SOLUTIONS TO

MATH 277 MIDTERM TEST

WINTER 2016

1. Find the arc length of the space curve given by the vector equation :

$\vec{r}(t) = t\,\vec{i} + 2\ln(t)\,\vec{j} + (1 - \frac{2}{t})\vec{k}$, $1 \le t \le 2$ is equal to

solution $\quad \vec{r}(t) = \left(t, 2\ln(t), 1 - \frac{2}{t}\right)$

$$\vec{v}(t) = \left(1, \frac{2}{t}, \frac{2}{t^2}\right)$$

$$\|\vec{v}\| = \sqrt{1^2 + \left(\frac{2}{t}\right)^2 + \left(\frac{2}{t^2}\right)^2} = \sqrt{1 + \frac{4}{t^2} + \frac{4}{t^4}}$$

$$= \sqrt{\left(1 + \frac{2}{t^2}\right)^2} = \left|1 + \frac{2}{t^2}\right| = 1 + \frac{2}{t^2}$$

Recall : Arc length $\quad L = \int_a^b \|\vec{v}(t)\|\,dt$

$$= \int_1^2 \left(1 + \frac{2}{t^2}\right) dt = \int_1^2 \left(1 + 2t^{-2}\right) dt$$

$$= t + 2\frac{t^{-1}}{-1}\Big|_1^2 = t - \frac{2}{t}\Big|_{t=1}^{t=2}$$

$$= \left(2 - \frac{2}{2}\right) - \left(1 - \frac{2}{1}\right) = 1 - (-1) = 2$$

2. Find the Standard parametric representation of the plane curve $4(x-4)^2 + 9(y+5)^2 = 36$

solution : $\quad 4(x-4)^2 + 9(y+5)^2 = 36 \quad (\div 36)$

$$\frac{(x-4)^2}{9} + \frac{(y+5)^2}{4} = 1$$

This is an equation of an Ellipse centred at $(\alpha, \beta) = (4, -5)$ and has semi-axes of length $a = \sqrt{9} = 3$, $b = \sqrt{4} = 2$

A parametric representation is thus given by

$$\vec{r}(t) = (\alpha + a\cos(t))\,\vec{i} + (\beta + b\sin(t))\,\vec{j}, \quad 0 \le t \le 2\pi$$

$$\therefore \vec{r}(t) = (4 + 3\cos(t))\,\vec{i} + (-5 + 2\sin(t))\,\vec{j}, \quad 0 \le t \le 2\pi$$

3. Find an equation of the straight line tangent to the space curve :

$$\vec{r}(t) = (t^2+3t+1)\vec{i} + (2-7t)\vec{j} + (4\sin(t)-3)\vec{k}$$

at the point on the curve corresponding to $t = 0$ is given by :

Solution : $\vec{r}(t) = \left(t^2+3t+1,\ 2-7t,\ 4\sin(t)-3\right)$

At $t=0$, $\vec{r}_0 = (0+0+1,\ 2-0,\ 4\sin(0)-3) = (1, 2, -3)$

Next, $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = (2t+3,\ -7,\ 4\cos(t))$

A vector in the direction of the tangent line at $t=0$ is thus given by $\vec{v} = \vec{v}(0) = (0+3, -7, 4\cos(0)) = (3, -7, 4)$

Therefore, vector equation of Tangent line to curve at $t=0$ is thus given by $\vec{r}(t) = \vec{r}_0 + t\vec{v}$, $t \in \mathbb{R}$

$= (1, 2, -3) + t(3, -7, 4)$, $t \in \mathbb{R}$

4. Using $t = x$ as a parameter ,find the the parametric representation of the curve of intersection of the two surfaces $y + z - x^4 = 3$, and $z = x^2y + 4$ is given by the vector equation :

Solution : $y + z - x^4 = 3$ - - - (1)

$z = x^2y + 4$ - - - (2)

substituting $x = t$ into (1), (2) we obtain,

$y + z - t^4 = 3$ - - - - (3)

$z = t^2y + 4$ - - - (4)

Now, let us solve (3), (4) for y, and z

substituting $z = t^2y+4$ into (3), we obtain

$y + t^2y + 4 - t^4 = 3 \Rightarrow y(1+t^2) = t^4 - 1 = (t^2+1)(t^2-1)$

$\therefore y = \dfrac{\cancel{(t^2+1)}(t^2-1)}{\cancel{t^2+1}} = t^2 - 1$, hence $z = t^2y + 4$

$= t^2(t^2-1) + 4$

$= t^4 - t^2 + 4$

$\therefore \vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$

$= t\vec{i} + (t^2-1)\vec{j} + (t^4 - t^2 + 4)\vec{k}$, $t \in \mathbb{R}$

5. Let **C** be the space curve given by the vector equation $\vec{r}(t) = (t - e^t)\,\vec{i} + 3t\,\vec{j} + (2t - t^2)\,\vec{k}$.

Find the unit Binormal $\vec{\mathbf{B}}$ at $t = 0$.

solution: $\vec{r}(t) = (t - e^t, 3t, 2t - t^2)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (1 - e^t, 3, 2 - 2t),$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (-e^t, 0, -2)$$

At $t = 0$, $\vec{v} = (1 - e^0, 3, 2 - 2(0)) = (0, 3, 2)$

$$\vec{a} = (-e^0, 0, -2) = (-1, 0, -2)$$

$$\vec{v} \times \vec{a} = \left(+\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix}, +\begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix}\right) = (-6, -2, 3)$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{(-6)^2 + (-2)^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$$

Recall $\vec{B} = \dfrac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \dfrac{(-6, -2, 3)}{7} \overset{\text{or}}{=} \left(-\dfrac{6}{7}, -\dfrac{2}{7}, \dfrac{3}{7}\right)$

6. Determine the curvature of the space curve $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + \frac{2}{3}t^3\,\vec{k}$ at $t = 1$.

solution: $\vec{r}(t) = (t, t^2, \frac{2}{3}t^3)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (1, 2t, 2t^2),$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (0, 2, 4t)$$

At $t = 1$, $\vec{v} = (1, 2, 2)$, $\vec{a} = (0, 2, 4)$

$$\therefore \vec{v} \times \vec{a} = \left(+\begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix}, +\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}\right) = (4, -4, 2)$$

$\therefore \|\vec{v} \times \vec{a}\| = \sqrt{4^2 + (-4)^2 + 2^2} = \sqrt{16 + 16 + 4} = \sqrt{36} = 6$, and

$$v = \|\vec{v}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

Recall $\kappa = \dfrac{\|\vec{v} \times \vec{a}\|}{v^3} = \dfrac{6}{3^3} = \dfrac{6}{27} \overset{\text{or}}{=} \dfrac{2}{9}$

7. A frictionless highway turn has a constant curvature $1.96 \times 10^{-2}\ m^{-1}$, and is banked at an angle $\theta = \tan^{-1}(0.2)$. What will be the maximum safe speed for the turn in m/s (metre per second). You may assume that the gravitational acceleration $g = 9.8\ m/s^2$.

Solution

Maximum speed $v = \sqrt{\rho\, g \tan(\theta)}$

Here $\theta = \tan^{-1}(0.2) \Rightarrow \tan(\theta) = 0.2$

$g = 9.8\ m/s^2$, and

$K = 1.96 \times 10^{-2} = \frac{196}{100} \cdot \frac{1}{100} = \frac{196}{10{,}000}\ m^{-1}$

$\Rightarrow \rho = \frac{1}{K} = \frac{10{,}000}{196}$

$\therefore\ v = \sqrt{\frac{10{,}0\cancel{00}}{\cancel{196}} \cdot \frac{\cancel{98}}{\cancel{10}} \cdot \frac{\cancel{2}}{\cancel{10}}} = \sqrt{100} = 10\ m/s$

(that's $(10)(3.6) = 36\ Km/hr$).

8. A plane curve **C** is given parametrically by the functions :

$x(t) = \cosh(t) - 2, \quad y(t) = \sinh(t), \quad t \in \mathbb{R}$. Find a Cartesian equation of the curve **C**.

Solution :

$x = \cosh(t) - 2 \quad \cdots (1)$

$y = \sinh(t) \quad \cdots (2)$

Need to Eliminate "t" among (1), (2).

From (1) : $\cosh(t) = x + 2$

From (2) : $\sinh(t) = y$

Recall $\cosh^2(t) - \sinh^2(t) = 1$

$\therefore\ (x+2)^2 - y^2 = 1$

9. Find the domain of the function $f(x) = \dfrac{1}{\sqrt[3]{y^3 + x^3}}$.

solution : Here $f(x,y) = \dfrac{1}{\sqrt[3]{y^3 + x^3}}$

f is defined and is real provided

$$y^3 + x^3 \neq 0 \Rightarrow y^3 \neq -x^3 \Rightarrow y \neq -x$$

$\therefore$ Domain f consists of all ordered pairs (x, y) such that

$$y \neq -x$$

10. Which of the following statements is **True**?

(I) $x^2 - 4y^2 - 9z^2 + 36 = 0$ is an equation of a Hyperboloid of **Two Sheets**.

(II) $x^2 = 2 - y^2 - z^2$ is an equation of a **Sphere**.

(III) $z = \sqrt{1 - x^2 - y^2}$ is an equation of a **Circular Cone**

(IV) $z = y^2$ is an equation of a **Paraboloid**.

solution :

(I) $x^2 - 4y^2 - 9z^2 + 36 = 0 \Rightarrow x^2 - 4y^2 - 9z^2 = -36 \quad (\div 36)$

$$-\frac{x^2}{36} + \frac{y^2}{9} + \frac{z^2}{4} = +1 \Rightarrow \text{Hyperboloid of one sheet!}$$

So : statement is False.

(II) $x^2 = 2 - y^2 - z^2 \Rightarrow x^2 + y^2 + z^2 = 2$. This is an equation of a sphere centred at $(0,0,0)$, radius $\sqrt{2}$.

statement is True

(III) $z = \sqrt{1 - x^2 - y^2} \Rightarrow z^2 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 1$

This is an equation of the upper hemi-sphere with centre $(0,0,0)$, radius 1. Statement is False.

(IV) $z = y^2$ is an equation of a "parabolic" cylinder with generators parallel to x-axis. Statement is False

11. If $z = x\sin\left(\frac{y}{x}\right)$, find $\frac{\partial^2 z}{\partial x \partial y}$.

Solution : Recall $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$.

Now, $z = x\sin\left(\frac{y}{x}\right) = x\sin(yx^{-1})$

$$\frac{\partial z}{\partial y} = x \cdot \cos(yx^{-1}) \cdot x^{-1} = \cancel{x}\cos(yx^{-1}) \cdot \frac{1}{\cancel{x}}$$

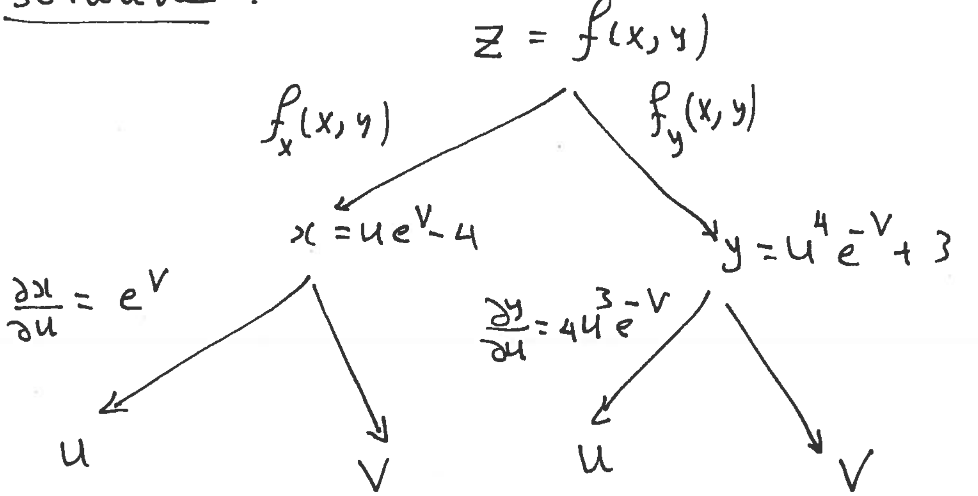
$$= \cos(yx^{-1})$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\cos(yx^{-1})\right) = -\sin(yx^{-1}) \cdot (-yx^{-2})$$

$$\overset{\text{or}}{=} + \frac{y}{x^2}\sin\left(\frac{y}{x}\right)$$

12. Let $z = f(x,y)$, where $x = ue^{v} - 4$, and $y = u^4 e^{-v} + 3$. Use the chain rule to find $\frac{\partial z}{\partial u}$ at $(u,v) = (1,0)$ given that $f_x(1,0) = 13$, $f_y(1,0) = -2$, $f_x(-3,4) = 4$, and $f_y(-3,4) = -11$.

Solution :



Note :
At $(u,v) = (1,0)$,
$x = 1e^0 - 4 = 1 - 4 = -3$
$y = 1^4 e^0 + 3 = 1 + 3 = 4$

$$\therefore \frac{\partial z}{\partial u} = f_x(x,y) \cdot e^{v} + f_y(x,y) \cdot (4u^3 e^{-v})$$

$$\therefore \left.\frac{\partial z}{\partial u}\right|_{(u,v)=(1,0)} = \left. f_x(x,y)\, e^{v} + f_y(x,y)\,(4u^3 e^{-v}) \right|_{\substack{x=-3,\ y=4 \\ u=1,\ v=0}}$$

$$= f_x(-3,4) \cdot e^0 + f_y(-3,4) \cdot 4 \cdot 1 \cdot e^0$$

$$= 4 + (-11)(4) = 4 - 44 = -40$$

13. If $w = g(x,y,z) = xy + yz + xz$, where $x(t) = \sin(2t-4)$, $y(t) = t^2$, and $z(t) = 2-t$, use the chain rule to find the value of $\frac{dw}{dt}$ at $t = 2$.

$$W = g(x,y,z) = xy + yz + xz$$

$$\frac{\partial g}{\partial x} = y + z \qquad \frac{\partial g}{\partial y} = x + z \qquad \frac{\partial g}{\partial z} = y + x$$

$$x = \sin(2t-4) \qquad y = t^2 \qquad z = 2-t$$

$$\frac{dx}{dt} = 2\cos(2t-4) \qquad \frac{dy}{dt} = 2t \qquad \frac{dz}{dt} = -1$$

$$t \qquad t \qquad t$$

Note: At $t = 2$,
$x = \sin(0) = 0$
$y = 2^2 = 4$
$z = 2-2 = 0$

$$\therefore \frac{dW}{dt} = (y+z)\cdot 2\cos(2t-4) + (x+z)\cdot 2t + (y+x)(-1)$$

At $t = 2$,

$$\left.\frac{dW}{dt}\right|_{t=2} = (y+z)\cdot 2\cos(2t-4) + (x+z)\cdot 2t + (y+x)(-1)\Big|_{x=0,\, y=4,\, z=0,\, t=2} = 4\cdot 2 + 0 - 4 = 8 - 4 = 4$$

14. Find an equation of the plane tangent to the surface $xy^3z^2 = 2$ at the point $P(2,1,-1)$ on the surface.

Solution: $xy^3z^2 = 2 \Rightarrow xy^3z^2 - 2 = 0$

Take $F(x,y,z) = xy^3z^2 - 2$

$$\vec{\nabla} f(P) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P = \left(y^3z^2,\ 3xy^2z^2,\ 2xy^3z\right)\Big|_{(x,y,z)=(2,1,-1)}$$

$$= (1, 6, -4)$$

$\therefore$ A vector Normal to Tangent plane at P is $\vec{N} = (1,6,-4)$

Equation of Tangent plane is thus given by

$$\vec{r}\cdot\vec{N} = \vec{r_0}\cdot\vec{N}\ ; \quad \vec{r} = (x,y,z), \quad \vec{r_0} = P = (2,1,-1)$$

$$\therefore (x,y,z)\cdot(1,6,-4) = (2,1,-1)\cdot(1,6,-4)$$

$$x + 6y - 4z = 2 + 6 + 4$$

$$\Rightarrow x + 6y - 4z = 12$$

15. The position vector of a moving particle in space is given by the vector equation

$\vec{r}(t) = \frac{1}{2}t^2\,\vec{i} + (2t+3)\,\vec{j} + \frac{4}{3}\,t^{3/2}\,\vec{k}$. When will the speed of the particle be 4 units?

Solution : $\vec{r}(t) = \left(\frac{1}{2}t^2,\ 2t+3,\ \frac{4}{3}t^{\frac{3}{2}}\right)$.

Note first that $t^{\frac{3}{2}} = t\sqrt{t}$. Hence $\vec{r}(t)$ is defined and is real only for $t \geq 0$.

Now, $\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(t,\ 2,\ \frac{4}{3}\cdot\frac{3}{2}t^{\frac{1}{2}}\right) = (t, 2, 2\sqrt{t})$

speed $v = \|\vec{v}\| = \sqrt{t^2 + 2^2 + (2\sqrt{t})^2} = \sqrt{t^2+4+4t}$

But $v = 4$, hence $4 = \sqrt{t^2+4+4t}$ ← square both sides

$\Rightarrow 16 = t^2+4+4t \Rightarrow t^2+4t-12 = 0$

$\therefore (t+6)(t-2) = 0$

$t = -6,\ t = 2$

But $t=-6$ must be rejected since $t \geq 0$, hence $t = 2$ (only)

16. The **Normal** component of acceleration of a moving object in space at time t is given by $a_N = \frac{2}{\sqrt{t^3+1}}$. If the Radius of Curvature at time t is $\rho = 2(t^3+1)^{3/2}$, find the **Tangential** component of the acceleration at $t = 2$.

Solution : Recall $a_N = \kappa v^2$. But $\kappa = \frac{1}{\rho}$

$$\therefore a_N = \frac{v^2}{\rho}$$

Here $a_N = \frac{2}{\sqrt{t^3+1}}$, $\rho = 2(t^3+1)^{\frac{3}{2}} = 2(t^3+1)\sqrt{t^3+1}$

$$\therefore \frac{2}{\cancel{\sqrt{t^3+1}}} = \frac{v^2}{2(t^3+1)\cancel{\sqrt{t^3+1}}} \Rightarrow v^2 = 4(t^3+1)$$

$$\therefore v = \sqrt{4(t^3+1)} = 2\sqrt{t^3+1}$$

$$\therefore a_T = \frac{dv}{dt} = \frac{d}{dt}\left(2\sqrt{t^3+1}\right) = 2\cdot\frac{1}{2}(t^3+1)^{-\frac{1}{2}}3t^2$$

$$= \frac{3t^2}{\sqrt{t^3+1}}$$

At $t = 2$, $a_T = \frac{3(2)^2}{\sqrt{2^3+1}} = \frac{\cancel{(3)}(4)}{\cancel{\sqrt{9}}} = 4$

17. Find the linear approximation of the function $f(x,y) = \ln(x^2 + y^2 + xy)$ at the point $(1,-1)$.

Solution: Recall: The linear approximation of $f(x,y)$ at the point (a,b) is given by

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Here $(a,b) = (1,-1)$, hence

$$L(x,y) = f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y+1)$$

Now, $f(x,y) = \ln(x^2+y^2+xy)$, $f_x(x,y) = \dfrac{2x+y}{x^2+y^2+xy}$, $f_y(x,y) = \dfrac{2y+x}{x^2+y^2+xy}$

At $(x,y) = (1,-1)$:

$f(1,-1) = \ln(1+1-1) = \ln 1 = 0$, $f_x(1,-1) = \dfrac{2-1}{1+1-1} = 1$, $f_y(1,-1) = \dfrac{-2+1}{1+1-1} = -1$

$\therefore$ $L(x,y) = 0 + 1(x-1) + (-1)(y+1) = x - 1 - y - 1$

$$\Rightarrow L(x,y) = x - y - 2$$

18. The Pressure **P**, Volume **V**, and Temperature **T** (in ° **K**) of a confined gas are related by the ideal gas law $PV = kT$, where k is a constant. If $P = 0.5\ lb/in^2$ when $V = 50\ in^3$ and $T = 360\ °K$, determine by approximately what percentage P changes if V and T change to $52\ in^3$ and $351\ °K$ respectively.

Solution: Know: $\Delta V = V - V_0 = 52 - 50 = +2\ in^3$,

$\Delta T = T - T_0 = 351 - 360 = -9\ °K$, want $\dfrac{\Delta P}{P}$?

Now, $PV = KT \Rightarrow P = \dfrac{KT}{V} \overset{\text{or}}{=} KTV^{-1}$, $P = P(T,V)$

Recall $\Delta P \approx dP = \dfrac{\partial P}{\partial T}\Delta T + \dfrac{\partial P}{\partial V}\Delta V$

$$= KV^{-1}\Delta T + (-KTV^{-2})\Delta V = \frac{K}{V}\Delta T - \frac{KT}{V^2}\Delta V$$

Dividing both sides by $P = \dfrac{KT}{V}$, we obtain

$$\frac{\Delta P}{P} \approx \frac{\frac{\cancel{K}}{\cancel{V}}\Delta T}{\frac{\cancel{K}T}{\cancel{V}}} - \frac{\frac{\cancel{KT}}{V^2}\Delta V}{\frac{\cancel{KT}}{V}} = \frac{\Delta T}{T} - \frac{\Delta V}{V}$$

$T = 360$
$V = 50$
$\Delta T = -9$, $\Delta V = +2$

$$\frac{\Delta P}{P} \approx -\frac{9}{360} - \frac{2}{50}$$

In percentage $\dfrac{\Delta P}{P} = -\left(\dfrac{9}{360} + \dfrac{2}{50}\right) \times 100\ \% \Rightarrow \dfrac{\Delta P}{P} \approx -6.5\ \%$

19. Describe the level curve of the function $f(x,y) = \frac{2}{x^2 - y^2 + 14}$ corresponding to $c = -1$.

Solution : Recall : Level Curves are given by $f(x,y) = c$. Here $c = -1$

$$\therefore \quad \frac{2}{x^2 - y^2 + 14} = -1 = \frac{1}{-1}$$

$$\Rightarrow x^2 - y^2 + 14 = -2 \Rightarrow x^2 - y^2 = -16 \quad (\div 16)$$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{16} = -1$$

This is an equation of a Hyperbola with Centre at $(0,0)$, and which opens up and down.

20. A rocket is fired vertically upward in a vacuum (that is Free Space where gravitational field is negligible)

During the burning process, the exhaust gases are ejected at a constant rate 1000 kg/s and at constant velocity with magnitude 400 m/s relative to the rocket.

Let M be the total initial mass of the rocket and assume it starts motion from rest.

In order to accelerate to the speed of 800 m/s , the rocket has to burn P % of the total initial mass M as fuel. Find the value of P.

Solution : Recall $\quad v = v_e \ln\left(\frac{M}{m(t)}\right), \quad m(t) = M - \alpha t$

Here $v_e = 400 \text{ m/s}$, hence

$$v = 400 \ln\left(\frac{M}{m}\right)$$

At $v = 800$, $\quad 800 = 400 \ln\left(\frac{M}{m}\right) \Rightarrow \ln\left(\frac{M}{m}\right) = 2$

$$\Rightarrow \frac{M}{m} = e^2 \quad \Rightarrow \quad m = \frac{M}{e^2}$$

$\therefore$ Amount of burnt fuel $= M - m = M - \frac{M}{e^2} = M\left(1 - \frac{1}{e^2}\right)$

$\therefore$ Required ratio $= \frac{\text{Amount Burnt}}{\text{Initial Mass}} = \frac{\cancel{M}\left(1 - \frac{1}{e^2}\right)}{\cancel{M}}$

$$= 1 - \frac{1}{e^2}$$

$$\text{or } \left(1 - \frac{1}{e^2}\right) \times 100 \ \%$$

$$\therefore P = 100\left(1 - \frac{1}{e^2}\right).$$

MATH 277 OFFICIAL FORMULA SHEET

A: BASIC INTEGRALS

Let r, a, $b \in \mathbb{R}$, $r \neq -1$, and $a \neq 0$.

1. $\int x^r \, dx = \dfrac{x^{r+1}}{r+1} + C$ 2. $\int \dfrac{1}{ax+b} \, dx = \dfrac{1}{a} \ln|ax+b| + C$ 3. $\int e^{ax} \, dx = \dfrac{1}{a} e^{ax} + C$

4. $\int \sin(ax) \, dx = -\dfrac{1}{a} \cos(ax) + C$ 5. $\int \cos(ax) \, dx = \dfrac{1}{a} \sin(ax) + C$

B: BASIC TRIGONOMETRIC IDENTITIES

(i) $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ (ii) $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$ (iii) $\sec(\theta) = \dfrac{1}{\cos(\theta)}$ (iv) $\csc(\theta) = \dfrac{1}{\sin(\theta)}$

(v) $\cos^2(\theta) + \sin^2(\theta) = 1$ (vi) $1 + \tan^2(\theta) = \sec^2(\theta)$ (vii) $\cot^2(\theta) + 1 = \csc^2(\theta)$

$(viii)$ $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ (ix) $\cos(2\theta) = 2\cos^2(\theta) - 1$ (x) $\cos(2\theta) = 1 - 2\sin^2(\theta)$

C: BASIC HYPERBOLIC IDENTITIES

(i) $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ (ii) $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$ (iii) $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$ (iv) $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$

(v) $\cosh^2(x) - \sinh^2(x) = 1$ (vi) $1 - \tanh^2(\theta) = \operatorname{sech}^2(\theta)$ (vii) $\coth^2(\theta) - 1 = \operatorname{csch}^2(\theta)$

$(viii)$ $\sinh(2x) = 2\sinh(x)\cosh(x)$ (ix) $\cosh(2x) = 2\cosh^2(x) - 1$ (x) $\cosh(2x) = 1 + 2\sinh^2(x)$

D: Other Formulae

Let $\vec{\mathbf{v}}(t)$, $\vec{\mathbf{a}}(t)$ and $\mathbf{v}(t)$ be respectively **velocity**, **acceleration** and **speed** of a moving object in three space.

The unit Tangent $\vec{\mathbf{T}}$, the Principal unit Normal $\vec{\mathbf{N}}$, the unit Binormal $\vec{\mathbf{B}}$, the curvature $\boldsymbol{\kappa}$, the radius of curvature $\boldsymbol{\rho}$ and the Torsion $\boldsymbol{\tau}$ are given by :

(i) $\vec{\mathbf{T}} = \dfrac{\vec{\mathbf{v}}(t)}{\mathbf{v}(t)}$ (ii) $\vec{\mathbf{N}} = \vec{\mathbf{B}} \times \vec{\mathbf{T}}$ (iii) $\vec{\mathbf{B}} = \dfrac{\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)}{\|\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)\|}$ (iv) $\boldsymbol{\kappa} = \dfrac{\|\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)\|}{\mathbf{v}^3}$

(v) $\boldsymbol{\rho} = \dfrac{1}{\boldsymbol{\kappa}}$ (vi) $\boldsymbol{\tau} = \dfrac{\left[\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)\right] \cdot \dfrac{d\vec{a}(t)}{dt}}{\|\vec{\mathbf{v}}(t) \times \vec{\mathbf{a}}(t)\|^2}$ (vii) $a_{\mathbf{T}} = \dfrac{d\mathbf{v}}{dt}$ $(viii)$ $a_{\mathbf{N}} = \boldsymbol{\kappa}\,\mathbf{v}^2$