MATHEMATICS 271 WINTER 2015
Solutions to Practice Problems 2

1. $\forall x,\ y \in \mathbb{R}$, if x and y are irrational then $x + y$ is irrational.

   Solution. This statement is **false**, and we will prove this by proving that its negation is true. Its negation is

   $\exists x, y \in \mathbb{R}$ such that x and y are irrational, and $x + y$ is rational.

   Proof. Let $x = \sqrt{2}$, and let $y = -\sqrt{2}$. Then x is irrational, and it is an easy proof by contradiction to show that y is irrational.

   Suppose $-\sqrt{2}$ is rational. Then there exist $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $-\sqrt{2} = \frac{a}{b}$. This implies that $\sqrt{2} = -\frac{a}{b} = \frac{-a}{b}$. Since $-a, b \in \mathbb{Z}$ and $b \neq 0$, $\sqrt{2}$ is rational, contradicting the fact that $\sqrt{2}$ is irrational. Therefore, $-\sqrt{2}$ is irrational.

   We now have $x = \sqrt{2}$ and $y = -\sqrt{2}$ irrational, but $x + y = \sqrt{2} + (-\sqrt{2}) = 0$, and 0 is rational. $\square$

2. $\forall x,\ y \in \mathbb{R}$, if x and y are irrational then xy is irrational.

   Solution. This statement is **false**, and we will prove this by proving that its negation is true. Its negation is

   $\exists x, y \in \mathbb{R}$ such that x and y are irrational, and xy is rational.

   Proof. Let $x = \sqrt{2}$, and let $y = \frac{1}{\sqrt{2}}$. Then x is irrational, and it is an easy proof by contradiction to show that y is irrational.

   Suppose $\frac{1}{\sqrt{2}}$ is rational. Then there exist $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $\frac{1}{\sqrt{2}} = \frac{a}{b}$. Note that since $\frac{1}{\sqrt{2}} \neq 0$, we also have $a \neq 0$. This implies that $\sqrt{2} = \frac{b}{a}$. Since $a, b \in \mathbb{Z}$ and $a \neq 0$, $\sqrt{2}$ is rational, contradicting the fact that $\sqrt{2}$ is irrational. Therefore, $\frac{1}{\sqrt{2}}$ is irrational.

   We now have $x = \sqrt{2}$ and $y = \frac{1}{\sqrt{2}}$ irrational, but $xy = \sqrt{2}(\frac{1}{\sqrt{2}}) = 1$, and 1 is rational. $\square$

3. $2 - \sqrt{2}$ is irrational.

   Solution. This statement is **true**, and we will use proof by contradiction.

   Proof. Suppose that $2 - \sqrt{2}$ is rational. Then there exist $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $2 - \sqrt{2} = \frac{a}{b}$. This implies that

   $$\sqrt{2} = 2 - \frac{a}{b} = \frac{2b - a}{b}.$$

   Since $2b - a, b \in \mathbb{Z}$ and $b \neq 0$, $\sqrt{2}$ is rational, contradicting the fact that $\sqrt{2}$ is irrational. Therefore, $2 - \sqrt{2}$ is irrational. $\square$

4. $3\sqrt{2}$ is irrational.

   Solution. This statement is **true**, and we will use proof by contradiction.

   Proof. Suppose that $3\sqrt{2}$ is rational. Then there exist $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $3\sqrt{2} = \frac{a}{b}$. This implies that
   $$\sqrt{2} = \frac{a}{3b}.$$
   Now $a, 3b \in \mathbb{Z}$, and since $b \neq 0$, $3b \neq 0$, implying that $\sqrt{2}$ is rational. This contradicts the fact that $\sqrt{2}$ is irrational. Therefore, $3\sqrt{2}$ is irrational. $\square$

5. $\forall x, y \in \mathbb{R},\ \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

   Solution. This statement is **false**, and we will prove this by proving that its negation is true. Its negation is

   $\exists x, y \in \mathbb{R}$ such that $\lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$

   Proof. Let $x = y = \frac{2}{3}$. Then $x + y = \frac{4}{3}$, so
   $$\lfloor x + y \rfloor = \left\lfloor \frac{4}{3} \right\rfloor = 1,$$
   while
   $$\lfloor x \rfloor + \lfloor y \rfloor = \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor = 0 + 0 = 0.$$
   Therefore for $x = y = \frac{2}{3}$, $\lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$. $\square$

6. $\exists a \in \mathbb{R}$ so that $a \notin \mathbb{Z}$, $a > 2015$, and $\lfloor a^2 \rfloor = \lfloor a \rfloor^2$.

   Solution. This statement is **true**.

   Proof. Let $a = 2015.0001$. Then $\lfloor a \rfloor = 2015$, and $\lfloor a \rfloor^2 = 2015^2 = 4,060,225$. Also, $a^2 = 4,060,225.40300001$, so $\lfloor a^2 \rfloor = 2015^2 = 4,060,225$.
   Therefore, for $a = 2015.0001$, $\lfloor a^2 \rfloor = \lfloor a \rfloor^2$. $\square$

7. $\forall n \in \mathbb{Z}^+,\ \exists a \in \mathbb{R}$ so that $a \notin \mathbb{Z}$, $a > n$ and $\lfloor a^2 \rfloor = \lfloor a \rfloor^2$.

   Solution. This statement is **true**.

   Proof. Let $n \in \mathbb{Z}$, $n \geq 1$, and let $a = n + \frac{1}{3n}$. Then $a > n$, $\lfloor a \rfloor = n$, and $\lfloor a \rfloor^2 = n^2$. Also,
   $$a^2 = \left(n + \frac{1}{3n}\right)^2 = n^2 + \frac{2}{3} + \frac{1}{9n^2}.$$
   Now, since $n \geq 1$, $n^2 \geq 1$ and $9n^2 \geq 9$. It follows that
   $$\frac{1}{9n^2} \leq \frac{1}{9} < \frac{1}{3}.$$

Therefore,

$$n^2 + \frac{2}{3} + \frac{1}{9n^2} < n^2 + \frac{2}{3} + \frac{1}{3} = n^2 + 1,$$

implying that $a^2 < n^2 + 1$. Also, $a > n > 0$ implies $a^2 > n^2$, so $n^2 < a^2 < n^2 + 1$, and thus $\lfloor a^2 \rfloor = n^2$.

Therefore, for $a = n + \frac{1}{3n}$, $\lfloor a^2 \rfloor = \lfloor a \rfloor^2$. □

8. For all real numbers x, there exists a real number y so that $x + y$ is rational.

   Solution. This statement is **true**.

   Proof. Let $x \in \mathbb{R}$ and let $y = -x$. Then $x + y = 0$, which is rational. □

9. For all real numbers x, there exists a real number y so that $x + y$ is irrational.

   Solution. This statement is **true**.

   Proof. Let $x \in \mathbb{R}$ and let $y = -x + \sqrt{2}$. Then $x + y = x + (-x + \sqrt{2}) = \sqrt{2}$ which is irrational. Thus we have proved that for each $x \in \mathbb{R}$ there is a $y \in \mathbb{R}$ such that $x + y$ is irrational. □

10. For all real numbers x, there exists a real number y so that xy is irrational.

    Solution. This statement is **false**, and we will prove this by proving that its negation is true. Its negation is

    $\exists x$ such that for all $y \in \mathbb{R}$, xy is rational.

    Proof. Let $x = 0$. Then for all $y \in \mathbb{R}$, $xy = (0)y = 0$ which is rational. □

11. For all real numbers x, if x is irrational then $\sqrt{x}$ is irrational.

    Solution. This statement is **true**, and we will prove it by contradiction.

    Proof. Suppose that $x \in \mathbb{R}$ is irrational and $\sqrt{x}$ is rational. Then there exist $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $\sqrt{x} = \frac{a}{b}$, and

    $$x = (\sqrt{x})^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}.$$

    Since $a^2, b^2 \in \mathbb{Z}$ and $b \neq 0$, $b^2 \neq 0$ so x is rational, contradicting the assumption that x is irrational.

    Therefore, $\sqrt{x}$ is irrational. □