# MATHEMATICS 271 WINTER 2011
# MIDTERM SOLUTIONS

[6] **1**. **Use the Euclidean algorithm** to find gcd (89, 36). Then use your work to write gcd (89, 36) in the form $89a + 36b$ where $a$ and $b$ are integers.
*Solution*: We have

$$\begin{aligned} 89 &= 2 \times 36 + 17 \\ 36 &= 2 \times 17 + 2 \\ 17 &= 8 \times 2 + 1, \\ 2 &= 2 \times 1 + 0 \end{aligned}$$

and so $\gcd(89, 36) = 1$, and
$\gcd(89, 36) = 1 = 17 - 8 \times 2 = 17 - 8 \times (36 - 2 \times 17) = 17 \times 17 - 8 \times 36 = 17 \times (89 - 2 \times 36) - 8 \times 36 = 17 \times 89 - 42 \times 36$.

Another way is to use the "table method" as follows.

| | | 89 | 36 |
|---|---|---|---|
| | 89 | 1 | 0 |
| | 36 | 0 | 1 |
| $R_1 - 2R_2$ | 17 | 1 | $-2$ |
| $R_2 - 2R_3$ | 2 | $-2$ | 5 |
| $R_3 - 8R_4$ | 1 | 17 | $-42$ |

Thus, $\gcd(89, 36) = 1$ and $\gcd(89, 36) = 89 \times 17 + 36 \times (-42)$, that is, $a = 17$ and $b = -42$.

[6] **2**. You are told that $X$ is a set, and that $\{1, 2\} \in \mathcal{P}(X)$ but $\{1, 2, 3\} \notin \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of $X$.
(a) Give an example of a set $A \neq \{1, 2\}$ that **must** be an element of $\mathcal{P}(X)$. Be sure to explain your answer.
*Solution*: Let $A = \emptyset$. Then it is clear that $A \neq \{1, 2\}$ and $\emptyset \subseteq X$, so $A = \emptyset \in \mathcal{P}(X)$.

The other choices for $A$ are the sets $\{1\}$ and $\{2\}$.

(b) Give an example of a set $B \neq \{1, 2, 3\}$ that **cannot** be an element of $\mathcal{P}(X)$. Be sure to explain your answer.
*Solution*: Let $B = \{3\}$. It is clear that $B \neq \{1, 2, 3\}$. It remains to explain why $B \notin \mathcal{P}(X)$. Since $\{1, 2\} \in \mathcal{P}(X)$, we know $\{1, 2\} \subseteq X$ and therefore 1 and 2 are elements of $X$. Since $\{1, 2, 3\} \notin \mathcal{P}(X)$, $\{1, 2, 3\} \nsubseteq X$ and so at least one of $1, 2, 3$ is not an element of $X$. Since we know 1 and 2 are elements of $X$, it follows that 3 is not an element of $X$. Since $3 \notin X$, $B = \{3\} \nsubseteq X$, that is, $B \notin \mathcal{P}(X)$.

[4] **3**. Use the definition of a rational number to prove:

for all real numbers $r$, if $2r + 1$ is rational then $r$ is rational.

*Solution*: Let $r$ be a real number and assume that $2r+1$ is rational, that is, $2r+1=\dfrac{m}{n}$ for some $m,\ n\in\mathbb{Z}$ where $n\neq 0$. Now, $r=\dfrac{1}{2}(2r+1-1)=\dfrac{1}{2}\left(\dfrac{m}{n}-1\right)=\dfrac{1}{2}\left(\dfrac{m-n}{n}\right)=\dfrac{m-n}{2n}$ where $m-n$ and $2n$ are integers (note that $2n\neq 0$ because $n\neq 0$). This implies that $r$ is rational.

[5] **4**. Prove the following statement, either by contradiction or by writing out the contrapositive and proving that. Use the element method.

$$\text{For all sets } A, B \text{ and } C, \text{ if } A\cap B=\emptyset \text{ then } (A\times C)\cap(B\times C)=\emptyset.$$

*Solution*: Let $A$, $B$ and $C$ be sets such that $A\cap B=\emptyset$. We prove that $(A\times C)\cap(B\times C)=\emptyset$ by contradiction. Suppose that $(A\times C)\cap(B\times C)\neq\emptyset$, that is, there exists an element $(x,y)\in(A\times C)\cap(B\times C)$. Since $(x,y)\in(A\times C)\cap(B\times C)$, we know $(x,y)\in A\times C$ and $(x,y)\in B\times C$, and so $x\in A$ and $x\in B$. Since $x\in A$ and $x\in B$, we get $x\in A\cap B$. Thus, there exists $x\in A\cap B$ and hence, $A\cap B\neq\emptyset$, which contradicts the assumption that $A\cap B=\emptyset$. Therefore, $(A\times C)\cap(B\times C)=\emptyset$.

[11] **5**. Let $\mathcal{S}$ be the statement:

$$\forall a,b\in\mathbb{Z}, \text{ if } 5\mid a \text{ and } 5\mid b \text{ then } 5\mid(2a-b).$$

(a) Prove $\mathcal{S}$, using the definition of $\mid$ ("divides into").
*Solution*: Suppose that $a,b\in\mathbb{Z}$ so that $5\mid a$ and $5\mid b$. Since $5\mid a$ and $5\mid b$, there are integers $k$ and $m$ so that $a=5k$ and $b=5m$. Then, $2a-b=2(5k)-5m=5(2k-m)$ where $(2k-m)\in\mathbb{Z}$, which means $5\mid(2a-b)$.

(b) Write out (as simply as possible) the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.
*Solution*: The converse of $\mathcal{S}$ is: "$\forall a,b\in\mathbb{Z}$, if $5\mid(2a-b)$ then $5\mid a$ and $5\mid b$."

The converse of $\mathcal{S}$ is false. We are to prove its negation which is: " There exist integers $a$ and $b$ such that $5\mid(2a-b)$, but $5\nmid a$ or $5\nmid b$." For example, let $a=1$ and $b=2$. Then $2a-b=0=5\times 0$ and $0\in\mathbb{Z}$, so $5\mid(2a-b)$, but $5\nmid a$ because $5\nmid 1$.

(c) Write out (as simply as possible) the *contrapositive* of statement $\mathcal{S}$. Is it true or false? Explain.
*Solution*: The contrapositive of $\mathcal{S}$ is: "$\forall a,b\in\mathbb{Z}$, if $5\nmid(2a-b)$ then $5\nmid a$ or $5\nmid b$."

The contrapositive of $\mathcal{S}$ is true because it is logically equivalent to $\mathcal{S}$ which is true by part (a).

[8] **6**. The sequence $b_1,\ b_2\ ,b_3,\ldots$ of integers is defined by: $b_1=3$, and

$$b_n=3b_{n-1}+2 \text{ for all integers } n\geq 2.$$

(a) **Using mathematical induction** and the definition of odd integers, prove that $b_n$ is odd for all integers $n \geq 1$.
*Solution*:
*Basis step*: $(n = 1)$
We have $b_1 = 3 = 2 \times 1 + 1$ where 1 is an integer, so $b_1$ is odd.

*Inductive step*: Let $k \geq 1$ be an integer and suppose that $b_k$ is odd. We want to show that $b_{k+1}$ is odd.

Now, since $b_k$ is odd, $b_k = 2a + 1$ for some integer $a$. Next, since $k + 1 \geq 2$, $b_{k+1} = 3b_k + 2 = 3\,(2a + 1) + 2 = 6a + 5 = 2\,(3a + 2) + 1$ where $3a + 2 \in \mathbb{Z}$, which means $b_{k+1}$ is odd.

Thus, we proved the inductive step.

Therefore, by the Principle of Mathematical Induction, we conclude that $b_n$ is odd for all integers $n \geq 1$.

(b) Is $b_n$ prime for all integers $n \geq 1$? Prove or disprove.
*Solution*: No, it is not true that $b_n$ is prime for all integers $n \geq 1$. Now, $b_2 = 3b_1 + 2 = 3 \times 3 + 2 = 11$, and $b_3 = 3b_2 + 2 = 3 \times 11 + 2 = 35$. We see that $b_3 = 35$ is not prime (because $35 = 5 \times 7$ and $1 < 5 < 35$).

# MATHEMATICS 271 L01/02 WINTER 2012
# MIDTERM SOLUTIONS

[6] 1. **Use the Euclidean algorithm** to find $\gcd(99, 31)$. Then use your work to write $\gcd(99, 31)$ in the form $99a + 31b$ where $a$ and $b$ are integers.
*Solution*: We have

$$\begin{aligned} 99 &= 3 \times 31 + 6 \\ 31 &= 5 \times 6 + 1 \\ 6 &= 6 \times 1 + 0, \end{aligned}$$

and so $\gcd(99, 31) = 1$, and
$\gcd(99, 31) = 1 = 31 - 5 \times 6 = 31 - 5 \times (99 - 3 \times 31) = 16 \times 31 - 5 \times 99$.

Another way is to use the "table method" as follows.

| | | 99 | 31 |
|---|---|---|---|
| | 99 | 1 | 0 |
| | 31 | 0 | 1 |
| $R_1 - 3R_2$ | 6 | 1 | $-3$ |
| $R_2 - 5R_3$ | 1 | $-5$ | 16 |

Thus, $\gcd(99, 31) = 1$ and $\gcd(99, 31) = 99 \times (-5) + 31 \times 16$, that is, $a = -5$ and $b = 16$.

[6] 2. $\mathcal{P}(\mathbb{Z})$ is the power set of the set $\mathbb{Z}$ of all integers. **Disprove** the following two statements:
(a) For all $A, B \in \mathcal{P}(\mathbb{Z})$, if $5 \in A$ and $3 \in B$ then $2 \in A - B$.
*Solution*: We prove the negation of the statement, which is: "There exist $A, B \in \mathcal{P}(\mathbb{Z})$ so that $5 \in A$ and $3 \in B$, but $2 \notin A - B$". For example, let $A = \{5\}$ and $B = \{3\}$. Then $A, B \in \mathcal{P}(\mathbb{Z})$ and $A - B = \{5\}$. It follows that $5 \in A$ and $3 \in B$, but $2 \notin A - B$.

(b) For all $A, B \in \mathcal{P}(\mathbb{Z})$, if $5 \in A$ and $3 \in B$ then $2 \notin A - B$.
*Solution*: We prove the negation of the statement, which is: "There exist $A, B \in \mathcal{P}(\mathbb{Z})$ so that $5 \in A$ and $3 \in B$, but $2 \in A - B$". For example, let $A = \{2, 5\}$ and $B = \{3\}$. Then $A, B \in \mathcal{P}(\mathbb{Z})$ and $A - B = \{2, 5\}$. Thus, $5 \in A$ and $3 \in B$, but $2 \in A - B$.

[9] 3. $\mathbb{Q}^+$ is the set of all positive rational numbers, and $\mathbb{Z}^+$ is the set of all positive integers. Two of the following three statements are true and one is false. Prove the true statements. Write out and prove the **negation** of the false statement.
(a) $\forall q \in \mathbb{Q}^+\ \exists n \in \mathbb{Z}^+$ so that $qn \in \mathbb{Z}$.
*Solution*: This statement is true and here is a proof. Let $q \in \mathbb{Q}^+$. Then $q = \dfrac{a}{b}$ for some positive integers $a$ and $b$. Choose $n = b$. Then $n \in \mathbb{Z}^+$ and $qn = \dfrac{a}{b} \times b = a \in \mathbb{Z}$.

(b) $\forall q \in \mathbb{Q}^+\ \exists n \in \mathbb{Z}^+$ so that $qn \notin \mathbb{Z}$.

*Solution*: This statement is false. Its negation is: "$\exists q \in \mathbb{Q}^+$ so that $\forall n \in \mathbb{Z}^+$, $qn \in \mathbb{Z}$". For example, let $q = 1$. Then $q \in \mathbb{Q}^+$ and for any $n \in \mathbb{Z}^+$, $qn = 1 \times n = n \in \mathbb{Z}$.

(c) $\forall q \in \mathbb{Q}^+ \ \exists n \in \mathbb{Z}^+$ so that $q/n \notin \mathbb{Z}$.
*Solution*: This statement is true and here is a proof. Let $q \in \mathbb{Q}^+$. Then $q = \dfrac{a}{b}$ for some positive integers $a$ and $b$. Choose $n = 2a$. Then $n \in \mathbb{Z}^+$ and $q/n = \dfrac{\frac{a}{b}}{2a} = \dfrac{1}{2b} \notin \mathbb{Z}$ (because $0 < \dfrac{1}{2b} < 1$ since $0 < 1 < 2b$ ).

Another solution is as follows: Let $q \in \mathbb{Q}^+$. Then we have two cases:
Case 1: $q \notin \mathbb{Z}$. We choose $n = 1$. Then $n \in \mathbb{Z}^+$ and $q/n = \dfrac{q}{1} = q \notin \mathbb{Z}$.
Case 2: $q \in \mathbb{Z}$. We choose $n = 2q$. Then $n \in \mathbb{Z}^+$ and $q/n = \dfrac{q}{2q} = \dfrac{1}{2} \notin \mathbb{Z}$.

[13] 4. Let $\mathcal{S}$ be the statement:

$$\forall a, b \in \mathbb{Z}, \text{ if } 3 \mid a \text{ and } 4 \mid b \text{ then } 12 \mid ab.$$

(a) Prove $\mathcal{S}$, using the definition of $\mid$ ("divides into").
*Solution*: Suppose $a, b \in \mathbb{Z}$ so that $3 \mid a$ and $4 \mid b$. Since $3 \mid a$ and $4 \mid b$, $a = 3m$ and $b = 4n$ for some integers $m$ and $n$. Then $ab = (3m)(4n) = 12mn$ where $mn \in \mathbb{Z}$, which implies $12 \mid ab$.

(b) Write out the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.
*Solution*: The converse of $\mathcal{S}$ is: "$\forall a, b \in \mathbb{Z}$, if $12 \mid ab$ then $3 \mid a$ and $4 \mid b$". The converse of $\mathcal{S}$ is false. For example, consider the case $a = 12$ and $b = 1$. It is clear that $a, b \in \mathbb{Z}$ and $12 \mid ab$ but $4 \nmid b$.

(c) Write out the *contrapositive* of statement $\mathcal{S}$. Is it true or false? Explain.
*Solution*: The contrapositive of $\mathcal{S}$ is: "$\forall a, b \in \mathbb{Z}$, if $12 \nmid ab$ then $3 \nmid a$ or $4 \nmid b$". The contrapositive of $\mathcal{S}$ is true since it is logically equivalent to $\mathcal{S}$, which is true as proven in (a).

(d) Write out the *negation* of statement $\mathcal{S}$. Is it true or false? Explain.
*Solution*: The negation of $\mathcal{S}$ is: "$\exists a, b \in \mathbb{Z}$ so that $3 \mid a$ and $4 \mid b$, but $12 \nmid ab$". The negation of $\mathcal{S}$ is false since it has the opposite truth value of $\mathcal{S}$, which is true as proven in (a).

[6] 5. **Use mathematical induction** to prove that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

for all integers $n \geq 1$.

*Solution*:

*Basis step*: $(n = 1)$

$\frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2!}$. Thus, the statement is true when $n = 1$.

*Inductive step*: Let $k \geq 1$ be an integer and suppose that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}. \qquad \text{[IH]}$$

We want to prove that $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$.

Now,

$$\begin{aligned}
\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k+1}{(k+2)!} &= \left(\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!}\right) + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} && \text{by [IH]} \\
&= 1 - \left(\frac{1}{(k+1)!} - \frac{k+1}{(k+2)!}\right) \\
&= 1 - \left(\frac{k+2}{(k+2)!} - \frac{k+1}{(k+2)!}\right) \\
&= 1 - \left(\frac{(k+2) - (k+1)}{(k+2)!}\right) \\
&= 1 - \frac{1}{(k+2)!}.
\end{aligned}$$

Thus, $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers $n \geq 1$.

# MATHEMATICS 271 L01/02 WINTER 2013
# MIDTERM SOLUTIONS

[6] 1. **Use the Euclidean algorithm** to find $\gcd(78, 59)$. Then use your work to write $\gcd(78, 59)$ in the form $78a + 59b$ where $a$ and $b$ are integers.
*Solution*: We have

$$\begin{aligned} 78 &= 1 \times 59 + 19 \\ 59 &= 3 \times 19 + 2 \\ 19 &= 9 \times 2 + 1 \\ 2 &= 2 \times 1 + 0, \end{aligned}$$

so $\gcd(78, 59) = 1$, and
$\gcd(78, 59) = 1 = 19 - 9 \times 2 = 19 - 9 \times (59 - 3 \times 19)$
$= 28 \times 19 - 9 \times 59 = 28 \times (78 - 19) - 9 \times 59 = 28 \times 78 - 37 \times 59.$

Another way is to use the "table method" as follows.

| | | 78 | 59 |
|---|---|---|---|
| | 78 | 1 | 0 |
| | 59 | 0 | 1 |
| $R_1 - R_2$ | 19 | 1 | $-1$ |
| $R_2 - 3R_3$ | 2 | $-3$ | 4 |
| $R_3 - 9R_4$ | 1 | 28 | $-37$ |

Thus, $\gcd(78, 59) = 1$ and $\gcd(78, 59) = 78 \times 28 + 59 \times (-37)$, that is, $a = 28$ and $b = -37$.

[4] 2. You are given that $A$ and $B$ are sets, and that $(2, 3) \in A \times B$, but $(2, 4) \notin A \times B$.
(a) Find an ordered pair that definitely is in $B \times A$. Explain.
*Solution*: The pair $(3, 2)$ is definitely an element of $B \times A$. Since $(2, 3) \in A \times B$, we know $2 \in A$ and $3 \in B$, and hence $(3, 2) \in B \times A$.

(b) Find another ordered pair ($\neq (2, 4)$) that definitely is **not** in $A \times B$. Explain.
*Solution*: The pair $(1, 4)$ is definitely not an element of $A \times B$. Since $(2, 3) \in A \times B$, we know $2 \in A$. Since $(2, 4) \notin A \times B$, we know $2 \notin A$ or $4 \notin B$, but $2 \in A$ as seen above, and so we know that $4 \notin B$. Since $4 \notin B$, the pair $(1, 4)$ is not an element of $A \times B$.

Note that $(a, b) \notin A \times B$ if and only if $a \notin A$ **or** $b \notin B$. Thus, if we know $b \notin B$ then we can conclude that $(a, b) \notin A \times B$ (it does not matter if $a \in A$ or $a \notin A$). In this case, we know $4 \notin B$, so we can conclude $(1, 4) \notin A \times B$, it does not matter whether 1 is an element of $A$ or not, and in fact we do not know whether 1 is an element of $A$.

Also, note that the empty set is **not** an element of every set. It is true that the empty set is a **subset** of every set.

[11] 3. Two of the following three statements are true, and one is false. Prove the true statements. Write out and prove the **negation** of the false statement. Use no properties of even and odd integers other than their definitions.
(a) $\forall n \in \mathbb{Z}\ \exists$ a prime number $p$ so that $pn$ is even.
*Solution*: This statement is true and here is a proof. Let $n \in \mathbb{Z}$. Choose $p = 2$. Then $p$ is a prime, and $pn = 2n$ where $n \in \mathbb{Z}$ and so $pn$ is even.

(b) $\forall n \in \mathbb{Z}\ \exists$ a prime number $p$ so that $pn$ is odd.
*Solution*: This statement is false. Its negation is: "$\exists n \in \mathbb{Z}$ so that $\forall$ prime numbers $p$, $pn$ is even". For example, let $n = 2$. Then $n \in \mathbb{Z}$ and for any prime number $p$, $pn = 2p$ is clearly even.

(c) $\forall n \in \mathbb{Z}\ \exists$ a prime number $p$ so that $p + n$ is odd.
*Solution*: This statement is true and here is a proof. Let $n \in \mathbb{Z}$.
Case 1: $n$ is odd, that is, $n = 2k+1$ for some $k \in \mathbb{Z}$. We choose $p = 2$. Then $p$ is a prime, and $p + n = 2 + 2k + 1 = 2(k+1) + 1$ where $k + 1 \in \mathbb{Z}$, and so $p + n$ is odd.
Case 2: $n$ is even, that is, $n = 2m$ for some $m \in \mathbb{Z}$. We choose $p = 3$. Then $p$ is a prime, and $p + n = 3 + 2m = 2(m+1) + 1$ where $m + 1 \in \mathbb{Z}$, and so $p + n$ is odd.

[13] 4. Let $\mathcal{S}$ be the statement:

for all sets $A, B$ and $C$, if $A \cap B = \emptyset$ then $(A - C) \cap (B - C) = \emptyset$.

(a) Write out the contrapositive of $\mathcal{S}$.
*Solution*: The contrapositive of $\mathcal{S}$ is
for all sets $A, B$ and $C$, if $(A - C) \cap (B - C) \neq \emptyset$ then $A \cap B \neq \emptyset$.

(b) Prove $\mathcal{S}$, using contradiction or the contrapositive.
*Solution (by contradiction)*: Suppose that there are sets $A$, $B$ and $C$ so that $A \cap B = \emptyset$ and $(A - C) \cap (B - C) \neq \emptyset$. Since $(A - C) \cap (B - C) \neq \emptyset$, there exist an element $x \in (A - C) \cap (B - C)$. It follows that $x \in (A - C)$ and $x \in (B - C)$, which implies that $x \in A$ and $x \in B$. Since $x \in A$ and $x \in B$, we know $x \in A \cap B$. Thus, there exists an element $x \in A \cap B$, which contradicts the assumption that $A \cap B = \emptyset$. Hence, for all sets $A, B$ and $C$, if $A \cap B = \emptyset$ then $(A - C) \cap (B - C) = \emptyset$.

*Solution (using the contrapositive)*: We prove the contrapositive of $\mathcal{S}$ (stated in part (a)). Suppose that $A$, $B$ and $C$ are sets so that $(A - C) \cap (B - C) \neq \emptyset$. Since $(A - C) \cap (B - C) \neq \emptyset$, there exist an element $x \in (A - C) \cap (B - C)$. It follows that $x \in (A - C)$ and $x \in (B - C)$, which implies that $x \in A$ and $x \in B$. Since $x \in A$ and $x \in B$, we know $x \in A \cap B$. Thus, there exists an element $x \in A \cap B$, and so $A \cap B \neq \emptyset$. Thus, the contrapositive of $\mathcal{S}$ is true and since $\mathcal{S}$ is logically equivalent to the contrapositive of $\mathcal{S}$, $\mathcal{S}$ is true.

(c) Write out the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.
*Solution*: The converse of $\mathcal{S}$ is:

for all sets $A, B$ and $C$, if $(A - C) \cap (B - C) = \emptyset$ then $A \cap B = \emptyset$.

The converse of $\mathcal{S}$ is false. For example, in the case $A = B = C = \{1\}$, we have $(A - C) \cap (B - C) = \emptyset \cap \emptyset = \emptyset$, but $A \cap B = \{1\} \neq \emptyset$.

(d) Write out the *negation* of statement $\mathcal{S}$. Is it true or false? Explain.
*Solution*: The negation of $\mathcal{S}$ is:

there exist sets $A, B$ and $C$ so that $A \cap B = \emptyset$ but $(A - C) \cap (B - C) \neq \emptyset$.

The negation of $\mathcal{S}$ is false since it has the opposite truth value of $\mathcal{S}$, which is true as proven in (b).

[6] 5. **Use mathematical induction** to prove that $2^n \geq 5n - 7$ for all integers $n \geq 3$.
*Solution*:
*Basis step*: $(n = 3)$

$2^n = 2^3 = 8 \geq 8 = 15 - 7 = 5 \times 3 - 7 = 5n - 7$. Thus, the statement is true when $n = 3$.

*Inductive step*: Let $k \geq 3$ be an integer and suppose that

$$2^k \geq 5k - 7. \qquad \text{[IH]}$$

We want to prove that $2^{k+1} \geq 5\,(k+1) - 7$.

Now,

$$\begin{aligned}
2^{k+1} &= 2 \times 2^k \\
&\geq 2\,(5k - 7) && \text{by [IH]} \\
&= 10k - 14 \\
&= 5\,(k+1) - 7 + (5k - 12) \\
&\geq 5\,(k+1) - 7. && \text{Note that } k \geq 3 \text{, so } 5k - 12 \geq 5 \times 3 - 12 = 3 \geq 0
\end{aligned}$$

Thus, $2^n \geq 5n - 7$ for all integers $n \geq 3$.

**MATHEMATICS 271 L01/02 WINTER 2014**
**MIDTERM SOLUTIONS**

[6] **1**. Use the Euclidean algorithm to find $\gcd(124, 54)$. Then use your work to write $\gcd(124, 54)$ in the form $124a + 54b$ where $a$ and $b$ are integers.
*Solution*: We have

$$\begin{aligned} 124 &= 2 \times 54 + 16 \\ 54 &= 3 \times 16 + 6 \\ 16 &= 2 \times 6 + 4 \\ 6 &= 1 \times 4 + 2 \\ 4 &= 2 \times 2 + 0 \end{aligned}$$

so $\gcd(124, 54) = 2$, and

$$\begin{aligned} \gcd(124, 54) &= 2 \\ &= 6 - 4 \\ &= 6 - (16 - 2 \times 6) \\ &= 3 \times 6 - 16 \\ &= 3 \times (54 - 3 \times 16) - 16 \\ &= 3 \times 54 - 10 \times 16 \\ &= 3 \times 54 - 10 \times (124 - 2 \times 54) \\ &= -10 \times 124 + 23 \times 54 \end{aligned}$$

Another way is to use the "table method" as follows.

| | | 124 | 54 |
|---|---|---|---|
| | 124 | 1 | 0 |
| | 54 | 0 | 1 |
| $R_1 - 2R_2$ | 16 | 1 | $-2$ |
| $R_2 - 3R_3$ | 6 | $-3$ | 7 |
| $R_3 - 2R_4$ | 4 | 7 | $-16$ |
| $R_4 - R_5$ | 2 | $-10$ | 23 |

Thus, $\gcd(124, 54) = 2$ and $\gcd(124, 54) = 124 \times (-10) + 54 \times 23$, that is, $a = -10$ and $b = 23$.

[4] **2**. Disprove the statement: "For all integers $a$, $b$ and $c$, if $a \mid bc$ then $a \mid b$ or $a \mid c$." by writing out its negation and then prove that.
*Solution*: The negation of the statement above is: "There are integers $a$, $b$ and $c$ so that $a \mid bc$, but $a \nmid b$ and $a \nmid c$." For example, consider the case that $a = 4$, and $b = c = 2$. Then $a \neq 0$ and $a = 4 = 1 \times 2 \times 2 = 1 \times bc$ where $1 \in \mathbb{Z}$ which implies that $a \mid bc$, but $a \nmid b$ and $a \nmid c$ because $4 \nmid 2$.

[12] **3**. Prove the following statements. You can use the fact that $\sqrt{2}$ is irrational. For irrational numbers other than $\sqrt{2}$, you need to explain why they are irrational.

(a) For all **non-zero** real numbers $a$ and $b$, if $a$ is rational and $b$ is irrational then $ab$ is irrational.

*Solution*: Let $a$ and $b$ be non-zero real numbers so that $a$ is rational and $b$ is irrational. We prove $ab$ is irrational by contradiction. Suppose that $ab$ is rational. Since $a$ and $ab$ are rational, there are integers $m$, $n$, $p$, $q$ where $n \neq 0 \neq q$ so that $a = \dfrac{m}{n}$ and $ab = \dfrac{p}{q}$. Since $a \neq 0$, we know $m \neq 0$ and so $b = \dfrac{ab}{a} = \dfrac{p/q}{m/n} = \dfrac{pn}{mq}$ where $pn,\ mq \in \mathbb{Z}$ and $mq \neq 0$ (because $m \neq 0$ and $q \neq 0$). This implies that $b$ is rational which contradicts the assumption that $b$ is irrational. Thus, $ab$ is irrational.

(b) There are irrational numbers $x$ and $y$ so that $x + y$ is rational.

*Solution*: Let $x = \sqrt{2}$ and $y = -\sqrt{2}$. Then $x$ is irrational because $\sqrt{2}$ is irrational. Also, $y = -\sqrt{2} = (-1)\sqrt{2}$ which is the product of the non-zero rational number $-1$ and the irrational number $\sqrt{2}$, so by part (a), $y$ is irrational. However, $x+y = \sqrt{2} - \sqrt{2} = 0$ which is rational.

(c) There are irrational numbers $x$ and $y$ so that $x + y$ is irrational.

*Solution*: Let $x = y = \frac{1}{2}\sqrt{2}$. Then $x$ and $y$ are the product of the non-zero rational number $\frac{1}{2}$ and the irrational number $\sqrt{2}$, so by part (a), $x$ and $y$ are irrational. Now, $x + y = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \left(\frac{1}{2} + \frac{1}{2}\right)\sqrt{2} = \sqrt{2}$ which is irrational.

[10] **4**. Of the two following statements, one is true and one is false. Prove the true statement, and for the false statement, write out its negation and prove that. Use the element method.

(a) For all sets $A$, $B$ and $C$, if $A - B = C$ then $A = B \cup C$.

*Solution*: This statement is false. The negation of this statement is:

There are sets $A$, $B$ and $C$ so that $A - B = C$ but $A \neq B \cup C$.

For example, consider the case $A = C = \emptyset$ and $B = \{1\}$. We have $A - B = \emptyset - B = \emptyset = C$. However, since $1 \in B$, we know $1 \in B \cup C$, but $1 \notin A$ (because $A = \emptyset$) which implies that $A \neq B \cup C$.

(b) For all sets $A$, $B$ and $C$, $(A - B) - C \subseteq A - (B - C)$.

*Solution*: This statement is true and here is a proof. Let $A$, $B$ and $C$ be sets. We prove that $(A - B) - C \subseteq A - (B - C)$. Let $x \in (A - B) - C$. Since $x \in (A - B) - C$, we know $x \in A - B$, and so $x \in A$ and $x \notin B$. Since $x \notin B$, we know $x \notin B - C$. Since $x \in A$ and $x \notin B - C$, we conclude that $x \in A - (B - C)$. Thus, $(A - B) - C \subseteq A - (B - C)$.

[8] **5**. Prove by induction on $n$ that $\sum_{i=1}^{n-1} i\,(i+1) = \dfrac{n\,(n-1)\,(n+1)}{3}$ for all integers $n \geq 2$.

*Solution*:

*Basis* $(n = 2)$

$\sum_{i=1}^{2-1} i\,(i+1) = \sum_{i=1}^{1} i\,(i+1) = 1 \times 2 = 2 = \dfrac{6}{3} = \dfrac{2 \times 1 \times 3}{3} = \dfrac{2 \times (2-1) \times (2+1)}{3}$.

Thus, $\sum_{i=1}^{n-1} i\,(i+1) = \dfrac{n\,(n-1)\,(n+1)}{3}$ when $n = 2$.

*Inductive Step:* Let $k \geq 2$ be an integer and suppose that

$$\sum_{i=1}^{k-1} i\,(i+1) = \frac{k\,(k-1)\,(k+1)}{3} \qquad [IH]$$

We want to prove that $\sum_{i=1}^{k} i\,(i+1) = \dfrac{(k+1)\,k(k+2)}{3}$.

Now,

$$\begin{aligned}
\sum_{i=1}^{k} i\,(i+1) &= \left(\sum_{i=1}^{k-1} i\,(i+1)\right) + k\,(k+1) \\
&= \frac{k\,(k-1)\,(k+1)}{3} + k\,(k+1) \qquad \text{by } [IH] \\
&= \frac{k\,(k-1)\,(k+1) + 3k\,(k+1)}{3} \\
&= \frac{k(k+1)\,(k-1+3)}{3} \\
&= \frac{k(k+1)\,(k+2)}{3} \\
&= \frac{(k+1)\,k(k+2)}{3}.
\end{aligned}$$

Thus, by the Principle of Mathematical Induction, $\sum_{i=1}^{n-1} i\,(i+1) = \dfrac{n\,(n-1)\,(n+1)}{3}$ for all integers $n \geq 2$.