MATHEMATICS 271 L01 FALL 2015
ASSIGNMENT 2 SOLUTION

1. Define the sequences $a_0, a_1, a_2, ...$ and $b_0, b_1, b_2, ...$ recursively as follows:
$a_0 = 0$, and for $n > 0$, $a_n = a_{\lfloor \frac{n}{5} \rfloor} + a_{\lfloor \frac{3n}{5} \rfloor} + n$, and
$b_0 = 2$, $b_1 = 3$ and for $n > 1$, $b_n = 3b_{n-1} - 2b_{n-2}$.
(a) Find $a_1,\ a_2,\ a_3,\ a_4$ and a_5.
(b) Prove that $a_n \leq 20n$ for all integers $n \geq 0$.
(c) Find $b_2,\ b_3,\ b_4$ and b_5.
(d) Guess a formula for b_n using part (c).
(e) Prove by induction on n that your guess in part (d) is correct for all integers $n \geq 1$.

Solution:
(a)

$$\begin{array}{lcccccl} a_1 &=& a_0 + a_0 + 1 &=& 0+0+1 &=& 1 \\ a_2 &=& a_0 + a_1 + 2 &=& 0+1+2 &=& 3 \\ a_3 &=& a_0 + a_1 + 3 &=& 0+1+4 &=& 4 \\ a_4 &=& a_0 + a_2 + 4 &=& 0+3+4 &=& 7 \\ a_5 &=& a_1 + a_3 + 5 &=& 1+4+5 &=& 10 \end{array}$$

(b) We prove $a_n \leq 20n$ for all integers $n \geq 0$ by induction on n (strong form).
Basis: $(n = 0)$
$a_0 = 0 \leq 0 = 20 \times 0$.
Inductive step: Let $k \geq 1$ be an integer and suppose that for all integers m where $0 \leq m < k$,
$a_m \leq 20m \qquad (IH)$
We want to show that $a_k \leq 20k$.
Now, since $k \geq 1$,

$$\begin{array}{lll} a_k &=& a_{\lfloor \frac{k}{5} \rfloor} + a_{\lfloor \frac{3k}{5} \rfloor} + k \\ &\leq& 20\left\lfloor \frac{k}{5} \right\rfloor + 20\left\lfloor \frac{3k}{5} \right\rfloor + k \qquad \text{by } (IH) \\ &\leq& 20\frac{k}{5} + 20\frac{3k}{5} + k \qquad \text{because } \lfloor x \rfloor \leq x \text{ for all } x \in \mathbb{R} \\ &=& 4k + 12k + k \\ &=& 17k \\ &\leq& 20k \end{array}$$

Thus, $a_n \leq 20n$ for all integers $n \geq 0$.

(c)

$$\begin{array}{lcccccr} b_2 &=& 3b_1 - 2b_0 &=& 3 \times 3 - 2 \times 2 &=& 5 \\ b_3 &=& 3b_2 - 2b_1 &=& 3 \times 5 - 2 \times 3 &=& 9 \\ b_4 &=& 3b_2 - 2b_1 &=& 3 \times 9 - 2 \times 5 &=& 17 \\ b_5 &=& 3b_2 - 2b_1 &=& 3 \times 17 - 2 \times 9 &=& 33 \end{array}$$

(d) From part (c), we see that:

$$\begin{array}{rcrcl} b_0 & = & 2 & = & 2^0+1 \\ b_1 & = & 2 & = & 2^1+1 \\ b_2 & = & 5 & = & 2^2+1 \\ b_3 & = & 9 & = & 2^3+1 \\ b_4 & = & 17 & = & 2^4+1 \\ b_5 & = & 32 & = & 2^5+1 \end{array}$$

Thus, we guess that $b_n = 2^n + 1$ for all integers $n \geq 0$.

(e) We prove by induction on n that $b_n = 2^n + 1$ for all integers $n \geq 0$.

Bases: $(n = 0, 1)$

We have shown $b_0 = 2^0 + 1$ and $b_1 = 2^1 + 1$ in part (d).

Inductive step: Let $k \geq 2$ be an integer and suppose that for all integers m where $0 \leq m < k$,

$b_m = 2^m + 1 \qquad (IH)$

We want to show that $b_k = 2^k + 1$.

Now, since $k \geq 2$,

$$\begin{array}{rcll} a_k & = & 3b_{k-1} - 2b_{k-2} & \\ & = & 3\left(2^{k-1}+1\right) - 2\left(2^{k-2}+1\right) & \text{by } (IH) \\ & = & 3 \times 2^{k-1} - 2 \times 2^{k-2} + 1 & \\ & = & 6 \times 2^{k-2} - 2 \times 2^{k-2} + 1 & \\ & = & 4 \times 2^{k-2} + 1 & \\ & = & 2^k + 1 & \end{array}$$

Thus, $b_n = 2^n + 1$ for all integers $n \geq 0$.

2. Define the Fibonacci sequence $f_1, f_2, f_3, \ldots$ recursively as follows:

$f_1 = f_2 = 1$, and for $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$.

(a) Prove that for all integers $n \geq 3$, $\gcd(f_n, f_{n-1}) = \gcd(f_{n-1}, f_{n-2})$. (You may want to use Lemma 4.8.2).

(b) Prove that $\gcd(f_n, f_{n-1}) = 1$ for all integers $n \geq 2$.

(c) Prove that $\sum_{i=1}^{n} f_i^2 = f_{n+1} f_n$ for all integers $n \geq 1$.

(d) Prove that $f_n < \left(\frac{7}{4}\right)^{n-1}$ for all integers $n \geq 2$.

Solution:

(a) Let $n \geq 2$ be an integer.

Lemma 4.8.2 says that if $a = bq + r$ then $\gcd(a, b) = \gcd(b, r)$. We know

$$f_n = f_{n-1} + f_{n-2}. \qquad (1)$$

Put $a = f_n$, $b = f_{n-1}$, $q = 1$ and $r = f_{n-2}$. We see that (1) has the form $a = bq + r$, and so by Lemma 4.8.2 we can conclude that $\gcd(f_n, f_{n-1}) = \gcd(f_{n-1}, f_{n-2})$.

(b) We prove this by (normal) induction on n.

Basis: $(n = 1)$

$\gcd(f_2, f_1) = \gcd(1, 1) = 1$.

Inductive step: Let $k \geq 1$ be an integer and suppose that

$\gcd(f_k, f_{k-1}) = 1.$ $[IH]$

We want to prove that $\gcd(f_{k+1}, f_k) = 1$.

Now, by part (a) we have

$$\begin{array}{rcll} \gcd(f_{k+1}, f_k) & = & \gcd(f_k, f_{k-1}) & \\ & = & 1 & \text{by } [IH] \end{array}.$$

Thus, $\gcd(f_n, f_{n-1}) = 1$ for all integers $n \geq 2$.

(c) We prove this by (normal) induction on n.

Basis: $(n = 1)$

$\sum\limits_{i=1}^{1} f_i^2 = 1^2 = 1 = 1 \times 1 = f_2 f_1 = f_{1+1} f_1.$

Inductive step: Let $k \geq 1$ be an integer and suppose that

$\sum\limits_{i=1}^{k} f_i^2 = f_{k+1} f_k$ (IH)

We want to prove that $\sum\limits_{i=1}^{k+1} f_i^2 = f_{k+2} f_{k+1}$.

Now,

$$\begin{array}{rcl} \sum\limits_{i=1}^{k+1} f_i^2 & = & \left(\sum\limits_{i=1}^{k} f_i^2\right) + f_{k+1}^2 \\ & = & f_{k+1} f_k + f_{k+1}^2 \\ & = & (f_k + f_{k+1}) f_{k+1} \\ & = & f_{k+2} f_{k+1}. \end{array}$$

Thus, $\sum\limits_{i=1}^{n} f_i^2 = f_{n+1} f_n$ for all integers $n \geq 1$.

(d) We prove this by (strong) induction on n.

Basis: $(n = 2, 3)$

$f_2 = 1 = \frac{4}{4} < \frac{7}{4} = \left(\frac{7}{4}\right)^{2-1}$, and

$f_3 = f_2 + f_1 = 1 + 1 = 2 = \frac{32}{16} < \frac{49}{16} = \left(\frac{7}{4}\right)^{3-1}$

Inductive step: Let $k \geq 4$ be an integer and suppose that for all integers m where $2 \leq m < k$, we have

$f_m < \left(\frac{7}{4}\right)^{m-1}$ (IH)

We want to prove that $f_k < \left(\frac{7}{4}\right)^{k-1}$.

Now, since $k \geq 3$,

$$\begin{array}{rcll} f_k & = & f_{k-1} + f_{k-2} & \\ & < & \left(\frac{7}{4}\right)^{k-2} + \left(\frac{7}{4}\right)^{k-3} & \text{by } (IH)\,,\ f_{k-1} < \left(\frac{7}{4}\right)^{k-2} \text{ and } f_{k-2} < \left(\frac{7}{4}\right)^{k-3} \\ & = & \left(\frac{7}{4}\right)^{-1}\left(\frac{7}{4}\right)^{k-1} + \left(\frac{7}{4}\right)^{-2}\left(\frac{7}{4}\right)^{k-1} & \\ & = & \left(\left(\frac{7}{4}\right)^{-1} + \left(\frac{7}{4}\right)^{-2}\right)\left(\frac{7}{4}\right)^{k-1} & \\ & = & \left(\frac{4}{7} + \frac{16}{49}\right)\left(\frac{7}{4}\right)^{k-1} & \\ & = & \left(\frac{28}{49} + \frac{16}{49}\right)\left(\frac{7}{4}\right)^{k-1} & \\ & = & \frac{44}{49}\left(\frac{7}{4}\right)^{k-1} & \\ & < & \left(\frac{7}{4}\right)^{k-1} & \end{array}$$

Thus, $f_n < \left(\frac{7}{4}\right)^{n-1}$ for all integers $n \geq 2$.

3. For any sets A and B, we define the *symmetric difference* $A \triangle B$ by $A \triangle B = (A \cup B) - (A \cap B)$. Note that it is also true that $A \triangle B = (A - B) \cup (B - A)$. Let $\mathcal{S}$ be the statement: "*For all sets A, B and C, if $A \subseteq B \cup C$ and $B \subseteq C \cup A$ then $A \triangle B = C$.*" and let $\mathcal{T}$ be the statement:"*For all sets A, B and C, if $A \triangle B = A \triangle C$ then $B \subseteq C$.*"

(a) Is $\mathcal{S}$ true? Prove your answer.

Solution: $\mathcal{S}$ is not true. For example, when $A = B = \emptyset$, and $C = \{1\}$ we have $A = \emptyset \subseteq B \cup C$ and $B = \emptyset \subseteq C \cup A$, but $A \triangle B = \emptyset \neq \{1\} = C$.

(b) Is $\mathcal{T}$ true? Prove your answer.

Solution: $\mathcal{T}$ true. Let A, B and C be sets and suppose that $A \triangle B = A \triangle C$. We prove that $B \subseteq C$. Let $x \in B$. We consider two cases $x \in A$ and $x \notin A$.

Case 1: $x \in A$. Since $x \in A$ and $x \in B$, $x \in A \cap B$ and hence $x \notin A \triangle B$. Since $A \triangle B = A \triangle C$ and $x \notin A \triangle B$, we get $x \notin A \triangle C = (A \cup C) - (A \cap C)$ and therefore $x \notin A \cup C$ or $x \in A \cap C$. However, since $x \in A$, $x \in A \cup C$ and it follows that $x \in A \cap C$ and so $x \in C$.

Case 1: $x \notin A$. Since $x \notin A$ and $x \in B$, $x \in B - A$ and hence $x \in A \triangle B$. Since $A \triangle B = A \triangle C$ and $x \in A \triangle B$, we get $x \in A \triangle C = (A \cup C) - (A \cap C)$ and therefore $x \in A \cup C$, which implies that $x \in A$ or $x \in C$. However, since $x \notin A$, we see that $x \in C$.

(c) Write the converse of $\mathcal{S}$. Is the converse of $\mathcal{S}$ true? Prove your answer.

Solution: The converse of $\mathcal{S}$ is "*For all sets A, B and C, if $A \triangle B = C$ then $A \subseteq B \cup C$ and $B \subseteq C \cup A$.*"

The converse of $\mathcal{S}$ is true. Let A, B and C be sets and suppose that $A \triangle B = C$.

First, we prove that $A \subseteq B \cup C$ by contradiction. Suppose that $A \not\subseteq B \cup C$. Then there exists an element $x \in A$ so that $x \notin B \cup C$, that is, $x \notin B$ and $x \notin C$. Since $x \in A$ and $x \notin B$, $x \in A - B \subseteq A \triangle B$. Since we have $x \in A \triangle B$ and $x \notin C$, $A \triangle B \neq C$ which contradicts the assumption that $A \triangle B = C$. Thus, $A \subseteq B \cup C$.

Next, we prove that $B \subseteq C \cup A$ by contradiction. Suppose that $B \not\subseteq C \cup A$. Then there exists an element $x \in B$ so that $x \notin C \cup A$, that is, $x \notin C$ and $x \notin A$. Since $x \in B$

and $x \notin A$, $x \in B - A \subseteq A \triangle B$. Since we have $x \in A \triangle B$ and $x \notin C$, $A \triangle B \neq C$ which contradicts the assumption that $A \triangle B = C$. Thus, $A \subseteq B \cup C$.