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Course Booklet

MATH277 Midterm

University of Calgary*

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Thanks a lot!

Antoine

Preface

This Course Booklet contains a summary of the key material that will be tested on the midterm of MATH 277. I encourage you to read through it, and write in any additional notes that you think will help you better understand the material.

The material is divided into 2 chapters. Each chapter begins with a summary of the theory and ends with practice exam-style questions. Some questions will be worked through during the Prep Session and some will be left for you to do on your own later.

At the end of the book is a list of Exam-Writing Tips. I hope you find these pointers useful.

Solutions

Solutions to all the questions will be posted at

www.prep101.com/solutions

after the Prep Session. I strongly recommend that you try to solve the questions on your own before looking at the solutions.

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Study List

This booklet was written based on your book, class notes, and previous midterm. Consequently this is your basic study list for your test. Now I must say this is not a complete MATH 277 course. Your notes and your book are good references too. Also whatever you teacher tells you about the test is golden! Remember he/she is the only one who knows for sure what's on the exam.

Before you go into your exam make sure you understand and can solve at least all the problems in this booklet! Go through the table of content and make sure you know exactly what each topic is about. Nothing beats being prepared all along but hey, we don't all do that, so now is your chance to recap! Work hard and I'm sure you'll be surprised by the results!

Best of luck!

Antoine

Contents

Chapter 1

Linear Algebra and Vector Calculus

1.1 Vector Functions and Curves

Review of Linear Algebra

Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be differentiable vector functions, and let $\lambda(t)$ be a differentiable function then

- $\dfrac{d}{dt}(\mathbf{u}(t) \pm \mathbf{v}(t)) = \mathbf{u}'(t) \pm \mathbf{v}'(t)$
- $\dfrac{d}{dt}(\lambda(t)\,\mathbf{u}(t)) = \lambda'(t)\,\mathbf{u}(t) + \lambda(t)\,\mathbf{u}'(t)$
- $\dfrac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
- $\dfrac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
- $\dfrac{d}{dt}\mathbf{u}(\lambda(t)) = \lambda'(t)\,\mathbf{u}'(t)$
- $\dfrac{d}{dt}|\mathbf{u}(t)| = \dfrac{\mathbf{u}(t) \cdot \mathbf{u}'(t)}{|\mathbf{u}(t)|}$

A consequence of the last property is that, if $\mathbf{u}(t)$ is a vector of constant magniture (unit vector for example) then $\mathbf{u}'(t)$ is perpendicular to $\mathbf{u}(t)$.

Vector Function

A vector function is a vector whose components are each functions of some parameter.

In general, a vector function that describes a curve in $\mathbb{R}^3$ is given by

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where the letter t describes the independent variable.

Derivatives of Vector Functions

Given a vector function

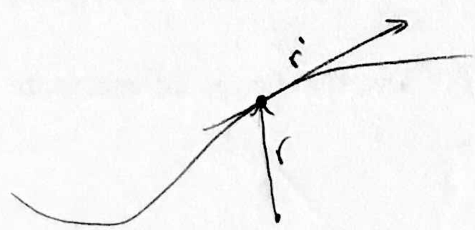
$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

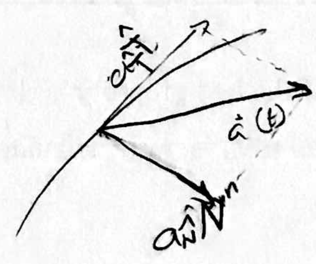
where $f(t)$, $g(t)$, and $h(t)$ are differentiable, then the derivative of $r(t)$ is

$$\mathbf{r}'(t) = \frac{d\mathbf{r}(t)}{dt} = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Position, Velocity, Speed, Acceleration

- Position Vector $\mathbf{r}(t)$.
- Velocity $\mathbf{v}(t) = r'(t)$.
- Speed $= v(t) = |\mathbf{v}(t)|$
- Acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$





Definite Integral of a Vector Function

The definite integral of a vector function is given by

$$\int_{t=a}^{b} \mathbf{r}(t)dt = \left(\int_{t=a}^{b} f(t)dt\right)\mathbf{i} + \left(\int_{t=a}^{b} g(t)dt\right)\mathbf{j} + \left(\int_{t=a}^{b} h(t)dt\right)\mathbf{k}$$

Arc Length

If the curve that has a vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $a \leqslant t \leqslant b$, ($f'(t)$, $g'(t)$ and $h'(t)$ are continuous functions) is traversed from $t = a$ to $t = b$ then the arc length is defined by

$$L = \int_{t=a}^{b} |\mathbf{r}'(t)|dt = \int_{t=a}^{b} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$$

Arc Length Function

The arc length function $s(t)$ is given by

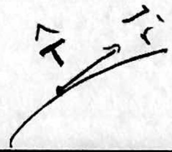
$$s(t) = \int_{s=a}^{t} |\mathbf{r}'(u)|du = \int_{s=a}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Note:

To re-parametrize a curve with respect to arc length we find $s(t)$ then solve for t as a function of s and replace in $(r)(t)$. This is useful because for $s = k$ then $r(t(k))$ is the position vector of the point that is k units of length along the curve from its starting point.

Unit Tangent Vector

If C is a smooth curve defined by the vector function $\mathbf{r}(t)$, then $\mathbf{r}(t) \neq \mathbf{0}$. The **unit tangent vector** $\mathbf{T}(t)$ to the curve C is defined by



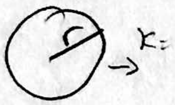
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

$\hat{T}$ - Unit vector

Curvature

The curvature $\kappa(t)$ of the curve C is

How fast the curve "curves"



$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$\rho = \frac{1}{\kappa}$

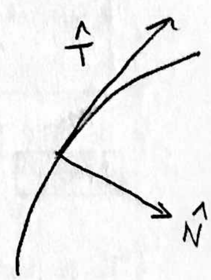
radius of curvature

In the special case where $y = f(x)$ defines a plane curve then

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

Binormal Vector

$\boxed{\text{(TNB Frame)}}$ Ferney Frame?



T - Thumb

N -

B - Sticks to plane

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}$$

Unit Normal Vector

T B N

The principal unit normal vector $\mathbf{N}(t)$ or simply the unit normal of a curve $\mathbf{r}(t)$ is given by

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$$

$T \times N = B$

$N \times B = T$

$B \times T = N$

$B \times N = -T$

$N \times T = -B$

$T \times B = -N$

Tangential and Normal Acceleration

We have that $\mathbf{a} = \mathbf{r}''(t) = \dfrac{d|\mathbf{r}'|}{dt}\,\mathbf{T} + |\mathbf{r}'|^2\kappa\,\mathbf{N}$ so

a_T $\dfrac{d|\mathbf{r}'|}{dt}$ is called the tangential component of the acceleration, and

a_N $|\mathbf{r}'|^2\kappa$ is called the normal component of the acceleration.

Torsion

The torsion of a vector is given by

$$\tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

Normal Plane

The normal plane of a curve C at the point P is the plane determined by the normal vector $(\mathbf{N}(t))$, binormal vector $(\mathbf{B}(t))$, and passing through the point P.

Osculating Plane

The osculating plane of a curve C at the point P is the plane determined by the tangent vector $(\mathbf{T}(t))$, normal vector $(\mathbf{N}(t))$, and passing through the point P.

Osculating Circle

The osculating circle (or circle of curvature) of a curve C at a point P is the circle that lies in the osculating plane of C at P, on the concave side of C (side toward which $\mathbf{N}$ points), has the same tangent as C at P, and has radius $\rho = 1/\kappa$. The osculating circle also shares the same normal and curvature as C at P.

Motion Involving Varying Mass

The **momentum p** of a moving object is the product of its mass ($m(t)$) and its velocity $\mathbf{v}(t)$. Newton's Second Law of motion sates that the rate of change of the momentum is equatl to the external force acting on the object *i.e.*

$$\mathbf{F} = \frac{d}{dt}\left(m(t)\,\mathbf{v}(t)\right)$$

The Changing Velocity of a Rocket

Suppose a rocket accelerates by burning its onboard fuel. If the exhaust gases are ejected with constant velocity $\mathbf{v_e}$ relative to the rocket, and if the rocket ejects $p\%$ of its initial mass while its engines are firing, by what amount will the rocket's velocity change? Assume the rocket is in deep space so that all external forces acting on it can be neglected.

Let $\mathbf{v}(t)$ be the velocity of the rocket at time t, and $m(t)$ the mass of the rocket at time t (rocket + fuel) then one can show that

$$\mathbf{v}(t) - \mathbf{v}(0) = -\mathbf{v_e} \ln\left(\frac{m(0)}{m(t)}\right)$$

So if $p\%$ of the initial mass is ejected then $(100-p)m(0)/100$ remains at time t and therefore, replace in the formula above to get the change in the rocket's velocity

$$\mathbf{v}(t) - \mathbf{v}(0) = -\mathbf{v_e} \ln\left(\frac{100}{100-p}\right)$$

Banking a Turn

Given a Frictionless banked turn, let ρ is the radius of curvature of the turn $\left(\rho = \frac{1}{\kappa}\right)$, θ the banking angle, and v the posted speed limit then

$$\boxed{\theta = \tan^{-1}\left(\frac{v^2}{\rho g}\right)}$$

Remember that $g \approx 9.8\ \text{m/s}^2$ or $32\ \text{ft/s}^2$.

Solving for v we may also write

$$\boxed{v = \sqrt{\rho g \tan\theta}}$$

Some Curves and Their Parameterizations

Lines in 3-Space

Vector Parametric Equations

$$\mathbf{r} = \mathbf{r_0} + t\,\mathbf{d}$$

$\mathbf{r}$ is the position vector,

$\mathbf{r_0}$ is the position at (x_0, y_0, z_0), and

$\mathbf{d}$ is a vector parallel to the line.

Scalar Parametric Equations of a Line

$$x = x_0 + a\,t \qquad y = y_0 + b\,t \qquad z = z_0 + c\,t$$

$\mathbf{d} =< a, b, c >$ is a vector parallel to the line, and

(x_0, y_0, z_0) is a point on the line.

Standard or Symmetric Form of the Equation of a Line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Planes in 3-Space

Standard Form of the Equation of a Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$\mathbf{n} =< a, b, c >$ is a vector perpendicular to the plane (normal), and

(x_0, y_0, z_0) is a point on the plane.

Circle in 2-Space

$\boxed{\cos^2\theta + \cos^2\theta = 1}$

Given a circle centred at (h, k) with radius r,

$r^2\cos^2\theta + r^2\sin^2\theta = 1r^2$

Cartesian Equation of a Circle

$$\boxed{(\underbrace{x-h}_{r\cos t})^2 + (\underbrace{y-k}_{r\sin t})^2 = r^2}$$

$x - h = r\cos t$ $\quad$ $y - k = r\sin t$

Parametric Equations of a Circle

$$x = h + r\cos t, \qquad y = k + r\sin t$$

Ellipse in 2-Space

Given an ellipse centred at (h, k) with radii a (along x-axis) and b (along y-axis),

Cartesian Equation of a ~~Circle~~ Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \underbrace{\left(\frac{x-h}{a}\right)^2}_{\cos t} + \underbrace{\left(\frac{y-k}{b}\right)^2}_{\sin t} = 1$$

Parametric Equations of an Ellipse

$$x = h + a\cos t, \qquad y = k + b\sin t$$

Parabola in 2-Space

Given an parabola with vertex (h, k),

Equation of a Parabola

$x(t) = t$

$y(t) = a(t-h)^2 + k$

If the parabola is open "up" or "down"

$$y = a(x-h)^2 + k$$ make $x = t$ get y

If the parabola is open to the "right" or "left"

$$x = a(y-k)^2 + h$$ make $y = t$ get x

Parametric Equations of a Parabola

If the parabola is open "up" or "down"

$$x = t + h, \qquad y = at^2 + k$$

If the parabola is open to the "right" or "left"

$$y = t + k, \qquad x = at^2 + h$$

Hyperbola in 2-Space

Given a hyperbola centred at (h, k),

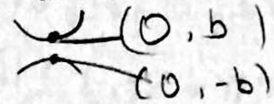
$\cosh^2 t - \sinh^2 t = 1$

Equation of a Hyperbola

If the parabola is open "up" and "down"

like ellipses but with minus.

The boss $$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$



If the parabola is open to the "right" and "left"

The boss $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Parametric Equations of a ~~Parabola~~ Hyperbola

If the parabola is open "up" and "down"

$$x = h + a\sinh t, \qquad y = k \pm b\cosh t$$

If the parabola is open to the "right" and "left"

$$y = k + b\sinh t, \qquad x = h \pm a\cosh t$$

Remember that $\cosh^2 x - \sinh^2 x = 1$

1.2 Exercises

Exercise 1

Determine the velocity, speed, and acceleration at time t of the particle with position

$\mathbf{r}(t) = \underbrace{2\cos t \sin t}_{\sin(2t)}\,\mathbf{i} + \overset{2\sin 2t}{2\sin^2 t}\,\mathbf{j} + 2\cos t\,\mathbf{k}$

$v(t) = \vec{r}\,'(t) = (2\cos(2t), 4\sin 2\cos t, -2\sin t)$ Velocity

$\|\vec{r}\,'(t)\| = \text{Speed} = \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 4\sin^2 t}$

$= \sqrt{4 + 4\sin^2 t}$

$= 2\sqrt{1 + \sin^2 t}$

$\vec{a}(t) = \vec{r}\,''(t) = (-4\sin(2t), 4\cos(2t), -2\cos t)$

acceleration.

Exercise 2

3D - Ask for normal plane.

only in 2D scenarious

3D no notion of perpendicular

Find the cartesian equations of the tangent and normal line to $\mathbf{r}(t) = < \cos t, 2\sin t >$ at the point corresponding to $t = \pi/6$

In 2D ($\mathbb{R}^2$)

slope & pt

$y - y_p = m(x - x_p)$

$\text{slope} = \frac{\Delta y}{\Delta x}$

All theory for lines in 2D

Special 1st Quadrant

	0	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$	π
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-

Point $(\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6})$

$(\frac{\sqrt{3}}{2}, 1)$

$\vec{r}'(t) = \langle -\sin t, 2\cos t \rangle$

$\vec{r}'(\frac{\pi}{6}) = \langle -\frac{1}{2}, \sqrt{3} \rangle$

$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\sqrt{3}}{-\frac{1}{2}} = -2\sqrt{3}$

Tan line: $m = -2\sqrt{3}$ pt $(\frac{\sqrt{3}}{2}, 1)$

$y - 1 = -2\sqrt{3}(x - \frac{\sqrt{3}}{2})$

$y = -2\sqrt{3}x + 4$

Normal line: $m_\perp = -\frac{1}{m}$ (in 2D)

in 3D no concept of slope only direction vector

$y - 1 = \frac{1}{2\sqrt{3}}(x - \frac{\sqrt{3}}{2})$

$\boxed{y = \frac{1}{2\sqrt{3}}x + \frac{3}{4}}$

Exercise 3 In 3D

Find the parametric equations of the tangent line to $\mathbf{r}(t) =< t, -2t^2, 2t^3 >$ at the point $(1, -2, 2)$.

Line in 3D - Needs a direction and a point

$\vec{d} = \langle a, b, c \rangle$ (x_0, y_0, z_0)

parametric equation

$$\left.\begin{aligned} x &= at + x_0 \\ y &= bt + y_0 \\ z &= ct + z_0 \end{aligned}\right\} \rightarrow \vec{r}(t) = \langle at + x_0, bt + y_0, ct + z_0 \rangle$$

$\langle 1, -2, 2 \rangle$

$(0,0,0)$

pt $(1, -2, 2) = t = \boxed{1}$

$t = 1$

$-2t^2 = -2$

$2t^3 = 2$

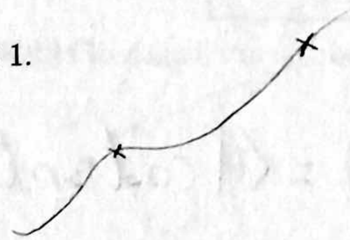
$(t=1)$ $\vec{r}'(1) = \langle 1, -4, 6 \rangle$

$Pt = (1, -2, 2)$

$$\begin{cases} x = t + 1 \\ y = -4t - 2 \\ z = 6t + 2 \end{cases}$$

Exercise 4

Find the arc length of $\mathbf{r}(t) = e^t\,\mathbf{i} + \sqrt{2}t\,\mathbf{j} + e^{-t}\,\mathbf{k}$ for $0 \leqslant t \leqslant 1$.



$$L = \int_{t=a}^{t=b} \underbrace{\|\vec{r}\,'(t)\|\, dt}_{ds}$$

$$\vec{r}\,'(t) = \langle e^t, \sqrt{2}, -e^{-t} \rangle$$

$$\|r'(t)\| = \sqrt{(e^t)^2 + (\sqrt{2})^2 + (-e^{-t})^2}$$

Look for the square

$$= \sqrt{\underbrace{e^{2t} + 2 + e^{-2t}}}$$

$$= \sqrt{(e^t + e^{-t})^2}$$

$$\int_0^1 (e^t + e^{-t})\,dt = (e^t - e^{-t})\Big|_0^1 = e^1 - e^{-1} - \cancel{0} = e - \frac{1}{e}$$

Exercise 5

Find the arc length of $\mathbf{r}(t) = 2\sin^2 t\,\mathbf{i} + \cos^3 t\,\mathbf{j} + \sin^3 t\,\mathbf{k}$ for $0 \leqslant t \leqslant \pi/4$.

$$r'(t) = \langle 4\cos t \sin t,\ -3\cos^2 t \sin t\ \ 3\cos t \sin^2 t \rangle$$

$$\|r'(t)\| = \sqrt{(4\cos t \sin t)^2 + (-3\cos^2 t \sin t)^2 + (3\cos t \sin^2 t)^2}$$

$$= \sqrt{16\cos^2 t \sin^2 t + 9\cos^4 t \sin^2 t + 9\cos t^2 \sin^4 t}$$

$$= \sqrt{\cos^2 t \sin^2 t\,(\underbrace{16 + 9\cos^2 t + 9\sin^2 t}_{25})}$$

$$= \sqrt{25}$$

$$= 5\sin t \cos t \quad \frac{5}{2}\sin(2t)$$

$$\frac{5}{2}\int_0^{\pi/4} \sin(2t)\,dt = -\frac{5}{2}\cdot\frac{1}{2}\cos(2t)\Big|_0^{\frac{\pi}{4}}$$

$$= \cancel{0} + \frac{5}{4}(1) = 5/4$$

Exercise 6

Give a parametrization of the segment line from $(0,1,2)$ to $(1,1,6)$.

$(\text{end} - \text{beg})t + \text{beg} \quad 0 \le t \le 1$

$A \longrightarrow B$

$(1, 0, 4)t + (0, 1, 2) \quad 0 \le t \le 1$

$\vec{r}(t) = \langle t, 1, 4t+2 \rangle \quad 0 \le t \le 1$

Exercise 7

h k

Give a parametrization of the ellipse centred at $(1, -2)$ and with radius on the x-axis equal to 2 and radius on the y-axis equal to 1. → b

a ←

$$1 = \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} \qquad \frac{(x-1)^2}{4} + \frac{(y+2)^2}{1} = 1$$

$$\frac{x-1}{2} = \cos t \qquad \begin{cases} x = 2\cos t + 1 \\ y = \sin t - 2 \end{cases}$$

$$y + 2 = \sin t$$

Exercise 8

Give a parametrization of the curve $x^2 + y^2 + 6y - 2x - 15 = 0$. Identify the curve.

$$(x^2 - 2x + 1) + (y^2 + 6y + 9) = 15 + 1 + 9$$

$$\underbrace{(x-1)}_{5\cos t}{}^2 + \underbrace{(y+3)}_{5\sin t}{}^2 = 25$$

centre $(1, -3)$ $r = 5$

$x = 5\cos t + 1$

$y = 5\sin t - 3$

$[0, 2\pi)$

Exercise 9

Give a parametrization of the intersection of $\boxed{x^2 + y^2 = 9}$ and $z = x + y$.

$x = 3\cos t$

$y = 3\sin t$

$z = 3\cos t + 3\sin t$

$\vec{r}(t) = \langle 3\cos t, 3\sin t, 3\cos t + 3\sin t \rangle$

#4 (Winter 2015) $4x^2 + y^2 + z^2 = 8$ $z = -\sqrt{4x^2 + y^2}$

$4x^2 + y^2 + 4x^2 + y^2 = 8$

$8x^2 + 2y^2 = 8$

$x^2 + \frac{1}{4}y^2 = 1$ Ellipse centered at $(0,0)$

$x^2 + \left(\frac{y}{2}\right)^2 = 1$ x-rad $= 1$

y-rad $= 2$

$x = \cos t$

$y = 2\sin t$

$z = -\sqrt{4\cos^2 t + 4\sin^2 t} = -2$

Exercise 10

Give a parametrization of the intersection of $yz + x = 1$ and $xz - x = 1$

$$\begin{cases} x = \dfrac{1}{t-1} \\ y = \dfrac{(t-2)}{t(t-1)} \\ z = t \end{cases}$$

$x(z-1) = 1$

$x = \dfrac{1}{z-1}$

$z = t$

$x = \dfrac{1}{t-1}$

$y(t) + \dfrac{1}{t-1} = 1$

$y(t) = 1 - \dfrac{1}{t-1} = \dfrac{t-2}{t-1}$

$y = \dfrac{(t-2)}{t(t-1)}$

Exercise 11

Find T, N, B, κ, ρ, and τ for $\mathbf{r}(t) = \sin t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{(2)}\cos(t)\,\mathbf{k}$ at $t = \pi/4$.

$\vec{r}' =$ ①

$\|r'\| =$ ②

$\vec{r}'' =$ ③

$\vec{r}' \times \vec{r}'' =$ ④

$\|r' \times r''\| =$ ⑤

$\vec{r}''' =$ ⑥

$\hat{T} = \dfrac{r'}{\|r'\|}$ $\hat{B} = \dfrac{r' \times r''}{\|r' \times r''\|}$

$a_{\hat{T}} = \dfrac{d}{dt} = \|r'\|$

$\hat{N} = \hat{B} \times \hat{T}$

$a_{\hat{N}} = \dfrac{\|r' \times r''\|}{\|r'\|}$

$\kappa = \dfrac{\|r' \times r''\|}{\|r'\|^3}$ $\rho = \dfrac{1}{\kappa}$

$\tau = \dfrac{(r' \times r'') \cdot r'''}{\|r' \times r''\|^2}$

Exercise 12

$t=1$

Find T, N, B, κ, ρ, and τ for $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k}$ at $(1,1,2)$. $\vec{r}(t) = \langle t, t^2, 2\rangle$

$\vec{r}' = \langle 1, 2t, 0\rangle \quad \|r'\| = \sqrt{1+4t^2}$

$\vec{r}'' = \langle 0, 2, 0\rangle$

$$\begin{matrix} \overset{+}{i} & \overset{-}{2t} & \overset{+}{0} \\ 0 & 2 & 0 \end{matrix}$$

$\vec{r}' \times \vec{r}'' = \langle 0, 0, 2\rangle \quad \|\vec{r}' \times \vec{r}''\| = 2$

$\vec{r}''' = \langle 0, 0, 0\rangle$

@$t=1$ $\vec{r}' = \langle 1, 2, 0\rangle \quad \|r'\| = \sqrt{5}$

$\vec{r}'' = \langle 0, 2, 0\rangle$

$r' \times r'' = \langle 0, 0, 2\rangle \quad \|r' \times r''\| = \sqrt{5} \quad \vec{r}''' = \vec{0}$

$\hat{T} = \frac{r'}{\|r'\|} = \frac{\langle 1, 2, 0\rangle}{\sqrt{5}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\rangle$

$\hat{B} = \frac{r' \times r''}{\|r' \times r''\|} = \langle 0, 0, 1\rangle \quad \hat{N} = \hat{B} \times \hat{T}$

$= \langle -\frac{2}{\sqrt{5}}, \frac{+1}{\sqrt{5}}, 0\rangle$

$$\begin{matrix} \overset{+}{0} & \overset{-}{0} & \overset{+}{1} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{matrix} \quad -\frac{2}{\sqrt{5}} \; -\frac{1}{\sqrt{5}}, \; 0$$

$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3} = \frac{2}{5\sqrt{5}} \quad \rho = \frac{5\sqrt{5}}{2}$

$\tau = 0$

Exercise 13

Find T, N, B, κ, ρ, and τ for $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\,\mathbf{k}$

$\vec{r}' = \langle \cos t, -\sin t, 1 \rangle \quad \|r'\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$

$\vec{r}'' = \langle -\sin t, -\cos t, 0 \rangle$

$\vec{r}' \times \vec{r}'' = \langle \cos t, \sin t, -1 \rangle \quad \|r' \times r''\| = \sqrt{2}$

$\vec{r}''' = \langle -\cos t, \sin t, 0 \rangle$

$\hat{T} = \left\langle \frac{\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \quad \hat{N} = \hat{B} \times \hat{T} = \langle -\sin t, -\cos t, 0 \rangle$

$\hat{B} = \left\langle \frac{\cos t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \quad \kappa = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} \quad \rho = 2$

$\tau = \frac{-1}{2}$

$a_{\hat{T}} = \frac{d}{dt}\|r'\|$

$a_{\hat{N}} = \frac{\|r' \times r''\|}{\|r'\|}$

Exercise 14

Find the tangential and normal components of the acceleration for $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k}$

$$r(t) = \langle t, t^2, \tfrac{1}{2}t^2 \rangle$$

$$r'(t) = \langle 1, 2t, t \rangle$$

$$\|r'\| = \sqrt{1+5t^2}$$

$$r''(t) = \langle 0, 2, 1 \rangle$$

$$r' \times r'' = \langle 0, -1, 2 \rangle \qquad \|r' \times r''\| = \sqrt{5}$$

$$a_{\hat{T}} = \frac{d}{dt}\left(\sqrt{1+5t^2}\right) = \frac{5t}{\sqrt{1+5t^2}}$$

$$a_{\hat{N}} = \frac{\sqrt{5}}{\sqrt{1+5t^2}}$$

Exercise 15

Find the tangential and normal components of the acceleration for

$\mathbf{r}(t) = \ln(t^2+1)\,\mathbf{j} + t - 2\tan^{-1} t\,\mathbf{k}$ at the point where $t = 0$.

Exercise 16

(Rocket)

$$v(t) - v(0) = V_e \ln\left(\frac{m(0)}{m(t)}\right)$$

$$v(t) - v(0) = V_e \ln\left(\frac{100}{100-p}\right)$$

(p% of the initial mass is gone.)

Q. Rocket... gases ejected at 800kg/s

at a constant $V_e = 200$ m/s (relative to rocket)

M initial mass, assume $v(0) = 0$

(a) what percentage of initial mass should be burned or dumped to accelerate to 500m/s?

$$v(t) = V_e \ln\left(\frac{100}{100-p}\right)$$

$$500 = 200 \ln\left(\frac{100}{100-p}\right)$$

$$\frac{5}{2} = \ln\left(\frac{100}{100-p}\right)$$

$$e^{5/2} = \frac{100}{100-p}$$

$$100e^{5/2} - e^{5/2}p = 100$$

$$p = \frac{100e^{5/2} - 100}{e^{5/2}}$$

Exercise 17

(Rocket)

Exercise 18

(Banking angle)

$\theta = \tan^{-1}\left(\frac{v^2}{\rho g}\right)$ v: speed, ρ: radius, θ: angle

$g = 9.8 m/s^2$ & $32 ft/s^2$

☆ Given v, ρ $\theta = \tan^{-1}\left(\frac{v^2}{\rho g}\right)$

☆ Given θ, v $\tan\theta = \frac{v^2}{\rho g} \Rightarrow \rho = \frac{v^2}{g\tan\theta}$

☆ Given θ, ρ $v^2 = \rho g \tan\theta \Rightarrow v = \sqrt{\rho g \tan\theta}$

☆ frictionless road turn with road $\frac{40}{9.8}$ m
Max speed is $2\sqrt{10}$ m/s. Banking angle?

$\rho = \frac{40}{9.8}$ $v = 2\sqrt{10}$ $\theta = \tan^{-1}\left(\frac{v^2}{\rho g}\right)$

$\theta = \tan^{-1}\left(\frac{40}{\frac{40}{9.8}9.8}\right) = \tan^{-1}(1)$

$\theta = \left(\frac{\pi}{4}\right)$

Given $v = \sqrt{9.8}$ m/s $\theta = \frac{\pi}{4}$ k-? $k = \frac{1}{\rho}$

$\tan\theta = \frac{v^2}{\rho g} \Rightarrow \tan\frac{\pi}{4} = \frac{(\sqrt{9.8})^2}{\rho(9.8)} \Rightarrow \frac{1}{\rho} = \frac{1}{\sqrt{3}}$

$k = \sqrt{3}$

Exercise 19

(Banking angle)

$$\Theta = \tan^{-1}\left(\frac{v^2}{\rho g}\right) = f(v, \rho)$$

$$d\theta = \frac{\partial f}{\partial v} + \frac{\partial f}{\partial \rho} \cdot d\rho \text{ (Exact differential)}$$

More exercises

Winter 2015 #1 $\vec{r}(t) = \langle t^3, \sqrt{3}t^2, (2t+1) \rangle$ $0 \le t \le 2$

arc length

$$\vec{r}'(t) = (3t^2, 2\sqrt{3}t, 2)$$

$$\|\vec{r}'\| = \sqrt{(3t^2)^2 + (2\sqrt{3}t)^2 + 4}$$

$$= \sqrt{9t^4 + 12t^2 + 4}$$

$$= \sqrt{(3t^2+2)(3t^2+2)}$$

$$= \sqrt{(3t^2+2)^2}$$

$$= 3t^2 + 2$$

$$\int_0^2 3t^2 + 2 = t^3 + 2t \Big|_0^{\boxed{2}} = 2^3 + 2(2)$$

$$= 12$$

Chapter 2

Partial Derivatives

2.1 A Review of the Theory

Function of Several Variables

A function f of n variables is a rule that assigns to each n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ in a set D a unique real number denoted by $f(x_1, x_2, \ldots, x_n)$.

Domain

The domain of f is the set of n-tuples $(x_1, x_2, \ldots, x_n)$ for which f is defined.

Range

The range of f is the set of values that f takes on when $(x_1, x_2, \ldots, x_n)$ is in the domain of f.

In what follows most of the theory will be presented with functions of two variables. The extension to three variables will only be mentioned if it is not trivial.

Partial Derivatives

Given $z = f(x, y)$, the partial derivative of f with respect to x is written as:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

The partial derivative of f with respect to y is written as:

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Higher Derivatives

The **second partial derivatives** of f are given by

$$(f_x)_x = \frac{\partial f}{\partial xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = \frac{\partial f}{\partial xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = \frac{\partial f}{\partial yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = \frac{\partial f}{\partial yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Clairaut's Theorem

If f is defined on a disk D and the functions f_{xy} and f_{yx} are continuous on D, then

$$\boxed{f_{xy} = f_{yx}}$$

Note:

The previous theorem is saying that if "everything" is smooth then it the order of differentiation does not matter.

Tangent Planes

Suppose that f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_o, y_o, z_o)$ is

$$z - z_o = f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)$$

Linear Approximation

Given a function $f(x, y)$, the linear approximation (or tangent plane approximation) of $f(x, y)$ at (a, b) is given by:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Total Differential

Given a function $z = f(x, y)$, the total differential is defined by:

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

The Chain Rule

Recall the chain rule from single variable calculus.

$$\text{If } y = f(x) \text{ and } x = g(t) \text{ then } \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Now consider a differentiable function $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$ are both differentiable functions. Therefore z is a differentiable function of t and

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Now consider a differentiable function $z = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions. Therefore z is a differentiable function of s and t and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

In general, suppose that u is a function of n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. This implies that u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} \quad \text{for each } i = 1, 2, \ldots, m.$$

Note:

The chain rule for function of multiple variables can best be presented using a "tree".

The Jacobian

The Jacobian of two functions f and g with respect to two variables x and y is given by the determinant

$$\boxed{\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\[2ex] \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{vmatrix}}$$

The Jacobian of three functions f, g and h with respect to three variables x, y and z is given by the determinant

$$\boxed{\frac{\partial(f,g,h)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\[2ex] \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \\[2ex] \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{vmatrix}}$$

This result may be generalized to any number of functions and variables.

Implicit Function Theorem

Consider a system of n equations ($F_i = 0, i = 1 \cdots n$) in $n+r$ variables $(x_1, \cdots, x_r, y_1, \cdots, y_n)$.

$$\text{If } \frac{\partial(F_1, F_2, \cdots, F_n)}{\partial(y_1, y_2, \cdots, y_n)} \neq 0 \text{ at point } P_0$$

then $F_1, F_2, \cdots, F_n$ can be solved for $y_1, y_2, \cdots, y_n$ as functions of $x_1, \cdots, x_r$ near P_0.

In other words $F_1, F_2, \cdots, F_n$ define $y_1, y_2, \cdots, y_n$ implicitly as functions of $x_1, \cdots, x_r$ near P_0.

Moreover,

$$\boxed{\left(\frac{\partial y_i}{\partial x_j}\right)_{x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_r} = -\frac{\dfrac{\partial(F_1, F_2, \cdots, F_n)}{\partial(y_1, \cdots, x_j, \cdots, y_n)}}{\dfrac{\partial(F_1, F_2, \cdots, F_n)}{\partial(y_1, \cdots, y_i, \cdots, y_n)}}}$$

Gradient

If $f(x, y)$ is a differentiable function then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Directional Derivative

If $f(x, y)$ is a differentiable function then its directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ is given by

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b = \nabla f(x, y) \cdot \mathbf{u}$$

The two above results are simply extended to a function of 3 variables or more.

Maximum/Minimum of the Directional Derivative

If f is a differentiable function of two or three variables, the maximum value of the directional derivative $D_{\mathbf{u}}f$ is $|\nabla f|$

The function increases most rapidly when $\mathbf{u}$ has the same direction as ∇f.

The function decreases most rapidly when $\mathbf{u}$ has the same direction as $-\nabla f$.

Tangent Planes to Level Surfaces

Given a surface S with equation $F(x, y, z) = k$ (a level surface). The equation of the tangent plane at a point $P(x_o, y_o, z_o)$ on the surface is given by

$$F_x(x_o, y_o, z_o)(x - x_o) + F_y(x_o, y_o, z_o)(y - y_o) + F_z(x_o, y_o, z_o)(z - z_o) = 0$$

Provided that $\nabla F(x_o, y_o, z_o) \neq \mathbf{0}$ and that $P(x_o, y_o, z_o)$ is on the surface.

Normal Lines to the Surface

The symmetric equations of the normal line to the surface S at the point $P(x_o, y_o, z_o)$ are

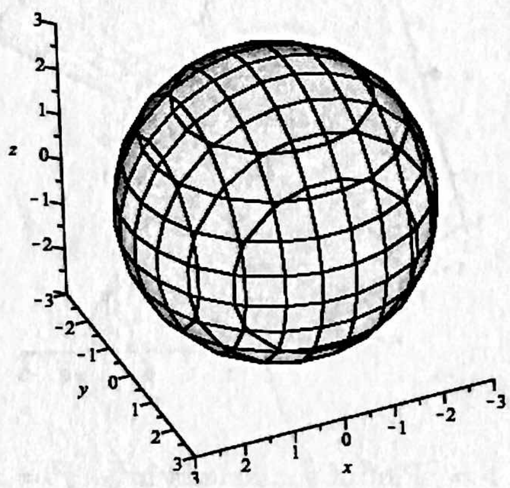
$$\frac{x - x_o}{F_x(x_o, y_o, z_o)} = \frac{y - y_o}{F_y(x_o, y_o, z_o)} = \frac{z - z_o}{F_z(x_o, y_o, z_o)}$$

Quadric Surfaces

Sphere

A sphere with radius r and centre (x_0, y_0, z_0) has equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$



Plot of the sphere $x^2 + y^2 + z^2 = 9$

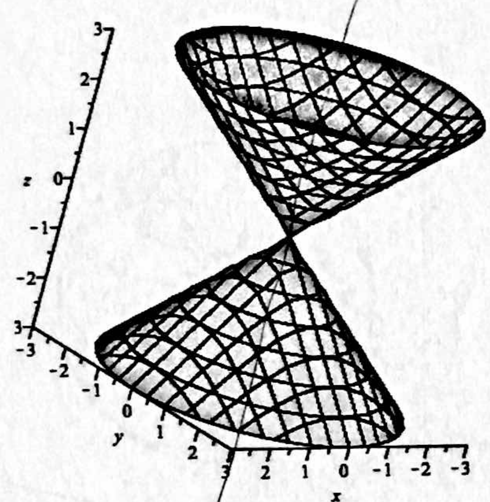
Right-Circular Cone

The equation of a right circular cone with centre axis along the z-axis is

$$x^2 + y^2 = z^2$$

z

$z = \sqrt{x^2+y^2}$

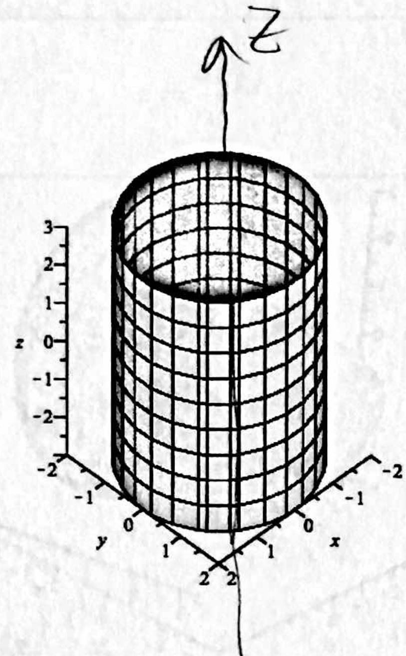


Plot of the cone by $x^2 + y^2 = z^2$

Right Cylinder

The equation of a right cylinder with centre axis along the z-axis is

$$x^2 + y^2 = a^2$$

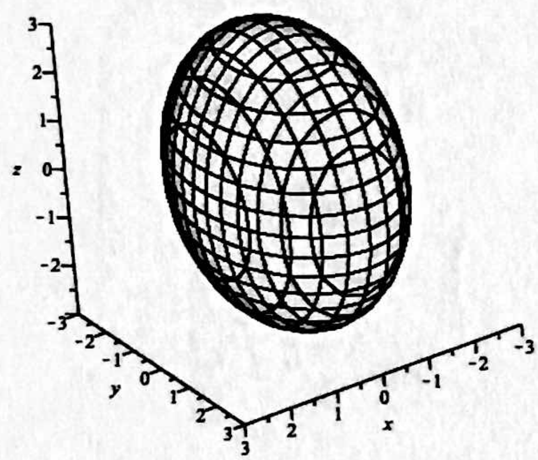


Plot of the cylinder $x^2 + y^2 = 4$

Ellipsoid

The equation of an ellipsoid centred at (x_0, y_0, z_0) and with semi axes $a, b,$ and c is

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$



Plot of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{9} = 1$

Paraboloid

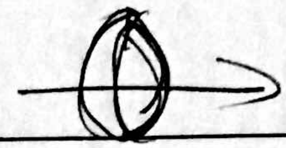
The equation of an **Elliptic Paraboloid** is

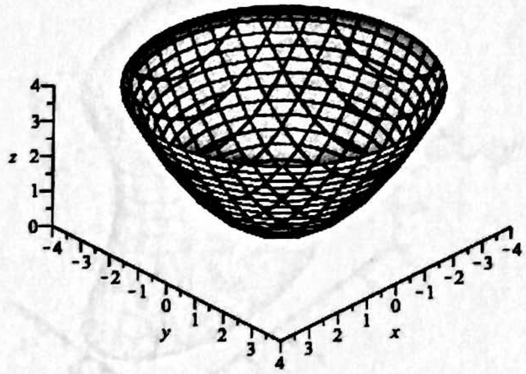
$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$

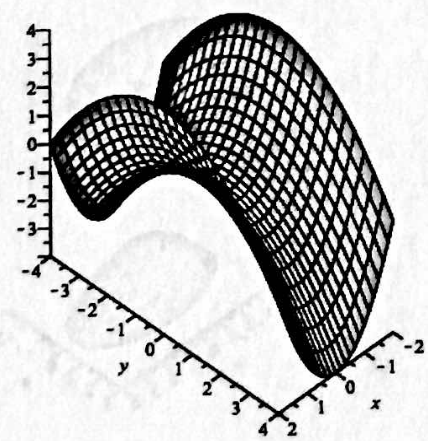
The equation of a **Hyperbolic Paraboloid** is

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$





Plot of the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} = z$



Plot of the hyperbolic paraboloid $x^2 - \frac{y^2}{4} = z$

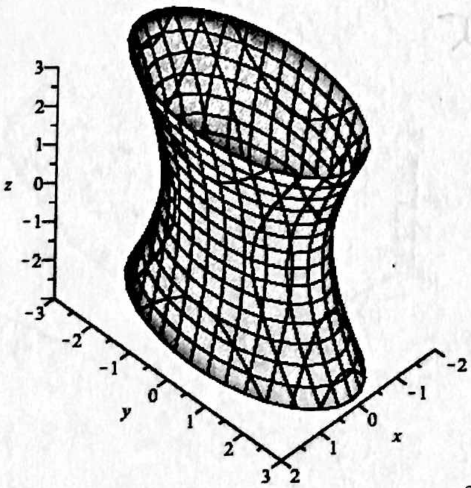
Hyperboloid

The equation of a **Hyperboloid of One Sheet** is

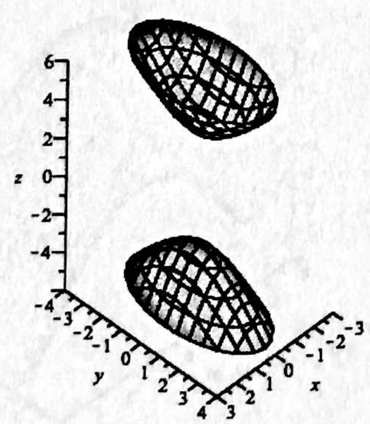
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

The equation of a **Hyperboloid of Two Sheets** is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$



Plot of the hyperboloid of one sheet $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1$



Plot of the elliptic paraboloid $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = -1$

Solutions posted at **www.prep101.com/solutions**

2.2 Exercises

Exercise 1

Find the domain of $f(x, y) = \sqrt{\ln(9 - x^2 - y^2)}$.

$9 - x^2 - y^2 > 0$

(inside a logarithm)

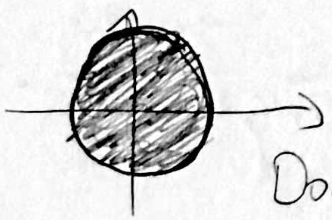
AND

$\ln(9 - x^2 - y^2) \geq 0$

$9 - x^2 - y^2 \geq 1 \leftarrow (e^0)$

$9 - x^2 - y^2 \geq 1$

$x^2 + y^2 \leq 8$



Domain inside boundary.

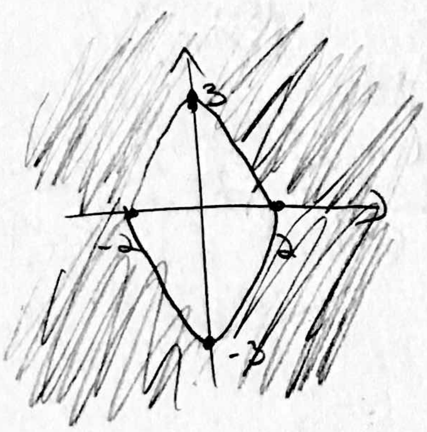
Exercise 2

Find the domain of $f(x, y) = \sqrt{9x^2 + 4y^2 - 36}$.

$9x^2 + 4y^2 - 36 \geq 0$

$9x^2 + 4y^2 \geq 36$

$\frac{x^2}{4} + \frac{y^2}{9} \geq 1$



outside + boundary

Exercise 3

Draw the level curves of $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2 + 1}$ for $k = 0, 1/2, -1/2$

$f(x,y) = k \leftarrow$ level curve

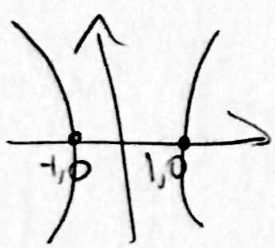
$(k=0)$ $\dfrac{x^2-y^2}{x^2+y^2+1} = 0 \rightarrow x^2 - y^2 = 0 \Rightarrow y^2 = x^2$

$y = x$ or $y = -x$

$\left(k = \frac{1}{2}\right)$ $\dfrac{x^2-y^2}{x^2+y^2+1} = \dfrac{1}{2} \Rightarrow x^2 - y^2 = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}$

$\dfrac{x^2}{2} - \dfrac{y^2}{2/3} = \dfrac{1}{2}$

$x^2 - \dfrac{y^2}{1/3} = 1$



$x = \cosh t$

$y = \sqrt{\frac{1}{3}}\sinh t$

$k = -\frac{1}{2}$

$\ldots \quad x^2 - y^2 = -\frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}$

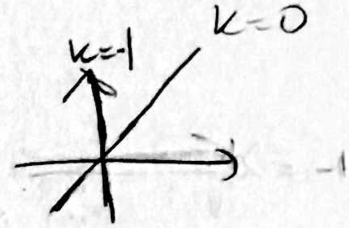
Exercise 4

Draw a few level curves of $f(x,y) = \dfrac{x-y}{x+y}$.

$k=0 \qquad x-y=0 \Rightarrow y=x$

$k=-1 \qquad x-y=-x-y \Rightarrow 2x=0$

$x=0$



Exercise 5

Find all the first partial derivatives of of $f(x,y) = x^{y \ln z}$.

$$\frac{\partial f}{\partial x} = y\ln z \, x^{y\ln z - 1}$$

$$\frac{\partial f}{\partial y} = x^{y\ln z} \ln z \cdot \ln x$$

$$\frac{\partial f}{\partial z} = x^{y\ln z} \ln x \cdot \frac{y}{z}$$

Note:

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} n^x = n^x \ln n$$

$$\frac{d}{dx} e^{4x^2} = e^{4x^2} \cdot 8x$$

$$\frac{d}{dx} 8^{4x^2} = 8^{4x^2} \cdot 8x \cdot \ln 8$$

11. $z = y^x$ $\frac{\partial^2 z}{\partial x \partial y}$

$$\frac{\partial z}{\partial y} = x \cdot y^{x-1}$$

$$\frac{\partial^2 z}{\partial x \partial y} = y^{x-1} + x \cdot y^{x-1} \cdot \ln y$$

☆A $f(x,y) = x^{\sin y}$

$$\frac{\partial^2 f}{\partial x \partial y} = ?$$

$$\frac{\partial f}{\partial y} = \underbrace{x^{\sin y}} \cdot \underbrace{\cos y \cdot \ln x}$$

$$\frac{\partial f}{\partial x \partial y} = \sin y \, x^{\sin y - 1} \cdot \cos y \ln x + x^{\sin y} \cdot \cos y \frac{1}{x}$$

Exercise 6 Find $\dfrac{\partial^2 f}{\partial x\, \partial y}$ given $f(x,y) = y^{\sin x}$

$$\frac{\partial f}{\partial x} = \underbrace{y^{\sin x}}\,\underbrace{\cos x \ln y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \sin x\, y^{\sin x - 1} \cos x \ln y + y^{\sin x} \cdot \cos x \cdot \frac{1}{y}$$

$$\frac{\partial f}{\partial y} = \underbrace{\sin x}\,\underbrace{y^{\sin x - 1}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \cos x\, y^{\sin x - 1} + \sin x\, y^{\sin x - 1} \cdot \cos x \cdot \ln y$$

$$\frac{1}{y} \cdot \frac{y^{\sin x}}{y} = y^{\sin x - 1}$$

Exercise 7

Find a so that $f(x, y, z) = e^{ax+y}\sin(2z)$ is harmonic is $\mathbb{R}^3$.

$F_{xx} + F_{yy} + F_{zz} = 0$

Exercise 8

Find the equation of tangent plane and the normal line to the $\boxed{\text{elliptic paraboloid } z = 2x^2 + y^2}$ at the point $(1, 1, 3)$.

$G = z - 2x^2 - y^2 = 0$

@ $(1,1,3)$

$\vec{\nabla} G = \langle -4x, -2y, 1 \rangle$

$\vec{\nabla} G(1,1,3) = \langle -4, -2, 1 \rangle = \vec{N}$ — Normal to the plane

$-4(x-1) - 2(y-1) + 1(z-3) = 0$

or $-4x - 2y + z + 3 = 0$

Plane $\vec{N} = \langle a, b, c \rangle$
Pt (x_0, y_0, z_0)
$a(x-x_0) + b(y-y_0) + c(z-z_0)$

Normal line $\vec{d} = \langle -4, -2, 1 \rangle$ or any multiple of this

Pt $(1,1,3)$

$$\begin{cases} x = -4t+1 \\ y = -2t+1 \\ z = t+3 \end{cases} \qquad \begin{cases} 4t+1 \\ 2t+1 \\ -t+3 \end{cases}$$

(#14) $5x^2 - 2y^2 + 2z = -9$ $(1, 2, -3)$ tan. plane.

$G = 5x^2 - 2y^2 + 2z + 9 = 0$ $\vec{\nabla} G = \langle 10x \ -4y + 2 \rangle$

$\vec{\nabla} G(1,2,-3) = \langle 10, -8, 2 \rangle$

$10(x-1) - 8(y-2) + 2(z+3) = 0$

Exercise 9

Find the equation of the tangent plane to $z = e^{x-y}$ at $(2, 2, 1)$.

Exercise 10

Find the equation of the tangent plane to $2x + 3y^2 + 2z^2 = 31$ at $(-2, 1, 4)$.

$G = 2x + 3y^2 + 2z^2 - 31 = 0$

$\vec{\nabla} G = \langle 2, 6y, 4z \rangle$ @ $(-2, 1, 4)$

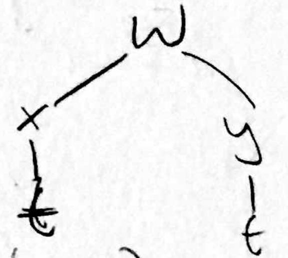
$\vec{\nabla} G = \langle 2, 6, 16 \rangle \parallel \vec{N} = \langle 1, 4, 8 \rangle$

pt $(-2, 1, 4)$

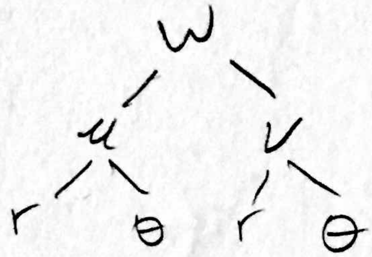
Exercise 11

(Chain Rule) $W = f(x, y)$ $x = m(t)$ $y = n(t)$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$



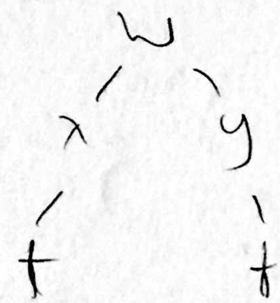
$W = f(u, v)$ $u = m(r, \theta)$ $v = n(r, \theta)$



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial \theta}$$

#13 $W = e^{2x+y}$ $x = t + \sin t$ $y = 2t - 1$

$\frac{dw}{dt}$ @ $t=0$



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \left(2e^{2x+y}\right)(1+\cos t) + e^{2x+y}(2)$$

$$= 2e^{-1}(1+\cos(0)) + e^{-1}(2) = \frac{6}{e}$$

Exercise 14

(Linearization)

Linearization
Exact Differential
Directional derivative
Identifying Curves.

$$f(x,y,z) = x^2 + y^2 + z^2$$

Linearization of $f(x,y,z)$ (called $L(x,y,z)$) near a point $(1,1,1)$

$$L(x,y,z) = f(a,b,c) + \frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c)$$

near (a,b,c)

$$\boxed{f(x,y,z) \simeq L(x,y,z)}$$

estimate $(0.9)^2 + (1.1)^2 + (0.9)^2 = f(0.9, 1.1, 0.9) \simeq L(0.9, 1.1, 0.9)$

$$L(x,y,c) = f(1,1,1) + \frac{\partial f}{\partial x}(x-1) + \frac{\partial f}{dy}(y-1) + \frac{\partial f}{\partial z}(z-1)$$

$$= 3 + 2(x-1) + 2(y-1) + 2(z-1)$$

$$L(0.9, 1.1, 0.9) = 3 + 2(-0.1) + 2(0.1) + 2(-0.1)$$
$$= 2.8$$

More exercises

$f(x,y,z) = 3x^2y + 4xe^z - 2y\ln z$

$\frac{\partial f}{\partial x} = 6xy + 4e^z$

$\frac{\partial f}{\partial y} = 3x^2 - 2\ln^z$

$\frac{\partial f}{\partial z} = 4xe^z - \frac{2y}{z}$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6y$ $\Big\{ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial x} f_y$

$= 6x \quad = 6x$

$\frac{\partial^2 y}{\partial z \partial y} = -\frac{2}{z}$

f harmonic if $f(x,y)$: $f_{xx} + f_{yy} = 0$

$f(x,y,z)$: $f_{xx} + f_{yy} + f_{zz} = 0$

harmonic = Sum of second partial derivatives = 0

8 $w(x,y,z) = \sin(\sqrt{5}x + 2y)\cosh(kz)$

$w_x = \cos(\sqrt{5}x+2y)(\sqrt{5})\cosh(kz)$

$\underline{w_{xx}} = -5\sin(\sqrt{5}x+2y)\cosh(kz)$

$w_y = 2\cos(\sqrt{5}x+2y)\cosh(kz)$

$\underline{w_{yy}} = -4\sin(\sqrt{5}x+2y)\cosh(kz)$

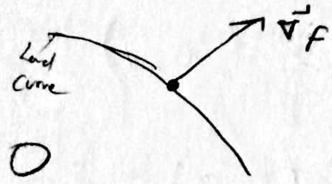
$w_z = k\sin(\sqrt{5}x+2y)\sinh(kz)$

$\underline{w_{zz}} = k^2\sin(\sqrt{5}x+2y)\cosh(kz)$

$-5-4+k^2=0$

$K = 3 \text{ or } -3$

Gradient Vector:



Level Curve $G(x,y)=0$

$\vec{\nabla} f = \left\langle \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right\rangle \perp$ to level curve

level surface $G(x,y,z)=0$

$\underbrace{\vec{\nabla} G = \left\langle \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right\rangle}_{\text{Normal to the tangent plane}} \perp$ to the level surface

$z = 2x^2 - 3y^2$

$G = z - 2x^2 + 3y^2 = 0$

$5x^2 - 2y^2 + 2z = -9$

☆ $5x^2 - 2y^2 + 2z + 9 = 0$ has a tangent plane which is parallel to $5x - 4y + z = 2$

$\langle 5, -4, 1 \rangle$

$\vec{\nabla} G = \langle 10x, -4y, 2 \rangle$

Which is it? $\langle 10, -4, 2 \rangle$

~~(A)~~ (B) $10x - 4y + 2 = 0$ (C) $5x - 4y + z = -6$

$\langle 5, -4, 1 \rangle$

~~(A)~~ (E) $5x - 4y + z = -9$

$\langle 5, -4, 1 \rangle$

$x = 1,\ y = 2$

$\cancel{5 - 8 - 3 = -9}$

$\cancel{-6 = -9}$

$5 - 8 - 3 = -6$ ✓

$5 - 8 + 2z + 9 = 0$

$2z = -6$

$z = -3$

Gradient parallel to normal vector.

$Z = f(u,v) \quad u = \ln\sqrt{x^2+y^2} \quad v = x + \arctan\left(\frac{y}{x}\right)$

$f_u(1,0) = 8 \quad f_v(1,0) = -9 \quad f_u(0,1) = 5$

$f_v(0,1) = -4$

$f(0,0) = 17$

$\frac{\partial z}{\partial y}$ at $(x,y) = (1,0)$

$$\begin{tikzcd}[column sep=small]
 & & z \arrow[dl, dash] \arrow[dr, dash] & & \\
 & u \arrow[dl, dash] \arrow[d, dash] & & v \arrow[d, dash] \arrow[dr, dash] & \\
x & y & & x & y
\end{tikzcd}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}$$

when $(x,y) = (1,0)$
$(u,v) = (0,1)$

$$= 5\,\frac{y}{x^2+y^2} + (-4)\,\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} \leftarrow \text{replace } x, y \text{ by } 1, 0$$

Use a proper linearization to estimate $\dfrac{1}{(3.9)^2 - 6.1}$

$f(x,y) = \dfrac{1}{x^2 - y}$ near $(4,6)$ $\quad f(4,6) = \dfrac{1}{10} = 0.1$

$f_x = \dfrac{-2x}{(x^2-y)^2}$ $\quad f_y = \dfrac{1}{(x^2-y)^2}$ @ $(4,6)$ $\quad f(x) = \dfrac{-8}{100} = -0.08$

$f_y = \dfrac{1}{100} = 0.01$

$L(x,y) = 0.1 - 0.08(x-4) + 0.01(y-6)$

$L(3.9, 6.1) = 0.1 - 0.08(-0.1) + 0.01(0.1)$

$= 0.1 + 0.008 + 0.001 = 0.109$

Exact Differential

$\boxed{dv \simeq \Delta v}$

$z = f(x,y)$
$dz = f_x dx + f_y dy$

$\star\, w = f(x,y,z)$
$dW = f_x dx + f_y dy + f_z dz$

Given $PV = \underset{\text{constant}}{k}T$ $P = 0.5$ $V = 64$ $T = 360$

By what % will P change if $V = 68$ $T = 350$

$\frac{dP}{P} = ?$ $\Delta V = 4 \simeq dV$ $\Delta t = -10 \simeq dT$

$P = k\frac{T}{V}$

$$dP = \frac{\partial P}{\partial V}dV + \frac{\partial P}{\partial T}dT$$

$$= -\frac{kT}{V^2}dV + \frac{k}{V}dT$$

$$\frac{dP}{P} = \frac{-kT/V^2}{kT/V}dV + \frac{k/V}{kT/V}dT = -1\frac{dV}{V} + 1\frac{dT}{T}$$

$$\frac{dP}{P} = -1\left(\frac{4}{64}\right) + 1\left(\frac{-10}{360}\right)$$

$$= -\frac{1}{16} - \frac{1}{36} = \_\_\_$$

Directional DERIVATIVE

$$D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u}$$

- f increases most rapidly in the direction of ∇f
- f decreases most rapidly in the direction of $-\nabla f$
- Max rate of change (inc & dec) is $\|\vec{\nabla} f\|$
- If $\hat{u}$ is $\perp \vec{\nabla} f$ No change in f (level curve)

Exam Writing Strategies

I have been writing exams for a really long time. Here are a few tips that Ive learned and heard over the years:

- Don't go to the exam sleep deprived!
- Don't leave studying till the last minute.
- We all feel we know very little right before a test, it is not true! It is normal to be a little afraid or a little stressed do NOT panic!

While writing the exam

- Start with the problems you know first. This sill boost your confidence and help you get rid of what you are sure of then you can clean your mind and work on those problems you are not 100% sure of.
- Make sure you read what is asked before you start working. Many do a lot more than they are asked to, some do less than they are asked or even forget to answer the question. Trust me when I say sometimes students solve a totally different question!!
- Always try a problem and do the best you can. Part marks (if given) can make the difference between an A and a B.
- I feel silly saying this but make sure your pens or pencils do actually write and that you have all the material you need for the test (eraser, ruler, calculator, etc.)
- Manage your time in a smart way. For example if you are stuck on a problem don't spend half the exam trying to solve it. Move on and come back to it if you have more time at the end.
- Write in a clear and understandable way. If the teacher cannot read or cannot find the answer you won't get the mark!