Banking a Turn

Problem 1:

The radius of curvature of a circle of radius "a" is $\rho = a$.

proof: A parametric representation of a circle of radius "a" Centred at (α, β) is given by the vector function

$$\vec{r}(t) = (\alpha + a\cos(t))\,\vec{i} + (\beta + a\sin(t))\,\vec{j}, \quad t \in [0, 2\pi]$$

For simplicity of computations, let us view the circle as a space curve by inserting z-component of zero value.

$$\vec{r}(t) = (\alpha + a\cos(t),\ \beta + a\sin(t),\ 0), \quad t \in [0, 2\pi]$$

$$\vec{v} = (-a\sin(t),\ a\cos(t),\ 0)$$

$$\vec{a} = (-a\cos(t),\ -a\sin(t),\ 0)$$

$$\vec{v} \times \vec{a} = \left(+ \begin{vmatrix} a\cos(t) & 0 \\ -a\sin(t) & 0 \end{vmatrix},\ - \begin{vmatrix} -a\sin(t) & 0 \\ -a\cos(t) & 0 \end{vmatrix},\ + \begin{vmatrix} -a\sin(t) & a\cos(t) \\ -a\cos(t) & -a\sin(t) \end{vmatrix} \right)$$

$$= (0,\ 0,\ a^2\sin^2(t) + a^2\cos^2(t))$$

But $a^2\sin^2(t) + a^2\cos^2(t) = a^2(\sin^2(t) + \cos^2(t))$

$$= a^2$$

$\therefore\ \vec{v} \times \vec{a} = (0, 0, a^2) \Rightarrow \|\vec{v} \times \vec{a}\| = a^2$

Next, speed $v = \|\vec{v}\|$

$$= \sqrt{(-a\sin(t))^2 + (a\cos(t))^2 + 0^2}$$

$$= \sqrt{a^2\sin^2(t) + a^2\cos^2(t)}$$

$$= \sqrt{a^2(\sin^2(t) + \cos^2(t))}$$

$$= \sqrt{a^2} = a$$

Curvature

$$K(t) = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{a^2}{a^3} = \frac{1}{a} \quad (\text{Constant})$$

Therefore, the radius of curvature at <u>any</u> point on the circle is given by

$$\rho = \frac{1}{K} = \frac{1}{\frac{1}{a}}$$

$$\Rightarrow \quad \rho = a \quad (\text{a Constant})$$

<u>Ex</u> : Find the radius of curvature of the circle given by

$$x^2 + y^2 - 4x + 20y + 23 = 0$$

<u>solution</u> : $(x^2 - 4x) + (y^2 + 20y) = -23$

Complete the square of x, and y - terms

$$[x^2 - 4x + \underline{(-2)^2}] + [y^2 + 20y + \underline{(10)^2}] = -23 + (-2)^2 + (10)^2$$

$$(x-2)^2 + (y+10)^2 = -23 + 100 + 4$$

$$= 81$$

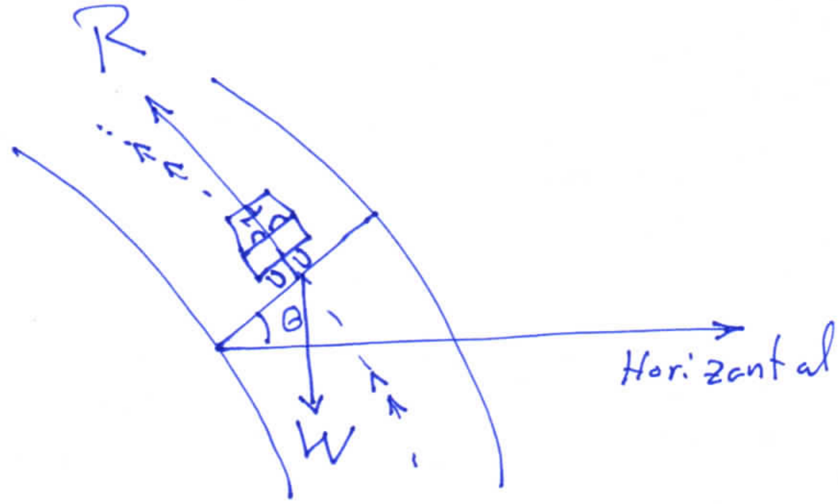
$\therefore$ Circle is of radius $a = \sqrt{81} = 9$

$\therefore$ $\rho = 9$ as well

Problem (2):

Rated Speed of a Banked Turn of a Road

When road is straight, its design is always horizonal. However, when entering a "Sharp Turn", it becomes "Angled".
This design is referred to as: "Banking of the Turn".
Usually banked roads have a posted (or rated) speed limit, that we all must not exceed in order to "safely" negotiate the turn, and hence prevent vehicle from being "pushed out" of the road!
In this problem, we shall discuss a Frictionless banked turn.
Let W, θ, and R be respectively the weight of vehicle, the banking angle, and the normal reaction force of vehicle on the road as shown in figure below.



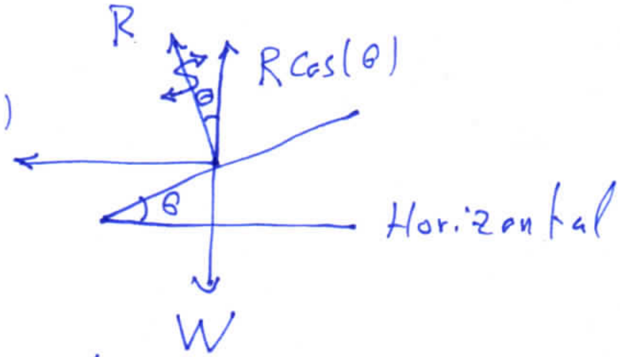
The Vertical and Horizontal Components of the reactoin force R are given by $R\cos(\theta)$, and $R\sin(\theta)$ respectively as shown.

let F_{Net} be the <u>net force</u> required to prevent the Vehicle from being "puched out" of the turn

Therefore

$R\cos(\theta) = W$ (Equilibrium in the Vertical direction)

$R\sin(\theta) = F_{Net}$



But $W = mg$, where

m is the mass of Vehicle,

and g is the gravitional acceleration

$(g \cong 9.8\ m/s^2$ or $32\ ft/s^2)$

$\therefore \quad R\cos(\theta) = mg \quad - - - - (1)$

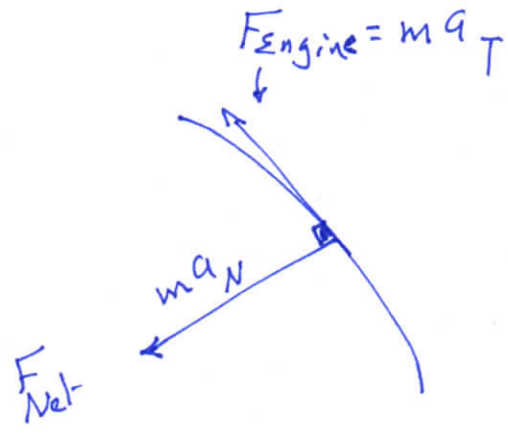
Next : F_{Net} = Centripital force

= (mass).(Normal Component of acceleration)

$= m a_N$

$F_{Net} = m K v^2$

where K is the Curvature of the turn



Hence

$$R \sin(\theta) = m K v^2 \quad \text{---- (2)}$$

Dividing both sides of Equations (1), (2) we obtain

$$\frac{R \sin(\theta)}{R \cos(\theta)} = \frac{m K v^2}{m g}$$

$$\Rightarrow \tan(\theta) = \frac{K v^2}{g}$$

But $K = \frac{1}{\rho}$, where ρ is the radius of curvature of the turn, and hence

$$\tan(\theta) = \frac{v^2}{\rho g}$$

$$\therefore \quad \boxed{\theta = \tan^{-1}\left(\frac{v^2}{\rho g}\right)}$$ "Banking angle"

Solving for v:

$$v^2 = \rho g \tan(\theta)$$

$$\boxed{v = \sqrt{\rho g \tan(\theta)}}$$ "Rated or posted speed limit"

Note: If the turn is approximately circular of radius a, then

$$\rho = a$$

as justified in Problem (1).

EX1 : A frictionless road turn is approximately circular of radius 50 metres is designed for a maximum speed of 10 m/s. Determine the banking angle of the turn to the nearest degree.

Solution :

Recall :

Banking angle $\theta = \tan^{-1}\left(\frac{v^2}{\rho g}\right)$

Here $v = 10$ m/s, $g = 9.8$ m/s^2, and $\rho = 50$ metres

$$\therefore \quad \theta = \tan^{-1}\left(\frac{(10)^2}{(50)(9.8)}\right) \cong 12^\circ$$

EX2: A frictionless turn is approximately circular of radius 41 metres is banked at an angle of 17°. Determine what will be the "Rated" speed of the turn to the nearest Kilometres per hour.

Solution :

Recall : Rated speed (speed Limit)

$$v = \sqrt{\rho g \tan(\theta)}$$

Here $\rho = 41$ m, $g = 9.8$ m/s^2, and $\theta = 17^\circ$

$$\therefore \quad v = \sqrt{(41)(9.8)(\tan(17^\circ))} \quad \text{m/s}$$

To find the speed in Kilometre per hour (Km/h), We multiply by $\frac{3600}{1000} = 3.6$

$$\therefore v = 3.6\sqrt{(41)(9.8)(\tan(17^\circ))}$$
$$\cong 40 \text{ Km/h}$$

- - - - - -

<u>EX3</u>: A frictionless turn is approximately Circular of radius 137 feet is banked at an angle of 17°. What will be the Rated Speed for the turn to the nearest Mile per hour (mi/h)?

<u>Solution</u>:

Recall: $v = \sqrt{\rho\, g \tan(\theta)}$

Here $\rho = 137$ ft., $g = 32$, $\theta = 17^\circ$

$$\therefore v = \sqrt{(137)(32)(\tan(17^\circ))} \text{ ft/s}$$

To Convert to mile per hour, we multiply by $\frac{3600}{5280} \approx 0.68$

$$\therefore v = 0.68\sqrt{(137)(32)(\tan(17^\circ))}$$
$$\cong 25 \text{ mi/h}$$