y = 0.1$. Then x and y are non-integer real numbers. Furthermore, $\lfloor xy \rfloor = \lfloor 0.01 \rfloor = 0 = 0 \times \lceil y \rceil = \lfloor 0.01 \rfloor \lceil y \rceil = \lfloor x \rfloor \lceil y \rceil$.
(c) There exists a real number x so that x is not an integer, $x > 271$ and $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.
Solution: This statement is true. Let $x = 271.001$. Then x is a real number, $x > 271$ and $\lfloor x^2 \rfloor = \lfloor 73441.542 \rfloor = 73441 = 271^2 = \lfloor x \rfloor^2$.

(d) For all positive integers N, there exists a real number x so that x is not an integer, $x > N$ and $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.
Solution: This statement is true. Let N be a positive integer. Let $x = N + \frac{1}{3N}$. Then x is a real number and $N < x = N + \frac{1}{3N} \leq N + \frac{1}{3} < N + 1$, so N is not an integer. Since N is an integer and $N \leq x = N + \frac{1}{3N} < N + 1$, we know that $\lfloor x \rfloor = N$ and so $\lfloor x \rfloor^2 = N^2$.Next,

$$\begin{aligned} x^2 &= \left(N + \tfrac{1}{3N}\right)^2 \\ &= N^2 + \tfrac{2}{3} + \tfrac{1}{9N^2} \\ &\leq N^2 + \tfrac{2}{3} + \tfrac{1}{9} && \text{because } \tfrac{1}{9N^2} \leq \tfrac{1}{9} \\ &= N^2 + \tfrac{5}{9} \\ &< N^2 + 1. && \text{because } \tfrac{5}{9} \leq 1 \end{aligned}$$

Since N^2 is an integer and $N^2 \leq x^2 < N^2 + 1$, $\lfloor x^2 \rfloor = N^2 = \lfloor x \rfloor^2$.

3. Let N be your University of Calgary ID number.
(a) Use the Euclidean Algorithm to find $\gcd(N, 271)$.
Solution: Let's say $N = 123456$. Then

$$\begin{aligned} 123456 &= 455 \times 271 + 151 \\ 271 &= 1 \times 151 + 120 \\ 151 &= 1 \times 120 + 31 \\ 120 &= 3 \times 31 + 27 \\ 31 &= 1 \times 27 + 4 \\ 27 &= 6 \times 4 + 3 \\ 4 &= 1 \times 3 + 1 \\ 3 &= 3 \times 1 + 0 \end{aligned}$$

Thus, $\gcd(N, 271) = 1$.

(b) Use the result in part (a) to find integers x and y so that $\gcd(N, 271) = Nx + 271y$.
Solution:

123456	1	0
271	0	1
151	1	-455
120	-1	456
31	2	-911
27	-7	3189
4	9	-4100
3	-61	27789
1	70	-31889

Thus, $\gcd(N, 271) = 1 = Nx + 271y$ where $x = 70$ and $y = -31889$.

(c) Suppose that M is an integer so that $\gcd(M, 271) = \gcd(M, 2015)$. Find $\gcd(M, 271)$. Explain how you get the answer.

Solution: We note that 271 is a prime, so its positive divisors are only 1 and 271, thus $\gcd(M, 271)$ can only be 1 or 271. Next, $2015 = 5 \times 13 \times 31$, so its positive divisors are 1, 5, 13, 31, $5 \times 13 = 65$, $5 \times 31 = 155$, $13 \times 31 = 403$ and 2015 and therefore, $\gcd(M, 2015)$ can only be one of the numbers 1, 5, 13, 31, 65, 155, 403 and 2015. Thus, the only way $\gcd(M, 271) = \gcd(M, 2015)$ is when $\gcd(M, 271) = \gcd(M, 2015) = 1$.

(d) Suppose that K is an integer between 800,000 and 900,000 so that $\gcd(K, 271) > \gcd(K, 2015) > 100$. Find all possible values of K. Explain how you get the answers.

Solution: From the condition $\gcd(K, 271) > \gcd(K, 2015) > 100$ and from part (c), we conclude that $\gcd(K, 271) = 271$ and $\gcd(K, 2015) = 155$, thus K must be a multiple of both 271 and 155 and so K is a multiple of $155 \times 271 = 42005$ (this is because $\gcd(271, 155) = 1$). The only multiples of 42005 that are between $800,000$ and $900,000$ are $20 \times 42005 = 840100$ and $21 \times 42005 = 882105$. Thus, $K = 840100$ or 882105.