MATH 277

Problem Set # 10 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. Use double integrals to find the $y-coordinate$ of the centroid of the lamina which occupies the planar region given by $-6y \leq x \leq 3y^2$, $0 \leq y \leq 2.$

2. Use double integrals to find the $x-coordinate$ of the centre of mass of the planar region $x^2 \leq y \leq x$, $0 \leq x \leq 1$ if the density function is given by $\delta(x,y) = 30(x+x^2).$

3. Use double integrals to find the mass of the planar region described by $-x \leq y \leq 2-x^2$, $1 \leq x \leq 2$ if the density function is given by $\delta(x,y) = \dfrac{1}{x^2}.$

4. Use double integrals to find the moment about the $y-axis$ of the lamina which occupies the planar region given by $x^2-x \leq y \leq x^2$, $0 \leq x \leq 1$ if the density function at the point (x,y) is given by $\delta(x,y) = e^x.$

5. Use double integrals to find the coordinates of the centroid of the planar region given by $x^2+y^2 \leq 4$, $x \geq 0$, $y \geq 0.$

6. Use double integrals to find the coordinates of the centre of mass of the planar region given by $x^2+y^2 \leq 1$, and $y \geq 0$ if the density function is given by $\delta(x,y) = x^2+y^2.$

7*Evaluate $\iint_R dA$, where R is the region enclosed by the Trapezoid with vertices $(0,0)$, $(0,6)$, $(3,5)$, and $(3,1).$

8. Evaluate $\iint_R xy^2 dA$, where R is a planar region with mass equal to 3 , Centre of mass at the point $(\bar{x},\bar{y}) = (1,4)$, and density $\delta(x,y) = xy.$

9. Use **Cylindrical Coordinates** to find the mass of the solid which occupies the region enclosed by the cones $z = 8 - \sqrt{x^2 + y^2}$, and $z = 3\sqrt{x^2 + y^2}$ if the density function $\delta(x, y, z) = 2 + \sqrt{x^2 + y^2}$.

10. Use **Spherical Coordinates** to the mass of the hemispherical solid $x^2 + y^2 + z^2 \leq 2$, $z \geq 0$ with density $\delta(x, y, z) = z^3 \sqrt{1 + (x^2 + y^2 + z^2)^3}$.

11. Use **Cylindrical Coordinates** to find the coordinates of the centroid of the region enclosed by $z = \sqrt{x^2 + y^2}$, and $z = 2$.

12. Use **Cylindrical Coordinates** to find the coordinates of the centroid of the solid enclosed by $z = \sqrt{x^2 + y^2}$, and the sphere $x^2 + y^2 + z^2 - 4z = 0$.

13* Re do problem # 7 using **Spherical Coordinates**.

14* Find the coordinates of the centroid of the hemispherical region described by $0 \leq z \leq \sqrt{64 - x^2 - y^2}$

15*Use spherical coordinates to calculate the moment $\mathbf{M}_{z=0}$ of the solid occupying the region $\mathbf{E}$ described by $0 \leq z \leq \sqrt{1 - x^2 - y^2}$ if the density function is given by $\delta(x, y, z) = (x^2 + y^2 + z^2)^{3/2}$.

MATH 277

Solutions to Problem Set #10

1. Use double integrals to find the y-coordinate of the centroid of the lamina which occupies the planar region given by $-6y \le x \le 3y^2$, $0 \le y \le 2$.

Solution: Recall $\bar{y} = \dfrac{M_{y=0}}{m}$

Note also that: The region is already described, hence there is no need to sketch

Now, $dm = \delta(x,y)\,dA$. For centroid $\delta(x,y) = $ a constant, say $\delta(x,y) = 1$.

$\therefore\quad dm = 1 \cdot dA \implies \boxed{dm = dA}$

$$\therefore \text{Mass } m = \iint_R dm = \iint_R dA = \int_0^2 \int_{-6y}^{3y^2} dx\,dy$$

$$= \int_0^2 x \Big|_{-6y}^{3y^2} dy = \int_0^2 [3y^2 - (-6y)]\,dy$$

$$= \int_0^2 (3y^2 + 6y)\,dy = y^3 + 3y^2 \Big|_0^2 = 2^3 + 3(2)^2$$

$$= 8 + 12 = 20$$

Next, $$M_{y=0} = \iint_R y\,dm = \iint_R y\,dA$$

$$= \int_0^2 y \left\{ \int_{-6y}^{3y^2} dx \right\} dy = \int_0^2 y[3y^2 + 6y]\,dy$$

$$= \int_0^2 (3y^3 + 6y^2)\,dy = \frac{3}{4}y^4 + 2y^3 \Big|_0^2$$

$$\therefore M_{y=0} = \frac{3}{4}(2)^4 + 2(2)^3 = 12 + 16 = 28$$

$$\therefore \bar{y} = \frac{28}{20} \quad \text{or} \quad \bar{y} = \frac{7}{5}$$

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2. Use double integrals to find the x-coordinate of the centre of mass of the planar region $x^2 \leq y \leq x$, $0 \leq x \leq 1$ if the density function is given by $\delta(x,y) = 30(x + x^2)$

<u>Solution</u>: Recall $\bar{x} = \dfrac{M_{x=0}}{m}$

Note also that: The region is already described. Hence, there is <u>no</u> need to sketch.

Now, $dm = \delta(x,y)\, dA = 30(x + x^2)\, dA$

$$\therefore \text{Mass} \quad m = \iint_R dm = \iint_R 30(x + x^2)\, dA$$

$$= \int_0^1 \int_{x^2}^{x} 30(x + x^2)\, dy\, dx$$

$$= \int_0^1 30(x + x^2) \left\{ \int_{x^2}^{x} dy \right\} dx$$

$$= \int_0^1 30(x + x^2)(x - x^2)\, dx$$

$$= 30 \int_0^1 (x^2 - x^4)\, dx = 30 \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1$$

$$= 30\left(\frac{1}{3} - \frac{1}{5}\right) = 10 - 6 = 4,$$

Next,

$$M_{x=0} = \iint_R x\,dm = \iint_R x \cdot 30(x+x^2)\,dA$$

$$= 30\int_0^1 \int_{x^2}^{x} x(x+x^2)\,dy\,dx$$

$$= 30\int_0^1 x(x+x^2)\left\{\int_{x^2}^{x} dy\right\}dx$$

$$= 30\int_0^1 x(x+x^2)(x-x^2)\,dx$$

$$= 30\int_0^1 x(x^2-x^4)\,dx = 30\int_0^1 (x^3-x^5)\,dx$$

$$= 30\left[\frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^1 = 30\left[\frac{1}{4}-\frac{1}{6}\right]$$

$$= \frac{15}{2} - 5 = \frac{5}{2}$$

$$\therefore \bar{x} = \frac{\frac{5}{2}}{4} = \frac{5}{8}$$

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3. Use Double integrals to find the mass of the planar region described by $-x \leq y \leq 2-x^2,\ 1 \leq x \leq 2$ if the density at the point (x,y) is given by $\delta(x,y) = \frac{1}{x^2}$.

Solution :

Recall mass $m = \iint_R dm$

Note also that : The region R is already described! Hence, there is no need to sketch.

Now, $dm = \delta(x,y)\,dA = \frac{1}{x^2}\,dA$

$$\therefore \quad m = \iint_R \frac{1}{x^2}\, dA = \int_1^2 \int_{-x}^{2-x^2} \frac{1}{x^2}\, dy\, dx$$

$$= \int_1^2 \frac{1}{x^2} \left\{ \int_{-x}^{2-x^2} dy \right\} dx = \int_1^2 \frac{1}{x^2} \left[(2-x^2) - (-x) \right] dx$$

$$= \int_1^2 x^{-2} \left[2 - x^2 + x \right] dx = \int_1^2 \left(2x^{-2} - 1 + \frac{1}{x} \right) dx$$

$$= -\frac{2}{x} - x + \ln|x| \Big|_1^2$$

$$= (-1 - 2 + \ln 2) - (-2 - 1 + \ln 1)$$

$$= -3 + \ln 2 - (-3 + 0) = \ln 2$$

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4. Use double integrals to find the moment about the y-axis of the Lamina which occupies the planar region given by $x^2 - x \leq y \leq x^2$, $0 \leq x \leq 1$ if the density at the point (x,y) is given by $\delta(x,y) = e^x$.

Solution :

Recall : Moment about the y-axis (which has the equation $x = 0$) is : $M_{x=0} = \iint_R x\, dm$

Note also that, there is no need to sketch region R Since it is already described (as a y-Simple !)

Now, $\quad dm = \delta(x,y)\, dA = e^x\, dA$

$$\therefore \quad M_{x=0} = \iint_R x \cdot e^x\, dA = \int_0^1 \int_{x^2-x}^{x^2} x e^x\, dy\, dx$$

$$\therefore\ M_{x=0} = \int_0^1 x e^{x} \left\{ \int_{x^2-x}^{x^2} dy \right\} dx$$

$$= \int_0^1 x e^{x} \left[x^2 - (x^2 - x) \right] dx = \int_0^1 x e^{x} \cdot x \, dx$$

$$= \int_0^1 x^2 e^{x} \, dx \leftarrow \text{By parts (Twice)}$$

$$= x^2 e^{x} - 2x e^{x} + 2e^{x} \Big|_0^1$$

$$= (e - 2e + 2e) - (0 - 0 + 2)$$

$$= e - 2$$

x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

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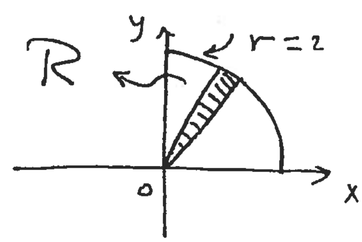
5. Use double integrals to find the coordinates of the Centroid of the planar region given by $x^2 + y^2 \leq 4$, $x \geq 0$, and $y \geq 0$.

<u>Solution</u>: The region R is the quarter Circle $x^2+y^2=4$ in first quadrant.



$0 \leq r \leq 2$

$0 \leq \theta \leq \frac{\pi}{2}$

In polar coordinates,

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2+y^2=r^2$, and $dA = r\,dr\,d\theta$

Therefore, $x^2+y^2=4 \Rightarrow r^2 = 4 \Rightarrow r = 2$

Now, for Centroid, $\delta(x,y) = 1$, hence $dm = 1 \cdot dA = dA$

$$\therefore \text{mass } m = \iint_R dm = \iint_R dA$$

$$= \text{area of region } R$$

$$= \frac{1}{4}\pi(2)^2 = \pi,$$

Next, $M_{x=0} = \iint_R x\,dm = \iint_R x\,dA$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 (r\cos(\theta)) \cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos(\theta)\,d\theta \cdot \int_0^2 r^2\,dr$$

$$= \sin(\theta)\Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{3} r^3\Big|_0^2$$

$$= \left[\sin\left(\tfrac{\pi}{2}\right) - \sin(0)\right] \cdot \frac{1}{3}\left[2^3 - 0^3\right]$$

$$= [1-0] \cdot \frac{1}{3}[8] = \frac{8}{3}$$

$$\therefore \quad \bar{x} = \frac{M_{x=0}}{m} = \frac{\frac{8}{3}}{\pi} = \frac{8}{3\pi}$$

From symmetry, $\bar{y} = \frac{8}{3\pi}$ as well

$\therefore$ Centroid is at the point $(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$.

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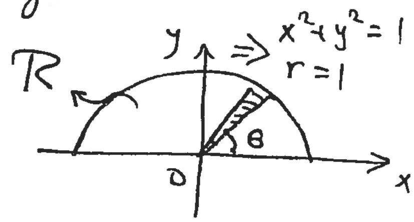
6. Use double integrals to find the coordinates of the Centre of mass of the planar region given by $x^2+y^2 \leq 1$, and $y \geq 0$ if the density function is given by $\delta(x,y) = x^2+y^2$.

Solution: We shall use polar Coordinates!

Clearly, R is the upper semi-circular region shown in figure.



$0 \leq r \leq 1$

$0 \leq \theta \leq \pi$

In polar Coordinates,

$x = r\cos(\theta)$, $y = r\sin(\theta)$,

$x^2+y^2 = r^2$, and $dA = r\,dr\,d\theta$

Now, $dm = \delta(x,y)\, dA$

$$= (x^2+y^2)\, dA$$

In polar Coordinates, $dm = r^2 \cdot r\,dr\,d\theta = r^3\,dr\,d\theta$

$\therefore$ mass $m = \iint_R dm = \int_0^{\pi}\int_0^1 r^3\,dr\,d\theta$

$$= \int_0^{\pi} d\theta \cdot \int_0^1 r^3\,dr = \pi \cdot \frac{1}{4} = \frac{\pi}{4}$$

Next, $M_{y=0} = \iint_R y\,dm = \int_0^{\pi}\int_0^1 r\sin(\theta) \cdot r^3\,dr\,d\theta$

$$= \int_0^{\pi} \sin(\theta)\,d\theta \cdot \int_0^1 r^4\,dr$$

$$= -\cos(\theta)\Big|_0^{\pi} \cdot \frac{1}{5} r^5\Big|_0^1$$

$$= -[\cos(\pi) - \cos(0)] \cdot \frac{1}{5} = -[-1-1] \cdot \frac{1}{5} = \frac{2}{5}$$

$\therefore \quad \bar{y} = \dfrac{M_{y=0}}{m} = \dfrac{\frac{2}{5}}{\pi/4} = \dfrac{8}{5\pi}$

Note that: Since Both density and region R are symmetric w.r.to y-axis, we have $\bar{x} = 0$

$\therefore$ Centre of mass is at the point $(\bar{x}, \bar{y}) = \left(0, \frac{8}{5\pi}\right)$

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7. Evaluate $\iint\limits_R dA$, where R is the region enclosed by the Trapezoid with vertices $(0,0)$, $(0,6)$, $(3,5)$, and $(3,1)$.

For students to do at home.

<u>Answer</u>: 15

8. Evaluate $\iint\limits_R xy^2\,dA$, where R is a planar region with mass equal to 3, Centre of mass at the point $(\bar{x},\bar{y}) = (1,4)$, and density $\delta(x,y) = xy$.

<u>Solution</u>: Here $\delta(x,y) = xy$

$$\therefore\ dm = \delta(x,y)\,dA = xy\,dA$$

Now, we may express $\iint\limits_R xy^2\,dA$ in the form

$$\iint\limits_R xy^2\,dA = \iint\limits_R y\,(xy\,dA) = \iint\limits_R y\,dm$$

$$= M_{y=0}$$

But $\bar{y} = \dfrac{M_{y=0}}{m}$

Hence $M_{y=0} = \bar{y}\,m = (4)(3) = 12$

$$\therefore\ \iint\limits_R xy^2\,dA = 12$$

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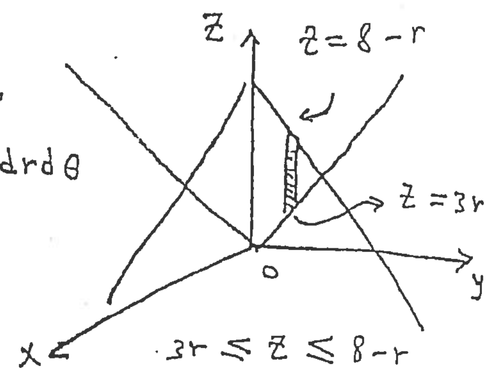
9. Use cylindrical coordinates to find the mass of the solid which occupies the region enclosed by the cones $z = 8 - \sqrt{x^2+y^2}$, and $z = 3\sqrt{x^2+y^2}$ if the density $\delta(x,y,z) = 2 + \sqrt{x^2+y^2}$.

<u>Solution</u>: In cylindrical coordinates

$x^2+y^2 = r^2$, $dV = dz\,dA$, $dA = r\,dr\,d\theta$



$3r \le z \le 8-r$

$\therefore$ Mass $m = \iiint\limits_E dm$

Now, $dm = \delta(x,y,z) \cdot dV$

$$= \left(2 + \sqrt{x^2+y^2}\right) dV = (2+r)\,dz\,dA$$

$$\therefore \quad m = \iiint\limits_E (2+r)\,dz\,dA = \iint\limits_{Base} \left\{ \int_{z=3r}^{z=8-r} dz \right\} (2+r)\,dA$$

$$= \iint\limits_{Base} (8-r-3r)(2+r)\,dA$$

$$= \int_0^{2\pi}\int_0^2 (8-4r)(2+r)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 (16r - 4r^3)\,dr$$

$$= 2\pi \left[8r^2 - r^4\right]_0^2$$

$$= 2\pi\left[8(2)^2 - 2^4\right]$$

$$= 2\pi[32 - 16]$$

$$= 2\pi \cdot 16$$

$$= 32\pi$$

<u>Base</u>:

$z = 8 - \sqrt{x^2+y^2}$

$= 8 - r \quad \text{---(1)}$

$z = 3\sqrt{x^2+y^2}$

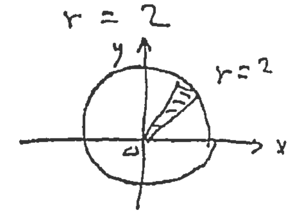
$= 3r \quad \text{---(2)}$

Eliminate "z" between (1), (2):

$8 - r = 3r$

$8 = 4r$

$r = 2$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

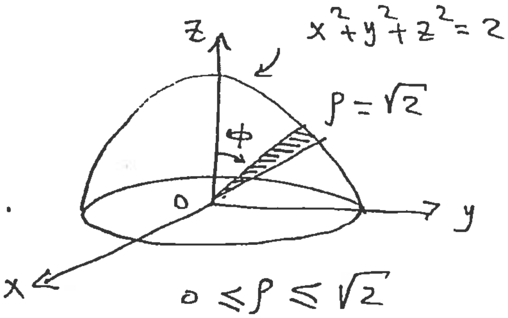
10. Use spherical coordinates to find the mass of the hemispherical solid $x^2+y^2+z^2 \leq 2$, $z \geq 0$ with density $\delta(x,y,z) = z^3\sqrt{1+(x^2+y^2+z^2)^3}$.

Solution: $dm = \delta(x,y,z)\,dV = z^3\sqrt{1+(x^2+y^2+z^2)^3}\,dV$

In spherical coordinates: $x^2+y^2+z^2 = \rho^2$, $z = \rho\cos(\phi)$, and $dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$

$$\therefore dm = (\rho\cos(\phi))^3\sqrt{1+(\rho^2)^3}\cdot\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$= \rho^5\sqrt{1+\rho^6}\,\cos^3(\phi)\sin(\phi)\,d\rho\,d\phi\,d\theta$$



$\therefore$ mass $m = \iiint_R dm$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} \rho^5\sqrt{1+\rho^6}\,\cos^3(\phi)\sin(\phi)\cdot d\rho\,d\phi\,d\theta$$

$0 \leq \rho \leq \sqrt{2}$
$0 \leq \phi \leq \frac{\pi}{2}$
$0 \leq \theta \leq 2\pi$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi)\,d\phi \cdot \int_0^{\sqrt{2}} \rho^5\sqrt{1+\rho^6}\,d\rho$$

$$= I \cdot J \cdot K$$

Clearly $I = \int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} = 2\pi$,

$J = \int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi)\,d\phi$ .. let $u = \cos(\phi)$
$\therefore du = -\sin(\phi)\,d\phi$

$$= -\int u^3\,du = -\frac{1}{4}u^4$$

$$= -\frac{1}{4}\cos^4(\phi)\Big|_0^{\frac{\pi}{2}} = -\frac{1}{4}[0-1] = \frac{1}{4},$$

and $K = \int_0^{\sqrt{2}} \rho^5\sqrt{1+\rho^6}\,d\rho$ .. let $t = 1+\rho^6$
$\therefore dt = 6\rho^5\,d\rho$

$$K = \frac{1}{6}\int \sqrt{t}\, dt = \frac{1}{6}\cdot\frac{2}{3}\, t^{\frac{3}{2}}$$

$$= \frac{1}{9}\left(1+\rho^6\right)^{\frac{3}{2}}\Bigg|_{\rho=0}^{\rho=\sqrt{2}}$$

$$= \frac{1}{9}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{1}{9}\left[9\sqrt{9}-1\right]$$

$$= \frac{1}{9}\left[27-1\right] = \frac{26}{9}$$

$$\therefore \text{ Mass } = I\cdot J\cdot K = 2\pi\cdot\frac{1}{4}\cdot\frac{26}{9} = \frac{13}{9}\pi$$

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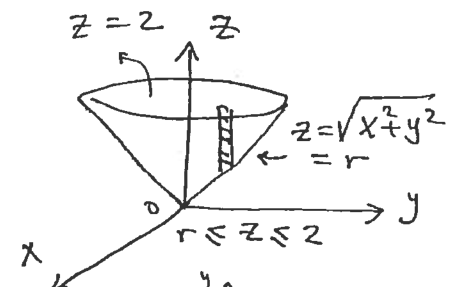
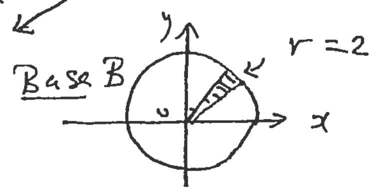
11. Use cylindrical coordinates to find the coordinates of the centroid of the region enclosed by $z=\sqrt{x^2+y^2}$, $z=2$.





$0 \le r \le 2$

$0 \le \theta \le 2\pi$

Solution: For centroid, $\delta = 1$, hence

$$dm = \delta\, dV = dV$$

In cylindrical coordinates,

$$x^2+y^2 = r^2,\quad dV = dA\,dz = r\,dr\,d\theta\,dz$$

$$\therefore z = \sqrt{x^2+y^2} \Rightarrow z = r$$

Need: z-limits: $r \le z \le 2$

Base. $z = r$, $z = 2$ .. Equate:

$$\therefore r = 2 \text{ (Circle)}$$

$$\therefore \text{ mass } m = \iiint_R dm = \iiint_R dV = \text{Volume of Cone of radius } r=2, \text{ height } h = 2$$

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (2)^2\cdot 2 = \frac{8\pi}{3},$$

$$M_{z=0} = \iiint_R z\, dm = \iiint_R z\, dV = \iint_B \left\{\int_r^2 z\, dz\right\} dA = \frac{1}{2}\iint_B (2^2 - r^2)\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (4-r^2)\, r\,dr\,d\theta = 4\pi$$

$$\therefore \bar{z} = \frac{M_{z=0}}{m} = \frac{4\pi}{8\pi/3} = \frac{3}{2}.$$ obviously $\bar{x} = \bar{y} = 0$, hence Centroid at $\left(0, 0, \frac{3}{2}\right)$.

12. Use Cylindrical Coordinates to find the Coordinates of the centroid of the solid enclosed by the Cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2-4z=0$.

Solution: let us 1st. sketch region E occupied by the solid.

Note 1st. that: Completing the square in z-terms we get: $x^2+y^2+(z-2)^2=4$

Hence the sphere is centred at the point $(0,0,2)$ and has radius 2-units.

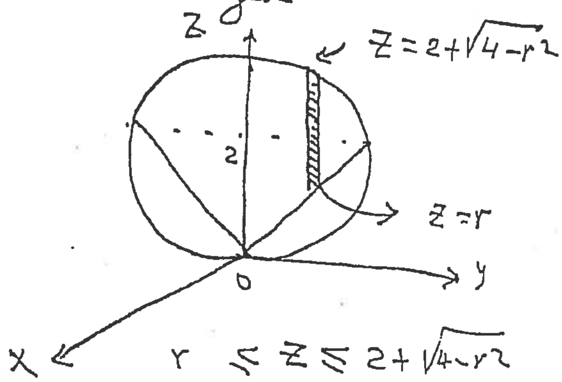
The Equation $z=\sqrt{x^2+y^2}$ is the top part of the circular Cone with vertex at the origin

In cylindrical Coordinates:

$x = r\cos(\theta)$, $y = r\sin(\theta)$,

$z=z$, $x^2+y^2=r^2$, and

$dV = dz\,dA$, $dA = r\,dr\,d\theta$



$r \leq z \leq 2+\sqrt{4-r^2}$

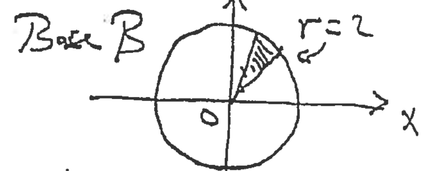
$(x,y) \in$ Base B

Now, In cylindrical coordinates

$z=\sqrt{x^2+y^2} \Rightarrow z=r$

$x^2+y^2+(z-2)^2=4 \Rightarrow$

$r^2+(z-2)^2=4 \Rightarrow (z-2)^2 = 4-r^2$



$0 \leq r \leq 2$
$0 \leq \theta \leq 2\pi$

$\therefore z-2=+\sqrt{4-r^2}$ or $z=2+\sqrt{4-r^2}$

The Base: let us first find out where surfaces

$$z = r$$
$$z = 2+\sqrt{4-r^2}$$

intersect?

Equating (1), (2): $r = 2+\sqrt{4-r^2}$ .. by inspection $r=2$.

$\therefore$ The base is the circular region centred at origin and is of radius 2

Recall $dm = \delta(x,y,z)\,dV$

For Centroid $\delta(x,y,z)$ = a constant say 1

$$\therefore \boxed{dm = dV}$$

$$\therefore \text{mass } m = \iiint_E dm = \iiint_E dV$$

$$= \iint_B \left\{ \int_{z=r}^{z=2+\sqrt{4-r^2}} dz \right\} dA$$

$$= \iint_B z \Big|_{z=r}^{z=2+\sqrt{4-r^2}} dA$$

$$= \iint_B \left\{ \left(2+\sqrt{4-r^2}\right) - r \right\} dA$$

$$= \int_0^{2\pi} \int_0^2 \left(2+\sqrt{4-r^2}-r\right) . r\,dr\,d\theta$$

$$m = \int_0^{2\pi} d\theta \cdot \int_0^2 (2 + \sqrt{4-r^2} - r)\, r\, dr$$

$$= 2\pi \int_0^2 (2r + r\sqrt{4-r^2} - r^2)\, dr$$

<u>Note</u>: For $\int r\sqrt{4-r^2}\, dr$ .. let $4 - r^2 = u$ .. you get

$$\int r\sqrt{4-r^2}\, dr = -\frac{1}{3}(4-r^2)^{\frac{3}{2}}$$

$$\therefore \quad m = 2\pi \left[r^2 - \frac{1}{3}(4-r^2)^{\frac{3}{2}} - \frac{1}{3}r^3 \right]_0^2$$

$$= 2\pi \left[\left(4 - 0 - \frac{8}{3}\right) - \left(0 - \frac{8}{3} - 0\right) \right] \qquad \underline{\text{Note}}\ 4^{\frac{3}{2}} = 8.$$

$$= 2\pi \left[4 \cancel{- \frac{8}{3} + \frac{8}{3}} \right] = 8\pi$$

Next, $M_{z=0} = \iiint_E z\, dm = \iiint_E z\, dV$

Recall E is described by: $r \le z \le 2 + \sqrt{4-r^2}$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$. We get

$$M_{z=0} = \int_0^{2\pi}\int_0^2 \left\{ \int_r^{2+\sqrt{4-r^2}} z\, dz \right\} r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{2} z^2 \Big|_{z=r}^{z=2+\sqrt{4-r^2}} \cdot r\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 \left\{ (2+\sqrt{4-r^2})^2 - r^2 \right\} r\, dr\, d\theta \leftarrow \text{Expand}$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta \cdot \int_0^2 (8 - 2r^2 + 4\sqrt{4-r^2})\, r\, dr$$

$$M_{z=0} = \frac{1}{2}\cdot 2\pi \int_0^2 \left(8r - 2r^3 + 4r\sqrt{4-r^2}\right) dr$$

$\hookrightarrow$ use substitution $u = 4 - r^2$

$$= \pi\left[4r^2 - \frac{1}{2}r^4 - \frac{4}{3}(4-r^2)^{\frac{3}{2}}\right]_0^2$$

$$= \pi\left[(16 - 8 - 0) - \left(0 - 0 - \frac{4}{3}\cdot 4^{\frac{3}{2}}\right)\right]$$

$$= \pi\left[8 + \frac{32}{3}\right] = 8\pi\left[1 + \frac{4}{3}\right] = 8\pi\cdot\frac{7}{3} = \frac{56\pi}{3}$$

$$\therefore \bar{z} = \frac{M_{z=0}}{m} = \frac{\frac{56\pi}{3}}{8\pi} = \frac{7}{3}$$

clearly: From symmetry: centroid lies on z-axis

$\therefore \bar{x} = 0, \ \bar{y} = 0$

$\therefore$ Centroid is at the point $\left(0, 0, \frac{7}{3}\right)$

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13. Re do problem #(12) using spherical coordinates.

<u>Hint</u>: In spherical coordinates:

$x = \rho\cos(\theta)\sin(\phi)$; $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$,

$x^2 + y^2 + z^2 = \rho^2$, $x^2 + y^2 = \rho^2\sin^2(\phi)$, $dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$

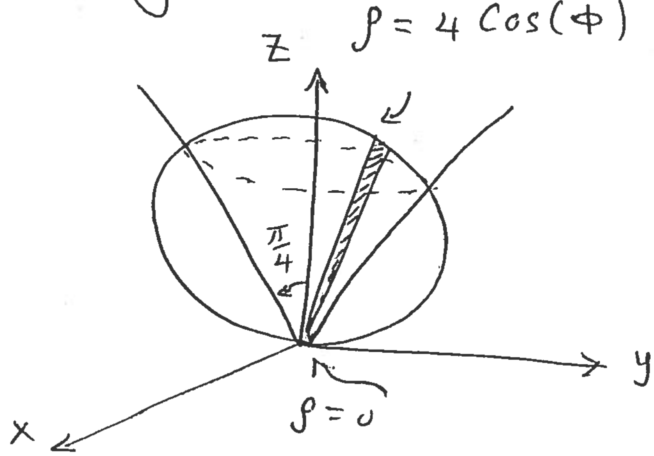
$\therefore$ surface $\underbrace{x^2 + y^2 + z^2}_{} - 4z = 0 \Rightarrow \rho^2 - 4\rho\cos(\phi) = 0$

$\Rightarrow \rho = 4\cos(\phi)$, $\rho = 0$

and $z = \sqrt{x^2 + y^2} \Rightarrow \rho\cos(\phi) = \rho\sin(\phi) \Rightarrow \tan(\phi) = 1 \Rightarrow \phi = \frac{\pi}{4}$
(for $\rho \neq 0$)

The region E is thus given by

$$0 \leq \rho \leq 4\cos(\phi)$$
$$0 \leq \phi \leq \frac{\pi}{4}$$
$$0 \leq \theta \leq 2\pi$$



as shown.

The problem is very easy to finish from this point.

<u>Answer</u>: $\bar{x} = 0$, $\bar{y} = 0$ (from symmetry), $\bar{z} = 7/3$

14. For students to do at home.

<u>Answer</u>: $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3)$

15. For students to do at home.

<u>Answer</u>: $M_{z=0} = \frac{\pi}{7}$

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