# MATH 277 - midterm booklet solutions

**Exercise 1**

$$\vec{r}' = \langle -2\sin^2 t + 2\cos^2 t,\ 4\sin t\cos t,\ -2\sin t \rangle = \vec{v}$$

$$\|\vec{v}\| = \|\vec{r}'\| = \sqrt{(-2\sin^2 t + 2\cos^2 t)^2 + 16\sin^2 t\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{(2\cos(2t))^2 + 16\sin^2 t\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{4\cos^2(2t) + 4\cdot(2\sin t\cos t)^2 + 4\sin^2 t}$$

$$= \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 4\sin^2 t} = 2\sqrt{1+\sin^2 t}$$

note : $\vec{r}' = \langle 2\cos(2t),\ 2\sin(2t),\ -2\sin t \rangle$

$$\vec{a}(t) = \vec{r}''(t) = \langle -4\sin(2t),\ 4\cos(2t),\ -2\cos t \rangle$$

Exercise 2

$\vec{r}' = -\sin t\ \hat{i} + 2\cos t\ \hat{j}$ $\qquad\qquad$ $\vec{r}\left(\frac{\pi}{6}\right) = \left\langle \frac{\sqrt{3}}{2}, 1 \right\rangle$

@ $\frac{\pi}{6}$ $\quad \vec{r}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\hat{i} + \sqrt{3}\,\hat{j}$

slope $m = \frac{\sqrt{3}}{-1/2} = -2\sqrt{3}$

Tan line $\quad y - 1 = -2\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right) \Rightarrow y = -2\sqrt{3}x + 4$

normal line has slope $\frac{-1}{-2\sqrt{3}} = \frac{1}{2\sqrt{3}}$

so $\quad y - 1 = \frac{1}{2\sqrt{3}}\left(x - \frac{\sqrt{3}}{2}\right) \Rightarrow y = \frac{1}{2\sqrt{3}}x + \frac{3}{4}$

**Exercise 3**

$(1,-2,2)$ corresponds to $t=1$.

$\vec{r}\,'(t) = \langle 1, -4t, 6t^2 \rangle$ $\qquad$ $\vec{r}\,'(1) = \langle 1, -4, 6 \rangle$

direction of tan. line

$$\begin{cases} x = t+1 \\ y = -4t-2 \\ z = 6t+2 \end{cases}$$

Exercise 4

$$\vec{r}'(t) = \langle e^t, \sqrt{2}, e^{-t} \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

$$L = \int_0^1 (e^t + e^{-t})\, dt = (e^t - e^{-t})\Big|_0^1 = e - \frac{1}{e}$$

Exercise 5

$$\vec{r}'(t) = \langle 4\sin t\cos t, -3\cos^2 t\sin t, 3\sin^2 t\cos t \rangle$$

$$\|r'\|^2 = 16\sin^2 t\cos^2 t + 9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t$$

$$= \sin^2 t\cos^2 t\,(16 + 9\cos^2 t + 9\sin^2 t)$$

$$= 25\sin^2 t\cos^2 t$$

$$\therefore \|r'(t)\| = 5\sin t\cos t$$

$$L = \int_0^{\frac{\pi}{4}} 5\sin t\cos t\,dt = \frac{5}{2}\sin^2 t\Big|_0^{\pi/4} = \frac{5}{2}\left(\frac{\sqrt{2}}{2}\right)^2 - 0$$

$$= \frac{5}{4}$$

**Exercise 6**

Note: $(\text{end} - \text{beg})t + \text{beg} \quad 0 \le t \le 1$

$$\vec{r}(t) = \langle 1, 0, 4 \rangle t + \langle 0, 1, 2 \rangle$$

$$= \langle t, 1, 4t + 2 \rangle \qquad 0 \le t \le 1$$

Exercise 7

$h = 1, \quad K = -2$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{1} = 1$$

Exercise 8

$$x^2 - 2x \qquad + y^2 + 6y \qquad = 15$$

$$x^2 - 2x + 1 + y^2 + 6y + 9 = 15 + 1 + 9$$

$$(x-1)^2 + (y+3)^2 = 25$$

circle of radius 5

centred @ $(1,-3)$

parametrization:

$$x = 5\cos t + 1$$
$$y = 5\sin t - 3 \qquad t \in [0, 2\pi)$$

**Exercise 9**

$$x = 3\cos t$$
$$y = 3\sin t$$
$$z = 3\cos t + 3\sin t$$

$$t \in [0, 2\pi)$$

Exercise 10

$$xz - x = 1$$

$$x(z-1) = 1$$

$$x = \frac{1}{z-1} \qquad z = t, \quad x = \frac{1}{t-1}$$

$$yz + x = 1$$

$$y = \frac{1-x}{z} = \frac{1 - \frac{1}{t-1}}{t} = \frac{\frac{t-1-1}{t-1}}{t} = \frac{t-2}{t(t-1)}$$

$$\vec{r}(t) = \left\langle \frac{1}{t-1}, \frac{t-2}{t(t-1)}, t \right\rangle \qquad (t \neq 1)$$

**Exercise 11**

$$\vec{r}' = \langle \cos t,\ \cos t,\ -\sqrt{2}\sin t \rangle \qquad \vec{r}'\left(\tfrac{\pi}{4}\right) = \left\langle \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, -1 \right\rangle$$

$$\left\|\vec{r}'\left(\tfrac{\pi}{4}\right)\right\| = \sqrt{\tfrac{1}{2}+\tfrac{1}{2}+1} = \sqrt{2}$$

$$\vec{r}''(t) = \langle -\sin t,\ -\sin t,\ -\sqrt{2}\cos t \rangle \qquad \vec{r}''\left(\tfrac{\pi}{4}\right) = \left\langle -\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, -1 \right\rangle$$

$$\vec{r}'\left(\tfrac{\pi}{4}\right) \times \vec{r}''\left(\tfrac{\pi}{4}\right) = \langle -\sqrt{2}, +\sqrt{2}, 0 \rangle \qquad \left\| r'\left(\tfrac{\pi}{4}\right) \times r''\left(\tfrac{\pi}{4}\right) \right\| = 2$$

$$\vec{r}'''(t) = \langle -\cos t,\ -\cos t,\ \sqrt{2}\sin t \rangle \qquad \vec{r}'''\left(\tfrac{\pi}{4}\right) = \left\langle -\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, 1 \right\rangle$$

$$\hat{T} = \frac{r'}{\|r'\|} = \left\langle \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-1}{\sqrt{2}} \right\rangle$$

$$\hat{B} = \frac{r' \times r''}{\|r' \times r''\|} = \left\langle -\tfrac{\sqrt{2}}{2}, +\tfrac{\sqrt{2}}{2}, 0 \right\rangle$$

$$\hat{N} = \hat{B} \times \hat{T} = \left\langle -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{\sqrt{2}} \right\rangle$$

$$\kappa = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}} \quad (\rho = \sqrt{2})$$

$$\tau = 0$$

Exercise 12

$\vec{r}' = \langle 1, 2t, 0 \rangle \qquad \|r'\| = \sqrt{1+4t^2}$

$\vec{r}'' = \langle 0, 2, 0 \rangle$

$\vec{r}' \times \vec{r}'' = \langle 0, 0, 2 \rangle \qquad \|\vec{r}' \times \vec{r}''\| = 2$

$\vec{r}''' = \langle 0, 0, 0 \rangle \qquad (\tau = 0)$

$\hat{T} = \dfrac{\langle 1, 2t, 0 \rangle}{\sqrt{1+4t^2}} \qquad \hat{B} = \langle 0, 0, 1 \rangle$

$\hat{N} = \hat{B} \times \hat{T} = \dfrac{\langle -2t, 1, 0 \rangle}{\sqrt{1+4t^2}}$

$\kappa = \dfrac{2}{(1+4t^2)^{3/2}} \qquad \rho = \dfrac{(1+4t^2)^{3/2}}{2}$

Exercise 13

$$\vec{r}' = \langle \cos t, -\sin t, 1 \rangle \qquad \|\vec{r}'\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$$

$$\vec{r}'' = \langle -\sin t, -\cos t, 0 \rangle$$

$$\vec{r}' \times \vec{r}'' = \langle +\cos t, -\sin t, -1 \rangle \qquad \|r' \times r''\| = \sqrt{2}$$

$$\vec{r}''' = \langle -\cos t, \sin t, 0 \rangle$$

$$\hat{T} = \left\langle \frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\hat{B} = \left\langle \frac{\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$

$$\hat{N} = \hat{B} \times \hat{T} = \langle -\sin t, -\cos t, 0 \rangle$$

$$\kappa = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \quad (\rho = \sqrt{2})$$

$$\tau = \frac{-\cos^2 t - \sin^2 t + 0}{2} = -\frac{1}{2}.$$

Exercise 14

$$\vec{r}' = \langle 1, 2t, t \rangle$$

$$\|r'\| = \sqrt{1+5t^2} \quad \rightarrow \quad a_{\hat{T}} = \frac{5t}{\sqrt{1+5t^2}}$$

$$\vec{r}'' = \langle 0, 2, 1 \rangle$$

$$\vec{r}' \times \vec{r}'' = \langle 0, -1, 2 \rangle \qquad \|r' \times r''\| = \sqrt{5}$$

$$a_{\hat{N}} = \frac{\sqrt{5}}{\sqrt{1+5t^2}}$$

**Exercise 15**

$$\vec{r}'(t)\left\langle 0, \frac{2t}{1+t^2}, 1-\frac{2}{1+t^2}\right\rangle = \left\langle 0, \frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2}\right\rangle.$$

$$\|r'(t)\| = \frac{1}{1+t^2}\sqrt{(2t)^2+(t^2-1)^2} = \frac{1}{1+t^2}\sqrt{4t^2+t^4-2t^2+1}$$

$$= \frac{\sqrt{(t^2+1)^2}}{1+t^2} = 1+t^2 \quad \rightarrow \quad a_{\hat{T}} = 2t$$

$$\vec{r}''(t) = \left\langle 0, \frac{2(1-t^2)}{(1+t^2)^2}, \frac{4t}{(1+t^2)^2}\right\rangle$$

$$\vec{r}' \times \vec{r}'' = \frac{1}{(1+t^2)^3}\left\langle 2(t^4+2t^2+1), 0, 0\right\rangle = \left\langle \frac{2}{1+t^2}, 0, 0\right\rangle$$

$$\|r' \times r''\| = \frac{2}{1+t^2} \quad \rightarrow \quad a_{\hat{N}} = \frac{2/_{1+t^2}}{1+t^2} = \frac{2}{(1+t^2)^2}$$

## 2.2 Exercises

**Exercise 1**

$$9 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 9$$

&

$$\ln(9 - x^2 - y^2) \geqslant 0 \Rightarrow 9 - x^2 - y^2 \geqslant 1$$

$$\Rightarrow x^2 - y^2 \leqslant 8$$

so $x^2 - y^2 \leq 8$

Inside + border of the disk centered at $(0,0)$ with radius $\sqrt{8}$

**Exercise 2**

$$9x^2 + 4y^2 - 36 \geqslant 0$$

$$9x^2 + 4y^2 \geqslant 36$$

$$\frac{x^2}{4} + \frac{y^2}{9} \geqslant 1$$

outside & border of ellipse centred at $(0,0)$

($x$_radius $= 2$, $y$_radius $= 3$)

Exercise 3

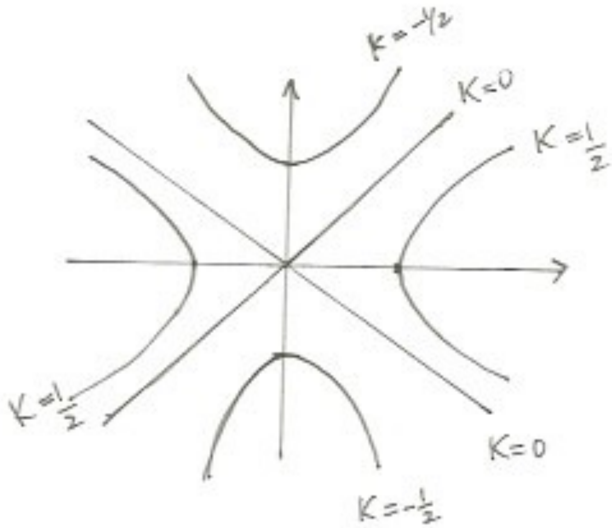
$$\frac{x^2 - y^2}{x^2 + y^2 + 1} = K \Rightarrow x^2 - y^2 = Kx^2 + Ky^2 + K$$

$$\Rightarrow x^2(1-K) - (1+K)y^2 = K$$

$K = 0 \rightarrow \quad x^2 - y^2 = 0 \rightarrow y^2 = x^2 \rightarrow y = \pm x$

$K = 1/2 \rightarrow \quad \frac{1}{2}x^2 - \frac{3}{2}y^2 = \frac{1}{2} \Rightarrow x^2 - 3y^2 = 1$

$K = -1/2 \rightarrow \quad \frac{3}{2}x^2 - \frac{1}{2}y^2 = -\frac{1}{2} \rightarrow y^2 - 3x^2 = 1$



Exercise 4

$$\frac{x-y}{x+y} = K \quad \Rightarrow \quad x - y = Kx + Ky$$

$$\Rightarrow \quad x(1-K) - y(1+K) = 0$$

$$\Rightarrow \quad y = \left(\frac{1-K}{1+K}\right)x$$

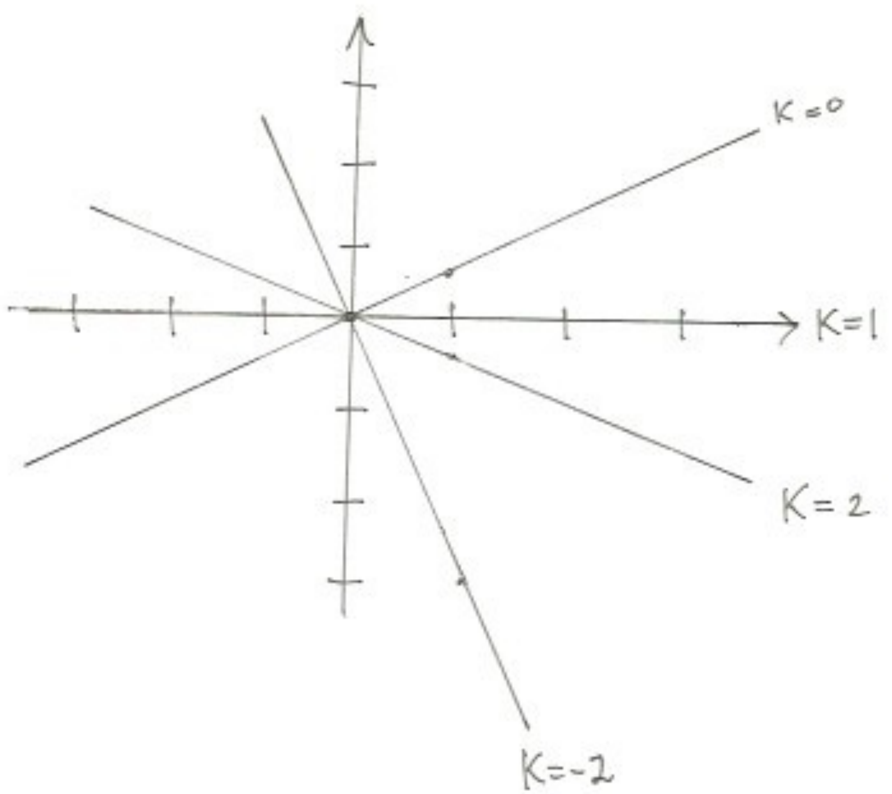
$K=0 \qquad y = \frac{1}{2}x$

$K=1 \qquad y = 0$

$K=-1$ (NO POSSIBLE)

$K=2 \qquad y = -\frac{1}{3}x$

$K=-2 \qquad y = -3x$



**Exercise 5**

$$\frac{\partial f}{\partial x} = y \ln z \; x^{y \ln z - 1}$$

$$\frac{\partial f}{\partial y} = \frac{\partial \left(x^{\ln z} \cdot x^{y}\right)}{\partial y} = x^{\ln z} \cdot x^{y} \cdot \ln x$$

$$\frac{\partial f}{\partial z} = \frac{\partial \left(x^{y}\, x^{\ln z}\right)}{\partial z} = x^{y} \cdot x^{\ln z} \cdot \ln x \cdot \frac{1}{z} .$$

Exercise 6

$$\frac{\partial f}{\partial y} = \sin x \cdot y^{\sin x - 1}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \cos x \ y^{\sin x - 1} + \sin x \cdot y^{\sin x - 1} \cdot \ln y \cdot \cos x .$$

Exercise 7

$$\frac{\partial f}{\partial x} = a\, e^{ax+y} \sin(2z) \qquad \frac{\partial^2 f}{\partial x^2} = a^2\, e^{ax+y} \sin(2z)$$

$$\frac{\partial f}{\partial y} = e^{ax+y} \sin(2z) \qquad \frac{\partial^2 f}{\partial y^2} = e^{ax+b} \sin(2z)$$

$$\frac{\partial f}{\partial z} = 2\, e^{ax+y} \cos(2z) \qquad \frac{\partial^2 f}{\partial z^2} = -4\, e^{ax+y} \sin(2z)$$

$$a^2 e^{ax+y} \sin(2z) + e^{ax+y} \sin(2z) - 4\, e^{ax+y} \sin(2z) = 0$$

$$\Rightarrow \quad a^2 + 1 - 4 = 0$$

$$\Rightarrow \quad a = \pm\sqrt{3}\,.$$

**Exercise 8**

$\vec{N} = \langle -4x, -2y, 1 \rangle \quad @ \ (1,1) \quad \langle -4, -2, 1 \rangle$

Pt $(1,1,3)$

tan. plane: $-4(x-1) - 2(y-1) + 1(z-3) = 0$

$$-4x - 2y + z + 3 = 0$$

$$4x + 2y - z = 3$$

Normal line: $\begin{cases} x = -4t + 1 \\ y = -2t + 1 \\ z = t + 3 \end{cases}$

Exercise 9

$\vec{N} = \langle -e^{x-y}, e^{x-y}, 1 \rangle$ @ $(2,2)$ $\quad \vec{N} = \langle -1, 1, 1 \rangle$

Pt $(2,2,1)$

tan. plane: $\quad -1(x-2) + 1(y-2) + 1(z-1) = 0$

$-x + y + z - 1 = 0$

Exercise 10

$$\vec{F} = 2x + 3y^2 + 2z^2 - 31 = 0$$

$$\vec{\nabla} F = \langle 2, 6y, 4z \rangle \quad @ \ (-2, 1, 4)$$

$$\vec{\nabla F} = \langle 2, 6, 16 \rangle$$

Choose $\vec{n} = \langle 1, 3, 8 \rangle$ pt $(-2, 1, 4)$

$$1(x+2) + 3(y-1) + 8(z-4) = 0$$

$$x + 3y + 8z - 33 = 0$$