MATHEMATICS 271 L01 FALL 2015
QUIZ 3 SOLUTIONS

1. The sequence $a_0,\ a_1,\ a_2, \ldots$is defined by $a_0 = a_1 = a_2 = 1$ and for integers $n \geq 3$, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

(a) Compute $a_3,\ a_4$ and a_5.

Solution:

$$\begin{array}{lllllll} a_3 & = & a_2 + a_1 + a_0 & = & 1+1+1 & = & 3 \\ a_4 & = & a_3 + a_2 + a_1 & = & 3+1+1 & = & 5 \\ a_5 & = & a_4 + a_3 + a_2 & = & 5+3+1 & = & 9 \end{array}$$

(b) Prove by strong induction on n that $a_n < 2^n$ for all integers $n \geq 1$.

Solution

Base cases $(n = 1,\ 2,\ 3)$

$$\begin{array}{lllllll} a_1 & = & 1 & < & 2 & = & 2^1 \\ a_2 & = & 1 & < & 4 & = & 2^2 \\ a_3 & = & 3 & < & 8 & = & 2^3 \end{array}$$

Inductive Step.

Suppose that $k \geq 3$ is an integer and suppose that

$a_m < 2^m$ for all integers m where $1 \leq m \leq k$. $\qquad (IH)$

We want to prove that $a_{k+1} < 2^{k+1}$.

Now, since $k + 1 \geq 4$, bu the defintion of the sequence, we have

$$\begin{array}{lll} a_{k+1} & = & a_k + a_{k-1} + a_{k-2} \\ & < & 2^k + 2^{k-1} + 2^{k-2} \qquad \text{by } (IH) \\ & = & (2^2 + 2 + 1)\, 2^{k-2} \\ & = & 7 \times 2^{k-2} \\ & < & 8 \times 2^{k-2} \\ & = & 2^3 \times 2^{k-2} \\ & = & 2^{k+1}. \end{array}$$

Thus, $a_n < 2^n$ for all integers $n \geq 1$.

2. Of the two following statements, one is true and one is false. Prove the true statement and disprove the false statement.

(a) For all sets A, B and C, if $A - B = C$ then $A = B \cup C$.

Solution: This statement is false. For example, when $A = C = \emptyset$ and $B = \{1\}$, we have $A - B = \emptyset - C = \emptyset = C$, but $A = \emptyset \neq \{1\} = B = B - \emptyset = B - C$.

(b) For all sets A, B and C, if $C \subseteq B - A$ then $A \cap C = \emptyset$.

Solution: This statement is true and here is a proof. Suppose that $A,\ B$ and C are sets so that $C \subseteq B - A$. We prove that $A \cap C = \emptyset$ by contradiction. Suppose that $A \cap C \neq \emptyset$, that is, there exists an element $a \in A \cap C$. Since $a \in A \cap C$, we know $a \in A$ and $a \in C$. Since $a \in A$, we know that $a \notin B - A$. Thus, there exists an element a so that $a \in C$ and $a \notin B - A$, this implies $C \nsubseteq B - A$, which contradicts the assumption that $C \subseteq B - A$. Thus, $A \cap C = \emptyset$.