MATH 277

Problem Set # 7 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. Given that the relation $2x^3y^2 + yz^4 - xz = 2$ implicitly defines x as a differentiable function of y and z ; that is $x = x(y,z)$. Find $\dfrac{\partial x}{\partial z}$.

2. The relation $x^5 + 2xy^3 + xyz - z^4 = -15$ implicitly defines y as a differentiable function of x ,and z. Find $\dfrac{\partial y}{\partial z}$ at $(x,z) = (1,2)$.

3. Given that $x = x(y,z)$ is implicitly defined by $y^2 + y\sqrt{z} = 2 - \sin(xz^2) + \dfrac{4}{z}$

   Compute $\dfrac{\partial x}{\partial y}$ at the point where $(x,y,z) = (0,1,4)$.

4. Consider the polar coordinates : $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$. Find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$ and $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

5. Consider the so called Spherical coordinates : $\begin{cases} x = \rho\cos(\theta)\sin(\phi) \\ y = \rho\sin(\theta)\sin(\phi) \\ z = \rho\cos(\phi) \end{cases}$.

   Calculate $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$ at $(\rho,\theta,\phi) = (1,0,\frac{\pi}{6})$.

6. Show that the system of equations $\begin{cases} xy^2 + xzu + yv^2 = 3 \\ u^3yz + 2xv - u^2v^2 = 2 \end{cases}$

   can be solved for u and v as functions of x,y,z near the point P where

   $(x,y,z;u,v) = (1,1,1;1,1)$. Compute $\dfrac{\partial v}{\partial y}$ at $(x,y,z) = (1,1,1)$.

7. Show whether the system of equations : $\begin{cases} u = x + xyz \\ v = y + xy \\ w = z + 2x + 3z^2 \end{cases}$

   can be solved for x , y , z in terms of u , v, and w near the point $(x,y,z) = (0,0,0)$.

8. Find $\left(\frac{\partial u}{\partial x}\right)_y$ near the point P where $(x, y; u, v) = (1, 1, 1, 1)$ if

$xu + yvu^2 = 2$, $xu^3 + y^2v^4 = 2$.

9. In each case , find the directional derivative of f at the point P in the indicated direction :

(a) $f(x, y) = 2x^2y - 3xy^2$; $P(3, -1)$, $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$

(b) $f(x, y, z) = x^2 - 2y^2 + 3z^2$; $P(2, 0, -1)$, $\vec{u} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{2}{3}\vec{k}$

(c) $f(x, y, z) = z^2\, e^{\,xy}$; $P(-1, 2, 3)$, $\vec{v} = (3, 1, -5)$

(d) $f(x, y) = \dfrac{x - y}{x + y}$; $P(2, -1)$, $\vec{v} = 3\vec{i} + 4\vec{j}$

(e) $f(x, y, z) = z^2 \tan^{-1}(x + y)$; $P(0, 0, 4))$, $\vec{v} = 6\vec{i} + \vec{k}$

10. Given $f(x, y, z) = \sqrt{x^2 + y^2 - 12z^2}$

(a) Find the directional Derivative of f at the point $P(-2, 3, 1)$ in the direction from the point P to the point $Q(1, 9, 3)$.

(b) Find the unit vector in the direction in which f increases most rapidly and find the rate of change in this direction.

(c) Find the unit vector in which the directional derivative is a minimum and give this minimum value.

11. In what direction from the point $(1, -1)$ is the instantaneous rate of change of the function $f(x, y) = 2x^2 + 2xy - 3y^2$ is equal to 2?

12*. Let F be a function such that $D_{\vec{u}}\, F(P) = 10$ and $D_{\vec{v}}\, F(P) = -5$ where $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in $\mathbb{R}^2$. Find $D_{\vec{n}}\, F(P)$ where $\vec{n}$ is the unit vector in the direction of the vector $\vec{w} = 7\,\vec{u} + 24\vec{v}$

Hint : $\vec{n} = \dfrac{\vec{w}}{\|\vec{w}\|}$, where $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w}$

MATH 277

Solutions to Problem Set # 7

1. $2x^3y^2 + yz^4 - xz = 2$

$\Rightarrow 2x^3y^2 + yz^4 - xz - 2 = 0$

Take $F(x,y,z) = 2x^3y^2 + yz^4 - xz - 2$

Since $x = x(y,z)$

$$\frac{\partial x}{\partial z} = - \frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial x}} = - \frac{4yz^3 - x}{6x^2y^2 - z}$$

provided $6x^2y^2 - z \neq 0$

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2. $x^5 + 2xy^3 + xyz - z^4 = -15$

$\Rightarrow x^5 + 2xy^3 + xyz - z^4 + 15 = 0 \quad (*)$

Take $F(x,y,z) = x^5 + 2xy^3 + xyz - z^4 + 15$

$$\frac{\partial y}{\partial z} = - \frac{F_z}{F_y} = - \frac{xy - 4z^3}{6xy^2 + xz}$$

Next, let us find y at $(x,z) = (1,2)$:

From $(*)$: $1 + 2y^3 + 2y - 16 + 15 = 0 \Rightarrow 2y^3 + 2y = 0$

$\Rightarrow 2y(y^2+1) = 0 \Rightarrow$ only $y = 0$

$$\therefore \left.\frac{\partial y}{\partial z}\right|_{\substack{x=1\\y=0\\z=2}} = - \frac{(1)(0) - 4(2)^3}{6(1)(0)^2 + (1)(2)} = - \frac{-32}{2} = 16$$

3. $y^2 + y\sqrt{z} = 2 - \sin(xz^2) + \frac{4}{z}$

$\Rightarrow y^2 + y\sqrt{z} - 2 + \sin(xz^2) - \frac{4}{z} = 0$

$\therefore$ Take $F(x,y,z) = y^2 + y\sqrt{z} - 2 + \sin(xz^2) - \frac{4}{z}$

$$\therefore \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{2y+\sqrt{z}}{z^2\cos(xz^2)}$$

At $(x,y,z) = (0,1,4)$, we have

$$\frac{\partial x}{\partial y} = -\frac{2(1)+\sqrt{4}}{4^2\cos(0)} = \frac{2+2}{4^2\cdot 1} = \frac{1}{4}$$

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4. $x = r\cos(\theta)$
$y = r\sin(\theta)$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$= r\cos^2(\theta) + r\sin^2(\theta)$$

$$= r(\cos^2(\theta) + \sin^2(\theta)) = r$$

Next, $x = r\cos(\theta) \Rightarrow r\cos(\theta) - x = 0$

$y = r\sin(\theta) \Rightarrow r\sin(\theta) - y = 0$

Take $F(x,y,r,\theta) = r\cos(\theta) - x$

$G(x,y,r,\theta) = r\sin(\theta) - y$

$$\therefore \frac{\partial(F,G)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial F}{\partial r} & \frac{\partial F}{\partial \theta} \\ \frac{\partial G}{\partial r} & \frac{\partial G}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$= r\cos^2(\theta) + r\sin^2(\theta)$$

$$= r$$

$$\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$$

But by chain rule,

$$\frac{\partial(F,G)}{\partial(x,y)} = \frac{\partial(F,G)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)}$$

$$1 = r \cdot \frac{\partial(r,\theta)}{\partial(x,y)}$$

$$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

Interesting observation:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r, \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r} \quad !!$$

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Here is a different way to compute $\frac{\partial(r,\theta)}{\partial(x,y)}$

Recall $x = r\cos(\theta)$, $y = r\sin(\theta)$ $\Rightarrow$

$$r = \sqrt{x^2 + y^2} \quad \text{--- (1)}$$

Next, $\frac{y}{x} = \frac{\cancel{r}\sin(\theta)}{\cancel{r}\cos(\theta)} = \tan(\theta)$, or

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{-- (2)}$$

$$\therefore \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \frac{1}{r} \quad (\text{Verify yourself!})$$

5. $x = \rho \cos(\theta) \sin(\phi)$

$y = \rho \sin(\theta) \sin(\phi)$

$z = \rho \cos(\phi)$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\theta)\sin(\phi) & -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{vmatrix}$$

At $\rho = 1$, $\theta = 0$, and $\phi = \frac{\pi}{2}$, we have

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin(\frac{\pi}{6}) & 0 & \cos(\frac{\pi}{6}) \\ 0 & \sin\frac{\pi}{6} & 0 \\ \cos(\frac{\pi}{6}) & 0 & -\sin(\frac{\pi}{6}) \end{vmatrix}$$

<u>Note</u>:
$\cos(0) = 1$
$\sin(0) = 0$

$$= \begin{vmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{vmatrix}$$

Using 2nd. row to expand determinant, we have

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \frac{1}{2}\begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}\left[-\frac{1}{4} - \frac{3}{4}\right] = -\frac{1}{2}$$

6. $xy^2 + xzu + yv^2 = 3$

$u^3yz + 2xv - u^2v^2 = 2$

Rewrite system in the form

$$F(x,y,z,u,v) = xy^2 + xzu + yv^2 - 3 = 0$$
$$G(x,y,z,u,v) = u^3yz + 2xv - u^2v^2 - 2 = 0$$

Need to compute $\dfrac{\partial(F,G)}{\partial(u,v)}$ at $(x,y,z;u,v) = (1,1,1;1,1)$

Now,
$$J = \frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} xz & 2yv \\ 3u^2yz - 2uv^2 & 2x - 2u^2v \end{vmatrix}$$

$\therefore$ At $(x,y,z;u,v) = (1,1,1;1,1)$, we have

$$J = \frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 0 - 2 = -2 \neq 0$$

Since $J \neq 0$, system can be solved for u, v as functions of x, y, z near the point given.

Next we need to compute $\frac{\partial v}{\partial y}$ (or $\left(\frac{\partial v}{\partial y}\right)_{x,z}$) at $(x,y,z) = (1,1,1)$

$$\frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}$$ Just replace "v" by "y"

$$= -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{J}$$ ← computed earlier

$$= -\frac{\begin{vmatrix} xz & 2xy + v^2 \\ 3u^2yz - 2uv^2 & u^3z \end{vmatrix}_{(x,y,z;u,v)=(1,1,1;1,1)}}{-2}$$

$$= -\frac{\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}}{-2} = -\frac{(-2)}{-2} = -1$$

7. $u = x + xyz \Rightarrow x + xyz - u = 0 \Rightarrow F = x + xyz - u$

$v = y + xy \Rightarrow y + xy - v = 0 \Rightarrow G = y + xy - v$

$w = z + 2x + 3z^2 \Rightarrow z + 2x + 3z^2 - w = 0 \Rightarrow$

$$H = z + 2x + 3z^2 - w$$

We need to compute Jacobian of F, G, and H w.r.t. Dependent variables x, y, z, namely

$$J = \frac{\partial(F, G, H)}{\partial(x, y, z)} = \begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \begin{vmatrix} 1+yz & xz & xy \\ y & 1+x & 0 \\ 2 & 0 & 1+6z \end{vmatrix}$$

At $(x,y,z) = (0,0,0)$, we have

$$J = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 1 \quad \text{as obvious!}$$

Since $J \neq 0$, system can be solved for x, y, z as functions of u, v, and w.

8. $xu + yvu^2 = 2$

$xu^3 + y^2v^4 = 2$

Take $F = xu + yvu^2 - 2$, and $G = xu^3 + y^2v^4 - 2$

The notation $\left(\frac{\partial u}{\partial x}\right)_y$ tells us:

(x, y) are the Independent variables,

and hence (u, v) are the Dependent variables!

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = - \frac{\frac{\partial(F, G)}{\partial(x, v)}}{\frac{\partial(F, G)}{\partial(u, v)}} = - \frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$= - \frac{\begin{vmatrix} u & yu^2 \\ u^3 & 4y^2v^3 \end{vmatrix}}{\begin{vmatrix} x + 2yvu & yu^2 \\ 3u^2x & 4y^2v^3 \end{vmatrix}}$$

At $x = 1$, $y = 1$, $u = 1$, and $v = 1$, we get,

$$\left(\frac{\partial u}{\partial x}\right)_y = - \frac{\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix}} = - \frac{4 - 1}{12 - 3} = -\frac{3}{9}$$

$$= -\frac{1}{3}$$

9. (a) $f(x,y) = 2x^2y - 3xy^2$, $P(3,1)$, $\vec{u} = (\frac{3}{5}, \frac{4}{5})$

$$\vec{\nabla} f(P) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_P$$

$$= (4xy - 3y^2, 2x^2 - 6xy)\Big|_{P(3,1)}$$

$$= (9, 0)$$

Note: $\vec{u} = (\frac{3}{5}, \frac{4}{5})$ is already a unit vector

Since $\|\vec{u}\| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = \sqrt{\frac{9+16}{25}} = \sqrt{\frac{25}{25}} = 1$

$$\therefore D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u} = (9,0) \cdot (\tfrac{3}{5}, \tfrac{4}{5})$$

$$= \frac{1}{5}(9,0) \cdot (3,4)$$

$$= \frac{1}{5}[27 + 0] = \frac{27}{5}$$

(b) $f(x,y,z) = x^2 - 2y^2 + 3z^2$, $P(2,0,-1)$, $\vec{u} = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})$

$$\vec{\nabla} f(P) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_P = (2x, -4y, 6z)\Big|_{P(2,0,-1)}$$

$$= (4, 0, -6)$$

Note: $\vec{u} = (\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}) = \frac{1}{3}(1,2,-2)$ is already a unit vector!

$$D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u} = (4,0,-6) \cdot \tfrac{1}{3}(1,2,-2)$$

$$= \frac{1}{3}(4,0,-6) \cdot (1,2,-2)$$

$$= \frac{1}{3}(4 + 0 + 12)$$

$$= \frac{16}{3}$$

(c) $f(x,y,z) = z^2 e^{xy}$

$$\vec{\nabla} f(P) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_P$$

$$= \left(y z^2 e^{xy}, x z^2 e^{xy}, 2z\, e^{xy}\right)\Big|_{P(-1,2,3)}$$

$$= \left(18\, e^{-2}, -9\, e^{-2}, 6\, e^{-2}\right)$$

A unit vector in the direction of $\vec{v} = (3,1,-5)$ is given by $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{(3,1,-5)}{\sqrt{3^2+1^2+(-5)^2}} = \dfrac{(3,1,-5)}{\sqrt{35}}$

$$\therefore D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$$

$$= \left(18 e^{-2}, -9 e^{-2}, 6 e^{-2}\right) \cdot \frac{(3,1,-5)}{\sqrt{35}}$$

$$= \frac{3 e^{-2}}{\sqrt{35}} (6, -3, 2) \cdot (3, 1, -5)$$

$$= \frac{3 e^{-2}}{\sqrt{35}} (18 - 3 - 10) = \frac{15 e^{-2}}{\sqrt{35}}$$

(d) $f(x,y) = \dfrac{x-y}{x+y}$. Hence $\dfrac{\partial f}{\partial x} = \dfrac{2y}{(x+y)^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{-2x}{(x+y)^2}$

$$\vec{\nabla} f(P) = (f_x, f_y)\Big|_P = \left(\frac{2y}{(x+y)^2}, \frac{-2x}{(x+y)^2}\right)\Big|_{P(2,-1)}$$

$$= \left(\frac{2(-1)}{(2-1)^2}, \frac{-2(2)}{(2-1)^2}\right)$$

$$= (-2, -4)$$

A unit vector in the direction of

$$\vec{v} = 3\vec{i} + 4\vec{j} \overset{or}{=} (3,4)$$

is $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{(3,4)}{\sqrt{3^2+4^2}} = \frac{1}{5}(3,4)$

$$\therefore \ D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$$

$$= (-2,-4) \cdot \frac{1}{5}(3,4)$$

$$= \frac{1}{5}[-6-16] = -\frac{22}{5}$$

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(e) $f(x,y,z) = z^2 \tan^{-1}(x+y)$

$\dfrac{\partial f}{\partial x} = z^2 \cdot \dfrac{1}{1+(x+y)^2}$, $\quad \dfrac{\partial f}{\partial y} = z^2 \cdot \dfrac{1}{1+(x+y)^2}$,

and $\dfrac{\partial f}{\partial z} = 2z \tan^{-1}(x+y)$

$$\therefore \ \vec{\nabla} f(P) = \left(\frac{z^2}{1+(x+y)^2}, \frac{z^2}{1+(x+y)^2}, 2z \tan^{-1}(x+y) \right)_{P(0,0,4)}$$

$$= (16, 16, 0)$$

A unit vector in the direction of

$$\vec{v} = 6\vec{i} + \vec{k} = 6\vec{i} + 0\vec{j} + 1\vec{k}$$

$$\overset{or}{=} (6,0,1)$$

is given by

$$\vec{u} = \frac{(6,0,1)}{\sqrt{6^2+0^2+1^2}} = \frac{1}{\sqrt{37}}(6,0,1)$$

$$\therefore D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$$

$$= (16,16,0) \cdot \frac{1}{\sqrt{37}}(6,0,1)$$

$$= \frac{16}{\sqrt{37}}(1,1,0) \cdot (6,0,1)$$

$$= \frac{16}{\sqrt{37}}(6+0+0) = \frac{96}{\sqrt{37}}$$

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10. (a) $f(x,y,z) = \sqrt{x^2+y^2-12z^2}$

$f_x = \frac{1}{2}(x^2+y^2-12z^2)^{-\frac{1}{2}} \cdot (2x) = \dfrac{x}{\sqrt{x^2+y^2-12z^2}}$. Similarly

$f_y = \dfrac{y}{\sqrt{x^2+y^2-12z^2}}$, $\quad f_z = \dfrac{-12z}{\sqrt{x^2+y^2-12z^2}}$

$$\therefore \vec{\nabla} f(P) = \left(\frac{x}{\sqrt{x^2+y^2-12z^2}}, \frac{y}{\sqrt{x^2+y^2-12z^2}}, \frac{-12z}{\sqrt{x^2+y^2-12z^2}} \right) \Bigg|_{P(-2,3,1)}$$

$$= \left(\frac{-2}{1}, \frac{3}{1}, \frac{-12}{1} \right)$$

$$= (-2, 3, -12)$$

Next, a vector $\vec{v}$ from P to Q is given by

$$\vec{v} = \vec{PQ} = Q - P$$
$$= (1,9,3) - (-2,3,1)$$
$$= (3,6,2)$$

$\therefore$ a unit vector in the direction of $\vec{v}$ is given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(3,6,2)}{\sqrt{3^2+6^2+2^2}} = \frac{(3,6,2)}{\sqrt{9+36+4}}$$

$$\therefore \vec{u} = \frac{1}{7}(3,6,2)$$

$$\therefore D_{\vec{u}}f(p) = \vec{\nabla}f(p) \cdot \vec{u}$$
$$= (-2,3,-12) \cdot \frac{1}{7}(3,6,2)$$
$$= \frac{1}{7}(-6+18-24)$$
$$= -\frac{12}{7}.$$

(b) The unit vector in the direction in which f increases most rapidly is given by

$$\vec{n}_1 = \frac{\vec{\nabla}f(p)}{\|\vec{\nabla}f(p)\|} = \frac{(-2,3,-12)}{\sqrt{(-2)^2+3^2+(-12)^2}} = \frac{1}{\sqrt{157}}(-2,3,-12)$$

and the rate in that direction is the Maximum rate, namely $\|\vec{\nabla} f(P)\| = \sqrt{(-2)^2 + 3^2 + (-12)^2}$

$$= \sqrt{4 + 9 + 144}$$

$$= \sqrt{157}$$

(c) The unit vector in which the directional derivative is a minimum is

$$\vec{n}_2 = -\vec{n}_1 = -\frac{1}{\sqrt{157}}(-2, 3, -12)$$

$$\overset{or}{=} \frac{1}{\sqrt{157}}(2, -3, 12)$$

and the minimum value is

$$-\|\vec{\nabla} f(P)\| = -\sqrt{157}.$$

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11. $f(x,y) = 2x^2 + 2xy - 3y^2$, $P(1,-1)$

$f_x(x,y) = 4x + 2y$, $f_y(x,y) = 2x - 6y$

$\therefore \vec{\nabla} f(P) = (f_x, f_y)\Big|_P = (4x+2y, 2x-6y)\Big|_{(x,y)=(1,-1)}$

$= (2, 8)$

let $\vec{u} = a\vec{i} + b\vec{j} \stackrel{or}{=} (a,b)$ be a unit vector in the direction where rate of change is 2

$\therefore \quad a^2 + b^2 = 1$

Now, $\quad D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$

$2 = (2,8) \cdot (a,b)$

$2 = 2a + 8b$

$\Rightarrow \quad a + 4b = 1 \quad --- \quad (1)$

Know $\quad a^2 + b^2 = 1 \quad --- \quad (2)$

let us solve system (1), (2) for a, and b:

From (1): $a = 1 - 4b$

Substituting $a = 1 - 4b$ into (2), we get

$(1-4b)^2 + b^2 = 1$

$\Rightarrow 1 - 8b + 16b^2 + b^2 = 1$

$\Rightarrow 17b^2 - 8b = 0$

$\therefore \quad b(17b - 8) = 0$

$\therefore$ Either $b = 0$ or $b = \frac{8}{17}$

<u>Case 1</u>: If $b = 0$, then

$$a = 1 - 4b = 1 - 4(0) = 1$$

$\therefore \quad \vec{u} = (a, b) = (1, 0)$

<u>Case 2</u>: If $b = \frac{8}{17}$, then

$$a = 1 - 4b = 1 - 4 \cdot \frac{8}{17} = \frac{17}{17} - \frac{32}{17} = -\frac{15}{17}$$

$\therefore \quad \vec{u} = (a, b) = \left(-\frac{15}{17}, \frac{8}{17}\right)$

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12. For students to do at Home

<u>Answer</u> $D_{\vec{n}} F(P) = -2$
