Implicit Differentiation : The Technique

Given a non linear system of "m" equations in "n" variables

Then, under certain conditions we may be able to solve for "m" variables as functions of the remaining variables $n - m \geq 0$.

Example For instance, consider a system of one equation in 3 variables say

$$F(x,y,z) = 0$$

This system can be solved in 3 different ways.

1) We can solve for x as a function of y and z written $x = x(y,z)$

2) $y = y(x,z)$

3) $z = z(x,y)$

Example Given a system of 2 equations in 5 variables

Say $F(x,y,z,u,v) = 0$
$G(x,y,z,u,v) = 0$

This system can be solved for 2 variables as a function of the remaining 3. $(5-2=3)$

There are $\binom{5}{2} = \frac{5\times 4}{2!} = \frac{20}{2} = $ 10 possibilities

① $x = x(z,u,v)$, $y = y(z,u,v)$

② $x = x(y,u,v)$, $z = z(y,u,v)$

③ $x = x(y,z,v)$, $u = u(y,z,v)$

④ $x = x(y,z,u)$, $v = v(y,z,u)$

⑤ $y = y(x,u,v)$, $z = z(x,u,v)$

⑥ $y = y(x,z,v)$, $u = u(x,z,v)$

⑦ $y = y(x, z, u)$
$v = v(x, z, u)$

⑧ $z = z(x, y, v)$
$u = u(x, y, v)$

⑨ $z = z(x, y, u)$
$v = v(x, y, u)$

⑩ $u = u(x, y, z)$
$v = v(x, y, z)$

Notation :

Suppose we are told to compute the partial derivative $\frac{\partial u}{\partial z}$. Then which u will we choose?

① $u = u(y, z, v)$

or ② $u = u(x, z, v)$

or ③ $u = u(x, y, z)$

→ To avoid this confusion, we shall write

$\left(\frac{\partial u}{\partial z}\right)_{x,y}$ to indicate that $u = u(x, y, z)$ is the correct choice.

Implicit Differentiation Formula

March 14

1. System of one equation in 3 variables

$$F(x, y, z) = 0$$

Assume that the equation can be solved for y as a function of the remaining x and z.

That is $y = y(x, z)$.

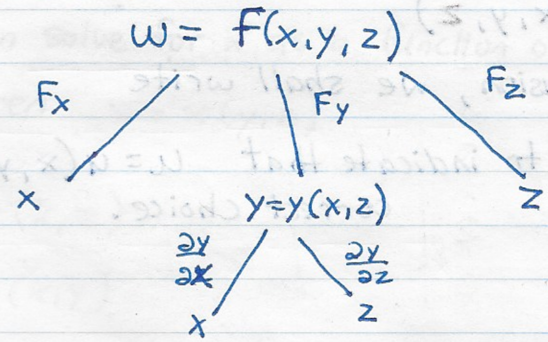
We want to compute $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$

or $\left(\frac{\partial y}{\partial x}\right)_z$ and $\left(\frac{\partial y}{\partial z}\right)_x$

By the chain Rule

Note: $w = 0$

Tree



$$\frac{\partial w}{\partial x} = F_x + F_y \frac{\partial y}{\partial x} \quad \therefore \quad 0 = F_x + F_y \frac{\partial y}{\partial x}$$

$$\therefore \quad F_y \frac{\partial y}{\partial x} = -F_x \qquad \boxed{\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}} \qquad F_y \text{ or } \frac{\partial F}{\partial y} \neq 0$$

Likewise

$$\boxed{\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}} \qquad F_y \neq 0$$

Remark: Implicit differentiation (from 275)

If $F(x, y) = 0 \longrightarrow y = y(x)$

then $\frac{dy}{dx} = -\frac{F_x}{F_y}$, $F_y \neq 0$

Example 1 Given $x^3y^2 + x\sin(y) + y^3e^x = 5$. Find $\frac{dy}{dx}$.

Solution

Rewrite equation in form where 1 side is zero ∃ $F(x,y) = 0$

$$x^3y^2 + x\sin(y) + y^3e^x - 5 = 0$$

Take $F = x^3y^2 + x\sin(y) + y^3e^x - 5$

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = \boxed{-\frac{3x^2y^2 + \sin(y) + e^xy^3}{2yx^3 + x\cos(y) + 3y^2e^x}}$$

Example 2 Given $\ln(x^6z^8) - 14y(x-1) + 8xz + y^3z = 0$ (1)

Find $\left(\frac{\partial y}{\partial z}\right)_x$ at the point $(x,z) = (1,1)$

Solution $\left(\frac{\partial y}{\partial z}\right)_x$ means ∃ system is solved for y as a function of x and z ∘ $y = y(x,z)$

$$\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

Take $F = \ln(x^6z^8) - 14y(x-1) + 8xz + y^3z$

simplify!

$F = 6\ln x + 8\ln z - 14y(x-1) + 8xz + y^3z$

$$\frac{\partial y}{\partial z} = -\frac{\frac{8}{z} + 8x + y^3}{-14(x-1) + 3y^2z}$$

→ Plug $x=1$ $z=1$ into equation (1) to find y

$$\ln(1) - 14(1)(0) + 8(1)(1) + (y^3)(1) = 0$$

$$0 - 0 + 8 + y^3 \qquad y = \sqrt[3]{-8} \quad \text{so} \quad \boxed{y = -2}$$

→ Sub into $\frac{\partial y}{\partial z}$, we obtain $\boxed{-2/3}$

2. Implicit Formula for a System of 2 Equations

Given a system of equations

$$F(x,y,u,v) = 0$$
$$G(x,y,u,v) = 0$$

Under certain contitions, we may be able to solve for u and v as functions of x and y

That is:

$$u = u(x,y)$$
$$v = v(x,y)$$

We need to compute $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$

Again by chain rule

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} = 0$$

$$\Longrightarrow \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = -\frac{\partial F}{\partial x} \quad (1)$$

$$\frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} = -\frac{\partial G}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\begin{matrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial v} \\ \frac{\partial F}{\partial x} & \frac{\partial G}{\partial v} \end{matrix}}{\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}}$$

← replace first colum of det. by R.H.S.

$$\frac{\partial v}{\partial x} = - \frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

Using Jacobian notation, we have

$$\frac{\partial u}{\partial x} = - \frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} \qquad \frac{\partial v}{\partial x} = - \frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}$$