MATHEMATICS 271 L02 FALL 2015
ASSIGNMENT 4 SOLUTIONS

1. Let $\mathcal{T}$ be the relation on $\mathbb{Z}$ defined by:

For all $x, y \in \mathbb{Z}$, $x\mathcal{T}y$ if and only if $3 \mid x + 2y$.

(a) Prove that $\mathcal{T}$ is an equivalence relation on $\mathbb{Z}$.

Solution: First, we prove that $\mathcal{T}$ is reflexive. Let $x \in \mathbb{Z}$. Since $x + 2x = 3x$ where x is an integer, $3 \mid x + 2x$ and so $x\mathcal{T}x$.

Next, we prove that $\mathcal{T}$ is symmetric. Let $x, y \in \mathbb{Z}$. Suppose that $x\mathcal{T}y$, that is, $3 \mid x + 2y$ and therefore, $x + 2y = 3m$ for some integer m. Now, $y + 2x = 3(x + y) - (x + 2y) = 3(x + y) - 3m = 3(x + y - m)$ where $x + y - 2m$ is an integer. Thus, $3 \mid y + 2x$ and so $y\mathcal{T}x$.

Now, we prove that $\mathcal{T}$ is transitive. Let $x, y, z \in \mathbb{Z}$. Suppose that $x\mathcal{T}y$ and $y\mathcal{T}z$ that is, $3 \mid x + 2y$ and $3 \mid y + 2z$ which implies that $x + 2y = 3m$ and $x + 2y = 3n$ for some integers m and n. Now, $x + 2z = (x + 2y) + (y + 2z) - 3y = 3m + 3n - 3y = 3(m + n - y)$ where $m + n - y$ is an integer. Thus, $3 \mid x + 2z$ and so $x\mathcal{T}z$.

Since $\mathcal{T}$ is reflexive, symmetric and transitive, it is an equivalence relation.

(b) List three negative elements and three positive elements of $[3]$.

Solution: -3, -6 and -9 are three negative elements of $[3]$, and 3, 6 and 9 are three positive elements of $[3]$.

(c) How many equivalence classes are there? Explain.

Solution: There are three equivalence classes, namely, $[0]$, $[1]$ and $[2]$. Note that $(0, 1) \notin \mathcal{T}$ because $3 \nmid 0 + 2 \times 1$ so $[0] \neq [1]$. Similarly, $(0, 2) \notin \mathcal{T}$ and $(1, 2) \notin \mathcal{T}$ because $3 \nmid 0 + 2 \times 2$ and $3 \nmid 1 + 2 \times 2$ so $[0] \neq [2]$ and $[1] \neq [2]$. Thus, $[0]$, $[1]$ and $[2]$ are three different equicalence classes.

Next, we prove that each integer must be in one of these equivalence classes and so these are all equivalence classes. Let $x \in \mathbb{Z}$. then by the Quotient-Remainder Theorem, $x = 3k$ or $x = 3k + 1$ or $x = 3k + 2$ for some integer k.

Case 1: $x = 3k$. Then $x + 2 \times 0 = x = 3k$ where $k \in \mathbb{Z}$, so $3 \mid x + 2 \times 0$ which means $x\mathcal{T}0$ and therefore, $x \in [0]$.

Case 2: $x = 3k + 1$. Then $x + 2 \times 1 = x + 2 = 3k + 3 = 3(k + 1)$ where $k + 1 \in \mathbb{Z}$, so $3 \mid x + 2 \times 1$ which means $x\mathcal{T}1$ and therefore, $x \in [1]$.

Case 2: $x = 3k + 2$. Then $x + 2 \times 2 = x + 4 = 3k + 6 = 3(k + 2)$ where $k + 2 \in \mathbb{Z}$, so $3 \mid x + 2 \times 2$ which means $x\mathcal{T}2$ and therefore, $x \in [2]$.

Thus, there are three equivalence classes.

2. Let $\mathbb{Z}^+$ be the set of all positive integers. Let $\mathcal{S}$ be the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined by:

For all (a, b), $(c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(a, b)\,\mathcal{S}\,(c, d)$ if and only if $a + 2b = c + 2d$.

(a) Prove that $\mathcal{S}$ is an equivalence relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$.

Solution:

First, we prove that $\mathcal{S}$ is reflexive. Let $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Since $a + 2b = a + 2b$, we get $x\mathcal{S}x$.

Next, we prove that $\mathcal{S}$ is symmetric. Let $(a,b)\,,\ (c,d)\ \in\ \mathbb{Z}^+\times\mathbb{Z}^+$. Suppose that $(a,b)\,\mathcal{S}\,(c,d)$, that is, $a+2b=c+2d$ and therefore, $c+2d=a+2b$ and so $(c,d)\,\mathcal{S}\,(a,b)$. Now, we prove that $\mathcal{S}$ is transitive. Let $(a,b)\,,\ (c,d)\,,\ (e,f)\in\mathbb{Z}^+\times\mathbb{Z}^+$. Suppose that $(a,b)\,\mathcal{S}\,(c,d)$ and $(c,d)\,\mathcal{S}\,(e,f)$ that is, $a+2b=c+2d$ and $c+2d=e+2f$ therefore, $a+2b=a+2bc+2d=e+2f$ and so $(a,b)\,\mathcal{S}\,(e,f)$.

Since $\mathcal{S}$ is reflexive, symmetric and transitive, it is an equivalence relation.

(b) List all elements of $[(3,3)]$.
Solution: The elements of $[(3,3)]$ are $(1,4)\,,\ (3,3)\,,\ (5,2)$ and $(7,1)$.

(c) List all elements of $[(4,4)]$.
Solution: The elements of $[(4,4)]$ are $(2,5)\,,\ (4,4)\,,\ (6,3)$, $(8,2)$and $(9,1)$.

(d) Is there an equivalence class that has exactly 271 elements? Explain.
Solution: Yes. there is an equivalence class that has exactly 271 elements, namely, the equivalence class of $(1,271)$ whose elements are $(1,271)\,,\ (3,270)\,,\ (5,269)\,,\ (7,268)\,,\ ...,(541,1)$.

The equivalence class of $(2,271)$ also has exactly 271 elements.

3. Let $S=\{1,2,3,...,2015\}$. Let $\mathcal{R}$ be the relation on $\mathcal{P}\,(S)$, the power set of S, defined by

For all $A,B\in\mathcal{P}\,(S)$, $A\mathcal{R}B$ if and only if $A\cup B=S$.

(a) Is $\mathcal{R}$ reflexive, symmetric, transitive? Prove your answers.
Solution:

$\mathcal{R}$ is not reflexive because $\emptyset\in\mathcal{P}\,(S)$ but $(\emptyset,\emptyset)\notin\mathcal{R}$ because $\emptyset\cup\emptyset=\emptyset\neq S$.

$\mathcal{R}$ is symmetric. Proof: Suppose $X,Y\in\mathcal{P}\,(S)$ so that $X\mathcal{R}Y$. Since $X\mathcal{R}Y$, $X\cup Y=S$ and so $Y\cup X=X\cup Y=S$ which implies $Y\mathcal{R}X$.

$\mathcal{R}$ is not transitive because $(\emptyset,S)\in\mathcal{R}$ and $(S,\emptyset)\in\mathcal{R}$, but $(\emptyset,\emptyset)\notin\mathcal{R}$. This is because $\emptyset\cup S=S\cup\emptyset=S$ but $\emptyset\cup\emptyset=\emptyset\neq S$.

(b) Is it true that for all $X\in\mathcal{P}\,(S)$, there exists $Y\in\mathcal{P}\,(S)$ so that $(X,Y)\notin\mathcal{R}$? Prove your answer.
Solution: No, it is not true that for all $X\in\mathcal{P}\,(S)$, there exists $Y\in\mathcal{P}\,(S)$ so that $(X,Y)\notin\mathcal{R}$. We will prove the negation of this statement, which is "There is $X\in\mathcal{P}\,(S)$ so that for all $Y\in\mathcal{P}\,(S)$, $(X,Y)\in\mathcal{R}$". Consider the case $X=S$. The for any $Y\in\mathcal{P}\,(S)$, since $Y\subseteq S$, $X\cup Y=S\cup Y=S$ so $(X,Y)\in\mathcal{R}$.

(c) Let $A=\{1,2,3,...,271\}$. How many elements X of $\mathcal{P}\,(S)$ are there so that $X\mathcal{R}A$? Explain.
Solution: There are 2^{271} elements X of $\mathcal{P}\,(S)$ are there so that $X\mathcal{R}A$. Since $X\mathcal{R}A$; that is, $X\cup A=S$, and so X must contains $271,\ 272,\ 273,...,2015$. However, for each of the remaining 271 elements of S. we have 2 choices (eithet choose it or not choose it for X. Thus, there are 2^{271} such X.