Solutions

(I) (i) $y = \tan^{-1}(\sqrt{x})$

$$y' = \frac{1}{1+(\sqrt{x})^2}\cdot(\sqrt{x})' = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\,(1+x)}$$

(ii) $f(x) = \sec(\sqrt{2x+35})$

$$f'(x) = \sec(\sqrt{2x+35})\tan(\sqrt{2x+35})\cdot(\sqrt{2x+35})'$$

$$= \sec(\sqrt{2x+35})\tan(\sqrt{2x+35})\cdot\frac{1}{\sqrt{2x+35}}$$

(iii) $g(x) = \sqrt[3]{e^{2x}} \overset{\text{or}}{=} (e^{2x})^{\frac{1}{3}}$

$$= e^{\frac{2}{3}x}$$

$$\therefore g'(x) = \frac{2}{3}e^{\frac{2}{3}x}.$$

(iv) $y = \cos(2\ln(x))$

$$y' = -\sin(2\ln(x)).(2\ln(x))'$$

$$= -\sin(2\ln(x)).2\left(\frac{1}{x}\right)$$

$$= -\frac{2}{x}\sin(2\ln(x)).$$

(v) $f(x) = x^{\ln(x)}$

<u>Note</u>: $a^b = e^{\ln a^b} = e^{b\ln a}$

$$\therefore \quad f(x) = e^{\ln(x)\ln(x)} = e^{(\ln(x))^2}$$

$$\therefore f'(x) = e^{(\ln(x))^2}\cdot 2(\ln(x)).\frac{1}{x}$$

$$\overset{\text{or}}{=} \frac{2}{x}\ln(x).\,x^{\ln(x)}. \quad \left(\text{Since } e^{(\ln(x))^2} = x^{\ln(x)}!\right)$$

(vi) $h(x) = 3^{\cosh^{-1}(x)}$

Note: If $u = u(x)$ is a differentiable function,

$$\frac{d}{dx}(a^u) = a^u \ln a . u', \quad a > 0, a \neq 1$$

$$\therefore \quad h'(x) = 3^{\cosh^{-1}(x)} . \ln 3 . (\cosh^{-1}(x))'$$

$$= 3^{\cosh^{-1}(x)} . \ln 3 . \frac{1}{\sqrt{x^2 - 1}} .$$

(vii) $y = \ln(x^2+1)^x \overset{\text{or}}{=} x \ln(x^2+1)$

$$\therefore \quad y' = 1 . \ln(x^2+1) + x . \frac{2x}{x^2+1}$$

$$= \ln(x^2+1) + \frac{2x^2}{x^2+1}$$

(viii) $y = \sec^{-1}\left(\frac{1}{x}\right), \quad 0 < x < 1$

$$\Rightarrow y = \cos^{-1}(x)$$

Hence $y' = \dfrac{-1}{\sqrt{1 - x^2}}$.

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(II) (i) $g(x) = \sqrt{\ln(x-2)}$

$g(x)$ is defined and is real provided

$$\ln(x-2) \geq 0$$

$$\Rightarrow \log_e(x-2) \geq 0$$

$$\Rightarrow x - 2 \geq e^0 = 1$$

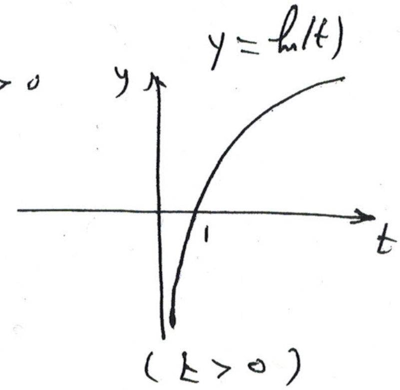
$$\Rightarrow x \geq 3$$

or $x \in [3, \infty)$

(ii) $f(x) = \sqrt[3]{\ln(x-2)}$

Since $\sqrt[3]{a}$ is defined for all $a \in \mathbb{R}$, we need only to insure $\ln(x-2)$ is defined.

From graph: $\ln(t)$ is defined for $t > 0$



Here $t = x - 2$

$\therefore$ $x - 2 > 0 \Rightarrow x > 2$

or $x \in (2, \infty)$

(III) (i) $\lim_{x \to 0^+} \frac{\sin(2x)}{e^x - 1}$ (form $\frac{\sin(0)}{e^0 - 1} = \frac{0}{0}$)

$\therefore$ Can apply L'Hôpital's Rule! We have,

$$\lim_{x \to 0^+} \frac{(\sin(2x))'}{(e^x - 1)'} = \lim_{x \to 0^+} \frac{2\cos(2x)}{e^x}$$

$$= \frac{2\cos(0)}{e^0} = \frac{2(1)}{1} = 2$$

(ii) $\lim_{x \to \infty} x \tan\left(\frac{2}{x}\right)$ (form $\infty \tan\left(\frac{2}{\infty}\right) = \infty \times 0$)

Can't apply L'Hôpital's rule!

Trick: Rewrite $x \tan\left(\frac{2}{x}\right)$ as $\frac{\tan\left(\frac{2}{x}\right)}{\frac{1}{x}}$

$\therefore$ Limit becomes

$$\lim_{x \to \infty} \frac{\tan\left(\frac{2}{x}\right)}{\frac{1}{x}} \quad \left(\text{ form is now } \frac{0}{0} !\right)$$

$$= \lim_{x \to \infty} \frac{\sec^2\left(\frac{2}{x}\right) \cdot \left(\cancel{-}\frac{2}{\cancel{x^2}}\right)}{\cancel{-\frac{1}{x^2}}}$$

$$= 2\sec^2\left(\frac{2}{\infty}\right) = 2\sec^2(0) = 2(1)^2 = 2$$

(iii) $\lim_{x \to 1^-} \frac{\ln(2x-1)}{1-x}$ (form $\frac{\ln 1}{1-1} = \frac{0}{0}$)

$$\therefore \lim_{x \to 1^-} \frac{\frac{2}{2x-1}}{-1} = \frac{\frac{2}{2-1}}{-1} = -2$$

(iv) $\lim_{x \to 0^-} \tan^{-1}\left(\frac{3}{x}\right) = \tan^{-1}\left(\lim_{x \to 0^-} \frac{3}{x}\right)$

$$= \tan^{-1}(-\infty) = -\tan^{-1}(\infty) = -\frac{\pi}{2}$$

(v) $\lim_{x \to 0^+} x \ln(x)$ (form $0 \times (-\infty)$)

Rewrite $x \ln(x)$ as $\frac{\ln(x)}{\frac{1}{x}}$. We have

$$\lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = 0$$

(vi) $\lim_{x \to -\infty} x^2 e^{5x}$ (form $\infty \times 0$)

$$= \lim_{x \to -\infty} \frac{x^2}{e^{-5x}} \quad \left(\text{Now form is } \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to -\infty} \frac{2x}{-5e^{-5x}} \quad \left(\text{still } \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to -\infty} \frac{2}{25 e^{-5x}} = \frac{2}{\infty} = 0$$

Note: you could use the fact that the exponential function wins over the power function at Infinity

$$\therefore \lim_{x \to -\infty} x^2 e^{5x} = \lim_{x \to -\infty} e^{5x} = e^{-\infty} = 0.$$

(IV) (1) Recall $\sin^2(\theta) = \frac{1}{2}[1-\cos(2\theta)]$

$$\therefore \quad I = \int \sin^2(\theta)\,d\theta = \frac{1}{2}\int[1-\cos(2\theta)]\,d\theta$$
$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$

(2) Recall $\cot^2(3x)+1 = \csc^2(3x)$. Hence

$$\cot^2(3x) = \csc^2(3x) - 1$$
$$I = \int \cot^2(3x)\,dx = \int(\csc^2(3x)-1)\,dx$$
$$= -\frac{1}{3}\cot(3x) - x + C$$

(3) $\int \sec^2(u)\,du = \tan(u) + C$

(4) Recall $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$

$$\therefore \quad I = \int[\cos^2(\theta)-\sin^2(\theta)]\,d\theta = \int \cos(2\theta)\,d\theta$$
$$= \frac{1}{2}\sin(2\theta) + C$$
$$\overset{\text{or}}{=} \frac{1}{2}\cdot 2\sin(\theta)\cos(\theta) + C$$
$$= \sin(\theta)\cos(\theta) + C$$

(5) Write $\dfrac{\sin(\theta)}{\cos^2(\theta)} = \dfrac{1}{\cos(\theta)}\cdot\dfrac{\sin(\theta)}{\cos(\theta)}$

$$= \sec(\theta)\tan(\theta)$$

$$\therefore \quad I = \int \frac{\sin(\theta)}{\cos^2(\theta)}\,d\theta = \int \sec(\theta)\tan(\theta)\,d\theta$$
$$= \sec(\theta) + C$$

(6) Recall $\cos^2(t) + \sin^2(t) = 1$. Hence

$$\cos^2(t) = 1 - \sin^2(t)$$

$$\therefore I = \int \frac{\cos^2(t)}{\sin(t)}\,dt = \int \frac{1-\sin^2(t)}{\sin(t)}\,dt$$

$$= \int \left(\frac{1}{\sin(t)} - \frac{\sin^2(t)}{\sin(t)}\right) dt$$

$$= \int (\csc(t) - \sin(t))\,dt$$

$$\underset{\text{Table}}{\overset{\text{By}}{=}} \ln|\csc(t) - \cot(t)| + \cos(t) + C$$

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(V) (1) $\int \frac{2}{5x-3}\,dx = 2\cdot\frac{1}{5}\ln|5x-3| + C$

$\overset{\text{or}}{=} \frac{2}{5}\ln|5x-3| + C$

(2) $\int \frac{1}{(7x-2)^2}\,dx = \int (7x-2)^{-2}\,dx$

$= \frac{1}{7}\frac{(7x-2)^{-1}}{-1} + C = -\frac{1}{7}(7x-2)^{-1} + C$

(3) $\int \frac{1}{\sqrt{2-9x}}\,dx = \int (2-9x)^{-\frac{1}{2}}\,dx$

$= \frac{1}{-9}\frac{(2-9x)^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= -\frac{2}{9}\sqrt{2-9x} + C$

(4) $\int e^{3x-4}\,dx = \frac{1}{3}e^{3x-4} + C$

(5) $\int (2z-y+3x)^{101}\,dy = \frac{(2z-y+3x)^{102}}{(-1)(102)} + C$

$= -\frac{1}{102}(2z-y+3x)^{102} + C$

(6) $\int x\cos(xy)\,dy = \cancel{x}\cdot\frac{1}{\cancel{x}}\sin(xy) + C$

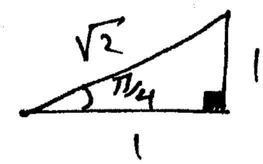
$= \sin(xy) + C.$

(7) $\displaystyle\int_0^3 \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big|_0^3$

$$= \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}\left(\frac{0}{3}\right)\right]$$

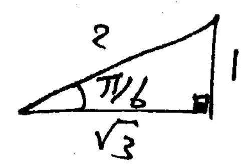
$$= \frac{1}{3}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] = \frac{1}{3}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{12}$$

<u>Note</u> : $\tan^{-1} 1$ = The acute angle (in radians) whose tan is 1.



$\tan\left(\frac{\pi}{4}\right) = 1 \Rightarrow$
$\tan^{-1}(1) = \frac{\pi}{4}$

(8) $\displaystyle\int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx = \sin^{-1}\left(\frac{x}{4}\right)\Big|_0^2$



$$= \sin^{-1}\left(\frac{2}{4}\right) - \sin^{-1}\left(\frac{0}{4}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

(9) $\displaystyle\int \frac{1}{\sqrt{4+t^2}}\,dt = \sinh^{-1}\left(\frac{t}{2}\right) + C$

(10) $\displaystyle\int \frac{1}{\sqrt{t^2-4}}\,dt = \cosh^{-1}\left(\frac{t}{2}\right) + C, \quad t > 2$

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(VI) (1) $\int \frac{\ln(x^2)}{x}\,dx$

First note that $\ln(x^2) = \ln|x|^2 = 2\ln|x| = 2\ln(x)$

Since $x>0$.

$\therefore\ I = \int \frac{2\ln(x)}{x}\,dx = 2\int \ln(x)\cdot\left(\frac{1}{x}\,dx\right)$

let $u = \ln(x)$, $du = \frac{1}{x}\,dx$

$I = 2\int u\,du = 2\cdot\frac{1}{2}u^2 + C = u^2 + C$

$= \ln^2(x) + C$

(2) $\int \frac{1}{e^{2x}+1}\,dx$

Multiplying numerator and denominator by e^{-2x}, we have

$$I = \int \frac{1}{(e^{2x}+1)}\cdot\frac{e^{-2x}}{e^{-2x}}\,dx$$

$$= \int \frac{e^{-2x}}{e^{0}+e^{-2x}}\,dx = \int \frac{e^{-2x}}{1+e^{-2x}}\,dx$$

Now, let $u = 1+e^{-2x}$, $\therefore\ du = -2e^{-2x}\,dx$

$\Rightarrow e^{-2x}\,dx = -\frac{1}{2}\,du$

$I = -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|u| + C$

$= -\frac{1}{2}\ln\left(1+e^{-2x}\right) + C$

Note: $1+e^{-2x} > 0$, so no absolute value needed!

(3) $\int x^3 e^{x^4} dx$

let $u = x^4$, $\therefore du = 4x^3 dx$ or $x^3 dx = \frac{1}{4} du$

$I = \int e^{x^4} \cdot (x^3 dx) = \int e^u \cdot \frac{1}{4} du$

$= \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C.$

(4) $\int \frac{\sin(\sqrt{x})}{2\sqrt{x}} dx$

let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$

$\therefore I = \int \sin(\sqrt{x}) \cdot \left(\frac{1}{2\sqrt{x}} dx\right)$

$= \int \sin(u) \cdot du = -\cos(u) + C$

$= -\cos(\sqrt{x}) + C$

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(VII) (a) $\int x \cos(4x)\,dx$ ← By parts Once

$$u = x \qquad dV = \cos(4x)\,dx$$

$$du = 1\,dx \xleftarrow{-\int} V = \frac{1}{4}\sin(4x)$$

$$I = \frac{x}{4}\sin(4x) - \frac{1}{4}\int \sin(4x)\,dx$$

$$= \frac{1}{4}x\sin(4x) + \frac{1}{16}\cos(4x) + C$$

(b) $\int 27x^2 e^{3x}\,dx$ → By parts Twice.

let us do it: The Fast Way!

$$I = (9x^2 - 6x + 2)e^{3x} + C$$

$27x^2$	$+$	e^{3x}
$54x$	$-$	$\frac{1}{3}e^{3x}$
54	$+$	$\frac{1}{9}e^{3x}$
0		$\frac{1}{27}e^{3x}$

(c) $\int \sqrt{x}\,\ln(\sqrt{x})\,dx$ ← By parts once.

Note $\ln\sqrt{x} = \frac{1}{2}\ln(x)$

$$I = \frac{1}{2}\int \sqrt{x}\,\ln(x)\,dx$$

$$u = \ln(x) \qquad dV = \sqrt{x}\, dx$$

$$du = \frac{1}{x}\, dx \qquad -\int \qquad V = \frac{2}{3}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$$

$$I = \frac{1}{2}\left[\frac{2}{3}x\sqrt{x}\ln(x) - \frac{2}{3}\int \frac{1}{x}\cdot x\sqrt{x}\, dx\right]$$

$$= \frac{1}{3}x\sqrt{x}\ln(x) - \frac{1}{3}\int \sqrt{x}\, dx$$

$$= \frac{1}{3}x\sqrt{x}\ln(x) - \frac{2}{9}x^{\frac{3}{2}} + C$$

$$\overset{or}{=} \frac{1}{3}x\sqrt{x}\ln(x) - \frac{2}{9}x\sqrt{x} + C$$

(d) $\int 2e^{x}\sinh(x)\, dx$

Use definition : $\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$

$$\therefore I = \int 2e^{x}\cdot\frac{1}{2}(e^{x} - e^{-x})\, dx$$

$$= \int e^{x}(e^{x} - e^{-x})\, dx = \int (e^{2x} - e^{0})\, dx$$

$$= \int (e^{2x} - 1)\, dx = \frac{1}{2}e^{2x} - x + C$$

(e) $\int \tan^{-1}(x)\,dx$ → By parts once.

$u = \tan^{-1}(x)$ $\qquad$ $dV = 1\,dx$

$du = \frac{1}{1+x^2}\,dx$ $\qquad$ $V = x$

$I = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx$

← adjust so that top = (bottom)'

$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$= x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C$

(f) $\int \frac{10}{x^2+4x-21}\,dx$ .. PFD

$$\frac{10}{x^2+4x-21} = \frac{10}{(x+7)(x-3)} = \frac{A}{x+7} + \frac{B}{x-3}$$

Multiplying both sides by $(x+7)(x-3)$:

$$10 = A(x-3) + B(x+7)$$

At $x=3$: $10 = 0 + B(3+7) \Rightarrow 10 = 10B$,

$$B = 1$$

$\underline{\underline{\text{At } x=-7}}$: $10 = A(-7-3) + 0$

$\Rightarrow 10 = -10A, \quad A = -1$

$$\therefore \frac{10}{(x+7)(x-3)} = -\frac{1}{x+7} + \frac{1}{x-3}$$

$$\therefore I = -\int \frac{1}{x+7}\,dx + \int \frac{1}{x-3}\,dx$$

$$= -\ln|x+7| + \ln|x-3| + C$$

$$\overset{\text{or}}{=} \ln\left|\frac{x-3}{x+7}\right| + C$$

(g) $\displaystyle\int \frac{x+2}{x^2+4x+20}\,dx$

Note: top can be made to equal to derivative of bottom!

Indeed

$$I = \int \frac{x+2}{x^2+4x+20}\,dx$$

$$= \frac{1}{2}\int \frac{2(x+2)}{x^2+4x+20}\,dx = \frac{1}{2}\int \frac{2x+4}{x^2+4x+20}\,dx$$

$$= \frac{1}{2}\ln|x^2+4x+20| + C$$

(h) $\displaystyle\int \frac{2x+5}{x^2+4x+20}\,dx$ ← Complete the Square! (why?)

$$x^2+4x+20 = (x+2)^2+16$$

$$I = \int \frac{2x+5}{(x+2)^2+16}\,dx$$

Now, let $t = x+2$, $dt = dx$, and $x = t-2$

$$I = \int \frac{2(t-2)+5}{t^2+16}\,dt = \int \frac{2t+1}{t^2+16}\,dt$$

← split

$$= \int \frac{2t}{t^2+16}\,dt + \int \frac{1}{t^2+16}\,dt$$

$$= \ln(t^2+16) + \frac{1}{4}\tan^{-1}\left(\frac{t}{4}\right) + C$$

Recall $t = x+2$

$$\therefore I = \ln\left((x+2)^2+16\right) + \frac{1}{4}\tan^{-1}\left(\frac{x+2}{4}\right) + C$$

$$\stackrel{\text{or}}{=} \ln\left(x^2+4x+20\right) + \frac{1}{4}\tan^{-1}\left(\frac{x+2}{4}\right) + C$$

(i) $\int \sqrt{\cosh(2x) - \sinh(2x)}\, dx$

Recall: $\cosh(\alpha) - \sinh(\alpha) = e^{-\alpha}$

$\therefore \cosh(2x) - \sinh(2x) = e^{-2x}$

$I = \int \sqrt{e^{-2x}}\, dx = \int (e^{-2x})^{\frac{1}{2}} dx = \int e^{-x} dx$

$= -e^{-x} + C$

(j) $\int_{-2}^{-1} \sqrt{4t^2 - 4t + 1}\, dt$

let us check whether $4t^2 - 4t + 1$ is a perfect square!

Indeed, $4t^2 - 4t + 1 = (2t-1)(2t-1)$

$= (2t-1)^2$ .. yes! it is

$\therefore I = \int_{-2}^{-1} \sqrt{(2t-1)^2}\, dt$

$= \int_{-2}^{-1} |2t-1|\, dt$

Warning $\sqrt{a^2} = |a|$

$= \begin{cases} -a & \text{if } a < 0 \\ +a & \text{if } a > 0 \end{cases}$

Now, we need to find the Sign of $2t-1$

$$2t-1 \quad - - - - - \overset{\bullet}{} + + + + +$$

$$-2 \qquad -1 \qquad \tfrac{1}{2}$$

$2t-1=0$

$\Rightarrow \boxed{t=\frac{1}{2}}$

so for $t \in [-2,-1]$, $2t-1<0$, hence

$$|2t-1| = -(2t-1) = 1-2t$$

$$I = \int_{-2}^{-1} (1-2t)\,dt = t - t^2 \Big|_{-2}^{-1}$$

$$= \left(-1-(-1)^2\right) - \left(-2-(-2)^2\right)$$

$$= (-1-1) - (-2-4)$$

$$= -2+6 = 4$$

(K) $\displaystyle\int_1^2 \sqrt{4t^2+12+\frac{9}{t^2}}\,dt$

Clearly $4t^2+12+\frac{9}{t^2} = \left(2t+\frac{3}{t}\right)\left(2t+\frac{3}{t}\right)$

$$= \left(2t+\frac{3}{t}\right)^2$$

$$I = \int_1^2 \sqrt{\left(2t+\frac{3}{t}\right)^2}\,dt = \int_1^2 \left|2t+\frac{3}{t}\right| dt$$

But $2t+\frac{3}{t} > 0$ for $t \in [1,2]$, hence

$$\left|2t+\frac{3}{t}\right| = +\left(2t+\frac{3}{t}\right)$$

$$\therefore I = \int_1^2 \left(2t+\frac{3}{t}\right) dt$$

$$= t^2 + 3\ln|t| \Big|_{t=1}^{t=2}$$

$$= (2^2 + 3\ln 2) - (1^2 + 3\cancel{\ln(1)}^{0})$$

$$= 4 + 3\ln 2 - 1 = 3 + 3\ln 2$$

(d) $\int \frac{1}{x^3+x}\, dx \to$ P.F.D

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by $x(x^2+1)$:

$$1 = A(x^2+1) + (Bx+C)x$$

At $x=0$: $1 = A(0+1) \Rightarrow \boxed{A=1}$

Comparing coefficients of x^2: $0 = A+B \Rightarrow B = -A$

$$\therefore \boxed{B=-1}$$

Comparing coefficients of x:

$$0 = C$$

$$\therefore \quad \frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-1x+0}{x^2+1}$$

$$= \frac{1}{x} - \frac{x}{x^2+1}$$

$$I = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx$$

$$= \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

Now: top = (bottom)'

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + C$$

$$\overset{or}{=} \ln|x| - \ln\sqrt{x^2+1} + C$$

$$= \ln\left\{\frac{|x|}{\sqrt{x^2+1}}\right\} + C$$

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