MATHEMATICS 271 L01 FALL 2015
MIDTERM DOLUTIONS

[8] **1**. Use the Euclidean algorithm to find gcd (134, 28). Then use your work to find integers a and b so that gcd (134, 28) is in the form $134a + 28b$.
Solution: We have

$$\begin{aligned} 134 &= 4 \times 28 + 22 \\ 28 &= 1 \times 22 + 6 \\ 22 &= 3 \times 6 + 4 \\ 6 &= 1 \times 4 + 2 \\ 4 &= 2 \times 2 + 0 \end{aligned}$$

so $\gcd(134, 28) = 2$, and

$$\begin{aligned} \gcd(134, 28) &= 2 \\ &= 6 - 4 \\ &= 6 - (22 - 3 \times 6) \\ &= 4 \times 6 - 22 \\ &= 4 \times (28 - 22) - 22 \\ &= 4 \times 28 - 5 \times 22 \\ &= 4 \times 28 - 5 \times (134 - 4 \times 28) \\ &= -5 \times 134 + 24 \times 54 \end{aligned}$$

Another way is to use the "table method" as follows.

	134	1	0
	28	0	1
$R_1 - 4R_2$	22	1	-4
$R_2 - R_3$	6	-1	5
$R_3 - 3R_4$	4	4	-19
$R_4 - R_5$	2	-5	24

Thus, $\gcd(134, 28) = 2$ and $\gcd(134, 28) = 134 \times (-5) + 54 \times 24$, that is, $a = -5$ and $b = 24$.

[10] **2**. Of the two following statements, one is true and one is false. Prove the true statement, and for the false statement, write out its negation and prove that.
(a) For all rational numbers r, there exists a positive integer m so that rm is an integer.
Solution: This statement is true and here is a proof. Let r be a rational number. Then $r = \dfrac{a}{b}$ for some integers a and b where $b \neq 0$. Choose $m = b^2$ then m is a positive integer and $rm = ab$ which is an integer.

(b) There exists a positive integer m so that for all rational numbers r, rm is an integer.
Solution: This statement is false. Its negation is "For all positive integers m, there is a rational number r so that rm is not an integer." Below is a proof of the negation.

Suppose that m is a positive integer. Choose $r = \dfrac{1}{2m}$ then r is a rational number and $rm = \dfrac{1}{2}$ which is not an integer.

[10] **3**. Let A be the set of all integers in the range from 0 to 100 inclusively so that the sum of its digits is even and let $B = \{x \in \mathbb{Z} \mid 0 \leq x \leq 100 \text{ and } 2 \mid x\}$.
(a) Is it true that $A \subseteq B$? Prove your answer.
Solution: No, $A \nsubseteq B$ because $11 \in A$ and $11 \notin B$.

(b) Find three elements of $A \cap B$.
Solution: $2,\ 22$ and 42 are elements of $A \cap B$.

(c) Find three elements of $B - A$.
Solution: $12,\ 32$ and 52 are elements of $B - A$.

(d) How many elements does $B - A$ have? Explain on how you got your answer.
Solution: Element of $B - A$ are even integers so that the sums of their digits are odd. For example, elements of $B - A$ can be 12, 14, 16, 100, etc.

Since they are even, their last digits must be even, and the sums of the digits is odd so they must have only one odd digit. We have 5×5 such two digit integers (there are 5 choices for the first digit and 5 choices for the last digit) and one such three digit integer, namely, 100.

Thus, the answer to this question is $5 \times 5 + 1 = 26$.

[8] **4**. Prove by induction on n that $\sum_{i=1}^{n} \dfrac{1}{i\,(i+1)} = \dfrac{n}{n+1}$ for all integers $n \geq 1$.
Solution:
Base case $(n = 1)$

$$\sum_{i=1}^{n} \frac{1}{i\,(i+1)} = \frac{1}{1\,(1+1)} = \frac{1}{1+1} = \frac{n}{n+1}.$$

Inductive step: Let $k \geq 1$ be an integer and suppose that

$$\sum_{i=1}^{k} \frac{1}{i\,(i+1)} = \frac{k}{k+1}. \qquad (IH)$$

We prove that $\sum_{i=1}^{k+1} \dfrac{1}{i\,(i+1)} = \dfrac{k+1}{k+2}$.
Now,

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i\,(i+1)} &= \left(\sum_{i=1}^{k} \frac{1}{i\,(i+1)}\right) + \frac{1}{(k+1)\,(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)\,(k+2)} && \text{by } (IH) \\
&= \frac{k\,(k+2)+1}{(k+1)\,(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)\,(k+2)} \\
&= \frac{(k+1)^2}{(k+1)\,(k+2)} \\
&= \frac{k+1}{k+2}.
\end{aligned}$$

Thus, $\sum_{i=1}^{n} \frac{1}{i\,(i+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.