MATHEMATICS 271 FALL 2015
Practice Problems 1 Solutions

1. $\forall n \in \mathbb{Z},\ n^2 + 2n$ is even.
Solution: This statement is false. Its negation is "There exists an integer n so that n^2+2n is odd." For example, in the case $n = 1$, we have $n^2+2n = 1+2 = 3$ which is odd, because $3 = 2 \times 1 + 1$ where $1 \in \mathbb{Z}$.

2. $\exists n \in \mathbb{Z}$ such that $n^3 + n$ is odd.
Solution: This statement is false. Its negation is "$\forall n \in \mathbb{Z},\ n^3 + n$ is even." What follows is a proof of the negation. Suppose that $n \in \mathbb{Z}$. We have two cases:

Case 1: n is even, that is, $n = 2k$ for some $k \in \mathbb{Z}$. Then $n^3 + n = n(n^2+1) = 2k(n^2+1)$ where $k(n^2+1)$ is an integer and therefore, $n^3 + n$ is even.

Case 2: n is odd, that is, $n = 2k+1$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^3 + n &= n\left(n^2+1\right) \\ &= n\left((2k+1)^2+1\right) \\ &= n\left(4k^2+4k+1+1\right) \\ &= n\left(4k^2+4k+2\right) \\ &= 2n\left(2k^2+2k+1\right) \end{aligned}$$

where $n(2k^2+2k+1)$ is an integer and therefore, $n^3 + n$ is even.

3. $\forall x \in \mathbb{R},\ x^2 - x \geq 0$.
Solution: This statement is false. Its negation is "There exists $x \in \mathbb{R}$ so that $x^2 - x < 0$." For example, consider the case $x = \frac{1}{3}$. Then $x \in \mathbb{R}$, and $x^2 - x = \frac{1}{9} - \frac{1}{3} = -\frac{2}{9} < 0$.

4. $\forall x \in \mathbb{Z},\ x^2 - x \geq 0$.
Solution: This statement is true. Suppose that $x \in \mathbb{Z}$. Then

$$\begin{aligned} x^2 - x &= x^2 - 2 \times \frac{1}{2} \times x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \\ &= \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} \\ &\geq -\frac{1}{4}. \end{aligned}$$

Thus, $x^2 - x$ is an **integer** larger or equals $-\dfrac{1}{4}$ and so $x^2 - x \geq 0$.

5. $\forall x, y \in \mathbb{Z}$, if $x^2 + 2x = y^2 + 2y$ then $x = y$.
Solution: This statement is false. Its negation is "$\exists x, y \in \mathbb{Z}$ so that $x^2 + 2x = y^2 + 2y$ but $x \neq y$." For example, consider the case $x = 0$ and $y = -2$. Then $x, y \in \mathbb{Z}$ and $x^2 + 2x = 0 = y^2 + 2y$ but $x \neq y$.

The converse of this statement is "$\forall x, y \in \mathbb{Z}$, if $x = y$ then $x^2 + 2x = y^2 + 2y$." and the contrapositive of this statement is "$\forall x, y \in \mathbb{Z}$, if $x \neq y$ then $x^2 + 2x \neq y^2 + 2y$."

It is clear that the converse is true. The contrapositive is false because it is logically equivalent to the original statement which is false as proven above.

6. $\forall x, y \in \mathbb{Z}$, if $2x^2 + x = 2y^2 + y$ then $x = y$.
Solution: This statement is true and here is a proof. Suppose that $x, y \in \mathbb{Z}$, and suppose that $2x^2 + x = 2y^2 + y$. Then $2x^2 + x - 2y^2 - y = 0$ which can be simplified as

$$(x - y)(2(x + y) + 1) = 0. \qquad (1)$$

We note that $2(x + y) + 1$ is an odd integer, and therefore, $2(x + y) + 1 \neq 0$ and hence from (1), we get $x - y = 0$ and so $x = y$.

The converse of this statement is "$\forall x, y \in \mathbb{Z}$, if $x = y$ then $2x^2 + x = 2y^2 + y$." and the contrapositive of this statement is "$\forall x, y \in \mathbb{Z}$, if $x \neq y$ then $2x^2 + x \neq 2y^2 + y$."

It is clear that the converse is true. The contrapositive is also true because it is logically equivalent to the original statement which is true as proven above.

7. $\forall a, b, c \in \mathbb{Z}$, if $a \mid b + c$ and $a \mid b - c$ then $a \mid b$ and $a \mid c$.
Solution: This statement is false. Its negation is "$\exists a, b, c \in \mathbb{Z}$ so that $a \mid b + c$ and $a \mid b - c$, but $a \nmid b$ or $a \nmid c$." For example, consider the case $a = 2$, $b = 1$ and $c = -1$. Then $a, b, c \in \mathbb{Z}$, $a \neq 0$ and $b + c = 0 = a \times 0$ and $b - c = 2 = a \times 1$ where $0,\ 1 \in \mathbb{Z}$. That implies $a \mid b + c$ and $a \mid b - c$, but it is clear that $a \nmid b$.

The converse of this statement is "$\forall a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid b + c$ and $a \mid b - c$." and the contrapositive of this statement is "$\forall a, b, c \in \mathbb{Z}$, if $a \nmid b$ or $a \nmid c$ then $a \nmid b + c$ or $a \nmid b - c$."

It is easy to prove that the converse is true. The contrapositive is false because it is logically equivalent to the original statement which is proven to be false above.

8. $\forall a, b, c \in \mathbb{Z}$, if $a \mid b + c$ and $a \mid 2b + c$ then $a \mid b$ and $a \mid c$.
Solution: This statement is true and here is a proof. Suppose that $a, b, c \in \mathbb{Z}$, and suppose that $a \mid b + c$ and $a \mid 2b + c$. Since $a \mid b + c$ and $a \mid 2b + c$, we know $a \neq 0$ and there are integers m and n so that $b + c = am$ and $2b + c = an$. Now,

$$b = (2b + c) - (b + c) = an - am = a(n - m), \text{ and}$$

Since $b = a(n - m)$, and $c = a(2m - n)$, where $a \neq 0$, and $n - m$ and $2m - n$ are integers, we can conclude that $a \mid b$ and $a \mid c$.

The converse of this statement is "$\forall a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid b + c$ and $a \mid 2b + c$." and the contrapositive of this statement is "$\forall a, b, c \in \mathbb{Z}$, if $a \nmid b$ or $a \nmid c$ then $a \nmid b + c$ or $a \nmid 2b + c$."

It is an easy exercise to prove that the converse is true. The contrapositive is also true because it is logically equivalent to the original statement which is true as proven above.

9. $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}$ such that $n + m$ is even.
Solution: This statement is true and here is a proof. Suppose $n \in \mathbb{Z}$. We choose $m = -n$. Then $m \in \mathbb{Z}$ and $n + m = 0$ which is even.

10. $\exists m \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$, $n + m$ is even.
Solution: This statement is false. Its negation is "$\forall m \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$ so that $n + m$ is odd." We now prove the negation. Suppose $m \in \mathbb{Z}$. We choose $n = 1 - m$. Then $n \in \mathbb{Z}$ and $n + m = 1$ which is odd.

11. $\forall r \in \mathbb{Q}, \exists m \in \mathbb{Z}$ such that $rm \in \mathbb{Z}$.
Solution: This statement is true and here is a proof. Suppose $r \in \mathbb{Q}$. Then $r = \frac{p}{q}$ for some integers p and q where $q \neq 0$. Now, put $m = q$. Then $m \in \mathbb{Z}$ and $rm = p \in \mathbb{Z}$.

12. $\exists m \in \mathbb{Z}$ such that $\forall r \in \mathbb{Q},\ rm \in \mathbb{Z}$.
Solution: This statement is true. Consider the case $m = 0$. Then $m \in \mathbb{Z}$ and for any $r \in \mathbb{Q},\ rm = 0 \in \mathbb{Z}$.

13. For all positive integers n, there exists a positive integer m so that $3 \mid n + m$.
Solution: This statement is true. Suppose that n is a positive integer. Put $m = 2n$. Then m is a positive integer and $n + m = 3n$ which is divisible by 3.

14. There exists a positive integer m so that for all positive integers n, $3 \mid n + m$.
Solution: This statement is false. Its negation is "For all positive integer m, there is a positive integers n so that $3 \nmid n + m$." We now prove the negation. Suppose that m is a positive integer. We choose $n = 2m + 1$. Then n is a positive integer and $n + m = 3m + 1$ which is not divisible by 3.