MATH 277

Problem Set # 8 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. Use Double integrals to find the volume of the region in the first octant bounded by the vertical plane $2x + y = 2$, and the surface $z = x^2$.

2. Use Double integrals to find the volume of the region in the first octant $(x, y, z \geq 0)$ below the plane $2x + y + z = 2$.

3. Find :

   (a) $\iint_R dA$, where R is the region enclosed by the straight lines $y = \frac{2}{5}x$, $y = -2x$, and $y = 2$.

   $(b)^*$ $\iint_D dA$, where D is the region $0 \leq x \leq \sqrt{4 - y^2}$.

4. (a) The iterated integral $\mathbf{J} = \int_0^4 dy \int_{3y^2/16}^{\sqrt{25-y^2}} f(x, y)\, dx$ is the double integral of $f(x, y)$ over a planar region D. Express the double integral $\mathbf{J}$ as a sum of two iterated integrals with the order of integration reversed.

   $(b)^*$ Let $\mathbf{T}$ be a planar region and let

   $$\mathbf{J} = \iint_{\mathbf{T}} g(x, y)\, dA = \int_0^2 dy \int_{y/3}^{y/2} g(x, y)\, dx + \int_2^3 dy \int_{y/3}^{1} g(x, y)\, dx.$$

   Express the double integral as an iterated integral with the order of integration reversed.

5. Evaluate $\iiint_R z\, dV$, where R is the region in $\mathbb{R}^3$ described by $0 \leq y \leq 1 - x^2$, $0 \leq z \leq x$.

6. Evaluate $\int_1^e \int_0^2 \int_z^2 \frac{2}{x} \sec^2(y^2)\, dy\, dz\, dx$ by first changing the order of the integration in an appropriate way.

7. Evaluate $\iiint_E 2y\,dV$, where E is the region in $\mathbb{R}^3$ given by :

$0 \le x \le 1$, $0 \le y \le \sqrt{10 - x^2 - z^2}$, $0 \le z \le 3x$.

8. Evaluate $\iiint_E 3y^2 dV$, where E is the solid enclosed by the planes $x = 0$, $y = 0$, $z = 0$,

and $x + y + z = 1$.

9. Evaluate :

(a) $\iint_T x\,dA$, where T is the triangular region enclosed by $y = -x$, $y = 2x$, and $y = 2$.

(b) $\iint_D x^2 y\,dA$, where D is the region enclosed by the line $y = x$, and the parabola $y = x^2$.

(c) $\iint_R xy\,dA$, where R is the rectangular region given by $-1 \le x \le 1$, $0 \le y \le 2$.

10. Evaluate $\iint_T \dfrac{\sin(\pi x)}{x+1}\,dA$, where T is the Trapezoidal region with vertices at the points

$(0,1)$, $(1,1)$, $(1,3)$, and $(0,2)$.

11. Find $\iint_R \cos(x^2)\,dA$, where R is the region enclosed by $y = 0$, $y = 2x$, and $x = 1$.

12* Compute $\mathbf{I} = \int_0^1 \left\{ \int_{\sqrt{x}}^1 e^{y^3} dy \right\} dx$, by first reversing order of integration.

13* Evaluate $\iiint_E 15x^2 dV$, where E is the solid described by :

$0 \le x \le 2 - y - z$, $0 \le z \le 2 - y$, $0 \le y \le 2$.

14*Use Triple integrals to find the volume of the solid occupied by the region described by

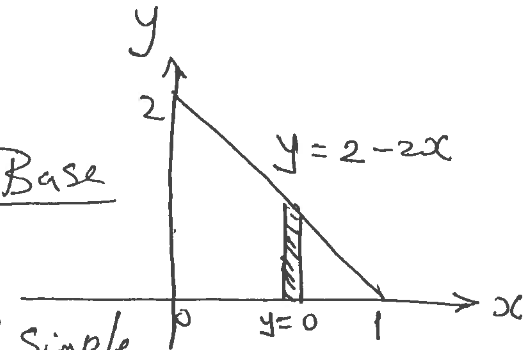
$x \ge 0$, $y \ge 0$, $z \ge 0$, $x + y \le 1$, and $z \le 4 - 3x^2$.

MATH 277

Solutions to Problem Set # 8

1.

Base



Y-Simple

$$0 \le y \le 2-2x$$
$$0 \le x \le 1$$

$$\text{Volume } V = \iint_{\text{Base}} \text{Height } dA$$

Here: Height $= x^2 - 0 = x^2$, and the base is the Triangular region in first quadrant bounded by $x=0$, $y=0$, and $y = 2-2x$ as shown in figure

$$\therefore \quad V = \int_0^1 \left\{ \int_0^{2-2x} x^2 \, dy \right\} dx = \int_0^1 x^2 \left\{ \int_0^{2-2x} dy \right\} dx$$

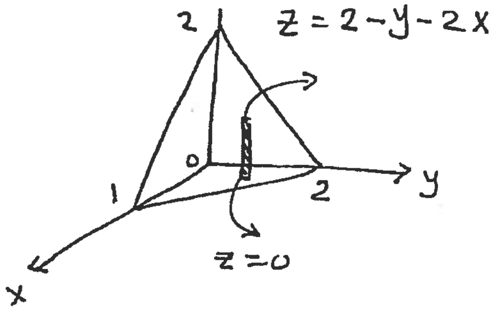
$$= \int_0^1 x^2 (2-2x)\, dx = \int_0^1 (2x^2 - 2x^3)\, dx$$

$$= \frac{2}{3}x^3 - \frac{1}{2}x^4 \Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6}$$

$$= \frac{1}{6}$$

2.

$$\text{Volume } V = \iint_{\text{Base}} \text{Height}\, dA$$



Now, $2x + y + z = 2 \Rightarrow$

$z = 2 - 2x - y$

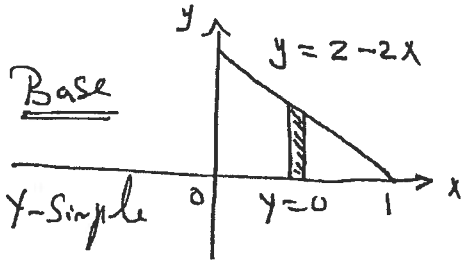
The other surface is $z = 0$

$\therefore$ Height $= z_{\text{upper}} - z_{\text{lower}}$

$= (2 - 2x - y) - 0$

$= 2 - 2x - y$

Base



Y-Simple

$0 \leq y \leq 2 - 2x$

$0 \leq x \leq 1$

The Base is the Triangular region R enclosed by $x = 0$, $y = 0$, and the Curve of intersection of the surfaces $z = 0$, and $z = 2 - 2x - y$, namely the line in $\mathbb{R}^2$ given by $0 = 2 - 2x - y$ or $y = 2 - 2x$

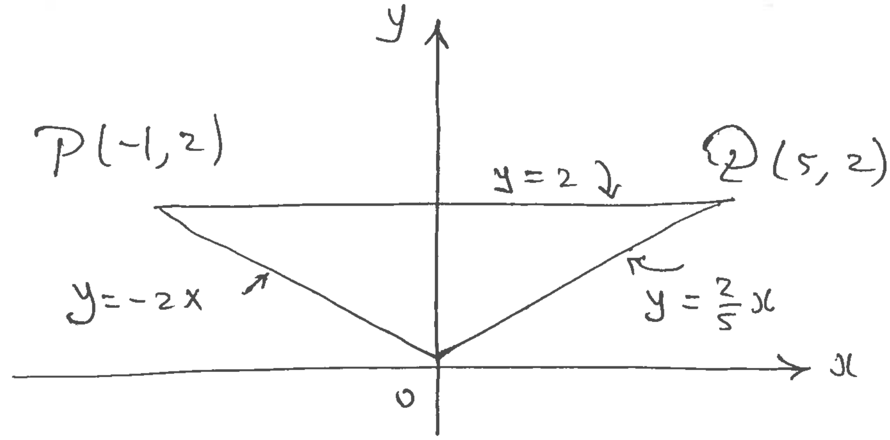
$$\therefore \quad V = \int_0^1 \left\{ \int_{y=0}^{y=2-2x} (2 - 2x - y)\, dy \right\} dx$$

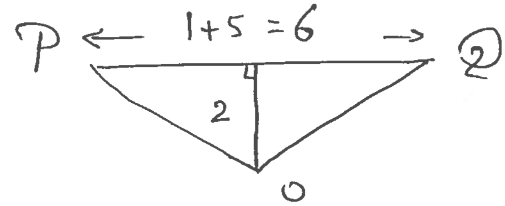
$$= \int_0^1 \left. \frac{(2-2x-y)^2}{(-1)2} \right|_{y=0}^{y=2-2x} dx = -\frac{1}{2} \int_0^1 \left[0^2 - (2-2x)^2 \right] dx$$

$$= \frac{1}{2} \int_0^1 (2-2x)^2\, dx = \frac{1}{2} \cdot \left. \frac{(2-2x)^3}{(-2) \cdot 3} \right|_{x=0}^{x=1}$$

$$= \frac{-1}{12} \left[0^3 - 2^3 \right] = \frac{8}{12} \text{ or } \frac{2}{3}$$

3. (a) First, let us sketch region R





<u>To find point P:</u>

Solve $y = 2$, and $y = -2x$:

$2 = -2x \Rightarrow x = -1$

$\therefore P(-1, 2)$

Similarly: To find Q:

Solve $y = 2$, $y = \frac{2}{5}x$

$\Rightarrow 2 = \frac{2}{5}x \Rightarrow x = 5$

$\therefore Q(5, 2)$

$\therefore I = \iint_R dA = \text{area of } \Delta\, OPQ$

$= \frac{1}{2}(\text{Base})(\text{height}) = \frac{1}{2}(6)(2) = 6$

$\therefore$ No <u>need</u> to Compute a Double Integral!

(b) <u>Answer:</u> 2π

4. (a) Let us first sketch the planar region D " as an x - Simple region:

$$\frac{3y^2}{16} \leq x \leq \sqrt{25-y^2}, \quad 0 \leq y \leq 4$$

Indeed, $x = \frac{3y^2}{16} \Rightarrow y^2 = \frac{16}{3}x$ is an equation of a parabola with vertex at $(0,0)$ and which opens to the right.

Next, $x = \sqrt{25-y^2}$. To identify curve we may square each side. But be careful:

Keep in mind that $x \geq 0$!!

Now, $x = \sqrt{25-y^2} \Rightarrow x^2+y^2 = 25$

$\therefore$ $x = \sqrt{25-y^2}$ is the right half of the circle centred at $(0,0)$ and has radius 5 units

Where curves intersect?

$$x = \frac{3y^2}{16} \quad \ldots (1)$$

$$x = \sqrt{25-y^2} \quad \ldots (2)$$

Equating (1), (2): $\frac{3y^2}{16} = \sqrt{25-y^2}$

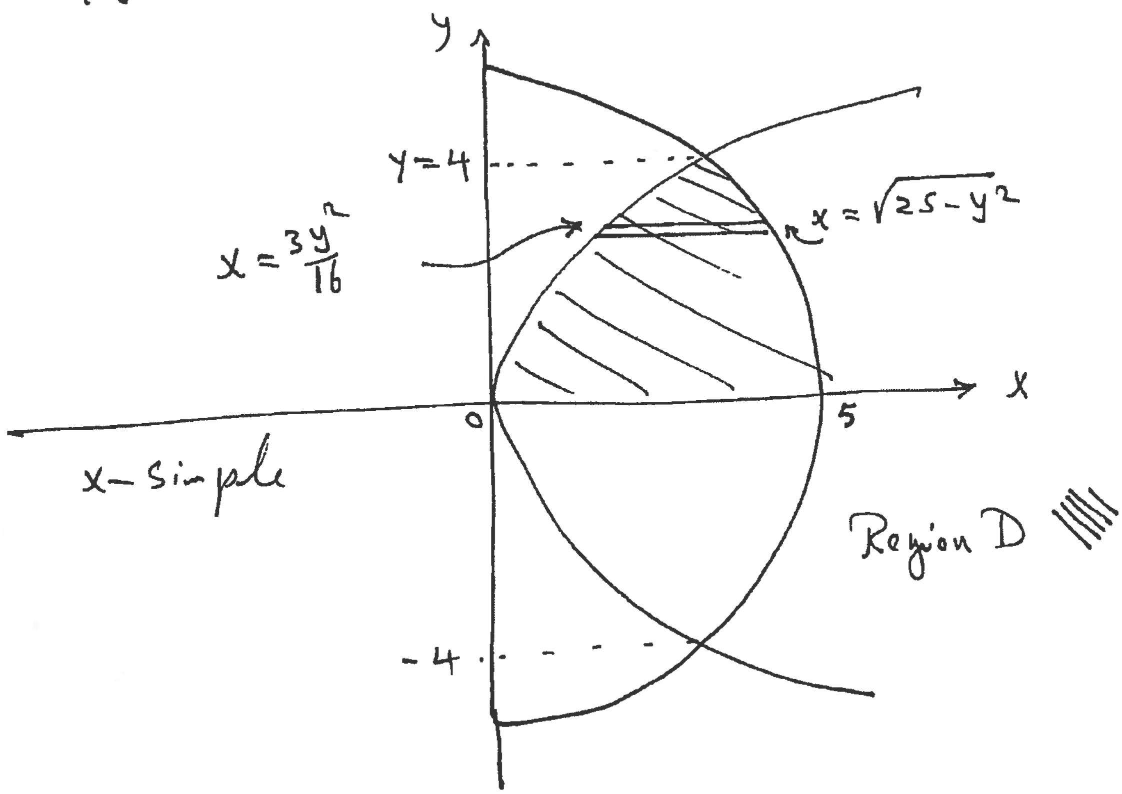
$\therefore \quad \dfrac{9y^4}{256} = 25 - y^2 \Rightarrow 9y^4 = 6400 - 256y^2$

$\Rightarrow 9y^4 + 256y^2 - 6400 = 0$

$(y^2 - 16)(9y^2 + 400) = 0$

$\therefore \quad y^2 - 16 = 0 \Rightarrow y = \pm 4$

or $9y^2 + 400 = 0$ .. No real solutions!



Now, $J = \iint_D f(x,y)\, dA$

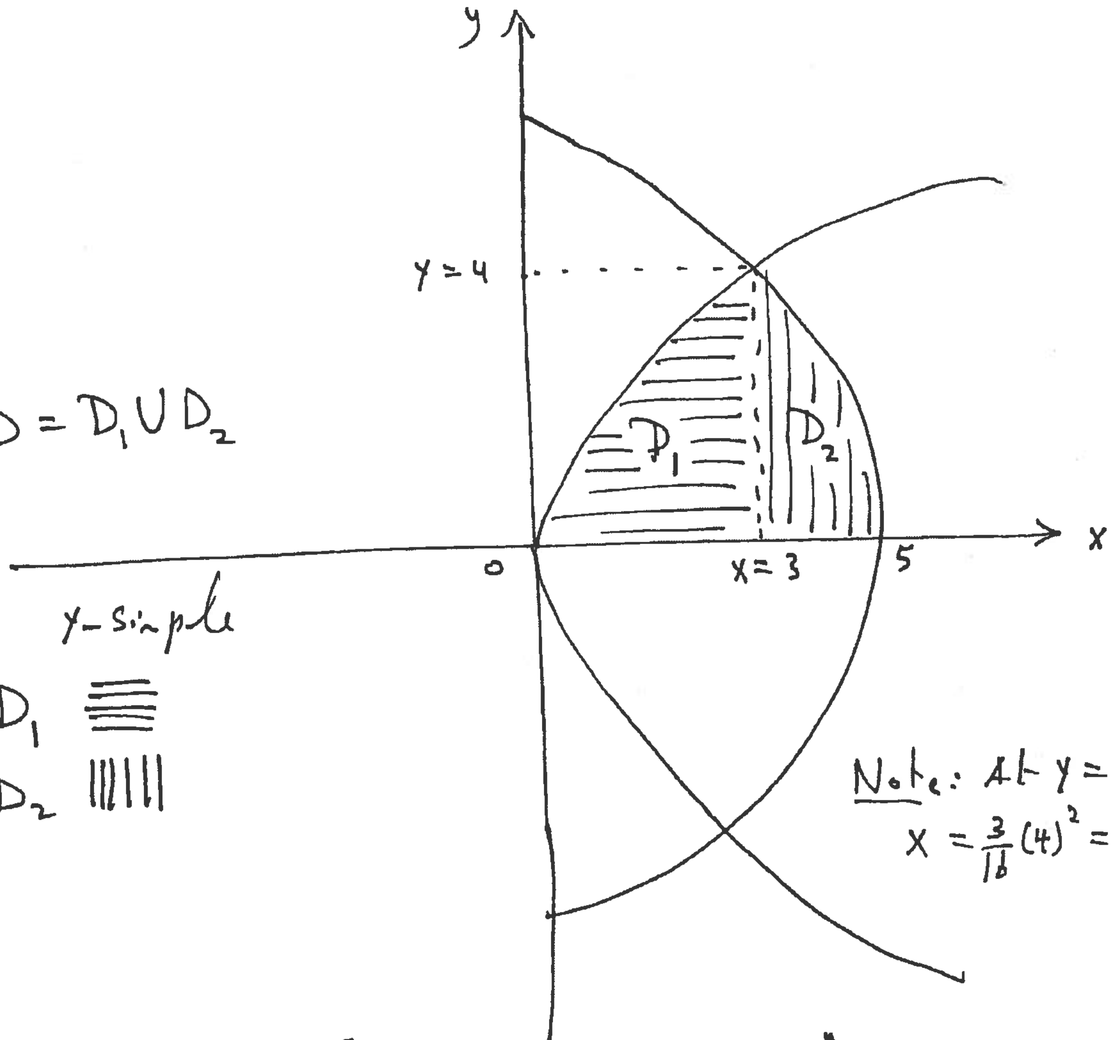
Clearly D is not a y- Simple region. However we may express D as a Union of two y- Simple regions D_1 and D_2 as shown in figure below.

$\therefore \quad J = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$

<u>Note</u>: $x = \sqrt{25 - y^2} \Rightarrow x^2 = 25 - y^2 \Rightarrow y^2 = 25 - x^2$

$\Rightarrow y = +\sqrt{25 - x^2}$ since $y \geq 0$

and $x = \dfrac{3y^2}{16} \Rightarrow y^2 = \dfrac{16x}{3} \Rightarrow y = 4\sqrt{\dfrac{x}{3}}$ since $y \geq 0$.



$D = D_1 \cup D_2$

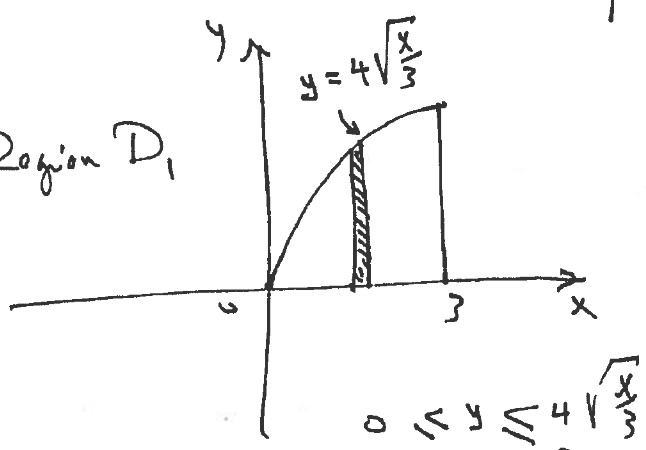
y-simple

D_1 ≡

D_2 |||||

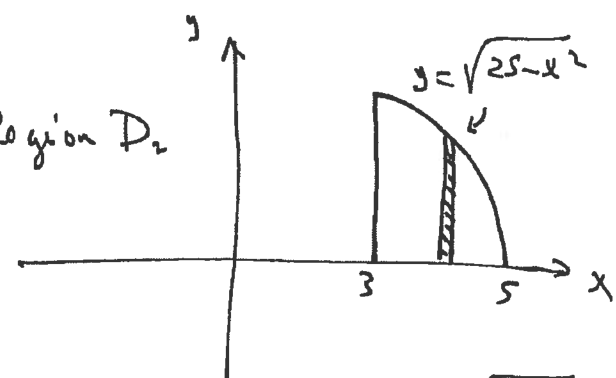
<u>Note</u>: At $y = 4$,
$x = \frac{3}{16}(4)^2 = 3$

Region D_1



$0 \leq y \leq 4\sqrt{\frac{x}{3}}$
$0 \leq x \leq 3$

Region D_2



$0 \leq y \leq \sqrt{25 - x^2}$
$3 \leq x \leq 5$

$$\therefore \; J = \int_0^3 \left\{ \int_0^{4\sqrt{\frac{x}{3}}} f(x,y)\,dy \right\} dx + \int_3^5 \left\{ \int_0^{\sqrt{25-x^2}} f(x,y)\,dy \right\} dx$$

(b) <u>Answer:</u>

$$J = \int_0^1 \left\{ \int_{2x}^{3x} g(x,y)\,dy \right\} dx$$

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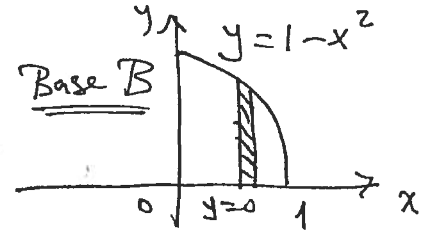
5. Let $I = \iiint_R z\,dV$. Integrate with respect to z-first.

The z-Limits : Given as $0 \leq z \leq x$

The Base: This is the region in xy-plane shown in figure.



$0 \leq y \leq 1 - x^2$

$0 \leq x \leq 1$

$$\therefore\ I = \iint_B \left\{ \int_{z=0}^{z=x} z\,dz \right\} dA$$

$$= \iint_B \frac{1}{2} z^2 \Big|_{z=0}^{z=x} dA$$

$$= \frac{1}{2} \iint_B x^2\,dA = \frac{1}{2} \int_0^1 \int_0^{1-x^2} x^2\,dy\,dx$$

$$= \frac{1}{2} \int_0^1 x^2 [y] \Big|_{y=0}^{y=1-x^2} dx = \frac{1}{2} \int_0^1 x^2 (1-x^2)\,dx$$

$$= \frac{1}{2} \int_0^1 (x^2 - x^4)\,dx = \frac{1}{2} \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_0^1$$

$$= \frac{1}{2} \left[\left(\frac{1}{3} - \frac{1}{5} \right) - (0 - 0) \right] = \frac{1}{2} \cdot \frac{2}{15}$$

$$= \frac{1}{15}.$$

6. Let $I = \int_1^e \int_0^2 \int_z^2 \frac{2}{x} \sec^2(y^2)\, dy\, dz\, dx = \iiint_E \frac{2}{x} \sec^2(y^2)\, dV$,

where E is the region in $\mathbb{R}^3$ described by

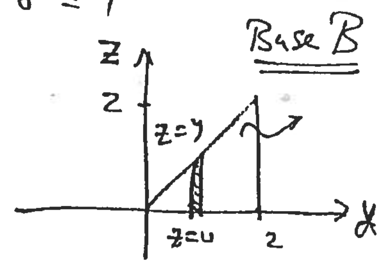
$z \leq y \leq 2$, $0 \leq z \leq 2$, and $1 \leq x \leq e$.

Since the first two inequalities are <u>independent of x</u>, we may integrate w.r.t. x first. We have

$I = \iint_B 2\sec^2(y^2) \left\{ \int_1^e \frac{1}{x}\, dx \right\} dA$, where $dA = dy\, dz$

and B is the Base of region E in the yz-plane given by $z \leq y \leq 2$, $0 \leq z \leq 2$ as shown in figure.

<u>Note</u>: $\int_1^e \frac{1}{x}\, dx = \ln|x| \Big|_1^e = \ln e - \ln 1 = 1 - 0 = 1$



$\therefore I = \iint_B 2\sec^2(y^2)\, dA$

To Compute this integral we must treat B as z-Simple!

$0 \leq z \leq y$

$0 \leq y \leq 2$

$I = \int_0^2 2\sec^2(y^2) \left\{ \int_0^y dz \right\} dy = \int_0^2 2y \sec^2(y^2)\, dy$

let $u = y^2$, $du = 2y\, dy$

$\therefore I = \int \sec^2(u)\, du = \tan(u) = \tan(y^2) \Big|_{y=0}^{y=2}$

$= \tan(4) - \tan(0)$

$= \tan(4)$

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7. We first observe that: There is no need to sketch region E since limits are already provided!

Now, since the second inequality involves all three variables, and the last involves two variables, the y-integration must be performed first, the z-integration second, and the x-integration last.

Hence we write $dV = dy\,dz\,dx$

Now, $I = \int_0^1 \int_0^{3x} \left\{ \int_0^{\sqrt{10-x^2-z^2}} 2y\,dy \right\} dz\,dx = \int_0^1 \int_0^{3x} y^2 \Big|_{y=0}^{y=\sqrt{10-x^2-z^2}} dz\,dx$

$$= \int_0^1 \int_0^{3x} \left[\left(\sqrt{10-x^2-z^2}\right)^2 - 0^2 \right] dz\,dx$$

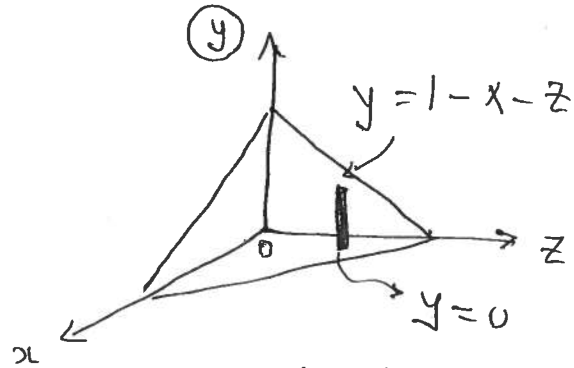
$$= \int_0^1 \left\{ \int_0^{3x} (10-x^2-z^2)\,dz \right\} dx = \int_0^1 10z - x^2 z - \frac{1}{3}z^3 \Big|_{z=0}^{z=3x} dx$$

$$= \int_0^1 \left[10(3x) - x^2(3x) - \frac{1}{3}(3x)^3 \right] dx$$

$$= \int_0^1 (30x - 12x^3)\,dx = 15x^2 - 3x^4 \Big|_0^1 = 15 - 3 = 12$$

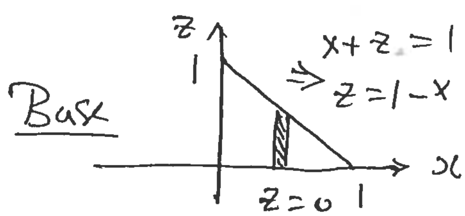
8. It is wise to integrate w.r. to y first.
(You may sketch the plane $x+y+z=1$ taking y-axis pointing upward for easy visualization!)

$$I = \iint_{\text{Base}} \left\{ \int_0^{1-x-z} 3y^2\,dy \right\} dA, \quad dA = dx\,dz \text{ or } dz\,dx$$



$0 \leq y \leq 1-x-z$

$$= \iint_{\text{Base}} y^3 \Big|_0^{1-x-z} dA$$



$0 \leq z \leq 1-x$
$0 \leq x \leq 1$

$$= \int_0^1 \left\{ \int_0^{1-x} (1-x-z)^3\,dz \right\} dx$$

$$= \int_0^1 \frac{(1-x-z)^4}{(-1)(4)} \Big|_0^{1-x} dx$$

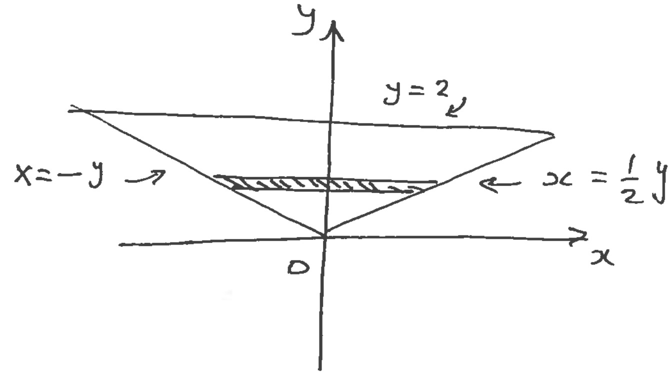
$$= -\frac{1}{4} \int_0^1 \left[0^4 - (1-x)^4 \right] dx$$

$$= \frac{1}{4} \int_0^1 (1-x)^4\,dx = \frac{1}{4} \cdot \frac{(1-x)^5}{(-1)(5)} \Big|_0^1$$

$$= -\frac{1}{20} \left[0^5 - 1^5 \right] = \frac{1}{20}$$

9. (a) First, let us sketch region T

We shall Treat region as an x-Simple region



$$T: \begin{array}{l} -y \le x \le \frac{1}{2}y \\ 0 \le y \le 2 \end{array}$$

$$\therefore I = \int_0^2 \left\{ \int_{-y}^{\frac{1}{2}y} x\,dx \right\} dy = \frac{1}{2}\int_0^2 x^2 \Big|_{x=-y}^{x=\frac{1}{2}y} dy$$

$$= \frac{1}{2}\int_0^2 \left[\left(\tfrac{1}{2}y\right)^2 - (-y)^2 \right] dy$$

$$= \frac{1}{2}\int_0^2 \left(\tfrac{1}{4}y^2 - y^2 \right) dy$$

$$= \frac{1}{2}\int_0^2 -\frac{3}{4}y^2\,dy = -\frac{3}{8}\cdot\frac{1}{3}y^3 \Big|_0^2$$

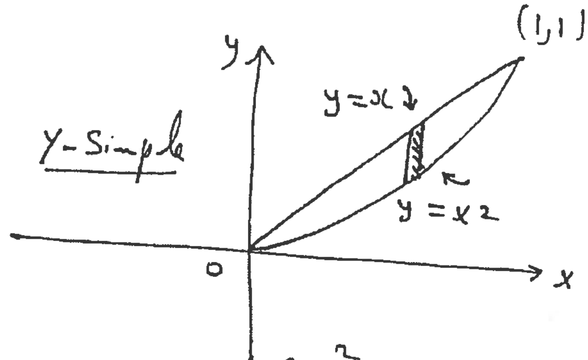
$$= -\frac{1}{8}\left[2^3 - 0^3\right] = -\frac{1}{8}\cdot 8$$

$$= -1$$

(b) let us first sketch region D

$$I = \iint_D x^2 y \, dA$$

Y-Simple



$$D: \begin{cases} x^2 \leq y \leq x \\ 0 \leq x \leq 1 \end{cases}$$

Note: Curves Intersect at (0,0), (1,1) as obvious!

$$= \int_0^1 x^2 \left\{ \int_{x^2}^{x} y \, dy \right\} dx$$

$$= \int_0^1 x^2 \cdot \frac{1}{2} y^2 \Big|_{y=x^2}^{y=x} dx$$

$$= \frac{1}{2} \int_0^1 x^2 \left[x^2 - (x^2)^2 \right] dx$$

$$= \frac{1}{2} \int_0^1 x^2 (x^2 - x^4) \, dx = \frac{1}{2} \int_0^1 (x^4 - x^6) \, dx$$

$$= \frac{1}{2} \left[\frac{1}{5} x^5 - \frac{1}{7} x^7 \right]_0^1 = \frac{1}{2} \left[\frac{1}{5} - \frac{1}{7} \right]$$

$$= \frac{1}{2} \left[\frac{7-5}{(5)(7)} \right]$$

$$= \frac{1}{2} \cdot \frac{2}{35} = \frac{1}{35}$$

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(c) The rectangular region may be Treated as an x-Simple or y-Simple!

There is no need to sketch!

We shall Treat region as an x-Simple!

$$I = \iint_R xy\, dA = \int_0^2 \left\{ \int_{-1}^1 xy\, dx \right\} dy$$

$$= \int_0^2 y \left\{ \int_{-1}^1 x\, dx \right\} dy$$

But $\displaystyle\int_{-1}^1 x\, dx = \frac{1}{2} x^2 \Big|_{-1}^1 = 0$

$$\therefore \quad I = 0$$

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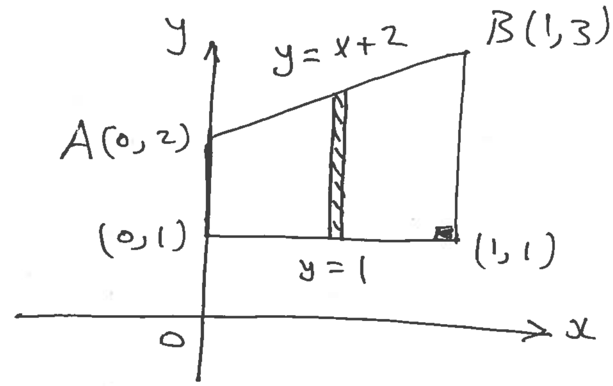
10. Note first that the equation of the line joining $A(0,2)$, and $B(1,3)$ is given by

$$y - 2 = m(x - 0),$$

where $m = \frac{3-2}{1-0} = 1$

$\therefore y - 2 = 1(x - 0)$ or

$$\boxed{y = x + 2}$$



$T: \quad 1 \le y \le x+2$
$\quad\;\; 0 \le x \le 1$

$$\therefore I = \iint_T \frac{\sin(\pi x)}{x+1}\, dA$$

$$= \int_0^1 \frac{\sin(\pi x)}{x+1} \left\{ \int_{y=1}^{y=x+2} dy \right\} dx = \int_0^1 \frac{\sin(\pi x)}{x+1} \left\{ y \Big|_{y=1}^{y=x+2} \right\} dx$$

$$= \int_0^1 \frac{\sin(\pi x)}{x+1} \left[(x+2) - 1 \right] dx$$

$$= \int_0^1 \frac{\sin(\pi x)}{\cancel{x+1}} \cancel{(x+1)}\, dx = \int_0^1 \sin(\pi x)\, dx$$

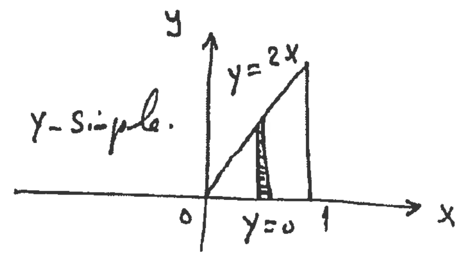
$$= -\frac{1}{\pi} \cos(\pi x) \Big|_{x=0}^{x=1}$$

$$= -\frac{1}{\pi}\left[\cos(\pi) - \cos(0)\right] = -\frac{1}{\pi}\left[-1 - 1\right]$$

$$= -\frac{1}{\pi}(-2) = \frac{2}{\pi}$$

11. Find $\iint\limits_R \cos(x^2)\, dA$, where R is the region enclosed by $y=0$, $y=2x$, and $x=1$.

<u>Solution</u>: First, sketch region R



$R: \quad 0 \leq y \leq 2x$
$\quad\;\; 0 \leq x \leq 1$

Note 1st. that region R is both x-simple and y-simple.
However since $\int \cos(x^2)\, dx$ is impossible to compute, we can't integrate w.r.t. x 1st. Hence we <u>must</u> treat region as a y-simple!

$$I = \int_0^1 \cos(x^2) \left\{ \int_0^{2x} dy \right\} dx = \int_0^1 2x \cos(x^2)\, dx$$

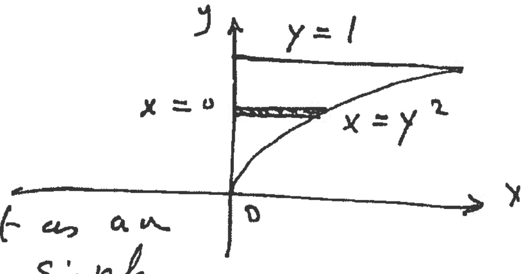
let $t = x^2$, $\therefore\ dt = 2x\, dx$

$$\therefore\ I = \int \cos(x^2) \cdot 2x\, dx = \int \cos(t)\, dt = \sin(t)$$
$$= \sin(x^2)\Big|_0^1 = \sin(1) - \sin(0)$$
$$= \sin(1).$$

*12. Compute $I = \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy\, dx$ by first reversing order of integration.

<u>Answer</u>: $I = \frac{1}{3}(e-1)$.

Hint $I = \iint\limits_R e^{y^3}\, dA$, where R is as shown in figure



$R: \begin{cases} 0 \leq x \leq y^2 \\ 0 \leq y \leq 1 \end{cases}$

Treat as an <u>x-Simple</u>

13. For students to do at home.

Answer : 8

14. For students to do at home.

Answer : Volume $V = \frac{7}{4}$
