MATH 277

Problem Set # 11 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. In each case find the x and y coordinates of the critical points of the given function $f(x,y)$:

   (i) $f(x,y) = x^4 - 4xy + 2y^2 + 9.$

   (ii) $f(x,y) = e^{xy^2 - 3x^3 + 24y - 1}$

   $(iii)^*$ $f(x,y) = xy^2 - 3x^3 + 24y - 1$

2. In each case find the critical points of the given function $f(x,y)$ and determine whether it is a local Maximum , a local Minimum or a Saddle point :

   (i) $f(x,y) = x^3 - 3x + y^2$

   (ii) $f(x,y) = x^3 - 3x + y^3 - 3y$

   $(iii)^*$ $f(x,y) = x^2 + xy + x + y^2 + 2y$

   (iv) $f(x,y) = x^3 - 3x^2y + 6y^2 + 24y$

3. In each case find the Maximum and Minimum values of the given function $f(x,y)$ over the indicated region $\mathbf{D}$:

   (i) $f(x,y) = x^3 - 3x + y^2$; $\mathbf{D}$ is the region bounded by the circle $(x-1)^2 + y^2 = 1.$

   (ii) $f(x,y) = x^2 + 2x + y^2 - 2y$; $\mathbf{D}$ is the region bounded by $x = 0$, $x = 3$, $y = 0$ and $y = 3$.

   $(iii)^*$ $f(x,y) = 3x^2 - 144y + 16y^3$; $\mathbf{D}$ is the region bounded by the ellipse $x^2 + 4y^2 = 25.$

   (iv) $f(x,y) = x^2 - 6x + y^2 - 4y$; $\mathbf{D}$ is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 7$

4. In each case find the extreme values of the given function $f(x,y)$ subject to the indicated constraint using Lagrange Multipliers

   (i) $f(x,y) = 3x - 4y$; $x^2 + y^2 = 25$

   (ii) $f(x,y) = xy$; $x^2 + 4y^2 = 8$

   (iii) $f(x,y) = x^2 + y^2$; $3x - 4y + 50 = 0$

   $(iv)^*$ $f(x,y) = x^2y$; $x^2 + 2y^2 = 6$

   $(v)^*$ $f(x,y) = 3x^2 + xy$; $y - x^2 + 9 = 0$

   $(vi)^*$ $f(x,y) = x^2y - 3x + y$; $xy = 4$

MATH 277

Solutions to Problem Set # 11

1. (i) $f(x,y) = x^4 - 4xy + 2y^2 + 9$

$$\frac{\partial f}{\partial x} = 4x^3 - 4y \quad , \quad \frac{\partial f}{\partial y} = -4x + 4y$$

Critical points occur where

$$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad 4x^3 - 4y = 0 , \quad \text{---} (1)$$

and $$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad -4x + 4y = 0 . \quad \text{---} (2)$$

From Equation (2): $-4x + 4y = 0 \quad (\div 4)$

$$-x + y = 0 \quad \text{or} \quad \boxed{y = x}$$

Substituting $y = x$ into Equation (1), we obtain,

$$4x^3 - 4x = 0$$

$$\Rightarrow 4x(x^2 - 1) = 0 \quad \Rightarrow x = 0, 1, -1$$

At $\underline{x = 0}$: $y = x = 0$; first critical point is $(0,0)$

At $\underline{x = 1}$: $y = x = 1$; Second critical point is $(1,1)$,

At $\underline{x = -1}$: $y = x = -1$; Third critical point is $(-1,-1)$

(ii) $f(x,y) = e^{xy^2-3x^3+24y-1}$

$$\frac{\partial f}{\partial x} = e^{xy^2-3x^3+24y-1} \cdot (y^2-9x^2)$$

$$\frac{\partial f}{\partial y} = e^{xy^2-3x^3+24y-1} \cdot (2xy+24)$$

Now, we need to solve $\frac{\partial f}{\partial x} = 0$, and $\frac{\partial f}{\partial y} = 0$;

namely

$$e^{xy^2-3x^3+24y-1} \cdot (y^2-9x^2) = 0$$

$$\Rightarrow \quad y^2-9x^2 = 0 \quad --(1)$$

and $$e^{xy^2-3x^3+24y-1}(2xy+24) = 0$$

$$\Rightarrow \quad 2xy+24 = 0 \quad --(2)$$

From (1) : $y^2-9x^2 = 0$

$$\Rightarrow (y-3x)(y+3x) = 0$$

$$\therefore \quad y = 3x \quad \text{or} \quad y = -3x$$

Case 1: If $y = 3x$, Equation (2) becomes

$$2x(3x)+24 = 0$$

$$6x^2+24 = 0 \quad \Rightarrow \boxed{x^2 = -4}$$

This equation has No real solutions!!

Case (2): If $y = -3x$, Equation (2) becomes

$$2x(-3x) + 24 = 0$$

$$-6x^2 + 24 = 0$$

$$x^2 = 4$$

$$\therefore \quad x = \pm 2$$

But $y = -3x$

$\therefore$ At $x = 2$,

$$y = -3(2) = 6$$

$$(x, y) = (2, 6)$$

At $x = -2$

$$y = -3(-2) = 6$$

$$\therefore (x, y) = (-2, 6)$$

Critical points are $(2, 6)$, $(-2, 6)$

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(iii) For students to do at Home.

Answer: Critical points are $(2, 6)$, $(-2, 6)$.

Hint: Identical to part (ii) above!!

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2. (i) $f(x,y) = x^3 - 3x + y^2$

First: let us find first and second order partial derivatives of f

$f_x(x,y) = 3x^2 - 3$, $f_y(x,y) = 2y$

$f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = 0$, $f_{yy}(x,y) = 2$

Critical points occur where

$f_x = 3x^2 - 3 = 0$ --- (1),

$f_y = 2y = 0$ --- (2)

From (1): $3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

From (2) $2y = 0 \Rightarrow y = 0$

$\therefore$ There are two critical points (both share same y-coordinate 0).

$(+1, 0), (-1, 0)$

Refer to Comparison Table below:

$f_{xx}(x,y)$

$f_{yy}(x,y)$

Critical Point	$(1, 0)$	$(-1, 0)$
$A = f_x(x,y)$	$6 > 0$	-6
$B = f_{xy}(x,y) = f_{yx}(x,y)$	0	0
$C = f_{xx}(x,y)$	2	2
$D(x,y) = B^2 - AC$	$-12 < 0$	$12 > 0$
Conclusion	local Minimum	saddle point
value $f(x,y)$	-2	2

f has a local Minimum at the point $(1,0)$ of value $f(1,0) = -2$

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(ii) $f(x,y) = x^3 - 3x + y^3 - 3y$

let us find first and second order partial derivatives of f

$$\frac{\partial f}{\partial x} = 3x^2 - 3, \quad \frac{\partial f}{\partial y} = 3y^2 - 3$$

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 6y$$

Critical points occur where

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \quad \text{---} (1)$$

and $$\frac{\partial f}{\partial y} = 3y^2 - 3 = 0 \quad \text{---} (2)$$

From (1): $3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

From (2) $3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

There are four Critical points

$(x,y) = (1,1), (1,-1), (-1,1)$, and $(-1,-1)$

Note: we took all possible Combinations of (x,y) where

$$x = \pm 1, \text{ and } y = \pm 1$$

Refer to Comparison Table below!

	Critical Point	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
$f_{xx}(x,y)$ ←	$A = f_x(x,y)$	$6 > 0$	6	-6	$-6 < 0$
	$B = f_{xy}(x,y) = f_{yx}(x,y)$	0	0	0	0
$f_{yy}(x,y)$ ←	$C = f_{xx}(x,y)$	6	-6	6	-6
	$D(x,y) = B^2 - AC$	$-36 < 0$	$36 > 0$	$36 > 0$	$-36 < 0$
	Conclusion	Local Minimum	saddle point	saddle point	Local Maximum
	value $f(x,y)$	-4	—	—	4

f has a local Minimum at $(1,1)$ of value -4, and a local Maximum at $(-1,-1)$ of value 4.

At $(1,-1)$, $(-1,1)$, f has saddle point.

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(iii) For students to do at Home.

<u>Answer</u>

f has a local Minimum at $(0,-1)$ of value -1

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(iv) $f(x,y) = x^3 - 3x^2 y + 6y^2 + 24y$

Let us first find first and second order partial derivatives of f:

$$\frac{\partial f}{\partial x} = 3x^2 - 6xy, \quad \frac{\partial f}{\partial y} = -3x^2 + 12y + 24$$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -6x,$$

$$\frac{\partial^2 f}{\partial y^2} = 12$$

critical points occur where

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 6xy = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow -3x^2 + 12y + 24 = 0 \quad \text{--- (2)}$$

From (1): $3x^2 - 6xy = 0 \Rightarrow 3x(x - 2y) = 0$

$\therefore$ Either $x = 0$ or $x = 2y$

Case (1): If $x = 0$, Eq. (2) becomes $12y + 24 = 0 \Rightarrow y = -2$

$\therefore$ Critical point is $(0,-2)$

Case (2): If $x = 2y$, then Eq. (2) becomes

$$-3(2y)^2 + 12y + 24 = 0 \Rightarrow y^2 - y - 2 = 0, \; y = -1, 2$$

hence: At $y = -1$, $x = 2(-1) = -2$; $(-2,-1)$

At $y = 2$, $x = 2(2) = 4$; $(4,2)$

Critical points are $(0,-2)$, $(-2,-1)$, and $(4,2)$.

Refer to Comparison Table below:

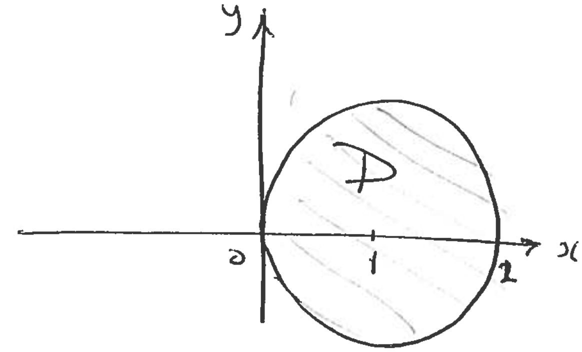
Critical Point	$(0,-2)$	$(-2,-1)$	$(4,2)$
$f_{xx}(x,y) \leftarrow$ $A = f_x(x,y)$	$12 > 0$	-6	12
$B = f_{xy}(x,y) = f_{yx}(x,y)$	0	12	-24
$f_{yy}(x,y) \leftarrow$ $C = f_{xx}(x,y)$	12	12	12
$D(x,y) = B^2 - AC$	$-144 < 0$	$216 > 0$	$432 > 0$
Conclusion	local Minimum	saddle point	saddle point
value $f(x,y)$	-24	—	—

f has a local Minimum at $(0,-2)$. f value -24.

f has Two saddle points at $(-2,-1)$, and at $(4,2)$

3. (i) $f(x,y) = x^3 - 3x + y^2$

D: Region enclosed by the circle $(x-1)^2+y^2=1$ Centred at $(1,0)$, and is of radius 1 as shown in figure.



$0 \leq x \leq 2$

Problem 1: Interior of D

$$f(x,y) = x^3 - 3x + y^2$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3, \quad \frac{\partial f}{\partial y} = 2y$$

Critical points occur where

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3 = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0 \quad \text{--- (2)}$$

From (1): $3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

From (2): $2y = 0 \Rightarrow y = 0$

There are two critical points $(1,0)$, $(-1,0)$. However only $(1,0)$ lies in the interior of the circle, and that

$$f(1,0) = 1^3 - 3(1) + 0^2 = -2$$

Problem 2: On the Boundary

On the Boundary, we have

$$(x-1)^2 + y^2 = 1 \Rightarrow y^2 = 1 - (x-1)^2 = 2x - x^2$$

substituting $y^2 = 2x - x^2$ into

$$f(x,y) = x^3 - 3x + y^2$$

we obtain a function of only x, say

$$g(x) = x^3 - 3x + (2x - x^2)$$
$$= x^3 - x^2 - x \quad , \quad \boxed{0 \leq x \leq 2}$$

Now, $g'(x) = 3x^2 - 2x - 1$

$g'(x) = 0 \Rightarrow 3x^2 - 2x - 1 = 0 \Rightarrow (x-1)(3x+1) = 0$

$x = 1$, $x = -\frac{1}{3}$ (Reject since $x \in (0,2)$)

So: Critical point $x = 1$, and End points $x = 0, 2$

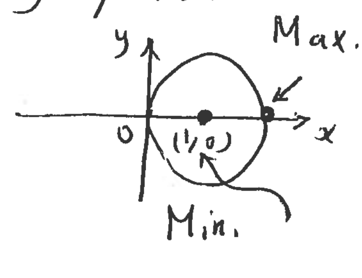
$\therefore$ $g(1) = 1 - 1 - 1 = -1$

$g(0) = 0 - 0 - 0 = 0$,

$g(2) = 2^3 - 2^2 - 2 = 2$

Comparing the Four circled values we conclude that f has a minimum value of -2 and a maximum value of 2.

Note: Minimum occurs at the interior point $(1,0)$ and the Maximum occur at the Boundary point $x = 2$, $y = 0$; $(2,0)$



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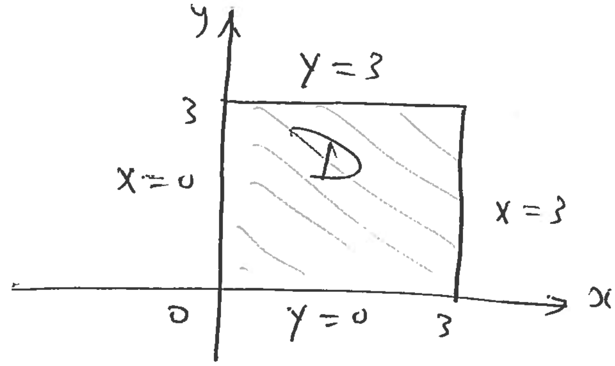
(ii) $f(x,y) = x^2 + 2x + y^2 - 2y$

D: Region enclosed by the lines $x = 0$, $x = 3$, $y = 0$, and $y = 3$ (which is a square as shown in figure).



<u>Problem 1</u>: Interior of D:

$$f(x,y) = x^2 + 2x + y^2 - 2y$$

$$\frac{\partial f}{\partial x} = 2x + 2, \quad \frac{\partial f}{\partial y} = 2y - 2$$

critical points occur where

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 2 = 0 \Rightarrow x = -1$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y - 2 = 0 \Rightarrow y = 1$$

only critical point is $(x,y) = (-1,1)$ which lies <u>outside</u> region D and hence must be rejected

$\therefore$ No critical points in D

<u>Problem 2</u>: Boundary of D

There are four boundaries. let us Examine Each separately

(a) Along the line $y=0$, $0 \leq x \leq 3$, the function $f(x,y)$ reduces to a function of the single variable x, say

$$g_1(x) = f(x,0) = x^2 + 2x + 0^2 - 2(0)$$
$$= x^2 + 2x, \qquad 0 \leq x \leq 3$$

Now, $g_1'(x) = 2x + 2$

$$g_1'(x) = 0 \Rightarrow 2x + 2 = 0$$
$$\Rightarrow x = -1 \quad (\text{Reject since } x \in (0,3))$$

Therefore there are No critical points. The end points are $x = 0, 3$

$$\therefore \quad g(0) = 0^2 + 2(0) = 0,$$
$$g(3) = 3^2 + 2(3) = 15$$

(b) Along the line $x=0$, $0 \leq y \leq 3$, the function $f(x,y)$ reduces to a function of the single variable y, say

$$g_2(y) = f(0,y) = 0^2 + 2(0) + y^2 - 2y$$
$$= y^2 - 2y, \qquad 0 \leq y \leq 3$$

Now, $g_2'(y) = 2y - 2$

$$g_2'(y) = 0 \Rightarrow 2y - 2 = 0 \Rightarrow y = 1$$

So: critical point is $y=1$, and end points are $y=0$, and $y=3$

$g_2(0) = 0^2 - 2(0) = 0$

$g_2(1) = 1^2 - 2(1) = -1$

$g_2(3) = 3^2 - 2(3) = 3$

(c) Along the line $y = 3$, $0 \leq x \leq 3$, the function $f(x, y)$ reduces to a function of the Single Variable x, say

$$g_3(x) = f(x,3) = x^2 + 2x + 3^2 - 2(3)$$
$$= x^2 + 2x + 3, \qquad 0 \leq x \leq 3$$

Now, $g_3'(x) = 2x + 2$

$g_3'(x) = 0 \Rightarrow 2x + 2 = 0$

$\Rightarrow x = -1$ (Reject since $x \in (0,3)$)

$\therefore$ No critical points, and End points are $x = 0, 3$

$\therefore$ $g_3(0) = 0^2 + 2(0) = 0$

$g_3(3) = 3^2 + 2(3) = 15$

(d) Along the line $x = 3$, $0 \leq y \leq 3$, the function $f(x,y)$ reduces to a function of the Single Variable "y" say

$$g_4(y) = f(3,y) = 3^2 + 2(3) + y^2 - 2y$$
$$= y^2 - 2y + 15, \qquad 0 \leq y \leq 3$$

Now, $g_4'(y) = 2y - 2$

$g_4'(y) = 0 \Rightarrow 2y - 2 = 0$

$\Rightarrow y = 1$

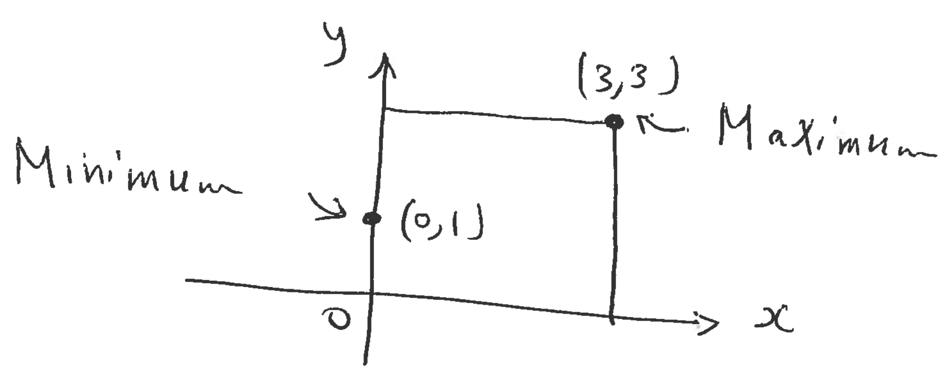
$\therefore$ Critical point at $y = 1$, and End points at $y = 0$, $y = 3$

$g_4(0) = 0^2 - 2(0) + 15 = 15$

$g_4(1) = 1^2 - 2(1) + 15 = 14$

$g_4(3) = 3^2 - 2(3) + 15 = 18$

Comparing the Ten Circled values we conclude that f has a minimum value of -1 which occur at the Boundary point $(0,1)$ and f has a maximum value of 18 which occur at the Boundary point $(3,3)$.



(iii) For students to do at home.

f has a minimum value of $-96\sqrt{3}$ (≈ -166.3) which occur at the interior point $(0, \sqrt{3})$ and f has a maximum value of 210 which occur at the boundary points $(-4, -\frac{3}{2})$, $(4, -\frac{3}{2})$.

<u>Note</u>: This Problem is Very Similar to part (i).

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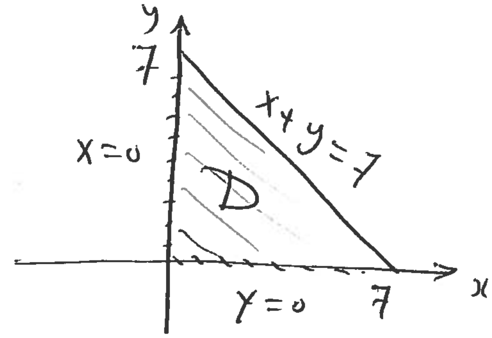
(iv) $f(x,y) = x^2 - 6x + y^2 - 4y$

D : The Triangular region bounded by the lines
$x = 0$, $y = 0$, and $x + y = 7$ as shown.



$0 \leq x \leq 7$

Problem 1 : Interior of D

$$f(x,y) = x^2 - 6x + y^2 - 4y$$
$$\frac{\partial f}{\partial x} = 2x - 6, \quad \frac{\partial f}{\partial y} = 2y - 4$$

Critical points occur where

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x - 6 = 0, \quad x = 3$$
$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y - 4 = 0, \quad y = 2$$

only critical point is $(x,y) = (3,2)$ which lies in the interior of D, and that

$$f(3,2) = 3^2 - 6(3) + 2^2 - 4(2)$$
$$= 9 - 18 + 4 - 8 = -13$$

Problem 2 : Boundary of D

There are three Boundaries. let us Examine each separately.

(a) Along the line $y = 0$, $0 \leq x \leq 7$, the function $f(x,y)$ reduces to a function of the Single Variable "x", say

$$g_1(x) = f(x,0) = x^2 - 6x + 0^2 - 4(0)$$
$$= x^2 - 6x, \qquad 0 \leq x \leq 7$$

Now, $g_1'(x) = 2x - 6$

$g_1'(x) = 0 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3$

$\therefore$ Critical point at $x = 3$, and End points at $x = 0$, and $x = 7$

$$\therefore \quad g_1(0) = 0^2 - 6(0) = 0$$
$$g_1(3) = 3^2 - 6(3) = 9 - 18 = -9$$
$$g_1(7) = 7^2 - 6(7) = 7$$

(b) Along the line $x = 0$, $0 \leq y \leq 7$, the function $f(x,y)$ reduces to a function of the single variable "y" say

$$g_2(y) = 0^2 - 6(0) + y^2 - 4y$$
$$= y^2 - 4y, \qquad 0 \leq y \leq 7$$

Now, $g_2'(y) = 2y - 4$

$g_2'(y) = 0 \Rightarrow 2y - 4 = 0 \Rightarrow y = 2$

$\therefore$ Critical point at $y = 2$, and End points at $y = 0$, and $y = 7$

$$g_2(0) = 0^2 - 4(0) = 0$$

$$g_2(2) = 2^2 - 4(2) = 4 - 8 = -4$$
$$g_2(7) = 7^2 - 4(7) = 49 - 28 = 21$$

(c) Along the line $x + y = 7$, we have

$$y = 7 - x, \qquad 0 \leq x \leq 7$$

Therefore the function $f(x, y)$ reduces to a function of the Single Variable "x" say

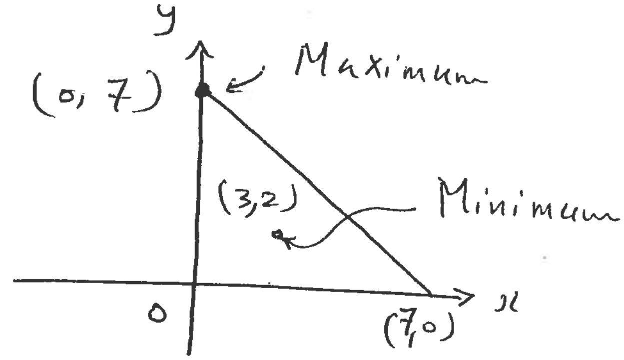
$$g_3(x) = f(x, 7-x) = x^2 - 6x + (7-x)^2 - 4(7-x)$$
$$= 2x^2 - 16x + 21, \qquad 0 \leq x \leq 7$$

Now,
$$g_3'(x) = 4x - 16$$
$$g_3'(x) = 0 \Rightarrow 4x - 16 = 0, \quad x = 4$$

$\therefore$ Critical point at $x = 4$, and End points at $x = 0$, and $x = 7$

$$\therefore \quad g_3(0) = 2(0)^2 - 16(0) + 21 = 21$$
$$g_3(4) = 2(4)^2 - 16(4) + 21 = -11$$
$$g_3(7) = 2(7)^2 - 16(7) + 21 = 7$$

Comparing the Ten Circled values we Conclude that f has a minimum value of -13 which occur at the interior point $(3, 2)$ and f has a maximum value of 21 which occur at the Boundary point $(0, 7)$



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4. (i) Recall: The Lagrangian $L = L(x, y; \lambda)$ is given by

$$L = f(x,y) + \lambda\, g(x,y)$$

Here $f(x,y) = 3x - 4y$, and constraint is

$$x^2 + y^2 = 25$$

$$\Rightarrow g(x,y) = x^2 + y^2 - 25 = 0$$

(Note: R.H.S of constraint must be zero)

$$\therefore L = 3x - 4y + \lambda(x^2 + y^2 - 25)$$

Need to solve the system:

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \text{ and } \frac{\partial L}{\partial \lambda} = 0$$

Now,

$$\frac{\partial L}{\partial x} = 3 + 2\lambda x = 0 \quad \text{---} (1)$$

$$\frac{\partial L}{\partial y} = -4 + 2\lambda y = 0 \quad \text{---} (2)$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 25 = 0 \quad \text{---} (3)$$

From (1): $3 + 2\lambda x = 0 \Rightarrow x = \frac{-3}{2\lambda}$

From (2): $-4 + 2\lambda y = 0 \Rightarrow y = \frac{2}{\lambda}$

substituting $x = \frac{-3}{2\lambda}$, $y = \frac{2}{\lambda}$ into (3), we obtain:

$$\left(-\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 25 = 0$$

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} - 25 = 0 \qquad (* \, 4\lambda^2)$$

$$9 + 16 - 100\lambda^2 = 0 \Rightarrow 100\lambda^2 = 25$$

$$\lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm\frac{1}{2}$$

<u>Case1</u>: If $\lambda = \frac{1}{2}$, then

$$x = \frac{-3}{2\lambda} = -\frac{3}{2\left(\frac{1}{2}\right)} = -3,$$

$$y = \frac{2}{\lambda} = \frac{2}{\left(\frac{1}{2}\right)} = 4$$

$\therefore$ First critical point is $(-3,4)$, and

$$f(-3,4) = 3(-3) - 4(4) = -25$$

<u>Case2</u>: If $\lambda = -\frac{1}{2}$, then

$$x = -\frac{3}{2\left(-\frac{1}{2}\right)} = 3, \quad y = \frac{2}{\left(-\frac{1}{2}\right)} = -4$$

$\therefore$ Second critical point is $(3,-4)$ and

$$f(3,-4) = 3(3) - 4(-4) = 25$$

<u>Conclusion</u>: f has a minimum at $(-3,4)$ of value $f(-3,4) = -25$ and has a maximum at $(3,-4)$ of value $f(3,-4) = 25$

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(ii) $L(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$

Here $f(x,y) = xy$, and $x^2 + 4y^2 = 8 \Rightarrow$

$$g(x,y) = x^2 + 4y^2 - 8 = 0$$

$$\therefore\ L = xy + \lambda(x^2 + 4y^2 - 8)$$

Critical points occur where

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

Now, $\frac{\partial L}{\partial x} = y + 2\lambda x = 0$ --- (1)

$\frac{\partial L}{\partial y} = x + 8\lambda y = 0$ --- (2)

$\frac{\partial L}{\partial \lambda} = x^2 + 4y^2 - 8 = 0$ -- (3)

From (1) $\quad 2\lambda x = -y$

From (2) $\quad 8\lambda y = -x$

) Divide both sides

$$\frac{\cancel{2}\cancel{\lambda} x}{{}_4\cancel{8}\cancel{\lambda} y} = \frac{-y}{-x}$$

$\Rightarrow x^2 = 4y^2$ --- (*)

Substituting into (3):

$$4y^2 + 4y^2 - 8 = 0$$

$$8y^2 = 8$$

$$y^2 = 1 \Rightarrow y = \pm 1$$

To find "x", we substitute into (*):

At $\underline{y=1}$, $x^2 = 4(1)^2 = 4$

$\Rightarrow x = \pm 2$

$\therefore$ critical points are $(2,1), (-2,1)$

At $\underline{y=-1}$, $x^2 = 4(-1)^2 = 4$

$\Rightarrow x = \pm 2$

other critical points are $(2,-1), (-2,-1)$

Now, compare values of f at these critical points:

$f(x,y) = xy$

$\therefore f(2,1) = (2)(1) = 2$

$f(-2,1) = -2$,

$f(2,-1) = -2$, and

$f(-2,-1) = 2$

Therefore f has a maximum value of 2 occuring at $(2,1), (-2,-1)$; and has a minimum value of -2 which occur at $(-2,1), (2,-1)$

(iii) $L(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$

Here $f(x,y) = x^2+y^2$, and $g(x,y) = 3x-4y+50 = 0$

$\therefore\ L = x^2+y^2+\lambda(3x-4y+50)$

$\frac{\partial L}{\partial x} = 2x+3\lambda$, $\frac{\partial L}{\partial y} = 2y-4\lambda$, and

$\frac{\partial L}{\partial \lambda} = 3x-4y+50$

Need to solve system:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2x+3\lambda = 0 \quad \cdots \quad (1)$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow 2y-4\lambda = 0 \quad \cdots \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 3x-4y+50 = 0 \quad \cdots \quad (3)$$

From (1): $2x+3\lambda = 0 \Rightarrow x = -\frac{3}{2}\lambda$

From (2) $2y-4\lambda = 0 \Rightarrow y = 2\lambda$

substituting x, y into (3), we obtain

$$3\left(-\tfrac{3}{2}\lambda\right) - 4(2\lambda) + 50 = 0 \qquad (*2)$$

$$-9\lambda - 16\lambda + 100 = 0$$

$$25\lambda = 100 \Rightarrow \lambda = 4$$

It follows that $x = -\frac{3}{2}\lambda = -\frac{3}{2}(4) = -6$

$y = 2\lambda = 2(4) = 8$

only critical point is $(x,y) = (-6, 8)$ and

that $f(-6,8) = (-6)^2 + 8^2 = 36+64 = 100$

Question: How do we know if f has a Maximum or a Minimum at the point $(-6, 8)$?

There are two simple ways!

First: pick any point on the constraint $3x - 4y + 50 = 0$ other than $(-6, 8)$!

pick say $x = 2$,

$\therefore 3(2) - 4y + 50 = 0 \Rightarrow 56 = 4y, \quad y = 14$

point is $(2, 14)$

Now compute $f(2,14) = x^2 + y^2 \Big|_{\substack{x=2 \\ y=14}} = 2^2 + (14)^2 = 200$

clearly $f(2,14) = 200 > 100$

$\therefore$ f has a Minimum value of 100 at the point $(-6, 8)$.

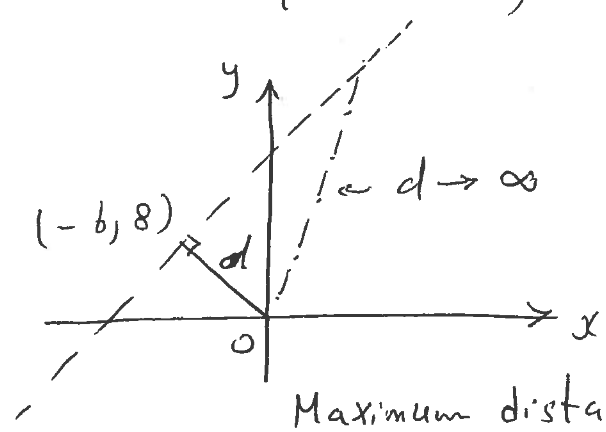
Another Method: sketch constraint (st. line)

Note that

$f(x,y) = x^2 + y^2 = d^2$,

where d is the distance from O to line

Clearly there can only be a minimum distance



Maximum distance from O to line is $+\infty$

(iv) For students to do at Home.

<u>Answer:</u> f has a minimum value -4 at the points $(2,-1)$, $(-2,-1)$ and a maximum value 4 at the points $(2,1)$, $(-2,1)$.

(v) For students to do at Home.

<u>Answer:</u> f has a minimum of value -5 at the point $(1,-8)$ and a maximum of value 27 at the point $(-3,0)$.

(vi) For students to do at Home.

<u>Answer:</u> f has a minimum of value -4 at the point $(-2,-2)$ and a maximum of value 4 at the point $(2,2)$

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