MATH 277

Problem Set # 9 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. In each case , express the iterated integral as an equivalent iterated integral in

   the order specified :

   (a) $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{y^2+z^2}^1 g(x,y,z)\, dx\, dz\, dy \quad ; \quad z\ ,\ y\ ,\ x.$

   $(b)^*$ $\int_0^4 \int_0^{4-x} \int_{x+y}^4 g(x,y,z)\, dz\, dy\, dx \quad ; \quad y\ ,\ z\ ,\ x.$

   $(c)^*$ $\int_0^1 \int_z^1 \int_0^{x-z} g(x,y,z)\, dy\, dx\, dz \quad ; \quad z\ ,\ y\ ,\ x.$

   (d) $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y f(x,y,z)\, dx\, dz\, dy \quad ; \quad y\ ,\ x\ ,\ z.$

   $(e)^*$ $\int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\, dx\, dy\, dz \quad ; \quad z\ ,\ y\ ,\ x.$

2. Let $\mathbf{J} = \iiint_E f(x,y,z)\, dV$, where E is the region in $\mathbb{R}^3$ enclosed by the cone

   $z = \sqrt{x^2+y^2}$ and the plane $z = 1$. Express $\mathbf{J}$ as an iterated integral in which the

   x – integration is performed first , the y – integration second , and the z – integration Last.

3. Let R be the planar region enclosed by $y = 0$, and $y = \sqrt{1-y^2}$.

   Use polar coordinates to find $\iint_R y\, dA.$

4. Use Double integrals to find the volume of the region enclosed by the surfaces $z = \sqrt{x^2+y^2}$,

   and $z = 6 - (x^2+y^2).$

5. Use polar coordinates to find $\iint_D e^{-(x^2+y^2)}\, dA$, where D is the region described by

   $1 \le x^2 + y^2 \le 3.$

6. Find a **spherical coordinate** equation for each of the following surfaces:

   (a) The sphere $x^2 + y^2 + (z-3)^2 = 9.$ $\qquad$ (b) The cone $z = \sqrt{x^2+y^2}$.

7. Sketch the region enclosed by $\rho = 4\cos(\varphi)$, and $\rho = 9$, $z \geq 0$.

8. Use **Spherical Coordinates** to evaluate $\iiint_E \frac{6z^3}{\sqrt{1+(x^2+y^2+z^2)^3}}\, dV$, where E is the region

above the $xy-plane$ below the sphere $x^2+y^2+z^2 = 2$.

9. Use **Cylindrical Coordinates** to evaluate $\iiint_E (2+\sqrt{x^2+y^2})\, dV$, where E is the region

enclosed by the cones $z = 8-\sqrt{x^2+y^2}$, and $z = 3\sqrt{x^2+y^2}$.

10. Given $\mathbf{J} = \iiint_E g(x,y,z)\, dV$, where E be the Region described by $\sqrt{x^2+y^2} \leq z \leq \sqrt{36-x^2-y^2}+6$

(a) Express the integral $\mathbf{J}$ in Cartesian coordinates in the order : z , y , x.

(b) Express the integral $\mathbf{J}$ in Cylindrical coordinates in the order : z , r , θ.

(c) Express the integral $\mathbf{J}$ in Spherical coordinates in the order : ρ , ϕ , θ.

(d) Find the volume of the region E.

11* Find the volume enclosed by the two surfaces $z = 5-x^2-y^2$, and $z = 4$.

12* Let R be the upper semi circular region centred at $(0,0)$ and has radius 2 units.

Use polar coordinates to compute $\iint_R y^2\, dA$.

13* Use **Cylindrical Coordinates** to evaluate $\iiint_E z\, dV$ where E is the region enclosed

by the cone $z = \sqrt{x^2+y^2}$, and the plane $z = 2$.

14* Evaluate $\iiint_H \frac{\sqrt{x^2+y^2+z^2}}{1+(x^2+y^2+z^2)^2}\, dV$, where H is the hemispherical region above the

$xy-plane$ and below the sphere centred at the origin and is of unit radius.

15* Use **Cylindrical Coordinates** to to evaluate $\iiint_E z\, dV$ where E is the region enclosed

by $z = \sqrt{x^2+y^2}$, and the sphere $x^2+y^2+z^2-4z = 0$.

16* Re do problem # 15 using **Spherical Coordinates**.

MATH 277

Solutions to Problem Set # 9

1. (a)

$$I = \int_{y=0}^{y=1} \int_{z=0}^{z=\sqrt{1-y^2}} \int_{x=y^2+z^2}^{x=1} g(x,y,z)\, dx\, dz\, dy$$

$$= \iiint_E g(x,y,z)\, dV$$

Where E is the region in $\mathbb{R}^3$ given by

$$y^2 + z^2 \leq x \leq 1 \quad \ldots (1)$$

$$0 \leq z \leq \sqrt{1-y^2} \quad \ldots (2)$$

$$0 \leq y \leq 1 \quad \ldots (3)$$

We simply need to Rearrage this set of Inequalities.

The requested order is z, y, and x. Therefore

(i) Limits for "z" could be functions of <u>both</u> y, x.
(ii) Limits for "y" could be functions of <u>only</u> x.
(iii) Limits for "x" <u>must</u> be constant real numbers.

Accordingly, we begin with inequality #1 (because it contains <u>all</u> three variables!). We have

$$y^2 + z^2 \leq x \quad \Rightarrow \quad z^2 \leq x - y^2$$

$$\Rightarrow \quad \boxed{z \leq \sqrt{x - y^2}}$$

On the other hand, from inequality #2:

$$\boxed{0 \leq z}$$

Combining, we get:

$$\boxed{0 \leq z \leq \sqrt{x - y^2}} \quad --- \quad *$$

(Inner most limits)

Next, from #(1): $y^2 + z^2 \leq x$

$$\Rightarrow y^2 \leq x - z^2$$

$$\Rightarrow y^2 \leq x$$

$$\Rightarrow \boxed{y \leq \sqrt{x}}$$

On the other hand, from #(3): $\boxed{0 \leq y}$

Combining, we get:

$$\boxed{0 \leq y \leq \sqrt{x}} \quad ---- \quad **$$

(Middle limits)

Finally, from (4): $\boxed{x \leq 1}$, and

$$y^2 + z^2 \leq x \Rightarrow x \geq y^2 + z^2 \Rightarrow \boxed{x \geq 0}$$

Combining, we get

$$0 \leq x \leq 1 \quad --- \quad (***)$$

(outermost Limits)

From (*), (**), (***):

$$I = \int_0^1 \int_0^{\sqrt{x}} \int_0^{\sqrt{x-y^2}} g(x,y,z)\, dz\, dy\, dx$$

— — — — — —

<u>Another solution</u>:

write $I = \iiint_E g(x,y,z)\, dV$, where E is the region given by $y^2 + z^2 \leq x \leq 1$, $0 \leq z \leq \sqrt{1-y^2}$, $0 \leq y \leq 1$.

We need to treat region E as z-simple!

We have shown earlier: $0 \leq z \leq \sqrt{x-y^2}$

$$\therefore \quad I = \iint_B \left\{ \int_{z=0}^{z=\sqrt{x-y^2}} g(x,y,z)\, dz \right\} dA, \text{ where } B \text{ is the}$$

Base of region E in xy-plane and $dA = \underline{\underline{dy\, dx}}$

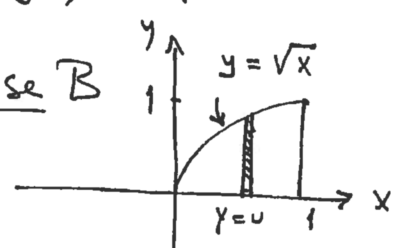
<u>To find Base</u>: let us eliminate "z" between

$$z = 0, \quad z = \sqrt{x-y^2}$$

we have $0 = \sqrt{x-y^2} \Rightarrow x - y^2 = 0 \Rightarrow x = y^2$, $0 \leq y \leq 1$

$$\therefore \quad I = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{x}} \int_{z=0}^{z=\sqrt{x-y^2}} g(x,y,z)\, dz\, dy\, dx$$

<u>Base B</u>



$0 \leq y \leq \sqrt{x}$

$0 \leq x \leq 1$

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(b)* At Home.

Answer $I = \int_0^4 \int_x^4 \int_0^{z-x} g(x,y,z)\,dy\,dz\,dx$

(c)* At Home.

Answer $I = \int_0^1 \int_0^x \int_0^{x-y} g(x,y,z)\,dz\,dy\,dx$

(d) $I = \int_{y=0}^{y=1} \int_{z=0}^{z=\sqrt{1-y^2}} \int_{x=0}^{x=y} f(x,y,z)\,dx\,dz\,dy$

$= \iiint_E f(x,y,z)\,dV$

where E is the region given by

$0 \le x \le y$ - - - (1)

$0 \le z \le \sqrt{1-y^2}$ - - - (2)

$0 \le y \le 1$ - - - (3)

The requested order is y, x, and z. Therefore

(i) Limits of "y" could be a function of both x, z.

(ii) Limits of "x" could be a function of only z.

(iii) Limits of "z" <u>must</u> be constant real Numbers.

From (1): $\boxed{x \le y}$, and from (2):

$z \le \sqrt{1-y^2} \Rightarrow z^2 \le 1-y^2$

$\Rightarrow y^2 \le 1-z^2 \Rightarrow \boxed{y \le \sqrt{1-z^2}}$

Combining, we get:

$$x \leq y \leq \sqrt{1-z^2} \quad --- \quad (*)$$

Next, from (1): $\boxed{0 \leq x}$, and from (*),

$$\boxed{x \leq \sqrt{1-z^2}}$$

Combining, we get:

$$0 \leq x \leq \sqrt{1-z^2} \qquad (**)$$

Finally from (2): $\boxed{0 \leq z}$, and

$$z \leq \sqrt{1-y^2}$$
$$\Rightarrow z \leq \sqrt{1-0^2} = 1 \quad \Rightarrow \quad \boxed{z \leq 1}$$

Combining, we have

$$0 \leq z \leq 1 \qquad (***)$$

From (*), (**), (***),

$$I = \int_0^1 \int_0^{\sqrt{1-z^2}} \int_x^{\sqrt{1-z^2}} f(x,y,z)\, dy\, dx\, dz$$

*(e) At Home.

Answer:

$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\, dz\, dy\, dx$$

— — — — — —

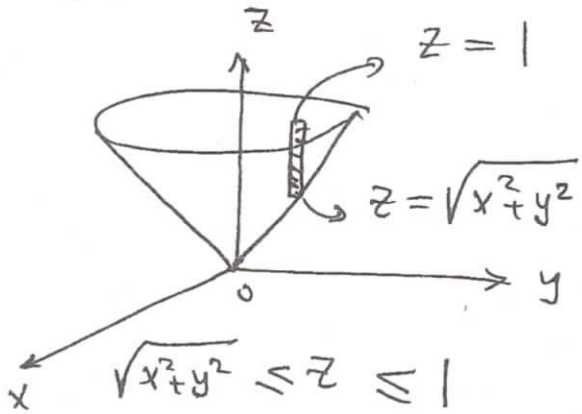
2. $I = \iiint_E f(x,y,z)\,dV$

where z is described by

$\sqrt{x^2+y^2} \le z \le 1 \quad \cdots (1)$

The requested order is:

x, y, and z.



Therefore:

(a) Limits for "x" Could be functions of <u>both</u> y, and z

(b) limits for "y" Could be functions of only z.

(c) Limits for "z" <u>must</u> be Constant real numbers

We need to treat region E as an x-Simple!

From (1):

$$\sqrt{x^2+y^2} \le z$$

$$\Rightarrow \quad 0 \le \sqrt{x^2+y^2} \le z \quad \text{(hence Can Square!)}$$

$$x^2+y^2 \le z^2$$

$$\Rightarrow x^2 \le z^2 - y^2$$

$$|x| \leq \sqrt{z^2 - y^2}$$

$$\Rightarrow \boxed{-\sqrt{z^2-y^2} \leq x \leq +\sqrt{z^2-y^2}} \quad --- (*)$$

It follows that

$$J = \iint_{B_{yz}} \left\{ \int_{x=-\sqrt{z^2-y^2}}^{x=+\sqrt{z^2-y^2}} f(x,y,z)\, dx \right\} dA, \quad dA = dy\,dz$$

where B_{yz} is the base of region in yz-plane.

<u>To find Base</u> : let us first eliminate "x"

between

$$x = \sqrt{z^2-y^2} \quad --- (2)$$

$$x = -\sqrt{z^2-y^2} \quad -- (3)$$

equating, we get:

$$\sqrt{z^2-y^2} = -\sqrt{z^2-y^2}$$

or

$$2\sqrt{z^2-y^2} = 0$$

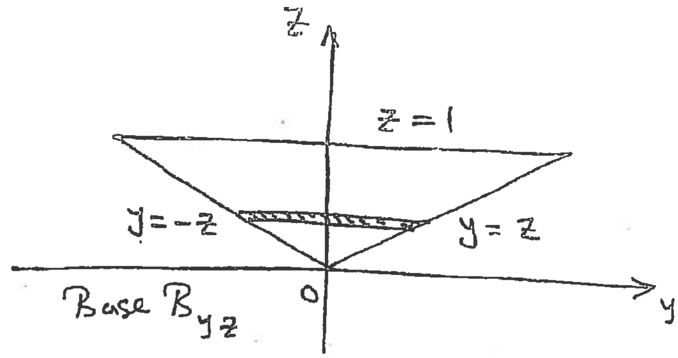
$$\Rightarrow z^2 - y^2 = 0$$

$$\Rightarrow y^2 = z^2 \quad \text{or}$$

$$y = \pm z$$

on the other hand we know: $0 \leq z \leq 1$

so B_{yz} is the Triangular region shown in figure.



$$-z \le y \le z$$
$$0 \le z \le 1$$

$$\therefore J = \int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\, dx\, dy\, dz$$

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3. Let us first sketch region R.

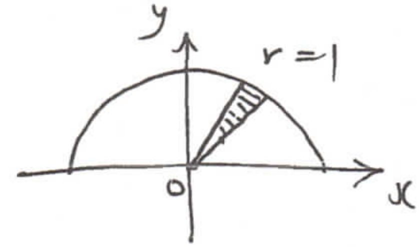
Note: $y = +\sqrt{1-x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$

$\therefore y = +\sqrt{1-x^2}$ is an Equation of the upper Semi-Circle Centred at $(0,0)$, and has radius 1 unit.

In polar Coordinates:

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2+y^2 = r^2$, and $dA = r\,dr\,d\theta$

$\therefore x^2 + y^2 = 1 \Rightarrow r^2 = 1 \Rightarrow r = +1$



$0 \le r \le 1$

$0 \le \theta \le \pi$

Now, $I = \iint_R y\,dA$

$$= \int_0^{\pi}\int_0^{1} r\sin(\theta)\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi} \sin(\theta)\cdot \int_0^1 r^2\,dr = -\cos(\theta)\Big|_0^{\pi} \cdot \frac{1}{3}r^3\Big|_0^1$$

$$= -[\cos(\pi) - \cos(0)]\cdot \frac{1}{3}[1^3 - 0^3]$$

$$= -[-1-1]\cdot \frac{1}{3} = \underline{2/3}$$

4. Volume $V = \iint_{Base} \text{Height}\, dA$

We shall make use of polar Coordinates

$x = r\cos(\theta)$,

$y = r\sin(\theta)$

$x^2 + y^2 = r^2$, and $dA = r\,dr\,d\theta$

In polar Coordinates:

$$Z = \sqrt{x^2+y^2} \Rightarrow z = r \quad \text{--- (1)}$$
$$Z = 6-(x^2+y^2) \Rightarrow z = 6-r^2 \quad \text{-- (2)}$$

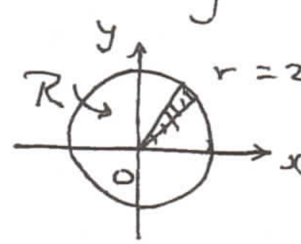
Eliminating "Z" between (1), (2) we get

$$r = 6 - r^2 \quad \text{or} \quad r^2 + r - 6 = 0$$
$$\Rightarrow (r-2)(r+3) = 0$$
$$\Rightarrow r = 2,\ r = -3 \ (\text{Reject Since } r \geq 0)$$

∴ Base is the Circular region enclosed by $r=2$

Next, height $= Z_{upper} - Z_{lower}$

$$= (6-r^2) - r$$
$$= 6 - r^2 - r$$



$$V = \iint_R (6 - r^2 - r)\, dA = \int_0^{2\pi}\int_0^2 (6-r^2-r)\, r\,dr\,d\theta \qquad \begin{matrix} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{matrix}$$

$$= \int_0^{2\pi}\int_0^2 (6r - r^3 - r^2)\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^2 (6r - r^3 - r^2)\,dr$$

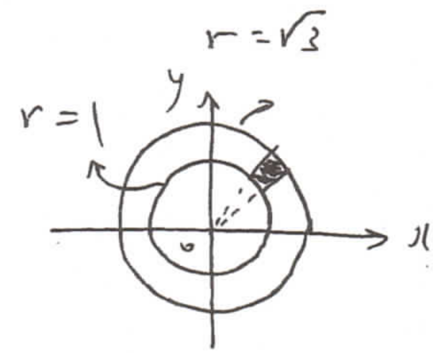
$$= 2\pi\left[3r^2 - \frac{1}{4}r^4 - \frac{1}{3}r^3\right]_0^2 = 2\pi\left[8 - \frac{8}{3}\right] = \frac{32}{3}\pi$$

5. The region D is the planar region enclosed by the two Circles $x^2+y^2=1$, and $x^2+y^2=3$

In polar Coordinates:

$$x^2+y^2=1 \Rightarrow r=1,$$

$$x^2+y^2=3 \Rightarrow r=\sqrt{3}$$



$1 \le r \le \sqrt{3}$

$0 \le \theta \le 2\pi$

Now, $I=\iint_D e^{-(x^2+y^2)}\,dA$

$$=\int_0^{2\pi}\int_1^{\sqrt{3}} e^{-r^2}\cdot r\,dr\,d\theta$$

$$=\int_0^{2\pi} d\theta\cdot\int_1^{\sqrt{3}} e^{-r^2} r\,dr \leftarrow \text{let } t=-r^2$$

$$=2\pi\cdot\left(-\frac{1}{2}\int e^t\,dt\right)$$

$$=-\pi\, e^{-r^2}\Big|_{r=1}^{r=\sqrt{3}} = -\pi\left[e^{-(\sqrt{3})^2}-e^{-(1)^2}\right]$$

$$=-\pi\left[e^{-3}-e^{-1}\right]=\pi\left[e^{-1}-e^{-3}\right]$$

6. (a) $x^2 + y^2 + (z-3)^2 = 9$

$\Rightarrow x^2 + y^2 + z^2 - 6z + 9 = 9$

$\Rightarrow x^2 + y^2 + z^2 = 6z$

In spherical coordinates

$x^2 + y^2 + z^2 = \rho^2, \quad z = \rho \cos(\phi)$

It follows that $\rho^2 = 6\rho\cos(\phi), \quad \rho > 0$

$\Rightarrow \rho = 6\cos(\phi)$

(b) $z = \sqrt{x^2 + y^2}$

In spherical coordinates

$x = \rho\cos(\theta)\sin(\phi), \quad y = \rho\sin(\theta)\sin(\phi)$, and

$z = \rho\cos(\phi)$

Now, $x^2 + y^2 = \rho^2\cos^2(\theta)\sin^2(\phi) + \rho^2\sin^2(\theta)\sin^2(\phi)$

$= \rho^2\sin^2(\phi)\left(\underbrace{\cos^2(\theta) + \sin^2(\theta)}_{\text{one}}\right)$

$= \rho^2\sin^2(\phi)$

$\therefore \sqrt{x^2 + y^2} = \sqrt{\rho^2\sin^2(\phi)} = \rho\,|\sin(\phi)|$

But $0 \leq \phi \leq \pi$, hence $|\sin(\phi)| = \sin(\phi)$

$\therefore \sqrt{x^2 + y^2} = \rho\sin(\phi)$

It follows that

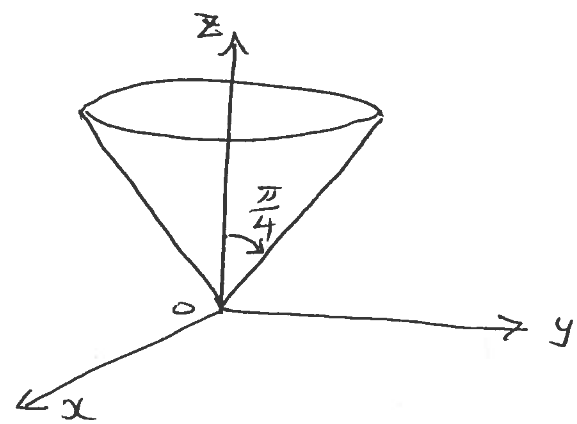
It follows that

$$Z = \sqrt{x^2 + y^2}$$

$$\Rightarrow \quad \rho \cos(\phi) = \rho \sin(\phi), \quad \rho > 0$$

$$\Rightarrow \quad \cos(\phi) = \sin(\phi)$$

or $\tan(\phi) = 1 \quad \Rightarrow \quad \phi = \pi/4$



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7. First $\rho = 9$ is obviously an equation of a sphere centred at $(0,0,0)$ and is of radius 9 units. But since $z \geq 0$, only the hemi-spherical region above the xy-plane is considered.

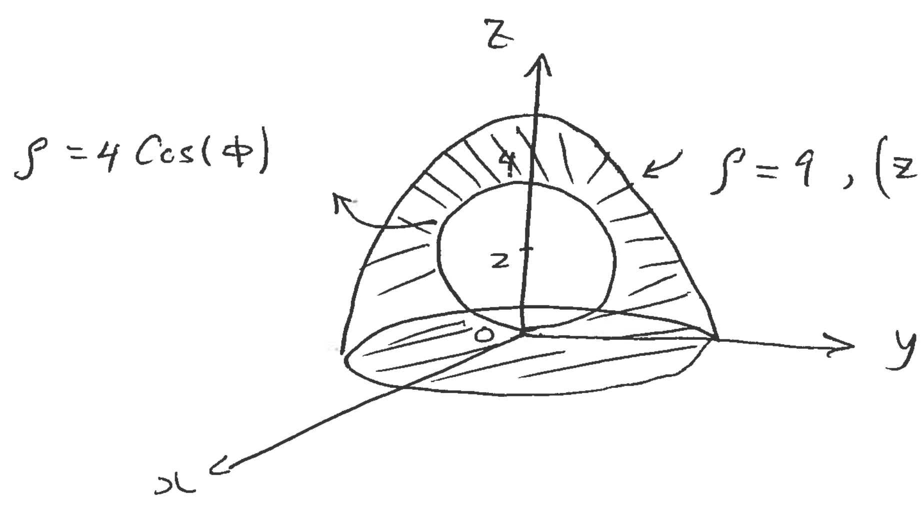
Next, $\rho = 4\cos(\phi) \Rightarrow$

$$\rho^2 = 4(\rho\cos(\phi))$$

$$\Rightarrow x^2 + y^2 + z^2 = 4z$$

or $$x^2 + y^2 + (z-2)^2 = 4$$

which is an equation of a sphere centred at $(0,0,2)$, and is of radius 2. Note that it passes through the origin $(0,0,0)$!



8. $$I = \iiint_E \frac{6z^3}{\sqrt{1+(x^2+y^2+z^2)^3}}\, dV$$

In spherical coordinates:

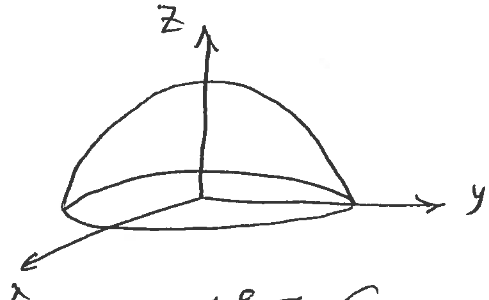
$$z = \rho\cos(\phi), \quad x^2+y^2+z^2 = \rho^2, \text{ and}$$

$$dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$I = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} \frac{6(\rho\cos(\phi))^3}{\sqrt{1+(\rho^2)^3}}\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi)\, d\phi \cdot \int_0^{\sqrt{2}} \frac{6\rho^5}{\sqrt{1+\rho^6}}\, d\rho$$

$$x^2+y^2+z^2 = 2$$
$$\Rightarrow \rho^2 = 2$$
$$\Rightarrow \rho = \sqrt{2}$$



$$0 \le \rho \le \sqrt{2}$$
$$0 \le \phi \le \pi/2$$
$$0 \le \theta \le 2\pi$$

Now, $$\int_0^{2\pi} d\theta = 2\pi,$$

$$\int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi)\, d\phi \quad \text{.. let } t = \cos(\phi)$$

$$= -\frac{1}{4}\cos^4(\phi)\Big|_0^{\frac{\pi}{2}} = -\frac{1}{4}[0-1] = \frac{1}{4}, \text{ and}$$

$$\int_0^{\sqrt{2}} \frac{6\rho^5}{\sqrt{1+\rho^6}}\, d\rho \quad \text{.. use the substitution } u = 1+\rho^6$$

$$= 2\sqrt{1+\rho^6}\Big|_0^{\sqrt{2}} = 2\left[\sqrt{9}-\sqrt{1}\right] = 4$$

$$I = 2\pi \cdot \frac{1}{4} \cdot 4 = 2\pi$$

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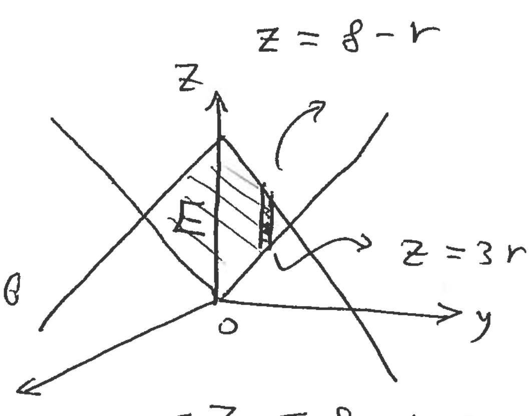
9. $I = \iiint_E \left(2+\sqrt{x^2+y^2}\right) dV$



$3r \leq z \leq 8-r$

In Cylindrical Coordinates:

$x^2+y^2=r^2, \; dV = dz\, dA, \; dA = r\,dr\,d\theta$

$\therefore I = \iiint (2+r)\, dz\, dA$

$= \iint_{Base} \left\{ \int_{z=3r}^{z=8-r} dz \right\} (2+r)\, dA$

$= \iint_{Base} (8-r-3r)(2+r)\, dA$

$= \int_0^{2\pi}\int_0^2 (8-4r)(2+r)\, r\, dr\, d\theta$

$= \int_0^{2\pi} d\theta \int_0^2 (16r-4r^3)\, dr$

$= 2\pi \left[8r^2 - r^4\right]_0^2$

$= 2\pi \left[32-16\right]$

$= 2\pi \cdot 16$

$= 32\pi$

Base

$z = 8-\sqrt{x^2+y^2}$

$= 8 - r \quad --- (1)$

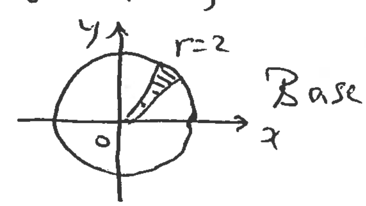
$z = 3\sqrt{x^2+y^2}$

$= 3r \quad --- (2)$

Eliminate "z" between (1), (2):

$8 - r = 3r$

$8 = 4r, \; r = 2$



$0 \leq r \leq 2$

$0 \leq \theta \leq 2\pi$

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10. <u>part (a)</u>

Clearly the region E described by

$$\sqrt{x^2+y^2} \le z \le \sqrt{36-x^2-y^2} + 6$$

is a z-simple region.

$$\therefore \quad J = \iiint_E g(x,y,z)\, dV = \iint_{Base} \left\{ \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{36-x^2-y^2}+6} g\, dz \right\} dA$$

<u>To find Base:</u>

we need to Eliminate "z" among the two surfaces:

$$z = \sqrt{x^2+y^2} \quad --- \quad (1)$$

$$z = \sqrt{36-x^2-y^2} + 6 \quad --- \quad (2)$$

But this is a Very Tedious Task!

Let us instead Eliminate the Block x^2+y^2.

From (1): $z = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = z^2$

substitute into (2)

$$z = \sqrt{36-(x^2+y^2)} + 6$$

$$z = \sqrt{36-z^2} + 6$$

$$\Rightarrow z - 6 = \sqrt{36-z^2}$$

$$(z-6)^2 = 36 - z^2$$

$$\Rightarrow z^2 - 12z + 36 = 36 - z^2$$

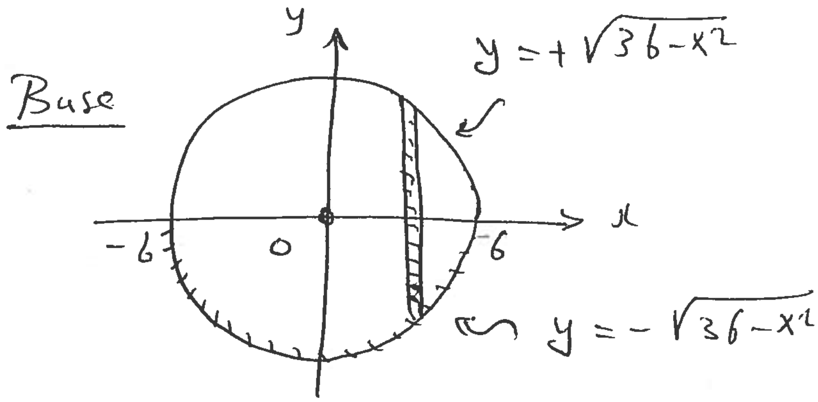
$$\Rightarrow 2z^2 = 12z$$

$$\Rightarrow z = 0, \quad z = 6$$

But $x^2+y^2=z^2$, hence

$x^2+y^2=0$, and $x^2+y^2=6^2=36$

$\therefore$ The Base is the region enclosed by the circles Centred at $(0,0)$ ad has radii 0, 6



Note: $x^2+y^2=36 \Rightarrow$

$y^2=36-x^2$

$\Rightarrow y=\pm\sqrt{36-x^2}$

So Base: $\begin{cases} -\sqrt{36-x^2} \leq y \leq \sqrt{36-x^2} \\ -6 \leq x \leq 6 \end{cases}$

$$\therefore J = \int_{x=-6}^{x=6} \int_{y=-\sqrt{36-x^2}}^{+\sqrt{36-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{36-x^2-y^2}+6} g(x,y,z)\, dz\, dy\, dx$$

part (b):

In cylindrical coordinates:

$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$

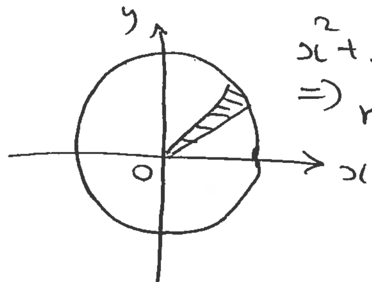
$x^2 + y^2 = r^2$, and $dV = dz\, dA$, where $dA = r\, dr\, d\theta$

Now, $z = \sqrt{x^2+y^2} \Rightarrow z = \sqrt{r^2} \Rightarrow z = r$

$z = \sqrt{36 - x^2 - y^2} + 6 \Rightarrow z = \sqrt{36 - r^2} + 6$

$$\therefore \quad r \leq z \leq \sqrt{36 - r^2} + 6$$

Base



$x^2 + y^2 = 36$
$\Rightarrow r = 6$

Note:
$g(x, y, z)$
$= g(r\cos\theta, r\sin(\theta), z)$
$=$ say $f(r, \theta, z)$

$0 \leq r \leq 6$
$0 \leq \theta \leq 2\pi$

$$\therefore \quad J = \iint_{\text{Base}} \left\{ \int_{z=r}^{z=\sqrt{36-r^2}+6} f(r, \theta, z)\, dz \right\} dA$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=6} \left\{ \int_{z=r}^{z=\sqrt{36-r^2}} f(r, \theta, z)\, dz \right\} \cdot r\, dr\, d\theta$$

part (c):

In spherical coordinates

$x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$,

$x^2 + y^2 = \rho^2\sin^2(\phi)$, $x^2 + y^2 + z^2 = \rho^2$, and

$dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$

Now, $z = \sqrt{x^2 + y^2} \Rightarrow$

$$\rho\cos(\phi) = \sqrt{\rho^2\sin^2(\phi)} = \rho\sin(\phi)$$

If $\rho \neq 0$, $\cos(\phi) = \sin(\phi) \Rightarrow \tan(\phi) = 1 \Rightarrow \phi = \frac{\pi}{4}$.

Next, $z = \sqrt{36 - x^2 - y^2} + 6$

$$\Rightarrow z - 6 = \sqrt{36 - x^2 - y^2}$$

$$\Rightarrow (z-6)^2 = 36 - x^2 - y^2$$

or $x^2 + y^2 + (z-6)^2 = 36$

(sphere: centre $(0,0,6)$, radius 6 -- It passes through $(0,0,0)$)

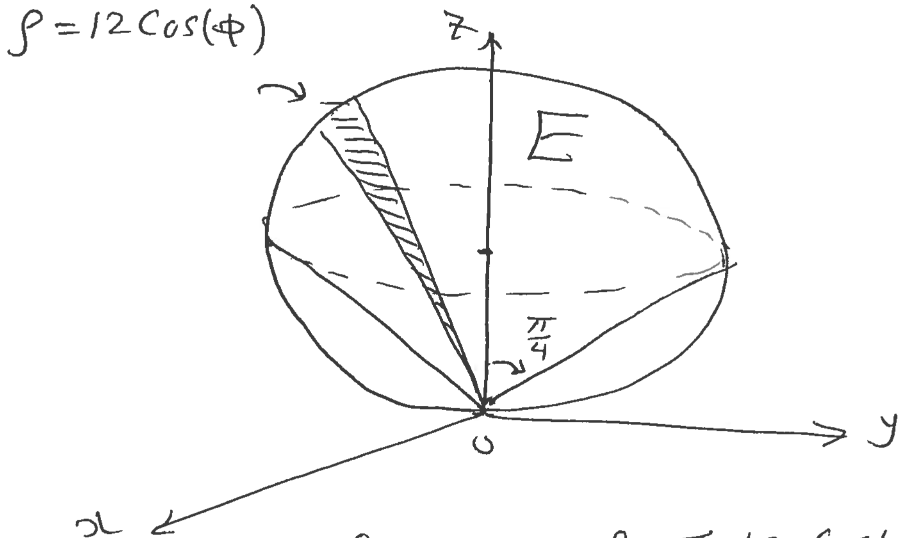
$$\therefore \quad x^2 + y^2 + z^2 - 12z + \cancel{36} = \cancel{36}$$

$$x^2 + y^2 + z^2 = 12z$$

$$\Rightarrow \rho^2 = 12\rho\cos(\phi), \quad \rho > 0$$

or $\rho = 12\cos(\phi)$

let us sketch the sphere $\rho = 12\cos(\phi)$, and the cone $\phi = \frac{\pi}{4}$.



$$E: \begin{cases} 0 \leq \rho \leq 12\cos(\phi) \\ 0 \leq \phi \leq \frac{\pi}{4} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

Note also that

$$g(x, y, z) = g\big(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)\big)$$

$$= \text{a new function say } h(\rho, \phi, \theta)$$

$$\therefore J = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{4}} \int_{\rho=0}^{\rho=12\cos(\phi)} h(\rho, \phi, \theta) \cdot \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

(d) If $g(x,y,z) = 1$, then

$$J = \iiint_E 1 \cdot dV = \text{volume of region } E$$

We have three choices to compute J.

part (c) is Easiest!

$$\therefore \text{ volume } V = J = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{12\cos(\phi)} 1 \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{4}} \sin(\phi) \cdot \frac{\rho^3}{3} \Big|_{\rho=0}^{\rho=12\cos(\phi)} d\phi$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sin(\phi) \cdot \frac{1}{3} \left(12\cos(\phi)\right)^3 d\phi$$

$$= 2\pi \cdot \frac{1}{3} \cdot (12)^3 \int_0^{\frac{\pi}{4}} \cos^3(\phi) \sin(\phi)\, d\phi$$

let $t = \cos(\phi)$, $dt = -\sin(\phi)\, d\phi$

$$\therefore V = 2\pi \cdot \frac{1}{3} \cdot (12)^3 \int (-t^3)\, dt$$

$$= -\frac{2\pi}{3} (12)^3 \frac{t^4}{4}$$

$$= -\frac{2\pi}{12} (12)^3 \left[\cos^4(\phi)\right]_0^{\frac{\pi}{4}} = -2\pi (12)^2 \left[\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right]$$

$$= 216\pi$$

11. <u>Answer</u>: Volume $V = \frac{\pi}{2}$

12. <u>Answer</u>: $\iint_R y^2 \, dA = 2\pi$

13. <u>Answer</u>: 4π

14. <u>Answer</u>: $\frac{\pi}{2} \ln(2)$

15. <u>Answer</u>: $\frac{56}{3}\pi$

<u>Hint</u>: Read Problem #(10)

16. <u>Answer</u>: $\frac{56}{3}\pi$

<u>Hint</u>: Read Problem #(10)

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