The Three Important Equations of

Engineering & Mathematical physics

The Heat Conduction Equation

The Wave Equation

The Potential Equation

The Three Important Equations of Mathematical physics

(1) The Heat Conduction Equation

The 3-dimensional Heat Equation is of the form

$$K\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial u}{\partial t} \quad \cdots (*)$$

Here $u = u(x, y, z; t)$ is the Temperature at position (x, y, z) in a solid at time t, and K is a Constant Called "Diffusivity".

<u>Notation</u>: <u>Laplace Operator</u>, <u>Laplacian</u>, and <u>Laplace Equation</u>:

The Differential operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

is the Three-Dimensional Laplace operator and is denoted by ∇^2 (del-squared).

The quantity $\nabla^2 u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$ is Called:

The Laplacian of $\vec{u}$, and the equation

$$\nabla^2 u = 0$$

is referred to as: Laplace Equation.

<u>Note</u>: A function u that satisfies Laplace Equation is Called: A "harmonic function".

Using these notations above, the three-dimensional heat conduction equation takes the simple form

$$K \nabla^2 u = \frac{\partial u}{\partial t}$$

Note that the two-dimensional and the one-dimensional heat equation are respectively given by

$$K \nabla^2 u = \frac{\partial u}{\partial t} \quad , \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{and}$$

$$K \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

(2) The Wave Equation

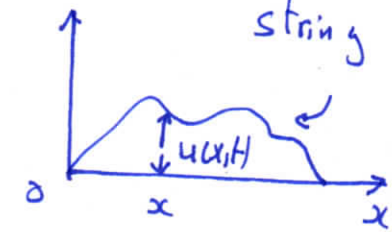
The one-dimensional wave equation is of the form

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

or $$c^2 u_{xx} = u_{tt}$$

This equation is applicable to the small transverse vibration of an elastic string stretched between two fixed points on the x-axis (as shown) and which no external forces act on the string.

The function $u = u(x,t)$ is the displacement of a point "x" of the string at time t, and c is a positive constant.



The wave Equation can be easily generalized to higher dimensions. For instance the two-dimensional wave Equation

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}$$

(or $c^2\nabla^2 u=\frac{\partial^2 u}{\partial t^2}$, $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$)

describe the small vibration of a membrane initially located on a rigid fram in the xy-plane.

(3) The potential Equation

The Laplace Equation $\nabla^2 u=0$ occur in many fields

In the theory of gravitation or Electricity, u represents the gravitational or Electrical potential respectively.

For this reason the Equation is often called: The Potential Equation.

Note also that in the theory of heat Conduction, the steady-state temperature u (that is the temperature after a long time has elapsed) satisfies the Laplace Equation $\kappa\nabla^2 u=0 \Rightarrow \boxed{\nabla^2 u=0}$

(This is because u is independent of time t, and hence $\frac{\partial u}{\partial t}\equiv 0$).
