OPTIONAL TOPICS

LAGRANGE MULTIPLIERS

Lagrange Multipliers

In extremum problems, scientists, Engineers, Economists, Enviornmentalists, and other professionals often seek to Maximize or Minimize the values of some function say $f(x,y)$ subject to a given Constraint, say $g(x,y)=0$. That is to find the Extrema of $f(x,y)$ for points (x,y) on the curve given by the equation $g(x,y)=0$.

For instance, an Economist may be interested in the following Problem:

Maximize the function

$$f(x,y) = y^2 + 5xy + 20x \quad \text{-----} \quad (1)$$

subject to the Constraint: $x + y - 60 = 0$ ---(2)

Where:

$f(x,y)$: The Monthly Revenue from the Sale of a Certain Product

x : The amount spent on Newspaper Advertisements (in thousands of Dollars)

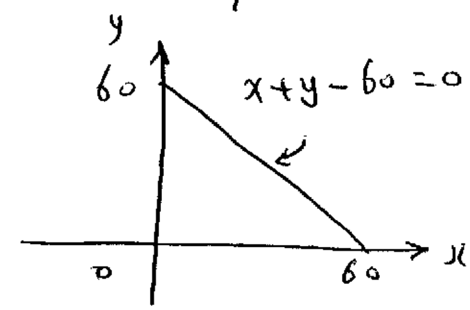
y : The amount spent on Television Commercials (in thousands of Dollars)

The economist can certainly decide how the \$60,000 should be divided.

One way to solve the economist problem is the "Substitution Method".

In this method, we solve the constraint equation

$$x + y - 60 = 0$$



$0 \leq x \leq 60$

for say y. We obtain

$$y = 60 - x$$

Then substituting into

$$f(x,y) = y^2 + 5xy + 20x$$

we obtain a new function of the <u>single</u> variable x, say

$$h(x) = (60-x)^2 + 5x(60-x) + 20x$$

$$= 3600 - 120x + x^2 + 300x - 5x^2 + 20x$$

$$h(x) = -4x^2 + 200x + 3600, \qquad 0 \leq x \leq 60$$

Now, we can determine the absolute maximum of $h(x)$ on the closed interval $[0,60]$ using Calculus I.

<u>Two steps</u>:

1. <u>Critical points</u>

These occur where $h'(x) = 0$, that is where

$$-8x + 200 = 0$$

$$\Rightarrow x = \frac{200}{8} = 25$$

2. Comparison Table

	x	$h(x)$	Conclusion
End points	0	3600	—
c.p →	25	6100	Absolute Max.
End points	60	1200	Absolute Min.

$h(x) = -4x^2 + 200x + 3600$

$h(0) = 3600$

$h(25) = -4(25)^2 + 200(25) + 3600 = 6100$

$h(60) = -4(60)^2 + 200(60) + 3600 = 1200$

Therefore the absolute maximum of Revenue \$6100 K (or \$6,100,000) occur at $x = 25$ K,

hence $y = 60 - x = 60 - 25 = 35$ K

∴ Economist must spend

\$25,000 on Newspapers advertisements, and
\$35,000 on TV Commercials.

Note that, Depending on the nature of the Constraint Equation, it may be difficult or Imposible to use the substitution method discussed above.

Now, we introduce an Alternative method of solving such problem. It is called:

The Method of Lagrange Multipliers

Consider the following Problem:

Maximize/Minimize $f(x,y)$ subject to the Constraint $g(x,y) = 0$

Define the function $L(x,y,\lambda)$ by

$$L(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$$

The function $L(x,y,\lambda)$ is called: Lagrange function and λ (Lambda) is referred to as: Lagrange multiplier.

It will be assumed that $f(x,y)$, $g(x,y)$ are differentiable functions and $\vec{\nabla} g(x,y) \neq \vec{0}$ at points on the curve $g(x,y) = 0$.

Now, let us outline the method of Lagrange Multipliers

<u>Two steps</u>:

1. Find Critical points of Lagrange function $L(x,y,\lambda)$. That is solve the vector eq.

$$\vec{\nabla} L = \vec{0}$$

But $\vec{\nabla} L = \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial \lambda}\right) = (0,0,0)$

Therefore, we need to solve the system

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0 \\ \dfrac{\partial L}{\partial y} = 0 \\ \dfrac{\partial L}{\partial \lambda} = 0 \end{cases}$$

for x, y, and λ simultaneously.

2. For each (x, y, λ) obtained in step (1), compute $f(x, y)$. The largest value is the absolute maximum and the smallest value is the absolute minimum

Remark: The method of Lagrange Multipliers can be generalized in a straightforward way to functions of three or more variables subject to two or more constraints.

EX1: Do the Economist problem using Lagrange Multipliers Technique.

Solution

Problem: Maximize

$$f(x, y) = y^2 + 5xy + 20x,$$

subject to:

$$x + y - 60 = 0$$

Here $f(x, y) = y^2 + 5xy + 20x$,

$g(x, y) = x + y - 60$

$\therefore \quad L(x, y, \lambda) = f(x, y) + \lambda\, g(x, y)$

That is

$$L(x,y,\lambda) = y^2 + 5xy + 20x + \lambda(x+y-60)$$

<u>step 1: Critical points</u>

solve system

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0 \Rightarrow 5y + 20 + \lambda = 0 \quad \text{---} (1) \\ \dfrac{\partial L}{\partial y} = 0 \Rightarrow 2y + 5x + \lambda = 0 \quad \text{--} (2) \\ \dfrac{\partial L}{\partial \lambda} = 0 \Rightarrow x + y - 60 = 0 \quad \text{---} (3) \end{cases}$$

Now, from (1): $\lambda = -5y - 20$

substituting into (2):

$$2y + 5x + (-5y - 20) = 0$$

$$5x - 3y - 20 = 0$$

Next, from (3): $y = 60 - x$. substituting into above Eq. we obtain

$$5x - 3(60 - x) - 20 = 0$$

$$8x - 200 = 0, \quad x = 25$$

$\therefore \quad y = 60 - x = 60 - 25 = 35$

<u>step 2</u>: There is only one Critical point found at

$x = 25, \quad y = 35$

$\therefore f(25,35) = (35)^2 + 5(25)(35) + 20(25)$

$=$

Maximum Revenue \$ 6100K (or \$6,100,000)

EX2: Maximize / Minimize

$f(x,y) = x^2 + y^2$ subject to the Constraint

$17x^2 + 12xy + 8y^2 = 100$

Solution

First rewrite Constraint equation in the form

$$17x^2 + 12xy + 8y^2 - 100 = 0$$

$$\therefore\ g(x,y) = 17x^2 + 12xy + 8y^2 - 100$$

Recall: Lagrange function

$$L(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$$

$$= x^2 + y^2 + \lambda\left(17x^2 + 12xy + 8y^2 - 100\right)$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2x + \lambda(34x + 12y) = 0 \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow 2y + \lambda(12x + 16y) = 0 \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 17x^2 + 12xy + 8y^2 - 100 = 0 \quad \text{-- (3)}$$

From (1): $\lambda(34x + 12y) = -2x$

From (2) $\lambda(12x + 16y) = -2y$

Dividing both sides

$$\frac{\cancel{\lambda}(34x + 12y)}{\cancel{\lambda}(12x + 16y)} = \frac{\cancel{-}2x}{\cancel{-}2y}$$

$$(34x + 12y)y = (12x + 16y)\,x$$

$$34xy + 12y^2 = 12x^2 + 16xy$$

$$\Rightarrow \quad 12x^2 - 18xy - 12y^2 = 0 \qquad (\div 12)$$

$$2x^2 - 3xy - 2y^2 = 0 \quad ---- \quad (4)$$

From (4) : $\quad 2y^2 = 2x^2 - 3xy$

$$\Rightarrow \quad y^2 = x^2 - \frac{3}{2}xy \quad --- \quad (*)$$

substituting into (3):

$$17x^2 + 12xy + 8\left(x^2 - \frac{3}{2}xy\right) - 100 = 0$$

$$17x^2 + \cancel{12xy} + 8x^2 - \cancel{12xy} - 100 = 0$$

$$25x^2 = 100, \quad x^2 = 4$$

$$x = \pm 2$$

If $\boxed{x = 2}$, $(*) \Rightarrow$

$y^2 = 2^2 - \frac{3}{2}\cdot 2y$

$y^2 = 4 - 3y$

$y^2 + 3y - 4 = 0$

$(y+4)(y-1) = 0$

$y = -4, 1$

$\therefore$ There are Two points.

$P(2,-4)$, $Q(2,1)$

If $\boxed{x = -2}$, $(*) \Rightarrow$

$y^2 = (-2)^2 - \frac{3}{2}(-2)y$

$y^2 = 4 + 3y$

$y^2 - 3y - 4 = 0$

$(y-4)(y+1) = 0$

$y = 4, -1$

There are two points

$R(-2,4)$, $S(-2,-1)$

Need to compute the values of $f(x,y) = x^2 + y^2$ at these four points.

$$f(2,-4) = 2^2 + (-4)^2 = 4 + 16 = 20$$
$$f(2,1) = 2^2 + 1^2 = 5$$
$$f(-2,4) = (-2)^2 + (4)^2 = 20$$
$$f(-2,-1) = (-2)^2 + (-1)^2 = 5$$

$\therefore$ Maximum value is 20, and minimum value is 5

<u>Ex3</u>: Read problem (4) in Lab (11)
