MATHEMATICS 271 L01 FALL 2015
ASSIGNMENT 3 SOLUTIONS

1. Let $S = \{1000, 1001, 1002, ..., 9999\}$. For each of the following questions, you must simplify your answers and explain how you got the answers.
(a) How many numbers in S have at least one digit that is a 2 or a 5?
Solution: The answer to this question is $9 \times 10^3 - 7 \times 8^3 = 5416$, which is the number of numbers in S minus the number of numbers in S which do not have a 2 nor a 5 (for there are 7 choices (1,3,4,6,7,8, or 9) for the first digit and there are 8 choices (0,1,3,4,6,7,8, or 9) for each of the 3 remaining digits).

(b) How many numbers in S have at least one digit that is a 2 and at least one digit that is a 5?
Solution: Let A be the set of numbers in S which have no 2's and let B be the set of numbers in S which have no 5's. Then, with the reasons similar to part (a), we see that $|A| = |B| = 8 \times 9^3$ and $|A \cap B| = 7 \times 8^3$. The answer to part (b) is

$$|S| - |A \cup B| = |S| - (|A| + |B| - |A \cap B|) = 9 \times 10^3 - (2 \times 8 \times 9^3 - 7 \times 8^3) = 920.$$

(c) How many numbers in S have the property that the sum of its digits is even?
Solution: The answer to (c) is $9 \times 10 \times 10 \times 5 = 4500$. The recipe is (1) Choose the first digit, (2) Choose the second digit, (3) Choose the third digit, and (4) Choose the last digit. There are 9 choices for the first step, 10 choices for the second and third step. For the last step, we have 5 choices (if the sum of the first three is odd then the last digit must be one of the 5 odd digits, and if the sum of the first three is even then the last digit must be one of the 5 even digits).

(d) How many numbers in S have the property that the digits appear in increasing order (that is, the first digit is smaller than the second digit, the second digit is smaller than the third digit, and the third digit is smaller than the fourth digit)?
Solution: The answer to this question is $\binom{9}{4}$, that is, we choose 4 digits from 1 to 9 and then arrange them in the increasing order.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2\lfloor x \rfloor - x$ for each $x \in \mathbb{R}$.
(a) Prove that f is one-to-one.
Solution: Let $a, b \in \mathbb{R}$ and suppose that $f(a) = f(b)$; that is,

$$2\lfloor x \rfloor - x = 2\lfloor y \rfloor - y. \qquad (1)$$

From the definition of floors, we see that $x = \lfloor x \rfloor + x_1$ and $y = \lfloor y \rfloor + x_1$ for some real numbers x_1 and y_1 where $0 \le x_1 < 1$ and $0 \le y_1 < 1$. From this we know that

$$-1 < x_1 - y_1 < 1 \qquad (2)$$

Now, (1) becomes $2\lfloor x \rfloor - (\lfloor x \rfloor + x_1) = 2\lfloor y \rfloor - (\lfloor y \rfloor + y_1)$ which implies:

$$x_1 - y_1 = \lfloor x \rfloor - \lfloor y \rfloor \qquad (3)$$

From (2) and (3), we see that $x_1 - y_1$ is an integer strictly between -1 and 1, and therefore $x_1 - y_1 = 0 = \lfloor x \rfloor - \lfloor y \rfloor$. Thus, $x = \lfloor x \rfloor + x_1 = \lfloor y \rfloor + x_1 = y$. Therefore, f is one-to-one.

(b) Prove that f is onto.
Solution: Let $y \in \mathbb{R}$. Put $x = 2\lceil y \rceil - y$. We will show that $f(x) = y$. From the definition of ceiling, we know $\lceil y \rceil - 1 < y \leq \lceil y \rceil$ which implies that

$$0 \leq \lceil y \rceil - y < 1 \qquad (4)$$

Now, by adding $\lceil y \rceil$ to (4), we get $\lceil y \rceil \leq 2\lceil y \rceil - y < \lceil y \rceil + 1$, but we know $x = 2\lceil y \rceil - y$, and so we have

$$\lceil y \rceil \leq x < \lceil y \rceil + 1 \qquad (5)$$

Now using the fact that $\lceil y \rceil$ is an integer and so (5) says that $\lfloor x \rfloor = \lceil y \rceil$. Since $x = 2\lceil y \rceil - y$ and $\lfloor x \rfloor = \lceil y \rceil$, we get $f(x) = 2\lfloor x \rfloor - x = 2\lceil y \rceil - (2\lceil y \rceil - y) = y$. Thus, f is onto.

(c) From (a) and (b) we see that f is invertible. Find a formula for $f^{-1}(x)$ for $x \in \mathbb{R}$. Verify that $f \circ f^{-1} = f^{-1} \circ f = I_{\mathbb{R}}$.
Solution: From part (b), we can see that for each $x \in \mathbb{R}$, $f^{-1}(x) = 2\lceil x \rceil - x$. Now, to verify $f \circ f^{-1} = f^{-1} \circ f = I_{\mathbb{R}}$, we need to show that $f \circ f^{-1}(x) = f^{-1} \circ f(x) = I_{\mathbb{R}}(x)$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. We have seen in part (b) that $f(2\lceil y \rceil - y) = y$ for all $y \in \mathbb{R}$. Hence $f(2\lceil x \rceil - x) = x$ and so $f \circ f^{-1}(x) = f(f^{-1}(x)) = f(2\lceil x \rceil - x) = x = I_{\mathbb{R}}(x)$. Thus, $f \circ f^{-1} = i_{\mathbb{R}}$.

Next, we want to show that $f^{-1} \circ f = I_{\mathbb{R}}$. Since $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, we get $0 \leq x - \lfloor x \rfloor < 1$ and so $-1 < \lfloor x \rfloor - x \leq 0$. Hence by adding $\lfloor x \rfloor$ we get $\lfloor x \rfloor - 1 < 2\lfloor x \rfloor - x \leq \lfloor x \rfloor$. which together with the fact that $\lfloor x \rfloor$ is an integer implies that $\lceil 2\lfloor x \rfloor - x \rceil = \lfloor x \rfloor$, that is

$$\lceil f(x) \rceil = \lfloor x \rfloor \qquad (6)$$

Now,

$$\begin{aligned} f^{-1} \circ f(x) &= f^{-1}(f(x)) \\ &= 2\lceil f(x) \rceil - f(x) \\ &= 2\lfloor x \rfloor - f(x) && \text{by (6)} \\ &= 2\lfloor x \rfloor - (2\lfloor x \rfloor - x) \\ &= x \\ &= I_{\mathbb{R}}(x). \end{aligned}$$

Thus, $f^{-1} \circ f = i_{\mathbb{R}}$.

3. Let $f : A \to B$ and $g : B \to C$ be functions. Prove or disprove the following statements:
(a) If both f and g are onto then $g \circ f$ is onto.

Solution: This statement is true and here is a proof. Suppose that $f : A \to B$ and $g : B \to C$ are onto functions. We prove that $g \circ f$ is onto. Let $c \in C$. Since g is onto, there exists $b \in B$ so that $g(b) = c$. Since $b \in B$ and f is onto, there exists $a \in A$ so that $f(a) = b$. Thus, $g \circ f(a) = g(f(a)) = g(b) = c$ and therefore $g \circ f$ is onto.

(b) If $g \circ f$ is onto then f is onto.
Solution: This statement is false. For example, let $A = C = \{1\}$, $B = \{1, 2\}$. Let $f = \{(1, 1)\}$ and $g = \{(1, 1), (2, 1)\}$. Then $g \circ f = \{(1, 1)\}$ which in an onto function from A to C, but clearly f is not onto B.

(c) If $g \circ f$ is onto then g is onto.
Solution: This statement is true and here is a proof. Suppose that $g \circ f$ is onto. We prove that g is onto C. Let $c \in C$. Since $g \circ f$ is onto, there exists $a \in A$ so that $g \circ f(a) = c$. Since $a \in A$ and f is a function from A to B, $b = f(a)$ is an element of B, and $g(b) = g(f(a)) = g \circ f(a) = c$. Thus, g is onto.

(d) If $g \circ f$ is onto and g is one-to-one then f is onto.
Solution: This statement is true and here is a proof. Suppose that $g \circ f$ is onto and that g is one-to-one. We prove that f is onto B. Let $b \in B$. Since $b \in B$ and g is a function from B to C, $c = g(b)$ is an element of C. Since $c \in C$ and $g \circ f$ is onto, there exists $a \in A$ so that $g \circ f(a) = c$. Now, since $g(f(a)) = g \circ f(a) = c = g(b)$, and g is one-to-one, we get $f(a) = b$. Thus, f is onto.