MATH 277 FINAL EXAM REVIEW SHEET

QUESTIONS

WINTER 2016

MATH 277 Final Exam Review Sheet

1. In each case , find the arc length of the given curve :

   (a) $\vec{r}(t) = (3t\ ,\ 2t^{3/2},\ 4)\ ,\quad 0 \le t \le 8.$

   (b) $\vec{r}(t) = (2\sin^2(t)\ ,\ \cos^3(t)\ ,\ \sin^3(t)),\quad 0 \le t \le \frac{\pi}{2}.$

   (c) $\vec{r}(t) = 2\,e^t\,\vec{i} + e^{-t}\,\vec{j} + 2t\,\vec{k}\quad ,\quad -1 \le t \le 1.$

   (d) $\vec{r}(t) = \frac{1}{2}\sin(t^2)\,\vec{i} + \frac{1}{2}\cos(t^2)\,\vec{j} + \frac{1}{3}(2t+1)^{3/2}\vec{k}\quad ,\quad 0 \le t \le 2$

2. In each case , find a parametrization of the curve of intersection of the given surfaces :

   (a) $4x^2 + y^2 = 16\ ,\quad 2x + 3y + 2z = 1.$

   (b) $x^2 + 2y + z = 3\ ,\quad xz + y = -2.$

   (c) $z = x^2 + y^2\quad ,\quad 2x - 4y - z + 4 = 0.$

   (d) $xy + xz = 6\ ,\quad x = -3.$

   (e) $x^2 - y^2 - z = 0\ ,\quad 2y^2 + z = 1.$

3. A rocket has mass $52,000$ kilogram (kg), which includes $39,000\ kg$ of fuel mixture is fired vertically upward in a vacuum (that is Free Space where gravitational field is negligible) During the burning process the exhaust gases are ejected at a constant rate $1300\ kg/s$ and at constant velocity with magnitude $500\ metre/s$ relative to the rocket.
   If the rocket was initially at rest , find its speed after 15 , 20 , 30 and 35 seconds.

4. For each of the following curves find the unit Tangent $\vec{\mathbf{T}}$, the Principal unit Normal $\vec{\mathbf{N}}$, the unit Binormal $\vec{\mathbf{B}}$, the curvature κ , the radius of curvature ρ and the Torsion τ at the indicated value :

   (a) $\vec{r}(t) = 3\sin(t)\,\vec{i} + 3\cos(t)\,\vec{j} + 4t\,\vec{k}\quad ;\quad t = 0$

   (b) $\vec{r}(t) = \sin(t)\,\vec{i} + \sqrt{2}\cos(t)\,\vec{j} + \sin(t)\,\vec{k}\quad ;\quad t = \frac{\pi}{4}$

   (c) $\vec{r}(t) = \cosh(t)\,\vec{i} - \sinh(t)\,\vec{j} + t\,\vec{k}\quad ;\quad t = 0$

5. In each case the position $\vec{r}(t)$ of a moving object at time t is given. Find the **Tangential** and **Normal** components of the acceleration at the indicated time :

(a) $\vec{r}(t) = t^2\,\vec{i} + t\,\vec{j} + \frac{1}{2}t^2\,\vec{k}$; $t = 4$

(b) $\vec{r}(t) = \ln(t^2+1)\,\vec{i} + (t - 2\tan^{-1}(t))\,\vec{j}$; $t = 2$

(c) $\vec{r}(t) = t\cos(t)\,\vec{i} + t\sin(t)\,\vec{j} + t^2\,\vec{k}$; $t = 0$

6. In each case , find the **Domain** of the given function and sketch :

(a) $f(x,y) = \dfrac{3-x}{x+y-5}$ (b) $f(x,y) = \sqrt{4x^2+9y^2-36}$

(c) $f(x,y) = \sqrt{1+x^2+y^2}$ (d) $f(x,y) = \sqrt{\ln(5-x^2-y^2}$

(e) $f(x,y) = \ln\sqrt{x^2+y^2-4}$ (f) $f(x,y) = \ln|x^2+y^2-4|$

7. In each case , draw level curves of $f(x,y)$ for the indicated values of c :

(a) $f(x,y) = x\,e^{-y}$, $c = 0\,,1\,,-1$

(b) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2+1}$, $c = 0\,,\frac{1}{2}\,,-\frac{1}{2}$

(c) $f(x,y) = \tan^{-1}(x+y)$, $c = 0\,,\frac{\pi}{4}\,,-\frac{\pi}{6}$

8. Identify each of the following surfaces:

(i) $z = 1+3\sqrt{x^2+y^2}$ (ii) $x = 2-y^2-z^2$ (iii) $2-x^2-3y^2-2z^2 = 0$

(iv) $\dfrac{x^2}{4} - \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1$ (v) $x = z^2$ (vi) $3x-2y+1 = 0$

(vii) $x^2+y^2+z^2-2x = 0$ (viii) $x^2+y^2-z^2-4z = 3$

9. (a) If $z = \ln(xy)^{\sin(xy)}$, $x > 0\,, y > 0$, find $\dfrac{\partial z}{\partial y}$. Hint : First , simplify Logarithm.

(b) Let $f(x,y) = y^{\tan(x)} + \cosh(x^2)$, find $f_{yx}(x,y)$.

10. In each case , find an equations of the tangent plane and the normal line to the given surface at the specified point P on the surface :

(a) $z = \sqrt{x^2+y^2}$, $P(3,-4,5)$.

(b) $xy + z^3 + e^{x-y+z} = 4$, $P(1,2,1)$.

11. In case , use the chain rule to find the specified derivatives computed at the indicated values :

(a) $\frac{dz}{dt}$ at $t = \frac{\pi}{6}$, if $z = \cot(3x + \frac{1}{12}y)$, where $x = \frac{1}{\pi}t^2$, and $y = \frac{\pi^2}{6t}$.

(b) $\frac{\partial z}{\partial v}$ at $u = 0,\ v = 0$, if $z = \ln(x^2 + 3xy)^{-4}$, where $x = \cosh(u)$, and $y = \sinh(v)$.

(c) $\frac{\partial w}{\partial s}$, if $w = f(t^2 - 3s, t^{-1}s^3, t + 3s)$, for some differentiable function $f(x, y, z)$.

Hint : Let $x = t^2 - 3s$, $y = t^{-1}s^3$, and $z = t + 3s$.

(d) $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$ $at\ (r, \theta) = (1, \frac{\pi}{6})$ if $z = \sqrt{x^2 - y^2}$, where $x = r\cos(\theta)$, and $y = r\sin(\theta)$.

(e) $\frac{\partial z}{\partial y}$, at $(x, y) = (1, 0)$ if $z = f(u, v)$, where $u = \ln\sqrt{x^2 + y^2}$, and $v = x + \arctan(\frac{y}{x})$,

given that $f_u(1,0) = 8$, $f_v(1,0) = -9$, $f_u(0,1) = 5$, $f_v(0,1) = -4$, and $f(0,0) = 17$.

(f) $\frac{\partial w}{\partial u}$, and $\frac{\partial w}{\partial v}$ at $(u, v) = (-2, 0)$ if $w = \ln(x^2 + y^2 + z^2)$, where $x = ue^v \sin(v)$,

$y = ue^v \cos(v)$, and $z = ue^v$.

12. (a) Find an equation of the plane tangent to the ellipsoid $4x^2 + 3y^2 + z^2 = 25$ at the point $P(1, 2, -3)$.

(b) Find an equation of the plane tangent to the paraboloid $2x + 3y^2 + 2z^2 = 31$ at the point $P(-2, 1, 4)$.

(c) Find a **unit vector** normal (orthogonal) to the surface $\sin(xyz - 6) + 2x - x^2 = 0$ at the point $Q\ (1, 2, 3)$ on the surface.

13. In each case , find the **Differential** of given function :

(a) $f(x, y) = e^{3x}\cos(2y) + 2x - y + 1$ (b) $g(x, y) = \sin^{-1}(\frac{y}{x})$, $x > 0$.

(c) $F(x, y) = e^{x+2y+3z}$ (d) $G(x, y) = \ln(x^2 + 2y - z)$

14. The Pressure **P** , Volume **V** , and Temperature **T** (in $^\circ$**K**) of a confined gas are related by the ideal gas law $PV = kT$, where k is a constant. If $P = 0.5\ lb/in^2$ when $v = 64\ in^3$ and $T = 360\ ^\circ K$, determine by approximately what percentage P change if V and T change to $68\ in^3$ and $351\ ^\circ K$ respectively.

15. Refer to problem (14) above. Determine by approximately what percentage the volume change if the Temperature is decreased by 0.8% and the pressure is increased by 0.5% (due to errors in their measurements).

16. The flow of blood in an arteriole is given by $F = \dfrac{\pi P R^4}{8 v l}$, where l is the length of the arteriole , R is the radius, P is the pressure difference between the two ends , and v is the viscosity of the blood. Suppose that v and l are constants. Use differentials to determine by approximately what percentage the flow change if the radius is decreased by 2% and the pressure is increased by 3%.

17. Find the parametric representation of the (space) curve of intersection of the surfaces $4x^3 - 5y^3 - 3z + 10 = 0$, and $y^3 + x^3 = 2$ using $t = \frac{1}{3}z$ as a parameter.

18. A rocket moves forward in a straight line by expelling particles of a fuel mixture backward (that is in the opposite direction of motion).Assume the exhaust gases are ejected at a constant rate $1000\ kg/s$ and at constant velocity with magnitude $400\ metre/s$ relative to the rocket. Let M be the total initial mass of rocket and assume it starts motion from rest.

(a) What percentage of the total initial mass M would the rocket have to burn as fuel in order to accelerate to the speed of $800\ metre/s$?

(b) What is the speed of rocket when only 40% of its initial mass remains?

(c) What is the speed of rocket when 40% of its initial mass is ejected during the burn?

You may assume that there are no external forces acting on the rocket as it travels in deep space.

19. (i) Let $z = \ln(x^3 + 2y)$, where $x = x(r,s)$, and $y = y(r,s)$. Find $\dfrac{\partial z}{\partial s}$ at $r = 1$, $s = 3$ given that $x(1,3) = 0$, $y(1,3) = \frac{1}{2}$, $\dfrac{\partial x}{\partial s}(1,3) = -1$, and $\dfrac{\partial y}{\partial s}(1,3) = 2$.

(ii) Let $z = g(u,v)$, where $u = x^2 - y^2$, and $v = 2xy$. Determine $\dfrac{\partial z}{\partial x}$ at $x = 2$, $y = -1$ given that $\dfrac{\partial g}{\partial u}(3,-4) = 7$, and $\dfrac{\partial g}{\partial v}(3,-4) = -5$.

(iii) Let $w(x,y,z) = x^2 + 2xyz$, where $x(t) = e^t$, $y(t) = \tan(3t) + 1$, and $z(t) = \cos^{-1}(t)$. Find $\dfrac{dw}{dt}$ at $t = 0$.

(iv) Let $z = x^4 + 2xy$, where $x(t) = 1 - \sin(2t)$, and $y(t) = t\ln(1+t)$. Find $\dfrac{dz}{dt}$ at $t = 0$.

20. In each case , find the directional derivative of the function f at the given point P in the direction specified :

(a) $f(x,y) = \sin(x+2y)$, $P(0,\frac{\pi}{2})$, $\vec{u}=(-\frac{3}{5},\frac{4}{5})$.

(b) $f(x,y,z) = e^{x^2+y-2z}$, $P(1,1,1)$, $\vec{v}=(0,-1,1)$.

(c) $f(x,y,z) = xy+2xz+3yz-2x-y+1$, $P(1,2,-3)$, in the direction from the point P towards the point $Q(0,0,-1)$.

21. Let $f(x,y,z) = \ln\sqrt{x^2+y^2+z^2)}$, and $P(1,-2,2\sqrt{5})$.

(i) Find the unit vector $\vec{u}$ for which $D_u f(P)$ is a maximum and give this maximum value.

(ii) Find the unit vector $\vec{v}$ for which $D_v f(P)$ is a minimum and give this minimum value .

22. (a) Assume that the relation $3\,e^{z+2y+1}+\sin(3xyz) = 2$ defines z as a differentiable function of x, y on some domain containing the point $(x,y) = (\frac{\pi}{6},-1)$. Find $\frac{\partial z}{\partial y}$ at $(x,y,z) = (\frac{\pi}{6},-1,1)$.

(b) Assume that the relation $x^2+3yz-\frac{2}{\ln(x+z)} = 5$ defines x implicitly as a differentiable function of y, z in some domain. Find $\frac{\partial x}{\partial y}$.

23. (i) The relation $x^5+2xy^3+xyz-z^4 = -15$ implicitly defines y as a differentiable function of x ,and z. Find $\frac{\partial y}{\partial z}$ at $(x,z) = (1,2)$. Hint : First , substitute $x = 1$, and $z = 2$ into the equation of the relation to find the $y-coordinate$.

(ii) Given that $x = x(y,z)$ is implicitly defined by $y^2+y\sqrt{z} = 2-\sin(xz^2)+\frac{4}{z}$

Compute $\frac{\partial x}{\partial y}$ at the point where $(x,y,z) = (0,1,4)$.

24. The equations $u = x^2+y^2$, $v = x^2-2xy^2$ define x , y implicitly a s functions of u , and v for values of (x,y) near $(1,2)$ and values of (u,v) near $(5,-7)$.

(a) Find $\frac{\partial x}{\partial u}$, and $\frac{\partial y}{\partial u}$ at $(u,v) = (5,-7)$.

(b) if $z = \ln(y^2-x^2)$, find $\frac{\partial z}{\partial u}$ at $(u,v) = (5,-7)$. Hint : Use the chain rule!

25. Show that the equations : $\begin{cases} u\,e^{v} + xw - \cos(y) = 2 \\ x\cos(v) + u^2 y - vw^2 = 1 \end{cases}$

can be solved for x, and y as functions of u, v, and w near the point P where $(u,v,w\,;\,x,y) = (2,0,1;1,0)$, and find $\left(\frac{\partial x}{\partial w}\right)_{u,v}$, and $\left(\frac{\partial y}{\partial v}\right)_{u,w}$ at $(u,v,w) = (2,0,1)$.

26. Use double integrals to find the volume of the solid which lies vertically above the planar region $0 \le y \le 1 - x^2$, $0 \le x \le 1$ below the plane $z = 1 - x$.

27. Find the volume enclosed by the surfaces $z = 13 - x^2 - y^2$, and $z = 4\sqrt{x^2 + y^2} + 1$.

28. Find the volume enclosed by the surfaces $z = \sqrt{x^2 + y^2 + 1}$, and $z = \dfrac{6}{\sqrt{2x^2 + 2y^2 + 3}}$.

29. Find the volume of the solid enclosed by the surfaces $z = x^2 + y^2 - 6$, $z = 4 + 3\sqrt{x^2 + y^2}$.

30. Evaluate $\int_0^1 \int_{\sqrt{x}}^1 3\ln(1 + y^3)\, dy\, dx$ by first **reversing** the order of integrations.

31. Let $\mathbf{J} = \iint_R f(x,y)\, dA$, where R is the planar region enclosed by $y = \sin(x)$, $y = \frac{1}{2}$, $x = 0$, and $x = \frac{\pi}{6}$.

(a) Express the double integral $\mathbf{J}$ as an iterated integral in which the y – integration is performed first.

(b) Express the double integral $\mathbf{J}$ as an iterated integral in which the x – integration is performed first.

32. Evaluate $\iint_R 4x\, dA$, where R is the planar region given by $0 \le y \le \sin(2x)$, $0 \le x \le \frac{\pi}{4}$.

33. Evaluate $\iint_R 4y\, dA$, where R is the region in the plane described by $0 \le y \le \sin(2x)$, $0 \le x \le \frac{\pi}{4}$.

34. Find the Cartesian equation of each of the following surfaces whose equation is given in **Spherical Coordinates** (ρ,ϕ,θ) :

(i) $\rho\cos(\phi) = 4$. Hint : In spherical coordinates : $z = \rho\cos(\phi)$.

(ii) $\rho\cos(\phi) = 2-\rho^2\sin^2(\phi)$. Hint : In spherical coordinates : $z = \rho\cos(\phi)$ and $x^2+y^2 = \rho^2\sin^2(\phi)$

(iii) $\rho = 4\cos(\phi)$. Hint : Multiplying both sides by ρ , we get :

$\rho^2 = 4\rho\cos(\phi)$. Now use the facts : $x^2+y^2+z^2 = \rho^2$, and $z = \rho\cos(\phi)$.

(iv) $\phi = \frac{\pi}{4.}$. Hint : Recall $x^2+y^2 = \rho^2\sin^2(\phi)$, and $z = \rho\cos(\phi)$.

Now show that if $\rho \neq 0$, $\dfrac{\sqrt{x^2+y^2}}{z} = \tan(\phi) = \tan(\frac{\pi}{4})$.

35. Find the equation of each of the following surfaces in **Cylindrical Coordinates** (r,θ,z).

(i) $z = \sqrt{16-x^2-y^2}$, $x \geq 0$, $y \geq 0$.

(ii) $z = \sqrt{5(x^2+y^2)}$

(iii) $x^2+y^2 = 1$, $y \geq 0$.

36. Use Cylindrical coordinates to find the mass of the solid having the shape of the region enclosed by the paraboloid $z = 2(x^2+y^2)$ and the hemisphere $z = \sqrt{5-x^2-y^2}$ if the density function is given by $\delta(x,y,z) = 12z$.

37. Use double or triple integrals to find the volume enclosed by the paraboloid $z = 13-x^2-y^2$, and the cone $z = 4\sqrt{x^2+y^2}+1$.

38. Use spherical coordinates to calculate the moment $\mathbf{M}_{z=0}$ of the solid occupying the region $\mathbf{E}$ described by $0 \leq z \leq \sqrt{1-x^2-y^2}$ if the density function is given by $\delta(x,y,z) = (x^2+y^2+z^2)^{3/2}$.

39. Use spherical coordinates to find the x, y , and z coordinates of the centroid of the solid enclosed by the cone $\sqrt{3}\,z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$.

Hint : In spherical coordinates : $x^2+y^2+z^2 = 1 \Rightarrow \rho = 1$, and

$\sqrt{3}\,z = \sqrt{x^2+y^2} \Rightarrow \sqrt{3}\,\rho\cos(\phi) = \rho\sin(\phi)$, hence $\tan(\phi) = \sqrt{3}$, that is $\phi = \frac{\pi}{3}$.

Therefore $0 \leq \rho \leq 1$, $0 \leq \phi \leq \frac{\pi}{3}$, $0 \leq \theta \leq 2\pi$.

40. (a) Express the iterated integral $\mathbf{J} = \int_0^4 \int_0^{4-y} \int_0^{\sqrt{y}} g(x,y,z)\; dx\; dz\; dy$ as an equivalent integral in which the y – integration is performed first , the z – integration second and the x – integration last.

(b) Express the iterated integral $\mathbf{I} = \int_0^1 \int_0^{\sqrt{1-y}} \int_0^{2x} f(x,y,z)\; dz\; dx\; dy$ as an equivalent integral in which the y – integration is performed first , the x – integration second and the z – integration last.

(c). Express the iterated integral $\int_0^1 \int_z^1 \int_0^z g(x,y,z)\; dx\; dy\; dz$ as an equivalent integral in which the z- integration is performed first , the y- integration second , and the x - integration last.

(d). Express the iterated integral $\int_0^1 \int_z^1 \int_0^y g(x,y,z)\; dx\; dy\; dz$ as an equivalent integral in which the z- integration is performed first , the y- integration second , and the x- integration last.

41. Determine $\iint_R xy^2\; dA$, where R is the planar region with mass equal to 3 ,centre of mass at $(\bar{x},\bar{y}) = (1,4)$, and density function $\delta(x,y) = xy.$

42. The centroid of a planar region $\mathbf{D}$ occupied by a thin uniform plate is at the point $(3,-5)$.

Determine the area of the region $\mathbf{D}$ given that $\iint_{\mathbf{D}} (3x - 4y + 2)\; dA = 124.$

Note : You may assume the uniform density $\delta(x,y) = 1.$

43. Evaluate $\int_0^3 \int_x^3 \sqrt{9 - y^2}\; dy\; dx$

Hint : Sketch the triangular region enclosed by the straight lines $x = 0$, $y = x$, and $y = 3$ and hence Reverse order (that is , treat region as an $x - simple$ instead!)

44. Find the coordinates of the centre of mass of the planar region R enclosed by $y = 2x^2 + 4x$, $y = 0$ from $x = 0$ to $x = 1$ if the density function $\delta(x,y) = x.$

45. Use double integrals to find the x and y – coordinates of the centroid of the planar region R enclosed by $y = \sqrt{x}$, $x = 0$, and $y = 1.$

46. Use double integrals to find the x and y – coordinates of the centroid of the planar region R enclosed by $y = \sqrt{36 - x^2}$, $y = x$, and $y = -x$.

Hint : In polar coordinates : The equation of the line $y = x$ is $\theta = \frac{\pi}{4}$, and the equation of the line $y = -x$ is $\theta = \frac{3\pi}{4}$.

47. Use **Cylindrical Coordinates** to find the the moment about the plane $y = 0$ of the solid in the first octant $(x, y, z \geq 0)$ enclosed by the cones $z = \sqrt{x^2 + y^2}$ and $z = 2 - \sqrt{x^2 + y^2}$ if the density function is given by $\delta(x, y, z) = 20xy$.

48. Evaluate $\iiint_E 9z^2 \, dV$, where E is the region in $\mathbb{R}^3$ given by $0 \leq x \leq \sqrt{1 - y}$, $0 \leq y \leq 1$, $0 \leq z \leq 2x$.

49. Use **Cylindrical Coordinates** to evaluate $\iiint_E \left(2 + \sqrt{x^2 + y^2}\right) dV$ where E is the region enclosed by the cones $z = 8 - \sqrt{x^2 + y^2}$ and $z = 3\sqrt{x^2 + y^2}$.

50. Use **Triple Integrals** to find the the moment about the plane $y = 0$ of the solid which occupies the region E in $\mathbb{R}^3$ given by $0 \leq x \leq 2$, $0 \leq y \leq \sqrt{4 - x^2}$, $0 \leq z \leq 3$. if the density function is given by $\delta(x, y, z) = 2x$.

51. Use **Spherical Coordinates** to find the moment about the plane $z = 0$ of the solid $\mathbf{S}$ occupying the region above the xy – plane and below the sphere $x^2 + y^2 + z^2 = 1$ if the density function is given by $\delta(x, y, z) = (x^2 + y^2)$.

52. Use **Spherical Coordinates** to find the mass of the solid $\mathbf{S}$ occupying the region above the xy – plane and below the sphere $x^2 + y^2 + z^2 = 1$ if the density function is given by

$$\delta(x, y, z) = \frac{\sqrt{x^2 + y^2 + z^2}}{1 + (x^2 + y^2 + z^2)^2}.$$

53. Let E be the region in $\mathbb{R}^3$ occupied by a uniform solid of volume 2 units. If the centroid of the region E is at the point $(\bar{x}, \bar{y}, \bar{z}) = (8, 11, 6)$, find the value of $\iiint_E (2x + z) \, dV$.

54. Let E be the region in $\mathbb{R}^3$ occupied by a solid of mass $\frac{1}{3}$ unit and moment about the $xz-plane$ equal to -5 units. If the centre of mass of the region E is at the point $(\bar{x}, \bar{y}, \bar{z})$, find $\bar{y}$.

55. Find the **volume** of the region below the surface $z = 3y^2,$ and above the triangular region in the $xy-$plane bounded by the straight lines $x = 0$, $y = 0$, and $x + 2y = 2.$

56. . Use double integrals to find the **mass** and the coordinates of the centre of mass of the lamina which occupies the planar region given by $-y \le x \le y^2$, $0 \le y \le 2$ if the density function $\delta(x, y) = 3y.$

57. Use **Spherical Coordinates** to find the $x-coordinate$ of the centroid of the solid **S** occupying the region that satisfies $x \ge 0$, $y \ge 0$, $z \ge 0$, and $x^2 + y^2 + z^2 \le 4.$

58. In each case find the x and y coordinates of the critical points of the given function $f(x, y)$:

(i) $f(x, y) = x^3 - xy + y^3$

(ii) $f(x, y) = x^3 + 2xy - 2y^2 - 10x$

(iii) $f(x, y) = x^2y + xy^2 + x + y - 17$

(iv) $f(x, y) = y^3 + x^2 - 6xy + 3x + 6y - 27$

59. In each case find the critical points of the given function $f(x, y)$ and determine whether it is a local Maximum , a local Minimum or a Saddle point :

(i) $f(x, y) = x^2 - 4xy + y^3 + 4y$ (ii) $f(x, y) = x^4 - 2x^2 + y^2 - 2$

(iii) $f(x, y) = (x + y)(xy + 1) - 17$ (iv) $f(x, y) = x^3 + y^2 - 6xy + 6x + 3y - 2$

60. In each case find the Maximum and Minimum values of the given function $f(x, y)$ over the indicated region **D** :

(i) $f(x, y) = x^2 - 12x + (y - 1)^2$; **D** is the region bounded by the ellipse $4x^2 + y^2 = 36.$

(ii) $f(x, y) = 2y^2 + x^2$; **D** is the region bounded by the circle $x^2 + y^2 + 2x - 3 = 0.$

(iii) $f(x, y) = 2x^3 - 24x - 9y^2$; **D** is the region bounded by the circle $x^2 + y^2 = 25.$

(iv) $f(x, y) = x^2 + y^2 - 4x - 6y$; **D** is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 7$