SOLUTIONS TO

MATH 277 FINAL EXAM REVIEW SHEET

WINTER 2016

1. (a) $\vec{r}(t) = (3t, 2t^{\frac{3}{2}}, 4), \qquad 0 \le t \le 8$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (3, 3t^{\frac{1}{2}}, 0)$$

$$v = \|\vec{v}(t)\| = \sqrt{3^2 + (3t^{\frac{1}{2}})^2} = \sqrt{9+9t}$$

$$= 3\sqrt{1+t}$$

Arc length

$$L = \int_a^b v\,dt = \int_0^8 3\sqrt{1+t}\,dt$$

$$= 3\int_0^8 (1+t)^{\frac{1}{2}}\,dt = 3 \cdot \frac{(1+t)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^8$$

$$= 2\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= 2\left[9\sqrt{9} - 1\right] = 2[27-1] = 52$$

(b) $\vec{r} = (2\sin^2(t), \cos^3(t), \sin^3(t)), \qquad 0 \le t \le \frac{\pi}{2}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(4\sin(t)\cos(t), -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t)\right)$$

$$v = \|\vec{v}\| = \sqrt{16\sin^2(t)\cos^2(t) + 9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}$$

$$= \sqrt{16\sin^2(t)\cos^2(t) + 9\sin^2(t)\cos^2(t)\underbrace{[\cos^2(t)+\sin^2(t)]}_{\text{one}}}$$

$$= \sqrt{25\sin^2(t)\cos^2(t)}$$

$$= 5\sin(t)\cos(t) \qquad \text{for } t \in \left[0, \tfrac{\pi}{2}\right]$$

$$\overset{\text{or}}{=} \frac{5}{2}\sin(2t)$$

Arc length
$$L = \int_0^{\frac{\pi}{2}} \frac{5}{2}\sin(2t)\,dt = -\frac{5}{2}\cdot\frac{1}{2}\cos(2t)\Big|_0^{\frac{\pi}{2}}$$
$$= -\frac{5}{4}\left[\cos(\pi)-\cos(0)\right]$$
$$= -\frac{5}{4}[-1-1] = -\frac{5}{4}(-2) = \frac{5}{2}$$

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(c) $\vec{r}(t) = (2e^{t}, e^{-t}, 2t)$

$\vec{v}(t) = \frac{d\vec{r}}{dt} = (2e^{t}, -e^{-t}, 2)$

speed $v = \|\vec{v}\| = \sqrt{4e^{2t} + e^{-2t} + 4} \leftarrow$ perfect square!

$$= \sqrt{(2e^{t} + e^{-t})^2}$$

$$= 2e^{t} + e^{-t}$$

$\therefore$ Arc length
$$L = \int_{-1}^{1} (2e^{t} + e^{-t})\,dt$$
$$= 2e^{t} - e^{-t}\Big|_{-1}^{1}$$
$$= (2e - e^{-1}) - (2e^{-1} - e) = 3e - 3e^{-1}$$
or
$$= 3e - \frac{3}{e}$$

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(d) $\vec{r}(t) = \left(\frac{1}{2}\sin(t^2),\ \frac{1}{2}\cos(t^2),\ \frac{1}{3}(2t+1)^{\frac{3}{2}} \right)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(\frac{1}{2}\cos(t^2)\cdot 2t,\ -\frac{1}{2}\sin(t^2)\cdot 2t,\ \frac{1}{3}\cdot\frac{3}{2}(2t+1)^{\frac{1}{2}}\cdot 2 \right)$$

$$= \left(t\cos(t^2),\ -t\sin(t^2),\ \sqrt{2t+1} \right)$$

$$v = \text{speed} = \sqrt{t^2\cos^2(t^2) + t^2\sin^2(t^2) + 2t+1}$$

$$= \sqrt{t^2\left(\cos^2(t) + \sin^2(t)\right) + 2t+1}$$

$$= \sqrt{\underline{t^2 + 2t+1}} \leftarrow \text{perfect square.}$$

$$= \sqrt{(t+1)^2}$$

$$= t+1, \quad 0 \leq t \leq 2$$

$$\therefore\ L = \int_0^2 (t+1)\,dt$$

$$= \frac{1}{2}t^2 + t \Big|_0^2 = 2+2 = 4$$

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2 (a) $$4x^2 + y^2 = 16 \quad \text{--- (1)}$$

$$2x + 3y + 2z = 1 \quad \text{--- (2)}$$

From (1): Dividing both sides by 16:

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

If this equation is Viewed in $\mathbb{R}^2$, it represents an equation of an Ellipse with centre $(h, k) = (0, 0)$, and semi-axes of length $a = \sqrt{4} = 2$, $b = \sqrt{16} = 4$.

$\therefore$ Its standard parametric equations are thus given by

$$x(t) = h + a\cos(t)$$
$$y(t) = k + b\sin(t), \quad t \in [0, 2\pi]$$

$$\therefore \quad x(t) = 0 + 2\cos(t) = 2\cos(t)$$
$$y(t) = 0 + 4\sin(t) = 4\sin(t), \quad t \in [0, 2\pi]$$

Substituting x, y into

$$2x + 3y + 2z = 1$$

We obtain:

$$2\left(2\cos(t)\right)+3\left(4\sin(t)\right)+2z=1$$

$\therefore\ 4\cos(t)+12\sin(t)+2z=1$

$\Rightarrow\ 2z=1-4\cos(t)-12\sin(t)$

Hence $\quad z=\frac{1}{2}-2\cos(t)-6\sin(t)$

Curve of intersection is given parametrically by

$$x(t)=2\cos(t),$$
$$y(t)=4\sin(t), \qquad , \quad t\in[0,2\pi]$$
$$z(t)=\frac{1}{2}-2\cos(t)-6\sin(t)$$

OR: Curve is given by the Vector Equation

$$\vec{r}(t)=x\vec{i}+y\vec{j}+z\vec{k}$$

$$=2\cos(t)\,\vec{i}+4\sin(t)\,\vec{j}$$
$$+\left(\frac{1}{2}-2\cos(t)-6\sin(t)\right)\vec{k},$$

$t\in[0,2\pi]$

(b) $x^2 + 2y + z = 3 \quad \ldots (1)$

$xz + y = -2 \quad \ldots (2)$

Clearly "y" is easy to Eliminate!

From (2): $y = -2 - xz$

Substituting $y = -2 - xz$ into (1):

$$x^2 + 2[-2 - xz] + z = 3$$

or $$x^2 - 4 - 2xz + z = 3$$

$$\Rightarrow x^2 - 2xz + z = 7$$

To parametrize, we have too many choices!

let say $x = t$

(This is best choice because we can easily find z)

$$\therefore \quad t^2 - 2tz + z = 7$$

$$z(1-2t) = 7 - t^2$$

$$z = \frac{7-t^2}{1-2t}$$

$$\therefore \quad y = -2 - xz = -2 - t \cdot \left(\frac{7-t^2}{1-2t}\right)$$

$$y \overset{\text{or}}{=} \frac{-2(1-2t) - t(7-t^2)}{1-2t}$$

$$\therefore \quad y = \frac{t^3 - 3t - 2}{1 - 2t}$$

$\therefore$ Curve of intersection is given parametrically by

$$x(t) = t$$

$$y(t) = \frac{t^3 - 3t - 2}{1-2t}, \quad t \in \mathbb{R}, \ t \neq \frac{1}{2}$$

$$z(t) = \frac{7 - t^2}{1 - 2t}$$

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<u>Note</u>: There are Infinitely-Many possible answers!

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(c) $z = x^2 + y^2$, $2x - 4y - z + 4 = 0$

The idea is to use the two equations to obtain a 3rd. equation containing only two variables which is much easier to parametrize!

Indeed
$$z = x^2 + y^2 \quad \cdots (1)$$

Now, $2x - 4y - z + 4 = 0$

$$\Rightarrow \quad z = 2x - 4y + 4 \quad -- \quad (2)$$

Equating (1), (2):

$$x^2 + y^2 = 2x - 4y + 4$$

$$\Rightarrow \quad x^2 - 2x + y^2 + 4y = 4$$

Now, complete the squares (in x, and y-terms)

$$(x^2 - 2x) + (y^2 + 4y) = 4$$

$\downarrow$ add $\left(-\frac{2}{2}\right)^2 = 1$ to both sides

$\downarrow$ add $\left(\frac{4}{2}\right)^2 = 4$ to both sides

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = 4 + 1 + 4$$

$$(x-1)^2 + (y+2)^2 = 9$$

If this equation is <u>Viewed</u> in $\mathbb{R}^2$, it is an

Equation of a circle with centre at $(h,k)=(1,-2)$ and is of radius $a=\sqrt{9}=3$

Its parametric equations are thus given by

$$x = h + a\cos(t)$$
$$y = k + a\sin(t) \quad , \quad t \in [0, 2\pi]$$

$$\therefore \quad x = 1 + 3\cos(t)$$
$$y = -2 + 3\sin(t) \quad , \quad t \in [0, 2\pi]$$

Recall $\quad z = 2x - 4y + 4$

$$= 2[1 + 3\cos(t)] - 4[-2 + 3\sin(t)] + 4$$

$$\Rightarrow \quad z = 14 + 6\cos(t) - 12\sin(t)$$

$\therefore$ Curve of Intersection is given parametrically by

$$\begin{cases} x(t) = 1 + 3\cos(t) \\ y(t) = -2 + 3\sin(t) \\ z(t) = 14 + 6\cos(t) - 12\sin(t) \end{cases} \quad , \quad t \in [0, 2\pi]$$

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(d) $xy + xz = 6$, $x = -3$

This is an Easy one!

substitute $x = -3$ into $xy + xz = 6$ to get

$$-3y - 3z = 6$$

$$\Rightarrow \quad y + z = -2$$

Now, let say $z = t$, hence $y + t = -2$

$$\therefore \quad y = -2 - t$$

$\therefore$ Curve of intersection is given parametrically by

$$\begin{aligned} x(t) &= -3 \\ y(t) &= -2 - t \quad , \quad t \in \mathbb{R} \\ z(t) &= t \end{aligned}$$

OR : Curve is given by the vector Equation

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= -3\vec{i} + (-2-t)\vec{j} + t\vec{k} \, , \quad t \in \mathbb{R}$$

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Note: Answer above is not unique. There are infinitely-many possible Answers!!

(e) $x^2 - y^2 - z = 0 \;,\; 2y^2 + z = 1$

Let us attempt to obtain a 3rd. Equation Containing only Two Variables!

$$x^2 - y^2 - z = 0 \implies$$

$$z = x^2 - y^2 \quad \ldots \quad (1)$$

$$2y^2 + z = 1 \implies$$

$$z = 1 - 2y^2 \quad \ldots \quad (2)$$

Equate (1), (2) (to Eliminate "z"!):

$$x^2 - y^2 = 1 - 2y^2$$

$$\implies x^2 + y^2 = 1$$

If $x^2 + y^2 = 1$ is Viewed in $\mathbb{R}^2$, it represents an Eq. of a Circle with Centre $(h,k) = (0,0)$, and radius $a = 1$

The standard parametric equations of Circle are thus given by

$$x = \cos(t)$$
$$y = \sin(t) \quad , \quad t \in [0, 2\pi]$$

Recall $z = 1 - 2y^2$

$\therefore \quad z = 1 - 2\sin^2(t)$

Hence, the curve of intersection is given parametrically by

$$x(t) = \cos(t)$$
$$y(t) = \sin(t) \qquad t \in [0, 2\pi]$$
$$z(t) = 1 - 2\sin^2(t)$$

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3. The speed v of a Rocket moving in a straight line only under the forces of its ejected gases is given by

$$v = v_e \ln\left(\frac{M}{m(t)}\right), \quad m(t) = M - \alpha t$$

Where v_e is the speed of ejected gases (assumed Constant), M is the total initial mass, α is the rate of ejected gases (assumed Constant), and $m(t)$ is the mass of rocket at time t.

Here $V_e = 500$ m/s, $\alpha = 1300$ Kg./s, $M = 52{,}000$ Kg.; $m(t) = 52{,}000 - 1300t$

$$\therefore \quad v = 500 \ln\left(\frac{52{,}000}{52{,}000 - 1300t}\right)$$

<u>At $t = 15$</u> $\quad V = 500 \ln\left(\frac{52{,}000}{52{,}000 - (1300)(15)}\right)$

$$= 500 \ln(1.6) \approx 235 \text{ m/s}$$

<u>At $t = 20$</u> $\quad V = 500 \ln\left(\frac{52{,}000}{52{,}000 - (1300)(20)}\right)$

$$= 500 \ln(2) \approx 347 \text{ m/s}$$

<u>At $t = 30$</u> $\quad V = 500 \ln\left(\frac{52{,}000}{52{,}000 - (1300)(30)}\right)$

$$= 500 \ln(4) = 693 \text{ m/s}$$

Note: The rocket burns its entire 39,000 Kg of fuel in $\frac{39,000}{1300} = 30$ seconds!

Therefore after 30 second, the speed of the rocket remains Constant at 693 m/s

$\therefore$ At $t = 35$, $v = 693$ m/s as well.

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4 . (a) $\vec{r}(t) = (3\sin(t),\ 3\cos(t),\ 4t)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (3\cos(t),\ -3\sin(t),\ 4)$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (-3\sin(t),\ -3\cos(t),\ 0)$$

$$\frac{d\vec{a}}{dt} = (-3\cos(t),\ 3\sin(t),\ 0)$$

At $t=0$:

$$\vec{v} = (3\cos(0), -3\sin(0), 4) = (3,0,4) \qquad (1)$$

$$\vec{a} = (-3\sin(0), -3\cos(0), 0) = (0,-3,0) \qquad (2)$$

$$\vec{v}\times\vec{a} = (3,0,4)\times(0,-3,0) = (12,0,-9) \qquad (3)$$

$$\|\vec{v}\times\vec{a}\| = \sqrt{(12)^2+(0)^2+(-9)^2} = \sqrt{225} = 15, \qquad (4)$$

$$\text{speed } \|\vec{v}\| = v = \sqrt{3^2+0^2+4^2} = \sqrt{25} = 5 \qquad (5)$$

Quantities (1) – (5) can be used to find $\vec{T}, \vec{N}, \vec{B}, \vec{K}$, ρ, and τ.

$$\vec{T} = \frac{\vec{v}}{v} = \frac{1}{5}(3,0,4)$$

$$\vec{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \frac{1}{15}(12,0,-9) = \frac{1}{5}(4,0,-3)$$

$$\vec{N} = \vec{B}\times\vec{T} = \frac{1}{5}(4,0,-3)\times\frac{1}{5}(3,0,4) = \frac{1}{25}(0,-25,0)$$

$$= (0,-1,0)$$

$$\kappa = \frac{\|\vec{v}\times\vec{a}\|}{v^3} = \frac{15}{5^3} = \frac{3}{25}$$

$$\rho = \frac{1}{\kappa} = \frac{25}{3}, \quad \text{and} \quad \left.\frac{d\vec{a}}{dt}\right|_{t=0} = (-3,0,0), \text{ hence}$$

$$\tau = \frac{(\vec{v}\times\vec{a})\cdot\frac{d\vec{a}}{dt}}{\|\vec{v}\times\vec{a}\|^2} = \frac{(12,0,-9)\cdot(-3,0,0)}{(15)^2} = \frac{-36+0+0}{225} = \frac{-4}{25}$$

(b) $\vec{r}(t) = (\sin(t), \sqrt{2}\cos(t), \sin(t))$

$\vec{v}(t) = \frac{d\vec{r}}{dt} = (\cos(t), -\sqrt{2}\sin(t), \cos(t))$

$\vec{a}(t) = \frac{d\vec{v}}{dt} = (-\sin(t), -\sqrt{2}\cos(t), -\sin(t))$

$\frac{d\vec{a}}{dt} = (-\cos(t), \sqrt{2}\sin(t), -\cos(t))$

<u>At $t = \frac{\pi}{4}$</u> : Note $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, $\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\therefore \quad \vec{v} = \left(\frac{1}{\sqrt{2}}, -\sqrt{2}\cdot\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(1, -\sqrt{2}, 1) \;\ldots (1)$

Similarly $\vec{a} = -\frac{1}{\sqrt{2}}(1, \sqrt{2}, 1) \;--- (2)$

$\frac{d\vec{a}}{dt} = \frac{1}{\sqrt{2}}(-1, \sqrt{2}, -1) \;.\;.\;--- (3)$

$\vec{v} \times \vec{a} = \frac{1}{\sqrt{2}}(1, -\sqrt{2}, 1) \times -\frac{1}{\sqrt{2}}(1, \sqrt{2}, 1)$

$= -\frac{1}{2}(1, -\sqrt{2}, 1) \times (1, \sqrt{2}, 1)$

$= -\frac{1}{2}(-2\sqrt{2}, 0, 2\sqrt{2}) = (\sqrt{2}, 0, -\sqrt{2}) \quad (4)$

$\|\vec{v} \times \vec{a}\| = \sqrt{(\sqrt{2})^2 + (0)^2 + (-\sqrt{2})^2} = \sqrt{4} = 2 \quad (5)$

and $v = \|\vec{v}\| = \frac{1}{\sqrt{2}}\sqrt{(1)^2 + (-\sqrt{2})^2 + (1)^2} = \frac{1}{\sqrt{2}}\sqrt{4} = \frac{2}{\sqrt{2}} = \sqrt{2}$ --- (6)

$\Rightarrow \quad v = \sqrt{2} \;--- \;(6)$

Using (1) $\longrightarrow$ (6) we can find the six quantities $\vec{T}$, $\vec{N}$, $\vec{B}$, κ, ρ, and τ.

$$\vec{T} = \frac{\vec{v}}{v} = \frac{\frac{1}{\sqrt{2}}(1, -\sqrt{2}, 1)}{\sqrt{2}} = \frac{1}{2}(1, -\sqrt{2}, 1)$$

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \frac{(\sqrt{2}, 0, -\sqrt{2})}{2} = \frac{\sqrt{2}}{2}(1, 0, -1)$$

$$= \frac{1}{\sqrt{2}}(1, 0, -1)$$

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\sqrt{2}}(1, 0, -1) \times \frac{1}{2}(1, -\sqrt{2}, 1)$$

$$= \frac{1}{2\sqrt{2}}(1, 0, -1) \times (1, -\sqrt{2}, 1)$$

$$= \frac{1}{2\sqrt{2}}(-\sqrt{2}, -2, -\sqrt{2})$$

$$\overset{or}{=} -\frac{1}{2}(1, \sqrt{2}, 1)$$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{2}{(\sqrt{2})^3} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\rho = \frac{1}{\kappa} = \sqrt{2}$, and

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \frac{d\vec{a}}{dt}}{\|\vec{v} \times \vec{a}\|^2} = \frac{(\sqrt{2}, 0, -\sqrt{2}) \cdot \frac{1}{\sqrt{2}}(-1, \sqrt{2}, -1)}{2^2}$$

$$= \frac{\frac{1}{\sqrt{2}}(-\sqrt{2} + 0 + \sqrt{2})}{4} = 0$$

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(c) $\vec{r}(t) = (\cosh(t), -\sinh(t), t)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (\sinh(t), -\cosh(t), 1)$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (\cosh(t), -\sinh(t), 0)$$

$$\frac{d\vec{a}}{dt} = (\sinh(t), -\cosh(t), 0)$$

At $t=0$, noting that $\sinh(0)=0$, $\cosh(0)=1$, we obtain:

$$\vec{v} = (0, -1, 1) \quad --- \quad (1)$$

$$\vec{a} = (1, 0, 0) \quad --- \quad (2)$$

$$\frac{d\vec{a}}{dt} = (0, -1, 0) \quad ---- \quad (3)$$

$$\vec{v} \times \vec{a} = (0,-1,1) \times (1,0,0)$$

$$= (0, 1, 1) \quad --- \quad (4)$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \quad (5)$$

$$v = \|\vec{v}\| = \sqrt{0 + (-1)^2 + 1^2} = \sqrt{2} \quad (6)$$

From (1) — (6) we can determine: $\vec{T}$, $\vec{N}$, $\vec{B}$, κ, ρ, and τ.

$$\vec{T} = \frac{\vec{v}}{v} = \frac{(0,-1,1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(0,-1,1)$$

$$\vec{B} = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|} = \frac{1}{\sqrt{2}}(0, 1, 1)$$

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\sqrt{2}}(0,1,1) \times \frac{1}{\sqrt{2}}(0,-1,1)$$

$$= \frac{1}{2}(0,1,1) \times (0,-1,1)$$

$$= \frac{1}{2}(2,0,0) = (1,0,0)$$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$$

$\rho = \dfrac{1}{\kappa} = 2$, and

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \frac{d\vec{a}}{dt}}{\|\vec{v} \times \vec{a}\|^2} = \frac{(0,1,1) \cdot (0,-1,0)}{(\sqrt{2})^2}$$

$$= \frac{0-1+0}{2} = -\frac{1}{2}$$

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5. (a) $\vec{r}(t) = (t^2, t, \frac{1}{2}t^2)$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (2t, 1, t)$$

$$\vec{a}(t) = (2, 0, 1)$$

speed $v = \|\vec{v}\| = \sqrt{(2t)^2 + 1^2 + t^2} = \sqrt{5t^2+1}$

Next, $\vec{v} \times \vec{a} = (2t, 1, t) \times (2, 0, 1)$

$$= \left(+ \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix}, - \begin{vmatrix} 2t & t \\ 2 & 1 \end{vmatrix}, + \begin{vmatrix} 2t & 1 \\ 2 & 0 \end{vmatrix} \right)$$

$$= (1, 0, -2)$$

$\therefore \|\vec{v} \times \vec{a}\| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5}$

Therefore:

Tangential Component of acceleration

$$a_T = \frac{dv}{dt} = \frac{d}{dt}\left(\sqrt{5t^2+1}\right)$$

$$= \frac{1}{2}(5t^2+1)^{-\frac{1}{2}} \cdot 10t = \frac{5t}{\sqrt{5t^2+1}}$$

At $t = 4$, $a_T = \frac{5(4)}{\sqrt{5(4)^2+1}} = \frac{20}{9}$

Normal Component of acceleration $a_N = \frac{\|\vec{v} \times \vec{a}\|}{v}$

$$\therefore a_N = \frac{\sqrt{5}}{\sqrt{5t^2+1}}$$

At $t = 4$, $a_N = \frac{\sqrt{5}}{\sqrt{80+1}} = \frac{\sqrt{5}}{9}$

(b) $\vec{r}(t) = \ln(t^2+1)\vec{i} + (t - 2\tan^{-1}(t))\vec{j}$

For Simplicity, let us "View" the curve in $\mathbb{R}^3$ by having the z-Component equal to 0.

$\therefore \vec{r}(t) = (\ln(t^2+1),\ t-2\tan^{-1}(t),\ 0)$

$$\vec{v}(t) = \left(\frac{2t}{t^2+1},\ 1 - \frac{2}{t^2+1},\ 0\right)$$

$$= \left(\frac{2t}{t^2+1},\ \frac{t^2-1}{t^2+1},\ 0\right) = \frac{1}{t^2+1}(2t, t^2-1, 0)$$

$$\vec{a}(t) = \left(\frac{2-2t^2}{(t^2+1)^2},\ \frac{4t}{(t^2+1)^2},\ 0\right) = \frac{1}{(t^2+1)^2}(2-2t^2, 4t, 0)$$

$$v(t) = \text{speed} = \|\vec{v}(t)\|$$

$$= \frac{1}{t^2+1}\sqrt{(2t)^2 + (t^2-1)^2 + 0^2}$$

$$= \frac{1}{t^2+1}\sqrt{4t^2 + t^4 - 2t^2 + 1}$$

$$= \frac{1}{t^2+1}\sqrt{t^4+2t^2+1} = \frac{1}{t^2+1}\sqrt{(t^2+1)^2}$$

$$\therefore v(t) = 1$$

$\therefore$ Tangential Component $a_T = \frac{d}{dt}(v(t)) = 0$

$\therefore a_T = 0$ at $t = 2$ as well!

Next, at $t = 2$,

$$\vec{v} = \frac{1}{5}(4, 3, 0),$$

$$\vec{a} = \frac{1}{25}(-6, 8, 0)$$

$$\vec{v} \times \vec{a} = \frac{1}{125}(4,3,0) \times (-6,8,0)$$

$$= \frac{1}{125}(0, 0, 50) = \frac{50}{125}(0,0,1)$$

$$= \frac{2}{5}(0,0,1)$$

$$\|\vec{v} \times \vec{a}\| = \frac{2}{5}\|(0,0,1)\| = \frac{2}{5}$$

$\therefore$ Normal Component $a_N = \dfrac{\|\vec{v} \times \vec{a}\|}{v} = \dfrac{\frac{2}{5}}{1} = \dfrac{2}{5}$

(c) $\vec{r}(t) = t\cos(t)\,\vec{i} + t\sin(t)\,\vec{j} + t^2\,\vec{k}$

$$\overset{\text{or}}{=} \left(t\cos(t),\ t\sin(t),\ t^2\right)$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(\cos(t) - t\sin(t),\ \sin(t) + t\cos(t),\ 2t\right)$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \left(-2\sin(t) - t\cos(t),\ 2\cos(t) - t\sin(t),\ 2\right)$$

$$v(t) = \|\vec{v}(t)\| = \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2 + (2t)^2}$$

<u>Note</u>: $(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2$

$$= \cos^2(t) - \cancel{2t\sin(t)\cos(t)} + t^2\sin^2(t) + \sin^2(t)$$
$$+ \cancel{2t\sin(t)\cos(t)} + t^2\cos^2(t)$$
$$= \left(\cos^2(t) + \sin^2(t)\right) + t^2\left(\sin^2(t) + \cos^2(t)\right)$$
$$= 1 + t^2$$

$$\therefore\ v(t) = \sqrt{1 + t^2 + 4t^2} = \sqrt{1 + 5t^2}$$

$$\therefore\ a_T = \frac{dv}{dt} = \frac{d}{dt}(1 + 5t^2)^{\frac{1}{2}}$$
$$= \frac{1}{2}(1 + 5t^2)^{-\frac{1}{2}}(10t)$$

At $t = 0$,

$$a_T = 0$$

Next, at $t = 0$,

$$\vec{v} = (1, 0, 0)$$

$$\vec{a} = (0, 2, 2)$$

and $$v = \sqrt{1 + 5(0)^2} = \sqrt{1} = 1$$

$$\therefore \quad \vec{v} \times \vec{a} = (1, 0, 0) \times (0, 2, 2) = (0, -2, 2)$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{0 + 4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$\therefore$ Normal component of acceleration

$$a_N = \frac{\|\vec{v} \times \vec{a}\|}{v} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

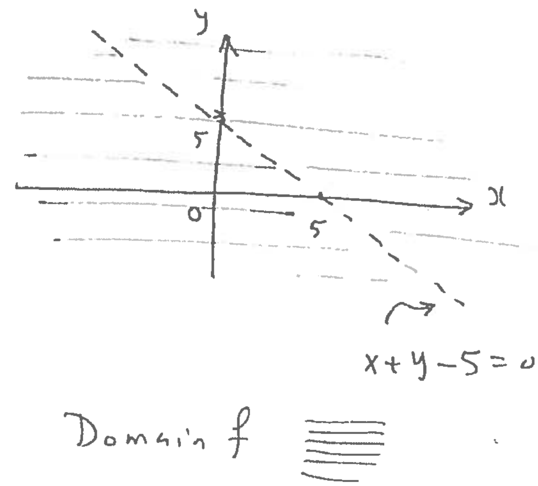
— — — — — — —

6. (a) $f(x,y) = \dfrac{3-x}{x+y-5}$

Let D be the domain of f.
Then D Consists of all (x,y) in $\mathbb{R}^2$ such that

$$x+y-5 \neq 0$$

That is D Consists of all points (x,y) in $\mathbb{R}^2$
Except points on the line $x+y-5=0$

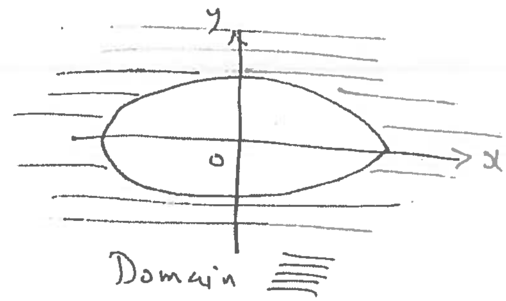


(b) $f(x,y) = \sqrt{4x^2+9y^2-36}$

The domain D Consists of all points (x,y) in $\mathbb{R}^2$
such that : $\quad 4x^2+9y^2-36 \geq 0$

$$\Rightarrow \quad 4x^2+9y^2 \geq 36 \qquad (\div 36)$$

$$\frac{x^2}{9} + \frac{y^2}{4} \geq 1$$

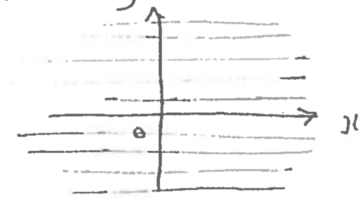


Note : $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is an Equation of an Ellipse
with Centre at $(0,0)$, and Semi-axes 3, 2

Hence $\quad \frac{x^2}{9} + \frac{y^2}{4} \geq 1$ is the region outside the
Ellipse !

(c) $f(x,y) = \sqrt{1+x^2+y^2}$

Since $1+x^2+y^2$ is always positive, then domain of f consists of all points (x,y) in $\mathbb{R}^2$



Domain : All of $\mathbb{R}^2$

(d) $f(x,y) = \sqrt{\ln(5-x^2-y^2)}$

Domain D consists of all points (x,y) in $\mathbb{R}^2$ such that

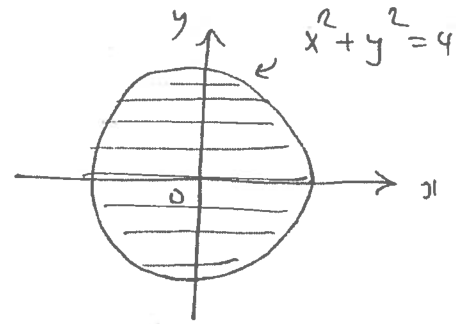
$$\ln(5-x^2-y^2) \geq 0$$

$$\Rightarrow \quad 5-x^2-y^2 \geq e^0$$

$$\Rightarrow \quad 5-x^2-y^2 \geq 1 \quad \text{or} \quad x^2+y^2 \leq 4$$

Note: $x^2+y^2=4$ is an equation of a circle centred at $(0,0)$ and has radius 4

Therefore $x^2+y^2 \leq 4$ is the region inside the circle



Domain

(e) $f(x,y) = \ln\sqrt{x^2+y^2-4}$

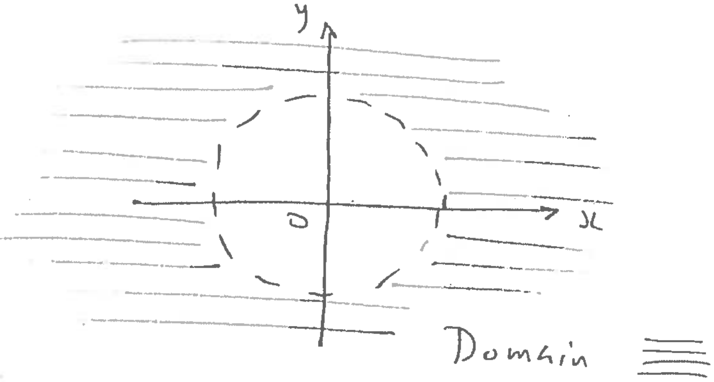
Note first that $\ln(t)$ is defined and is real only if $t > 0$.

Therefore the domain D consists of all (x,y) in $\mathbb{R}^2$ such that $t = x^2+y^2-4 > 0$

$$\Rightarrow x^2+y^2 > 4$$

Note: $x^2+y^2=4$ is an equation of a circle with centre $(0,0)$ and radius 2

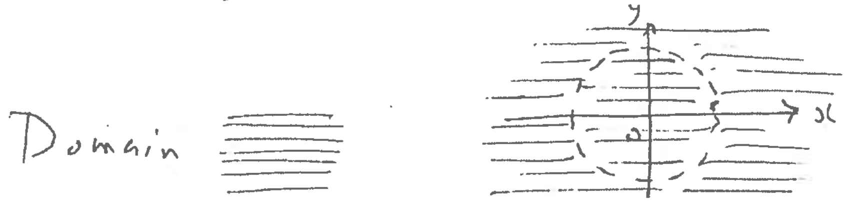
$\therefore$ $x^2+y^2>4$ is the region "strictly" outside the circle



(f) $f(x,y) = \ln|x^2+y^2-4|$. Clearly $|x^2+y^2-4| \geq 0$

Domain D consists of all (x,y) in $\mathbb{R}^2$ except where $x^2+y^2-4=0 \Rightarrow x^2+y^2=4$

So: Domain consists of all (x,y) in $\mathbb{R}^2$ except points on the circumference of the circle $x^2+y^2=4$



7. (a) $f(x,y) = x e^{-y}$

Level Curves are given by $f(x,y) = C$, that is

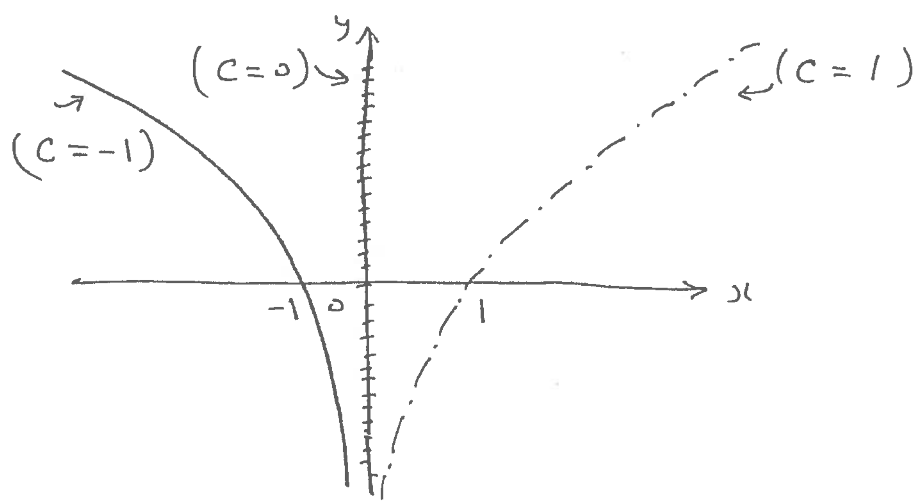
$$x e^{-y} = C$$

<u>$C=0$</u>: $\quad x e^{-y} = 0 \quad (\div e^{-y})$

$\Rightarrow x = 0$ .. The y-axis

<u>$C=1$</u>: $\quad x e^{-y} = 1 \Rightarrow e^{y} = x$ or $y = \ln(x)$, $x>0$

<u>$C=-1$</u>: $\quad x e^{-y} = -1 \Rightarrow e^{y} = -x$ or $y = \ln(-x)$, $x<0$



(b) $f(x,y) = \dfrac{x^2 - y^2}{x^2+y^2+1}$

Level curves are given by $f(x,y) = C$, that is

$$\frac{x^2 - y^2}{x^2+y^2+1} = C$$

$\underline{\underline{C = 0}}$:

$$\frac{x^2 - y^2}{x^2+y^2+1} = 0 \Rightarrow x^2 - y^2 = 0$$

$$\therefore \quad (x-y)(x+y) = 0$$

$$y = x, \quad y = -x \quad (\text{pair of lines})$$

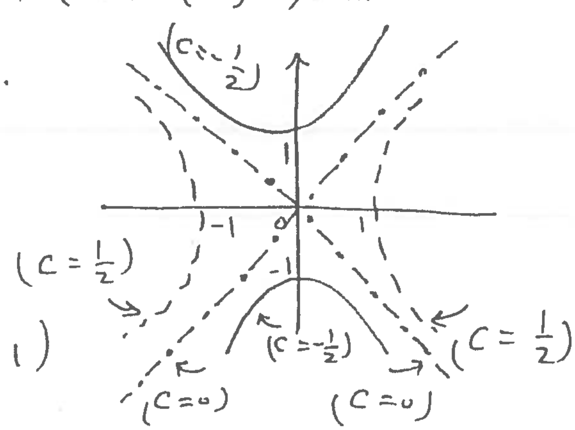
$\underline{\underline{C = \frac{1}{2}}}$:

$$\frac{x^2 - y^2}{x^2+y^2+1} = \frac{1}{2}$$

$$\Rightarrow \quad x^2+y^2+1 = 2(x^2-y^2)$$

$$\Rightarrow \quad x^2 - 3y^2 = 1$$

(An Eq. of a Hyperbola with centre at $(0,0)$ and which opens to the left & right).



$\underline{\underline{C = -\frac{1}{2}}}$:

$$\frac{x^2 - y^2}{x^2+y^2+1} = -\frac{1}{2}$$

$$\Rightarrow \quad 2(x^2-y^2) = -(x^2+y^2+1)$$

$$\Rightarrow \quad y^2 - 3x^2 = 1$$

(An Equation of a Hyperbola with centre $(0,0)$ and which opens up & down).

(c) $f(x,y) = \tan^{-1}(x+y)$

Level curves are given by

$$f(x,y) = C$$

That is $\quad \tan^{-1}(x+y) = C$

or $\quad x+y = \tan(c)$ (easier!)

$\underline{\underline{C=0}}$ $\quad x+y = \tan(0)$

$\Rightarrow x+y=0$ (line through origin)

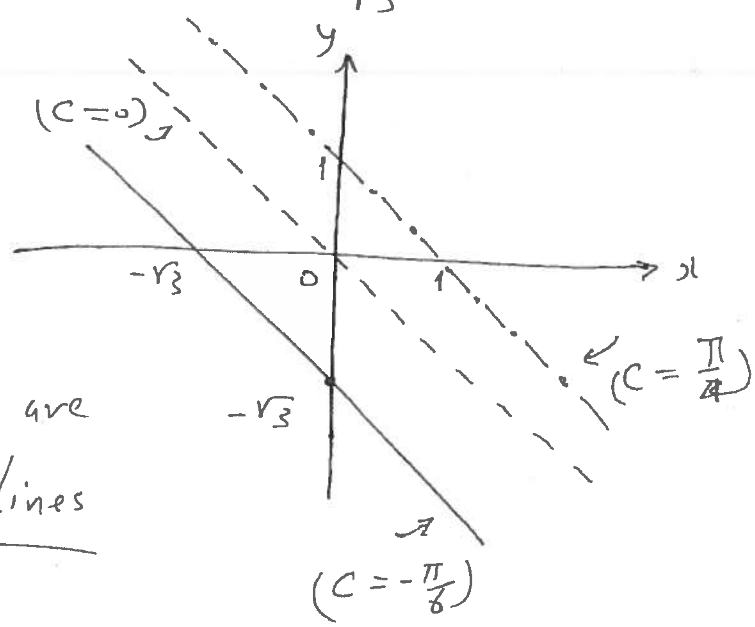
$\underline{\underline{C=\frac{\pi}{4}}}$ $\quad x+y = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow x+y=1$ (st. line)

$\underline{\underline{C=-\frac{\pi}{6}}}$ $\quad x+y = \tan\left(-\frac{\pi}{6}\right)$

$= -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

$\Rightarrow x+y = -\frac{1}{\sqrt{3}}$ (st. line)



Level curves are parallel lines

8. (i) $z = 1 + 3\sqrt{x^2 + y^2}$

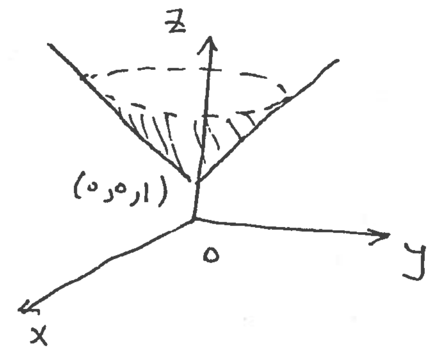
$$z - 1 = 3\sqrt{x^2 + y^2}$$

To Identify surface, let us first square each side:

$$(z-1)^2 = 9\,(x^2 + y^2)$$

or $$(z-1)^2 = \frac{x^2}{\frac{1}{9}} + \frac{y^2}{\frac{1}{9}}$$

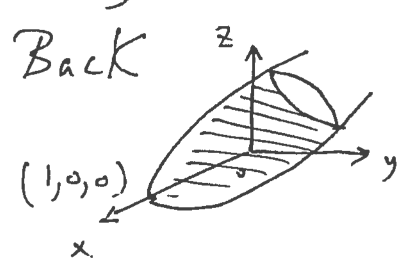
This is an equation of a Circular Cone with vertex at $(0,0,1)$ and axis of symmetry is the z-axis. However $z - 1 = +3\sqrt{x^2 + y^2}$ represents only the upper nappe of Cone.



(ii) $x = 2 - y^2 - z^2$

$$\Rightarrow x - 2 = -\,(y^2 + z^2)$$

This is an Equation of a Circular paraboloid with vertex at $(2,0,0)$, axis of symmetry is the x-axis and which opens towards the Back

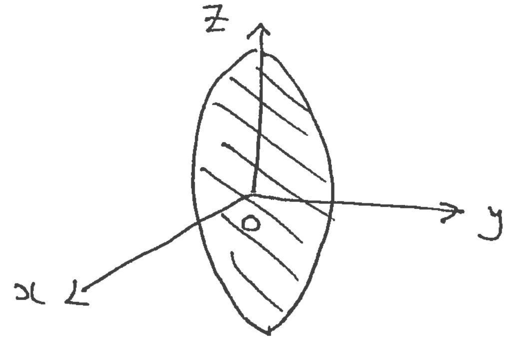


(iii) $2 - x^2 - 3y^2 - 2z^2 = 0$

$$\Rightarrow \quad x^2 + 3y^2 + 2z^2 = 2 \qquad (\div 2)$$

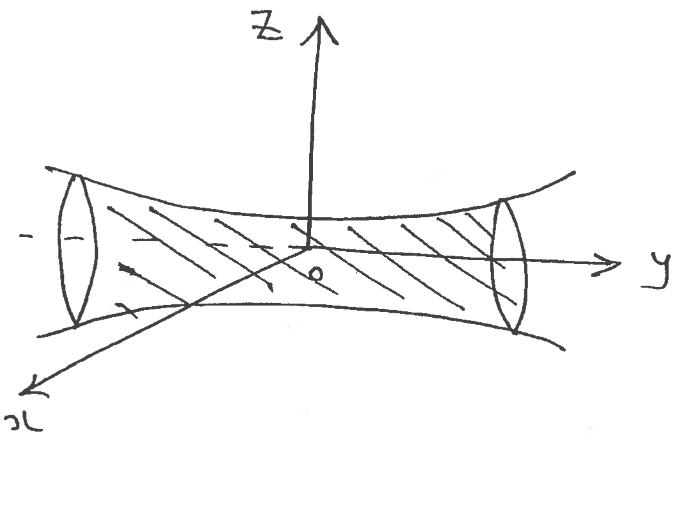
$$\frac{x^2}{2} + \frac{y^2}{\frac{2}{3}} + \frac{z^2}{1} = 1$$

This is an Equation of an Ellipsoid with Centre at $(0,0,0)$, and Semi-axes of length $a = \sqrt{2}$, $b = \sqrt{\frac{2}{3}}$, and $c = \sqrt{1} = 1$



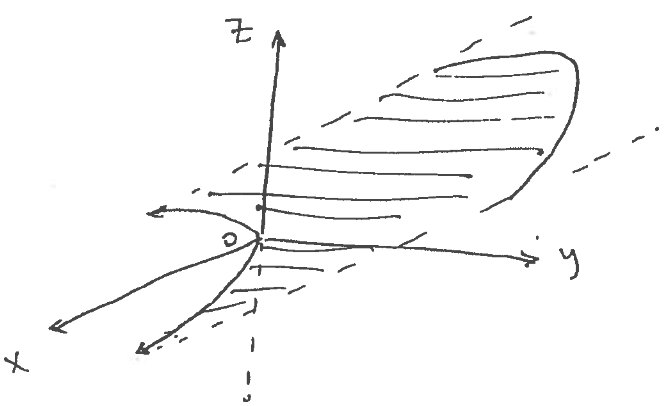
(iv) $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{25} = + 1$

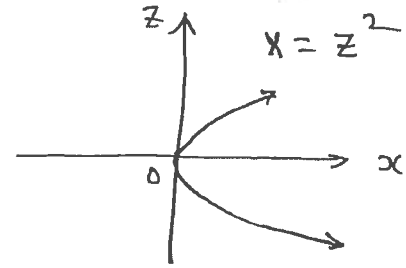
This is an equation of a Hyperboloid of <u>One</u> sheet Centred at $(0,0,0)$, and axis of symmetry is the y-axis



(V) $x = z^2$

This is an equation of a "parabolic" Cylindre generated by a line parallel to y-axis (why?) and its cross section by a plane perpendicular to y-axis is the parabola $x = z^2$ (which may be thought of as "The Base".

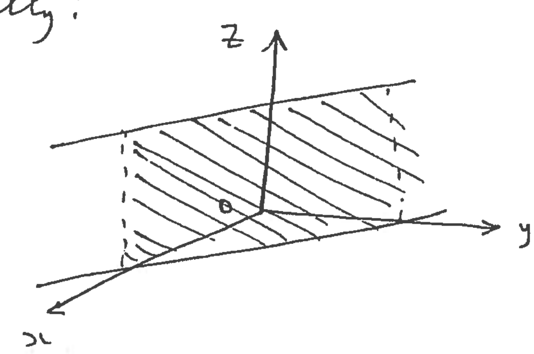


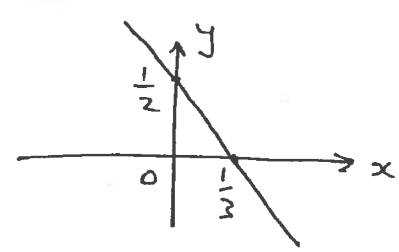


(Vi) $3x - 2y + 1 = 0$

This is an equation of a "plane" in $\mathbb{R}^3$.

<u>Note</u>: To sketch the plane, we first sketch the <u>line</u> $3x - 2y + 1$ in xy-plane, then pile the lines vertically!





(vii) $x^2+y^2+z^2-2x=0$

let us first Complete the square of x^2-2x to get

$$x^2-2x = (x-1)^2-1$$

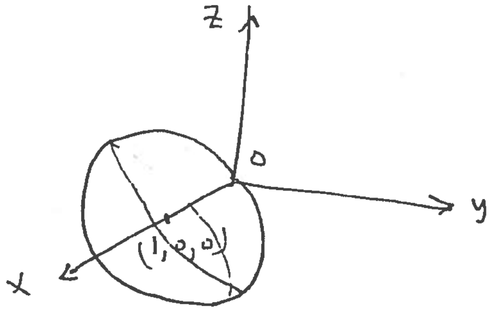
$\therefore$ Equation of surfaces becomes

$$(x-1)^2-1+y^2+z^2=0$$

or $(x-1)^2+y^2+z^2=1$

This is an Equation of a Sphere with Centre at $(1,0,0)$ and radius 1 unit.

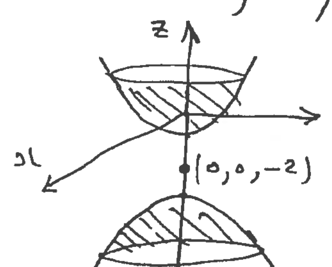
Note also that: It passes through origin $(0,0,0)$



(viii) $x^2+y^2-z^2-4z=3$ ← Complete square in z-terms

$$x^2+y^2-(z+2)^2=-1$$

This is an Equation of a Hyperboloid of Two sheets with Centre at $(0,0,-2)$, and axis of symmetry is the z-axis.



Note: Don't worry about the sketches in Problem (14).
I drew them Just for FUN!

— — — — — — — —

9. (a) $\quad Z = \ln(xy)^{\sin(xy)}, \quad x > 0,\ y > 0$

Simplify first

$$Z = \sin(xy)\ln(xy)$$

$$= \sin(xy)\left[\ln(x) + \ln(y)\right]$$

$$\therefore \frac{\partial z}{\partial y} = x\cos(xy)\left[\ln(x) + \ln(y)\right) + \sin(xy)\left[0 + \frac{1}{y}\right]$$

$$= x\cos(xy).\ln(xy) + \frac{\sin(xy)}{y}$$

(b) $\quad f(x,y) = y^{\tan(x)} + \cosh(x^2)$

$$f_y(x,y) = \tan(x).\,y^{\tan(x)-1} + 0$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(f_y(x,y)\right)$$

$$= \frac{\partial}{\partial x}\left[\tan(x).\,y^{\tan(x)-1}\right] \leftarrow \text{Apply product Rule}$$

Note : $\frac{d}{dt}\left(a^{u(t)}\right) = a^{u(t)}.(\ln a).\,u'(t)$

$$\therefore f_{yx}(x,y) = \sec^2(x).\,y^{\tan(x)-1} + \tan(x).\,y^{\tan(x)-1}.\ln(y).\left(\sec^2(x)\right)$$

$$= \sec^2(x).\,y^{\tan(x)-1}\left[1 + \tan(x).\ln(y)\right]$$

---

10. (a) $z = \sqrt{x^2 + y^2}$, $P(3,-4,5)$

Rewrite equation of surface in the form

$$z^2 = x^2 + y^2$$

or $F(x,y,z) = x^2 + y^2 - z^2 = 0$

For equation of tangent plane, we need:

1. <u>A point</u>: Given as $P(3,-4,5)$, viewed as a position vector $\vec{r_0} = (3,-4,5)$

2. <u>A vector Normal to Tangent plane</u>

This is $\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P$, $F = x^2 + y^2 - z^2$

$$= (2x, 2y, -2z)\Big|_{(x,y,z)=(3,-4,5)}$$

$$= (6, -8, -10)$$

Eq. of Tangent plane is thus given by

$$\vec{r} \cdot \vec{N} = \vec{r_0} \cdot \vec{N} \quad ; \quad \vec{r} = (x,y,z)$$

$$(x,y,z)\cdot(6,-8,-10) = (3,-4,5)\cdot(6,-8,-10)$$

$$6x - 8y - 10z = 18 + 32 - 50$$

$$\Rightarrow \quad 3x - 4y - 5z = 0$$

A parametric equation of normal line is thus given by

$$\vec{r}(t) = \vec{r}_0 + t\vec{N}, \quad t \in \mathbb{R}$$

$$(x,y,z) = (3,-4,5) + t(6,-8,-10), \quad t \in \mathbb{R}$$

(b) $xy + z^3 + e^{x-y+z} = 4 \Rightarrow$

$$F(x,y,z) = xy + z^3 + e^{x-y+z} - 4 = 0$$

Need: (i) A point: Given as $P(1,2,1) \Rightarrow \vec{r}_0 = (1,2,1)$

(ii) A normal vector $\vec{N}$:

$$\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P, \quad F = xy + z^3 + e^{x-y+z} - 4$$

$$= \left(y + e^{x-y+z},\ x - e^{x-y+z},\ 3z^2 + e^{x-y+z}\right)\Big|_{(x,y,z)=(1,2,1)}$$

$$= (3, 0, 4)$$

Eq. of Tangent plane:

$$(x,y,z)\cdot(3,0,4) = (1,2,1)\cdot(3,0,4)$$

$$\Rightarrow 3x - 4z = 3 + 0 + 4$$

$$3x + 4z = 7$$

Eq. of normal line: $(x,y,z) = (1,2,1) + t(3,0,4), \quad t \in \mathbb{R}$

---

11 (a) Let $z = f(x,y) = \cot\left(3x + \frac{1}{12}y\right)$

$$x = \frac{1}{\pi}t^2, \quad y = \frac{\pi^2}{6t} = \frac{\pi^2}{6}t^{-1}$$

$$z = f(x,y) = \cot\left(3x + \frac{1}{12}y\right)$$

$$\frac{\partial f}{\partial x} = -3\csc^2\left(3x+\frac{1}{12}y\right) \qquad \frac{\partial f}{\partial y} = -\frac{1}{12}\csc^2\left(3x+\frac{1}{12}y\right)$$

$$x = \frac{1}{\pi}t^2 \qquad y = \frac{\pi^2}{6}t^{-1}$$

$$\frac{dx}{dt} = \frac{2}{\pi}t \qquad \frac{dy}{dt} = -\frac{\pi^2}{6}t^{-2} = -\frac{\pi^2}{6t^2}$$

$$t \qquad t$$

Note: At $t = \pi/6$,

$$X = \frac{1}{\pi}t^2 = \frac{1}{\pi}\left(\frac{\pi}{6}\right)^2 = \frac{1}{\pi}\cdot\frac{\pi^2}{36} = \frac{\pi}{36},$$

$$y = \frac{\pi^2}{6t} = \frac{\pi^2}{\pi(\pi/6)} = \pi \text{, and hence}$$

$$3x + \frac{1}{12}y = 3\left(\frac{\pi}{36}\right) + \frac{1}{12}(\pi) = \frac{\pi}{12} + \frac{\pi}{12} = \frac{2\pi}{12} = \pi/6$$

$$\therefore \left.\frac{dz}{dt}\right|_{t=\pi/6} = \left. -3\csc^2\left(3x+\frac{1}{12}y\right)\cdot\left(\frac{2}{\pi}t\right) - \frac{1}{12}\csc^2\left(3x+\frac{1}{12}y\right)\cdot\left(-\frac{\pi^2}{6t^2}\right)\right|_{\substack{t=\pi/6 \\ X=\pi/36 \\ Y=\pi}}$$

$$= -3\csc^2\left(\frac{\pi}{6}\right)\left(\frac{2}{\pi}\cdot\frac{\pi}{6}\right) - \frac{1}{12}\csc^2\left(\frac{\pi}{6}\right)\cdot\left(-\frac{\pi^2}{6(\pi/6)^2}\right)$$

$$= -3\cdot 2^2\cdot\frac{1}{3} - \frac{1}{12}\cdot 2^2(-6) = -4 + 2 = -2$$

(b) $Z = f(x,y) = \ln(x^2+3xy)^{-4} = -4\ln(x^2+3xy)$,

$x = \cosh(u)$, $y = \sinh(v)$

$$\begin{tikzcd}[column sep=small]
 & Z = f(x,y) = -4\ln(x^2+3xy) \arrow[dl, "\frac{\partial f}{\partial x} = -4\cdot\frac{2x+3y}{x^2+3xy}"'] \arrow[dr, "\frac{\partial f}{\partial y} = -4\cdot\frac{3x}{x^2+3xy}"] & \\
x = \cosh(u) \arrow[d, "\frac{dx}{du} = \sinh(u)"'] & & y = \sinh(v) \arrow[d, "\frac{dy}{dv} = \cosh(v)"] \\
u & & v
\end{tikzcd}$$

<u>Note</u>: At $u=0, v=0$, we have

$x = \cosh(u) = \cosh(0) = 1$, $y = \sinh(v) = \sinh(0) = 0$

$$\therefore \left.\frac{\partial z}{\partial v}\right|_{\substack{u=0\\ v=0}} = -4\cdot\left.\frac{3x}{x^2+3xy}\cdot\cosh(v)\right|_{\substack{u=0\\ v=0\\ x=1\\ y=0}}$$

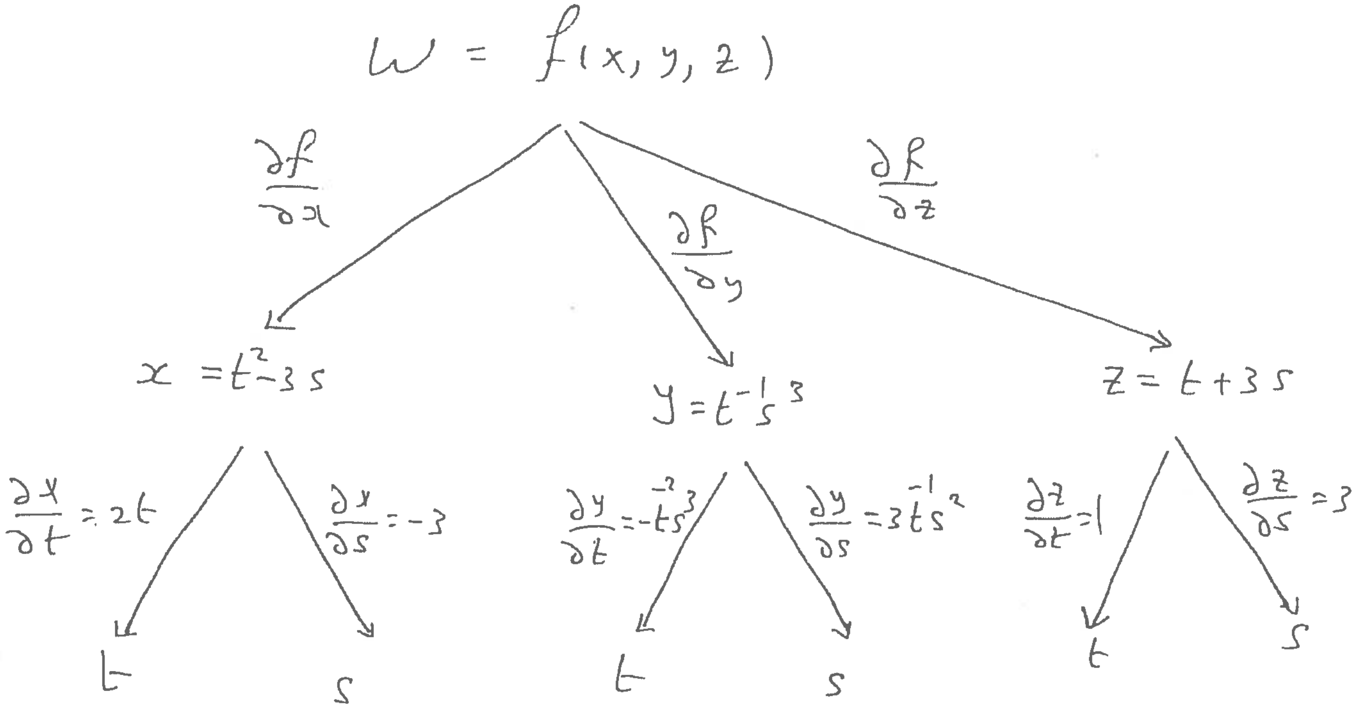
$$= -4\cdot\frac{3(1)}{(1)^2+3(1)(0)}\cosh(0)$$

$$= -4\cdot 3\cdot 1 = -12$$

(c) $W = f(t^2 - 3s,\ t^{-1}s^3,\ t + 3s)$

$= f(x, y, z)$, where

$x = t^2 - 3s$, $y = t^{-1}s^3$, and $z = t + 3s$



$$\frac{\partial W}{\partial s} = \frac{\partial f}{\partial x}\cdot(-3) + \frac{\partial f}{\partial y}\cdot 3t^{-1}s^2 + \frac{\partial f}{\partial z}\cdot 3$$

or $$\frac{\partial W}{\partial s} = -3 f_x(x,y,z) + 3t^{-1}s^2 f_y(x,y,z) + 3 f_z(x,y,z)$$

where x, y, and z are as above.

— — — — — —

(d) Let $z = f(x,y) = \sqrt{x^2 - y^2} = (x^2-y^2)^{\frac{1}{2}}$

$x = r\cos(\theta)$, $\quad y = r\sin(\theta)$

$$\therefore \frac{\partial f}{\partial x} = \frac{1}{2}(x^2-y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2-y^2}},$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2-y^2)^{-\frac{1}{2}}(-2y) = -\frac{y}{\sqrt{x^2-y^2}}$$

$$\frac{\partial x}{\partial r} = \cos(\theta) \overset{\text{or}}{=} \frac{x}{r},$$

$$\frac{\partial x}{\partial \theta} = -r\sin(\theta) \overset{\text{or}}{=} -y,$$

$$\frac{\partial y}{\partial r} = \sin(\theta) \overset{\text{or}}{=} \frac{y}{r}, \text{ and}$$

$$\frac{\partial y}{\partial \theta} = r\cos(\theta) \overset{\text{or}}{=} x$$

Refer to Tree Diagram below:

$$\begin{tikzcd}
& & z = f(x,y) \arrow[dl, "\frac{\partial f}{\partial x}"'] \arrow[dr, "\frac{\partial f}{\partial y}"] & & \\
& x \arrow[dl, "\frac{\partial x}{\partial r}"'] \arrow[d, "\frac{\partial x}{\partial \theta}"] & & y \arrow[d, "\frac{\partial y}{\partial r}"'] \arrow[dr, "\frac{\partial y}{\partial \theta}"] & \\
r & \theta & & r & \theta
\end{tikzcd}$$

<u>Note</u>:

At $(r,\theta) = (1, \frac{\pi}{6})$,

$x = r\cos(\theta) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$,

$y = r\sin(\theta) = \sin(\frac{\pi}{6}) = \frac{1}{2}$

From the diagram, we have

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\therefore \frac{\partial z}{\partial r} = \frac{x}{\sqrt{x^2 - y^2}} \cdot \frac{x}{r} + \frac{-y}{\sqrt{x^2 - y^2}} \cdot \frac{y}{r}$$

$$= \frac{x^2 - y^2}{r\sqrt{x^2 - y^2}} \leftarrow \text{Rationalize}$$

$$= \frac{\cancel{(x^2 - y^2)}}{r\cancel{\sqrt{x^2 - y^2}}} \cdot \frac{\sqrt{x^2 - y^2}}{\cancel{\sqrt{x^2 - y^2}}} = \frac{\sqrt{x^2 - y^2}}{r}$$

$$\therefore \left.\frac{\partial z}{\partial r}\right|_{\substack{r=1 \\ \theta = \frac{\pi}{6}}} = \left.\frac{\sqrt{x^2 - y^2}}{r}\right|_{\substack{r=1 \\ x = \frac{\sqrt{3}}{2} \\ y = \frac{1}{2}}} = \frac{\sqrt{\frac{3}{4} - \frac{1}{4}}}{1} = \frac{\sqrt{\frac{1}{2}}}{1} = \frac{1}{\sqrt{2}}$$

Next, $\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta}$

$$= \frac{x}{\sqrt{x^2 - y^2}} \cdot (-y) + \frac{-y}{\sqrt{x^2 - y^2}} \cdot x$$

$$= \frac{-2xy}{\sqrt{x^2 - y^2}}$$

$$\therefore \left.\frac{\partial z}{\partial \theta}\right|_{\substack{r=1 \\ \theta = \frac{\pi}{6}}} = \left.\frac{-2xy}{\sqrt{x^2 - y^2}}\right|_{\substack{x = \frac{\sqrt{3}}{2} \\ y = \frac{1}{2}}} = \frac{-2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\sqrt{\frac{3}{4} - \frac{1}{2}}}$$

$$= -\frac{\frac{\sqrt{3}}{2}}{\sqrt{\frac{1}{2}}} = -\frac{\sqrt{6}}{2}$$

(e) $\quad z = f(u, v),$

$$u = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2), \quad v = x + \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = f(u, v)$$

$f_u(u,v)$ $\qquad\qquad$ $f_v(u,v)$

$u = \frac{1}{2}\ln(x^2+y^2)$ $\qquad\qquad$ $v = x + \tan^{-1}\left(\frac{y}{x}\right)$

No Need! $\qquad$ $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$ $\qquad$ No Need! $\qquad$ $\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} \overset{\text{or}}{=} \frac{x}{x^2+y^2}$

x $\qquad$ y $\qquad$ x $\qquad$ y

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=0}} = \left. f_u(u,v)\cdot\frac{y}{x^2+y^2} + f_v(u,v)\cdot\frac{x}{x^2+y^2}\right|_{\substack{x=1\\y=0\\u=0\\v=1}}$$

Note:
at $x=1,\ y=0$
$u = \frac{1}{2}\ln 1 = 0,$
$V = \cancel{1} + \tan^{-1} 0 = 0$

$$= f_u(0,1)\cdot\frac{0}{1+0} + f_v(0,1)\cdot\frac{1}{1+0}$$

$$= f_v(0,1) = -4 \qquad \text{(from Data Given)}$$

(f) Let $W = f(x,y,z) = \ln(x^2+y^2+z^2)$,

$x = ue^{v}\sin(v)$, $y = ue^{v}\cos(v)$, $z = ue^{v}$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2},$$

$$\frac{\partial f}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

$$\frac{\partial x}{\partial u} = e^{v}\sin(v), \quad \frac{\partial x}{\partial v} = ue^{v}\sin(v) + ue^{v}\cos(v)$$

$$\frac{\partial y}{\partial u} = e^{v}\cos(v), \quad \frac{\partial y}{\partial v} = ue^{v}\cos(v) - ue^{v}\sin(v)$$

$$\frac{\partial z}{\partial u} = e^{v}, \quad \frac{\partial z}{\partial v} = ue^{v}$$

Let us construct the Tree Diagram:

$$\begin{tikzcd}
 & & & & W = f(x,y,z) \arrow[dlll, "\frac{\partial f}{\partial x}"'] \arrow[d, "\frac{\partial f}{\partial y}"] \arrow[drrr, "\frac{\partial f}{\partial z}"] & & & & \\
 & x \arrow[dl, "\frac{\partial x}{\partial u}"'] \arrow[dr, "\frac{\partial x}{\partial v}"] & & & y \arrow[dl, "\frac{\partial y}{\partial u}"'] \arrow[dr, "\frac{\partial y}{\partial v}"] & & & z \arrow[dl, "\frac{\partial z}{\partial u}"'] \arrow[dr, "\frac{\partial z}{\partial v}"] & \\
u & & v & u & & v & u & & v
\end{tikzcd}$$

Note:

At $(u,v) = (-2,0)$:

$x = -2e^{0}\sin(0) = \underline{\underline{0}}$

$y = -2e^{0}\cos(0) = \underline{\underline{-2}}$

and

$z = -2e^{0} = -2$

$\therefore (x,y,z) = (0,-2,-2)$

From Diagram, we obtain:

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{2x}{x^2+y^2+z^2}\cdot e^{v}\sin(v) + \frac{2y}{x^2+y^2+z^2}\cdot e^{v}\cos(v) + \frac{2z}{x^2+y^2+z^2}\left(e^{v}\right)\Bigg|_{\substack{(u,v)=(-2,0)\\(x,y,z)=(0,-2,-2)}}$$

$$= 0 + \frac{-4}{0+4+4}\left(e^{0}\cos(0)\right) + \frac{-4}{0+4+4}e^{0}$$

$$= -\frac{1}{2} - \frac{1}{2} = -1,$$

$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{2x}{x^2+y^2+z^2}\left(ue^{v}\sin(v) + ue^{v}\cos(v)\right) + \frac{2y}{x^2+y^2+z^2}\left(ue^{v}\cos(v) - ue^{v}\sin(v)\right) + \frac{2z}{x^2+y^2+z^2}\cdot ue^{v}\Bigg|_{\substack{(u,v)=(-2,0)\\(x,y,z)=(0,-2,-2)}}$$

$$= 0 + \frac{-4}{0+4+4}\left(-2e^{0}\cos(0) - 0\right) + \frac{-4}{0+4+4}\left(-2e^{0}\right)$$

$$= 1+1 = 2$$

12 (a) $4x^2+3y^2+z^2=25$, $P(1,2,-3)$

Let $F(x,y,z) = 4x^2+3y^2+z^2-25 = 0$

A vector normal to surface at P is thus given by

$$\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P , \quad F = 4x^2+3y^2+z^2-25$$

$$= (8x, 6y, 2z)\Big|_{(x,y,z)=(1,2,-3)}$$

$$= (8, 12, -6)$$

Eq. of tangent plane:

$$\vec{r}\cdot\vec{N} = \vec{r_0}\cdot\vec{N}$$

$$(x,y,z)\cdot(8,12,-6) = (1,2,-3)\cdot(8,12,-6)$$

$$8x+12y-6z = 8+24+18$$

$$= 50$$

or $\quad 4x+6y-3z=25$

— — — — — —

(b) $2x+3y^2+2z^2=31$, $P(-2,1,4)$

$$\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P , \quad F = 2x+3y^2+2z^2-31$$

$$= (2, 6y, 4z)\Big|_{(x,y,z)=(-2,1,4)}$$

$$= (2, 6, 16)$$

Equation of tangent plane:

$$(x,y,z)\cdot(2,6,16) = (-2,1,4)\cdot(2,6,16)$$
$$2x+6y+16z = -4+6+64$$
$$= 66$$

$$\Rightarrow x+3y+8z = 33$$

(c) $\sin(xyz-6) + 2x - x^2 = 0$, @ $(1,2,3)$

let $F(x,y,z) = \sin(xyz-6) + 2x - x^2$

A vector normal to surface at @ is thus given by

$$\vec{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_{@}$$

$$= \left(yz\cos(xyz-6)+2-2x,\ xz\cos(xyz-6),\ xy\cos(xyz-6)\right)\Big|_{(x,y,z)=(1,2,3)}$$

$$= (6+2-2,\ 3,\ 2)$$
$$= (6,3,2)$$

$\therefore$ A unit vector $\perp$ to surface is given by

$$\vec{n} = \pm\frac{\vec{N}}{\|\vec{N}\|} = \pm\frac{(6,3,2)}{\|(6,3,2)\|} = \pm\frac{1}{7}(6,3,2)$$

---

13. (a) Recall: The differential of f is denoted and defined by

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Here $f(x,y) = e^{3x}\cos(2y) + 2x - y + 1$

$$\frac{\partial f}{\partial x} = 3e^{3x}\cos(2y) + 2,$$

$$\frac{\partial f}{\partial y} = -2e^{3x}\sin(2y) - 1$$

$$\therefore\ df = \left[3e^{3x}\cos(2y) + 2\right]dx + \left[-2e^{3x}\sin(2y) - 1\right]dy$$

(b) $g(x,y) = \sin^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial g}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2\sqrt{1-\frac{y^2}{x^2}}}$$

$$= \frac{-y}{x^2\frac{\sqrt{x^2-y^2}}{x}} = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial g}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(\frac{1}{x}\right) = \frac{1}{\cancel{x}\,\frac{\sqrt{x^2-y^2}}{\cancel{x}}} = \frac{1}{\sqrt{x^2-y^2}}$$

$$\therefore\ dg = g_x\,dx + g_y\,dy = -\frac{y}{x\sqrt{x^2-y^2}}\,dx + \frac{1}{\sqrt{x^2-y^2}}\,dy$$

(c) Recall:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

Here $F(x,y,z) = e^{x+2y+3z}$

$\therefore \frac{\partial F}{\partial x} = e^{x+2y+3z}$, $\frac{\partial F}{\partial y} = 2e^{x+2y+3z}$, and

$\frac{\partial F}{\partial z} = 3\, e^{x+2y+3z}$

$\therefore dF = e^{x+2y+3z}\left[dx + 2dy + 3dz\right]$

(d) $G(x,y,z) = \ln(x^2+2y-z)$

$\frac{\partial G}{\partial x} = \frac{2x}{x^2+2y-z}$, $\frac{\partial G}{\partial y} = \frac{2}{x^2+2y-z}$, $\frac{\partial G}{\partial z} = \frac{-1}{x^2+2y-z}$

$$\therefore dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy + \frac{\partial G}{\partial z} dz$$

$$= \frac{1}{x^2+2y-z}\left[2x\,dx + 2dy - dz\right]$$

— — — — —

14. $PV = KT \Rightarrow P = \dfrac{KT}{V}$

or $P = KTV^{-1}$

$\therefore \dfrac{\partial P}{\partial T} = KV^{-1}$, $\dfrac{\partial P}{\partial V} = -KTV^{-2}$

$\therefore dP = \dfrac{\partial P}{\partial T} dT + \dfrac{\partial P}{\partial V} dV$

$= KV^{-1} dT - KTV^{-2} dV$

But $\Delta P \approx dP$

$\therefore \Delta P \approx KV^{-1} dT - KTV^{-2} dV$

or $\Delta P \approx KV^{-1} \Delta T - KTV^{-2} \Delta V$

Dividing both sides by $P = KTV^{-1}$:

$$\frac{\Delta P}{P} \approx \frac{KV^{-1} \Delta T}{KTV^{-1}} - \frac{KTV^{-2}}{KTV^{-1}} \Delta V$$

$$\Rightarrow \frac{\Delta P}{P} \approx \frac{\Delta T}{T} - \frac{\Delta V}{V}$$

Now, $V = 64$, $\Delta V = 68 - 64 = 4$,

$T = 360$, $\Delta T = 351 - 360 = -9$

$$\therefore \quad \frac{\Delta P}{P} \approx -\frac{9}{360} - \frac{4}{64}$$

$$\approx -\left(\frac{1}{40} + \frac{1}{16}\right) = -0.0875$$

$$\therefore \quad \frac{\Delta P}{P} \approx (-0.0875)(100)\ \%$$

$$\approx 8.75\ \%$$

so, the Pressure decreases by approximately 8.75%.

15. From problem (14)

$$\frac{\Delta P}{P} \approx \frac{\Delta T}{T} - \frac{\Delta V}{V}$$

We know: $\frac{\Delta T}{T} = -0.8\%$

$\frac{\Delta P}{P} = +0.5\%$

It follows that

$$0.5\% \approx -0.8\% - \frac{\Delta V}{V}$$

$$\Rightarrow \quad \frac{\Delta V}{V} \approx -0.8\% - 0.5\% = -1.3\%$$

so, the volume decreases by approximately 1.3%.

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16. $$F = \frac{\pi P R^4}{8 v l}$$

let $\frac{\pi}{8 v l}$ = a constant K

$$\therefore \quad F = K P R^4$$

$$\frac{\partial F}{\partial P} = K R^4 \quad , \quad \frac{\partial F}{\partial R} = 4 K P R^3$$

$$\therefore \quad dF = \frac{\partial F}{\partial P} dP + \frac{\partial F}{\partial R} dR$$

$$= K R^4 dP + 4 K P R^3 dR$$

Dividing both sides by $F = KPR^4$, we obtain

$$\frac{dF}{F} = \frac{K R^4 dP}{K P R^4} + \frac{4 K P R^3}{K P R^4} dR$$

$$\therefore \quad \frac{dF}{F} = \frac{dP}{P} + 4 \left(\frac{dR}{R} \right)$$

But $\Delta F \approx dF$,

and $dP = \Delta P$, $dR = \Delta R$

$\therefore \quad \frac{\Delta F}{F} \approx \frac{\Delta P}{P} + 4 \frac{\Delta R}{R}$

Know: $\frac{\Delta R}{R} = -2\%$, and $\frac{\Delta P}{P} = 3\%$

$$\therefore \quad \frac{\Delta F}{F} = 3\% + 4(-2\%)$$

$$= -5\%$$

$\therefore$ The Blood flow decrease by approximately 5%

— — — — — —

17. $4x^3 - 5y^3 - 3z + 10 = 0$ ---- (1)

$x^3 + y^3 = 2$ ---- (2)

Given $t = \frac{z}{3}$, hence $z = 3t$

Substituting $z = 3t$ into equation (1), we obtain

$4x^3 - 5y^3 - 9t + 10 = 0$ ---- (3)

But from equation (2): $y^3 = 2 - x^3$.

Therefore, equation (3) reduces to

$$4x^3 - 5(2 - x^3) - 9t + 10 = 0$$

$$4x^3 - \cancel{10} + 5x^3 - 9t + \cancel{10} = 0$$

$$\Rightarrow 9x^3 - 9t = 0$$

$$x^3 = t$$

$$\Rightarrow x = \sqrt[3]{t}$$

To find y: $\quad y^3 = 2 - x^3$

$$= 2 - (\sqrt[3]{t})^3$$

$$= 2 - t$$

$$\therefore y = \sqrt[3]{2 - t}$$

$\therefore$ A parametric representation of curve of intersection is given by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$$= \sqrt[3]{t}\,\vec{i} + \sqrt[3]{2 - t}\,\vec{j} + 3t\,\vec{k}, \quad t \in \mathbb{R}$$

— — — — — — —

18. Recall: As in problem #(8):

$$v = v_e \ln\left(\frac{M}{m(t)}\right), \quad m(t) = M - \alpha t$$

Here $v_e = 400$ m/s, hence

$$v = 400 \ln\left(\frac{M}{m}\right) \quad --- (*)$$

(a) Let $v = 800$, we obtain

$$800 = 400 \ln\left(\frac{M}{m}\right)$$

$$\Rightarrow \ln\left(\frac{M}{m}\right) = 2 \Rightarrow$$

$$\frac{M}{m} = e^2 \Rightarrow m = \frac{M}{e^2}$$

$\therefore$ Amount of burnt fuel $= M - m = M - \frac{M}{e^2} = M\left(1 - \frac{1}{e^2}\right)$

Hence the required ratio:

$$\frac{M-m}{M} = \frac{M\left(1-\frac{1}{e^2}\right)}{M} = 1 - \frac{1}{e^2}$$

$$\overset{\text{or}}{=} 100\left(1 - \frac{1}{e^2}\right)\%$$

$$\cong 86.5\%$$

(b) Here: Remaining Mass $m(t) = 40\%$ of $M = 0.4M$

$$\therefore (*) \Rightarrow V = 400 \ln\left(\frac{\cancel{M}}{0.4\cancel{M}}\right)$$

$$= 400 \ln(2.5) \cong 367 \text{ m/s}$$

(c) Here: Amount of burnt fuel is 40% of M, that is $0.4\,M$. Hence remaining mass $m(t) = M - 0.4M = 0.6M$

$\therefore$ (*) $\Rightarrow$ $v = 400 \ln\left(\frac{\cancel{M}}{0.6\cancel{M}}\right)$

$$= 400 \ln\left(\frac{1}{0.6}\right) = 400 \ln\left(\frac{5}{3}\right)$$

$$\cong 204 \text{ m/s}$$

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19. For students to do at home.

Answers

(i) $\dfrac{\partial z}{\partial s} = 4$

(ii) $\dfrac{\partial z}{\partial x} = 38$

(iii) $\dfrac{dW}{dt} = 4\pi$

(iv) $\dfrac{dz}{dt} = -8$

20. (a) Recall: The Directional Derivative of f at the point P in the direction of the unit vector $\vec{u}$ is denoted and given by

$$D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$$

Here $f(x,y) = \sin(x+2y)$, $P(0, \frac{\pi}{2})$, $\vec{u} = (-\frac{3}{5}, \frac{4}{5})$

$$\vec{\nabla} f(P) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_P$$

$$= \left(\cos(x+2y), 2\cos(x+2y)\right)\Big|_{(x,y)=(0,\frac{\pi}{2})}$$

$$= \left(\cos(\pi), 2\cos(\pi)\right) = (-1, -2)$$

$$\therefore D_{\vec{u}} f(P) = (-1, -2) \cdot \left(-\frac{3}{5}, \frac{4}{5}\right)$$

$$= \frac{3}{5} - \frac{8}{5} = -\frac{5}{5} = -1$$

(b) $f(x,y,z) = e^{x^2+y-2z}$, $P(1,1,1)$, $\vec{v} = (0,-1,1)$

$$\vec{\nabla} f(P) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_P$$

$$= \left(e^{x^2+y-2z} \cdot 2x, e^{x^2+y-2z} \cdot 1, e^{x^2+y-2z} \cdot (-2)\right)\Big|_{P(1,1,1)}$$

$$= (2, 1, -2)$$

<u>Warning</u> $\vec{v} = (0,-1,1)$

A <u>unit</u> vector in the direction of $\vec{v}$ is given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(0,-1,1)}{\sqrt{0+1+1}} = \frac{1}{\sqrt{2}}(0,-1,1)$$

$$\therefore \; D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$$

$$= (2,1,-2) \cdot \frac{1}{\sqrt{2}}(0,-1,1)$$

$$= \frac{1}{\sqrt{2}} (2,1,-2) \cdot (0,-1,1)$$

$$= \frac{1}{\sqrt{2}}(0-1-2) = -\frac{3}{\sqrt{2}}$$

(c) $f(x,y,z) = xy + 2xz + 3yz - 2x - y + 1$, $P(1,2,-3)$

$$\vec{\nabla} f(P) = (f_x, f_y, f_z)\Big|_P$$

$$= (y+2z-2,\; x+3z-1,\; 2x+3y)\Big|_{\substack{x=1\\ y=2\\ z=-3}}$$

$$= (-6,-9,8)$$

Now, a vector in the direction from $P(1,2,-3)$ to $Q(0,0,-1)$ is $\vec{v} = (0,0,-1) - (1,2,-3) = (-1,-2,2)$

$\therefore$ A unit vector in the direction of $\vec{v}$ is thus given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(-1,-2,2)}{\sqrt{1+4+4}} = \frac{1}{3}(-1,-2,2)$$

$$\therefore D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$$

$$= (-6,-9,8) \cdot \frac{1}{3}(-1,-2,2)$$

$$= \frac{1}{3}(6+18+16) = \frac{40}{3}$$

— — — — — —

21 $f(x,y,z) = \ln\left(\sqrt{x^2+y^2+z^2}\right)$ ← Simplify

$$= \frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\vec{\nabla} f(P) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_P$$

$$= \left(\frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}, \frac{1}{2}\cdot\frac{2y}{x^2+y^2+z^2}, \frac{1}{2}\cdot\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{1}{x^2+y^2+z^2}(x, y, z)\Big|_{\substack{x=1\\y=-2\\z=2\sqrt{5}}}$$

$$= \frac{1}{1+4+20}(1,-2,2\sqrt{5}) = \frac{1}{25}(1,-2,2\sqrt{5})$$

$$\|\vec{\nabla} f(P)\| = \frac{1}{25}\sqrt{1^2+(-2)^2+(2\sqrt{5})^2} = \frac{1}{25}\sqrt{25} = \frac{5}{25} = \frac{1}{5}$$

(i) The unit vector $\vec{u}$ for which $D_u f(P)$ is a maximum is given by

$$\vec{u} = \frac{\vec{\nabla} f(P)}{\|\vec{\nabla} f(P)\|} = \frac{\frac{1}{25}(1,-2,2\sqrt{5})}{\frac{1}{5}} = \frac{1}{5}(1,-2,2\sqrt{5})$$

and the Maximum value is $\|\vec{\nabla} f(P)\| = \frac{1}{5}$

(ii) The unit vector $\vec{v}$ for which $D_{\vec{v}} f(P)$ is a minimum is given by $\vec{v} = -\frac{1}{5}(1,-2,2\sqrt{5}) \overset{\text{or}}{=} \frac{1}{5}(-1,2,-2\sqrt{5})$,

and Minimum value is $-\frac{1}{5}$

22. (a) $3e^{z+2y+1} + \sin(3xyz) = 2$

$$\Rightarrow F(x,y,z) = 3e^{z+2y+1} + \sin(3xyz) - 2 = 0$$

$$\therefore \frac{\partial F}{\partial y} = 6e^{z+2y+1} + 3xz\cos(3xyz)$$

$$\frac{\partial F}{\partial z} = 3e^{z+2y+1} + 3xy\cos(3xyz)$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{6e^{z+2y+1} + 3xz\cos(3xyz)}{3e^{z+2y+1} + 3xy\cos(3xyz)}$$

At $(x,y,z) = (\frac{\pi}{6}, -1, 1)$, we obtain

$$\frac{\partial z}{\partial y} = -\frac{6e^0 + 3(\frac{\pi}{6})(1)\cos(-\frac{\pi}{2})}{3e^0 + 3(\frac{\pi}{6})(-1)\cos(-\frac{\pi}{2})}$$

But $e^0 = 1$, $\cos(-\frac{\pi}{2}) = 0$

$$\therefore \frac{\partial z}{\partial y} = -\frac{6}{3} = -2$$

(b) $$x^2 + 3yz - \frac{2}{\ln(x+z)} = 5$$

$$\Rightarrow F(x,y,z) = x^2 + 3yz - \frac{2}{\ln(x+z)} - 5 = 0$$

or $$F(x,y,z) = x^2 + 3yz - 2\left[\ln(x+z)\right]^{-1} - 5$$

$$\therefore \frac{\partial F}{\partial x} = 2x + 2\left[\ln(x+z)\right]^{-2} \cdot \frac{1}{x+z},$$

$$\frac{\partial F}{\partial y} = 3z$$

$$\therefore \frac{\partial x}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} = -\frac{3z}{2x + 2\left[\ln(x+z)\right]^{-2} \cdot \frac{1}{x+z}}$$

— — — — — — —

23. (i) $x^5 + 2xy^3 + xyz - z^4 = -15$

let us first find "y" at $x = 1$, $z = 2$:

$$1 + 2y^3 + 2y - 16 = -15$$

$$\Rightarrow \quad 2y^3 + 2y = 0 \quad \Rightarrow 2y(y^2+1) = 0$$

$$\therefore \quad \boxed{y = 0} \quad \text{or} \quad y^2 + 1 = 0 \ (\text{No Solution})$$

$\therefore$ point is $(x, y, z) = (1, 0, 2)$

Next, let $F(x,y,z) = x^5 + 2xy^3 + xyz - z^4 + 15$

$$\therefore \quad \frac{\partial F}{\partial y} = 6xy^2 + xz \quad ,$$

$$\frac{\partial F}{\partial z} = xy - 4z^3$$

$$\therefore \quad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y} = -\frac{xy - 4z^3}{6xy^2 + xz}$$

at $(x,y,z) = (1,0,2)$, we obtain,

$$\frac{\partial y}{\partial z} = -\frac{0 - 4(2)^3}{0 + (1)(2)} = -\frac{-(4)(8)}{2}$$

$$= 16$$

(ii) $y^2 + y\sqrt{z} = 2 - \sin(xz^2) + \frac{4}{z}$

$$\Rightarrow \quad y^2 + y\sqrt{z} - 2 + \sin(xz^2) - \frac{4}{z} = 0$$

Take $F(x,y,z) = y^2 + y\sqrt{z} - 2 + \sin(xz^2) - \frac{4}{z}$

$$F_x(x,y,z) = z^2 \cos(xz^2),$$

$$F_y(x,y,z) = 2y + \sqrt{z}$$

$$\therefore \quad \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{2y + \sqrt{z}}{z^2 \cos(xz^2)}$$

At $(x,y,z) = (0,1,4)$, we obtain

$$\frac{\partial x}{\partial y} = -\frac{2(1) + \sqrt{4}}{4^2 \cos(0)} = -\frac{4}{16}$$

$$= -\frac{1}{4}$$

— — — — — —

24. $u = x^2 + y^2 \Rightarrow x^2 + y^2 - u = 0$

$v = x^2 - 2xy^2 \Rightarrow x^2 - 2xy^2 - v = 0$

Take $F = x^2 + y^2 - u$, $G = x^2 - 2xy^2 - v$

Here $x = x(u,v)$, and $y = y(u,v)$. That is x, and y are the Dependent Variables.

(a) $$\frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} -1 & 2y \\ 0 & -4xy \end{vmatrix}}{\begin{vmatrix} 2x & 2y \\ 2x-2y^2 & -4xy \end{vmatrix}}$$

At $(x,y) = (1,2)$, we obtain

$$\frac{\partial x}{\partial u} = -\frac{\begin{vmatrix} -1 & 4 \\ 0 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ -6 & -8 \end{vmatrix}} = -\frac{8}{8} = -1$$

Next $$\frac{\partial y}{\partial u} = -\frac{\dfrac{\partial(F,G)}{\partial(x,u)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} 2x & -1 \\ 2x-2y^2 & 0 \end{vmatrix}}{\begin{vmatrix} 2x & 2y \\ 2x-2y^2 & -4xy \end{vmatrix}}$$

At $(x, y) = (1, 2)$, we obtain

$$\frac{\partial y}{\partial u} = - \frac{\begin{vmatrix} 2 & -1 \\ -6 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ -6 & -8 \end{vmatrix}} = - \quad \frac{-6}{8} = \frac{3}{4}$$

(b) $z = f(x, y) = \ln(y^2 - x^2)$

By chain rule

$$\begin{tikzcd}[column sep=small]
& & & z = f(x,y) = \ln(y^2-x^2) \arrow[dll, "\frac{\partial f}{\partial x} = \frac{-2x}{y^2-x^2}"'] \arrow[drr, "\frac{\partial f}{\partial y} = \frac{2y}{y^2-x^2}"] & & & \\
& x = x(u,v) \arrow[dl, "\frac{\partial x}{\partial u}"'] \arrow[dr, "\frac{\partial x}{\partial v}"] & & & & y = y(u,v) \arrow[dl, "\frac{\partial y}{\partial u}"'] \arrow[dr, "\frac{\partial y}{\partial v}"] & \\
u & & v & & u & & v
\end{tikzcd}$$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{-2x}{y^2 - x^2} \cdot \frac{\partial x}{\partial u} + \frac{2y}{y^2 - x^2} \frac{\partial y}{\partial u}$$

At $(x, y) = (1, 2)$, we know from part (a) that

$\frac{\partial x}{\partial u} = -1$, and $\frac{\partial y}{\partial u} = \frac{3}{4}$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{-2(1)}{2^2 - 1^2} (-1) + \frac{2(2)}{2^2 - 1^2} \left(\frac{3}{4}\right)$$

$$= \frac{2}{3} + 1 = \frac{5}{3}$$

25. $ue^{v} + xw - \cos(y) = 2$

$\Rightarrow ue^{v} + xw - \cos(y) - 2 = 0$

Take $F = ue^{v} + xw - \cos(y) - 2$

Next: $x\cos(v) + u^2 y - vw^2 = 1$

$\Rightarrow x\cos(v) + u^2 y - vw^2 - 1 = 0$

Take $G = x\cos(v) + u^2 y - vw^2 - 1$

Here $x = x(u, v, w)$, and $y = y(u, v, w)$. That is x, and are the Dependent Variables.

Now,
$$\frac{\partial(F, G)}{\partial(x, y)} = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}$$

$$= \begin{vmatrix} w & -\sin(y) \\ \cos(v) & u^2 \end{vmatrix}$$

At $(u, v, w; x, y) = (2, 0, 1; 1, 0)$ we obtain

$$\frac{\partial(F, G)}{\partial(x, y)} = \begin{vmatrix} 1 & -\sin(0) \\ \cos(0) & 2^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = 4 \neq 0,$$

Hence system is solvable for x, and y as functions of u, v, and w near the point P where
$P = (u, v, w; x, y) = (2, 0, 1; 1, 0)$

Next,

$$\left(\frac{\partial x}{\partial w}\right)_{u,v} = -\frac{\dfrac{\partial(F,G)}{\partial(w,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_w & F_y \\ G_w & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} x & -\sin(y) \\ -2vw & u^2 \end{vmatrix}}{\begin{vmatrix} w & -\sin(y) \\ \cos(v) & u^2 \end{vmatrix}}$$

At $(u,v,w;x,y) = (2,0,1;1,0)$, we obtain

$$\left(\frac{\partial x}{\partial w}\right)_{u,v} = -\frac{\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix}} = -\frac{4}{4} = -1$$

Finally

$$\left(\frac{\partial y}{\partial v}\right)_{u,w} = -\frac{\dfrac{\partial(F,G)}{(x,v)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} w & ue^v \\ \cos(v) & -x\sin(v) - w^2 \end{vmatrix}}{\begin{vmatrix} w & -\sin(y) \\ \cos(v) & u^2 \end{vmatrix}}$$

At $(u,v,w;x,y) = (2,0,1;1,0)$, we obtain

$$\left(\frac{\partial y}{\partial v}\right)_{u,w} = -\frac{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix}} = -\frac{-3}{4} = 3/4$$

---

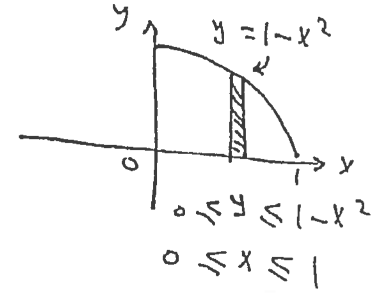
26. Use double integrals to find the volume of the solid which lies vertically above the planar region $0 \le y \le 1-x^2$, $0 \le x \le 1$ below the plane $z = 1-x$.

Solution:

$$\text{Volume } V = \iint_{\text{Base}} \text{height}\, dA$$

Here: The Base is the region shown in figure!

Height is $z_{top} - z_{bottom} = (1-x) - 0 = 1-x$



$$\therefore V = \iint_{\text{Base}} (1-x)\, dA = \int_0^1 \int_0^{1-x^2} (1-x)\, dy\, dx$$

$$= \int_0^1 (1-x) \left\{ \int_0^{1-x^2} dy \right\} dx = \int_0^1 (1-x)(1-x^2)\, dx$$

$$= \int_0^1 (1 - x - x^2 + x^3)\, dx = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 \Big|_0^1$$

$$= 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{12-6-4+3}{12}$$

$$= \frac{5}{12}$$

---

27. Find the volume enclosed by the surfaces $z = 13 - x^2 - y^2$, and $z = 4\sqrt{x^2+y^2} + 1$

Answer: $V = \frac{56}{3}\pi$.

Hint: Use polar coordinates!

28. Find the Volume enclosed by the surfaces

$$z=\sqrt{x^2+y^2+1}\,,\quad z=\frac{6}{\sqrt{2x^2+2y^2+3}}\,.$$

<u>solution</u> : we shall use polar coordinates:

$$x=r\cos(\theta),\ y=r\sin(\theta),\ x^2+y^2=r^2,\ dA=r\,dr\,d\theta$$

In polar coordinates: $z=\sqrt{x^2+y^2+1} \Rightarrow z=\sqrt{r^2+1}$,

$$z=\frac{6}{\sqrt{2x^2+2y^2+3}}=\frac{6}{\sqrt{2(x^2+y^2)+3}}=\frac{6}{\sqrt{2r^2+3}}\,.$$

<u>Base</u> : let us Eliminate "z" between

$$z=\sqrt{r^2+1}\,,\quad z=\frac{6}{\sqrt{2r^2+3}}$$

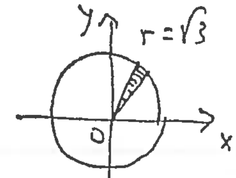
we get $\sqrt{r^2+1}=\dfrac{6}{\sqrt{2r^2+3}} \Rightarrow r^2+1=\dfrac{36}{2r^2+3}$

$$(r^2+1)(2r^2+3)=36 \Rightarrow 2r^4+5r^2-33=0$$

$$(r^2-3)(2r^2+11)=0 \Rightarrow r=+\sqrt{3}$$

clearly for $r\in[0,\sqrt{3}]$, $z_{top}=\dfrac{6}{\sqrt{2r^2+3}}$, $z_{bottom}=\sqrt{r^2+1}$

$\therefore$ Height $=z_{top}-z_{bottom}=\dfrac{6}{\sqrt{2r^2+3}}-\sqrt{r^2+1}$



$0\leq r\leq\sqrt{3}$

$0\leq\theta\leq 2\pi$

Volume $V=\displaystyle\iint_{Base}\text{height}\,dA$

$$=\iint_{Base}\left(\frac{6}{\sqrt{2r^2+3}}-\sqrt{r^2+1}\right)r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\left(\frac{6r}{\sqrt{2r^2+3}}-r\sqrt{r^2+1}\right)dr\,d\theta$$

$$\therefore\ V = \int_0^{2\pi} d\theta \cdot \int_0^{\sqrt{3}} \left(\frac{6r}{\sqrt{2r^2+3}} - r\sqrt{r^2+1} \right) dr$$

$$= 2\pi \left[\int_0^{\sqrt{3}} \frac{6r}{\sqrt{2r^2+3}}\, dr - \int_0^{\sqrt{3}} r\sqrt{r^2+1}\, dr \right]$$

$$= 2\pi \left[I - J \right], \text{ where}$$

$$I = \int_0^{\sqrt{3}} \frac{6r}{\sqrt{2r^2+3}}\, dr \longrightarrow \text{let } u = 2r^2+3$$

$$\therefore du = 4r\,dr \Rightarrow r\,dr = \tfrac{1}{4}\, du$$

$$= \frac{6}{4}\int \frac{1}{\sqrt{u}}\, du = \frac{3}{2}\cdot 2\sqrt{u} = 3\sqrt{2r^2+3}\,\Big|_0^{\sqrt{3}}$$

$$= 3\left[\sqrt{9} - \sqrt{3}\right] = 3\left[3 - \sqrt{3}\right]$$

Similarly $J = \int_0^{\sqrt{3}} r\sqrt{r^2+1}\, dr$ -- let $u = r^2+1$

$$du = 2r\,dr \Rightarrow r\,dr = \tfrac{1}{2}\, du$$

$$= \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\cdot\frac{2}{3}\, u^{\frac{3}{2}}$$

$$= \frac{1}{3}\left(r^2+1\right)^{\frac{3}{2}}\Big|_0^{\sqrt{3}} = \frac{1}{3}\left[4^{\frac{3}{2}} - 1\right]$$

$$= \frac{1}{3}\left[4\sqrt{4} - 1\right] = \frac{1}{3}\left[8-1\right] = 7/3$$

$$\therefore \text{ Volume } V = 2\pi\left[I - J\right]$$

$$= 2\pi\left[3(3-\sqrt{3}) - \tfrac{7}{3}\right]$$

$$= 2\pi\left[9 - 3\sqrt{3} - 7/3\right]$$

$$= 2\pi\left[\tfrac{20}{3} - 3\sqrt{3}\right].$$

---

29. Find the volume of the solid enclosed by the surfaces $z = x^2 + y^2 - 6$, and $z = 4 + 3\sqrt{x^2+y^2}$.

Answer: $V = \frac{375}{2}\pi$.

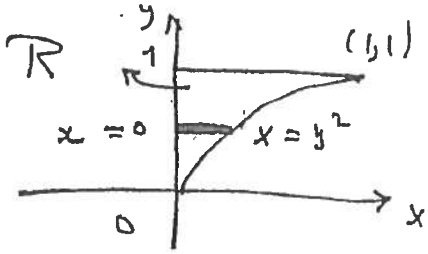
30. Evaluate $\int_0^1 \int_{\sqrt{x}}^1 3\ln(1+y^3)\, dy\, dx$ by first reversing the order of integration.

<u>solution</u>: let $I = \int_0^1 \int_{\sqrt{x}}^1 3\ln(1+y^3)\, dy\, dx = \iint_R 3\ln(1+y^3)\, dA$

where R is the region described by $\sqrt{x} \leq y \leq 1$, $0 \leq x \leq 1$

<u>Note</u>: $y = \sqrt{x} \Rightarrow y^2 = x$. The region R is shown in figure below. Treating R as an x-simple, we get

$$I = \int_0^1 \int_0^{y^2} 3\ln(1+y^3)\, dx\, dy$$



$0 \leq x \leq y^2$

$0 \leq y \leq 1$

$$= \int_0^1 3\ln(1+y^3) \cdot \left\{ \int_0^{y^2} dx \right\} dy$$

$$= \int_0^1 3\ln(1+y^3) \cdot y^2\, dy$$

let $t = 1 + y^3$, $\therefore dt = 3y^2 dy$.

$\therefore I = \int \ln(t)\, dt \rightarrow$ by parts

$\ln(t) \quad 1$

$\frac{1}{t} \xleftarrow{\int} t$

$$= t\ln(t) - \int \frac{1}{t} t\, dt$$

$$= t\ln(t) - t = t(\ln(t) - 1)$$

$$= (1+y^3)\left[\ln(1+y^3) - 1\right]\Big|_0^1$$

$$= 2[\ln(2) - 1] - 1[\overset{0}{\ln(1)} - 1]$$

$$= 2\ln(2) - 2 + 1 = 2\ln(2) - 1$$

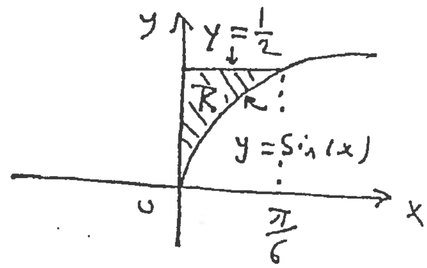
31. Let $J = \iint_R f(x,y)\,dA$, where R is the planar region enclosed by $y = \sin(x)$, $y = \frac{1}{2}$, $x = 0$, and $x = \frac{\pi}{6}$.

(a) Express the double integral J as an iterated integral in which the y-integration is performed first.

(b) Express the double integral J as an iterated integral in which the x-integration is performed first.

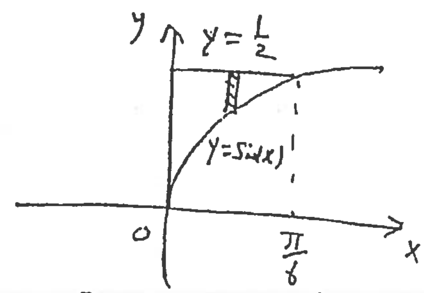
Solution: let us first sketch region R

Note: $\sin(\frac{\pi}{6}) = \frac{1}{2}$



(a) We must treat R as a y-simple. We have

$$J = \int_0^{\frac{\pi}{6}} \left\{ \int_{\sin(x)}^{\frac{1}{2}} f(x,y)\,dy \right\} dx$$

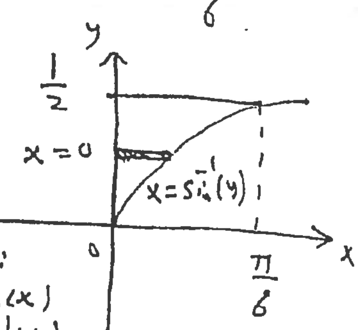


$\sin(x) \leq y \leq \frac{1}{2}$

$0 \leq x \leq \frac{\pi}{6}$

(b) We must treat R as an x-simple. We have

$$J = \int_0^{\frac{1}{2}} \left\{ \int_0^{\sin^{-1}(y)} f(x,y)\,dx \right\} dy$$



Note:
$y = \sin(x)$
$\Rightarrow x = \sin^{-1}(y)$

$0 \leq x \leq \sin^{-1}(y)$

$0 \leq y \leq \frac{1}{2}$

— — — — — —

32. Evaluate $\iint_R 4x\,dA$, where R is the planar region given by $0 \le y \le \sin(2x)$, $0 \le x \le \frac{\pi}{4}$.

<u>Solution</u>: Note 1st. that, there is <u>no</u> need to sketch region R since limits are already provided.

In fact: R is a y-simple region

$$\therefore \quad I = \iint_R 4x\,dA = \int_0^{\frac{\pi}{4}} \left\{ \int_0^{\sin(2x)} 4x\,dy \right\} dx$$

$$= \int_0^{\frac{\pi}{4}} 4x \left\{ \int_0^{\sin(2x)} dy \right\} dx = \int_0^{\frac{\pi}{4}} 4x \sin(2x)\,dx$$

Integrating by parts, we have

$$I = -2x\cos(2x) + \sin(2x) \Big|_0^{\frac{\pi}{4}}$$

$$= \left[-2 \cdot \frac{\pi}{4} \overbrace{\cancel{\cos}\left(\frac{\pi}{2}\right)}^{0} + \sin\left(\frac{\pi}{2}\right)\right] - \left[0 + \overbrace{\cancel{\sin}(0)}^{0}\right]$$

$$= 1$$

33. Evaluate $\iint_R 4y\,dA$, where R is the region in the plane described by $0 \le y \le \sin(2x)$, $0 \le x \le \frac{\pi}{4}$.

<u>Answer</u>: $I = \frac{\pi}{4}$.

— — — — — — —

34.

(i) $\rho \cos(\phi) = 4$

But in spherical coordinates $z = \rho \cos(\phi)$

$\therefore$ $\boxed{z = 4}$

(ii) $\rho \cos(\phi) = 2 - \rho^2 \sin^2(\phi)$

Recall: In spherical coordinates

$z = \rho \cos(\phi)$, $\quad x^2 + y^2 = \rho^2 \sin^2(\phi)$

Hence $\boxed{z = 2 - (x^2 + y^2)}$

(iii) $\rho = 4 \cos(\phi)$

Multiplying both sides by ρ:

$\rho^2 = 4 \rho \cos(\phi)$

But $\rho^2 = x^2 + y^2 + z^2$, and $z = \rho \cos(\phi)$

$\therefore$ $\boxed{x^2 + y^2 + z^2 = 4z}$

(iv) $\phi = \frac{\pi}{4} \Rightarrow \tan\phi = \tan\left(\frac{\pi}{4}\right) = 1$

$\therefore$ $\frac{\sin(\phi)}{\cos(\phi)} = 1$

If $\rho \neq 0$, $\frac{\rho \sin(\phi)}{\rho \cos(\phi)} = 1$

But $\rho \cos(\phi) = z$, and $x^2 + y^2 = \rho^2 \sin^2(\phi) \Rightarrow$

$\sqrt{x^2 + y^2} = \rho \sin(\phi)$

Therefore $\frac{\sqrt{x^2+y^2}}{z} = 1 \Rightarrow$ $\boxed{z = \sqrt{x^2 + y^2}}$

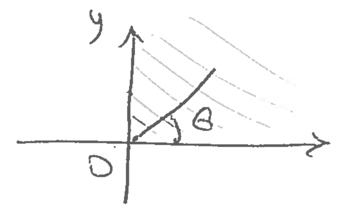
35. (i) $z = \sqrt{16 - x^2 - y^2}$, $x \geqslant 0$, $y \geqslant 0$

In cylindrical coordinates

$$x^2 + y^2 = r^2$$

observe also that since $x \geqslant 0$, $y \geqslant 0$ (first quadrant),

$$0 \leqslant \theta \leqslant \frac{\pi}{2}$$



$0 \leqslant \theta \leqslant \frac{\pi}{2}$

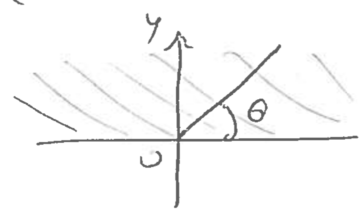
$$\therefore \quad z = \sqrt{16 - (x^2 + y^2)}$$

$$\Rightarrow \quad z = \sqrt{16 - r^2}, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

(ii) $z = \sqrt{5(x^2 + y^2)}$

$$\Rightarrow \quad z = \sqrt{5r^2}$$

$$= \sqrt{5}\, r, \qquad 0 \leqslant \theta \leqslant 2\pi \quad \text{(No restriction here)}$$



$0 \leqslant \theta \leqslant \pi$

(iii) $x^2 + y^2 = 1$, $y \geqslant 0$

$$\Rightarrow \quad r^2 = 1, \qquad 0 \leqslant \theta \leqslant \pi$$

$$\Rightarrow \quad r = 1, \qquad 0 \leqslant \theta \leqslant \pi$$

36. Recall

$$dm = \delta(x,y,z)\,dV$$

Here $\quad \delta(x,y,z) = 12z$

$$\therefore \quad dm = 12z\,dV$$

Mass $\quad m = \iiint_E dm$

$$= \iiint_E 12z\,dV$$

In Cylindrical coordinates

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$,

$x^2 + y^2 = r^2$, $dV = dz\,dA$, $dA = r\,dr\,d\theta$

$$\therefore \quad m = \iint_{Base} \left\{ \int_{z_{bottom}}^{z_{top}} 12z\,dz \right\} dA$$

let us find Base, z_{top}, and z_{bottom}.

Indeed, $\quad z = 2(x^2+y^2) \Rightarrow z = 2r^2 \quad \ldots (1)$

$z = \sqrt{5-(x^2+y^2)} \Rightarrow z = \sqrt{5-r^2} \quad \ldots (2)$

Eliminating z among (1), (2) (by Equating), we obtain

$$2r^2 = \sqrt{5-r^2}$$

$$4r^4 = 5 - r^2$$

$$4r^4 + r^2 - 5 = 0$$

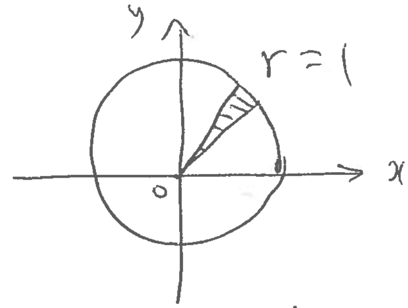
$$(4r^2+5)(r^2-1) = 0$$

$\therefore$ Either $4r^2+5=0$ (has no solution)

or $r^2-1=0 \Rightarrow r=+1$ (Since $r \geq 0$)

$\therefore$ The Base is a Circle with Centre at $(0,0)$ and radius 1 unit.

Clearly, $z=\sqrt{5-r^2}$ is z_{top} (check!)



$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$\therefore z_{bottom} = 2r^2$

$$\therefore m = \int_0^{2\pi}\int_0^1 \left\{ \int_{2r^2}^{\sqrt{5-r^2}} 12z\,dz \right\} r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 \left. 6z^2 \right|_{z=2r^2}^{z=\sqrt{5-r^2}} \cdot r\,dr$$

$$= 2\pi \cdot 6 \int_0^1 [(5-r^2)-4r^4]\,r\,dr$$

$$= 12\pi \int_0^1 (5r-r^3-4r^5)\,dr$$

$$= 12\pi \left[\frac{5}{2}r^2 - \frac{1}{4}r^4 - \frac{4}{6}r^6 \right]_0^1$$

$$= 12\pi \left[\frac{5}{2} - \frac{1}{4} - \frac{2}{3} \right]$$

$$= \pi [30-3-8] = 19\pi$$

37. For students to do at home.

Volume $V = \frac{56}{3}\pi$

38. Recall $M_{z=0} = \iiint_E z \, dm$

Here $dm = \delta(x,y,z)\, dV$
$= (x^2+y^2+z^2)^{\frac{3}{2}}$

$\therefore M_{z=0} = \iiint_E z\,(x^2+y^2+z^2)^{\frac{3}{2}}\, dV$

In spherical coordinates:

$x^2+y^2+z^2 = \rho^2$, $z = \rho\cos(\phi)$, and $dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

$\therefore M_{z=0} = \iiint_E \rho\cos(\phi)\,(\rho^2)^{\frac{3}{2}}\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

The region E is described by

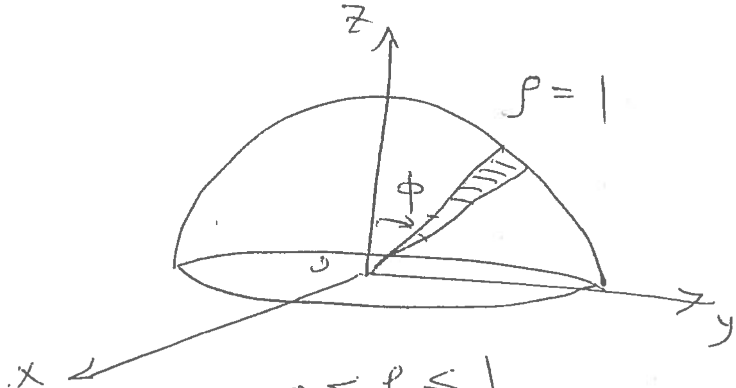
$0 \leq z \leq \sqrt{1-x^2-y^2}$

Now $z=0$ (The xy-plane)

$z = \sqrt{1-x^2-y^2} \Rightarrow z^2 = 1-x^2-y^2$
$\Rightarrow x^2+y^2+z^2 = 1$

$\therefore z = \sqrt{1-x^2-y^2}$ is the upper hemi-sphere with centre $(0,0,0)$, radius 1

In spherical: $x^2+y^2+z^2 = 1 \Rightarrow \rho^2 = 1 \Rightarrow \rho = 1$.



$0 \leqslant \rho \leqslant 1$
$0 \leqslant \phi \leqslant \frac{\pi}{2}$
$0 \leqslant \theta \leqslant 2\pi$

$$\therefore M_{z=0} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{1} \rho^6 \cos(\phi)\sin(\phi)\, d\rho\, d\phi\, d\theta$$

(split : why?)

$$M_{z=0} = \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \cos(\phi)\sin(\phi)\, d\phi \cdot \int_0^1 \rho^6\, d\rho$$

$$= 2\pi \cdot \int_0^{\frac{\pi}{2}} \cos(\phi)\sin(\phi)\, d\phi \cdot \frac{1}{7}\rho^7 \Big|_0^1$$

$$= \frac{2\pi}{7} \int_0^{\frac{\pi}{2}} \underline{\cos(\phi)}\sin(\phi)\, \underline{d\phi}$$

let $t = \sin(\phi)$ (or $\cos(\phi)$ in this case!)

$\therefore dt = \underline{\cos(\phi)\, d\phi}$

$t = \sin(\phi)$
$\phi = 0 \Rightarrow t = 0$
$\phi = \frac{\pi}{2} \Rightarrow t = 1$

$$M_{z=0} = \frac{2\pi}{7} \int_0^1 t\, dt$$

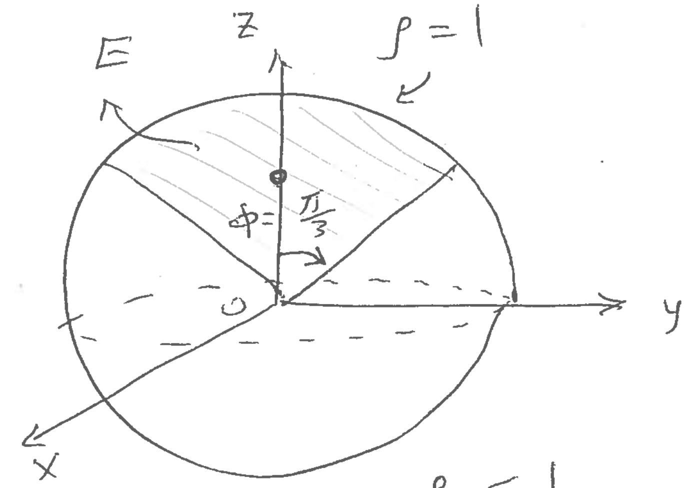
$$= \frac{2\pi}{7} \cdot \frac{1}{2} t^2 \Big|_0^1$$

$$= \frac{\pi}{7}$$

39. First, let us sketch:

$x^2+y^2+z^2=1$ (a sphere with centre $(0,0,0)$, radius 1)

$\sqrt{3}z=\sqrt{x^2+y^2}$ or $z=\sqrt{\frac{x^2+y^2}{3}}$ is an equation of the upper nappe of a cone with vertex $(0,0,0)$, and axis of symmetry being z-axis



Clearly, from symmetry of solid E about z-axis, the centroid lies on z-axis. Hence it's coordinates are

$0\leq\rho\leq 1$
$0\leq\phi\leq\frac{\pi}{3}$
$0\leq\theta\leq 2\pi$

$\bar{x}=0,\ \bar{y}=0,\ \bar{z}=\frac{M_{z=0}}{m}$

let us calculate $\bar{z}$.

Recall $dm=\delta(x,y,z)\,dV$

For centroid δ = a constant, say 1

$\therefore\quad dm=dV$

or $\quad dm=\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$ (in spherical)

$\therefore\quad m=\iiint\limits_E dm$

$=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$

(split into three single integrals! .. why?)

$$m = \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{3}} \sin(\phi)\, d\phi \cdot \int_0^1 \rho^2\, d\rho$$

$$= 2\pi \left[-\cos(\phi)\right]_0^{\frac{\pi}{3}} \cdot \frac{1}{3}\rho^3 \Big|_0^1$$

$$= 2\pi \left[-\frac{1}{2}+1\right] \cdot \frac{1}{3} = 2\pi \cdot \frac{1}{2} \cdot \frac{1}{3} = \pi/3$$

Next, $M_{z=0} = \iiint_E z\, dm$

But $z = \rho\cos(\phi)$, $dm = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

$$\therefore M_{z=0} = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho\cos(\phi) \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{3}} \cos(\phi)\sin(\phi)\, d\phi \cdot \int_0^1 \rho^3\, d\rho$$

$$= 2\pi \cdot \frac{1}{2}\sin^2(\phi)\Big|_0^{\frac{\pi}{3}} \cdot \frac{1}{4}\rho^4\Big|_0^1$$

Verify!

$$= 2\pi \cdot \frac{1}{2}\left[\left(\frac{\sqrt{3}}{2}\right)^2 - 0^2\right] \cdot \frac{1}{4}$$

$$= 2\pi \cdot \frac{1}{2}\left[\frac{3}{4}\right] \cdot \frac{1}{4} = \frac{3\pi}{16}$$

$$\therefore \bar{z} = \frac{M_{z=0}}{m} = \frac{\frac{3\pi}{16}}{\pi/3} = \frac{9}{16}$$

Centroid is at the point $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{9}{16})$

(40)

(a) $$J = \int_0^4 \int_0^{4-y} \int_0^{\sqrt{y}} g(x,y,z)\,dx\,dz\,dy$$

$$= \iiint_E g(x,y,z)\,dV$$

where E is the region in $\mathbb{R}^3$ described by

$$0 \le x \le \sqrt{y} \quad \cdots \quad (1)$$

$$0 \le z \le 4-y \quad \cdots \quad (2)$$

$$0 \le y \le 4 \quad \cdots \quad (3)$$

The new order is y, z, then x.

From (1) $\quad x \le \sqrt{y}$

$$\Rightarrow \boxed{x^2 \le y}$$

From (2) $\quad z \le 4-y \Rightarrow \boxed{y \le 4-z}$

Hence $\quad \boxed{x^2 \le y \le 4-z}$ Inner-most limits

Next $\quad x^2 \le 4-z \Rightarrow \boxed{z \le 4-x^2}$

on the other hand $\quad 0 \le z$

$$\therefore \quad \boxed{0 \le z \le 4-x^2}$$ Middle limits

Finally, $\quad 0 \le x \le \sqrt{y} \quad \cdots$ But $y \le 4$

$$\therefore \quad 0 \le x \le \sqrt{4}$$

$$\Rightarrow \boxed{0 \le x \le 2}$$ outer-most limits

$$\therefore J = \int_0^2 \int_0^{4-x^2} \int_{x^2}^{4-z} g(x,y,z)\,dy\,dz\,dx$$

(b) $I = \int_0^1 \int_0^{\sqrt{1-y}} \int_0^{2x} f(x,y,z)\, dz\, dx\, dy$

$= \iiint_E f(x,y,z)\, dV$

where E is the region in $\mathbb{R}^3$ described by

$$0 \le z \le 2x \quad \text{---} \quad (1)$$
$$0 \le x \le \sqrt{1-y} \quad \text{---} \quad (2)$$
$$0 \le y \le 1 \quad \text{---} \quad (3)$$

The new order is y, x, then z

From (2) $\quad x \le \sqrt{1-y}$

$\Rightarrow x^2 \le 1-y$

$\Rightarrow y \le 1-x^2$

and from (3) $0 \le y$. Hence

$\boxed{0 \le y \le 1-x^2}$ Inner-most limits

Next, from (1): $\quad z \le 2x \Rightarrow \frac{z}{2} \le x$

and from (2): $\quad x \le \sqrt{1-y}$ .. But $y \ge 0$

$\therefore \quad x \le \sqrt{1} = 1$

$\therefore \quad \boxed{\frac{z}{2} \le x \le 1}$ middle limits

Finally, from (1) $\quad 0 \le z \le 2x$, but $x \le 1$

$\therefore \quad 0 \le z \le 2(1)$

$\Rightarrow \boxed{0 \le z \le 2}$ outer-most limits

$\therefore I = \int_0^2 \int_{z/2}^1 \int_0^{1-x^2} f(x,y,z)\, dy\, dx\, dz$

(c) $J = \int_0^1 \int_z^1 \int_0^z g(x,y,z)\, dx\, dy\, dz$

$= \iiint_E g(x,y,z)\, dV$

where E is the region in $\mathbb{R}^3$ described by

$0 \leq x \leq z$ --- (1)

$z \leq y \leq 1$ --- (2)

$0 \leq z \leq 1$ --- (3)

New order is z, y, then x

From (1): $x \leq z$, and from (2) $z \leq y$

$\therefore$ $\boxed{x \leq z \leq y}$ Inner-most limits

Next, from Inequality above,

$x \leq y$

But from (2) $y \leq 1$, hence

$\boxed{x \leq y \leq 1}$ Middle limits

Finally, from (1):

$0 \leq x \leq z$ -- but $z \leq 1$ (from (3))

$\therefore$ $\boxed{0 \leq x \leq 1}$ outer-most limits

$\therefore$ $J = \int_0^1 \int_x^1 \int_x^y g(x,y,z)\, dz\, dy\, dx$

(d) For students to do at home.

<u>Answer</u>: $\displaystyle\int_0^1 \int_x^1 \int_0^y g(x,y,z)\, dz\, dy\, dx$

---

(41) $\iint_R x y^2 \, dA$

Recall $dm = \delta(x,y)\, dA = xy\, dA$

$$\therefore \iint_R x y^2 \, dA = \iint_R y\,(xy\, dA)$$

$$= \iint_R y\, dm = M_{y=0}$$

But $\bar{y} = \dfrac{M_{y=0}}{m}$

Hence $M_{y=0} = m\bar{y}$

Here $m = 3$, $\bar{y} = 4$

$$\therefore M_{y=0} = (3)(4) = 12$$

$$\therefore \iint_R x y^2 \, dA = 12$$

(42) Recall: For Centroid $\delta(x,y) =$ a Constant, say 1

$$\therefore \quad dm = \delta(x,y)\, dA$$

$$\Rightarrow \quad dm = dA$$

That is to say: Mass and Area are <u>Numerically</u> Equal. Hence

$$m = A$$

Now, $$\iint_D (3x - 4y + 2)\, dA = 124$$

$$\Rightarrow \quad \iint_D (3x - 4y + 2)\, dm = 124$$

$$\Rightarrow \quad 3\iint_D x\, dm - 4\iint_D y\, dm + 2\iint_D dm = 124$$

or $$3M_{x=0} - 4M_{y=0} + 2m = 124 \quad (*)$$

$$\bar{x} = \frac{M_{x=0}}{m} \Rightarrow M_{x=0} = m\bar{x} = 3m$$

$$\bar{y} = \frac{M_{y=0}}{m} \Rightarrow M_{y=0} = m\bar{y} = -5m$$

substituting into (*):

$$3(3m) - 4(-5m) + 2m = 124$$

$$31m = 124 \Rightarrow m = \frac{124}{31} = 4$$

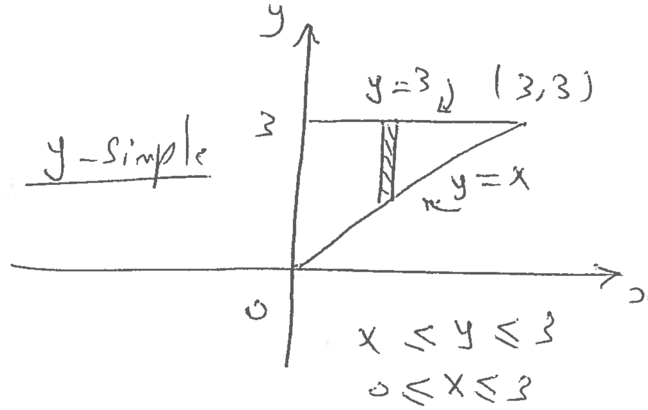
$\therefore$ area $A = 4$ as well

(43) $I = \int_0^3 \int_x^3 \sqrt{9-y^2}\, dy\, dx = \iint_R \sqrt{9-y^2}\, dA$

where R is the region bounded by the lines

$y = x$, $y = 3$ from $x = 0$ to $x = 3$ as shown.

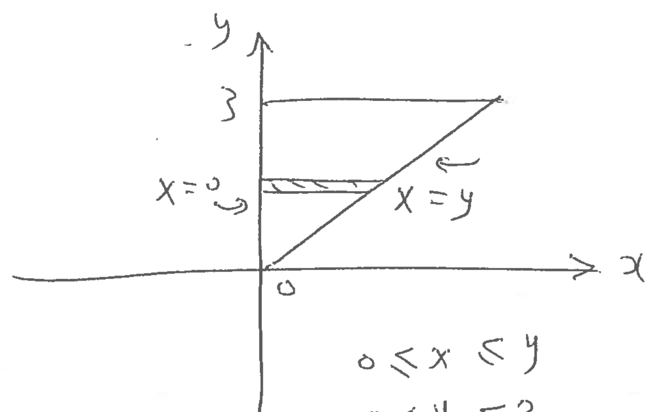
Let us treat R as an x-simple instead!



$$I = \int_0^3 \int_0^y \sqrt{9-y^2}\, dx\, dy$$

$$= \int_0^3 \sqrt{9-y^2} \left\{ \int_0^y dx \right\} dy$$

$$= \int_0^3 \sqrt{9-y^2} \cdot \underline{y\, dy}$$



Let $t = 9 - y^2$

$dt = -2y\,dy \Rightarrow \underline{y\,dy} = -\frac{1}{2} dt$

New limits

$t = 9 - y^2$

At $y = 0$, $t = 9$

At $y = 3$, $t = 9 - 9 = 0$

$$\therefore \quad I = -\frac{1}{2} \int_9^0 \sqrt{t}\, dt$$

$$= +\frac{1}{2} \int_0^9 t^{\frac{1}{2}}\, dt$$

$$= \frac{1}{2} \cdot \frac{2}{3} t^{\frac{3}{2}} \Big|_0^9$$

$$= \frac{1}{3} t\sqrt{t} \Big|_0^9 = \frac{1}{3}\left[9\sqrt{9}\right] = \frac{1}{\cancel{3}} \cdot 9 \cdot \cancel{3}$$

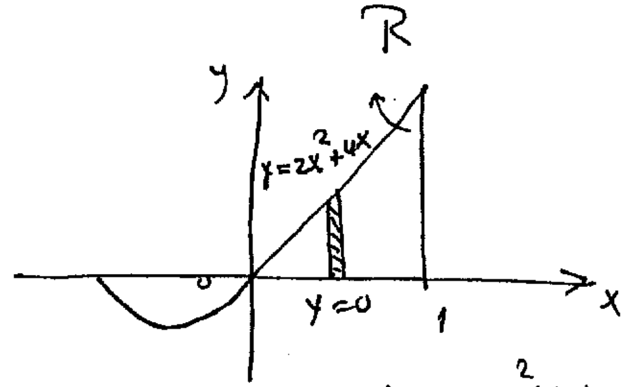
$$= 9$$

44. Find the coordinates of the centre of mass of the planar region R enclosed by $y = 2x^2+4x$, $y=0$ from $x=0$ to 1 if density $\delta(x,y) = x$.

Solution: First note that $y = 2x^2+4x$ is an equation of a parabola which opens upward.

Its x-intercepts are given by

$$0 = 2x^2+4x$$
$$\Rightarrow 2x(x+2) = 0$$
$$x = 0, -2$$



$0 \le y \le 2x^2+4x$
$0 \le x \le 1$

Here $dm = \delta(x,y)\, dA$

$\therefore \boxed{dm = x\, dA}$

$$\therefore \text{mass } m = \iint_R dm = \iint_R x\, dA$$
$$= \int_0^1 x\left\{\int_0^{2x^2+4x} dy\right\} dx = \int_0^1 x(2x^2+4x)\,dx = 11/6$$

$$M_{x=0} = \iint_R x\, dm = \iint_R x.x\, dA = \int_0^1 x^2\left\{\int_0^{2x^2+4x} dy\right\} dx$$
$$= \int_0^1 x^2(2x^2+4x)\, dx = \tfrac{2}{5}x^5 + x^4\Big|_0^1 = 7/5$$

$$\text{Finally } M_{y=0} = \iint_R y\, dm = \iint_R y.x\, dA = \iint_R xy\, dA$$
$$= \int_0^1 x\left\{\int_0^{2x^2+4x} y\, dy\right\} dx = \frac{1}{2}\int_0^1 x\, y^2\Big|_0^{2x^2+4x} dx$$
$$= \frac{1}{2}\int_0^1 x(2x^2+4x)^2\, dx = \frac{59}{15}$$

$$\therefore \bar{x} = \frac{M_{x=0}}{m} = \frac{42}{55}, \quad \bar{y} = \frac{M_{y=0}}{m} = \frac{118}{55}$$

— — — — — —

45. Use Double integrals to find the x and y-coordinates of the centroid of the planar region R enclosed by $y = \sqrt{x}$, $x = 0$, and $y = 1$.

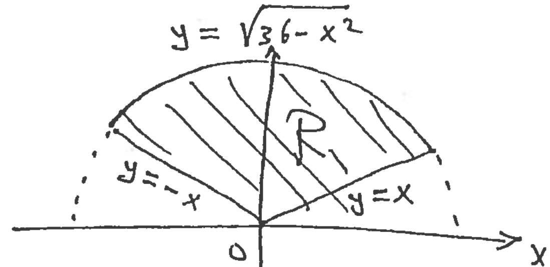
Ans.: $\bar{x} = \frac{3}{10}$, $\bar{y} = \frac{3}{4}$.

46. Use Double integrals to find the x, and y-coordinate of the Centroid of the planar region R enclosed by $y = \sqrt{36 - x^2}$, $y = x$, and $y = -x$.

Solution: First: sketch region R:

$y = \sqrt{36 - x^2} \Rightarrow x^2 + y^2 = 36$

$\therefore$ $y = \sqrt{36 - x^2}$ is the upper semi-circle centred at $(0,0)$, and has radius 6.



We shall use polar coordinates!

In polar coordinates:

$x = r\cos(\theta)$, $y = r\sin(\theta)$,

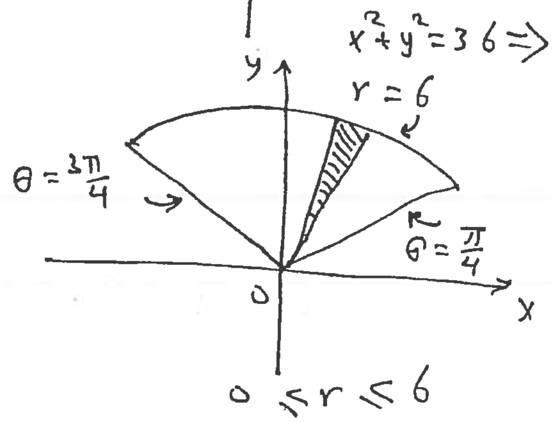
$x^2 + y^2 = r^2$, and $dA = r\,dr\,d\theta$

Note: In polar coordinates,

$y = \pm x \Rightarrow r\sin(\theta) = \pm r\cos(\theta)$

If $r \neq 0$, $\sin(\theta) = \pm\cos(\theta)$ or $\tan(\theta) = \pm 1$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$



$0 \leq r \leq 6$

$\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

Now, $dm = \delta\, dA$

For centroid δ = a constant, say 1

$\therefore$ $dm = dA$

$\therefore$ mass $m = \iint_R dm = \iint_R dA$

$= \text{area of region } R$ (Numerically!)

Clearly area of $R = \frac{1}{4}$ (area of a circle of radius 6)

$= \frac{1}{4}\pi(6)^2 = \frac{1}{4}\cdot\pi\cdot 36 = 9\pi$

Next, $M_{y=0}$ = moment about x-axis

$$= \iint_R y\, dm = \iint_R y\, dA$$

$$= \int_{\pi/4}^{3\pi/4}\int_0^6 r\sin(\theta)\cdot r\, dr\, d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \sin(\theta)\, d\theta \cdot \int_0^6 r^2\, dr$$

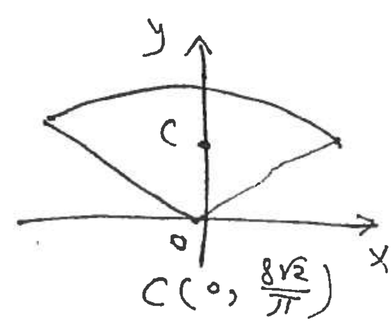
$$= -\cos(\theta)\Big|_{\pi/4}^{3\pi/4} \cdot \frac{1}{3} r^3\Big|_0^6$$

$$= -\left[\cos\left(\tfrac{3\pi}{4}\right) - \cos\left(\tfrac{\pi}{4}\right)\right]\cdot\frac{1}{3}\left[6^3 - 0^3\right]$$

$$= -\left[\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]\cdot\frac{1}{\cancel{3}}\cdot\overset{2}{\cancel{6}}\cdot 6\cdot 6$$

$$= \frac{2}{\sqrt{2}}\cdot 72 = 72\sqrt{2}$$

$$\therefore \bar{y} = \frac{M_{y=0}}{m} = \frac{72\sqrt{2}}{9\pi} = \frac{8\sqrt{2}}{\pi}$$



From symmetry, it is obvious that $\bar{x} = 0$.

— — — — — — —

(47)

Recall: Moment about the plane $y=0$ is given by

$$M_{y=0} = \iiint_E y \, dm$$

where $dm = \delta(x,y,z)\, dV$

$$= 20xy \, dV$$

$$\therefore M_{y=0} = \iiint_E y(20xy)\, dV$$

$$M_{y=0} = \iiint_E 20xy^2 \, dV$$

In cylindrical coordinates:

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2+y^2=r^2$, and

$dV = dz\,dA$, $dA = r\,dr\,d\theta$

$$\therefore M_{y=0} = \iint_{Base} \left\{ \int_{z_{bottom}}^{z_{top}} 20(r\cos(\theta))(r\sin(\theta))^2 \, dz \right\} dA$$

<u>To find Base:</u>

Region E is the region in first octant $(x,y,z \geq 0)$ enclosed by $z = \sqrt{x^2+y^2}$,

$$z = 2 - \sqrt{x^2+y^2}$$

Converting to cylindrical coordinates, we obtain:

$$z = r \quad \cdots (1)$$

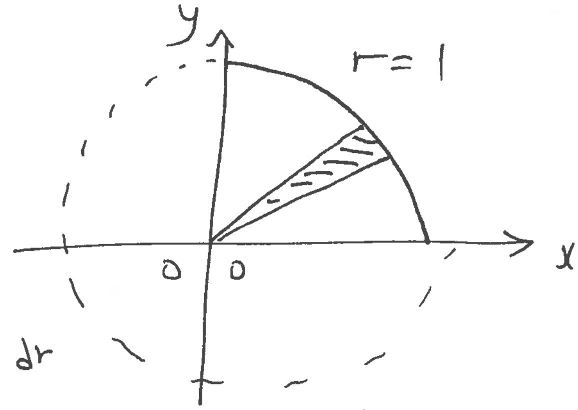
$$z = 2 - r \quad \cdots (2)$$

Eliminating "z" (by Equating),

$$r = 2 - r \Rightarrow 2r = 2 \Rightarrow r = 1$$

Base is a circle with centre at (0,0), radius 1

However because $x \geq 0$, $y \geq 0$ .. only the part of circle in first quadrant is considered!



Next, obviously

$z = 2-r$ is z_{top} and

$z = r$ is z_{bottom}

$$\therefore M_{y=0} = \int_0^{\frac{\pi}{2}}\int_0^1 20\cos(\theta)\sin^2(\theta)r^4\left\{\int_r^{2-r} dz\right\}dr$$

$0 \leq r \leq 1$

$0 \leq \theta \leq \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 20\cos(\theta)\sin^2(\theta).r^4(2-r-r)\,dr$$

$$= 20\int_0^{\frac{\pi}{2}} \cos(\theta)\sin^2(\theta)\,d\theta \cdot \int_0^1 r^4(2-2r)\,dr$$

Use substitution

$t = \sin(\theta)$, $0 \leq t \leq 1$

$\therefore dt = \cos(\theta)d\theta$

$$M_{y=0} = 20\int_0^1 t^2 dt \cdot \left(\frac{2}{5}r^5 - \frac{2}{6}r^6\right)\Big|_0^1 = 20 \cdot \frac{1}{3}\left(\frac{2}{5} - \frac{2}{6}\right)$$

$$= \frac{4}{9}$$

(48) $\iiint_E 9z^2\,dV$

where E is the region given by

$0 \le x \le \sqrt{1-y}\,, \quad 0 \le y \le 1, \quad 0 \le z \le 2x$

Clearly: The region can be treated as an z-simple

$\therefore$ order is z, x, then y

$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{1-y}} \left\{ \int_{z=0}^{z=2x} 9z^2\,dz \right\} dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y}} 3z^3 \Big|_0^{2x} dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y}} 3(2x)^3\,dx\,dy$$

$$= \int_0^1 \left\{ \int_0^{\sqrt{1-y}} 24x^3\,dx \right\} dy$$

$$= \int_0^1 6x^4 \Big|_{x=0}^{x=\sqrt{1-y}} dy$$

$$= 6\int_0^1 \left[\left(\sqrt{1-y}\right)^4 - 0^4 \right] dy$$

$$= 6\int_0^1 (1-y)^2\,dy = 6\,\frac{(1-y)^3}{-3} \Big|_{y=0}^{y=1}$$

$$= -2\left[0^3 - 1^3\right] = \underline{2}$$

(49) For students to do at Home.

Answer: 32π

(50) For students to do at Home.

Answer: 12

(51) Recall

$$M_{z=0} = \iiint_E z \, dm$$

where $dm = \delta(x,y,z)\, dV$

$$= (x^2+y^2)\, dV$$

$$\therefore M_{z=0} = \iiint_E z(x^2+y^2)\, dV$$

In spherical coordinates:

$$z = \rho\cos(\phi)$$

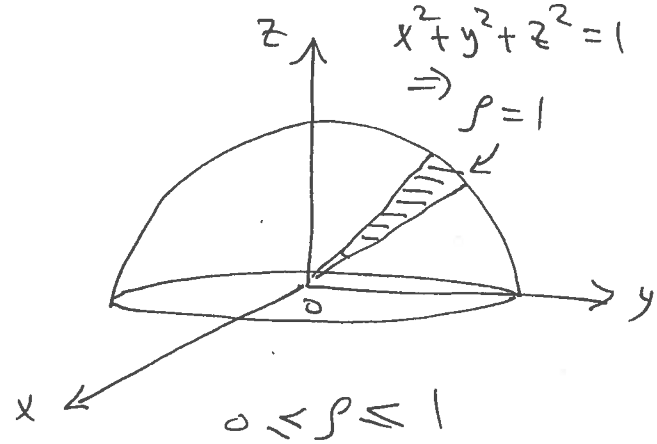
$$x^2+y^2 = \rho^2\sin^2(\phi), \text{ and}$$

$$dV = \rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$M_{z=0} = \iiint_E \rho\cos(\phi) \cdot \rho^2\sin^2(\phi) \cdot \rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \iiint_E \cos(\phi)\sin^3(\phi)\rho^5\, d\rho\, d\phi\, d\theta$$

E: The Hemi-spherical region above xy-plane shown in figure below:



$0 \leq \rho \leq 1$
$0 \leq \phi \leq \frac{\pi}{2}$
$0 \leq \theta \leq 2\pi$

$$M_{z=0} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \cos(\phi) \sin^3(\phi) \rho^5 \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^5 d\rho \cdot \int_0^{\frac{\pi}{2}} \sin^3(\phi) \cos(\phi) \, d\phi$$

$\leftarrow$ Let $t = \sin(\phi)$
$dt = \cos(\phi)\, d\phi$
$0 \leq t \leq 1$

$$= 2\pi \cdot \frac{1}{6} \int_0^1 t^3 dt$$

$$= 2\pi \cdot \frac{1}{6} \cdot \frac{1}{4} = \frac{\pi}{12}$$

(52) For students to do at home.

Answer: mass $m = \frac{\pi}{2} \ln 2$

(53) $I = \iiint_E (2x + z)\, dV$

Recall $dm = \delta(x,y,z)\, dV$

For Centroid $\delta = 1$, hence

$$dm = dV$$

That is Volume and mass are numerically Equal.

$$I = \iiint_E (2x + z)\, dm$$

$$= 2\iiint_E x\, dm + \iiint_E z\, dm$$

$$= 2 M_{x=0} + M_{z=0}$$

But $\bar{x} = \dfrac{M_{x=0}}{m} \Rightarrow M_{x=0} = m\bar{x}$,

$\bar{z} = \dfrac{M_{z=0}}{m} \Rightarrow M_{z=0} = m\bar{z}$

substituting $\bar{x} = 8$, $\bar{z} = 6$, and $m = V = 2$,

we obtain:

$M_{x=0} = (2)(8) = 16$, $M_{z=0} = (2)(6) = 12$

Hence $I = 2M_{x=0} + M_{z=0}$

$$= 2(16) + 12 = 44$$

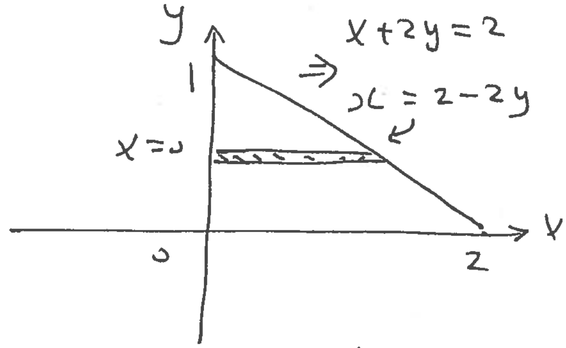
(54) The moment about the xz-plane which has equation $y=0$ is $M_{y=0}$.

Now, $$\bar{y} = \frac{M_{y=0}}{m} = \frac{-5}{\frac{1}{3}} = (-5)(3) = -15$$

— — — — —

(55) Volume $$V = \iint_{\text{Base}} \text{height}\, dA$$

Here: Base is the region enclosed by the Triangle shown in figure



$0 \le x \le 2-2y$
$0 \le y \le 1$

Height $= z_{top} - z_{bottom}$

$z_{top} = 3y^2$

z_{bottom} is the xy-plane $= 0$

$\therefore$ height $= 3y^2 - 0 = 3y^2$

$$V = \int_0^1 \int_0^{2-2y} 3y^2\, dx\, dy$$

$$= \int_0^1 3y^2 \left\{ \int_0^{2-2y} dx \right\} dy$$

$$= \int_0^1 3y^2(2-2y)\, dy = \int_0^1 (6y^2 - 6y^3)\, dy$$

$$= \frac{6}{3}y^3 - \frac{6}{4}y^4 \Big|_0^1 = 2 - \frac{3}{2} = \frac{1}{2}$$

(56)

$$\text{Mass } m = \iint_R dm$$

where $dm = \delta(x,y)\,dA = 3y\,dA$

$$\therefore \quad m = \iint_R 3y\,dA$$

R is described by:

$$-y \le x \le y^2, \quad 0 \le y \le 2$$

(An x-simple region).. No need to sketch!

$$m = \int_0^2 \int_{-y}^{y^2} 3y\,dx\,dy = \int_0^2 3y \left\{ \int_{-y}^{y^2} dx \right\} dy$$

$$= \int_0^2 3y\,(y^2+y)\,dy = \int_0^2 (3y^3 + 3y^2)\,dy$$

$$= \frac{3}{4}y^4 + y^3 \Big|_0^2 = \frac{3}{4}(2)^4 + 2^3 = \frac{3}{4}\cdot 16 + 8$$

$$= 12 + 8$$

$$= 20$$

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(57) For students to do at Home

<u>Answer:</u> $\bar{x} = \frac{3}{4}$

(58)

(i) $f(x,y) = x^3 - xy + y^3$

$f_x = 3x^2 - y, \quad f_y = -x + 3y^2$

Critical points occur where

$f_x = 0 \Rightarrow 3x^2 - y = 0 \quad \ldots (1)$

$f_y = 0 \Rightarrow -x + 3y^2 = 0 \quad \ldots (2)$

From (1) $y = 3x^2$

substituting into (2):

$-x + 3(3x^2)^2 = 0$

$-x + 27x^4 = 0$

$x(-1 + 27x^3) = 0$

$\therefore$ Either $x = 0$, or $-1 + 27x^3 = 0$

$\Rightarrow 27x^3 = 1, \quad x^3 = \frac{1}{27}$, hence $x = \frac{1}{3}$

At $x = 0$,

$y = 3x^2$
$= 3(0)^2 = 0$

$(0, 0)$

At $x = \frac{1}{3}$

$y = 3x^2 = 3\left(\frac{1}{3}\right)^2$
$= 3 \cdot \frac{1}{9} = \frac{1}{3}$

$\left(\frac{1}{3}, \frac{1}{3}\right)$

C.ps are $(0, 0)$, $\left(\frac{1}{3}, \frac{1}{3}\right)$

(ii) $f(x,y) = x^3 + 2xy - 2y^2 - 10x$

$f_x = 3x^2 + 2y - 10,$

$f_y = 2x - 4y$

Critical points occur where

$f_x = 0 \Rightarrow \quad 3x^2 + 2y - 10 = 0 \quad \cdots (1)$

$f_y = 0 \Rightarrow \quad 2x - 4y = 0 \quad \cdots (2)$

From (2): $\quad x = 2y$

substituting into (1)

$3(2y)^2 + 2y - 10 = 0$

$12y^2 + 2y - 10 = 0 \qquad (\div 2)$

$6y^2 + y - 5 = 0$

$(y+1)(6y-5) = 0$

$y = -1, \quad y = \frac{5}{6}$

At $y = -1$,

$x = 2y = 2(-1) = -2$

$(-2, -1)$

At $y = \frac{5}{6}$

$x = 2y = 2\left(\frac{5}{6}\right) = \frac{5}{3}$

$\left(\frac{5}{3}, \frac{5}{6}\right)$

C.ps are $(-2,-1)$, $\left(\frac{5}{3}, \frac{5}{6}\right)$

(iii) For students to do at home.

<u>Answer</u>: Two critical points at $(1,-1)$, $(-1,1)$.

(iv) $f(x,y) = y^3 + x^2 - 6xy + 3x + 6y - 27$

$$f_x = 2x - 6y + 3$$
$$f_y = 3y^2 - 6x + 6$$

Critical points occur where

$$f_x = 0 \Rightarrow 2x - 6y + 3 = 0, \quad \cdots (1)$$
$$f_y = 0 \Rightarrow 3y^2 - 6x + 6 = 0 \quad \cdots (2)$$

From (1): $\quad 2x - 6y + 3 = 0$

$$2x = 6y - 3 \Rightarrow x = \tfrac{1}{2}(6y-3)$$

substituting into (2):

$$3y^2 - 6\cdot\tfrac{1}{2}(6y-3) + 6 = 0$$
$$3y^2 - 3(6y-3) + 6 = 0$$
$$3y^2 - 18y + 15 = 0 \qquad (\div 3)$$
$$y^2 - 6y + 5 = 0, \quad (y-5)(y-1) = 0, \quad y = 1, 5$$

At $y = 1$,

$x = \frac{1}{2}(6y-3) = \frac{1}{2}(6-3) = \frac{3}{2}$

$\left(\frac{3}{2}, 1\right)$

At $y = 5$,

$x = \frac{1}{2}(6y-3)$

$= \frac{1}{2}(30-3) = \frac{27}{2}$

$\left(\frac{27}{2}, 5\right)$

C.Ps are $\left(\frac{3}{2}, 1\right)$, $\left(\frac{27}{2}, 5\right)$

(59)

(i) $f(x,y) = x^2 - 4xy + y^3 + 4y$

Let us first find all first and second order partials:

$$f_x = 2x - 4y \quad , \quad f_y = -4x + 3y^2 + 4$$

$$A = f_{xx} = 2, \quad B = f_{xy} = -4, \quad C = f_{yy} = 6y$$

$$\therefore D = B^2 - AC = (-4)^2 - (2)(6y) = 16 - 12y$$

Two steps:

$\boxed{1}$ Find critical points:

solve system

$$f_x = 0 \Rightarrow 2x - 4y = 0 \quad --- \quad (1)$$

$$f_y = 0 \Rightarrow -4x + 3y^2 + 4 = 0 \quad -- \quad (2)$$

From (1): $2x - 4y = 0 \Rightarrow x = 2y$

Substituting into (2):

$$-4(2y) + 3y^2 + 4 = 0$$

$$3y^2 - 8y + 4 = 0$$

$$(3y-2)(y-2) = 0$$

$$y = \tfrac{2}{3}, \quad y = 2$$

At $y = \frac{2}{3}$,

$x = 2y = 2\left(\frac{2}{3}\right) = \frac{4}{3}$

$\left(\frac{4}{3}, \frac{2}{3}\right)$

At $y = 2$

$x = 2y = 2(2) = 4$

$(4, 2)$

There are two Critical points

$\left(\frac{4}{3}, \frac{2}{3}\right), \ (4,2)$

$\boxed{2}$ Testing: Construct Table as shown

Critical points	$\left(\frac{4}{3}, \frac{2}{3}\right)$	$(4,2)$
$A = 2$	2	$2 > 0$
$B = -4$	-4	-4
$C = 6y$	4	$12 > 0$
$D = 16 - 12y$	$8 > 0$	$-8 < 0$
Conclusion	Saddle point	local Minimum
value of $f(x,y)$	——	0

$$f(x,y) = x^2 - 4xy + y^3 + 4y$$
$$f(4,2) = 4^2 - 4(4)(2) + 2^3 + 4(2)$$
$$= 16 - 32 + 8 + 8 = 0$$

So: f has a local Min. at $(4,2)$ of value 0, and a saddle point at $\left(\frac{4}{3}, \frac{2}{3}\right)$.

(ii) For students to do at home.

Answer: f has a local minimum at $(1,0)$ of value -3, a local minimum at $(-1,0)$ of value -3 as well and a saddle point at $(0,0)$.

(iii) $f(x,y) = (x+y)(xy+1)-17$ ← Expand

$$= x^2y + xy^2 + x + y - 17$$

Let us first find first and second order partials:

$$f_x = 2xy + y^2 + 1, \quad f_y = x^2 + 2xy + 1$$

$$A = f_{xx} = 2y, \quad B = f_{xy} = 2x+2y, \quad C = f_{yy} = 2x$$

$$\therefore D = B^2 - AC = (2x+2y)^2 - (2y)(2x)$$

$$= 4x^2 + 8xy + 4y^2 - 4xy$$

$$= 4x^2 + 4xy + 4y^2$$

step1: Critical points

We have already shown earlier (problem # 58-part (iii)) that there are Two Critical points:

$$(1,-1), \quad (-1,1)$$

step2: Testing

Construct Table as shown below:

Critical points	$(1,-1)$	$(-1,1)$
$A=2y$	-2	2
$B=2x+2y$	0	0
$C=2x$	2	-2
$D=B^2-AC$	$4>0$	$4>0$
Conclusion	saddle point	saddle point
value of $f(x,y)$	———	———

(iv) For students to do at home

<u>Answer:</u>

f has a local minimum at $(5, \frac{27}{2})$ of value $-\frac{117}{4}$ and a saddle point at $(1, \frac{3}{2})$

(6 a)

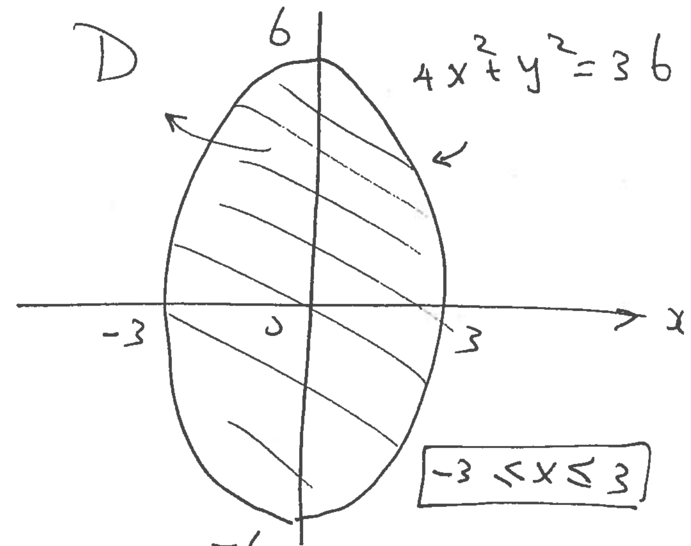
(i) $f(x,y) = x^2 - 12x + y^2$

D: Region enclosed by the Ellipse: $4x^2+y^2=36$
(shown in figure)

Note: $4x^2+y^2=36 \Rightarrow \dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$

so: Centre $(0,0)$, Semi-axes: $\sqrt{9}=3$, $\sqrt{36}=6$

<u>problem 1</u>: The interior of D



$f(x,y) = x^2 - 12x + y^2$

$f_x = 2x - 12, \quad f_y = 2y$

Critical points occur where

$f_x = 0 \Rightarrow 2x - 12 = 0$

$f_y = 0 \Rightarrow 2y = 0$

$\therefore x = 6, \quad y = 0$

only c.p is at $(6,0)$ which lies OUTSIDE of D!

Nothing further to Compute!

<u>problem 2</u>: On the boundary of D

$f(x,y) = x^2 - 12x + y^2$

On the Boundary of D, we have

$4x^2 + y^2 = 36$

or $y^2 = 36 - 4x^2$

Substituting $y^2 = 36 - 4x^2$ into $f(x,y)$, we obtain a new function of the single variable x, say

$$g(x) = x^2 - 12x + (36 - 4x^2)$$

$$g(x) = -3x^2 - 12x + 36, \qquad -3 \leq x \leq 3$$

(Refer to figure).

Critical points occur where

$$g'(x) = 0$$

$$\Rightarrow \quad -6x - 12 = 0, \quad x = -2$$

$\therefore$ Critical point at $x = -2$, and end points $x = -3, 3$

Now,

$$g(-2) = -3(-2)^2 - 12(-2) + 36 = 48$$

$$g(3) = -3(3)^2 - 12(3) + 36 = -27$$

$$g(-3) = -3(-3)^2 - 12(-3) + 36 = 45$$

$\therefore$ Extreme values are

Maximum 48 occurs at $x = -2$, hence

$$y^2 = 36 - 4x^2 = 36 - 4(-2)^2 = 20$$

$$\therefore y = \pm 2\sqrt{5}$$

$\therefore$ Maximum 48 at the points $(-2, 2\sqrt{5})$, $(-2, -2\sqrt{5})$

Minimum value -27 occur at the point $(3, 0)$

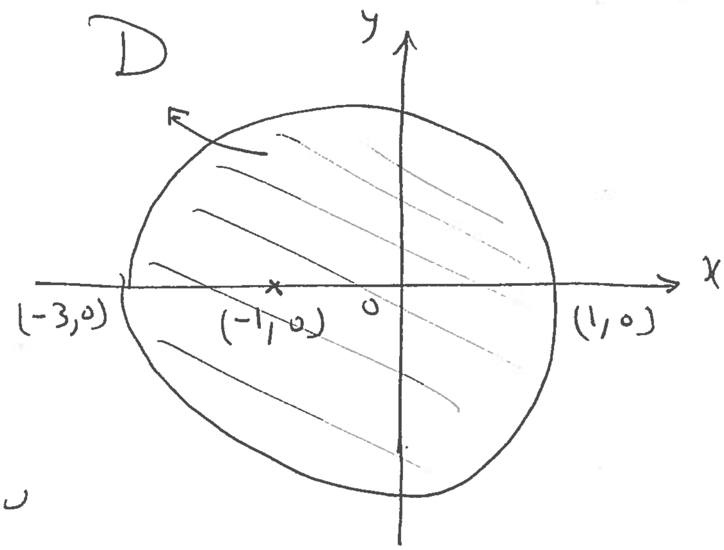
(ii) $f(x,y) = 2y^2 + x^2$

D: Region bounded by the Circle $x^2+y^2+2x-3=0$

(shown in figure)

Note: $x^2+y^2+2x-3=0 \Rightarrow (x+1)^2+y^2=4$

So: Centre $(-1,0)$, radius $\sqrt{4}=2$



$\boxed{-3 \le x \le 1}$

problem 1: Interior of D

$f(x,y) = 2y^2 + x^2$

$f_x = 2x,\ f_y = 4y$

Critical points:

solve

$f_x = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$

$f_y = 0 \Rightarrow 4y = 0 \Rightarrow y = 0$

The only c.p occur at origin $(0,0)$ which lies within D

$\therefore\ f(0,0) = 2(0)^2 + 0^2 = 0$

problem 2: On the Boundary of D

$f(x,y) = 2y^2 + x^2$

on the Boundary of D, we have

$x^2+y^2+2x-3=0$

or $y^2 = 3-2x-x^2$

Substituting $y^2 = 3 - 2x - x^2$ into $f(x,y)$, we obtain a new function of the single variable x, say

$$g(x) = 2(3-2x-x^2) + x^2$$
$$= 6 - 4x - x^2, \quad \boxed{-3 \le x \le 1}$$

$\uparrow$
Refer to figure

<u>Critical points</u>

$g'(x) = 0 \Rightarrow$

$-4 - 2x = 0, \quad x = -2$

Now let us calculate $g(x)$ at $x = -2$ and at End points $x = -3$, $x = 1$:

$$g(-2) = 6 - 4(-2) - (-2)^2 = 10$$
$$g(-3) = 6 - 4(-3) - (-3)^2 = 9$$
$$g(1) = 6 - 4(1) - 1^2 = 1$$

Comparing the <u>four</u> circled values of problems (1), (2):

0, 10, 9, and 1 we conclude:

<u>Maximum is 10</u>
(occurs at $x=-2$, $\therefore y^2 = 3 - 2(-2) - 2^2 = 3 \Rightarrow y = \pm\sqrt{3}$ .. points $(-2, \sqrt{3}), (-2, -\sqrt{3})$)

Minimum is 0 which occur at $(0,0)$.

(iii) For students to do at home

Answer:

Maximum 130 occurs at the point $(5, 0)$

Minimum -238 occurs at the points $(1, 2\sqrt{6})$, and $(1, -2\sqrt{6})$.

(iv) For student to do at home.

Read a similar problem in Lab (11)

Answer:

Maximum 21 occurs at the point $(7, 0)$

Minimum -13 occurs at the point $(2, 3)$

— — — — — —