MATH 277

Problem Set # 4 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. In each case , find the **Domain** of f and **sketch** :

   (a) $f(x,y) = \dfrac{1}{x^2 - y}$ (b) $f(x,y) = \ln(2x - y)$

   (c) $f(x,y) = \sqrt[7]{x^2 + y^2 - 1}$ (d) $f(x,y) = \sqrt{\ln(2x - y)}$

2. Draw a **Contour Map with Four Level Curves** using positive , negative or zero values of $\mathbf{c}$ where appropriate!

   (a) $f(x,y) = y - \ln(x)$ (b) $f(x,y) = y\, e^{-x}$

   (c) $f(x,y) = y - \cos(x)$ (d) $f(x,y) = \dfrac{1}{x^2 - y^2}$

3. In each case determine $f_x(x,y)$ and $f_y(x,y)$ at the indicated point :

   (a) $f\,(x,y) = \ln\left(\dfrac{y^x}{x^y}\right)$; $(x,y) = (e^2, 1)$

   $(b)^*$ $f\,(x,y) = e^{x^2 - y^2}$; $(x,y) = (2, -2)$

   $(c)^*$ $f\,(x,y) = \dfrac{xy}{\sqrt{x^2 + y - 12}}$; $(x,y) = (3, 4)$

   (d) $f\,(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$; $(x,y) = (1, 1)$

4. Show that the function $u(x,t) = \tan^{-1}(3 - \dfrac{\sqrt{x}}{t^2})$ satisfies the equation

   $4x\dfrac{\partial u}{\partial x} + t\dfrac{\partial u}{\partial t} = 0.$

5. Find the slope of the line tangent to the curve of intersection of the surface $z = \sin(x + 2y^3)$ and the plane $x = -2$ at the point $(-2, 1, 0)$ on the surface.

6. Find all second order partial derivatives of :

   (a) $f(x,y) = x^2\, e^{xy}$ (b) $W = \ln(x^3 y^5 z^6)$

7*. (a) If $z = y^x$, find $\dfrac{\partial^2 z}{\partial x \partial y}$

(b) If $z = \dfrac{(x+y)^{y+1}}{y+1} + \cosh^3(y) + 2$, find $\dfrac{\partial^2 z}{\partial y \partial x}$

8. Find constant real numbers A , and B such that the function

$w(x,y) = x^5 + Ax^3y^2 + Bxy^4$ satisfies *Laplace Equation* : $w_{xx} + w_{yy} = 0$ in $\mathbb{R}^2$.

8. Show that the function $W(x,y,z) = e^{3x+4y}\sin(5z)$ is **Harmonic** in $\mathbb{R}^3$, that is it satisfies

Laplace Equation $W_{xx} + W_{yy} + W_{zz} = 0.$

10. In each case, answer **True** or **False**. If statement is **False**, write a **Correction**.

(a) $-2x^2 - 4y^2 + 3z^2 = -24$ is an equation of a **Hyperboloid of Two Sheets**.

(b) $z = \sqrt{16 - x^2 - y^2}$ is an equation of a **Cone**.

(c) $x = \sqrt{y^2 + z^2 + 1}$ is an equation of a **Cone**.

(d) $y^2 = z$ is an equation of a **Parabola** in $\mathbb{R}^3$.

(e) $x + 3y = 0$ is an equation of a **Straight Line** in $\mathbb{R}^3$.

(f) $3y^2 = x^2 - 4z^2$ is an equation of a **Hyperboloid of One Sheet**.

(g) $x^2 + y^2 = 1$ is an equation of a **Circle** in $\mathbb{R}^3$.

(h) $z = x^2 + y^2 + z^2$ is an equation of a **Paraboloid**.

(i) $y = 3 - x^2 - z^2$ is an equation of a **Paraboloid**

(j) $2x^2 + 4y^2 + 15z^2 = 120$ is an equation of an **Ellipsoid**.

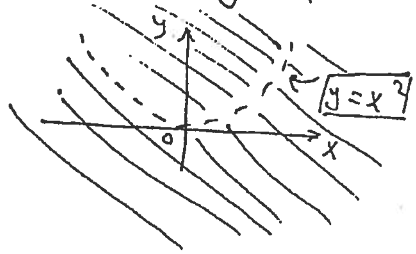
Solutions

1. (a) $f(x,y) = \dfrac{1}{x^2 - y}$

Domain f consists of all ordered pairs (x,y) except where $x^2 - y = 0 \Rightarrow y = x^2$

In other words domain f consists of all points (x,y) in $\mathbb{R}^2$ except those which lie on the parabola $y = x^2$

$\therefore$ Domain $f = \{(x,y) \in \mathbb{R}^2 : y \neq x^2\}$



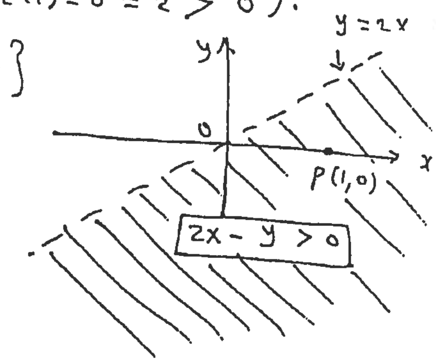
(b) $f(x,y) = \ln(2x - y)$

f is defined and is real provided $2x - y > 0$ ---- (1)

First sketch the st. line $2x - y = 0$ (or $y = 2x$) and use a test point to determine whether the region described by (1) is above or below the line!

(Use say $P(1,0)$ for Test : $2x - y = 2(1) - 0 = 2 > 0$).

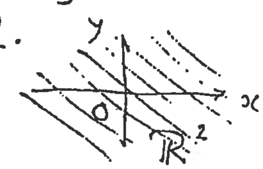
$\therefore$ Domain $f = \{(x,y) \in \mathbb{R}^2 : 2x - y > 0\}$



(c) $f(x,y) = \sqrt[7]{x^2 + y^2 - 1}$

First note that if n is an odd positive integer, then $\sqrt[n]{a}$ is <u>defined</u> and is <u>real</u> for all $a \in \mathbb{R}$.

$\therefore$ Domain f is all of $\mathbb{R}^2$!

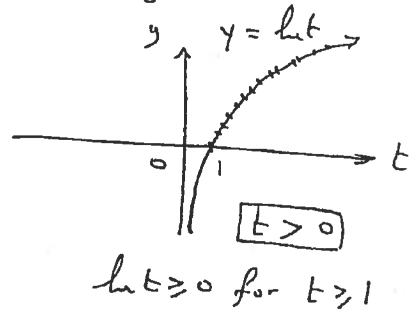


(d) $f(x,y) = \sqrt{\ln(2x-y)}$

Two restrictions need to be made: Refer to figure below

<u>First</u>: $\ln(2x-y)$ is defined and is real provided

$$2x - y > 0 \quad \text{--- (1)}$$



Next, $\sqrt{a}$ is defined and is real provided $a \geq 0$.

putting $a = \ln(2x-y)$, we have

$$\ln(2x-y) \geq 0$$

$$\Rightarrow \quad 2x - y \geq 1 \quad \text{--- (2)}$$

Clearly condition (2) alone is sufficient!

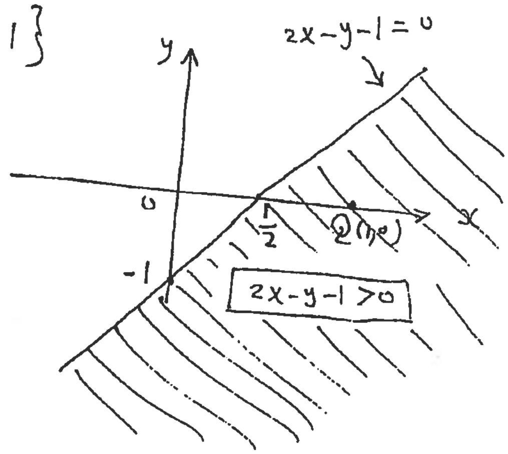
$\therefore$ Domain $f = \{(x,y) \in \mathbb{R}^2 : 2x - y \geq 1\}$

sketch line $2x - y - 1 = 0$:

x	0	$\frac{1}{2}$
y	-1	0

$(0,-1)$ $(\frac{1}{2}, 0)$

use $Q(1,0)$ for Testing: $2x - y - 1 = 2(1) - 0 - 1$
$= 1 > 0$ ✓



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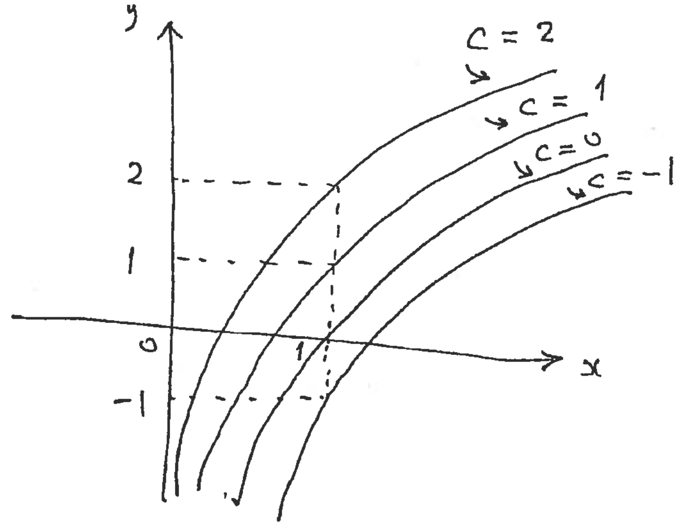
2. (a) $f(x,y) = y - \ln(x)$. Level curves are given by

$$y - \ln(x) = c, \quad c \in \mathbb{R}$$

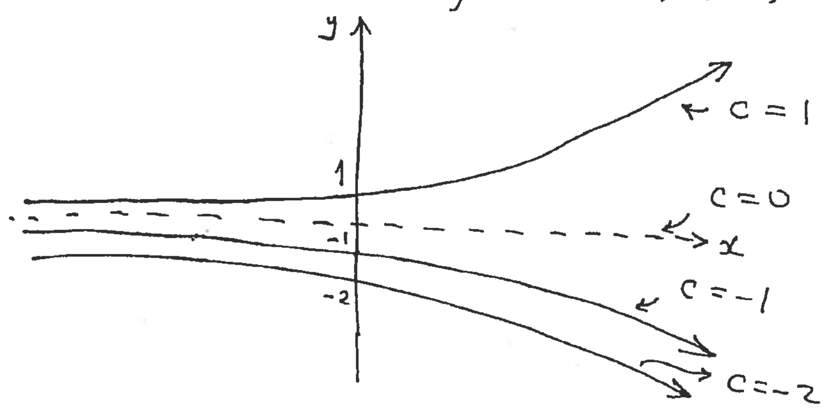
$$\Rightarrow y = c + \ln(x)$$

let us sketch level curves for $c = 0, 1, -1, 2$.



(b) $f(x,y) = y e^{-x}$. Level curves are given by $f(x,y) = c$,
that is $y \cdot e^{-x} = c \Rightarrow y = c e^{x}$, $c \in \mathbb{R}$

let us sketch level curves for $c = 0, 1, -1, -2$.

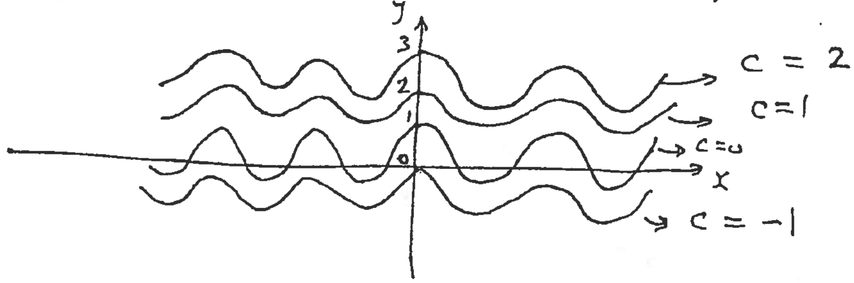


(c) $f(x,y) = y - \cos(x)$. Level curves are given by

$$y - \cos(x) = c \quad , \quad c \in \mathbb{R}$$

$$\Rightarrow \quad y = c + \cos(x) \quad , \quad c \in \mathbb{R}$$

let us sketch level curves for $c = 0, \pm 1$, ad 2.



3. (a) $f(x,y) = \ln\left(\dfrac{y^x}{x^y}\right)$ .. Simplify first

$$= \ln y^x - \ln x^y$$

$$= x \ln y - y \ln x$$

$$\therefore \frac{\partial f}{\partial x} = \ln(y) - \frac{y}{x},$$

$$\frac{\partial f}{\partial y} = \frac{x}{y} - \ln(x)$$

At $(x,y) = (e^2, 1)$, we have

$$f_x(e^2,1) = \ln 1 - \frac{1}{e^2} = 0 - \frac{1}{e^2} = -\frac{1}{e^2},$$

$$f_y(e^2,1) = \frac{e^2}{1} - \ln e^2 = e^2 - 2\ln e = e^2 - 2$$

(b) For students to do at home.

$$f_x(2,-2) = 4 \qquad , f_y(2,-2) = 4$$

(c) For students to do at home.

$$f_x(3,4) = -32, \quad f_y(3,4) = -3$$

(d) $f(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right), \quad \frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right)$$

$$= \frac{-y}{x^2+y^2} \qquad = \frac{x}{x^2+y^2}$$

$$\therefore f_x(1,1) = -\frac{1}{2}, \quad f_y(1,1) = \frac{1}{2}$$

4. We need to calculate $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial t}$!

Now, $u(x,y) = \tan^{-1}\left(3 - \sqrt{x}\, t^{-2}\right)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{1+\left(3-\sqrt{x}\,t^{-2}\right)^2} \cdot \left(0 - \frac{t^{-2}}{2\sqrt{x}}\right)$$

Multiplying each side by $4x$ we get:

$$4x\frac{\partial u}{\partial x} = -\frac{2\sqrt{x}\,t^{-2}}{1+\left(3-\sqrt{x}\,t^{-2}\right)^2} \quad \text{-----} (1)$$

Next, $$\frac{\partial u}{\partial t} = \frac{1}{1+\left(3-\sqrt{x}\,t^{-2}\right)^2}\left(0+2\sqrt{x}\,t^{-3}\right)$$

Multiplying each side by t we get,

$$t\frac{\partial u}{\partial t} = \frac{2\sqrt{x}\,t^{-2}}{1+\left(3-\sqrt{x}\,t^{-2}\right)^2} \quad \text{-----} (2)$$

adding both sides of (1), (2):

$$4x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = \frac{-2\sqrt{x}\,t^{-2}}{1+\left(3-\sqrt{x}\,t^{-2}\right)^2} + \frac{2\sqrt{x}\,t^{-2}}{1+\left(3-\sqrt{x}\,t^{-2}\right)^2}$$

$= \text{zero}!$ .. Verified!

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5. slope $m = \frac{\partial f}{\partial y}(-2,1)$

Here $z = f(x,y) = \sin\left(x+2y^3\right)$

$$\therefore \frac{\partial f}{\partial y} = 6y^2\cos\left(x+2y^3\right)$$

$$\left.\frac{\partial f}{\partial y}\right|_{x=-2,\,y=1} = 6(1)^2\cos\left(-2+2(1)\right) = 6\cos(0) = 6$$

$\therefore$ slope $m = 6$

6. (a) $f(x,y) = x^2 e^{xy}$

$$\frac{\partial f}{\partial x} = 2x e^{xy} + x^2 y e^{xy} = e^{xy}(2x + x^2 y),$$

$$\frac{\partial f}{\partial y} = x^2 \cdot x e^{xy} = x^3 e^{xy}$$

$$\frac{\partial^2 f}{\partial x^2} = y e^{xy}(2x + x^2 y) + e^{xy}(2 + 2xy) = e^{xy}(x^2 y^2 + 4xy + 2)$$

$$\frac{\partial^2 f}{\partial y^2} = x^4 e^{xy},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^3 e^{xy}) = 3x^2 e^{xy} + x^3 \cdot y e^{xy}$$

$$= x^2 e^{xy}(xy + 3), \text{ and}$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left[e^{xy}(2x + x^2 y)\right]$$

$$= x e^{xy}(2x + x^2 y) + e^{xy} x^2 = x^2 e^{xy}(xy + 3)$$

<u>Note</u>: $f_{xy}(x,y) = f_{yx}(x,y)$ as expected.

(b) $W = \ln(x^3 y^5 z^6) = \ln x^3 + \ln y^5 + \ln z^6 = 3\ln x + 5\ln y + 6\ln z$

$W_x = \frac{3}{x}$, $W_y = \frac{5}{y}$, and $W_z = \frac{6}{z}$

$\therefore W_{xx} = -\frac{3}{x^2}$, $W_{yy} = -\frac{5}{y^2}$, $W_{zz} = -\frac{6}{z^2}$, $W_{xy} = W_{yx} = 0$,

$W_{yz} = W_{zy} = 0$, and $W_{xz} = W_{zx} = 0$.

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7. For students to do at home.

<u>Answer</u>:

(a) $$\frac{\partial^2 z}{\partial x \partial y} = y^{x-1}[x \ln(y) + 1]$$

(b) $$\frac{\partial^2 z}{\partial y \partial x} = \left[\frac{y}{x+y} + \ln(x+y)\right](x+y)^y$$

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8. $W_{xx} + W_{yy} = 0$ ---- (*)

Here $W(x,y) = x^5 + Ax^3y^2 + Bxy^4$

$\therefore W_x = 5x^4 + 3Ax^2y^2 + By^4$

$W_{xx} = 20x^3 + 6Axy^2 + 0 = 20x^3 + 6Axy^2$ ... (1)

Likewise $W_y = 0 + 2Ax^3y + 4Bxy^3$

$W_{yy} = 2Ax^3 + 12Bxy^2$ --- (2)

substituting (1), (2) into (*)

$(20x^3 + 6Axy^2) + (2Ax^3 + 12Bxy^2) = 0$

$\therefore (20 + 2A)x^3 + (6A + 12B)xy^2 = 0$

Equating Coefficients to zero:

$20 + 2A = 0 \Rightarrow 2A = -20 \Rightarrow \boxed{A = -10}$

and $6A + 12B = 0 \Rightarrow 12B = -6A \Rightarrow 2B = -A$

But $A = -10$,

$\therefore 2B = -(-10)$

$= 10$

$\therefore \boxed{B = 5}$

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9. $W(x,y,z) = e^{3x+4y} \sin(5z)$

$W_x = 3e^{3x+4y} \sin(5z)$,

$W_{xx} = 9e^{3x+4y} \sin(5z)$ or $\boxed{W_{xx} = 9W}$ ... (1)

Next, $W_y = 4e^{3x+4y} \sin(5z)$

$W_{yy} = 16e^{3x+4y} \sin(5z)$ or $\boxed{W_{yy} = 16W}$ ... (2)

Finally, $W_z = 5e^{3x+4y} \cdot \cos(5z)$

$W_{zz} = -25e^{3x+4y} \sin(5z)$, or

$\boxed{W_{zz} = -25W}$ ... (3)

Adding both sides of (1), (2), and (3), we get:

$W_{xx} + W_{yy} + W_{zz} = 9W + 16W - 25W$

$= 0$ .. Verified!

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10. In each Case, answer <u>True</u> or <u>False</u>. If statement is false, write a Correction

(a) $-2x^2-4y^2+3z^2=-24$ is an equation of a Hyperboloid of two sheets.

(b) $z=\sqrt{16-x^2-y^2}$ is an equation of a Cone.

(c) $x=\sqrt{y^2+z^2+1}$ is an equation of a Cone.

(d) $y^2=z$ is an equation of a parabola in $\mathbb{R}^3$.

(e) $x+3y=0$ is an equation of a st. Line in $\mathbb{R}^3$

(f) $3y^2=x^2-4z^2$ is an equation of a Hyperboloid of one sheet.

(g) $x^2+y^2=1$ is an equation of a Circle in $\mathbb{R}^3$

(h) $z=x^2+y^2+z^2$ is an equation of a paraboloid.

(i) $y=3-x^2-z^2$ is an Equation of a paraboloid

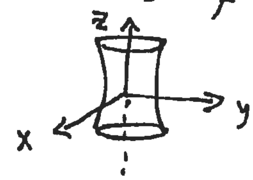
(j) $2x^2+4y^2+15z^2=120$ is an Equation of an Ellipsoid.

Solution :

(a) $-2x^2 - 4y^2 + 3z^2 = -24$ $\quad (\div -24)$

$$\frac{x^2}{12} + \frac{y^2}{6} - \frac{z^2}{8} = 1$$

This is an equation of a Hyperboloid of one sheet with centre at $(0,0,0)$ and axis of symmetry being the z-axis.

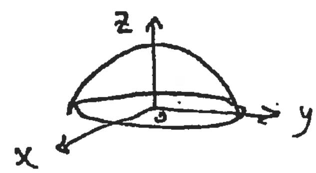


Answer: False!

(b) $z = \sqrt{16 - x^2 - y^2}$ ... $(*)$

$\Rightarrow z^2 = 16 - x^2 - y^2$

$\Rightarrow x^2 + y^2 + z^2 = 16$



This is an equation of a sphere with centre $(0,0,0)$, and radius 4. However $(*)$ is an equation of the Upper hemi-sphere (since $z \geq 0$).

Answer: False!

(c) $x = \sqrt{y^2 + z^2 + 1}$

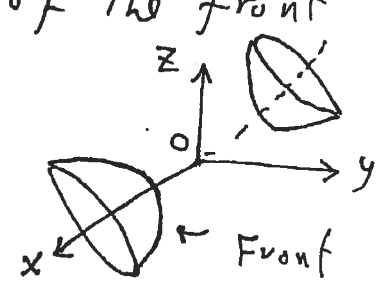
$\Rightarrow x^2 = y^2 + z^2 + 1$

$\Rightarrow -x^2 + y^2 + z^2 = -1$

This is an equation a Hyperboloid of two sheets.

Note : $x = \sqrt{y^2 + z^2 + 1}$ is the Eq. only of the front sheet.

Answer: False!



(d) $y^2 = z$ is an equation of a (parabolic) cylinder with generators parallel to the x-axis axis.

Answer: False!

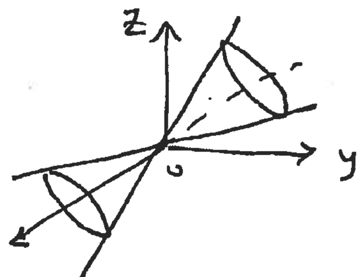
(e) $x + 3y = 0$ is an equation of a plane in 3-space!

Answer: False!

(f) $3y^2 = x^2 - 4z^2$

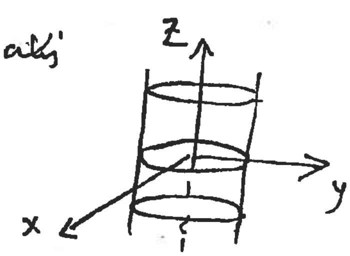
$\Rightarrow x^2 = 3y^2 + 4z^2$

This is an equation of an (elliptic) cone with vertex at $(0,0,0)$, and axis of symmetry being the x-axis



Answer: False!

(g) $x^2 + y^2 = 1$ is an equation of a (circular) cylinder with generators // to z-axis



Answer: False!

(h) $z = x^2 + y^2 + z^2$

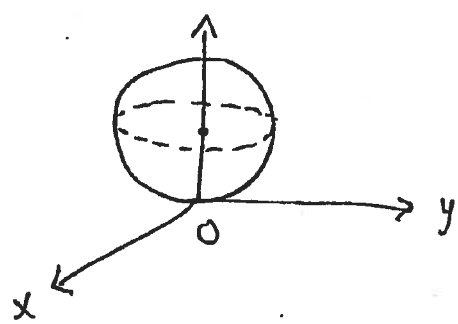
$\Rightarrow x^2 + y^2 + z^2 - z = 0$

Completing the square in "z" we have

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

This is an equation of a sphere with centre at $(0, 0, \frac{1}{2})$, and radius $\frac{1}{2}$ unit.

Answer: False!

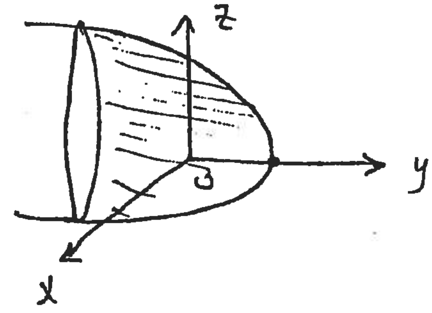


(i) $y = 3 - x^2 - z^2$

$\Rightarrow y - 3 = (-)(x^2 + z^2)$

This is an equation of a paraboloid (with vertex at $(0, 3, 0)$, axis: y-axis and which opens to the left).

Answer: True!



(j) $2x^2 + 4y^2 + 15z^2 = 120$

$\Rightarrow \frac{x^2}{60} + \frac{y^2}{30} + \frac{z^2}{8} = 1$

This is an equation of an ellipsoid -- centred at O.

Answer: True!