SOLUTIONS TO REVIEW SHEET PROBLEMS

1. (a) $\vec{r}(t) = (a\cos(t),\ a\sin(t),\ b)$, $a, b > 0$

$\vec{v}(t) = (-a\sin(t),\ a\cos(t),\ 0)$

$\vec{a}(t) = (-a\cos(t),\ -a\sin(t),\ 0)$

speed $v = \|\vec{v}\| = \sqrt{[a\sin(t)]^2 + [a\cos(t)]^2 + 0^2}$

$= \sqrt{a^2(\sin^2(t) + \cos^2(t))}$

$= \sqrt{a^2} = a$

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(b) $\vec{r}(t) = (t^2,\ -t^2,\ 1)$, $t > 0$

$\vec{v}(t) = (2t,\ -2t,\ 0)$,

$\vec{a}(t) = (2,\ -2,\ 0)$

$v = \|\vec{v}(t)\| = \sqrt{(2t)^2 + (-2t)^2 + 0^2} = \sqrt{4t^2 + 4t^2}$

$= \sqrt{8t^2} = 2\sqrt{2}\,t$

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(c) $\vec{r}(t) = \left(\ln(t),\ \sin^2(t),\ \frac{1}{2}\sin(2t)\right)$, $t > 0$

Note : For Simplicity: $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$

$\therefore\ \vec{r}(t) = \left(\ln(t),\ \frac{1}{2}(1 - \cos(2t)),\ \frac{1}{2}\sin(2t)\right)$

$\vec{v}(t) = \left(\frac{1}{t},\ \sin(2t),\ \cos(2t)\right)$

$\vec{a}(t) = \left(-\frac{1}{t^2},\ 2\cos(2t),\ -2\sin(2t)\right)$

$v = \|\vec{v}\| = \sqrt{\left(\frac{1}{t}\right)^2 + \sin^2(2t) + \cos^2(2t)}$

$= \sqrt{\frac{1}{t^2} + 1} = \sqrt{\frac{1 + t^2}{t^2}} = \frac{\sqrt{1 + t^2}}{t}$

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2. (i) $x(t) = 2t^3 + 4, \quad y(t) = 6e^t - 6t - 3t^2 - 7$

At $t = 0$, $x = 0 + 4 = 4$, $y = 6e^0 - 0 - 0 - 7 = 6 - 7 = -1$

$\therefore$ A point on curve is $(x_1, y_1) = (4, -1)$

Next

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6e^t - 6 - 6t}{6t^2}$$

At $t = 0$, $m = \frac{6 - 6 - 0}{0} = \frac{0}{0}$ .. Use limit!

$$m = \lim_{t \to 0} \frac{6e^t - 6 - 6t}{6t^2} \leftarrow \text{Apply L'Hôpital's Rule}$$

$$= \lim_{t \to 0} \frac{6e^t - 6}{12t} \quad \left(\text{still } \frac{0}{0}\right)$$

$$\stackrel{L'H}{=} \lim_{t \to 0} \frac{6e^t}{12} = \frac{6e^0}{12} = \frac{1}{2}$$

Equation of tangent line is thus given by

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = \frac{1}{2}(x - 4)$$

$$y + 1 = \frac{1}{2}(x - 4)$$

and equation of normal line (having slope -2) is given by

$$y + 1 = -2(x - 4)$$

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(ii) $\vec{r} = (t^2-2t+31, t^2-1)$

$\therefore x = t^2-2t+31 \quad , \quad y = t^2-1$

First, let us find the value of "t" corresponding to the point $P(39,3)$

$39 = t^2-2t+31 \Rightarrow t^2-2t-8=0 \Rightarrow (t-4)(t+2)=0$

$\therefore t = 4 \quad \text{or} \quad -2$

Next, $3 = t^2-1 \Rightarrow t^2 = 4$

$\therefore t = 2 \quad \text{or} \quad -2$

The common value of "t" is $t=-2$

$$\therefore \quad m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2t-2}$$

At $t=-2$, $m = \dfrac{2(-2)}{2(-2)-2} = -\dfrac{4}{-6} = \dfrac{2}{3}$

$\therefore$ slope of Tangent line is $\frac{2}{3}$, and slope of normal line is $-\frac{3}{2}$

Eq. of Tangent Line: $y-3 = \frac{2}{3}(x-39)$

Eq. of normal line: $y-3 = -\frac{3}{2}(x-39)$

3. $\vec{r}(t) = (e^{t}, 2e^{-t}, e^{2t})$

<u>We need</u>:

(i) <u>A point</u>: At $t = \ln(2)$,

$$\bar{r} = (e^{\ln(2)}, 2e^{-\ln(2)}, e^{2\ln(2)})$$

$$= (e^{\ln(2)}, 2e^{\ln(2^{-1})}, e^{\ln(2^2)})$$

$$= (2, 2.2^{-1}, 2^2) = (2, 1, 4)$$

(ii) <u>A direction vector</u>:

This is $\vec{v} = \frac{d\vec{r}}{dt} = (e^{t}, -2e^{-t}, 2e^{2t})$

At $t = \ln(2)$,

$$\vec{v} = (e^{\ln(2)}, -2e^{-\ln(2)}, 2e^{2\ln(2)})$$

$$= (2, -2.2^{-1}, 2.2^{2})$$

$$= (2, -1, 8)$$

Parametric Equations of Tangent Line are thus given by

$$x = 2 + 2s$$
$$y = 1 - 1s \quad , \quad s \in \mathbb{R}$$
$$z = 4 + 8s$$

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4. (a) $\vec{r}(t) = (3t, 2t^{\frac{3}{2}}, 4)$, $\quad 0 \leq t \leq 8$

$\vec{v}(t) = \dfrac{d\vec{r}}{dt} = (3, 3t^{\frac{1}{2}}, 0)$

$v = \|\vec{v}(t)\| = \sqrt{3^2 + (3t^{\frac{1}{2}})^2} = \sqrt{9+9t}$

$= 3\sqrt{1+t}$

Arc length

$$L = \int_a^b v\,dt = \int_0^8 3\sqrt{1+t}\,dt$$

$$= 3\int_0^8 (1+t)^{\frac{1}{2}}\,dt = 3 \cdot \frac{(1+t)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^8$$

$$= 2\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= 2\left[9\sqrt{9} - 1\right] = 2[27-1] = 52$$

(b) $\vec{r} = (2\sin^2(t), \cos^3(t), \sin^3(t))$, $\quad 0 \leq t \leq \frac{\pi}{2}$

$\vec{v} = \dfrac{d\vec{r}}{dt} = \left(4\sin(t)\cos(t), -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t)\right)$

$v = \|\vec{v}\| = \sqrt{16\sin^2(t)\cos^2(t) + 9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}$

$= \sqrt{16\sin^2(t)\cos^2(t) + 9\sin^2(t)\cos^2(t)[\underbrace{\cos^2(t) + \sin^2(t)}_{\text{one}}]}$

$= \sqrt{25\sin^2(t)\cos^2(t)}$

$= 5\sin(t)\cos(t)$ for $t \in [0, \frac{\pi}{2}]$

$\overset{\text{or}}{=} \dfrac{5}{2}\sin(2t)$

Arc length

$$L = \int_0^{\frac{\pi}{2}} \frac{5}{2} \sin(2t)\, dt = -\frac{5}{2} \cdot \frac{1}{2} \cos(2t) \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{5}{4}\left[\cos(\pi) - \cos(0)\right]$$

$$= -\frac{5}{4}[-1-1] = -\frac{5}{4}(-2) = \frac{5}{2}$$

(c) $\vec{r}(t) = (2e^{t}, e^{-t}, 2t)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (2e^{t}, -e^{-t}, 2)$$

speed $v = \|\vec{v}\| = \sqrt{4e^{2t} + e^{-2t} + 4}$ ← perfect square!

$$= \sqrt{(2e^{t} + e^{-t})^2}$$

$$= 2e^{t} + e^{-t}$$

∴ Arc length

$$L = \int_{-1}^{1} (2e^{t} + e^{-t})\, dt$$

$$= 2e^{t} - e^{-t} \Big|_{-1}^{1}$$

$$= (2e - e^{-1}) - (2e^{-1} - e) = 3e - 3e^{-1}$$

or

$$= 3e - \frac{3}{e}$$

(d) $\vec{r}(t) = \left(\frac{1}{2}\sin(t^2),\ \frac{1}{2}\cos(t^2),\ \frac{1}{3}(2t+1)^{\frac{3}{2}} \right)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(\frac{1}{2}\cos(t^2) \cdot 2t,\ -\frac{1}{2}\sin(t^2) \cdot 2t,\ \frac{1}{3} \cdot \frac{3}{2}(2t+1)^{\frac{1}{2}} \cdot 2 \right)$$

$$= \left(t\cos(t^2),\ -t\sin(t^2),\ \sqrt{2t+1} \right)$$

$$v = \text{speed} = \sqrt{t^2\cos^2(t^2) + t^2\sin^2(t^2) + 2t+1}$$

$$= \sqrt{t^2\left(\cos^2(t) + \sin^2(t)\right) + 2t+1}$$

$$= \sqrt{t^2 + 2t + 1} \leftarrow \text{perfect square.}$$

$$= \sqrt{(t+1)^2}$$

$$= t+1, \qquad 0 \leq t \leq 2$$

$$\therefore \quad L = \int_0^2 (t+1)\,dt$$

$$= \frac{1}{2}t^2 + t \Big|_0^2 = 2 + 2 = 4$$

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5. (a) Recall: The parametric equations of a line segment joining $P(x_1, y_1)$, and $Q(x_2, y_2)$ are given by

$$x(t) = x_1 + t\,(x_2 - x_1)$$
$$y(t) = y_1 + t\,(y_2 - y_1) \quad , \quad 0 \leq t \leq 1$$

Here $P(1,-4)$, $Q(2,-3)$

$$\therefore \quad x(t) = 1 + t(2-1) \Rightarrow x = 1+t$$
$$y(t) = -4 + (-3-(-4)) \Rightarrow y = -4+t \quad , \quad 0 \leq t \leq 1$$

(b) parametric equations of line segment are given by

$$x(t) = 0 + t(1-0) \Rightarrow x = t$$
$$y(t) = 1 + t(1-1) \Rightarrow y = 1 \quad , \quad 0 \leq t \leq 1$$
$$z(t) = 2 + t(-1-2) \Rightarrow z = 2-3t$$

(c) Recall: parametric equations of a circle centred at (h,k) and has radius a are given by

$$x = h + a\cos(t)$$
$$y = k + a\sin(t) \quad , \quad t \in [0, 2\pi]$$

Here $(h,k) = (1,0)$, $a = 4$

$$\therefore \quad x = 1 + 4\cos(t), \quad y = 0 + 4\sin(t) \quad , \quad t \in [0, 2\pi]$$

or $\vec{r}(t) = (1 + 4\cos(t))\,\vec{i} + 4\sin(t)\,\vec{j}, \quad t \in [0, 2\pi]$

6. (i) $(3x+1)^2 + (5y-2)^2 = 900 \quad \ldots (*)$

Let us first express equation in standard form.

<u>Note</u>: $3x+1 = 3\left(x+\frac{1}{3}\right)$

$5y-2 = 5\left(y-\frac{2}{5}\right)$

Eq. $(*)$ becomes:

$$3^2\left(x+\frac{1}{3}\right)^2 + 5^2\left(y-\frac{2}{5}\right)^2 = 900 \qquad (\div 900)$$

$$\frac{\left(x+\frac{1}{3}\right)^2}{100} + \frac{\left(y-\frac{2}{5}\right)^2}{36} = 1$$

This is an equation of an Ellipse centred at $(h,k) = \left(-\frac{1}{3}, \frac{2}{5}\right)$ and has semi-axes of length $a = \sqrt{100} = 10$, $b = \sqrt{36} = 6$

Parametric equation is thus given by

$$\vec{r}(t) = (h + a\cos(t))\vec{i} + (k + b\sin(t))\vec{j}, \quad t \in [0, 2\pi]$$

$$= \left(-\frac{1}{3} + 10\cos(t)\right)\vec{i} + \left(\frac{2}{5} + 6\sin(t)\right)\vec{j}, \quad t \in [0, 2\pi]$$

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(ii) $x^2+y^2-2x+6y-15=0$

$$(x^2-2x)+(y^2+6y) = 15$$

Let us Complete the square in both the x, and y-terms.

$$\underbrace{\left[x^2-2x+(-1)^2\right]}_{\text{add: }\left(\frac{\text{Coefficient of } x}{2}\right)^2 \text{ To both sides}} + \underbrace{\left[y^2+6y+3^2\right]}_{\text{add: }\left(\frac{\text{Coefficient of } y}{2}\right)^2 \text{ To both sides}} = 15+(-1)^2+3^2$$

$$(x-1)^2+(y+3)^2 = 15+1+9$$

$$= 25$$

This is an Equation of a Circle Centred at $(h,k) = (1,-3)$ and has radius $\sqrt{25} = 5$

The parametric Equation is thus given by

$$\vec{r}(t) = (1+5\cos(t))\,\vec{i} + (-3+5\sin(t))\,\vec{j}, \quad t\in[0,2\pi]$$

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7. (a) $$4x^2 + y^2 = 16 \quad \cdots (1)$$

$$2x + 3y + 2z = 1 \quad \cdots (2)$$

From (1): Dividing both sides by 16:

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

If this equation is <u>Viewed</u> in $\mathbb{R}^2$, it represents an equation of an Ellipse with centre $(h, k) = (0, 0)$, and semi-axes of length $a = \sqrt{4} = 2$, $b = \sqrt{16} = 4$.

$\therefore$ Its standard parametric equations are thus given by

$$\begin{aligned} x(t) &= h + a\cos(t) \\ y(t) &= k + b\sin(t) \end{aligned}, \quad t \in [0, 2\pi]$$

$$\therefore \begin{aligned} x(t) &= 0 + 2\cos(t) = 2\cos(t) \\ y(t) &= 0 + 4\sin(t) = 4\sin(t) \end{aligned}, \quad t \in [0, 2\pi]$$

Substituting x, y into

$$2x + 3y + 2z = 1$$

We obtain:

$$2\left(2\cos(t)\right) + 3\left(4\sin(t)\right) + 2z = 1$$

$$\therefore \; 4\cos(t) + 12\sin(t) + 2z = 1$$

$$\Rightarrow 2z = 1 - 4\cos(t) - 12\sin(t)$$

Hence $\quad z = \frac{1}{2} - 2\cos(t) - 6\sin(t)$

Curve of intersection is given parametrically by

$$x(t) = 2\cos(t),$$

$$y(t) = 4\sin(t), \qquad , \quad t \in [0, 2\pi]$$

$$z(t) = \frac{1}{2} - 2\cos(t) - 6\sin(t)$$

OR: Curve is given by the Vector Equation

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= 2\cos(t)\,\vec{i} + 4\sin(t)\,\vec{j}$$

$$+ \left(\frac{1}{2} - 2\cos(t) - 6\sin(t)\right)\vec{k},$$

$t \in [0, 2\pi]$

(b) $x^2 + 2y + z = 3 \quad \ldots (1)$

$xz + y = -2 \quad \ldots (2)$

Clearly "y" is easy to Eliminate!

From (2): $y = -2 - xz$

substituting $y = -2 - xz$ into (1):

$$x^2 + 2[-2 - xz] + z = 3$$

or $x^2 - 4 - 2xz + z = 3$

$\Rightarrow \quad x^2 - 2xz + z = 7$

To parametrize, we have too many choices!

let say $x = t$

(This is best choice because we can easily find z)

$$\therefore \quad t^2 - 2tz + z = 7$$

$$z(1-2t) = 7 - t^2$$

$$z = \frac{7-t^2}{1-2t}$$

$$\therefore \quad y = -2 - xz = -2 - t \cdot \left(\frac{7-t^2}{1-2t}\right)$$

$$y \overset{\text{or}}{=} \frac{-2(1-2t) - t(7-t^2)}{1-2t}$$

$$\therefore \quad y = \frac{t^3 - 3t - 2}{1 - 2t}$$

$\therefore$ curve of intersection is given parametrically by

$$x(t) = t$$

$$y(t) = \frac{t^3 - 3t - 2}{1 - 2t}, \quad t \in \mathbb{R}, \ t \neq \frac{1}{2}$$

$$z(t) = \frac{7 - t^2}{1 - 2t}$$

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Note: There are Infinitely-Many possible answers!

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(c) $z = x^2 + y^2$, $2x - 4y - z + 4 = 0$

The idea is to use the two equations to obtain a 3rd. equation containing only two variables which is much easier to parametrize!

Indeed
$$z = x^2 + y^2 \quad \ldots (1)$$
Now, $2x - 4y - z + 4 = 0$
$$\Rightarrow \quad z = 2x - 4y + 4 \quad -- \quad (2)$$

Equating (1), (2):
$$x^2 + y^2 = 2x - 4y + 4$$
$$\Rightarrow \quad x^2 - 2x + y^2 + 4y = 4$$
Now, complete the squares (in x, and y-terms)
$$(x^2 - 2x) + (y^2 + 4y) = 4$$
$$\downarrow \qquad\qquad \downarrow$$
add $\left(-\frac{2}{2}\right)^2 = 1$ to both sides add $\left(\frac{4}{2}\right)^2 = 4$ to both sides
$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = 4 + 1 + 4$$
$$(x-1)^2 + (y+2)^2 = 9$$
If this equation is Viewed in $\mathbb{R}^2$, it is an

Equation of a circle with centre at $(h,k)=(1,-2)$ and is of radius $a=\sqrt{9}=3$

Its parametric equations are thus given by

$$x = h + a\cos(t)$$
$$y = k + a\sin(t), \quad t\in[0,2\pi]$$

$$\therefore \quad x = 1 + 3\cos(t)$$
$$y = -2 + 3\sin(t), \quad t\in[0,2\pi]$$

Recall $\quad z = 2x - 4y + 4$

$$= 2\left[1+3\cos(t)\right] - 4\left[-2+3\sin(t)\right] + 4$$

$$\Rightarrow \quad z = 14 + 6\cos(t) - 12\sin(t)$$

$\therefore$ Curve of Intersection is given parametrically by

$$\begin{cases} x(t) = 1 + 3\cos(t) \\ y(t) = -2 + 3\sin(t) \\ z(t) = 14 + 6\cos(t) - 12\sin(t) \end{cases}, \quad t\in[0,2\pi]$$

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(d) $xy + xz = 6$, $x = -3$

This is an Easy one!

substitute $x = -3$ into $xy + xz = 6$ to get

$$-3y - 3z = 6$$

$$\Rightarrow \quad y + z = -2$$

Now, let say $z = t$, hence $y + t = -2$

$$\therefore \quad y = -2 - t$$

$\therefore$ Curve of intersection is given parametrically by

$$\begin{aligned} x(t) &= -3 \\ y(t) &= -2 - t \quad , \quad t \in \mathbb{R} \\ z(t) &= t \end{aligned}$$

OR : Curve is given by the Vector Equation

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= -3\vec{i} + (-2-t)\vec{j} + t\vec{k} \, , \quad t \in \mathbb{R}$$

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<u>Note</u>: Answer above is <u>not</u> unique. There are infinitely-many possible Answers!!

(e) $x^2 - y^2 - z = 0$, $2y^2 + z = 1$

Let us attempt to obtain a 3rd. Equation Containing only Two Variables!

$$x^2 - y^2 - z = 0 \Rightarrow$$

$$z = x^2 - y^2 \quad \ldots \quad (1)$$

$$2y^2 + z = 1 \Rightarrow$$

$$z = 1 - 2y^2 \quad \ldots \quad (2)$$

Equate (1), (2) (to Eliminate "z"!):

$$x^2 - y^2 = 1 - 2y^2$$

$$\Rightarrow x^2 + y^2 = 1$$

If $x^2+y^2=1$ is Viewed in $\mathbb{R}^2$, it represents an Eq. of a Circle with Centre $(h,k) = (0,0)$, and radius $a = 1$

The standard parametric equations of Circle are thus given by

$$x = \cos(t)$$
$$y = \sin(t) \quad , \quad t \in [0, 2\pi]$$

Recall $z = 1 - 2y^2$

$\therefore \quad z = 1 - 2\sin^2(t)$

Hence, the curve of intersection is given parametrically by

$$x(t) = \cos(t)$$
$$y(t) = \sin(t) \qquad t \in [0, 2\pi]$$
$$z(t) = 1 - 2\sin^2(t)$$

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8. The speed v of a Rocket moving in a straight line only under the forces of its ejected gases is given by

$$v = v_e \ln\left(\frac{M}{m(t)}\right), \quad m(t) = M - \alpha t$$

Where v_e is the speed of ejected gases (assumed Constant), M is the total initial mass, α is the rate of ejected gases (assumed Constant), and $m(t)$ is the mass of rocket at time t.

Here $v_e = 500$ m/s, $\alpha = 1300$ Kg./s, $M = 52{,}000$ Kg.; $m(t) = 52{,}000 - 1300t$

$$\therefore \quad v = 500 \ln\left(\frac{52{,}000}{52{,}000 - 1300t}\right)$$

At $t = 15$ $\quad v = 500 \ln\left(\frac{52{,}000}{52{,}000 - (1300)(15)}\right)$

$$= 500 \ln(1.6) \approx 235 \text{ m/s}$$

At $t = 20$ $\quad v = 500 \ln\left(\frac{52{,}000}{52{,}000 - (1300)(20)}\right)$

$$= 500 \ln(2) \approx 347 \text{ m/s}$$

At $t = 30$ $\quad v = 500 \ln\left(\frac{52{,}000}{52{,}000 - (1300)(30)}\right)$

$$= 500 \ln(4) = 693 \text{ m/s}$$

<u>Note</u>: The rocket burns its entire 39,000 Kg of fuel in $\frac{39,000}{1300} = 30$ seconds!

Therefore after 30 second, the speed of the rocket remains constant at 693 m/s

$\therefore$ At $t = 35$, $v = 693$ m/s as well.

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9. (a) $\vec{r}(t) = (t, \ln(\cos(t)))$

For Simplicity, let us "View" $\vec{r}(t)$ as a Curve in 3-space by inserting a zero z-Component

$$\vec{r}(t) = (t, \ln(\cos(t)), 0)$$

$$\vec{v}(t) = \left(1, -\frac{\sin(t)}{\cos(t)}, 0\right) = (1, -\tan(t), 0)$$

$$\vec{a}(t) = (0, \sec^2(t), 0)$$

At $t = \frac{\pi}{4}$,

$$\vec{V} = (1, -\tan(\tfrac{\pi}{4}), 0) = (1, -1, 0)$$

$$\vec{a} = (0, -\sec^2(\tfrac{\pi}{4}), 0) = (0, -2, 0)$$

$$\vec{v} \times \vec{a} = (1, -1, 0) \times (0, -2, 0) = (0, 0, -2)$$

$$\text{speed} = v = \|\vec{v}\| = \sqrt{1+1+0} = \sqrt{2}, \quad \|\vec{v}\times\vec{a}\| = 2$$

$$\therefore \vec{T} = \frac{\vec{V}}{v} = \frac{(1,-1,0)}{\sqrt{2}} \overset{\text{or}}{=} \frac{1}{\sqrt{2}}(1,-1,0)$$

$$\vec{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \frac{(0,0,-2)}{2} = (0,0,-1)$$

$$\vec{N} = \vec{B} \times \vec{T} = (0,0,-1) \times \frac{1}{\sqrt{2}}(1,-1,0)$$

$$= -\frac{1}{\sqrt{2}}(1,1,0)$$

$$\kappa = \frac{\|\vec{v}\times\vec{a}\|}{v^3} = \frac{2}{(\sqrt{2})^2} = \frac{1}{\sqrt{2}}.$$

(b) $\vec{r}(t) = (2t+3, 5-t^2)$

let us view $\vec{r}(t)$ as a curve in $\mathbb{R}^3$ for simplicity sake!

$$\vec{r}(t) = (2t+3, 5-t^2, 0)$$
$$\vec{v}(t) = (2, -2t, 0)$$
$$\vec{a}(t) = (0, -2, 0)$$

At $t = \sqrt{3}$,

$$\vec{v} = (2, -2\sqrt{3}, 0)$$
$$\vec{a} = (0, -2, 0)$$

$\therefore$ $\vec{v} \times \vec{a} = (2, -2\sqrt{3}, 0) \times (0, -2, 0)$

$$= (0, 0, -4) = -4(0, 0, 1)$$

$$v = \|\vec{v}\| = \sqrt{2^2 + (-2\sqrt{3})^2 + 0^2} = \sqrt{16} = 4$$

$$\|\vec{v} \times \vec{a}\| = |-4| \sqrt{0+0+1} = 4$$

$\therefore$ $\vec{T} = \dfrac{\vec{v}}{v} = \dfrac{(2, -2\sqrt{3}, 0)}{4} = \dfrac{2}{4}(1, -\sqrt{3}, 0) = \dfrac{1}{2}(1, -\sqrt{3}, 0)$

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \frac{-4(0,0,1)}{4} = -(0, 0, 1)$$

$$\vec{N} = \vec{B} \times \vec{T} = -(0,0,1) \times \frac{1}{2}(1, -\sqrt{3}, 0) = -\frac{1}{2}(\sqrt{3}, 1, 0)$$

$$K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{4}{4^3} = \frac{1}{4^2} = \frac{1}{16}$$

10. (a) $\vec{r}(t) = (3\sin(t), 3\cos(t), 4t)$

$\vec{v}(t) = \dfrac{d\vec{r}}{dt} = (3\cos(t), -3\sin(t), 4)$

$\vec{a}(t) = \dfrac{d\vec{v}}{dt} = (-3\sin(t), -3\cos(t), 0)$

$\dfrac{d\vec{a}}{dt} = (-3\cos(t), 3\sin(t), 0)$

At $t=0$: $\vec{v} = (3\cos(0), -3\sin(0), 4) = (3,0,4)$ (1)

$\vec{a} = (-3\sin(0), -3\cos(0), 0) = (0,-3,0)$ (2)

$\vec{v}\times\vec{a} = (3,0,4)\times(0,-3,0) = (12,0,-9)$ (3)

$\|\vec{v}\times\vec{a}\| = \sqrt{(12)^2+(0)^2+(-9)^2} = \sqrt{225} = 15,$ (4)

speed $\|\vec{v}\| = v = \sqrt{3^2+0^2+4^2} = \sqrt{25} = 5$ (5)

Quantities (1) – (5) Can be used to find $\vec{T}, \vec{N}, \vec{B}, \vec{K}$, ρ, and τ.

$\vec{T} = \dfrac{\vec{v}}{v} = \dfrac{1}{5}(3,0,4)$

$\vec{B} = \dfrac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \dfrac{1}{15}(12,0,-9) = \dfrac{1}{5}(4,0,-3)$

$\vec{N} = \vec{B}\times\vec{T} = \dfrac{1}{5}(4,0,-3)\times\dfrac{1}{5}(3,0,4) = \dfrac{1}{25}(0,-25,0)$
$= (0,-1,0)$

$\kappa = \dfrac{\|\vec{v}\times\vec{a}\|}{v^3} = \dfrac{15}{5^3} = \dfrac{3}{25}$

$\rho = \dfrac{1}{\kappa} = \dfrac{25}{3}$, and $\left.\dfrac{d\vec{a}}{dt}\right|_{t=0} = (-3,0,0)$, hence

$\tau = \dfrac{(\vec{v}\times\vec{a})\cdot\frac{d\vec{a}}{dt}}{\|\vec{v}\times\vec{a}\|^2} = \dfrac{(12,0,-9)\cdot(-3,0,0)}{(15)^2} = \dfrac{-36+0+0}{225} = \dfrac{-4}{25}$

(b) $\vec{r}(t) = (\sin(t), \sqrt{2}\cos(t), \sin(t))$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (\cos(t), -\sqrt{2}\sin(t), \cos(t))$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (-\sin(t), -\sqrt{2}\cos(t), -\sin(t))$$

$$\frac{d\vec{a}}{dt} = (-\cos(t), \sqrt{2}\sin(t), -\cos(t))$$

<u>At $t = \frac{\pi}{4}$</u> : Note $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, $\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$$\therefore \quad \vec{v} = \left(\frac{1}{\sqrt{2}}, -\sqrt{2}\cdot\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(1, -\sqrt{2}, 1) \quad \text{..(1)}$$

Similarly $\vec{a} = -\frac{1}{\sqrt{2}}(1, \sqrt{2}, 1)$ --- (2)

$$\frac{d\vec{a}}{dt} = \frac{1}{\sqrt{2}}(-1, \sqrt{2}, -1) \quad \text{---- (3)}$$

$$\vec{v}\times\vec{a} = \frac{1}{\sqrt{2}}(1, -\sqrt{2}, 1) \times \frac{-1}{\sqrt{2}}(1, \sqrt{2}, 1)$$

$$= -\frac{1}{2}(1, -\sqrt{2}, 1)\times(1, \sqrt{2}, 1)$$

$$= -\frac{1}{2}(-2\sqrt{2}, 0, 2\sqrt{2}) = (\sqrt{2}, 0, -\sqrt{2}) \quad (4)$$

$$\|\vec{v}\times\vec{a}\| = \sqrt{(\sqrt{2})^2 + (0)^2 + (-\sqrt{2})^2} = \sqrt{4} = 2 \quad (5)$$

and $v = \|\vec{v}\| = \frac{1}{\sqrt{2}}\sqrt{(1)^2 + (-\sqrt{2})^2 + (1)^2} = \frac{1}{\sqrt{2}}\sqrt{4} = \frac{2}{\sqrt{2}} = \sqrt{2}$ --- (6)

$$\Rightarrow \quad v = \sqrt{2} \quad \text{--- (6)}$$

Using (1) $\longrightarrow$ (6) we can find the six quantities $\vec{T}$, $\vec{N}$, $\vec{B}$, K, ρ, and τ.

$$\vec{T} = \frac{\vec{v}}{v} = \frac{\frac{1}{\sqrt{2}}(1, -\sqrt{2}, 1)}{\sqrt{2}} = \frac{1}{2}(1, -\sqrt{2}, 1)$$

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \frac{(\sqrt{2}, 0, -\sqrt{2})}{2} = \frac{\sqrt{2}}{2}(1, 0, -1)$$

$$= \frac{1}{\sqrt{2}}(1, 0, -1)$$

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\sqrt{2}}(1, 0, -1) \times \frac{1}{2}(1, -\sqrt{2}, 1)$$

$$= \frac{1}{2\sqrt{2}}(1, 0, -1) \times (1, -\sqrt{2}, 1)$$

$$= \frac{1}{2\sqrt{2}}(-\sqrt{2}, -2, -\sqrt{2})$$

$$\overset{\text{or}}{=} -\frac{1}{2}(1, \sqrt{2}, 1)$$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{2}{(\sqrt{2})^3} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\rho = \frac{1}{\kappa} = \sqrt{2}, \quad \text{and}$$

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \frac{d\vec{a}}{dt}}{\|\vec{v} \times \vec{a}\|^2} = \frac{(\sqrt{2}, 0, -\sqrt{2}) \cdot \frac{1}{\sqrt{2}}(-1, \sqrt{2}, -1)}{2^2}$$

$$= \frac{\frac{1}{\sqrt{2}}(-\sqrt{2} + 0 + \sqrt{2})}{4} = 0$$

— — — — — —

(c) $\vec{r}(t) = (\cosh(t), -\sinh(t), t)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (\sinh(t), -\cosh(t), 1)$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (\cosh(t), -\sinh(t), 0)$$

$$\frac{d\vec{a}}{dt} = (\sinh(t), -\cosh(t), 0)$$

At $t=0$, noting that $\sinh(0)=0$, $\cosh(0)=1$, we obtain:

$$\vec{v} = (0, -1, 1) \quad --- \quad (1)$$

$$\vec{a} = (1, 0, 0) \quad --- \quad (2)$$

$$\frac{d\vec{a}}{dt} = (0, -1, 0) \quad ---- \quad (3)$$

$$\vec{v} \times \vec{a} = (0,-1,1) \times (1,0,0)$$

$$= (0, 1, 1) \quad --- \quad (4)$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \qquad (5)$$

$$v = \|\vec{v}\| = \sqrt{0 + (-1)^2 + 1^2} = \sqrt{2} \qquad (6)$$

From (1) – (6) we can determine: $\vec{T}$, $\vec{N}$, $\vec{B}$, κ, ρ, and τ.

$$\vec{T} = \frac{\vec{v}}{v} = \frac{(0,-1,1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(0,-1,1)$$

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \frac{1}{\sqrt{2}}(0, 1, 1)$$

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\sqrt{2}}(0,1,1) \times \frac{1}{\sqrt{2}}(0,-1,1)$$

$$= \frac{1}{2}(0,1,1) \times (0,-1,1)$$

$$= \frac{1}{2}(2,0,0) = (1,0,0)$$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$$

$\rho = \frac{1}{\kappa} = 2$, and

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \frac{d\vec{a}}{dt}}{\|\vec{v} \times \vec{a}\|^2} = \frac{(0,1,1) \cdot (0,-1,0)}{(\sqrt{2})^2}$$

$$= \frac{0-1+0}{2} = -\frac{1}{2}$$

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11. (a) $\vec{r}(t) = (t^2, t, \frac{1}{2}t^2)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (2t, 1, t)$$

$$\vec{a}(t) = (2, 0, 1)$$

Speed $v = \|\vec{v}\| = \sqrt{(2t)^2 + 1^2 + t^2} = \sqrt{5t^2+1}$

Next, $\vec{v} \times \vec{a} = (2t, 1, t) \times (2, 0, 1)$

$$= \left(+\begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 2t & t \\ 2 & 1 \end{vmatrix}, +\begin{vmatrix} 2t & 1 \\ 2 & 0 \end{vmatrix} \right)$$

$$= (1, 0, -2)$$

$\therefore \|\vec{v} \times \vec{a}\| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5}$

Therefore:

Tangential Component of acceleration

$$a_T = \frac{dv}{dt} = \frac{d}{dt}\left(\sqrt{5t^2+1}\right)$$

$$= \frac{1}{2}(5t^2+1)^{-\frac{1}{2}} \cdot 10t = \frac{5t}{\sqrt{5t^2+1}}$$

At $t=4$, $a_T = \frac{5(4)}{\sqrt{5(4)^2+1}} = \frac{20}{9}$

Normal Component of acceleration $a_N = \frac{\|\vec{v} \times \vec{a}\|}{v}$

$$\therefore a_N = \frac{\sqrt{5}}{\sqrt{5t^2+1}}$$

At $t=4$, $a_N = \frac{\sqrt{5}}{\sqrt{80+1}} = \frac{\sqrt{5}}{9}$

(b) $\vec{r}(t) = \ln(t^2+1)\,\vec{i} + \left(t - 2\tan^{-1}(t)\right)\vec{j}$

For simplicity, let us "View" the curve in $\mathbb{R}^3$ by having the z-Component equal to 0.

$\therefore\ \vec{r}(t) = \left(\ln(t^2+1),\ t - 2\tan^{-1}(t),\ 0\right)$

$$\vec{v}(t) = \left(\frac{2t}{t^2+1},\ 1 - \frac{2}{t^2+1},\ 0\right)$$

$$= \left(\frac{2t}{t^2+1},\ \frac{t^2-1}{t^2+1},\ 0\right) = \frac{1}{t^2+1}\left(2t,\ t^2-1,\ 0\right)$$

$$\vec{a}(t) = \left(\frac{2-2t^2}{(t^2+1)^2},\ \frac{4t}{(t^2+1)^2},\ 0\right) = \frac{1}{(t^2+1)^2}\left(2-2t^2,\ 4t,\ 0\right)$$

$$v(t) = \text{speed} = \|\vec{v}(t)\|$$

$$= \frac{1}{t^2+1}\sqrt{(2t)^2 + (t^2-1)^2 + 0^2}$$

$$= \frac{1}{t^2+1}\sqrt{4t^2 + t^4 - 2t^2 + 1}$$

$$= \frac{1}{t^2+1}\sqrt{t^4 + 2t^2 + 1} = \frac{1}{t^2+1}\sqrt{(t^2+1)^2}$$

$\therefore\ v(t) = 1$

$\therefore$ Tangential Component $a_T = \frac{d}{dt}\left(v(t)\right) = 0$

$\therefore\ a_T = 0$ at $t = 2$ as well!

Next, at $t = 2$,

$$\vec{v} = \frac{1}{5}(4, 3, 0),$$

$$\vec{a} = \frac{1}{25}(-6, 8, 0)$$

$$\vec{v} \times \vec{a} = \frac{1}{125}(4,3,0) \times (-6,8,0)$$

$$= \frac{1}{125}(0, 0, 50) = \frac{50}{125}(0,0,1)$$

$$= \frac{2}{5}(0,0,1)$$

$$\|\vec{v} \times \vec{a}\| = \frac{2}{5}\|(0,0,1)\| = \frac{2}{5}$$

$\therefore$ Normal Component $a_N = \dfrac{\|\vec{v} \times \vec{a}\|}{v} = \dfrac{\frac{2}{5}}{1} = \dfrac{2}{5}$

(c) $\vec{r}(t) = t\cos(t)\,\vec{i} + t\sin(t)\,\vec{j} + t^2\,\vec{K}$

$$\overset{\text{or}}{=} \left(t\cos(t),\ t\sin(t),\ t^2\right)$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(\cos(t) - t\sin(t),\ \sin(t) + t\cos(t),\ 2t\right)$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \left(-2\sin(t) - t\cos(t),\ 2\cos(t) - t\sin(t),\ 2\right)$$

$$v(t) = \|\vec{v}(t)\| = \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2 + (2t)^2}$$

Note: $(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2$

$$= \cos^2(t) - \cancel{2t\sin(t)\cos(t)} + t^2\sin^2(t) + \sin^2(t)$$
$$+ \cancel{2t\sin(t)\cos(t)} + t^2\cos^2(t)$$

$$= \left(\cos^2(t) + \sin^2(t)\right) + t^2\left(\sin^2(t) + \cos^2(t)\right)$$

$$= 1 + t^2$$

$$\therefore\ v(t) = \sqrt{1 + t^2 + 4t^2} = \sqrt{1 + 5t^2}$$

$$\therefore\ a_T = \frac{dv}{dt} = \frac{d}{dt}(1 + 5t^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1 + 5t^2)^{-\frac{1}{2}}(10t)$$

At $t = 0$,

$$a_T = 0$$

Next, at $t = 0$,

$$\vec{v} = (1, 0, 0)$$

$$\vec{a} = (0, 2, 2)$$

and $v = \sqrt{1 + 5(0)^2} = \sqrt{1} = 1$

$$\therefore \quad \vec{v} \times \vec{a} = (1, 0, 0) \times (0, 2, 2) = (0, -2, 2)$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{0 + 4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$\therefore$ Normal Component of acceleration

$$a_N = \frac{\|\vec{v} \times \vec{a}\|}{v} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

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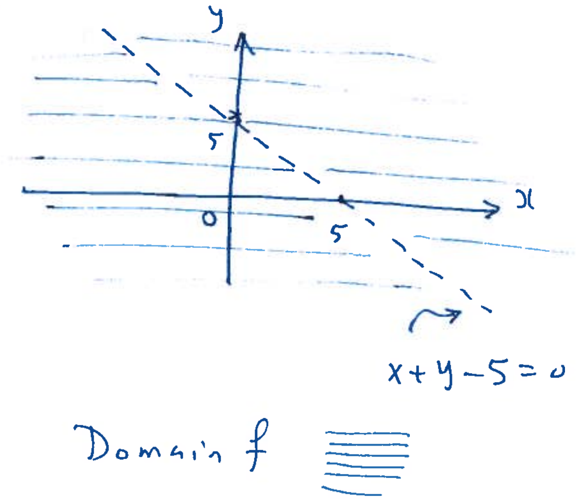
12. (a) $f(x,y) = \dfrac{3-x}{x+y-5}$

Let D be the domain of f.
Then D consists of all (x,y) in $\mathbb{R}^2$ such that

$$x+y-5 \neq 0$$

That is D consists of all points (x,y) in $\mathbb{R}^2$ except points on the line $x+y-5=0$

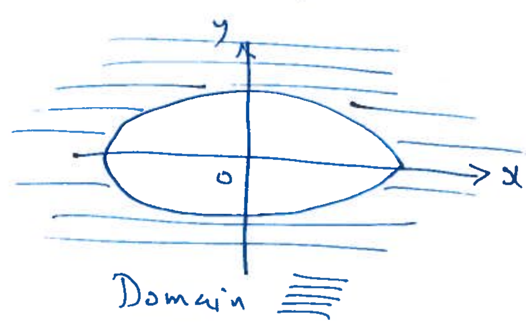


(b) $f(x,y) = \sqrt{4x^2+9y^2-36}$

The domain D consists of all points (x,y) in $\mathbb{R}^2$ such that: $4x^2+9y^2-36 \geq 0$

$\Rightarrow 4x^2+9y^2 \geq 36 \quad (\div 36)$

$\dfrac{x^2}{9} + \dfrac{y^2}{4} \geq 1$



Note: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ is an equation of an ellipse with centre at $(0,0)$, and semi-axes 3, 2

Hence $\dfrac{x^2}{9} + \dfrac{y^2}{4} \geq 1$ is the region outside the ellipse!

(c) $f(x,y) = \sqrt{1 + x^2 + y^2}$

Since $1 + x^2 + y^2$ is always positive, then domain of f consists of all points (x,y) in $\mathbb{R}^2$

Domain : All of $\mathbb{R}^2$

(d) $f(x,y) = \sqrt{\ln(5 - x^2 - y^2)}$

Domain D consists of all points (x,y) in $\mathbb{R}^2$ such that

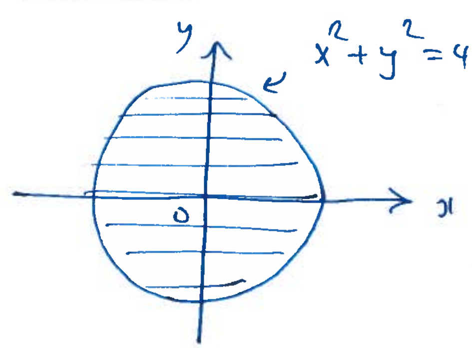
$$\ln(5 - x^2 - y^2) \geq 0$$

$$\Rightarrow \quad 5 - x^2 - y^2 \geq e^0$$

$$\Rightarrow \quad 5 - x^2 - y^2 \geq 1 \quad \text{or} \quad x^2 + y^2 \leq 4$$

Note : $x^2 + y^2 = 4$ is an Equation of a Circle Centred at $(0,0)$ and has radius 4

Therefore $x^2 + y^2 \leq 4$ is the region inside the Circle



Domain

(e) $f(x,y) = \ln\sqrt{x^2+y^2-4}$

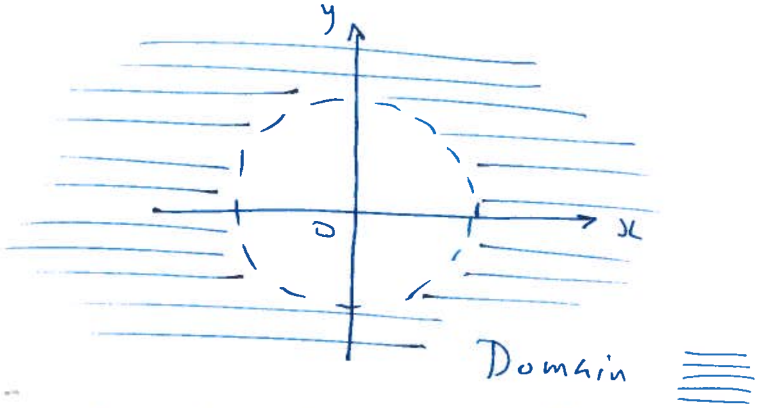
Note first that $\ln(t)$ is defined and is real only if $t > 0$.

Therefore the domain D consists of all (x,y) in $\mathbb{R}^2$ such that $t = x^2+y^2-4 > 0$

$$\Rightarrow x^2+y^2 > 4$$

Note: $x^2+y^2 = 4$ is an equation of a circle with centre $(0,0)$ and radius 2

$\therefore$ $x^2+y^2 > 4$ is the region "strictly" outside the circle



(f) $f(x,y) = \ln|x^2+y^2-4|$. Clearly $|x^2+y^2-4| \geq 0$

Domain D consists of all (x,y) in $\mathbb{R}^2$ except where

$x^2+y^2-4 = 0 \Rightarrow x^2+y^2 = 4$

So: Domain consists of all (x,y) in $\mathbb{R}^2$ except points on the circumference of the circle $x^2+y^2=4$

13. (a) $f(x,y) = x e^{-y}$

Level Curves are given by $f(x,y) = c$, that is

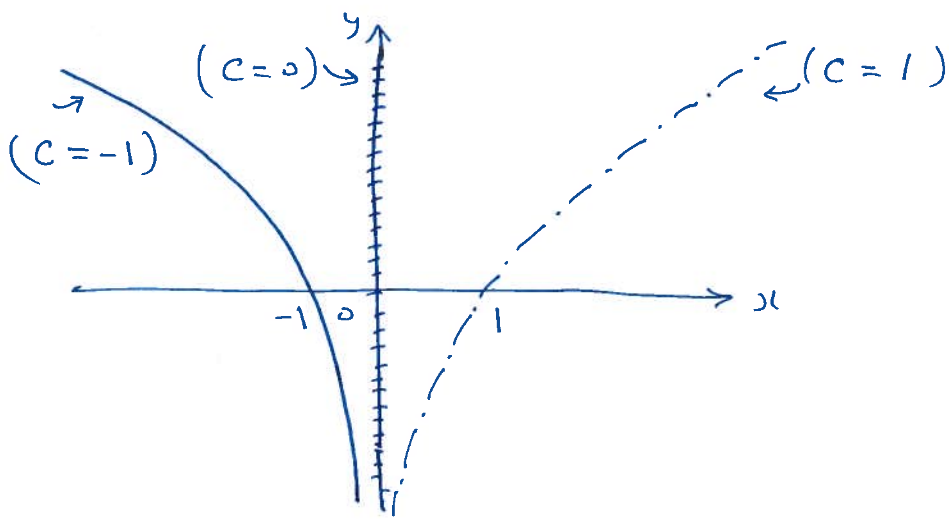
$$x e^{-y} = c$$

$\underline{\underline{c = 0}}$: $\quad x e^{-y} = 0 \quad (\div e^{-y})$

$\Rightarrow x = 0$ ∴ The y-axis

$\underline{c = 1}$: $\quad x e^{-y} = 1 \Rightarrow e^{y} = x$ or $y = \ln(x)$, $x > 0$

$\underline{c = -1}$: $\quad x e^{-y} = -1 \Rightarrow e^{y} = -x$ or $y = \ln(-x)$, $x < 0$



(b) $f(x,y) = \dfrac{x^2 - y^2}{x^2+y^2+1}$

Level curves are given by $f(x,y) = C$, that is

$$\frac{x^2 - y^2}{x^2+y^2+1} = C$$

$\underline{\underline{C=0}}$:

$$\frac{x^2 - y^2}{x^2+y^2+1} = 0 \Rightarrow x^2 - y^2 = 0$$

$$\therefore \quad (x-y)(x+y) = 0$$

$$y = x, \quad y = -x \quad \text{(pair of lines)}$$

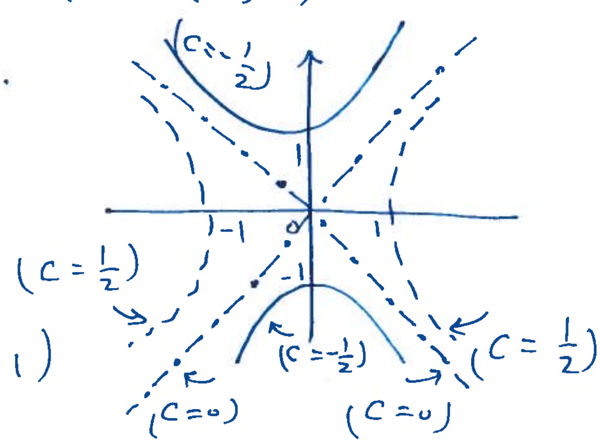
$\underline{\underline{C=\frac{1}{2}}}$:

$$\frac{x^2 - y^2}{x^2+y^2+1} = \frac{1}{2}$$

$$\Rightarrow x^2+y^2+1 = 2(x^2-y^2)$$

$$\Rightarrow x^2 - 3y^2 = 1$$

(An eq. of a Hyperbola with centre at $(0,0)$ and which opens to the left & right).



$\underline{\underline{C=-\frac{1}{2}}}$:

$$\frac{x^2 - y^2}{x^2+y^2+1} = -\frac{1}{2}$$

$$\Rightarrow 2(x^2-y^2) = -(x^2+y^2+1)$$

$$\Rightarrow y^2 - 3x^2 = 1$$

(An Equation of a Hyperbola with centre $(0,0)$ and which opens up & down).

(c) $f(x,y) = \tan^{-1}(x+y)$

Level curves are given by

$$f(x,y) = c$$

That is $\quad \tan^{-1}(x+y) = c$

or $\quad x+y = \tan(c)$ (easier!)

$\underline{\underline{c=0}}$ $\quad x+y = \tan(0)$

$\Rightarrow x+y=0$ (line through origin)

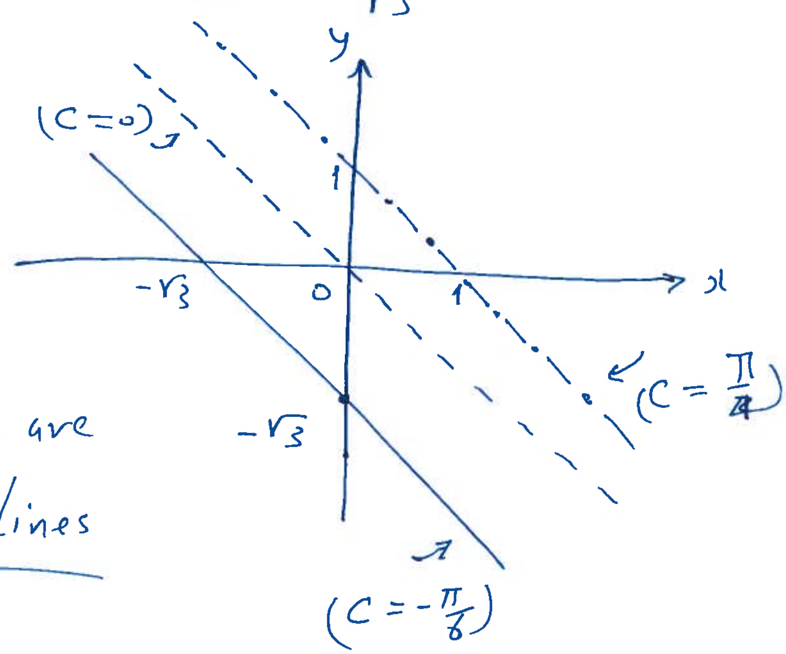
$\underline{\underline{c=\frac{\pi}{4}}}$ $\quad x+y = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow x+y=1$ (st. line)

$\underline{\underline{c=-\frac{\pi}{6}}}$ $\quad x+y = \tan\left(-\frac{\pi}{6}\right)$

$= -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

$\Rightarrow x+y = -\frac{1}{\sqrt{3}}$ (st. line)



Level curves are parallel lines

14. (i) $z = 1 + 3\sqrt{x^2+y^2}$

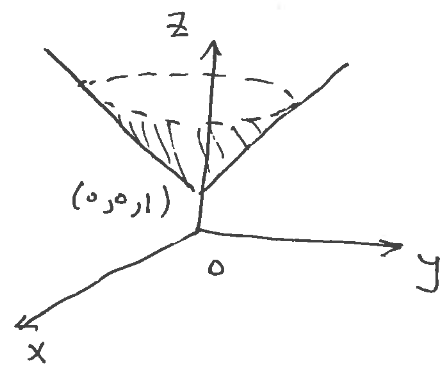
$$z - 1 = 3\sqrt{x^2+y^2}$$

To Identify surface, let us first square each side:

$$(z-1)^2 = 9(x^2+y^2)$$

or $$(z-1)^2 = \frac{x^2}{\frac{1}{9}} + \frac{y^2}{\frac{1}{9}}$$

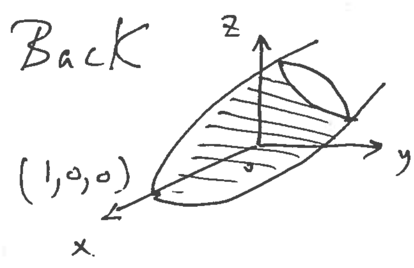
This is an equation of a circular cone with vertex at (0,0,1) and axis of symmetry is the z-axis. However $z-1 = +3\sqrt{x^2+y^2}$ represents only the upper nappe of cone.



(ii) $x = 2 - y^2 - z^2$

$$\Rightarrow x - 2 = -(y^2+z^2)$$

This is an equation of a circular paraboloid with vertex at (2,0,0), axis of symmetry is the x-axis and which opens towards the Back

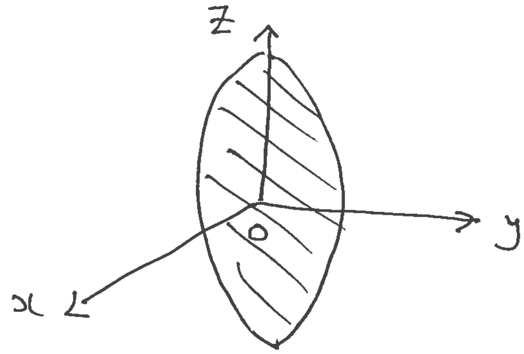


(iii) $2 - x^2 - 3y^2 - 2z^2 = 0$

$$\Rightarrow \quad x^2 + 3y^2 + 2z^2 = 2 \qquad (\div 2)$$

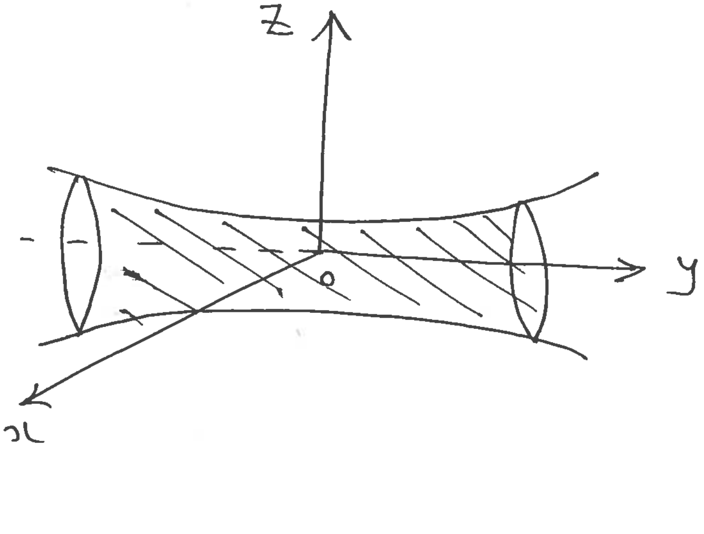
$$\frac{x^2}{2} + \frac{y^2}{\frac{2}{3}} + \frac{z^2}{1} = 1$$

This is an Equation of an Ellipsoid with Centre at $(0,0,0)$, and Semi-axes of length $a = \sqrt{2}$, $b = \sqrt{\frac{2}{3}}$, and $c = \sqrt{1} = 1$



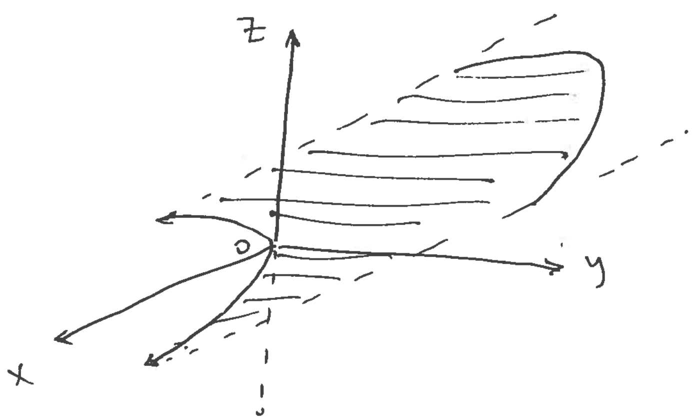
(iv) $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{25} = + 1$

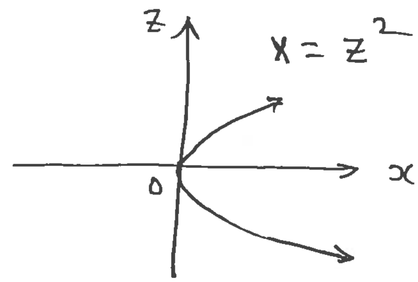
This is an equation of a Hyperboloid of One sheet Centred at $(0,0,0)$, and axis of symmetry is the y-axis



(v) $x = z^2$

This is an equation of a "parabolic" Cylindre generated by a line parallel to y-axis (why?) and its cross section by a plane perpendicular to y-axis is the parabola $x = z^2$ (which may be thought of as "The Base".

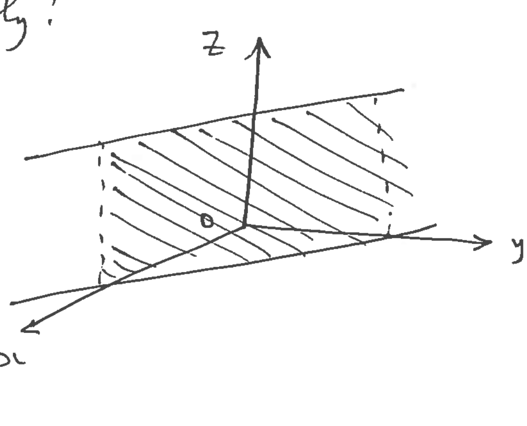


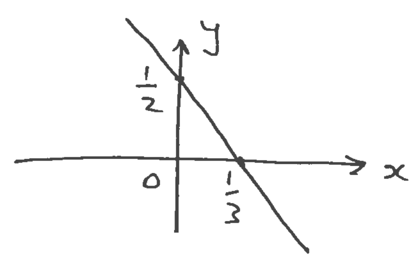


(vi) $3x - 2y + 1 = 0$

This is an equation of a "plane" in $\mathbb{R}^3$.

<u>Note</u>: To sketch the plane, we first sketch the <u>line</u> $3x-2y+1$ in xy-plane, then pile the lines vertically!





$3x - 2y + 1 = 0$

(vii) $x^2+y^2+z^2-2x=0$

let us first Complete the square of x^2-2x to get

$$x^2-2x = (x-1)^2-1$$

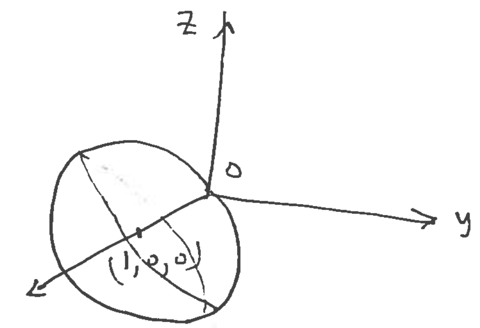
$\therefore$ Equation of surfaces becomes

$$(x-1)^2-1+y^2+z^2=0$$

or $(x-1)^2+y^2+z^2=1$

This is an Equation of a Sphere with Centre at $(1,0,0)$ and radius 1 unit.

Note also that: It passes through origin $(0,0,0)$

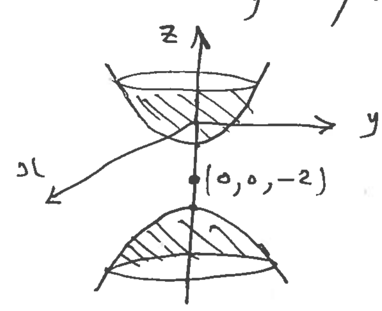


(viii) $x^2+y^2-z^2-4z=3$ ← Complete square in z-terms

$$x^2+y^2-(z+2)^2=-1$$

This is an Equation of a Hyperboloid of <u>Two</u> sheets with Centre at $(0,0,-2)$, and axis of symmetry is the z-axis.

<u>Note</u>: Don't worry about the sketches in Problem (14).
I drew them Just for FUN!



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15. (a) $Z = \ln (xy)^{\sin(xy)}$, $x>0$, $y>0$

Simplify first

$$Z = \sin(xy) \ln(xy)$$

$$= \sin(xy)[\ln(x) + \ln(y)]$$

$$\therefore \frac{\partial z}{\partial y} = x\cos(xy)[\ln(x) + \ln(y)) + \sin(xy)[0 + \frac{1}{y}]$$

$$= x\cos(xy) . \ln(xy) + \frac{\sin(xy)}{y}$$

(b) $f(x,y) = y^{\tan(x)} + \cosh(x^2)$

$$f_y(x,y) = \tan(x) . y^{\tan(x)-1} + 0$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y(x,y))$$

$$= \frac{\partial}{\partial x}[\tan(x) . y^{\tan(x)-1}] \leftarrow \text{Apply Product Rule}$$

Note: $\frac{d}{dt}(a^{u(t)}) = a^{u(t)} . (\ln a) . u'(t)$

$$\therefore f_{yx}(x,y) = \sec^2(x) . y^{\tan(x)-1} + \tan(x) . y^{\tan(x)-1} . \ln(y) . (\sec^2(x))$$

$$= \sec^2(x) . y^{\tan(x)-1}[1 + \tan(x) . \ln(y)]$$

---

16. $W(x,y,z) = x^4 + y^4 + z^4 + A(x^2y^2 + x^2z^2 + y^2z^2)$

$$\frac{\partial W}{\partial x} = 4x^3 + A(2xy^2 + 2xz^2),$$

$$\frac{\partial^2 W}{\partial x^2} = 12x^2 + A(2y^2 + 2z^2) = 12x^2 + 2A(y^2 + z^2)$$

Similarly: $$\frac{\partial^2 W}{\partial y^2} = 12y^2 + 2A(x^2 + z^2), \text{ and}$$

$$\frac{\partial^2 W}{\partial z^2} = 12z^2 + 2A(x^2 + y^2)$$

Since W is harmonic, it satisfies Laplace Equation

$$W_{xx} + W_{yy} + W_{zz} = 0$$

$$\therefore \quad 12x^2 + 2A(y^2 + z^2) + 12y^2 + 2A(x^2 + z^2) + 12z^2 + 2A(x^2 + y^2) = 0$$

$$\therefore \quad 12(x^2 + y^2 + z^2) + 2A\left[(y^2 + z^2) + (x^2 + z^2) + (x^2 + y^2)\right] = 0$$

$$12(x^2 + y^2 + z^2) + 4A(x^2 + y^2 + z^2) = 0 \quad (\div\, x^2 + y^2 + z^2)$$

$$12 + 4A = 0, \quad \boxed{A = -3}$$

— — — — — — —

17. $f(x,y,z) = e^{mz} \cos(2\sqrt{5}x) \cosh(2my)$

$$f_x = -2\sqrt{5}\, e^{mz} \sin(2\sqrt{5}x) \cosh(2my)$$

$$f_{xx} = -2\sqrt{5} \cdot 2\sqrt{5}\, \underbrace{e^{mx} \cos(2\sqrt{5}x) \cosh(2my)}_{\hookrightarrow \text{This is } f}$$

$$f_{xx} = -20 f \quad \text{---} \quad (1)$$

Next, $f_y = 2m e^{mz} \cos(2\sqrt{5}x) \sinh(2my)$

$$f_{yy} = 2m \cdot 2m\, e^{mz} \cos(2\sqrt{5}x) \cosh(2my)$$

$$f_{yy} = 4m^2 f \quad \text{---} \quad (2)$$

Finally, $f_z = m e^{mz} \cos(2\sqrt{5}x) \cosh(2my)$

$$f_{zz} = m \cdot m\, e^{mz} \cos(2\sqrt{5}x) \cosh(2my)$$

$$\therefore \quad f_{zz} = m^2 f \quad \text{---} \quad (3)$$

Now, f is harmonic $\Rightarrow$ $f_{xx} + f_{yy} + f_{zz} = 0$

That is
$$-20f + 4m^2 f + m^2 f = 0 \qquad (\div f)$$

$$-20 + 5m^2 = 0,$$

$$5m^2 = 20 \Rightarrow m^2 = 4$$

$$\therefore \quad m = \pm 2$$

18. (a) $z = \sqrt{x^2+y^2}$, $P(3,-4,5)$

Rewrite equation of surface in the form

$$z^2 = x^2 + y^2$$

or $F(x,y,z) = x^2 + y^2 - z^2 = 0$

For equation of tangent plane, we need:

1. A point: Given as $P(3,-4,5)$, viewed as a position vector $\vec{r_0} = (3,-4,5)$

2. A vector Normal to Tangent plane

This is $\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P$, $F = x^2+y^2-z^2$

$$= (2x, 2y, -2z)\Big|_{(x,y,z)=(3,-4,5)}$$

$$= (6, -8, -10)$$

Eq. of Tangent plane is thus given by

$$\vec{r} \cdot \vec{N} = \vec{r_0} \cdot \vec{N} \quad ; \quad \vec{r} = (x,y,z)$$

$$(x,y,z)\cdot(6,-8,-10) = (3,-4,5)\cdot(6,-8,-10)$$

$$6x - 8y - 10z = 18 + 32 - 50$$

$$\Rightarrow \quad 3x - 4y - 5z = 0$$

A parametric equation of normal line is thus given by

$$\vec{r}(t) = \vec{r_0} + t\vec{N}, \quad t \in \mathbb{R}$$

$$(x, y, z) = (3, -4, 5) + t(6, -8, -10), \quad t \in \mathbb{R}$$

(b) $xy + z^3 + e^{x-y+z} = 4 \Rightarrow$

$$F(x,y,z) = xy + z^3 + e^{x-y+z} - 4 = 0$$

<u>Need</u>: (i) <u>A point</u>: Given as $P(1,2,1) \Rightarrow \vec{r_0} = (1,2,1)$

(ii) <u>A normal vector $\vec{N}$</u>:

$$\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P, \quad F = xy + z^3 + e^{x-y+z} - 4$$

$$= \left(y + e^{x-y+z}, x - e^{x-y+z}, 3z^2 + e^{x-y+z}\right)\Big|_{(x,y,z)=(1,2,1)}$$

$$= (3, 0, 4)$$

<u>Eq. of Tangent plane</u>:

$$(x,y,z)\cdot(3,0,4) = (1,2,1)\cdot(3,0,4)$$

$$\Rightarrow 3x - 4z = 3 + 0 + 4$$

$$3x + 4z = 7$$

Eq. of normal line: $(x,y,z) = (1,2,1) + t(3,0,4), \quad t \in \mathbb{R}$

---

19. (a) Let $z = f(x,y) = \cot\left(3x + \frac{1}{12}y\right)$

$$x = \frac{1}{\pi}t^2, \quad y = \frac{\pi^2}{6t} = \frac{\pi^2}{6}t^{-1}$$

$$z = f(x,y) = \cot\left(3x + \frac{1}{12}y\right)$$

$$\frac{\partial f}{\partial x} = -3\csc^2\left(3x+\frac{1}{12}y\right) \qquad \frac{\partial f}{\partial y} = -\frac{1}{12}\csc^2\left(3x+\frac{1}{12}y\right)$$

$$x = \frac{1}{\pi}t^2 \qquad y = \frac{\pi^2}{6}t^{-1}$$

$$\frac{dx}{dt} = \frac{2}{\pi}t \qquad \frac{dy}{dt} = -\frac{\pi^2}{6}t^{-2} = -\frac{\pi^2}{6t^2}$$

$$t \qquad t$$

Note: At $t = \pi/6$,

$$x = \frac{1}{\pi}t^2 = \frac{1}{\pi}\left(\frac{\pi}{6}\right)^2 = \frac{1}{\pi}\cdot\frac{\pi^2}{36} = \frac{\pi}{36},$$

$$y = \frac{\pi^2}{6t} = \frac{\pi^2}{\pi(\pi/6)} = \pi \text{ , and hence}$$

$$3x + \frac{1}{12}y = 3\left(\frac{\pi}{36}\right) + \frac{1}{12}(\pi) = \frac{\pi}{12} + \frac{\pi}{12} = \frac{2\pi}{12} = \pi/6$$

$$\therefore \left.\frac{dz}{dt}\right|_{t=\pi/6} = \left. -3\csc^2\left(3x+\frac{1}{12}y\right)\cdot\left(\frac{2}{\pi}t\right) - \frac{1}{12}\csc^2\left(3x+\frac{1}{12}y\right)\cdot\left(-\frac{\pi^2}{6t^2}\right)\right|_{\substack{t=\pi/6\\ x=\pi/36\\ y=\pi}}$$

$$= -3\csc^2\left(\frac{\pi}{6}\right)\left(\frac{2}{\pi}\cdot\frac{\pi}{6}\right) - \frac{1}{12}\csc^2\left(\frac{\pi}{6}\right)\cdot\left(-\frac{\pi^2}{6(\pi/6)^2}\right)$$

$$= -3\cdot 2^2\cdot\frac{1}{3} - \frac{1}{12}\cdot 2^2(-6) = -4 + 2 = -2$$

(b) $Z = f(x,y) = \ln(x^2+3xy)^{-4} = -4\ln(x^2+3xy)$,

$x = \cosh(u)$, $y = \sinh(v)$

$$Z = f(x,y) = -4\ln(x^2+3xy)$$

$$\frac{\partial f}{\partial x} = -4 \cdot \frac{2x+3y}{x^2+3xy} \qquad \frac{\partial f}{\partial y} = -4 \cdot \frac{3x}{x^2+3xy}$$

$$\begin{tikzcd}
 & Z = f(x,y) \arrow[dl] \arrow[dr] & \\
x = \cosh(u) \arrow[d, "\frac{dx}{du} = \sinh(u)"'] & & y = \sinh(v) \arrow[d, "\frac{dy}{dv} = \cosh(v)"] \\
u & & v
\end{tikzcd}$$

Note: At $u = 0$, $v = 0$, we have

$x = \cosh(u) = \cosh(0) = 1$, $y = \sinh(v) = \sinh(0) = 0$

$$\therefore \left.\frac{\partial z}{\partial v}\right|_{\substack{u=0\\v=0}} = -4 \cdot \frac{3x}{x^2+3xy} \cdot \cosh(v) \Bigg|_{\substack{u=0\\v=0\\x=1\\y=0}}$$

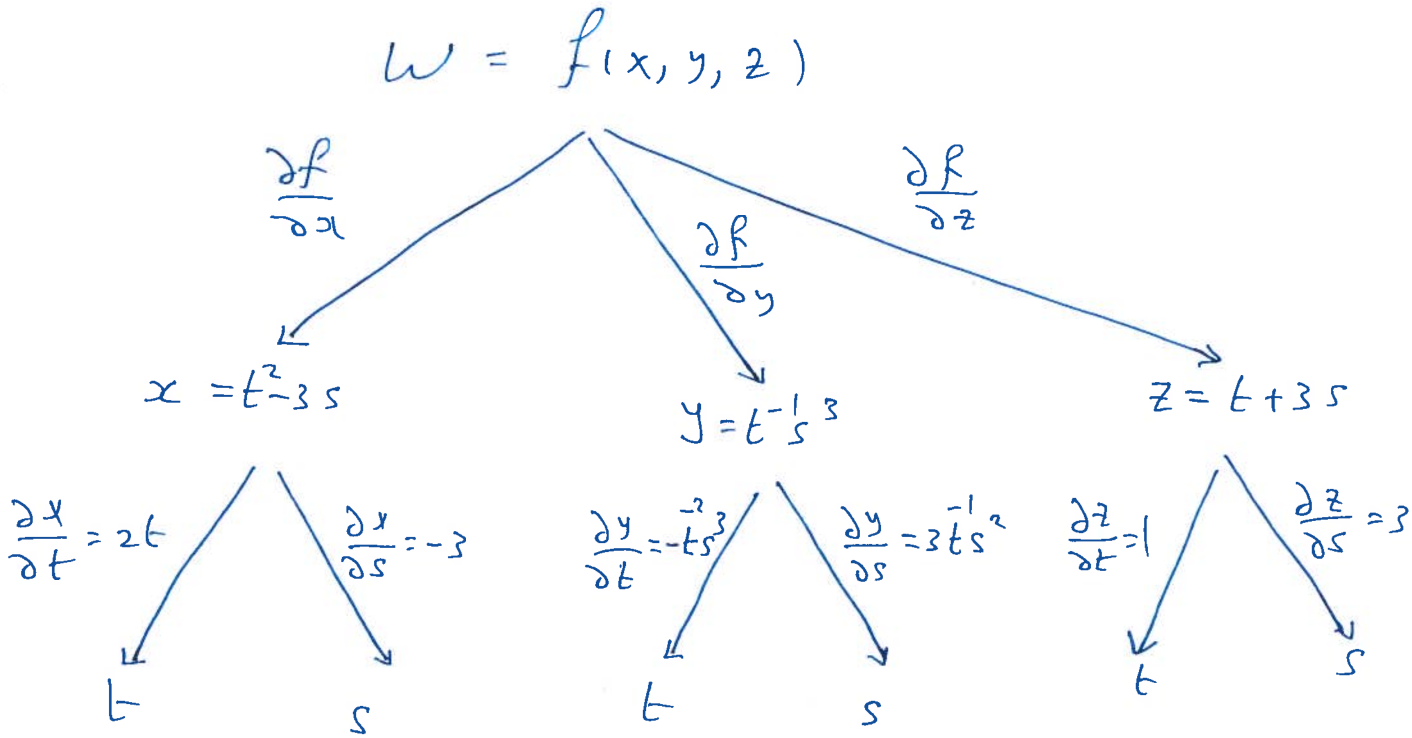
$$= -4 \cdot \frac{3(1)}{(1)^2+3(1)(0)} \cosh(0)$$

$$= -4 \cdot 3 \cdot 1 = -12$$

(c) $w = f(t^2 - 3s,\ t^{-1}s^3,\ t + 3s)$

$= f(x, y, z)$, where

$x = t^2 - 3s$, $y = t^{-1}s^3$, and $z = t + 3s$



$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot(-3) + \frac{\partial f}{\partial y}\cdot 3t^{-1}s^2 + \frac{\partial f}{\partial z}\cdot 3$$

or $$\frac{\partial w}{\partial s} = -3 f_x(x,y,z) + 3t^{-1}s^2 f_y(x,y,z) + 3f_z(x,y,z)$$

where x, y, and z are as above.

— — — — — —

(d) Let $Z = f(x,y) = \sqrt{x^2 - y^2} = (x^2-y^2)^{\frac{1}{2}}$

$x = r\cos(\theta)$, $\quad y = r\sin(\theta)$

$$\therefore \frac{\partial f}{\partial x} = \frac{1}{2}(x^2-y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2-y^2}},$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2-y^2)^{-\frac{1}{2}}(-2y) = -\frac{y}{\sqrt{x^2-y^2}}$$

$$\frac{\partial x}{\partial r} = \cos(\theta) \overset{\text{or}}{=} \frac{x}{r},$$

$$\frac{\partial x}{\partial \theta} = -r\sin(\theta) \overset{\text{or}}{=} -y,$$

$$\frac{\partial y}{\partial r} = \sin(\theta) \overset{\text{or}}{=} \frac{y}{r}, \text{ and}$$

$$\frac{\partial y}{\partial \theta} = r\cos(\theta) \overset{\text{or}}{=} x$$

Refer to Tree Diagram below:

$$\begin{tikzcd}
& & & Z = f(x,y) \arrow[dll, "\frac{\partial f}{\partial x}"'] \arrow[drr, "\frac{\partial f}{\partial y}"] & & & \\
& x \arrow[dl, "\frac{\partial x}{\partial r}"'] \arrow[dr, "\frac{\partial x}{\partial \theta}"] & & & & y \arrow[dl, "\frac{\partial y}{\partial r}"'] \arrow[dr, "\frac{\partial y}{\partial \theta}"] & \\
r & & \theta & & r & & \theta
\end{tikzcd}$$

Note:

At $(r,\theta) = (1, \frac{\pi}{6})$,

$x = r\cos(\theta) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$,

$y = r\sin(\theta) = \sin(\frac{\pi}{6}) = \frac{1}{2}$

From the diagram, we have

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\therefore \frac{\partial z}{\partial r} = \frac{x}{\sqrt{x^2-y^2}}\cdot\frac{x}{r} + \frac{-y}{\sqrt{x^2-y^2}}\cdot\frac{y}{r}$$

$$= \frac{x^2-y^2}{r\sqrt{x^2-y^2}} \leftarrow \text{Rationalize}$$

$$= \frac{\cancel{(x^2-y^2)}}{r\cancel{\sqrt{x^2-y^2}}}\cdot\frac{\sqrt{x^2-y^2}}{\cancel{\sqrt{x^2-y^2}}} = \frac{\sqrt{x^2-y^2}}{r}$$

$$\therefore \left.\frac{\partial z}{\partial r}\right|_{\substack{r=1\\ \theta=\frac{\pi}{6}}} = \left.\frac{\sqrt{x^2-y^2}}{r}\right|_{\substack{r=1\\ x=\frac{\sqrt{3}}{2}\\ y=\frac{1}{2}}} = \frac{\sqrt{\frac{3}{4}-\frac{1}{4}}}{1} = \frac{\sqrt{\frac{1}{2}}}{1} = \frac{1}{\sqrt{2}}$$

Next, $\displaystyle\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$

$$= \frac{x}{\sqrt{x^2-y^2}}\cdot(-y) + \frac{-y}{\sqrt{x^2-y^2}}\cdot x$$

$$= \frac{-2xy}{\sqrt{x^2-y^2}}$$

$$\therefore \left.\frac{\partial z}{\partial \theta}\right|_{\substack{r=1\\ \theta=\frac{\pi}{6}}} = \left.\frac{-2xy}{\sqrt{x^2-y^2}}\right|_{\substack{x=\frac{\sqrt{3}}{2}\\ y=\frac{1}{2}}} = \frac{-2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\sqrt{\frac{3}{4}-\frac{1}{2}}}$$

$$= \frac{-\frac{\sqrt{3}}{2}}{\sqrt{\frac{1}{2}}} = -\frac{\sqrt{6}}{2}$$

(e) $z = f(u, v)$,

$$u = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2), \quad v = x + \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = f(u, v)$$

$f_u(u, v)$ $\qquad$ $f_v(u, v)$

$u = \frac{1}{2}\ln(x^2+y^2)$ $\qquad$ $v = x + \tan^{-1}\left(\frac{y}{x}\right)$

No Need! $\to x$ $\qquad$ $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \to y$

No Need! $\to x$ $\qquad$ $\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} \overset{\text{or}}{=} \frac{x}{x^2+y^2} \to y$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=0}} = f_u(u,v)\cdot\frac{y}{x^2+y^2} + f_v(u,v)\cdot\left.\frac{x}{x^2+y^2}\right|_{\substack{x=1\\y=0\\u=0\\v=1}}$$

Note:
at $x = 1, y = 0$
$u = \frac{1}{2}\ln 1 = 0$,
$V = $ [illegible] $+ \tan^{-1} 0 = 0$

$$= f_u(0,1)\cdot\frac{0}{1+0} + f_v(0,1)\cdot\frac{1}{1+0}$$

$$= f_v(0,1) = -4 \qquad \text{(From Data Given)}$$

(f) Let $W = f(x,y,z) = \ln(x^2+y^2+z^2)$,

$x = u e^{v}\sin(v)$, $y = u e^{v}\cos(v)$, $z = u e^{v}$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2},$$

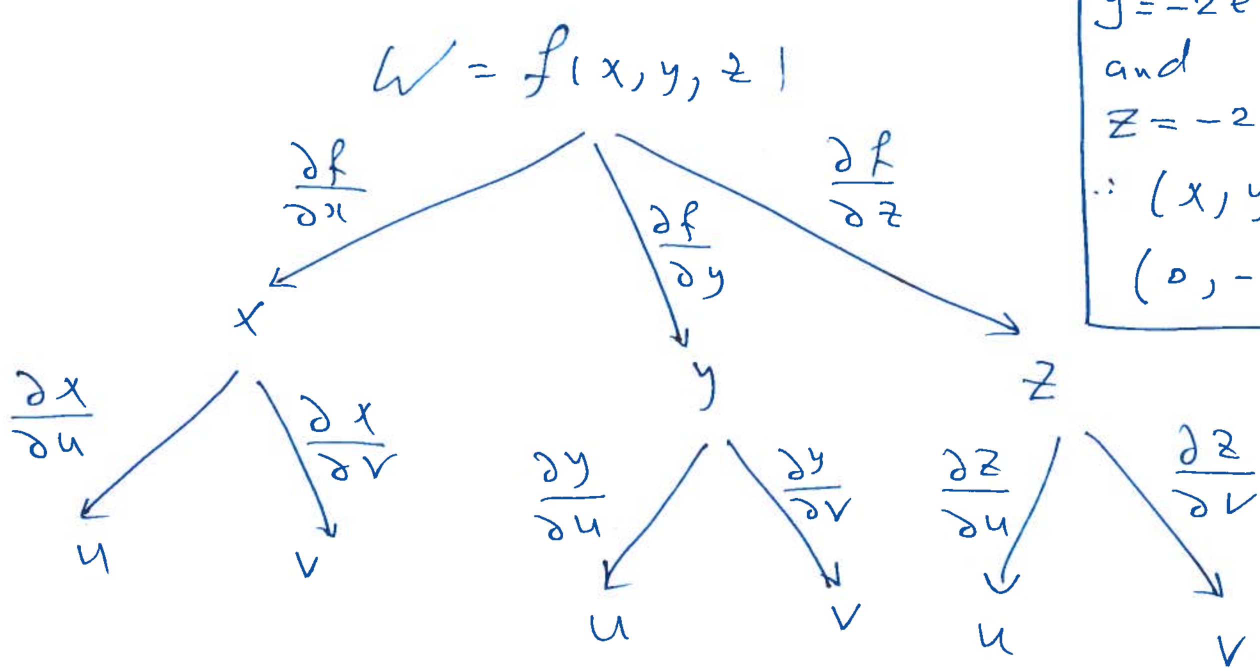
$$\frac{\partial f}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

$$\frac{\partial x}{\partial u} = e^{v}\sin(v), \quad \frac{\partial x}{\partial v} = u e^{v}\sin(v) + u e^{v}\cos(v)$$

$$\frac{\partial y}{\partial u} = e^{v}\cos(v), \quad \frac{\partial y}{\partial v} = u e^{v}\cos(v) - u e^{v}\sin(v)$$

$$\frac{\partial z}{\partial u} = e^{v}, \quad \frac{\partial z}{\partial v} = u e^{v}$$

Let us Construct the Tree Diagram:



Note:

At $(u,v) = (-2,0)$:

$x = -2e^{0}\sin(0) = \underline{\underline{0}}$

$y = -2e^{0}\cos(0) = \underline{\underline{-2}}$

and

$z = -2e^{0} = -2$

$\therefore (x,y,z) = (0,-2,-2)$

From Diagram, we obtain:

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{2x}{x^2+y^2+z^2}\cdot e^{v}\sin(v) + \frac{2y}{x^2+y^2+z^2}\cdot e^{v}\cos(v) + \frac{2z}{x^2+y^2+z^2}\left(e^{v}\right)\Bigg|_{\substack{(u,v)=(-2,0)\\(x,y,z)=(0,-2,-2)}}$$

$$= 0 + \frac{-4}{0+4+4}\left(e^{0}\cos(0)\right) + \frac{-4}{0+4+4}e^{0}$$

$$= -\frac{1}{2} - \frac{1}{2} = -1,$$

$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{2x}{x^2+y^2+z^2}\left(ue^{v}\sin(v) + ue^{v}\cos(v)\right) + \frac{2y}{x^2+y^2+z^2}\left(ue^{v}\cos(v) - ue^{v}\sin(v)\right) + \frac{2z}{x^2+y^2+z^2}\cdot ue^{v}\Bigg|_{\substack{(u,v)=(-2,0)\\(x,y,z)=(0,-2,-2)}}$$

$$= 0 + \frac{-4}{0+4+4}\left(-2e^{0}\cos(0) - 0\right) + \frac{-4}{0+4+4}\left(-2e^{0}\right)$$

$$= 1 + 1 = 2$$

(20) (a) $4x^2+3y^2+z^2=25$, $P(1,2,-3)$

Let $F(x,y,z) = 4x^2+3y^2+z^2-25 = 0$

A vector normal to surface at P is thus given by

$$\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P , \quad F = 4x^2+3y^2+z^2-25$$

$$= (8x, 6y, 2z)\Big|_{(x,y,z)=(1,2,-3)}$$

$$= (8, 12, -6)$$

Eq. of tangent plane:

$$\vec{r}\cdot\vec{N} = \vec{r_0}\cdot\vec{N}$$

$$(x,y,z)\cdot(8,12,-6) = (1,2,-3)\cdot(8,12,-6)$$

$$8x+12y-6z = 8+24+18$$

$$= 50$$

or $\quad 4x+6y-3z = 25$

— — — — — —

(b) $2x+3y^2+2z^2=31$, $P(-2,1,4)$

$$\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P , \quad F = 2x+3y^2+2z^2-31$$

$$= (2, 6y, 4z)\Big|_{(x,y,z)=(-2,1,4)}$$

$$= (2, 6, 16)$$

Equation of tangent plane:

$$(x,y,z)\cdot(2,6,16) = (-2,1,4)\cdot(2,6,16)$$
$$2x+6y+16z = -4+6+64$$
$$= 66$$

$$\Rightarrow \quad x+3y+8z = 33$$

— — — — —

(c) $\sin(xyz-6)+2x-x^2=0$, @ $(1,2,3)$

let $F(x,y,z) = \sin(xyz-6)+2x-x^2$

A vector normal to surface at @ is thus given by

$$\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_{@}$$

$$= \Big(yz\cos(xyz-6)+2-2x,\ xz\cos(xyz-6),\ xy\cos(xyz-6)\Big)\Big|_{(x,y,z)=(1,2,3)}$$

$$= (6+2-2,\ 3,\ 2)$$
$$= (6,3,2)$$

$\therefore$ A unit vector $\perp$ to surface is given by

$$\vec{n} = \pm\frac{\vec{N}}{\|\vec{N}\|} = \pm\frac{(6,3,2)}{\|(6,3,2)\|} = \pm\frac{1}{7}(6,3,2)$$

— — — — — —

21. (a) Recall: The differential of f is denoted and defined by

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Here $\quad f(x,y) = e^{3x}\cos(2y) + 2x - y + 1$

$$\frac{\partial f}{\partial x} = 3e^{3x}\cos(2y) + 2,$$

$$\frac{\partial f}{\partial y} = -2e^{3x}\sin(2y) - 1$$

$$\therefore\ df = \left[3e^{3x}\cos(2y) + 2\right]dx + \left[-2e^{3x}\sin(2y) - 1\right]dy$$

(b) $g(x,y) = \sin^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial g}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2\sqrt{1-\frac{y^2}{x^2}}}$$

$$= \frac{-y}{x^2\frac{\sqrt{x^2-y^2}}{x}} = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial g}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(\frac{1}{x}\right) = \frac{1}{\cancel{x}\frac{\sqrt{x^2-y^2}}{\cancel{x}}} = \frac{1}{\sqrt{x^2-y^2}}$$

$$\therefore\ dg = g_x\,dx + g_y\,dy = -\frac{y}{x\sqrt{x^2-y^2}}\,dx + \frac{1}{\sqrt{x^2-y^2}}\,dy$$

(c) Recall:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

Here $F(x,y,z) = e^{x+2y+3z}$

$\therefore \frac{\partial F}{\partial x} = e^{x+2y+3z}$, $\frac{\partial F}{\partial y} = 2e^{x+2y+3z}$, and

$\frac{\partial F}{\partial z} = 3\, e^{x+2y+3z}$

$\therefore dF = e^{x+2y+3z}\left[dx + 2dy + 3dz\right]$

(d) $G(x,y,z) = \ln(x^2 + 2y - z)$

$\frac{\partial G}{\partial x} = \frac{2x}{x^2+2y-z}$, $\frac{\partial G}{\partial y} = \frac{2}{x^2+2y-z}$, $\frac{\partial G}{\partial z} = \frac{-1}{x^2+2y-z}$

$$\therefore dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz$$

$$= \frac{1}{x^2+2y-z}\left[2x\,dx + 2\,dy - dz\right]$$

— — — — —

22. (a) Recall: The linearization of $f(x,y)$ at (a,b) is given by

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Here $f(x,y) = \sqrt{x-2y+30}$, $(a,b) = (4,-1)$

$$f(x,y) = \sqrt{x-2y+30}$$

$$f_x(x,y) = \frac{1}{2\sqrt{x-2y+30}}$$

$$f_y(x,y) = \frac{-1}{\sqrt{x-2y+30}}$$

$$f(4,-1) = \sqrt{4+2+30} = 6$$

$$f_x(4,-1) = \frac{1}{2\sqrt{36}} = \frac{1}{12}$$

$$f_y(4,-1) = -\frac{1}{\sqrt{36}} = -\frac{1}{6}$$

$$L(x,y) = f(4,-1) + f_x(4,-1)(x-4) + f_y(4,-1)(y-(-1))$$

$$= 6 + \frac{1}{12}(x-4) - \frac{1}{6}(y+1)$$

(b) $f(x,y) = \ln(x^2+y^2+xy)$, $(a,b) = (1,-1)$

$\therefore L(x,y) = f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y-(-1))$

$f(x,y) = \ln(x^2+y^2+xy)$

$f_x(x,y) = \dfrac{2x+y}{x^2+y^2+xy}$

$f_y(x,y) = \dfrac{2y+x}{x^2+y^2+xy}$

$f(1,-1) = \ln(1+1-1) = \ln 1 = 0$

$f_x(1,-1) = \dfrac{2-1}{1+1-1} = 1$

$f_y(1,-1) = \dfrac{-2+1}{1+1-1} = -1$

$\therefore L(x,y) = 0 + 1(x-1) - 1(y+1)$

or $L(x,y) = x - y - 2$

(c) Recall: The Linearization of $f(x,y,z)$ at (a,b,c) is given by

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

Here $(a,b,c) = (1,1,1)$

$$\therefore L(x,y,z) = f(1,1,1) + f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1)$$

$f(x,y,z) = xy + yz + zx$	$f(1,1,1) = 1+1+1 = 3$
$f_x(x,y,z) = y + z$	$f_x(1,1,1) = 1+1 = 2$
$f_y(x,y,z) = x + z$	$f_y(1,1,1) = 1+1 = 2$
$f_z(x,y,z) = y + x$	$f_z(1,1,1) = 1+1 = 2$

$$\therefore L(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1)$$

$$= 2x + 2y + 2z - 3$$

23. (a) From problem (22) - part (a):

$$L(x,y) = 6 + \frac{1}{12}(x-4) - \frac{1}{6}(y+1)$$

$\therefore \quad f(x,y) \sim L(x,y) \quad$ near $(a,b) = (4,-1)$

That is

$$\sqrt{x-2y+30} \sim 6 + \frac{1}{12}(x-4) - \frac{1}{6}(y+1) \quad \text{near } (4,-1)$$

putting $x = 4.12$, and $y = -0.88$, we obtain

$$\sqrt{4.12-2(-0.88)+30} \sim 6 + \frac{1}{12}(4.12-4) - \frac{1}{6}(-0.88+1)$$

$$\therefore \sqrt{35.88} \sim 6 + \frac{0.12}{12} - \frac{1}{6}(+0.12) = 6 + 0.01 - 0.02$$

$$\therefore \sqrt{35.88} \approx 5.99$$

(b) From problem (22) - part (b):

$$L(x,y) = x - y - 2$$

$$\therefore \quad f(x,y) = \ln(x^2 + y^2 + xy) \sim x - y - 2 \quad \text{near } (a,b) = (1,-1)$$

putting $x = 1.05$, $y = -1.03$, we obtain

$$\ln((1.05)^2 + (-1.03)^2 + (1.05)(-1.03)) \sim 1.05 - (-1.03) - 2$$

That is $\ln(1.0819) \sim 0.08$

24. $f(x,y) = \dfrac{1}{x^2+8y}$

To estimate $f(x,y)$ at $(x,y) = (2.9, -0.9)$, we shall use $L(x,y)$ at $(a,b) = (3,-1)$

$\therefore L(x,y) = f(3,-1) + f_x(3,-1)(x-3) + f_y(3,-1)(y+1)$

$f(x,y) = \dfrac{1}{x^2+8y} = (x^2+8y)^{-1}$

$f_x(x,y) = -(x^2+8y)^{-2} \cdot 2x$

$= -\dfrac{2x}{(x^2+8y)^2}$

$f_y(x,y) = -(x^2+8y)^{-2} \cdot 8$

$= -\dfrac{8}{(x^2+8y)^2}$

$f(3,-1) = \dfrac{1}{9-8} = 1$

$f_x(3,-1) = -\dfrac{2(3)}{(9-8)^2} = -6$

$f_y(3,-1) = -\dfrac{8}{(9-8)^2} = -8$

$\therefore L(x,y) = 1 - 6(x-3) - 8(y+1)$

$\therefore f(x,y) = \dfrac{1}{x^2+8y} \sim L(x,y) = 1 - 6(x-3) - 8(y+1)$

$\Rightarrow \dfrac{1}{x^2+8y} \sim 1 - 6(x-3) - 8(y+1)$ near $(3,-1)$

putting $x = 2.9$, $y = -0.9$, we obtain

$\dfrac{1}{(2.9)^2+8(-0.9)} \sim 1 - 6(2.9-3) - 8(-0.9+1) =$

$\therefore \dfrac{1}{1.21} \sim 0.80$

25. $PV = KT \Rightarrow P = \dfrac{KT}{V}$

or $P = KTV^{-1}$

$\therefore \dfrac{\partial P}{\partial T} = KV^{-1}, \quad \dfrac{\partial P}{\partial V} = -KTV^{-2}$

$\therefore dP = \dfrac{\partial P}{\partial T} dT + \dfrac{\partial P}{\partial V} dV$

$= KV^{-1} dT - KTV^{-2} dV$

But $\Delta P \approx dP$

$\therefore \Delta P \approx KV^{-1} dT - KTV^{-2} dV$

or $\Delta P \approx KV^{-1} \Delta T - KTV^{-2} \Delta V$

Dividing both sides by $P = KTV^{-1}$:

$$\frac{\Delta P}{P} \approx \frac{KV^{-1} \Delta T}{KTV^{-1}} - \frac{KTV^{-2}}{KTV^{-1}} \Delta V$$

$$\Rightarrow \frac{\Delta P}{P} \approx \frac{\Delta T}{T} - \frac{\Delta V}{V}$$

Now, $V = 64$, $\Delta V = 68 - 64 = 4$,

$T = 360$, $\Delta T = 351 - 360 = -9$

$$\therefore \quad \frac{\Delta P}{P} \approx -\frac{9}{360} - \frac{4}{64}$$

$$\approx -\left(\frac{1}{40} + \frac{1}{16}\right) = -0.0875$$

$$\therefore \quad \frac{\Delta P}{P} \approx (-0.0875)(100)\ \%$$

$$\approx 8.75\ \%$$

So, the Pressure decreases by approximately 8.75%.

26. From problem (25)

$$\frac{\Delta P}{P} \approx \frac{\Delta T}{T} - \frac{\Delta V}{V}$$

We know: $\frac{\Delta T}{T} = -0.8\%$

$\frac{\Delta P}{P} = +0.5\%$

It follows that

$$0.5\% \approx -0.8\% - \frac{\Delta V}{V}$$

$$\Rightarrow \quad \frac{\Delta V}{V} \approx -0.8\% - 0.5\% = -1.3\%$$

So, the volume decreases by approximately 1.3%.

— — — — — —

27. $$F = \frac{\pi P R^4}{8 \nu l}$$

Let $\frac{\pi}{8 \nu l}$ = a constant K

$$\therefore \quad F = K P R^4$$

$$\frac{\partial F}{\partial P} = K R^4 \quad , \quad \frac{\partial F}{\partial R} = 4 K P R^3$$

$$\therefore \quad dF = \frac{\partial F}{\partial P} dP + \frac{\partial F}{\partial R} dR$$

$$= K R^4 dP + 4 K P R^3 dR$$

Dividing both sides by $F = KPR^4$, we obtain

$$\frac{dF}{F} = \frac{K R^4 dP}{K P R^4} + \frac{4 K P R^3}{K P R^4} dR$$

$$\therefore \quad \frac{dF}{F} = \frac{dP}{P} + 4 \left(\frac{dR}{R} \right)$$

But $\Delta F \approx dF$,

and $dP = \Delta P$, $dR = \Delta R$

$$\therefore \quad \frac{\Delta F}{F} \approx \frac{\Delta P}{P} + 4 \frac{\Delta R}{R}$$

Know: $\frac{\Delta R}{R} = -2\%$, and $\frac{\Delta P}{P} = 3\%$

$$\therefore \quad \frac{\Delta F}{F} = 3\% + 4(-2\%)$$

$$= -5\%$$

$\therefore$ The Blood flow decrease by approximately 5%

— — — — — —

(28) $4x^2+4y^2-4x+12y-39990=0 \cdots (*)$

Clearly: This plane curve is a Circle.

Recall: The radius of curvature of a Circle of radius "a" is given by

$$\rho = a$$

Hence: Curvature of the Circle $K = \frac{1}{\rho} = \frac{1}{a}$

There remains to determine the radius of the Circle $(*)$

$$4x^2+4y^2-4x+12y-39990=0 \qquad (\div 4)$$

$$x^2+y^2-x+3y = \frac{39990}{4}$$

let us Complete the square in x, and y-terms:

$$(x^2-x)+(y^2+3y) = \frac{39990}{4}$$

$\downarrow$ add $\left(-\frac{1}{2}\right)^2$ To both sides

$\downarrow$ add $\left(\frac{3}{2}\right)^2$ To both sides

$$\left[x^2-x+\left(-\tfrac{1}{2}\right)^2\right]+\left[y^2+3y+\left(\tfrac{3}{2}\right)^2\right] = \frac{39990}{4}+\left(-\tfrac{1}{2}\right)^2+\left(\tfrac{3}{2}\right)^2$$

$$\left(x-\tfrac{1}{2}\right)^2+\left(y+\tfrac{3}{2}\right)^2 = \frac{39990}{4}+\frac{1}{4}+\frac{9}{4}$$

$$= \frac{39990+1+9}{4} = \frac{40,000}{4}$$

$$\therefore \quad \left(x-\frac{1}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=10{,}000$$

Radius $\quad a=\sqrt{10{,}000}=100$

Hence $\quad K=\frac{1}{a}=\frac{1}{100}=0.01 \ m^{-1}$

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(29) Recall: The Banking Angle of a road turn is given by

$$\theta = \tan^{-1}\frac{v^2}{\rho g}$$

Here $g=9.8 \ m/s^2$, $\rho=100 \ m$ (from problem 28), and $v=54 \ Km/hr$; that is

$$\frac{54}{3.6}=15 \quad m/s$$

$$\therefore \quad \theta=\tan^{-1}\left(\frac{(15)^2}{(100)(9.8)}\right)\approx 13^\circ$$

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(30) $4x^3 - 5y^3 - 3z + 10 = 0$ ---- (1)

$x^3 + y^3 = 2$ ---- (2)

Given $t = z/3$, hence $z = 3t$

Substituting $z = 3t$ into Equation (1), we obtain

$$4x^3 - 5y^3 - 9t + 10 = 0 \quad \text{---- (3)}$$

But from Equation (2): $y^3 = 2 - x^3$.

Therefore, Equation (3) reduces to

$$4x^3 - 5(2 - x^3) - 9t + 10 = 0$$

$$4x^3 - \cancel{10} + 5x^3 - 9t + \cancel{10} = 0$$

$$\Rightarrow 9x^3 - 9t = 0$$

$$x^3 = t$$

$$\Rightarrow x = \sqrt[3]{t}$$

To find y: $y^3 = 2 - x^3$

$$= 2 - (\sqrt[3]{t})^3$$

$$= 2 - t$$

$$\therefore y = \sqrt[3]{2 - t}$$

$\therefore$ A parametric representation of curve of intersection is given by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$$= \sqrt[3]{t}\,\vec{i} + \sqrt[3]{2-t}\,\vec{j} + 3t\,\vec{k}, \quad t \in \mathbb{R}$$

— — — — — — —

(31) $\vec{v}(t) = -2\sin(t)\vec{i} + 3\cos(t)\vec{j}$, $t \in [0, 2\pi]$

But $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$, and hence

$$\vec{r}(t) = \int \vec{v}(t)\, dt$$

$$= \int \left(-2\sin(t), 3\cos(t)\right) dt$$

$$\vec{r}(t) = \left(2\cos(t) + C_1 ,\ 3\sin(t) + C_2\right)$$

Applying the initial condition

$$\vec{r}(0) = (-1, 1)$$

(which means $r = (-1,1)$ at $t = 0$),

$$(-1,1) = \left(2\cos(0) + C_1 ,\ 3\sin(0) + C_2\right)$$

$$(-1,1) = \left(2 + C_1 ,\ 0 + C_2\right)$$

$$\therefore\ -1 = 2 + C_1 \Rightarrow C_1 = -3$$

$$1 = 0 + C_2 \Rightarrow C_2 = 1$$

$$\therefore\ \vec{r}(t) = \left(-3 + 2\cos(t),\ 1 + 3\sin(t)\right),\ t \in [0, 2\pi]$$

This is an equation of an Ellipse with Centre at $(-3, 1)$ and semi-axes of length $a = 2$, $b = 3$.

Its Cartesian Eq. is thus given by

$$\frac{(x+3)^2}{2^2} + \frac{(y-1)^2}{3^2} = 1 \quad \text{or} \quad \frac{(x+3)^2}{4} + \frac{(y-1)^2}{9} = 1$$

— — — — — —

32. $x = \sin(t) \;\cdots\cdots (1)$ $\quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \cos(2t) \;\cdots\cdots (2)$

We need to Eliminate "t" between (1), (2):

Recall the Double Angle Identity:

$$\cos(2t) = 1 - 2\sin^2(t)$$

substituting (1), (2), we obtain

$$y = 1 - 2x^2$$

This is an equation of a parabola with vertex at $(0,1)$ and which opens downward

<u>End points:</u>

At $t = -\frac{\pi}{2}$,

$x = \sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$,

$y = \cos\left(2\left(-\frac{\pi}{2}\right)\right) = \cos(-\pi) = -1$

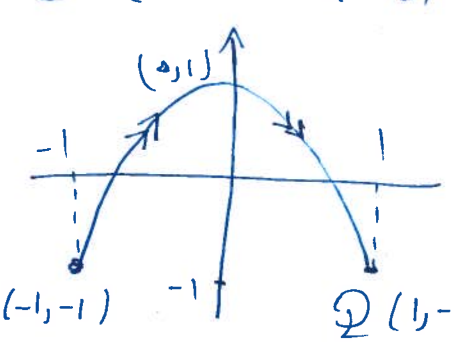
$\therefore$ Initial point is $P(-1,-1)$

At $t = \frac{\pi}{2}$,

$x = \sin\left(\frac{\pi}{2}\right) = 1$, $\quad y = \cos\left(2\left(\frac{\pi}{2}\right)\right) = \cos(\pi) = -1$

$\therefore$ Terminal point is $Q(1,-1)$

Orientation: from P to Q (Indicated by arrow heads)



33. $x(t) = 2\cosh^2(t) - 2$ ... (1)

$y(t) = 4\sinh(t)$ ... (2)

To find Cartesian Equation of the curve, we need to "Eliminate" t among (1), (2):

We shall use the Identity:

$$\cosh^2(t) - \sinh^2(t) = 1 \quad \ldots (*)$$

Now, from (1):

$$x + 2 = 2\cosh^2(t)$$

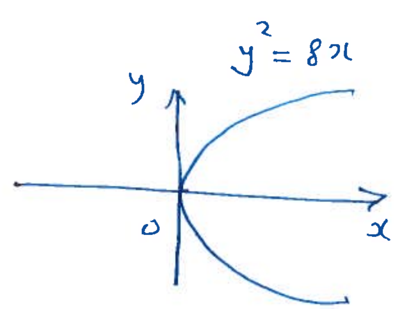
$$\Rightarrow \cosh^2(t) = \frac{x+2}{2} = \frac{x}{2} + 1 \quad \ldots (3)$$

From (2): $y = 4\sinh(t)$

$$\Rightarrow \sinh(t) = \frac{y}{4}$$

$$\Rightarrow \sinh^2(t) = \frac{y^2}{16} \quad \ldots (4)$$

Substituting (3), (4) into (*):



$$\frac{x}{2} + 1 - \frac{y^2}{16} = 1$$

$$\Rightarrow \frac{y^2}{16} = \frac{x}{2} \quad \text{or} \quad \boxed{y^2 = 8x}$$

This is an equation of a parabola with vertex at (0,0), axis of symmetry is the x-axis and which opens to the Right.

34. Recall: As in problem #(8):

$$v = v_e \ln\left(\frac{M}{m(t)}\right), \quad m(t) = M - \alpha t$$

Here $v_e = 400$ m/s, hence

$$v = 400 \ln\left(\frac{M}{m}\right) \quad --- (*)$$

(a) Let $v = 800$, we obtain

$$800 = 400 \ln\left(\frac{M}{m}\right)$$

$$\Rightarrow \ln\left(\frac{M}{m}\right) = 2 \quad \Rightarrow$$

$$\frac{M}{m} = e^2 \quad \Rightarrow \quad m = \frac{M}{e^2}$$

$\therefore$ Amount of burnt fuel $= M - m = M - \frac{M}{e^2} = M\left(1 - \frac{1}{e^2}\right)$

Hence the required ratio:

$$\frac{M-m}{M} = \frac{M\left(1-\frac{1}{e^2}\right)}{M} = 1 - \frac{1}{e^2}$$

$$\overset{\text{or}}{=} 100\left(1 - \frac{1}{e^2}\right)\%$$

$$\cong 86.5\%$$

(b) Here: Remaining Mass $m(t) = 40\%$ of $M = 0.4M$

$\therefore (*) \Rightarrow$

$$V = 400 \ln\left(\frac{\cancel{M}}{0.4\cancel{M}}\right)$$

$$= 400 \ln(2.5) \cong 367 \text{ m/s}$$

(c) Here : Amount of burnt fuel is 40% of M, that is $0.4\,M$. Hence remaining mass $m(t) = M - 0.4M = 0.6M$

$$\therefore (*) \Rightarrow \quad v = 400 \ln\left(\frac{\cancel{M}}{0.6\cancel{M}}\right)$$

$$= 400 \ln\left(\frac{1}{0.6}\right) = 400 \ln\left(\frac{5}{3}\right)$$

$$\cong 204 \text{ m/s}$$

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