MATHEMATICS 271 WINTER 2015
Practice Problems 3 Solutions

For each of the statements from1 to 10, prove or disprove the statement.

1. $\forall A \subseteq \mathbb{Z},\ \exists B \subseteq \mathbb{Z}$ so that $1 \in B - A$.
Solution: This statement is false. Its negation is: "$\exists A \subseteq \mathbb{Z}$ so that $\forall B \subseteq \mathbb{Z}$, $1 \notin B - A$.", and a proof of the negation is as follows. Let $A = \{1\}$. Then $A \subseteq \mathbb{Z}$, and for any set $B \subseteq \mathbb{Z}$, we see that $1 \notin B - A$ (because $1 \in A$). Actually, we can choose A to be any subset of $\mathbb{Z}$ which contains the element 1.

Comment: To prove the existence of such a set A, you must choose a specific subset A of $\mathbb{Z}$ first and then prove that for all subsets B of $\mathbb{Z}$, $1 \notin B - A$.

2. $\forall A \subseteq \mathbb{Z},\ \exists B \subseteq \mathbb{Z}$ so that $1 \notin B - A$.
Solution: This statement is true and here is a proof. Let $A \subseteq \mathbb{Z}$. Let $B = \emptyset$. Then $B - A = \emptyset - A = \emptyset$, and $1 \notin \emptyset$ and so $1 \notin B - A$. Actually, we can choose B to be any subset of $\mathbb{Z}$ which does not contain the element 1.

3. For all sets $A,\ B$ and C, if $A \cup B = C$ then $C \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
Solution: This statement is false. Its negation is: "There exist sets $A,\ B$ and C so that $A \cup B = C$ but $C \notin \mathcal{P}(A) \cup \mathcal{P}(B)$", and a proof of the negation is as follows. Let $A = \{1\}\,,\ B = \{2\}$ and $C = \{1,2\}$. In this case, $A \cup B = C$, but $C \nsubseteq A$ and $C \nsubseteq B$, so $C \notin \mathcal{P}(A)$ and $C \notin \mathcal{P}(B)$ and therefore $C \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

4. For all sets A, B and C, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.
Solution: This statement is true and here is a proof. Let A, B and C be sets. We prove that $(A \cup B) \cap C \subseteq A \cup (B \cap C)$. Let $x \in (A \cup B) \cap C$; that is, $x \in A \cup B$ and $x \in C$. Now, from $x \in A \cup B$ we have $x \in A$ or $x \in B$. Thus, we have two cases:

Case 1: $x \in A$ and $x \in C$. Since $x \in A$, we get $x \in A \cup (B \cap C)$.

Case 2: $x \in B$ and $x \in C$. Since $x \in B$ and $x \in C$, we get $x \in B \cap C$ and so $x \in A \cup (B \cap C)$.

Thus, for all sets A, B and C, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

5. For all sets A, B and C, $A \cup (B \cap C) \subseteq (A \cup B) \cap C$.
Solution: This statement is false. Its negation is: "There exist sets $A,\ B$ and C so that $A \cup (B \cap C) \nsubseteq (A \cup B) \cap C$", and a proof of the negation is as follows. Let $A = \{1\}$ and $B = C = \emptyset$. Then

$$A \cup (B \cap C) = A \cup (\emptyset \cap C) = A \cup (\emptyset \cap C) = A = \{1\} \nsubseteq \emptyset = (A \cup B) \cap \emptyset = (A \cup B) \cap C.$$

6. For all sets A, B and C, if $A \times B = A \times C$ then $B = C$.
Solution: This statement is false. Its negation is: "There exist sets $A,\ B$ and C so that $A \times B = A \times C$ but $B \neq C$", and a proof of the negation is as follows. Let $A = B = \emptyset$ and $C = \{1\}$. Then $A \times B = \emptyset \times B = \emptyset = \emptyset \times C = A \times C$ but $B \neq C$.

7. For all sets A, B and C, if $A - B \subseteq C$ then $A - C \subseteq B$.
Solution: This statement is true and here is a proof. Let A, B and C be sets so that $A - B \subseteq C$. We prove that $A - C \subseteq B$. Let $x \in A - C$. Then $x \in A$ and $x \notin C$. We prove that $x \in B$ by contradiction. Suppose that $x \notin B$. Then from $x \in A$ and $x \notin B$ we get $x \in A - B$. Now, since $x \in A - B$ and $A - B \subseteq C$, we get $x \in C$. Thus, we have the contradiction $x \in C$ and $x \notin C$. Thefore, $x \in B$.

8. For all sets A, B and C, if $A \cap B \subseteq C$ and $B \cap C \subseteq A$ then $C \cap A \subseteq B$.
Solution: This statement is false. Its negation is: "There exist sets A, B and C so that $A \cap B \subseteq C$ and $B \cap C \subseteq A$ but $C \cap A \nsubseteq B$", and a proof of the negation is as follows. Let $A = C = \{1\}$ and $B = \emptyset$. Then $A \cap B = A \cap \emptyset = \emptyset \subseteq C$ and $B \cap C = \emptyset \cap C = \emptyset \subseteq A$, but $A \cap C = \{1\} \nsubseteq \emptyset = B$.

9. For all sets A, B and C, if $A - (B \cap C) = \emptyset$ then $A - C \subseteq B$.
Solution: This statement is true and here is a proof. Let A, B and C be sets so that $A - (B \cap C) = \emptyset$. We prove that $A - C \subseteq B$. Let $x \in A - C$. Then $x \in A$ and $x \notin C$. We prove that $x \in B$ by contradiction. Suppose that $x \notin B$. Then from $x \notin B$ we get $x \notin B \cap C$. Now, since $x \in A$ and $x \notin B \cap C$, we get $x \in A - (B \cap C)$, which contradicts $A - (B \cap C) = \emptyset$. Therefore, $x \in B$.

10. For all sets A, B and C, if $A - C \subseteq B$ then $A - (B \cap C) = \emptyset$.
Solution: This statement is false. Its negation is: "There exist sets A, B and C so that $A - C \subseteq B$ but $A - (B \cap C) \neq \emptyset$", and a proof of the negation is as follows. Let $A = B = \{1\}$ and $C = \emptyset$. Then $A - C = A - \emptyset = A = \{1\} = B \subseteq B$, but $A - (B \cap C) = A - (B \cap \emptyset) = A - \emptyset = \{1\} \neq \emptyset$.

11. You are given that A and B are arbitrary subsets of the set $\mathbb{Z}$ of all integers such that $A \cap B = \{1\}$.
(a) Find an element of $A \times B$. Explain why it is an element of $A \times B$.
Solution: An element of $A \times B$ is $(1, 1)$ because from $A \cap B = \{1\}$, we know $1 \in A \cap B$, so $1 \in A$ and $1 \in B$ and therefore $(1, 1) \in A \times B$.

(b) Find an element of the complement $(A \times B)^c$. (Here, assume the universal set is $\mathbb{Z} \times \mathbb{Z}$.) Explain.
Solution: An element of $(A \times B)^c$ is $(2, 2)$ because from $A \cap B = \{1\}$, we know $2 \notin A \cap B$, so $2 \notin A$ or $2 \notin B$ and therefore $(2, 2) \notin A \times B$, that is, $(2, 2) \in (A \times B)^c$.