MATH 277

Problem Set # 1 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. The position vector of a particle at time t is given by $\vec{r}(t) = (t^2 - 2t)\,\vec{i} + (6t - 10)\,\vec{j} + \sqrt{2}\,t^{3/2}\,\vec{k}$.

   Find the velocity , acceleration , and the speed of the particle at the point $(0, 2, 4)$.

2. Find the Cartesian equation of the line tangent to the plane curve given parametrically

   by the vector function : $\vec{r}(t) = 3\sqrt{2}\,\sin(t)\,\vec{i} + (1 + 2\cos(2t))\,\vec{j}$ at the point on curve

   corresponding to $t = \frac{\pi}{4}$.

3. Find the Cartesian equation of the straight line tangent to the plane curve given parametrically

   by $x(t) = 2t^2 + 12t + 17$, $y(t) = t^3 + 6t^2 + 9t$ at the point $(-1, 0)$ on the curve.

4. Find the arc length of the space curve given by the vector equation :

   $\vec{r}(t) = \frac{1}{3}(2 + t^2)^{\frac{3}{2}}\,\vec{i} + 3t\,\vec{j} + t^2\,\vec{k}$, $0 \le t \le 3.$

5. Find the arc length of the space curve given by the vector equation :

   $\vec{r}(t) = (t^3 - 3t)\,\vec{i} + (t^3 + 3t)\,\vec{j} + 3t^2\,\vec{k}$, $0 \le t \le 3.$

6. Find the parametric equations of the straight line tangent to the space curve :

   $\vec{r}(t) = (3\cos(4\sqrt{t})\ ,\ 2\ ,1 - \sqrt{2}\,\sin(2\sqrt{t}))$ at the point on the curve corresponding to $t = \frac{\pi^2}{64}$.

7* The position vector of a particle at time is given by :

   $\vec{r}(t) = (\frac{1}{4}t^4\ ,\ \frac{2}{5}\sqrt{6}\,t^{\frac{5}{2}}\ ,\ 3t - 1).$

   (a) Determine the speed of the particle at time $t \ge 0.$

   (b) When will the speed of the particle be 67 units?

   (c) Find the arc length of the curve from $t = 0$ to $t = 2.$

8. The position of a particle in space is given by $\vec{r}(t) = t^2\,\vec{i} + \frac{2}{3}t^{\frac{3}{2}}\,\vec{j} + 2t\,\vec{k}.$

   Find the time(s) where the speed of the particle is 3 units.

9. The position vector of a particle at time $t \geq 0$ is given by $\vec{r}(t) = \frac{1}{2}t^2\vec{i} + (2t+1)\vec{j} + 2t\sqrt{t}\,\vec{k}$.

When will the speed of the particle be $2\sqrt{10}$?

10. The acceleration of a moving particle in three space is given by

$\vec{a}(t) = 4t\vec{i} + 6t\vec{j} + \vec{k}$, $t > 0$. Find its velocity and position at time $t > 0$

if its initial velocity and its initial position are respectively given by

$\vec{v}(0) = \vec{i} - \vec{j} + \vec{k}$, $\vec{r}(0) = \vec{i}$.

11. The position of a particle moving in three space is given by

$\vec{r}(t) = 4t\vec{i} + 3\cos(t)\vec{j} + 2\sin(t)\vec{k}$

(a) Find the Maximum and Minimum values of the speed of particle.

(b) Find the Maximum and Minimum values of the magnitude of the particle's acceleration.

12. A particle moves to the right along the curve $y = \frac{3}{x}$ in the $xy-plane$. If its speed is 10 units as it passes the point $(2, \frac{3}{2})$. What is its velocity at that time?

MATH 277

Problem Set # (1) for Labs

1. The position vector of a particle at time t is given by $\vec{r}(t) = (t^2 - 2t)\vec{i} + (6t-10)\vec{j} + \sqrt{2}\, t^{\frac{3}{2}} \vec{k}$.
Find the velocity, acceleration, and the speed of the particle at the point $(0, 2, 4)$.

<u>solution</u> : Let us first find the value(s) of t corresponding to the point $(0, 2, 4)$.

Here $x = t^2 - 2t = 0 \Rightarrow t = 2, 0$

$y = 6t - 10 = 2 \Rightarrow t = 2$, and

$z = \sqrt{2}\, t^{\frac{3}{2}} = 4 \Rightarrow 2t^3 = 16 \Rightarrow t^3 = 8 \Rightarrow t = 2$

$\therefore \quad t = 2$ (the common value!)

Now, $\vec{r}(t) = (t^2 - 2t,\ 6t - 10,\ \sqrt{2}\, t^{\frac{3}{2}})$

velocity $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = (2t - 2,\ 6,\ \frac{3}{2}\sqrt{2}\, t^{\frac{1}{2}})$

acceleration $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \left(2,\ 0,\ \dfrac{3\sqrt{2}}{4\sqrt{t}}\right)$

At $t = 2$, $\vec{v}(2) = (2, 6, 3)$,

$\vec{a}(2) = (2, 0, \frac{3}{4})$.

speed $S = \|\vec{v}(2)\| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{4 + 36 + 9}$

$= \sqrt{49} = 7$

2. Find the Cartesian equation of the line tangent to the plane curve given parametrically by the vector function $\vec{r}(t) = 3\sqrt{2}\,\sin(t)\,\vec{i} + (1+2\cos(2t))\,\vec{j}$ at the point on curve corresponding to $t=\frac{\pi}{4}$.

<u>Solution</u>: Here $x(t) = 3\sqrt{2}\sin(t)$, $x'(t) = 3\sqrt{2}\cos(t)$

$y(t) = 1+2\cos(2t)$, $\therefore y'(t) = -4\sin(2t)$

At $t=\frac{\pi}{4}$,

$$x = 3\sqrt{2}\sin\left(\frac{\pi}{4}\right) = 3\sqrt{2}\cdot\frac{1}{\sqrt{2}} = 3,$$

$$y = 1+2\cos\left(2\cdot\frac{\pi}{4}\right) = 1+2\cos\left(\frac{\pi}{2}\right)$$

$$= 1+0 = 1$$

The point on curve is $P(3,1)$

Next At $t=\frac{\pi}{4}$, $x' = 3\sqrt{2}\cos\left(\frac{\pi}{4}\right) = 3\sqrt{2}\cdot\frac{1}{\sqrt{2}} = 3,$

$$y' = -4\sin\left(2\cdot\frac{\pi}{4}\right) = -4\sin\left(\frac{\pi}{2}\right)$$

$$= -4\cdot 1 = -4$$

$\therefore$ slope of tangent line is thus given by

$$m = \frac{y'}{x'} = \frac{-4}{3}.$$

Equation of tangent line:

$$y-1 = -\frac{4}{3}(x-3)$$

$$\Rightarrow y = -\frac{4}{3}x + 5$$

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3. Find the Cartesian equation of the straight line tangent to the plane curve given parametrically by $x(t) = 2t^2 + 12t + 17$, $y(t) = t^3 + 6t^2 + 9t$ at the point $(-1, 0)$ on the curve.

<u>Solution</u> :

First, let us find the value of the parameter "t" corresponding to the point $(x, y) = (-1, 0)$.

$$x = 2t^2 + 12t + 17 = -1 \Rightarrow 2t^2 + 12t + 18 = 0$$
$$\Rightarrow t^2 + 6t + 9 = 0 \Rightarrow (t+3)^2 = 0 \Rightarrow \underline{t = -3}$$

Next, $y = t^3 + 6t^2 + 9t = 0 \Rightarrow t(t^2 + 6t + 9) = 0$
$$\Rightarrow t(t+3)^2 = 0 \Rightarrow t = 0, \underline{\underline{-3}}$$

The common value is $t = -3$.

Now, slope of tangent line

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 + 12t + 9}{4t + 12}$$

At $t = -3$; $\quad m = \dfrac{3(-3)^2 + 12(-3) + 9}{4(-3) + 12} = \dfrac{0}{0}$

$\therefore$ Use L'Hôpital's Rule,

$$m = \lim_{t \to -3} \frac{3t^2 + 12t + 9}{4t + 12} = \lim_{t \to -3} \frac{6t + 12}{4} = \frac{6(-3) + 12}{4} = -\frac{3}{2}$$

Equation of tangent line : $y - 0 = -\frac{3}{2}(x - (-1))$
$$\Rightarrow y = -\frac{3}{2}(x+1)$$

4. Find the arc length of the space curve given by the vector equation

$$\vec{r}(t) = \frac{1}{3}(2+t^2)^{\frac{3}{2}}\vec{i} + 3t\vec{j} + t^2\vec{k} \quad , \quad 0 \le t \le 3$$

<u>Solution</u>:

Velocity $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\frac{1}{3}(2+t^2)^{\frac{3}{2}}, 3t, t^2\right)$

$$= \left(\frac{1}{3}\cdot\frac{3}{2}(2+t^2)^{\frac{1}{2}}\cdot 2t,\ 3,\ 2t\right)$$

$$= \left(t\sqrt{2+t^2},\ 3,\ 2t\right)$$

$$\therefore \|\vec{v}(t)\| = \sqrt{\left(t\sqrt{2+t^2}\right)^2 + 3^2 + (2t)^2}$$

$$= \sqrt{t^2(2+t^2) + 9 + 4t^2}$$

$$= \sqrt{2t^2 + t^4 + 9 + 4t^2}$$

$$= \sqrt{t^4 + 6t^2 + 9} = \sqrt{(t^2+3)^2}$$

$$= |t^2+3| = t^2 + 3$$

$\therefore$ Arc length $L = \int_a^b \|\vec{v}(t)\|\,dt$

$$= \int_0^3 (t^2+3)\,dt = \frac{1}{3}t^3 + 3t\Big|_0^3$$

$$= \left(\frac{1}{3}(3)^3 + 3(3)\right) - (0+0)$$

$$= 9 + 9 = 18$$

5. Find the arc length of the space curve given by the vector equation

$$\vec{r}(t) = (t^3-3t)\vec{i} + (t^3+3t)\vec{j} + 3t^2\vec{k}, \quad 0 \leq t \leq 3$$

<u>Solution</u>:

Velocity $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t^3-3t,\ t^3+3t\ \ 3t^2\right)$

$$= \left(3t^2-3,\ 3t^2+3,\ 6t\right)$$

$$= 3\left(t^2-1,\ t^2+1,\ 2t\right)$$

$$\|\vec{v}(t)\| = 3\sqrt{(t^2-1)^2+(t^2+1)^2+(2t)^2}$$

$$= 3\sqrt{t^4-\cancel{2t^2}+1+t^4+\cancel{2t^2}+1+4t^2}$$

$$= 3\sqrt{2t^4+4t^2+2}$$

$$= 3\sqrt{2(t^4+2t^2+1)} = 3\sqrt{2(t^2+1)^2}$$

$$= 3\sqrt{2}\,|t^2+1| = 3\sqrt{2}\,(t^2+1)$$

Arc length $L = \int_a^b \|\vec{v}(t)\|\,dt$

$$= \int_0^3 3\sqrt{2}\,(t^2+1)\,dt = 3\sqrt{2}\left(\frac{1}{3}t^3+t\right)\Big|_0^3$$

$$= 3\sqrt{2}\left[\left(\frac{1}{3}(3)^3+3\right)-(0+0)\right] = 3\sqrt{2}\,(9+3)$$

$$= 36\sqrt{2}$$

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6. Find parametric equations of the straight line tangent to the space curve

$$\vec{r}(t) = \left(3\cos(4\sqrt{t}),\ 2,\ 1-\sqrt{2}\,\sin(2\sqrt{t})\right)$$

at the point on curve corresponding to $t = \frac{\pi^2}{64}$.

<u>Solution</u>: At $t = \frac{\pi^2}{64}$,

$$\vec{r} = (x, y, z) = \left(3\cos\left(\tfrac{\pi}{2}\right),\ 2,\ 1-\sqrt{2}\sin\left(\tfrac{\pi}{4}\right)\right)$$

$$= \left(0,\ 2,\ 1-\sqrt{2}\cdot\tfrac{1}{\sqrt{2}}\right) = (0, 2, 0)$$

$$= \text{say } \vec{r_0}$$

Next. velocity vector

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(-\frac{6}{\sqrt{t}}\sin(4\sqrt{t}),\ 0,\ -\frac{\sqrt{2}}{t}\cos(2\sqrt{t})\right)$$

At $t = \frac{\pi^2}{64}$,

$$\vec{v} = \left(-\frac{48}{\pi}\sin\left(\tfrac{\pi}{2}\right),\ 0,\ -\frac{8\sqrt{2}}{\pi}\cdot\cos\left(\tfrac{\pi}{4}\right)\right)$$

$$= \left(-\frac{48}{\pi}\cdot 1,\ 0,\ -\frac{8\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{2}}\right)$$

$$= \left(-\frac{48}{\pi},\ 0,\ -\frac{8}{\pi}\right)$$

$$= -\frac{8}{\pi}(6, 0, 1),$$ so direction is $(6,0,1)$

parametric equations of tangent line are thus given by

$$\begin{aligned} x &= 0 + 6t_1 \\ y &= 2 + 0t_1, \quad t_1 \in \mathbb{R} \\ z &= 0 + 1t_1 \end{aligned}$$

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7. For students to do at Home.

<u>Answer</u>:

(a) Speed $v = t^3 + 3$

(b) $t = 4$

(c) Arc length $= 10$

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8. The position of a particle in space is given by

$$\vec{r}(t) = t^2\,\vec{i} + \frac{2}{3}t^{\frac{3}{2}}\,\vec{j} + 2t\,\vec{k}$$

Find the time(s) where the speed of the particle is 3 units.

<u>Solution</u>: write

$$\vec{r}(t) = \left(t^2, \frac{2}{3}t^{\frac{3}{2}}, 2t\right)$$

$\therefore$ velocity $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \left(2t, \frac{2}{3}\cdot\frac{3}{2}t^{\frac{1}{2}}, 2\right)$

$$= (2t, \sqrt{t}, 2) \quad \cdots (*)$$

speed $v = \|\vec{v}(t)\| = \sqrt{(2t)^2 + (\sqrt{t})^2 + 2^2}$

$$= \sqrt{4t^2 + t + 4}$$

But speed $= 3$, hence

$$3 = \sqrt{4t^2 + t + 4}$$

Squaring, we have

$$9 = 4t^2 + t + 4 \Rightarrow$$

$$4t^2 + t - 5 = 0$$

$$(4t+5)(t-1) = 0 \Rightarrow t = -\frac{5}{4} \text{ or } t = 1$$

But from $(*)$, Domain: $t \geq 0 \therefore$ reject $t = -\frac{5}{4}$

$\therefore$ Required Time is $t = 1$ unit

9. $\vec{r}(t) = \left(\frac{1}{2}t^2, 2t+1, 2t\sqrt{t}\right), \quad t \geq 0$

$= \left(\frac{1}{2}t^2, 2t+1, 2t^{\frac{3}{2}}\right)$

$\therefore$ velocity $\vec{V}(t) = \frac{d\vec{r}}{dt} = \left(t, 2, 3t^{\frac{1}{2}}\right)$

speed $V = \|\vec{V}(t)\| = \sqrt{t^2 + 2^2 + (3t^{\frac{1}{2}})^2}$

$V = \sqrt{t^2 + 4 + 9t}$

But speed $V = 2\sqrt{10}$, hence

$2\sqrt{10} = \sqrt{t^2 + 4 + 9t}$

squaring each side:

$(2\sqrt{10})^2 = \left(\sqrt{t^2 + 4 + 9t}\right)^2$

$(4)(10) = t^2 + 4 + 9t$

$t^2 + 9t - 36 = 0$

$(t+12)(t-3) = 0$

$t = -12$ .. Reject, Since $t \geq 0$,

or $t = 3$

$\therefore$ Required time is $t = 3$ units

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10. The acceleration of a moving particle in three space is given by $\vec{a}(t) = 4t\vec{i} + 6t\vec{j} + \vec{K}$, $t>0$. Find its velocity and position at time $t>0$ if its initial velocity and its initial position are respectively given by $\vec{v}(0) = \vec{i} - \vec{j} + \vec{K}$, $\vec{r}(0) = \vec{i}$

<u>Solution</u>: $\vec{a}(t) = (4t, 6t, 1)$

Recall $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = (4t, 6t, 1)$

$$\therefore \vec{v}(t) = \int (4t, 6t, 1)\,dt + \text{Constant } \vec{C}$$

$$= \left(\int 4t\,dt, \int 6t\,dt, \int dt\right) + \vec{C}$$

$$\vec{v}(t) = (2t^2, 3t^2, t) + \vec{C}$$

Applying $\vec{v}(0) = \vec{i} - \vec{j} + \vec{K} \overset{\text{or}}{=} (1,-1,1)$, we have

$$(1,-1,1) = (0,0,0) + \vec{C} = \vec{0} + \vec{C}$$

$$\Rightarrow \vec{C} = (1,-1,1)$$

$$\therefore \vec{v}(t) = (2t^2, 3t^2, t) + (1,-1,1)$$

$$= (2t^2+1, 3t^2-1, t+1)$$

$$\overset{\text{or}}{=} (2t^2+1)\vec{i} + (3t^2-1)\vec{j} + (t+1)\vec{K}$$

Next,

$$\frac{d\vec{r}}{dt} = \vec{v}(t) = (2t^2+1, 3t^2-1, t+1)$$

$$\therefore \vec{r}(t) = \int (2t^2+1, 3t^2-1, t+1)\,dt + \vec{d}$$

$$= \left(\frac{2}{3}t^3 + t, t^3 - t, \frac{1}{2}t^2 + t\right) + \vec{d}$$

Applying $\vec{r}(0) = \vec{i} = (1,0,0)$, we have

$$(1,0,0) = (0,0,0) + \vec{d}$$

$$\Rightarrow \vec{d} = (1,0,0)$$

$$\therefore \vec{r}(t) = \left(\frac{2}{3}t^3 + t, t^3 - t, \frac{1}{2}t^2 + t\right) + (1,0,0)$$

$$= \left(\frac{2}{3}t^3 + t + 1\right)\vec{i} + (t^3 - t)\vec{j} + \left(\frac{1}{2}t^2 + t\right)\vec{k}$$

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11. The position of a particle moving in 3-space is given by $\vec{r}(t) = 4t\vec{i} + 3\cos(t)\vec{j} + 2\sin(t)\vec{k}$

(a) Find the Maximum and Minimum values of the speed of the particle.

(b) Find the Maximum and Minimum Values of the magnitude of the particle's acceleration.

<u>Solution</u> $\vec{r}(t) = (4t, 3\cos(t), 2\sin(t))$

velocity $\vec{v}(t) = \frac{d\vec{r}}{dt} = (4, -3\sin(t), 2\cos(t))$

Acceleration $\vec{a}(t) = \frac{d\vec{v}}{dt} = (0, -3\cos(t), -2\sin(t))$

$\therefore$ speed $v = \|\vec{v}(t)\| = \sqrt{4^2 + (-3\sin(t))^2 + (2\cos(t))^2}$

$$= \sqrt{16 + 9\sin^2(t) + 4\cos^2(t)}$$

But $\cos^2(t) = 1 - \sin^2(t)$

$\therefore$ $v = \sqrt{16 + 9\sin^2(t) + 4(1 - \sin^2(t))}$

$$= \sqrt{20 + 5\sin^2(t)}$$

Recall: Minimum value of $\sin^2(t)$ is Zero

Maximum value of $\sin^2(t)$ is One

$\therefore$ Minimum speed $= \sqrt{20 + 5(0)} = \sqrt{20} = 2\sqrt{5}$ units

Maximum speed $= \sqrt{20 + 5(1)} = \sqrt{25} = 5$ units

Next, let "a" be the magnitude of $\vec{a}$

$$\therefore \quad a = \|\vec{a}\| = \|(0, -3\cos(t), -2\sin(t))\|$$

$$= \sqrt{0^2 + (-3\cos(t))^2 + (-2\sin(t))^2}$$

$$= \sqrt{9\cos^2(t) + 4\sin^2(t)}$$

Recall $\sin^2(t) = 1 - \cos^2(t)$

$$\therefore \quad a = \sqrt{9\cos^2(t) + 4(1 - \cos^2(t))}$$

$$= \sqrt{5\cos^2(t) + 4}$$

But Minimum value of $\cos^2(t)$ is zero

$$\therefore \quad a_{min.} = \sqrt{5(0) + 4} = 2 \quad \text{units}$$

and Maximum value of $\cos^2(t)$ is one

$$\therefore \quad a_{Max.} = \sqrt{5(1) + 4} = \sqrt{9} = 3 \quad \text{units}$$

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12. A particle moves to the <u>right</u> along the curve $y = \frac{3}{x}$ in the xy-plane. If its speed is 10 units as it passes the point $(2, \frac{3}{2})$. What is its velocity at that time.

<u>Solution</u>:

position $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

Here $y = \frac{3}{x}$

$$\therefore \quad \vec{r}(t) = \left(x(t), \frac{3}{x(t)} \right)$$

velocity $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x, \frac{3}{x} \right)$ ← Apply chain Rule.

$$= \frac{d}{dx}\left(x, \frac{3}{x} \right) \cdot \frac{dx}{dt}$$

$$= \left(1, -\frac{3}{x^2} \right) \frac{dx}{dt} \quad \text{--- } (*)$$

speed $v = \|\vec{v}\| = \left|\frac{dx}{dt}\right| \sqrt{1^2 + \left(-\frac{3}{x^2}\right)^2}$

$$= \left|\frac{dx}{dt}\right| \sqrt{1 + \frac{9}{x^4}}$$

But particle moves to the right meaning "x" increases as "t" increase, hence

$$\frac{dx}{dt} > 0$$

$$\Rightarrow \left|\frac{dx}{dt}\right| = \frac{dx}{dt}$$

$\therefore \quad v = \frac{dx}{dt}\sqrt{1+\frac{9}{x^4}}$

But $v = 10$ at $x = 2$

$\therefore \quad 10 = \frac{dx}{dt}\sqrt{1+\frac{9}{2^4}}$

$10 = \frac{dx}{dt}\sqrt{1+\frac{9}{16}} = \frac{dx}{dt}\sqrt{\frac{25}{16}}$

$10 = \frac{5}{4}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{(10)(4)}{5} = 8$

Substituting $\frac{dx}{dt} = 8$ into $(*)$:

Velocity

$\vec{V} = \left(1, -\frac{3}{x^2}\right)\frac{dx}{dt}\Big|_{x=2}$

$= \left(1, -\frac{3}{2^2}\right)(8)$

$= (8, -6)$

or $\vec{V} = 8\vec{i} - 6\vec{j}$
