MATHEMATICS 271 L01 FALL 2015
QUIZ 2 SOLUTIONS

1. Use the Euclidean Algorithm to find $\gcd(156, 115)$ and find integers x and y such that $\gcd(156, 115) = 156x + 115y$.
Solution: We have

$$\begin{aligned} 156 &= 1 \times 115 + 41 \\ 115 &= 2 \times 41 + 33 \\ 41 &= 1 \times 33 + 8 \\ 33 &= 4 \times 8 + 1 \\ 8 &= 8 \times 1 + 0 \end{aligned}$$

so $\gcd(156, 115) = 1$, and using the "table method":

	156	1	0
	115	0	1
$R_1 - R_2$	41	1	-1
$R_2 - 2R_3$	33	-2	3
$R_3 - R_4$	8	3	-4
$R_4 - 4R_5$	1	-14	19

Thus, $\gcd(156, 115) = 1$ and $\gcd(156, 115) = 156 \times (-14) + 115 \times (19)$, that is, $x = -14$ and $y = 19$.

2. Prove that for all **non-zero** real numbers x and y, if x is rational and y is irrational then xy is irrational.
Solution: Suppose that x and y are non-zero real numbers. Suppose that x is rational and y is irrational. We prove that xy is irrational by contradiction. Suppose that xy is rational. Since x and xy are rational, there are integers m, n, p, q so that $x = \frac{m}{n}$ and $xy = \frac{p}{q}$ where $n \neq 0 \neq q$. Note that $m \neq 0$ also because $x \neq 0$. Now, since $x \neq 0$, $y = \dfrac{xy}{x} = \dfrac{\frac{p}{q}}{\frac{m}{n}} = \dfrac{pn}{mq}$ where pn and mq are integers, and $mq \neq 0$ because $m \neq 0 \neq q$. Thus, y is rational, which contradicts the assumption that y is irrational. Thus, xy is irrational.

3. Prove by induction on n that $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.
Solution: We prove this by induction on n.
Basis: $(n = 1)$
When $n = 1$, $5^n - 4n - 1 = 5^1 - 4 \times 1 - 1 = 0 = 16 \times 0$ where $0 \in \mathbb{Z}$, so $5^n - 4n - 1$ is divisible by 16 when $n = 1$.
Inductive Hypothesis: Suppose that for some integer $k \geq 1$,

$$5^k - 4k - 1 \text{ is divisible by 16.} \qquad (IH)$$

We want to show that $5^{k+1} - 4(k+1) - 1$ is divisible by 16.

From (IH), there is an integer m so that $5^k - 4k - 1 = 16m$.

Now,

$$\begin{aligned}
5^{k+1} - 4(k+1) - 1 &= 5 \times 5^k - 4k - 4 - 1 \\
&= 5 \times 5^k - 4k - 5 \\
&= 5 \times 5^k - 20k - 5 + 16k \\
&= 5\left(5^k - 4k - 1\right) + 16k \\
&= 5 \times 16m + 16k \qquad \text{because } 5^k - 4k - 1 = 16m \\
&= 16(5m + k).
\end{aligned}$$

Since $5^{k+1} - 4(k+1) - 1 = 16(5m + k)$ where $5m + k$ is an integer, $5^{k+1} - 4(k+1) - 1$ is divisible by 16, as required.

Thus, $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.