MATHEMATICS 271 WINTER 2011
MIDTERM

[6] **1**. **Use the Euclidean algorithm** to find $\gcd(89, 36)$. Then use your work to write $\gcd(89, 36)$ in the form $89a + 36b$ where a and b are integers.

[6] **2**. You are told that X is a set, and that $\{1, 2\} \in \mathcal{P}(X)$ but $\{1, 2, 3\} \notin \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X.
(a) Give an example of a set $A \neq \{1, 2\}$ that **must** be an element of $\mathcal{P}(X)$. Be sure to explain your answer.

(b) Give an example of a set $B \neq \{1, 2, 3\}$ that **cannot** be an element of $\mathcal{P}(X)$. Be sure to explain your answer.

[4] **3**. Use the definition of a rational number to prove:

for all real numbers r, if $2r + 1$ is rational then r is rational.

[5] **4**. Prove the following statement, either by contradiction or by writing out the contrapositive and proving that. Use the element method.

For all sets A, B and C, if $A \cap B = \emptyset$ then $(A \times C) \cap (B \times C) = \emptyset$.

[11] **5**. Let $\mathcal{S}$ be the statement:

$$\forall a, b \in \mathbb{Z}, \text{ if } 5 \mid a \text{ and } 5 \mid b \text{ then } 5 \mid (2a - b).$$

(a) Prove $\mathcal{S}$, using the definition of $\mid$ ("divides into").

(b) Write out (as simply as possible) the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.

(c) Write out (as simply as possible) the *contrapositive* of statement $\mathcal{S}$. Is it true or false? Explain.

[8] **6**. The sequence $b_1,\ b_2\ , b_3, \ldots$ of integers is defined by: $b_1 = 3$, and

$$b_n = 3b_{n-1} + 2 \text{ for all integers } n \geq 2.$$

(a) **Using mathematical induction** , and the definition of odd integers, prove that b_n is odd for all integers $n \geq 1$.

(b) Is b_n prime for all integers $n \geq 1$? Prove or disprove.

MATHEMATICS 271 L01/02 WINTER 2012
MIDTERM

[6] 1. **Use the Euclidean algorithm** to find $\gcd(99, 31)$. Then use your work to write $\gcd(99, 31)$ in the form $99a + 31b$ where a and b are integers.

[6] 2. $\mathcal{P}(\mathbb{Z})$ is the power set of the set $\mathbb{Z}$ of all integers. **Disprove** the following two statements:
(a) For all $A, B \in \mathcal{P}(\mathbb{Z})$, if $5 \in A$ and $3 \in B$ then $2 \in A - B$.

(b) For all $A, B \in \mathcal{P}(\mathbb{Z})$, if $5 \in A$ and $3 \in B$ then $2 \notin A - B$.

[9] 3. $\mathbb{Q}^+$ is the set of all positive rational numbers, and $\mathbb{Z}^+$ is the set of all positive integers. Two of the following three statements are true and one is false. Prove the true statements. Write out and prove the **negation** of the false statement.
(a) $\forall q \in \mathbb{Q}^+ \ \exists n \in \mathbb{Z}^+$ so that $qn \in \mathbb{Z}$.

(b) $\forall q \in \mathbb{Q}^+ \ \exists n \in \mathbb{Z}^+$ so that $qn \notin \mathbb{Z}$.

(c) $\forall q \in \mathbb{Q}^+ \ \exists n \in \mathbb{Z}^+$ so that $q/n \notin \mathbb{Z}$.

[13] 4. Let $\mathcal{S}$ be the statement:

$$\forall a, b \in \mathbb{Z}, \text{ if } 3 \mid a \text{ and } 4 \mid b \text{ then } 12 \mid ab.$$

(a) Prove $\mathcal{S}$, using the definition of $\mid$ ("divides into").

(b) Write out the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.

(c) Write out the *contrapositive* of statement $\mathcal{S}$. Is it true or false? Explain.

(d) Write out the *negation* of statement $\mathcal{S}$. Is it true or false? Explain.

[6] 5. **Use mathematical induction** to prove that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

for all integers $n \geq 1$.

MATHEMATICS 271 L01/02 WINTER 2013
MIDTERM

[6] 1. **Use the Euclidean algorithm** to find $\gcd(78, 59)$. Then use your work to write $\gcd(78, 59)$ in the form $78a + 59b$ where a and b are integers.

[4] 2. You are given that A and B are sets, and that $(2, 3) \in A \times B$, but $(2, 4) \notin A \times B$.
(a) Find an ordered pair that definitely is in $B \times A$. Explain.

(b) Find another ordered pair ($\neq (2, 4)$) that definitely is **not** in $A \times B$. Explain.

[11] 3. Two of the following three statements are true, and one is false. Prove the true statements. Write out and prove the **negation** of the false statement. Use no properties of even and odd integers other than their definitions.
(a) $\forall n \in \mathbb{Z}\ \exists$ a prime number p so that pn is even.

(b) $\forall n \in \mathbb{Z}\ \exists$ a prime number p so that pn is odd.

(c) $\forall n \in \mathbb{Z}\ \exists$ a prime number p so that $p + n$ is odd.

[13] 4. Let $\mathcal{S}$ be the statement:

$$\text{for all sets } A, B \text{ and } C, \text{ if } A \cap B = \emptyset \text{ then } (A - C) \cap (B - C) = \emptyset.$$

(a) Write out the contrapositive of $\mathcal{S}$.

(b) Prove $\mathcal{S}$, using contradiction or the contrapositive.

(c) Write out the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.

(d) Write out the *negation* of statement $\mathcal{S}$. Is it true or false? Explain.

[6] 5. **Use mathematical induction** to prove that $2^n \geq 5n - 7$ for all integers $n \geq 3$.

MATHEMATICS 271 L01/02 WINTER 2014
MIDTERM

[6] **1**. Use the Euclidean algorithm to find $\gcd(124, 54)$. Then use your work to write $\gcd(124, 54)$ in the form $124a + 54b$ where a and b are integers.

[4] **2**. Disprove the statement: "For all integers a, b and c, if $a \mid bc$ then $a \mid b$ or $a \mid c$." by writing out its negation and then prove that.

[12] **3**. Prove the following statements. You can use the fact that $\sqrt{2}$ is irrational. For irrational numbers other than $\sqrt{2}$, you need to explain why they are irrational.
(a) For all **non-zero** real numbers a and b, if a is rational and b is irrational then ab is irrational.

(b) There are irrational numbers x and y so that $x + y$ is rational.

(c) There are irrational numbers x and y so that $x + y$ is irrational.

[10] **4**. Of the two following statements, one is true and one is false. Prove the true statement, and for the false statement, write out its negation and prove that. Use the element method.
(a) For all sets A, B and C, if $A - B = C$ then $A = B \cup C$.

(b) For all sets A, B and C, $(A - B) - C \subseteq A - (B - C)$.

[8] **5**. Prove by induction on n that $\sum_{i=1}^{n-1} i\,(i+1) = \dfrac{n\,(n-1)\,(n+1)}{3}$ for all integers $n \geq 2$.