MATH 271
Supplementary Notes – Induction

Revised 2015

1 Introduction

Proof by induction is a powerful technique that is used mainly to prove statements of the form

$$\text{“For all integers } n \geq n_0,\ \mathcal{P}(n)\text{”},$$

where $\mathcal{P}(n)$ is a statement involving the integer n. Here, n_0 is some integer; in the case that $n_0 = 1$, alternative wording might be “For all positive integers n, $\mathcal{P}(n)$.”

Consider the following scenario. Suppose that there are only 3¢ (tricent) and 5¢ (nickel) coins, and someone wishes to have exactly k¢ using only tricents and nickels.

total	# tricents	# nickels
1¢	?	?
2¢	?	?
3¢	1	0
4¢	?	?
5¢	0	1
6¢	2	0
7¢	?	?
8¢	1	1
9¢	3	0
10¢	0	2
11¢	2	1
12¢	4	0
13¢	1	2

Is it possible to have exactly k¢ with only tricents and nickels, provided $k \neq 1, 2, 4, 7$? The claim is that for $k \geq 8$, it is possible to have exactly k¢ using only tricents and nickels.

Here are a few other values:

total	# tricents	# nickels
37¢	4	5
56¢	2	10
101¢	2	19

These examples do not constitute a proof that k¢, $k \geq 8$, can always be obtained using tricents and nickels. In fact no number of specific examples, on their own, constitute a proof. There are at least two ways of *proving* this claim; the proof that is presented here is an *inductive* proof based on the following observation.

# tricents	total	# nickels	total	difference
2	6¢	1	5¢	−1¢
3	9¢	2	10¢	1¢

(The "difference" column is just the second total minus the first total.) Observe that replacing one nickel with two tricents results in a gain of 1¢; similarly, replacing three tricents with two nickels results in a gain of 1¢.

Why is this important?

Suppose that someone has exactly p¢ in tricents and nickels. If there is one nickel (or more), then exchanging one nickel for two tricents results in a gain of 1¢, giving a total of $(p+1)$¢; if there are three tricents (or more), then exchanging three tricents for two nickels results in a gain of 1¢, giving a total of $(p+1)$¢. **This is one of the key ideas in mathematical induction: finding a method to increment from one value (p¢) to the next value ($(p+1)$¢).**

2 The Principle of Mathematical Induction

The Principle of Mathematical Induction formalizes the ideas that have been discussed.

Principle of Mathematical Induction: *Let $\mathcal{P}(n)$ be a statement where n is an integer, and suppose that for some integer n_0:*

(A) $\mathcal{P}(n_0)$, and

(B) for all integers $k \geq n_0$, $\mathcal{P}(k) \to \mathcal{P}(k+1)$.

Then $\mathcal{P}(n)$ for all integers $n \geq n_0$.

Part (A) is the *basis* or *base case* for the induction; it is here that the truth of the statement is established for a particular integer. Part (B) is the *inductive step* and involves an implication: it says that the truth of $\mathcal{P}(k)$ implies the truth of $\mathcal{P}(k+1)$. The Principle of Mathematical Induction states that if both (A) and (B) are true, then the statement $\mathcal{P}(n)$ is true for all integers $n \geq n_0$.

Here now is a complete proof of the statement

For every integer $n \geq 8$, it is possible to obtain exactly n¢ using only tricents and nickels.

Proof. **Basis:** One tricent and one nickel give a total of 8 cents, so the result hold for $n = 8$.

(The basis is often quite easy to establish. For the inductive step, it is important to establish the hypothesis of the implication. This is call the **Inductive Hypothesis** *or* **I.H.** *for short.)*

Inductive Hypothesis (I.H.): Suppose that it is possible to have exactly k¢ in tricents and nickels for some $k \geq 8$.

(Notice that we say **some** *and not* **all***; this is an important distinction. The idea is that there is k¢ for a fixed but arbitrary value of k, for example, $k = 8$.)*

Show that it is possible to obtain exactly $(k+1)$¢ in tricents and nickels.

(The **Show** *statement is a clear indication of the conclusion of the implication for the inductive step. What follows is the technical aspect of the proof; the basic idea is just what was stated earlier: exchanging one nickel for two tricents, or exchanging three tricents for two nickels. Part of this*

involves establishing that there is always at least one nickel or at least three tricents, otherwise there would be problems.)

Suppose it is possible to have exactly $k¢$ in tricents and nickels **(I.H.)**. If there is at least one nickel, then exchanging one nickel for two tricents, gives a total of

$$k¢ - 5¢ + 6¢ = (k+1)¢,$$

as required. If there are no nickels, then because $k \geq 8$, it must be the case that there are at least three tricents. *(If there are no nickels and at most two tricents, then the total is at most* $6¢$*, a contradiction.)* Exchanging three tricents for two nickels give a total of

$$k¢ - 9¢ + 10¢ = (k+1)¢,$$

as required.

Therefore, for all $n \geq 8$, it is possible to obtain exactly $n¢$ using only tricents and nickels. □

Example 2.1. To prove by induction that

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1, \text{ for all integers } n \geq 0,$$

prove $\mathcal{P}(n)$ for all integers $n \geq n_0$, where $\mathcal{P}(n)$ is the statement:

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1,$$

and $n_0 = 0$. To do this, first prove the base case $\mathcal{P}(0)$; that is, prove $1 = 2^{0+1} - 1$. Then prove the inductive step; that is, prove

$$\text{For all integers } k \geq n_0, \mathcal{P}(k) \to \mathcal{P}(k+1),$$

or equivalently, prove: For all integers $k \geq 0$, if

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 1,$$

then

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{k+1} = 2^{(k+1)+1} - 1.$$

Thus in the inductive step, assume that $k \geq 0$ is an integer, and that $1 + 2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 1$ (that is, assume $\mathcal{P}(k)$, which is the inductive hypothesis (I.H.)); then show that $1 + 2 + 2^2 + 2^3 + \cdots + 2^{k+1} = 2^{(k+1)+1} - 1$ (that is, show $\mathcal{P}(k+1)$). Here is a proof by induction that

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1, \text{ for all integers } n \geq 0.$$

Proof. **Basis:** (when $n = 0$) $2^{0+1} - 1 = 2^1 - 1 = 2 - 1 = 1$.

Inductive Hypothesis: Suppose that for some integer $k \geq 0$,

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 1.$$

Show that $1 + 2 + 2^2 + 2^3 + \cdots + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$.

Observe that

$$\begin{aligned}
1 + 2 + 2^2 + 2^3 + \cdots + 2^{k+1} &= (1 + 2 + 2^2 + 2^3 + \cdots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} \qquad \text{(I.H.)} \\
&= 2^{k+1} + 2^{k+1} - 1 \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{k+2} - 1, \text{ as required.}
\end{aligned}$$

Thus, by the Principle of Mathematical Induction, $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$, for all integers $n \geq 0$. □

Outline of a Proof by Induction

1. **Basis:** The basis consists of verifying the statement in one or more specific cases, including the *smallest* integer for which the statement holds.

2. **Inductive Hypothesis**: A clear and explicit statement of what is to be assumed. (It is not sufficient to write something like "Assume that the statement is true for k," or "Assume $\mathcal{P}(k)$.")

3. **Show that . . .** : A clear and explicit statement of what needs to be shown. (It is not sufficient to write something like "Show that the statement is true for $k + 1$," or "Prove $\mathcal{P}(k + 1)$.")

4. The technical part of the proof, where the inductive hypothesis is used. *Indicate where the inductive hypothesis is invoked.*

5. **Conclusion**: A one sentence summary of what has been established.

Example 2.2. Prove that $5^n - 4n - 1$ is divisible by 8 for all integers $n \geq 0$.

Proof. **Basis:** (when $n = 0$) $5^0 - 4 \cdot 0 - 1 = 0 = 0 \cdot 8$ and so $5^0 - 4 \cdot 0 - 1$ is divisible by 8. Thus the result holds when $n = 0$.

Inductive Hypothesis: Suppose that for some integer $k \geq 0$, $5^k - 4k - 1$ is divisible by 8.

Show that $5^{k+1} - 4(k + 1) - 1$ is divisible by 8.

From the I.H., it follows that $5^k - 4k - 1 = 8m$ for some integer m, and so

$$\begin{aligned}
5^{k+1} - 4(k + 1) - 1 &= 5 \cdot 5^k - 4k - 4 - 1 \\
&= 5(5^k - 4k - 1) + 16k \\
&= 5(8m) + 16k \\
&= 8(5m + 2k).
\end{aligned}$$

Since m and k are integers, $5m - 2k$ is an integer, implying that $5^{k+1} - 4\,(k + 1) - 1$ is divisible by 8.

Therefore, by the Principle of Mathematical Induction, $5^n - 4n - 1$ is divisible by 8, for all integers $n \geq 0$. □

Example 2.3. Prove that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$.

Proof. **Basis:** (when $n = 2$) $1 + \frac{1}{4} = \frac{5}{4} < \frac{6}{4} = \frac{3}{2} = 2 - \frac{1}{2}$.

Inductive Hypothesis: Suppose that for some integer $k \geq 2$,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}.$$

Show that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}.$$

Now,

$$\begin{aligned}
1 + \frac{1}{4} + \cdots + \frac{1}{(k+1)^2} &= \left(1 + \frac{1}{4} + \cdots + \frac{1}{k^2}\right) + \frac{1}{(k+1)^2} \\
&< \left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2} \qquad \text{by the I.H.} \\
&= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \\
&= 2 - \frac{(k+1)^2 - k}{k(k+1)^2} \\
&= 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\
&< 2 - \frac{k^2 + k}{k(k+1)^2}, \text{ since } k^2 + k + 1 > k^2 + k \\
&= 2 - \frac{k(k+1)}{k(k+1)^2} \\
&= 2 - \frac{1}{k+1}, \text{ as required.}
\end{aligned}$$

Therefore, by the Principle of Mathematical Induction,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n},$$

for all integers $n \geq 2$. $\square$

Example 2.4. Prove that $n^3 < 2 \times 3^{n-1}$ for all integers $n \geq 5$.

An incorrect proof:

Proof. **Basis:** When $n = 5$, the inequality says

$$\begin{aligned} n^3 &< 2 \times 3^{n-1} \\ 5^3 &< 2 \times 3^{5-1} \\ 125 &< 2 \times 3^4 \\ 125 &< 2 \times 81 \\ 125 &< 162 \end{aligned}$$

which is true.

Inductive Hypothesis (I.H): $k^3 < 2 \times 3^{k-1}$.

$$\begin{aligned} (k+1)^3 &< 2 \times 3^{(k+1)-1} \\ k^3 + 3k^2 + 3k + 1 &< 2 \times 3^k \\ k^3 + 3k^2 + 3k + 1 &< 2 \times 3 \times 3^{k-1} \\ 3k^2 + 3k + 1 &< 6 \times 3^{k-1} - k^3 \\ 3k^2 + 3k + 1 &< 6 \times 3^{k-1} - 2 \times 3^{k-1} \text{ (I.H.)} \\ 3k^2 + 3k + 1 &< 4 \times 3^{k-1} \end{aligned}$$

Since $k \geq 5$, $k^3 \geq 3k^2 + 3k + 1$; by the I.H., $k^3 < 2 \times 3^{k-1}$ and thus $k^3 < 4 \times 3^{k-1}$, as required.

Therefore $n^3 < 2 \times 3^{n-1}$ for all $n \geq 5$. □

A correct proof:

Basis: When $n = 5$, $n^3 = 5^3 = 125$ and $2 \times 3^{n-1} = 2 \times 3^4 = 162$. Since $125 < 162$, the result holds when $n = 5$.

Inductive Hypothesis (I.H): Suppose that for some integer $k \geq 5$, $k^3 < 2 \times 3^{k-1}$.

Show that $(k+1)^3 < 2 \times 3^k$.

Starting with the binomial theorem

$$\begin{aligned} (k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\ &= k^3 \left(1 + \frac{3}{k} + \frac{3}{k^2} + \frac{1}{k^3}\right) \\ &< 2 \times 3^{k-1} \times \left(1 + \frac{3}{k} + \frac{3}{k^2} + \frac{1}{k^3}\right) \text{ (I.H.)} \\ &\leq 2 \times 3^{k-1} \times \left(1 + \frac{3}{5} + \frac{3}{25} + \frac{1}{125}\right) \\ &\qquad \text{(since } k \geq 5\text{)} \\ &= 2 \times 3^{k-1} \times \frac{216}{125} \\ &< 2 \times 3^{k-1} \times 3, \text{ since } \tfrac{216}{125} < 3 \\ &= 2 \times 3^k \text{ as required.} \end{aligned}$$

Therefore $n^3 < 2 \times 3^{n-1}$ for all integers $n \geq 5$. □

What is wrong with the first proof?

- In both the basis and the actual inductive step, we are beginning by assuming the statement that we are supposed to prove, and then going through a verification process by manipulating both sides of the inequality. This is **wrong**.
- In the inductive step, we end up proving that if $(k+1)^3 < 2 \times 3^{k+1-1}$, then $3k^2 + 3k + 1 < 4 \times 3^{k-1}$; what we should be showing is that if $k^3 < 2 \times 3^{k-1}$, then $(k+1)^3 < 2 \times 3^k$.
- The remainder of the proof is also flawed. For example, in the inductive step we state **without proof** that $k^3 \geq 3k^2 + 3k + 1$ because $k \geq 5$. This requires justification.
- There is a serious error in the step where the I.H. is invoked. (Since $k^3 < 2 \times 3^{k-1}$, $-k^3 > -2 \times 3^{k-1}$, so the substitution into the inequality is invalid.)
- The inductive hypothesis is incomplete. Hypothesis means assumption, so there should be an indication that we are *assuming* something.
- As part of the inductive hypothesis, we must state that $k \geq 5$ (since the result is false for some values of $k < 5$).
- After the inductive hypothesis, there is no statement of what we are now required to prove. This is important in that it helps you focus on exactly what you are supposed to do.

A slightly different correct proof.

Proof. **Basis:** When $n = 5$, $n^3 = 5^3 = 125$ and $2 \times 3^{n-1} = 2 \times 3^4 = 162$. Since $125 < 162$, the result holds when $n = 5$.

Inductive Hypothesis (I.H): Suppose that for some integer $k \geq 5$, $k^3 < 2 \times 3^{k-1}$.

Show that $(k+1)^3 < 2 \times 3^k$. Starting with the binomial theorem

$$\begin{aligned} (k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\ &= k^3 + 3k^2 + (3k+1) \end{aligned}$$

Since $k \geq 5$, $k^3 \geq 5k^2 > 3k^2$. Also, since $k \geq 5$,

$$k^3 \geq 25k > 4k = 3k + k > 3k + 1.$$

Therefore

$$\begin{aligned} k^3 + 3k^2 + (3k+1) &< k^3 + k^3 + k^3 \\ &= 3k^3 \\ &< 3 \times 2 \times 3^{k-1} \text{ (I.H.)} \\ &= 2 \times 3^k \text{ as required.} \end{aligned}$$

Therefore $n^3 < 2 \times 3^{n-1}$ for all integers $n \geq 5$. □

Exercises.
Prove the following statements by induction.

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.
2. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all integers $n \geq 1$.
3. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{(n+1)}$ for all integers $n \geq 1$.
4. $5^{n+1} + 2(3)^n + 1$ is divisible by 8 for all integers $n \geq 0$.
5. $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.
6. $(2n+1) + (2n+3) + (2n+5) + \cdots + (4n-1) = 3n^2$ for all integers $n \geq 1$.
7. $(n^2+1) + (n^2+2) + (n^2+3) + \cdots + (n+1)^2 = n^3 + (n+1)^3$ for all integers $n \geq 0$.
8. $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for all $n \geq 0$.

3 Strong Induction

There are times when mathematical induction seems like the appropriate proof technique, but the inductive step requires more than the truth of $\mathcal{P}(k)$ to prove $\mathcal{P}(k+1)$. In such circumstances, *Strong Induction* may be used.

Example 3.1. The sequence $a_0, a_1, a_2, \ldots$ is defined as follows: $a_0 = 7, a_1 = 4$, and for any integer $n \geq 2$,

$$a_n = -a_{n-1} + 6a_{n-2}.$$

Prove that $a_n = 2 \times (-3)^n + 5 \times 2^n$ for all integers $n \geq 0$.

In this case, we are asked to prove $\mathcal{P}(n)$ for all integers $n \geq n_0$, where $\mathcal{P}(n)$ is the statement:

$$a_n = 2 \times (-3)^n + 5 \times 2^n.$$

Proof. **Basis:** $2 \times (-3)^0 + 5 \times 2^0 = 7 = a_0$, so the result holds when $n = 0$.

Inductive Hypothesis: Let $k \geq 0$ be an integer, and suppose that $a_k = 2 \times (-3)^k + 5 \times 2^k$.

Show that $a_{k+1} = 2 \times (-3)^{k+1} + 5 \times 2^{k+1}$.

Using the definition of the sequence,

$$a_{k+1} = -a_k + 6a_{k-1}.$$

At this point, the next natural step is to apply the I.H. to a_k, and get

$$a_{k+1} = (2 \times (-3)^k + 5 \times 2^k) + 6a_{k-1}.$$

Now what? We can't complete this proof, because we have no way of dealing with a_{k-1}, which is where the problem arises. □

In more general terms, the inductive hypothesis $\mathcal{P}(k)$ is insufficient. What is required for the inductive step, in this case, is the assumption that **both** $\mathcal{P}(k)$ and $\mathcal{P}(k-1)$ are true; i.e.,

$$\mathcal{P}(k-1) \text{ and } \mathcal{P}(k) \to \mathcal{P}(k+1).$$

This requires a revision of the inductive hypothesis to allow for more than just one previous case. In addition, the basis for the induction may require more than the verification for one value of n. In this particular example, the inductive step

$$\mathcal{P}(k-1) \text{ and } \mathcal{P}(k) \to \mathcal{P}(k+1)$$

does not make sense when $k = 0$, since there is no $\mathcal{P}(-1)$. To overcome this problem, it is necessary to verify the cases $n = 0$ and $n = 1$ in the basis, so that the inductive step applies only to $n \geq 2$.

Principle of Mathematical Induction (Strong Form): *Let $\mathcal{P}(n)$ be a statement where n is an integer, and suppose that for some integers n_0 and n_1, where $n_0 \leq n_1$,*

(A) $\mathcal{P}(n_0), \mathcal{P}(n_0+1), \mathcal{P}(n_0+2), \ldots \mathcal{P}(n_1)$, and

(B) for all integers $k > n_1$, if $\mathcal{P}(m)$, $n_0 \leq m < k$, then $\mathcal{P}(k)$.

Then $\mathcal{P}(n)$ for all integers $n \geq n_0$.

Example 3.1 (continued).

Here now is a complete proof, using strong induction, that

$$a_n = 2 \times (-3)^n + 5 \times 2^n \text{ for all } n \geq 0.$$

Proof. **Basis:** $(n = 0, 1)$

$$\begin{aligned} 2 \times (-3)^0 + 5 \times 2^0 &= 2 + 5 = 7 = a_0 \\ 2 \times (-3)^1 + 5 \times 2^1 &= 2(-3) + 5(2) = -6 + 10 = 4 = a_1. \end{aligned}$$

Thus the result holds for $n = 0$ and $n = 1$.

Inductive Hypothesis: Suppose that $k > 1$ is an integer, and that for all integers m, $0 \leq m < k$,

$$a_m = 2 \times (-3)^m + 5 \times 2^m.$$

Show that $a_k = 2 \times (-3)^k + 5 \times 2^k$.

$$\begin{aligned} a_k &= -a_{k-1} + 6a_{k-2} && \text{– by the definition of } a_k,\ k \geq 2. \\ &= -(2 \times (-3)^{k-1} + 5 \times 2^{k-1}) + 6(2 \times (-3)^{k-2} + 5 \times 2^{k-2}) && \text{– by the I.H., which applies since} \\ &= -\frac{2}{-3} \times (-3)^k - \frac{5}{2} \times 2^k + \frac{12}{(-3)^2}(-3)^k + \frac{30}{2^2}2^k && 0 \leq k-2, k-1 < k. \\ &= \left(\frac{2}{3} + \frac{12}{9}\right) \times (-3)^k + \left(-\frac{5}{2} + \frac{30}{4}\right) \times 2^k \\ &= \left(\frac{2}{3} + \frac{4}{3}\right) \times (-3)^k + \left(-\frac{5}{2} + \frac{15}{2}\right) \times 2^k \\ &= \left(\frac{6}{3}\right) \times (-3)^k + \left(\frac{10}{2}\right) \times 2^k \\ &= 2 \times (-3)^k + 5 \times 2^k, \text{ as required.} \end{aligned}$$

Thus, by the Principle of Mathematical Induction (Strong Form), it follows that

$$a_n = 2 \times (-3)^n + 5 \times 2^n,$$

for all integers $n \geq 0$. □

Example 3.2. A sequence $a_1, a_2, a_3, \ldots$ is defined by: $a_1 = 2$, $a_2 = 2$, and for all integers $n \geq 3$, $a_n = a_{n-1} \cdot a_{n-2}$. Prove that $2^n | a_n$ for all integers $n \geq 5$.

An immediate observation is that this proof requires *strong* induction, **(WHY?)** and the basis for the induction consists of verifying the statement for $n = 5$ and $n = 6$ **(WHY?)**.

Proof. **Basis:** Using the recurrence relation,

$$\begin{aligned} a_3 &= a_2 \cdot a_1 = 2 \cdot 2 = 4 \\ a_4 &= a_3 \cdot a_2 = 4 \cdot 2 = 8 \\ a_5 &= a_4 \cdot a_3 = 8 \cdot 4 = 32 \\ a_6 &= a_5 \cdot a_4 = 32 \cdot 8 = 256 \end{aligned}$$

Since $32 = 2^5$, $a_5 = 2^5 \cdot 1$, so 2^5 divides a_5. Similarly $256 = 2^8$, so $a_6 = 2^6 \cdot 4$, and thus $2^6 | a_6$. Therefore the result holds for $n = 5$ and $n = 6$.

Comments.
Basis: As already noted, the basis requires verification of the result for $n = 5$ and $n = 6$. To do this, a_5 and a_6 need to be calculated. Because the definition is recursive, a_3 and a_4 must also be calculated, though they are never used in the proof.

Observe that although the statement is true for $n = 1$, it is false for $n = 2, 3$ and 4.

Inductive Hypothesis (I.H): Let $k > 6$, and suppose that for all integers m with $5 \leq m < k$, $2^m | a_m$.

Inductive Hypothesis: Think about what this is saying.

Show that $2^k | a_k$.

By the definition of a_k, $a_k = a_{k-1} \cdot a_{k-2}$. Using the inductive hypothesis, $2^{k-1} | a_{k-1}$ and $2^{k-2} | a_{k-2}$, and thus $a_{k-1} = 2^{k-1} \cdot f$ and $a_{k-2} = 2^{k-2} \cdot g$ for some integers f and g. Thus

Here we are using the inductive hypothesis with $m = k - 1$ and $m = k - 2$; this is valid because $k > 6$, so $k \geq 7$, implying that $5 \leq k-1 < k$ and $5 \leq k-2 < k$.

$$\begin{aligned} a_k &= a_{k-1} \cdot a_{k-2} \\ &= (2^{k-1} \cdot f)(2^{k-2} \cdot g) \\ &= 2^{2k-3} \cdot f \cdot g \\ &= 2^k (2^{k-3} \cdot f \cdot g) \end{aligned}$$

Since $k > 6$, 2^{k-3} is an integer; also, f and g are integers, and thus $2^{k-3} \cdot f \cdot g$ is an integer. Therefore, $2^k | a_k$ as required.

Therefore, for all integers $n \geq 5$, $2^n | a_n$. □

Example 3.3. The sequence $a_0, a_1, a_2, \cdots$ is defined by: $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, and for all integers $n \geq 3$,

$$a_n = a_{n-2} + 2a_{n-3}.$$

Prove by induction that $a_n > (\frac{3}{2})^n$ for all integers $n \geq 1$.

Proof. **Basis:** $(n = 1, 2, 3)$

$$\begin{aligned} a_1 &= 2 = \frac{4}{2} > \frac{3}{2} = \left(\frac{3}{2}\right)^1, \\ a_2 &= 3 = \frac{12}{4} > \frac{9}{4} = \left(\frac{3}{2}\right)^2, \text{ and} \\ a_3 &= a_1 + 2a_0 = 2 + 2 \times 1 = 4 = \frac{32}{8} > \frac{27}{8} = \left(\frac{3}{2}\right)^3. \end{aligned}$$

Inductive Hypothesis: Suppose that $k > 3$ is any integer, and that for all integers m with $1 \le m < k$,

$$a_m > \left(\frac{3}{2}\right)^m.$$

Show that $a_k > \left(\frac{3}{2}\right)^k$.

Now,

$$\begin{aligned}
a_k &= a_{k-2} + 2a_{k-3} && \text{– by the definition of } a_0, a_1, a_2, \ldots \\
&> \left(\frac{3}{2}\right)^{k-2} + 2\left(\frac{3}{2}\right)^{k-3} && \text{– by the I.H.} \\
&= \left(\frac{3}{2}\right)^k \left[\left(\frac{3}{2}\right)^{-2} + 2\left(\frac{3}{2}\right)^{-3}\right] \\
&= \left(\frac{3}{2}\right)^k \left[\left(\frac{2}{3}\right)^{2} + 2\left(\frac{2}{3}\right)^{3}\right] \\
&= \left(\frac{3}{2}\right)^k \left[\frac{4}{9} + 2\left(\frac{8}{27}\right)\right] \\
&= \left(\frac{3}{2}\right)^k \left[\frac{12}{27} + \frac{16}{27}\right] \\
&= \left(\frac{3}{2}\right)^k \frac{28}{27} > \left(\frac{3}{2}\right)^k && \text{– because } \tfrac{28}{27} > \tfrac{27}{27} = 1
\end{aligned}$$

Thus, by the Principle of Mathematical Induction (Strong Form), it follows that

$$a_n > \left(\frac{3}{2}\right)^n$$

for all integers $n \ge 1$. □

> In Example 3.1, there are two cases in the basis ($n = 0$ and $n = 1$), but in Example 3.3, there are three cases in the basis ($n = 1$, $n = 2$ and $n = 3$). WHY?

Example 3.4. The Fibonacci numbers are defined by $F_0 = F_1 = 1$, and for all integers $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that F_n is even if and only if $n = 3m - 1$ for some integer m.

Before beginning the proof, we list the first ten Fibonacci numbers just to see what is going on.

$$\begin{array}{rcrcrcr}
F_0 &=& 1 & \qquad & F_5 &=& 8 \\
F_1 &=& 1 & & F_6 &=& 13 \\
F_2 &=& 2 & & F_7 &=& 21 \\
F_3 &=& 3 & & F_8 &=& 34 \\
F_4 &=& 5 & & F_9 &=& 55
\end{array}$$

The even Fibonacci numbers are F_2, F_5 and F_8; and $2 = 3(1) - 1$, $5 = 3(2) - 1$, and $8 = 3(3) - 1$. The odd Fibonacci numbers are $F_0, F_1, F_3, F_4, F_6, F_7$ and F_9; and none of $\{0, 1, 3, 4, 6, 7, 9\}$ is of the form $3m - 1$ for some integer m.

Proof. Notice that it is sufficient to prove that if $n = 3m - 1$ for some integer m, then F_n is even, and otherwise F_n is odd.

Basis: $F_0 = 1$ and $F_1 = 1$ are both odd, and neither 0 nor 1 is of the form $3m - 1$ for any integer m. Thus, the result hold for $n = 0$ and 1.

Inductive Hypothesis: Let $k > 1$ be an integer, and suppose that for all integers j, $0 \leq j < k$, F_j is even if $j = 3m - 1$ for some integer m, and F_j is odd otherwise.

Show that F_k is even if $k = 3m - 1$ for some integer m, and F_k is odd otherwise.

From the definition of the Fibonacci numbers for $k \geq 2$, we have $F_k = F_{k-1} + F_{k-2}$.
Case 1. If $k = 3m - 1$ for some integer m,

$$k - 1 = 3m - 2 \text{ and } k - 2 = 3m - 3 = 3(m - 1).$$

Since $0 \leq k - 2, k - 1 < k$, we may apply the inductive hypothesis to F_{k-1} and F_{k-2}. Note that neither $k - 1 = 3m - 2$ nor $k - 2 = 3(m - 1)$ are of the form required for even Fibonacci numbers. Thus both F_{k-1} and F_{k-2} are odd. Since the sum of two odd integers is even, F_k is even.

Case 2. Suppose that $k \neq 3m - 1$ for any integer m. Then $k = 3m$ or $k = 3m - 2$ for some integer m. In either case, $0 \leq k - 2, k - 1 < k$, so the inductive hypothesis applies to F_{k-1} and F_{k-2}. In the first case, $k - 1 = 3m - 1$ and $k - 2 = 3m - 2$; by the I.H., F_{k-1} is even and F_{k-2} is odd (since $k - 2 = 3m - 2$ is not the form required for an even Fibonacci number). In the second case, $k - 1 = 3(m - 1)$ and $k - 2 = 3m - 4 = 3(m - 1) - 1$; by the I.H., F_{k-2} is even and F_{k-1} is odd (since $k - 1 = 3(m - 1)$ is not the form required for an even Fibonacci number). Therefore, $F_k = F_{k-1} + F_{k-2}$ tells us that F_k is the sum of one even integer and one odd integer, and therefore F_k is odd.

Thus, by the principle of mathematical induction, F_n is even if and only if $n = 3m - 1$ for some integer m. □

Exercises.

9. The sequence $a_0, a_1, a_2, \ldots$ is defined by: $a_0 = 0$, $a_1 = 1$, and for all integers $n \geq 2$, $a_n = 3a_{n-1} - 2a_{n-2}$. Prove by induction that $a_n = 2^n - 1$ for all integers $n \geq 0$.

10. The sequence $c_0, c_1, c_2, \ldots$ is defined by: $c_0 = 1$, $c_1 = 8$, and for all integers $k \geq 2$, $c_k = c_{k-1} + 2c_{k-2}$. Prove by induction that $c_n = 3 \cdot 2^n - 2 \times (-1)^n$ for all integers $n \geq 0$.

11. The sequence $b_0, b_1, b_2, \ldots$ is defined by: $b_0 = 12$, $b_1 = 29$ and for all integers $k \geq 2$, $b_k = 5b_{k-1} - 6b_{k-2}$. Prove by induction that $b_n = 5 \times 3^n + 7 \times 2^n$ for all integers $n \geq 0$.

12. The sequence $t_1, t_2, t_3, \ldots$ is defined by: $t_1 = 1$, and for all integers $k \geq 2$, $t_k = 2 \times t_{\lfloor k/2 \rfloor}$. Prove by induction that $t_n \leq n$ for all integers $n \geq 1$.

13. Prove that any integer $n > 1$ is divisible by a prime number.

14. Prove that every positive integer can be expressed as a sum of distinct powers of two. (For example, $43 = 2^5 + 2^3 + 2^1 + 2^0$.)

Answers to Selected Exercises

3. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{(n+1)}$ for all integers $n \geq 1$.

Proof. **Basis:** When $n = 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

and $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$, so equality holds.

Inductive Hypothesis: Suppose that for some integer $k \geq 1$,

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}.$$

Show that

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}.$$

Now

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \textbf{by the I.H.} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

as required.

Therefore, for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

$\square$

7. $(n^2+1)+(n^2+2)+(n^2+3)+\cdots+(n+1)^2 = n^3+(n+1)^3$ for all integers $n \geq 0$.

Proof. Note that $(n+1)^2 = n^2+2n+1$ so we may write the sum as

$$\sum_{i=1}^{2n+1}(n^2+i).$$

Basis: When $n=0$,

$$\sum_{i=1}^{2n+1}(n^2+i) = \sum_{i=1}^{1}(0^2+i) = 0^2+1 = 1,$$

and $n^3+(n+1)^3 = 0^3+(0+1)^3 = 1$, so equality holds.

Inductive Hypothesis: Suppose that for some integer $k \geq 0$,

$$\sum_{i=1}^{2k+1}(k^2+i) = k^3+(k+1)^3.$$

Show that

$$\sum_{i=1}^{2(k+1)+1}((k+1)^2+i) = (k+1)^3+((k+1)+1)^3,$$

or, equivalently

$$\sum_{i=1}^{2k+3}((k+1)^2+i) = (k+1)^3+(k+2)^3.$$

Now,

$$\begin{aligned}
\sum_{i=1}^{2k+3}((k+1)^2+i) &= \sum_{i=1}^{2k+3}(k^2+2k+1+i) \\
&= \sum_{i=1}^{2k+3}(k^2+i) + \sum_{i=1}^{2k+3}(2k+1) \\
&= \sum_{i=1}^{2k+3}(k^2+i) + (2k+3)(2k+1) \\
&= \sum_{i=1}^{2k+1}(k^2+i) + (k^2+2k+2)+(k^2+2k+3)+(4k^2+8k+3) \\
&= \sum_{i=1}^{2k+1}(k^2+i) + (6k^2+12k+8) \\
&= k^3+(k+1)^3+(4k^2+12k+8) \quad \textbf{by the I.H.} \\
&= (k+1)^3+(k^3+4k^2+12k+8) \\
&= (k+1)^3+(k+2)^3
\end{aligned}$$

as required.

Therefore, for all integers $n \geq 0$,

$$\sum_{i=1}^{2n+1} (n^2 + i) = n^3 + (n+1)^3.$$

$\square$

10. The sequence $c_0, c_1, c_2, \ldots$ is defined by: $c_0 = 1$, $c_1 = 8$, and for all integers $k \geq 2$, $c_k = c_{k-1} + 2c_{k-2}$. Prove by induction that $c_n = 3 \cdot 2^n - 2 \times (-1)^n$ for all integers $n \geq 0$.

   Proof. **Basis:** When $n = 0$, $c_n = c_0 = 1$ and

   $$3 \cdot 2^n - 2 \times (-1)^n = 3 \cdot 2^0 - 2 \times (-1)^0 = 3 \cdot 1 - 2 \cdot 1 = 3 - 2 = 1.$$

   When $n = 1$, $c_n = c_1 = 8$ and

   $$3 \cdot 2^n - 2 \times (-1)^n = 3 \cdot 2^1 - 2 \times (-1)^1 = 3 \cdot 2 - 2 \cdot (-1) = 6 + 2 = 8.$$

   Therefore the result holds for $n = 0, 1$.

   Inductive Hypothesis: Let $k > 1$ and suppose that for all integers i, $0 \leq i \leq k$,

   $$c_i = 3 \cdot 2^i - 2 \times (-1)^i.$$

   Show that $c_k = 3 \cdot 2^k - 2 \times (-1)^k$.

   Since $k \geq 2$,

   $$\begin{aligned}
   c_k &= c_{k-1} + 2c_{k-2} \\
   &= (3 \cdot 2^{k-1} - 2 \times (-1)^{k-1}) + 2(3 \cdot 2^{k-2} - 2 \times (-1)^{k-2}) \quad \textbf{by the I.H.} \\
   &\qquad\qquad \text{since } 0 \leq k-1 < k \text{ and } 0 \leq k-1 < k \\
   &= 3(2^{k-1} + 2^{k-1}) - 2 \times ((-1)^{k-1} + 2(-1)^{k-2}) \\
   &= 3 \cdot 2 \cdot 2^{k-1}) - 2 \times ((-1)^{k-1} - 2(-1)^{k-1}) \\
   &= 3 \cdot 2^k) - 2 \times (-(-1)^{k-1}) \\
   &= 3 \cdot 2^k - 2 \times (-1)^k
   \end{aligned}$$

   as required.

   Therefore, for all integers $n \geq 0$, $c_n = 3 \cdot 2^n - 2 \times (-1)^n$

   $\square$

12. The sequence $t_1, t_2, t_3, \ldots$ is defined by: $t_1 = 1$, and for all integers $k \geq 2$, $t_k = 2 \times t_{\lfloor k/2 \rfloor}$. Prove by induction that $t_n \leq n$ for all integers $n \geq 1$.

Proof. **Basis:** When $n = 2$,

$$t_n = t_2 = 2 \times t_{\lfloor 2/2 \rfloor} = 2 \times t_1 = 2 \times 1 = 2,$$

and $2 \leq 2$, so $t_2 \leq 2$.

When $n = 3$,

$$t_n = t_3 = 2 \times t_{\lfloor 3/2 \rfloor} = 2 \times t_1 = 2 \times 1 = 2,$$

and $2 \leq 3$, so $t_3 \leq 3$.

Thus, the result holds when $n = 2, 3$.

Inductive Hypothesis: Suppose that $k > 3$ and for all integers i, $2 \leq i < k$, $t_i \leq i$.

Show that $t_k \leq k$. Now,

$$t_k = 2 \times t_{\lfloor k/2 \rfloor}.$$

Since $k \geq 4$, $2 \leq \lfloor k/2 \rfloor < k$, **so we may apply the I.H.**

$$\begin{aligned} t_k &= 2 \times t_{\lfloor k/2 \rfloor} \\ &\leq 2 \times \lfloor k/2 \rfloor \\ &\leq 2 \times k/2 = k, \end{aligned}$$

as required.

Therefore, for all integers $n \geq 2$, $t_n \leq n$. $\square$