MIDTERM REVIEW SHEET

MATH 277 Midterm Review Sheet

1. In each case , the position of a moving object in space is given.

   Determine the velocity , the acceleration , and the speed of the object at time t.

   (a) $\vec{r}(t) = a\cos(t)\vec{i} + a\sin(t)\vec{j} + b\vec{k} \quad , \quad a, b > 0.$

   (b) $\vec{r}(t) = t^2\vec{i} - t^2\vec{j} + \vec{k} \quad , \quad t > 0.$

   (c) $\vec{r}(t) = (\ln(t) \ , \ \sin^2(t) \ , \ \frac{1}{2}\sin(2t)) \quad , \quad t > 0.$

2. Find the Cartesian equations of the <u>tangent</u> and <u>normal</u> lines to each of the following

   parametric curves at the indicated point :

   (i) $x(t) = 2t^3 + 4$, $y(t) = 6e^t - 6t - 3t^2 - 7$ at the point on the curve corresponding to $t = 0$.

   (ii) $\vec{r}(t) = (t^2 - 2t + 31)\vec{i} + (t^2 - 1)\ \vec{j}$ at the point $P(39, 3)$ on the curve.

3. Find parametric equations of the line tangent to the space curve

   $\vec{r}(t) = e^t\vec{i} + 2\ e^{-t}\vec{j} + e^{2t}\vec{k}$, at the point on the curve corresponding to $t = \ln(2)$.

4. In each case , find the arc length of the given curve :

   (a) $\vec{r}(t) = (3t \ , \ 2t^{3/2}, \ 4) \ , \quad 0 \le t \le 8.$

   (b) $\vec{r}(t) = (2\sin^2(t) \ , \cos^3(t) \ , \ \sin^3(t)), \quad 0 \le t \le \frac{\pi}{2}.$

   (c) $\vec{r}(t) = 2\ e^t\ \vec{i} + e^{-t}\ \vec{j} + 2t\ \vec{k} \quad , \quad -1 \le t \le 1.$

   (d) $\vec{r}(t) = \frac{1}{2}\sin(t^2)\ \vec{i} + \frac{1}{2}\cos(t^2)\ \vec{j} + \frac{1}{3}(2t+1)^{3/2}\vec{k} \quad , \quad 0 \le t \le 2$

   Hint : Use the identity : $\cos^2(t^2) + \sin^2(t^2) = 1$ to simplify $\| \vec{v}(t) \|$ for part (b) & (d).

5. Find parametric equations of :

   (a) The straight line segment in $\mathbb{R}^2$ from the point $P(1, -4)$ to the point $Q(2, -3)$.

   (b) The straight line segment in $\mathbb{R}^3$ from the point $A(0, 1, 2)$ to the point $B(1, 1, -1)$.

   (c) The circle centred at the point $(1, 0)$ and has radius 4 units.

6. Find a standard parametric representation of each of the following plane curves :

   (i) $(3x + 1)^2 + (5y - 2)^2 = 900.$ Hint : First , express equation in standard form. Identify curve.

   (ii) $x^2 + y^2 - 2x + 6y - 15 = 0.$ Hint : First , complete the square in both x and y terms. Identify curve

7. In each case , find a parametrization of the curve of intersection of the given surfaces :

(a) $4x^2 + y^2 = 16$, $2x + 3y + 2z = 1.$

(b) $x^2 + 2y + z = 3$, $xz + y = -2.$

(c) $z = x^2 + y^2$, $2x - 4y - z + 4 = 0.$

(d) $xy + xz = 6$, $x = -3.$

(e) $x^2 - y^2 - z = 0$, $2y^2 + z = 1.$

8. A rocket has mass $52,000$ kilogram (kg), which includes $39,000\ kg$ of fuel mixture is fired vertically upward in a vacuum (that is Free Space where gravitational field is negligible) During the burning process the exhaust gases are ejected at a constant rate $1300\ kg/s$ and at constant velocity with magnitude $500\ metre/s$ relative to the rocket. If the rocket was initially at rest , find its speed after 15 , 20 , 30 and 35 seconds.

9. For each of the following curves find the unit Tangent $\vec{\mathbf{T}}$ and the unit Normal $\vec{\mathbf{N}}$ and the curvature $\boldsymbol{\kappa}$ at the indicated value of t :

(a) $\vec{r}(t) = t\,\vec{i} + \ln(\cos(t))\,\vec{j}$; $t = \frac{\pi}{4}$

(b) $\vec{r}(t) = (2t+3)\,\vec{i} + (5 - t^2)\,\vec{j}$; $t = \sqrt{3}$

10. For each of the following curves find the unit Tangent $\vec{\mathbf{T}}$, the Principal unit Normal $\vec{\mathbf{N}}$, the unit Binormal $\vec{\mathbf{B}}$, the curvature $\boldsymbol{\kappa}$, the radius of curvature ρ and the Torsion $\boldsymbol{\tau}$ at the indicated value :

(a) $\vec{r}(t) = 3\sin(t)\,\vec{i} + 3\cos(t)\,\vec{j} + 4t\,\vec{k}$; $t = 0$

(b) $\vec{r}(t) = \sin(t)\,\vec{i} + \sqrt{2}\cos(t)\,\vec{j} + \sin(t)\,\vec{k}$; $t = \frac{\pi}{4}$

(c) $\vec{r}(t) = \cosh(t)\,\vec{i} - \sinh(t)\,\vec{j} + t\,\vec{k}$; $t = 0$

11. In each case the position $\vec{r}(t)$ of a moving object at time t is given. Find the **Tangential** and **Normal** components of the acceleration at the indicated time :

(a) $\vec{r}(t) = t^2\,\vec{i} + t\,\vec{j} + \frac{1}{2}t^2\,\vec{k}$; $t = 4$

(b) $\vec{r}(t) = \ln(t^2+1)\,\vec{i} + (t - 2\tan^{-1}(t))\,\vec{j}$; $t = 2$

(c) $\vec{r}(t) = t\cos(t)\vec{i} + t\sin(t)\,\vec{j} + t^2\,\vec{k}$; $t = 0$

12. In each case , find the **Domain** of the given function and sketch :

(a) $f(x,y) = \dfrac{3-x}{x+y-5}$ (b) $f(x,y) = \sqrt{4x^2+9y^2-36}$

(c) $f(x,y) = \sqrt{1+x^2+y^2}$ (d) $f(x,y) = \sqrt{\ln(5-x^2-y^2}$

(e) $f(x,y) = \ln\sqrt{x^2+y^2-4}$ (f) $f(x,y) = \ln|x^2+y^2-4|$

13. In each case , draw level curves of $f(x,y)$ for the indicated values of c :

(a) $f(x,y) = x\,e^{-y}$, $c = 0\,,1\,,-1$

(b) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2+1}$, $c = 0\,,\dfrac{1}{2}\,,-\dfrac{1}{2}$

(c) $f(x,y) = \tan^{-1}(x+y)$, $c = 0\,,\dfrac{\pi}{4}\,,-\dfrac{\pi}{6}$

14. Identify each of the following surfaces:

(i) $z = 1+3\sqrt{x^2+y^2}$ (ii) $x = 2-y^2-z^2$ (iii) $2-x^2-3y^2-2z^2 = 0$

(iv) $\dfrac{x^2}{4}-\dfrac{y^2}{9}+\dfrac{z^2}{25} = 1$ (v) $x = z^2$ (vi) $3x-2y+1 = 0$

(vii) $x^2+y^2+z^2-2x = 0$ (viii) $x^2+y^2-z^2-4z = 3$

15. (a) If $z = \ln(xy)^{\sin(xy)}$, $x > 0\,, y > 0$, find $\dfrac{\partial z}{\partial y}$. Hint : First , simplify Logarithm.

(b) Let $f(x,y) = y^{\tan(x)}+\cosh(x^2)$, find $f_{yx}(x,y)$.

16. Find all values of the constant real number A such that the function

$W(x,y,z) = x^4+y^4+z^4+A(x^2y^2+x^2z^2+y^2z^2)$ is Harmonic in $\mathbb{R}^3$.

Note : $W(x,y,z)$ is Harmonic in $\mathbb{R}^3$ if it satisfies Laplace Equation $\nabla^2 W = W_{xx}+W_{yy}+W_{zz} = 0.$

17. Find the constant real number m such that the function $f(x,y,z) = e^{mz}\cos(2\sqrt{5}\,x)\cosh(2my)$ is Harmonic in $\mathbb{R}^3$.

18. In each case , find an equations of the tangent plane and the normal line to the given surface at the specified point P on the surface :

(a) $z = \sqrt{x^2+y^2}$, $P(3,-4,5)$.

(b) $xy+z^3+e^{x-y+z} = 4$, $P(1,2,1)$.

19. In case , use the chain rule to find the specified derivatives computed at the indicated values :

(a) $\frac{dz}{dt}$ at $t = \frac{\pi}{6}$, if $z = \cot(3x + \frac{1}{12}y)$, where $x = \frac{1}{\pi}t^2$, and $y = \frac{\pi^2}{6t}$.

(b) $\frac{\partial z}{\partial v}$ at $u = 0$, $v = 0$, if $z = \ln(x^2 + 3xy)^{-4}$, where $x = \cosh(u)$, and $y = 2\sinh(v)$.

(c) $\frac{\partial w}{\partial s}$, if $w = f(t^2 - 3s, t^{-1}s^3, t + 3s)$, for some differentiable function $f(x, y, z)$.

Hint : Let $x = t^2 - 3s$, $y = t^{-1}s^3$, and $z = t + 3s$.

(d) $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$ $at\ (r, \theta) = (1, \frac{\pi}{6})$ if $z = \sqrt{x^2 - y^2}$, where $x = r\cos(\theta)$, and $y = r\sin(\theta)$.

(e) $\frac{\partial z}{\partial y}$, at $(x, y) = (1, 0)$ if $z = f(u, v)$, where $u = \ln\sqrt{x^2 + y^2}$, and $v = x + \arctan(\frac{y}{x})$,

given that $f_u(1,0) = 8$, $f_v(1,0) = -9$, $f_u(0,1) = 5$, $f_v(0,1) = -4$, and $f(0,0) = 17$.

(f) $\frac{\partial w}{\partial u}$, and $\frac{\partial w}{\partial v}$ at $(u, v) = (-2, 0)$ if $w = \ln(x^2 + y^2 + z^2)$, where $x = ue^v\sin(v)$,

$y = ue^v\cos(v)$, and $z = ue^v$.

20. (a) Find an equation of the plane tangent to the ellipsoid $4x^2 + 3y^2 + z^2 = 25$ at the point $P(1, 2, -3)$.

(b) Find an equation of the plane tangent to the paraboloid $2x + 3y^2 + 2z^2 = 31$ at the point $P(-2, 1, 4)$.

(c) Find a **unit vector** normal (orthogonal) to the surface $\sin(xyz - 6) + 2x - x^2 = 0$ at the point $Q\,(1, 2, 3)$ on the surface.

21. In each case , find the **Differential** of given function :

(a) $f(x, y) = e^{3x}\cos(2y) + 2x - y + 1$

(b) $g(x, y) = \sin^{-1}(\frac{y}{x})$, $x > 0$.

(c) $F(x, y) = e^{x+2y+3z}$

(d) $G(x, y) = \ln(x^2 + 2y - z)$

22. In each case , find the **Linearization** $\mathbf{L}(x, y)$ of given function at the indicated point :

(a) $f(x, y) = \sqrt{x - 2y + 30}$; $(4, -1)$

(b) $g(x, y) = \ln(x^2 + y^2 + xy)$; $(1, -1)$

(c) $f(x, y, z) = xy + yz + zx$; $(1, 1, 1)$

23. Refer to Question (22)

(i) Use the linearization of part (a) to estimate the value of $\sqrt{35.88} = f(4.12, -0.88)$

(ii) Use the linearization of part (b) to estimate the value of $\ln(1.0819 = f(1.05, -1.03)$

24. Let $f(x, y) = \dfrac{1}{x^2 + 8y}$. Use a suitable linearization to estimate the value of $f(2.9, -0.9)$.

25. The Pressure $\mathbf{P}$, Volume $\mathbf{V}$, and Temperature $\mathbf{T}$ (in $^\circ\mathbf{K}$) of a confined gas are related by the ideal gas law $PV = kT$, where k is a constant. If $P = 0.5\ lb/in^2$ when $v = 64\ in^3$ and $T = 360\ ^\circ K$, determine by approximately what percentage P change if V and T change to $68\ in^3$ and $351\ ^\circ K$ respectively.

26. Refer to problem (25) above. Determine by approximately what percentage the volume change if the Temperature is decreased by 0.8% and the pressure is increased by 0.5% (due to errors in their measurements).

27. The flow of blood in an arteriole is given by $F = \dfrac{\pi P R^4}{8vl}$, where l is the length of the arteriole , R is the radius, P is the pressure difference between the two ends , and v is the viscosity of the blood. Suppose that v and l are constants. Use differentials to determine by approximately what percentage the flow change if the radius is decreased by 2% and the pressure is increased by 3%.

28. In each case , find the directional derivative of the function f at the given point P in the direction specified :

(a) $f(x, y) = \sin(x + 2y)$, $P(0, \frac{\pi}{2})$, $\vec{u} = (-\frac{3}{5}, \frac{4}{5})$.

(b) $f(x, y, z) = e^{x^2 + y - 2z}$, $P(1, 1, 1)$, $\vec{v} = (0, -1, 1)$.

(c) $f(x, y, z) = xy + 2xz + 3yz - 2x - y + 1$, $P(1, 2, -3)$, in the direction from the point P towards the point $Q(0, 0, -1)$.

29. Let $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2)}$, and $P(1, -2, 2\sqrt{5})$.

(i) Find the unit vector $\vec{u}$ for which $D_u f(P)$ is a maximum and give this maximum value.

(ii) Find the unit vector $\vec{v}$ for which $D_v f(P)$ is a minimum and give this minimum value .

30. (a) Assume that the relation $3\, e^{z+2y+1} + \sin(3xyz) = 2$ defines z as a differentiable function of x, y on some domain containing the point $(x,y) = (\frac{\pi}{6}, -1)$. Find $\frac{\partial z}{\partial y}$ at $(x,y,z) = (\frac{\pi}{6}, -1, 1)$.

(b) Assume that the relation $x^2 + 3yz - \dfrac{2}{\ln(x+z)} = 5$ defines x implicitly as a differentiable function of y, z in some domain. Find $\frac{\partial x}{\partial y}$.

31. (i) The relation $x^5 + 2xy^3 + xyz - z^4 = -15$ implicitly defines y as a differentiable function of x ,and z. Find $\frac{\partial y}{\partial z}$ at $(x,z) = (1,2)$. Hint : First , substitute $x = 1$, and $z = 2$ into the equation of the relation to find the $y - coordinate$.

(ii) Given that $x = x(y,z)$ is implicitly defined by $y^2 + y\sqrt{z} = 2 - \sin(xz^2) + \frac{4}{z}$

Compute $\frac{\partial x}{\partial y}$ at the point where $(x,y,z) = (0,1,4)$.

32. Find the Cartesian equation of the plane curve given parametrically by :

$$x(t) = \sin(t) \quad , \quad y(t) = \cos(2t) \quad t \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

Identify the curve and sketch its graph indicating orientation.

33. Find the Cartesian equation of the plane curve given parametrically by :

$$x(t) = 2\cosh^2(t) - 2 \quad , \quad y(t) = 4\sinh(t) \quad t \in \mathbb{R}.$$

Identify the curve and sketch its graph.

34. A rocket moves forward in a straight line by expelling particles of a fuel mixture backward (that is in the opposite direction of motion).Assume the exhaust gases are ejected at a constant rate $1000\ kg/s$ and at constant velocity with magnitude $400\ metre/s$ relative to the rocket. Let M be the total initial mass of rocket and assume it starts motion from rest.

(a) What percentage of the total initial mass M would the rocket have to burn as fuel in order to accelerate to the speed of $800\ metre/s$?

(b) What is the speed of rocket when only 40% of its initial mass remains?

(c) What is the speed of rocket when 40% of its initial mass is ejected during the burn?

You may assume that there are no external forces acting on the rocket as it travels in deep space.