MATH 277

Problem Set # 2 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. In each case identify the plane curve given by the specified vector equation :

   (i) $\vec{r}(t) = (-2+t)\,\vec{i} + (4+8t)\,\vec{j}\ ,\quad 0 \le t \le 1.$

   (ii) $\vec{r}(t) = (-2+6\cos(t))\,\vec{i} + (4+8\sin(t))\,\vec{j}\ ,\quad 0 \le t \le 2\pi.$

   (iii) $\vec{r}(t) = (-2+6\cos(t))\vec{i} + (4+6\sin(t))\vec{j}\ ,\quad 0 \le t \le 2\pi.$

   (iv) $\vec{r}(t) = (2+6\cosh(t))\,\vec{i} + (-4+8\sinh(t))\,\vec{j}\ ,\quad t \in \mathbb{R}$

   (v) $\vec{r}(t) = (t-1, t^2+2)\ ,\ t \in \mathbb{R}$

2. In each case, find a parametric representation of the curve of intersection of the two

   given surfaces :

   (i) $4x^2 + y^2 = 16 \quad , \quad 2x + 3y + 2z = 1$

   (ii) $x^2 + y^2 = 4\ , \quad z = x + 1.$

   $(iii)^*$ $x^2 - y^2 - z = 3 \quad , \quad 2y^2 + z = 1.$

   $(iv)^*$ $z^2 + y - x = 2\ , \quad x + yz + 1 = 0$ (using $t = -z$ as a parameter).

3. Find the Cartesian equation of the plane curve $\mathbf{C}$ given parametrically by :

   $x(t) = -2 + 3\cos(t) \quad , \ y(t) = 4 + 3\sin(t) \quad , \quad 0 \le t \le 2\pi.$

   Name the curve and sketch.

4^* Find the Cartesian equation of the plane curve given by :

   $\vec{r}(t) = (2 + 7\cos(t)\ ,\ -1 + 2\sin(t))\ ,\ t \in [0, \pi].$

   Name the curve.

5^* Find the Cartesian equation of the plane curve given parametrically by :

   $x(t) = 2\cosh(t) \quad , \ y(t) = 4\sinh^2(t)\ \ t \in \mathbb{R}.$

   Identify the curve and sketch its graph.

6. In each case , find a parametrization of the curve of intersection of the two given surfaces :

(a) $z = x^2 + y^2$, $2x - 8y + z + 8 = 0$

$(b)^*$ $4x^2 - y = 1$, $4x + y - z = -2$

7. A ball of ice is having mass $100\ gram\ (g)$ at time $t = 0$ is melting and therefore losing mass at a steady rate $1\ g/s$. The ball has initial velocity $\vec{i} + 2\vec{j}$ and is subject to a constant force $\vec{F} = 3\vec{i}$ thereafter. What is the velocity of the ball after 1 min?

8. A rocket moves forward in a straight line by expelling particles of a fuel mixture backward (that is in the opposite direction of motion).Assume the exhaust gases are ejected at a constant rate α and at constant velocity with magnitude v_e relative to the rocket. Let M be the total initial mass of rocket and assume it starts motion from rest.
   (a) Find an expression for the speed $v(t)$ of the rocket at any time $t > 0$.
   (b) What percentage of the total initial mass M would the rocket have to burn as fuel in order to accelerate to the speed of its own exhaust gasses?
   (c) What percentage of the total initial mass M would the rocket have to burn as fuel in order to accelerate to twice the speed of its own exhaust gasses?
   (d) What is the speed of rocket when 50% of its initial mass is ejected during the burn?
   You may assume that there are no external forces acting on the rocket as it travels in deep space

9*. Refer to problem above.
   If the rocket is fired vertically upward in a constant gravitational field of magnitude g.
   Find an expression for the speed $v(t)$ of the rocket at any time $t > 0$.
   Find an expression for the distance travelled by the rocket at any time $t > 0$.

10. A rocket has mass $25,000$ kilogram (kg), which includes $20,000\ kg$ of fuel mixture is fired vertically upward in a constant gravitational field of magnitude $g = 9.8\ metre/s^2$.
   During the burning process the exhaust gases are ejected at a constant rate $1000\ kg/s$ and at constant velocity with magnitude $400\ metre/s$ relative to the rocket.
   If the rocket was initially at rest , find its velocity after 15 , 20 , and 30 seconds.

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Solutions to Problem Set # 2

1. (i) $\vec{r}(t) = (-2+t)\vec{i} + (4+8t)\vec{j}$, $\quad 0 \leq t \leq 1$

$= (-2,4) + t(1,8)$, $\quad 0 \leq t \leq 1$

This is an equation of a straight line segment in $\mathbb{R}^2$ joining the points $P = \vec{r}(0) = (-2,4)$, and the point $Q = \vec{r}(1) = (-2,4) + 1(1,8) = (-1,12)$

(ii) Recall: The parametric equations of an ellipse centred at (h,k) and has semi-axes of length a, b are

$$\begin{cases} x = h + a\cos(t) \\ y = k + b\sin(t) \end{cases}, \quad t \in [0, 2\pi]$$

Now, $\vec{r}(t) = (-2 + 6\cos(t))\vec{i} + (4 + 8\sin(t))\vec{j}$, $\quad 0 \leq t \leq 2\pi$

is equivalent to

$$\begin{cases} x = -2 + 6\cos(t) \\ y = 4 + 8\sin(t) \end{cases} \quad t \in [0, 2\pi]$$

Those are the parametric equations of an ellipse centred at the point $(h,k) = (-2,4)$, and has semi-axis of length $a = 6$, and $b = 8$

(iii) Recall: The vector equation

$$\vec{r}(t) = (h + a\cos(t))\vec{i} + (K + a\sin(t))\vec{j},$$

$0 \leq t \leq 2\pi$, $a > 0$ is an equation of a circle with centre at (h, K) and radius r.

Therefore

$$\vec{r}(t) = (-2 + 6\cos(t))\vec{i} + (4 + 6\sin(t))\vec{j}, \quad 0 \leq t \leq 2\pi$$

is an Eq. of a circle centred at $(-2, 4)$ and has radius 6 units

(iv) $\vec{r}(t) = (2 + 6\cosh(t))\vec{i} + (-4 + 8\sinh(t))\vec{j}, \quad t \in \mathbb{R}$

Here $x(t) = 2 + 6\cosh(t) \Rightarrow \cosh(t) = \dfrac{x-2}{6}$

$y(t) = -4 + 8\sinh(t) \Rightarrow \sinh(t) = \dfrac{y+4}{8}$

Recall: $\cosh^2(t) - \sinh^2(t) = 1, \quad t \in \mathbb{R}$

$$\therefore \frac{(x-2)^2}{6^2} - \frac{(y+4)^2}{8^2} = 1$$

This is an equation of a Hyperbola with centre at $(2, -4)$ and which opens to the left and right, and vertices at $\underline{(8, -4)}$, and $(-4, -4)$.

But since $\cosh(t) = \dfrac{x-2}{6} \geq 1 \Rightarrow x \geq 8$

$\therefore \vec{r}(t)$ represents: Only the Right-Hand Branch!

(v) $\vec{r}(t) = (t-1, t^2+2)$, $t \in \mathbb{R}$

$\therefore$ $x = t-1$, --- (1)

$y = t^2 + 2$ --- (2)

To identify curve, let us find its Cartesian Equation by Eliminating "t" among (1), (2)

From (1): $t = x+1$

substituting $t = x+1$ into (2):

$$y = (x+1)^2 + 2$$

or $$y - 2 = (x+1)^2$$

This is an Equation of a parabola with vertex at $(h,k) = (-1,2)$ and which opens upward.

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2. (i) $4x^2 + y^2 = 16 \Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1$

This equation Viewed in $\mathbb{R}^2$ is an equation of an ellipse centred at $(h, K) = (0, 0)$, and has semi-axes of length $a = \sqrt{4} = 2$, and $b = \sqrt{16} = 4$

A parametrization is thus given by

$$x = h + a\cos(t) \Rightarrow x = 0 + 2\cos(t),$$
$$y = K + b\sin(t) \Rightarrow y = 0 + 4\sin(t) \qquad t \in [0, 2\pi]$$

substituting x, y into 2nd. equation

$$2x + 3y + 2z = 1$$

we have

$$2(2\cos(t)) + 3(4\sin(t)) + 2z = 1$$
$$2z = 1 - 4\cos(t) - 12\sin(t)$$
$$\Rightarrow z = \frac{1}{2} - 2\cos(t) - 6\sin(t)$$

The curve of intersection is given parametrically by

$$\begin{cases} x = 2\cos(t) \\ y = 4\sin(t) \\ z = \frac{1}{2} - 2\cos(t) - 6\sin(t) \end{cases} \qquad t \in [0, 2\pi]$$

or $\vec{r}(t) = \left(2\cos(t), 4\sin(t), \frac{1}{2} - 2\cos(t) - 6\sin(t)\right),$

$t \in [0, 2\pi]$

(ii) $x^2+y^2=4, \quad z=x+1$

Consider the equation $x^2+y^2=4$. This equation may be thought of as an equation of a circle in $\mathbb{R}^2$ centred at $(0,0)$ and is of radius 2 units.

Therefore we may use the standard parametric equations

$$x = h + a\cos(t) \implies x = 0 + 2\cos(t) = 2\cos(t),$$
$$y = k + a\sin(t) \implies y = 0 + 2\sin(t) = 2\sin(t),$$

$t \in [0, 2\pi]$.

It follows that $z = x+1 = 2\cos(t)+1$

$\therefore$ Curve of intersection is given parametrically by

$$x = 2\cos(t),\ y = 2\sin(t),\ z = 2\cos(t)+1,\ t\in[0,2\pi],$$

or $\vec{r}(t) = 2\cos(t)\,\vec{i} + 2\sin(t)\,\vec{j} + (2\cos(t)+1)\,\vec{k}, \quad 0 \le t \le 2\pi.$

(iii), (iv): For students to do at Home. Very Important.

Answer :

(iii) A possible parametrization is given by

$$x = 2\cos(t),\ y = 2\sin(t),\ z = 1 - 8\sin^2(t),\ t\in[0,2\pi]$$

(iv) The parametric equations are

$$x = t^2+t-1,\ y = 1+t,\ z = -t,\ t\in\mathbb{R},\ t\neq 1$$

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3. Find the Cartesian equation of the plane Curve C given parametrically by

$$x(t) = -2 + 3\cos(t), \quad y(t) = 4 + 3\sin(t),$$

$0 \leq t \leq 2\pi$. Name Curve and sketch.

Solution:

$$x = -2 + 3\cos(t) \quad \cdots (1)$$
$$y = 4 + 3\sin(t) \quad \cdots\cdots (2)$$

From (1):

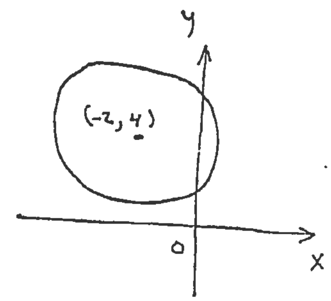
$$x + 2 = 3\cos(t)$$
$$\Rightarrow \cos(t) = \left(\frac{x+2}{3}\right)$$

From (2)

$$y - 4 = 3\sin(t)$$
$$\Rightarrow \sin(t) = \left(\frac{y-4}{3}\right)$$

Recall:

$$\cos^2(t) + \sin^2(t) = 1$$
$$\therefore \left(\frac{x+2}{3}\right)^2 + \left(\frac{y-4}{3}\right)^2 = 1$$
$$\therefore \frac{(x+2)^2}{9} + \frac{(y-4)^2}{9} = 1$$
$$\Rightarrow (x+2)^2 + (y-4)^2 = 9$$



This is an equation of a circle centred at $(-2,4)$, and is of radius 3 units.

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4. For students to do at home.

Answer: The upper half of the Ellipse with Centre at $(2,-1)$, and semi-axes of length $a = 7$, and $b = 2$.

5. For students to do at home.

Answer $y = x^2 - 4$ (in first quadrant!)

Hint: To Eliminate "t" between $x(t)$, and $y(t)$ use the identity: $\cosh^2(t) - \sinh^2(t) = 1$, $t \in \mathbb{R}$

Note also that $x(t) > 0$, $y(t) \geq 0$.

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6. In each case, find a parametrization of the curve of intersection of the two given surfaces:

(a) $z = x^2 + y^2$, $2x - 8y + z + 8 = 0$

(b)* $4x^2 - y = 1$, $4x + y - z = -2$

<u>Hint</u>: For simplicity, use $t = x$ as a parameter!

<u>Solution</u>:

(a) $z = x^2 + y^2$ --- (1)

$2x - 8y + z + 8 = 0$ --- (2)

The idea is to use equations (1), (2) to obtain an equation containing only two variables. Such an equation may be viewed as a curve in $\mathbb{R}^2$!

Let us eliminate "z" among (1), (2) as follows:

From Eq. (2): $z = -8 - 2x + 8y$

substituting into Eq. (1), we have

$$-8 - 2x + 8y = x^2 + y^2$$

$$\Rightarrow x^2 + 2x + y^2 - 8y = -8$$

Complete the square in both x, y-terms to get:

$$\left(x^2 + 2x + (1)^2\right) + \left(y^2 - 8y + (-4)^2\right) = (1)^2 + (-4)^2 - 8$$

$$(x+1)^2 + (y-4)^2 = 9$$

This is an equation of a circle (in $\mathbb{R}^2$) with centre at $(h, k) = (-1, 4)$, and radius $a = 3$

Parametric equations of circle are thus given by

$$x = h + a\cos(t)$$
$$y = k + a\sin(t) \quad , \quad t \in [0, 2\pi]$$

$\therefore$ $x = -1 + 3\cos(t)$, $y = 4 + 3\sin(t)$, $t \in [0, 2\pi]$

<u>To find z</u>: Recall $z = -8 - 2x + 8y$

$$\therefore \quad z = -8 - 2[-1 + 3\cos(t)] + 8[4 + 3\sin(t)]$$
$$= -8 + 2 - 6\cos(t) + 32 + 24\sin(t)$$
$$z = 26 - 6\cos(t) + 24\sin(t)$$

$\therefore$ parametric equation are:

$$\begin{cases} x = -1 + 3\cos(t) \\ y = 4 + 3\sin(t) \\ z = 26 - 6\cos(t) + 24\sin(t) \end{cases} , \quad t \in [0, 2\pi]$$

(b) <u>Answer</u>: $\begin{cases} x = t \\ y = 4t^2 - 1 \\ z = (2t+1)^2 \end{cases} , \quad t \in \mathbb{R}$

<u>Note</u>: Answer above is <u>not</u> unique. There are infinitely many possible parametrizations!!

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7. Recall: Newton's Second Law of Motion:

Let m be the mass of a moving object at time "t", $\vec{v}$ be its Velocity, and $\vec{F}$ be the net force acting on the object.

Then $\frac{d}{dt}(m\vec{v}) = \vec{F}$

Here, since ball is losing mass at a constant rate 1 g/s, the mass at time "t" is thus given by

$$m = m_0 \text{ (Initial mass) } - 1 \cdot t$$

$$\therefore \quad m = 100 - t$$

Let $\vec{v} = (x, y)$, hence $\frac{d\vec{V}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

Now, from (*):

$$\frac{d}{dt}\left[(100-t)\vec{v}\right] = \vec{F}$$

By product Rule

$$(100-t)\frac{dV}{dt} - 1 \cdot \vec{v} = \vec{F} = 3i \overset{or}{=} (3, 0)$$

$$\therefore \quad (100-t)\left(\frac{dx}{dt}, \frac{dy}{dt}\right) - (x, y) = (3, 0)$$

$$\therefore \quad (100-t)\frac{dx}{dt} - x = 3 \quad \cdots (1)$$

$$(100-t)\frac{dy}{dt} - y = 0 \quad \cdots (2)$$

Note also that

$$\vec{V}(0) = i + 2j = (1, 2) \Rightarrow x(0) = 1, \; y(0) = 2$$

Consider (1)

$$(100-t)\frac{dx}{dt} - x = 3 \quad , \quad x(0)=1$$

$$\therefore \quad (100-t)\frac{dx}{dt} = x+3$$

$$\therefore \quad \frac{dx}{x+3} = \frac{dt}{100-t}$$

Integrating:

$$\ln(x+3) = -\ln(100-t) + \text{constant say } \ln C$$

$$\ln(x+3) = \ln\left(\frac{C}{100-t}\right)$$

$$\text{or} \quad x+3 = \frac{C}{100-t}$$

$$\Rightarrow \quad x = \frac{C}{100-t} - 3$$

But $x=1$ at $t=0$, hence

$$1 = \frac{C}{100} - 3 \Rightarrow 4 = \frac{C}{100} \quad \text{or } C = 400$$

$$\therefore \quad \boxed{x(t) = \frac{400}{100-t} - 3}$$

Consider (2): $\quad (100-t)\frac{dy}{dt} - y = 0, \quad y(0) = 2$

Student may verify:

$$y(t) = \frac{200}{100-t}$$

$$\therefore \quad \vec{v} = (x(t), y(t)) = \left(\frac{400}{100-t} - 3, \frac{200}{100-t}\right)$$

After 1 minutes, that is $t = 60$ seconds, we have $\vec{v} = \left(\frac{400}{100-60} - 3, \frac{200}{100-60}\right) = (7, 5) \stackrel{\text{or}}{=} 7\vec{i} + 5\vec{j}$

8 (a) Let $\vec{v}(t)$ be the velocity of the rocket at time "t" and $\vec{v}_e$ be the velocity of the ejected gases relative to the rocket (assumed constant)

$$\therefore \quad m \frac{d\vec{v}}{dt} - \vec{v}_e \frac{dm}{dt} = \vec{F}$$

where $m(t)$ is the mass of the rocket at time "t".

$$\therefore \quad m(t) = M - \alpha t, \text{ and } \frac{dm}{dt} = -\alpha$$ (Rocket is losing Mass!!).

$$\therefore \quad (M - \alpha t) \frac{d\vec{v}}{dt} - \vec{v}_e(-\alpha) = \vec{F}$$

But for one-dimensional motion,

$$\vec{v}(t) = v(t)\vec{i}, \quad \vec{v}_e = -v_e\vec{i}, \text{ and } \vec{F} = F\vec{i}$$

$$\therefore \quad (M - \alpha t) \frac{dv}{dt}\vec{i} - v_e \alpha \vec{i} = F\vec{i}$$

or $$(M - \alpha t) \frac{dv}{dt} - \alpha v_e = F$$

Here $F = 0$,

$$\therefore \quad (M - \alpha t) \frac{dv}{dt} - \alpha v_e = 0$$

$$\Rightarrow \quad \frac{dv}{dt} = \frac{\alpha v_e}{M - \alpha t}, \quad v(0) = 0$$ (Rocket started from Rest)

Integrating, we get

$$V(t) = V_e \int \frac{\alpha}{M-\alpha t}\,dt \leftarrow$$ use substitution $u = M - \alpha t$

$$= -V_e \ln(M-\alpha t) + C$$

$V(0) = 0 \Rightarrow$

$$0 = -V_e \ln(M) + C \Rightarrow C = V_e \ln(M)$$

Hence $V(t) = -V_e \ln(M-\alpha t) + V_e \ln(M)$

$$= V_e [\ln(M) - \ln(M-\alpha t)]$$

$$\therefore \quad V(t) = V_e \ln\left(\frac{M}{M-\alpha t}\right)$$

But $m(t) = M - \alpha t$

$\therefore$ speed $V(t) = V_e \ln\left(\frac{M}{m(t)}\right) \quad (*)$

(b) If $V(t) = V_e$, $(*) \Rightarrow$

$$V_e = V_e \ln\left(\frac{M}{m}\right)$$

$$\therefore \ln\left(\frac{M}{m}\right) = 1 \Rightarrow \frac{M}{m} = e \Rightarrow \boxed{\frac{m}{M} = \frac{1}{e}}$$

It follows that:

Percentage of burnt fuel $= \dfrac{M-m}{M}$

$$= 1 - \frac{m}{M} = 1 - \frac{1}{e}$$

(c) percentage $= 1 - \dfrac{1}{e^2}$

(students may verify !,).

(d) Note that:
50% of Initial Mass is Ejected means
50% of Initial Mass remains

$\therefore \quad m(t) = 0.5\,M$

Now, $\quad V(t) = V_e \ln\left(\frac{M}{m(t)}\right) = V_e \ln\left(\frac{M}{0.5M}\right)$

$$\therefore \quad V(t) = V_e \ln 2$$

$$\approx 0.69\,V_e$$

$\therefore$ speed of rocket is approximately 0.69 of the speed of ejected gases.

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9. For students to do at home.

<u>Answer</u>:

(a) $\quad V(t) = V_0 \ln\left(\frac{M}{M-\alpha t}\right) - gt$

(b) Distance $x(t) = V_e t + \frac{V_e}{\alpha}(M-\alpha t)\ln\left(\frac{M-\alpha t}{M}\right) - \frac{1}{2}gt^2$

<u>Hint</u>: part (a) is identical to problem (8) except that force $F = -mg$.

For part (b): write $v = \frac{dx}{dt}$, then integrate!

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10. From problem (9) part (a)

$$V(t) = V_e \ln\left(\frac{M}{M-\alpha t}\right) - gt$$

Here $M = 25{,}000$, $\alpha = 1000$, $V_e = 400$, and $g = 9.8$

$$\therefore \quad V(t) = 400 \ln\left(\frac{25{,}000}{25{,}000 - 1000t}\right) - 9.8t$$

At $t = 15$

$$V(15) = 400 \ln\left(\frac{25{,}000}{25{,}000 - 1000(15)}\right) - (9.8)(15)$$

$$= 400 \ln(2.5) - 147$$

$$\approx 220 \text{ m/s}$$

At $t = 20$

$$V(20) = 400 \ln\left(\frac{25{,}000}{25{,}000 - 1000(20)}\right) - (9.8)(20)$$

$$= 400 \ln(5) - 196$$

$$\approx 448 \text{ m/s}$$

At $t = 30$:

Note: The 20,000 Kg of fuel will be completely burnt in 20 seconds!

So for $t \geq 20$ velocity

$$V(t) = V_0 - gt, \quad V_0 = 448 \text{ m/s}$$

$$= 448 - (9.8)t$$

At $t = 30$,

$$V(30) = 448 - (9.8)(30-20) = 448 - (9.8)(10)$$

$$= 350 \text{ m/s.}$$