MATH 277

Problem Set # 3 for Labs

Note : Problems marked with ($*$) are left for students to do at home.

1. For each of the following curves find the unit Tangent $\vec{\mathbf{T}}$ and the unit Normal $\vec{\mathbf{N}}$ at the indicated value of t :

   (i) $\vec{r}(t) = 2t\,\vec{i} + \frac{1}{t}\,\vec{j} + 2\ln(t)\,\vec{k}$; $t = 1$

   (ii)* $\vec{r}(t) = 2\cos(t)\,\vec{i} + 2\sin(t)\,\vec{j} + 4t\,\vec{k}$; $t = \frac{3\pi}{4}$

   (iii)* $\vec{r}(t) = t\cos(t)\,\vec{i} + t\sin(t)\,\vec{j}$; $t = 0$

   (iv) $y = \cos(x)$; $x = 0$

2. For each of the following curves find the curvature κ at the indicated value:

   (i) $\vec{r}(t) = e^t\sin(t)\,\vec{i} + e^t\cos(t)\,\vec{j} + e^t\,\vec{k}$; $t = 0$

   (ii) $\vec{r}(t) = t\ln(t)\,\vec{i} + \frac{1}{t}\,\vec{j}$; $t = 1$

   (iii)* $\vec{r}(t) = 2\cos(t)\,\vec{i} + \sin(t)\,\vec{k}$; $t = \frac{\pi}{3}$

   (iv)* $\vec{r}(t) = \sqrt{t^2-3}\,\vec{i} + \frac{t}{\sqrt{t^2-3}}\,\vec{j}$; $t = 2$

   (v)* $\vec{r}(t) = t\cos(t)\,\vec{i} + t\sin(t)\,\vec{j}$; $t = 0$

   (vi) $y = \frac{1}{x}$; $x = 1$

   (vii)* $y = x^3 - 2x^2 + 3$; $x = 1$

   (viii)* $y = \cos(x)$; $x = 0$

3. For each of the following curves find the unit Tangent $\vec{\mathbf{T}}$, the Principal unit Normal $\vec{\mathbf{N}}$, the unit Binormal $\vec{\mathbf{B}}$, the curvature κ , the radius of curvature ρ and the Torsion τ at the indicated value :

   (i) C_1 : $x(t) = \sin(t)\cos(t)$, $y(t) = \sin^2(t)$, $z(t) = \cos(t)$; $t = \frac{\pi}{4}$

   (ii)* C_2 : $x(t) = t$, $y(t) = \frac{1}{2}t^2$, $z(t) = t$; $t = 0$

$(iii)^*$ C_3 : The curve of intersection of the two surfaces $y = \frac{1}{2}x^2$, and $z = \frac{1}{3}x^3$; $x = 1$

4. For each of the following curves find the curvature κ and the Torsion τ at the indicated value :

   (i) $\vec{r}(t) = e^t\,\vec{i} + \sqrt{2}\,t\,\vec{j} + e^{-t}\,\vec{k}$; $t = \ln(2)$.

   (ii) $\vec{r}(t) = (2 + \sqrt{2}\cos(t))\,\vec{i} + (1 - \sin(t))\,\vec{j} + (3 + \sin(t))\,\vec{k}$; $t \in \mathbb{R}$

   $(iii)^*$ $\vec{r}(t) = (3t - t^3)\,\vec{i} + 3t^2\,\vec{j} + (3t + t^3)\,\vec{k}$; $t = \sqrt{3}$

5. In each case , a particle moves along the given parametric curve. Find the Tangential and Normal components of the acceleration at the indicated value of t :

   (i) $\vec{r}(t) = 3\cos(2t)\,\vec{i} + 3\sin(2t)\,\vec{j}$; $t \in \mathbb{R}$

   $(ii)^*$ $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j}$; $t \in \mathbb{R}$

   $(iii)^*$ $\vec{r}(t) = e^t\cos(t)\,\vec{i} + e^{-t}\sin(t)\,\vec{j} + t\,\vec{k}$; $t = 0$

   (iv) $\vec{r}(t) = 2\ln(t)\,\vec{i} + \frac{t-1}{t}\,\vec{j} + 2t\,\vec{k}$; $0 < t \in \mathbb{R}$

   $(v)^*$ $\vec{r}(t) = t^2\,\vec{i} + t^3\,\vec{j}$; $t = 1$

   (vi) $\vec{r}(t) = (\sin(t) - t\cos(t))\,\vec{i} + (\cos(t) + t\sin(t))\,\vec{j} + t^2\,\vec{k}$; $0 < t \in \mathbb{R}$

   $(vii)^*$ $\vec{r}(t) = 4\sqrt{t}\,\vec{i} + (1 - 2t^2)\,\vec{j} + \dfrac{8(t-1)}{\sqrt{t+3}}\,\vec{k}$; $t = 1$

6. Find the Maximum value of the curvature of the plane curve $y = \ln(x)$ and determine the point(s) on the curve where the curvature is Maximum.

7. Find the Maximum and Minimum values of the curvature of the ellipse $9x^2 + 4y^2 = 36$ and determine the points on the curve where the curvature is Maximum or Minimum.

8*. Find the Maximum and Minimum values of the curvature of the curve given by $\vec{r}(t) = 2t\,\vec{i} + 2\cosh(t)\,\vec{j}$ and determine the point(s) on the curve where the curvature is Maximum.

9. Show that the curvature of a straight line is equal to **zero**.

10*. Show that the torsion of a plane curve $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$ is equal to **zero**.

You may assume $x(t)$ and $y(t)$ have continuous derivatives of order up to and including third order.

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Solutions to Problem Set # 3

1. (i) $\vec{r}(t) = 2t\,\vec{i} + \frac{1}{t}\vec{j} + 2\ln(t)\,\vec{k}$, $t = 1$

$$= \left(2t, \frac{1}{t}, 2\ln(t)\right)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(2, -\frac{1}{t^2}, \frac{2}{t}\right)$$

$$v = \|\vec{v}\| = \sqrt{4 + \frac{1}{t^4} + \frac{4}{t^2}} \leftarrow \text{perfect square}$$

$$= \sqrt{\left(2 + \frac{1}{t^2}\right)^2} = 2 + \frac{1}{t^2}$$

$\therefore$ A unit Tangent is thus given by

$$\vec{T}(t) = \frac{\vec{v}}{v} = \frac{\left(2, -\frac{1}{t^2}, \frac{2}{t}\right)}{2 + \frac{1}{t^2}}$$

To simplify: Multiply top / bottom by t^2:

$$\vec{T}(t) = \frac{t^2\left(2, -\frac{1}{t^2}, \frac{2}{t}\right)}{t^2\left(2 + \frac{1}{t^2}\right)}$$

$$= \frac{(2t^2, -1, 2t)}{2t^2 + 1}$$

At $t = 1$,

$$\vec{T} = \frac{(2, -1, 2)}{3} \overset{\text{or}}{=} \frac{1}{3}(2, -1, 2)$$

Next, the unit normal $\vec{N}(t)$ is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|T'(t)\|}$$

Now, $\vec{T}(t) = \dfrac{(2t^2-1, 2t)}{2t^2+1} = (2t^2+1)^{-1}(2t^2-1, 2t)$

$\vec{T}'(t) \underset{\text{product Rule}}{\overset{\text{By}}{=}} -(2t^2+1)^{-2}(4t)(2t^2-1, 2t) + (2t^2+1)^{-1}(4t, 0, 2)$

Do <u>not</u> simplify now!

First: substitute $t=1$:

$$\vec{T}'(1) = -3^{-2}(4)(2,-1,2) + 3^{-1}(4,0,2)$$

$$= -\frac{4}{9}(2,-1,2) + \frac{1}{3}(4,0,2)$$

$$= -\frac{4}{9}(2,-1,2) + \frac{3}{9}(4,0,2)$$

$$= \frac{1}{9}\left[-4(2,-1,2) + 3(4,0,2)\right]$$

$$= \frac{1}{9}(4,4,-2)$$

$$\|\vec{T}'(1)\| = \frac{1}{9}\sqrt{4^2+4^2+(-2)^2} = \frac{1}{9}\sqrt{36} = \frac{6}{9}$$

$$\therefore \overline{N}(1) = \frac{\vec{T}'(1)}{\|\vec{T}'(1)\|} = \frac{\frac{1}{9}(4,4,-2)}{\frac{6}{9}} = \frac{(4,4,-2)}{6}$$

$$\overset{\text{or}}{=} \frac{1}{3}(2,2,-1)$$

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(ii) For students to do at home.

Answer: $\vec{T} = \left(-\frac{1}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{5}}\right)$

$\vec{N} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

(iii) For students to do at home.

Answer: $\vec{T} = (1, 0)$, $\vec{N} = (0, 1)$

(iv) $y = \cos(x)$

let us find a parametrization of the curve!

let $x(t) = t$

$\therefore y(t) = \cos(t)$

$\therefore \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$= t\vec{i} + \cos(t)\vec{j} \overset{\text{or}}{=} (t, \cos(t))$

Note also that $x = 0 \Rightarrow t = 0$

Now, $\vec{v} = \frac{d\vec{r}}{dt} = (1, -\sin(t))$

$v = \|\vec{v}\| = \sqrt{1 + \sin^2(t)}$

$\therefore \vec{T}(t) = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(1, -\sin(t))}{\sqrt{1+\sin^2(t)}}$

At $t = 0$, $\vec{T}(0) = \frac{(1, -\sin(0))}{\sqrt{1+\sin^2(0)}} = \frac{(1, 0)}{1}$

$= (1, 0) \overset{\text{or}}{=} \vec{i}$

Next, $\vec{N}(t) = \frac{\vec{T}'(t)}{\|T'(t)\|}$

$$\vec{T}(t) = \left(1 + \sin^2(t)\right)^{-\frac{1}{2}} \left(1, -\sin(t)\right)$$

By product rule,

$$\vec{T}'(t) = -\frac{1}{2}\left(1+\sin^2(t)\right) . 2\sin(t)\cos(t) \left(1, -\sin(t)\right)$$

$$+ \left(1+\sin^2(t)\right)^{-\frac{1}{2}} \left(0, -\cos(t)\right)$$

Again, do <u>not</u> Simplify now!

put $t=0$ and noting that $\cos(0)=1$, $\sin(0)=0$, we get,

$$\vec{T}'(0) = (0, -1)$$

$$\|\vec{T}'(0)\| = \sqrt{0+1} = 1$$

$$\therefore \vec{N}(0) = \frac{(0,-1)}{1} = (0,-1) \overset{or}{=} -\vec{j}$$

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2 (c) $\vec{r}(t) = \left(e^t \sin(t),\ e^t \cos(t),\ e^t\right)$, $t = 0$

$$\stackrel{or}{=} e^t \left(\sin(t), \cos(t), 1 \right)$$

$$\therefore \vec{v} = \frac{d\vec{r}}{dt} = e^t\left(\cos(t), -\sin(t), 0\right) + e^t\left(\sin(t), \cos(t), 1\right)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = e^t\left(-\sin(t), -\cos(t), 0\right) + e^t\left(\cos(t), -\sin(t), 0\right)$$
$$+ e^t\left(\cos(t), -\sin(t), 0\right) + e^t\left(\sin(t), \cos(t), 1\right)$$

Do not simplify!

At $\underline{t=0}$:

$$\vec{v}(0) = (1,0,0) + (0,1,1) = (1,1,1)$$
$$\vec{a}(0) = (0,-1,0) + (1,0,0) + (1,0,0) + (0,1,1)$$
$$= (2,0,1)$$

Recall $K = \dfrac{\|\vec{v} \times \vec{a}\|}{v^3}$

Now, $\vec{v} \times \vec{a} = (1,1,1) \times (2,0,1)$

$$= \left(\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \right)$$
$$= (1, 1, -2)$$

$$\Rightarrow \|\vec{v} \times \vec{a}\| = \sqrt{1+1+4} = \sqrt{6}$$

and $v = \|\vec{v}\| = \|(1,1,1)\| = \sqrt{1+1+1} = \sqrt{3}$

$$\therefore K = \frac{\sqrt{6}}{(\sqrt{3})^3} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{1}{3}\sqrt{\frac{6}{3}} = \frac{\sqrt{2}}{3}$$

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(ii) $\vec{r}(t) = \left(t\ln(t), \frac{1}{t}\right)$, $t = 1$

<u>Trick</u>: you may consider this plane curve as a space curve with z-component being zero!

Let $\vec{r}(t) = \left(t\ln(t), \frac{1}{t}, 0\right)$

$$\therefore \vec{v} = \frac{d\vec{r}}{dt} = \left(t \cdot \frac{1}{t} + 1 \cdot \ln(t), -\frac{1}{t^2}, 0\right)$$

$$= \left(1 + \ln(t), -\frac{1}{t^2}, 0\right)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\frac{1}{t}, \frac{2}{t^3}, 0\right)$$

<u>At $t = 1$</u>: $\vec{v} = (1 + \ln(1), -1, 0) = (1, -1, 0)$

$$\vec{a} = (1, 2, 0)$$

$$\vec{v} \times \vec{a} = (1, -1, 0) \times (1, 2, 0)$$

$$= (0, 0, 3) \Rightarrow \|\vec{v} \times \vec{a}\| = \sqrt{0+0+9} = 3$$

$$v = \|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$\therefore K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{3}{(\sqrt{2})^3} = \frac{3}{2\sqrt{2}}$$

(iii) For students to do at home.

<u>Answer</u>: $K = \dfrac{16}{13\sqrt{13}}$

(iv) For students to do at home

<u>Answer</u> $K = \dfrac{27}{13\sqrt{13}}$

(v) For students to do at home.

Answer : $K = 2$

(vi) $y = \frac{1}{x}$, $x = 1$

let us find a parametrization for the curve (in $\mathbb{R}^3$!)

let $x = t$, $\therefore y = \frac{1}{t}$, take $z = 0$

$$\therefore \vec{r}(t) = (x, y, z) = (t, \frac{1}{t}, 0)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (1, -\frac{1}{t^2}, 0),$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (0, \frac{2}{t^3}, 0)$$

At $x = 1$, we have $t = 1$

$$\therefore \vec{v} = (1, -1, 0), \quad \vec{a} = (0, 2, 0)$$

$$\vec{v} \times \vec{a} = (1, -1, 0) \times (0, 2, 0)$$

$$= (0, 0, 2) = 2(0, 0, 1)$$

$$\Rightarrow \|\vec{v} \times \vec{a}\| = 2$$

and $\|\vec{v}\| = v = \sqrt{1 + 1 + 0} = \sqrt{2}$

$$\therefore K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{2}{(\sqrt{2})^3} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

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(vii) For students to do at home.

Answer : $K = \frac{1}{\sqrt{2}}$

(viii) For students to do at home.

Answer: $K = 1$

3.(i) C_1: $x = \sin(t)\cos(t) \overset{\text{or}}{=} \frac{1}{2}\sin(2t)$

$y = \sin^2(t) \overset{\text{or}}{=} \frac{1}{2}(1-\cos(2t))$

$z = \cos(t)$

$\therefore \vec{r}(t) = \left(\frac{1}{2}\sin(2t),\ \frac{1}{2}-\frac{1}{2}\cos(2t),\ \cos(t)\right)$

$\vec{v} = \frac{d\vec{r}}{dt} = \left(\cos(2t),\ \sin(2t),\ -\sin(t)\right)$

$\vec{a} = \frac{d\vec{v}}{dt} = \left(-2\sin(2t),\ 2\cos(2t),\ -\cos(t)\right)$

$\vec{a}' = \frac{d\vec{a}}{dt} = \left(-4\cos(2t),\ -4\sin(2t),\ \sin(t)\right)$

At $t = \frac{\pi}{4}$ (Note $\sin\frac{\pi}{2} = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$),

$\vec{v} = \left(0, 1, -\frac{1}{\sqrt{2}}\right)$

$\vec{a} = \left(-2, 0, -\frac{1}{\sqrt{2}}\right)$

$\vec{a}' = \left(0, -4, \frac{1}{\sqrt{2}}\right)$

$v = \|\vec{v}\| = \sqrt{0^2+1^2+\left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1+\frac{1}{2}} = \sqrt{\frac{3}{2}}$

$\vec{v}\times\vec{a} = \left(0, 1, -\frac{1}{\sqrt{2}}\right)\times\left(-2, 0, -\frac{1}{\sqrt{2}}\right)$

$= \left(-\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 2\right)$, $\|\vec{v}\times\vec{a}\| = \sqrt{\frac{13}{2}}$

$\therefore \vec{T} = \frac{\vec{v}}{v} = \frac{\left(0, 1, -\frac{1}{\sqrt{2}}\right)}{\sqrt{\frac{3}{2}}} = \sqrt{\frac{2}{3}}\left(0, 1, -\frac{1}{\sqrt{2}}\right)$

$\vec{B} = \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} = \frac{\left(-\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 2\right)}{\sqrt{\frac{13}{2}}} = \frac{\sqrt{2}}{\sqrt{13}}\left(-\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 2\right)$

$\overset{\text{or}}{=} \frac{1}{\sqrt{13}}\left(-1, 2, 2\sqrt{2}\right)$

Next,

$$\vec{N} = \vec{B} \times \vec{T}$$

$$= \frac{1}{\sqrt{13}}\left(-1, 2, 2\sqrt{2}\right) \times \sqrt{\frac{2}{3}}\left(0, 1, -\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{13}}\sqrt{\frac{2}{3}}\left(-1, 2, 2\sqrt{2}\right) \times \left(0, 1, -\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{39}}\left(-3\sqrt{2}, -\frac{1}{\sqrt{2}}, -1\right)$$

$$\overset{\text{or}}{=} -\frac{1}{\sqrt{39}}\left(6, 1, \sqrt{2}\right)$$

Next,

$$K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\sqrt{\frac{13}{2}}}{\left(\sqrt{\frac{3}{2}}\right)^3} = \frac{\sqrt{13}}{\sqrt{2} \cdot \frac{3}{2}\frac{\sqrt{3}}{\sqrt{2}}}$$

$$= \frac{2\sqrt{13}}{3\sqrt{3}} \overset{\text{or}}{=} \frac{2\sqrt{13}}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{39}}{9}, \quad \rho = \frac{9}{2\sqrt{39}}$$

Finally,

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot a'}{\|\vec{v} \times \vec{a}\|^2}$$

$$= \frac{\left(-\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 2\right) \cdot \left(0, -4, \frac{1}{\sqrt{2}}\right)}{\left(\sqrt{\frac{13}{2}}\right)^2}$$

$$= \frac{0 - \frac{8}{\sqrt{2}} + \frac{2}{\sqrt{2}}}{\left(\frac{13}{2}\right)} = \frac{-\frac{6}{\sqrt{2}}}{\frac{13}{2}} = \frac{-12}{13\sqrt{2}} \overset{\text{or}}{=} \frac{-6\sqrt{2}}{13}$$

(ii) For students to do at home.

<u>Answer</u>: $\vec{T} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\vec{N} = (0, 1, 0)$, $\vec{B} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $K = \frac{1}{2}$, $\rho = 2$, and $\tau = 0$

(iii) For students to do at home.

<u>Answer</u>: $\vec{T} = \frac{1}{\sqrt{3}}(1,1,1)$, $\vec{N} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $\vec{B} = \frac{1}{\sqrt{6}}(1,-2,1)$, $K = \frac{\sqrt{2}}{3}$, $\rho = \frac{3}{\sqrt{2}}$, $\tau = \frac{1}{3}$

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4. (i) $\vec{r}(t) = \left(e^{t}, \sqrt{2}\,t, e^{-t} \right)$, $t = \ln(2)$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(e^{t}, \sqrt{2}, -e^{-t} \right)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(e^{t}, 0, e^{-t} \right)$$

$$\vec{a}' = \frac{d\vec{a}}{dt} = \left(e^{t}, 0, -e^{-t} \right)$$

At $t = \ln 2$,

$$\vec{v} = \left(e^{\ln 2}, \sqrt{2}, -e^{-\ln 2} \right)$$

Note : $e^{-\ln 2} = e^{\ln \frac{1}{2}} = \frac{1}{2}$

$$= \left(2, \sqrt{2}, -\frac{1}{2} \right)$$

$$\vec{a} = \left(e^{\ln 2}, 0, e^{-\ln 2} \right) = \left(2, 0, \frac{1}{2} \right)$$

$$\vec{a}' = \left(e^{\ln 2}, 0, -e^{-\ln 2} \right) = \left(2, 0, -\frac{1}{2} \right)$$

$$\therefore \quad \vec{v} \times \vec{a} = \left(2, \sqrt{2}, -\frac{1}{2} \right) \times \left(2, 0, +\frac{1}{2} \right)$$

$$= \left(\begin{vmatrix} \sqrt{2} & -\frac{1}{2} \\ 0 & +\frac{1}{2} \end{vmatrix}, -\begin{vmatrix} 2 & -\frac{1}{2} \\ 2 & +\frac{1}{2} \end{vmatrix}, \begin{vmatrix} 2 & \sqrt{2} \\ 2 & 0 \end{vmatrix} \right)$$

$$= \left(+\frac{\sqrt{2}}{2}, -2, -2\sqrt{2} \right)$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + (-2)^2 + (-2\sqrt{2})^2} = \sqrt{\frac{1}{2} + 4 + 8} = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$$

$$v = \|\vec{v}\| = \sqrt{2^2 + (\sqrt{2})^2 + \left(\frac{1}{2}\right)^2} = \sqrt{4 + 2 + \frac{1}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

$$\therefore \kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\frac{5}{\sqrt{2}}}{\left(\frac{5}{2}\right)^3} = \frac{4\sqrt{2}}{25}$$

Next, $\tau = \dfrac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{\|\vec{v} \times \vec{a}\|^2}$

$$\tau = \frac{\left(\frac{\sqrt{2}}{2}, -2, -2\sqrt{2}\right) \times \left(2, 0, -\frac{1}{2}\right)}{\left(\frac{5}{\sqrt{2}}\right)^2}$$

$$= \frac{\sqrt{2} + 0 + \sqrt{2}}{\frac{25}{2}} = \frac{2\sqrt{2}}{\frac{25}{2}} = \frac{4\sqrt{2}}{25}$$

(ii) $\vec{r} = \left(2 + \sqrt{2}\cos(t),\ 1 - \sin(t),\ 3 + \sin(t)\right)$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(-\sqrt{2}\sin(t),\ -\cos(t),\ \cos(t)\right)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-\sqrt{2}\cos(t),\ \sin(t),\ -\sin(t)\right)$$

$$\vec{a}' = \frac{d\vec{a}}{dt} = \left(\sqrt{2}\sin(t),\ \cos(t),\ -\cos(t)\right)$$

$$v = \|\vec{v}\| = \sqrt{\left[-\sqrt{2}\sin(t)\right]^2 + \left[-\cos(t)\right]^2 + \left[\cos(t)\right]^2}$$

$$= \sqrt{2\sin^2(t) + 2\cos^2(t)}$$

$$= \sqrt{2\left(\sin^2(t) + \cos^2(t)\right)} = \sqrt{2(1)} = \sqrt{2}$$

$$\vec{v} \times \vec{a} = \left(\begin{vmatrix} -\cos(t) & \cos(t) \\ \sin(t) & -\sin(t) \end{vmatrix},\ -\begin{vmatrix} -\sqrt{2}\cos(t) & -\sin(t) \\ \sqrt{2}\sin(t) & -\cos(t) \end{vmatrix},\ \begin{vmatrix} -\sqrt{2}\cos(t) & \sin(t) \\ \sqrt{2}\sin(t) & \cos(t) \end{vmatrix} \right)$$

$$= \left(0, -\sqrt{2}, -\sqrt{2}\right)$$

(Note: we have used the identity $\sin^2(t) + \cos^2(t) = 1$)

$$\therefore \|\vec{v} \times \vec{a}\| = \sqrt{0 + 2 + 2} = 2$$

$$\therefore \quad K = \frac{\|\vec{v}\times\vec{a}\|}{v^3} = \frac{2}{(\sqrt{2})^3} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

and
$$\tau = \frac{(\vec{v}\times\vec{a})\cdot\vec{a}'}{\|\vec{v}\times\vec{a}\|^2}$$

$$= \frac{(0,-\sqrt{2},-\sqrt{2})\cdot(\sqrt{2}\sin(t),\cos(t),-\cos(t))}{(2)^2}$$

$$= \frac{0-\sqrt{2}\cos(t)+\sqrt{2}\cos(t)}{4} = 0$$

(iii) For students to do at home

$K = \frac{1}{48}$, $\tau = \frac{1}{48}$

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5. Recall: The acceleration $\vec{a}(t)$ is given by

$$\vec{a}(t) = a_T \vec{T} + a_N \vec{N}$$

where $a_T = \frac{dv}{dt}$ is the Tangential Component

and $a_N = K v^2$ is the Normal Component of the acceleration.

(i) $\vec{r}(t) = (3\cos(2t), 3\sin(2t))$

$$\overset{\text{or}}{=} (3\cos(2t), 3\sin(2t), 0)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (-6\sin(2t), 6\cos(2t), 0)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (-12\cos(2t), -12\sin(2t), 0)$$

$$\vec{v} \times \vec{a} = (0, 0, 72(\sin^2(2t) + \cos^2(2t)))$$

$$= (0, 0, 72) = 72(0, 0, 1)$$

$$v = \|\vec{v}\| = \sqrt{36\sin^2(2t) + 36\cos^2(2t)} = \sqrt{36} = 6$$

$$K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{72\,\|(0,0,1)\|}{6^3} = \frac{1}{3}$$

$$\therefore\ a_T = \frac{dv}{dt} = \frac{d}{dt}(6) = 0,$$

and $a_N = K v^2 = \frac{1}{3}(6)^2 = 12$

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(ii) For students to do at home.

Answer: $a_T = \dfrac{4t}{\sqrt{1+4t^2}}$, $a_N = \dfrac{2}{\sqrt{1+4t^2}}$

(iii) For students to do at home.

Answer $a_T = -\dfrac{2}{\sqrt{3}}$, $a_N = \sqrt{\dfrac{8}{3}}$

(iv) $\vec{r}(t) = \left(2\ln(t),\ \dfrac{t-1}{t},\ 2t\right) \overset{\text{or}}{=} \left(2\ln(t), 1-\dfrac{1}{t}, 2t\right)$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{2}{t}, \frac{1}{t^2}, 2\right)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-\frac{2}{t^2}, -\frac{2}{t^3}, 0\right)$$

$$\vec{v} \times \vec{a} = \left(\frac{2}{t}, \frac{1}{t^2}, 2\right) \times \left(-\frac{2}{t^2}, -\frac{2}{t^3}, 0\right)$$

$$= \left(\frac{4}{t^3}, -\frac{4}{t^2}, -\frac{2}{t^4}\right)$$

$$= \frac{2}{t^4}\left(2t, -2t^2, -1\right)$$

$$\|\vec{v} \times \vec{a}\| = \frac{2}{t^4}\sqrt{4t^2 + 4t^4 + 1} = \frac{2}{t^4}\sqrt{(2t^2+1)^2}$$

$$= \frac{2}{t^4}(2t^2+1)$$

$$v = \|\vec{v}\| = \left\|\left(\frac{2}{t}, \frac{1}{t^2}, 2\right)\right\| = \frac{1}{t^2}\left\|(2t, 1, 2t^2)\right\|$$

$$= \frac{1}{t^2}\sqrt{4t^2+1+4t^4} = \frac{1}{t^2}\sqrt{(2t^2+1)^2} = \frac{1}{t^2}(2t^2+1)$$

$$\therefore K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\frac{2}{t^4}(2t^2+1)}{\left(\frac{2t^2+1}{t^2}\right)^3} = \frac{2t^2}{(2t^2+1)^2}$$

$$a_T = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{t^2}(2t^2+1)\right)$$

$$= \frac{d}{dt}\left(2 + \frac{1}{t^2}\right) = -\frac{2}{t^3}$$

$a_N = K v^2$. But $K = \dfrac{\|\vec{v} \times \vec{a}\|}{v^3}$

$$\therefore \; a_N = \frac{\|\vec{v} \times \vec{a}\|}{v^3} v^2 = \frac{\|\vec{v} \times \vec{a}\|}{v} \quad \text{(Much Easier!)}$$

$$= \frac{\frac{2}{t^4}\cancel{(2t^2+1)}}{\frac{1}{t^2}\cancel{(2t^2+1)}} = \frac{2}{t^2}$$

(v) For students to do at home.

<u>Answer:</u> $a_T = \dfrac{22}{\sqrt{13}}$, $a_N = \dfrac{6}{\sqrt{13}}$

(vi) $\vec{r}(t) = (\sin(t) - t\cos(t), \cos(t) + t\sin(t), t^2)$

$$\vec{v} = \frac{d\vec{r}}{dt} = (\cancel{\cos(t)} - \cancel{\cos(t)} + t\sin(t), -\cancel{\sin(t)} + \cancel{\sin(t)} + t\cos t, 2t)$$

$$= (t\sin(t), t\cos(t), 2t)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (\sin(t) + t\cos(t), \cos(t) - t\sin(t), 2)$$

$$\vec{v} \times \vec{a} = (2t^2\sin(t), 2t^2\cos(t), -t^2)$$

$$\therefore \|\vec{v} \times \vec{a}\| = \sqrt{4t^4\sin^2(t) + 4t^4\cos^2(t) + t^4}$$

$$\|\vec{v}\times\vec{a}\| = \sqrt{4t^4(\sin^2(t)+\cos^2(t)) + t^4}$$

$$= \sqrt{4t^4 + t^4} = \sqrt{5t^4} = \sqrt{5}\,t^2$$

$$v = \|\vec{v}\| = \sqrt{t^2\sin^2(t) + t^2\cos^2(t) + 4t^2} = \sqrt{5}\,t \quad , \quad t > 0$$

$$a_T = \frac{dv}{dt} = \frac{d}{dt}(\sqrt{5}\,t) = \sqrt{5}$$

$$a_N = \frac{\|\vec{v}\times\vec{a}\|}{v} = \frac{\sqrt{5}\,t^2}{\sqrt{5}\,t} = t$$

(vii) For students to do at home.

<u>Answer:</u> $a_T = \dfrac{5}{3}$, $a_N = \dfrac{\sqrt{137}}{3}$

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6. $y = \ln(x)$

let us parametrize the plane curve $y = \ln(x)$ as a curve in 3-space

let $x = t$, hence $y = \ln(t)$, take $z = 0$

$\therefore \vec{r}(t) = (t, \ln(t), 0)$, $t > 0$

$\vec{v} = \frac{d\vec{r}}{dt} = (1, \frac{1}{t}, 0)$

$\vec{a} = \frac{d\vec{v}}{dt} = (0, -\frac{1}{t^2}, 0)$

$\vec{v} \times \vec{a} = (1, \frac{1}{t}, 0) \times (0, -\frac{1}{t^2}, 0)$

$= (0, 0, -\frac{1}{t^2}) = \frac{1}{t^2}(0, 0, 1)$

$\| \vec{v} \times \vec{a} \| = \frac{1}{t^2} \| (0,0,1) \| = \frac{1}{t^2}$

$\| \vec{v} \| = v = \sqrt{1 + \frac{1}{t^2} + 0} = \sqrt{\frac{t^2+1}{t^2}} = \frac{\sqrt{t^2+1}}{t}$

$\therefore K = \frac{\| \vec{v} \times \vec{a} \|}{v^3} = \frac{\frac{1}{t^2}}{\left(\frac{\sqrt{t^2+1}}{t}\right)^3} = \frac{t}{(t^2+1)^{\frac{3}{2}}}$

$\therefore K(t) = \frac{t}{(t^2+1)^{\frac{3}{2}}} = t(t^2+1)^{-\frac{3}{2}}$

We need to Maximize $K(t)$.

Now, $K'(t) = (t^2+1)^{-\frac{3}{2}} + t.\left(-\frac{3}{2}(t^2+1)^{-\frac{5}{2}}.2t\right)$

$$K'(t) = \frac{1}{(t^2+1)^{\frac{3}{2}}} - \frac{3t^2}{(t^2+1)^{\frac{5}{2}}}$$

$$= \frac{(t^2+1) - 3t^2}{(t^2+1)^{\frac{5}{2}}} = \frac{1-2t^2}{(t^2+1)^{\frac{5}{2}}}$$

$$K'(t) = 0 \Rightarrow 1 - 2t^2 = 0, \quad 2t^2 = 1, \quad t = +\frac{1}{\sqrt{2}}$$

Sign of K'

$$\begin{array}{ccc} + + + + + & & - - - - \\ \hline 0 & \frac{1}{\sqrt{2}} & \end{array}$$

Clearly $K'(t)$ attains its absolute maximum at $t = \frac{1}{\sqrt{2}}$ (why?).

$\therefore$ Maximum Curvature

$$K = \left. \frac{t}{(t^2+1)^{\frac{3}{2}}} \right|_{t=\frac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{(\frac{1}{2}+1)^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{2}\,(\frac{3}{2})^{\frac{3}{2}}} = \frac{1}{\sqrt{2} \cdot \frac{3}{2} \frac{\sqrt{3}}{\sqrt{2}}} = \frac{2}{3\sqrt{3}}$$

Maximum occur at the point

$$(x, y) = (t, \ln(t)) \Big|_{t=\frac{1}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}}, \ln\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$\text{or } \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\ln 2\right)$$

7. $9x^2 + 4y^2 = 36 \Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

The ellipse is centred at the origin and has semi-axes of length $a = \sqrt{4} = 2$, and $b = \sqrt{9} = 3$

$\therefore$ A parametric representation of Ellipse (in $\mathbb{R}^3$) is given by

$$\vec{r}(t) = (2\cos(t), 3\sin(t), 0), \quad t \in [0, 2\pi]$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (-2\sin(t), 3\cos(t), 0)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (-2\cos(t), -3\sin(t), 0)$$

$$\therefore \vec{v} \times \vec{a} = (-2\sin(t), 3\cos(t), 0) \times (-2\cos(t), -3\sin(t), 0)$$

$$= (0, 0, 6)$$

$$v = \|\vec{v}\| = \sqrt{4\sin^2(t) + 9\cos^2(t)}$$

But $\sin^2(t) = 1 - \cos^2(t)$

$$\therefore v = \sqrt{4(1-\cos^2(t)) + 9\cos^2(t)} = \sqrt{4 + 5\cos^2(t)}$$

$$\therefore K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\|(0,0,6)\|}{(\sqrt{4+5\cos^2(t)})^3}$$

$$= \frac{6}{(4 + 5\cos^2(t))^{3/2}}$$

clearly K attains its Maximum value when $\cos(t) = 0$ which occurs at $t = \frac{\pi}{2}, \frac{3\pi}{2}$

The Maximum Value is

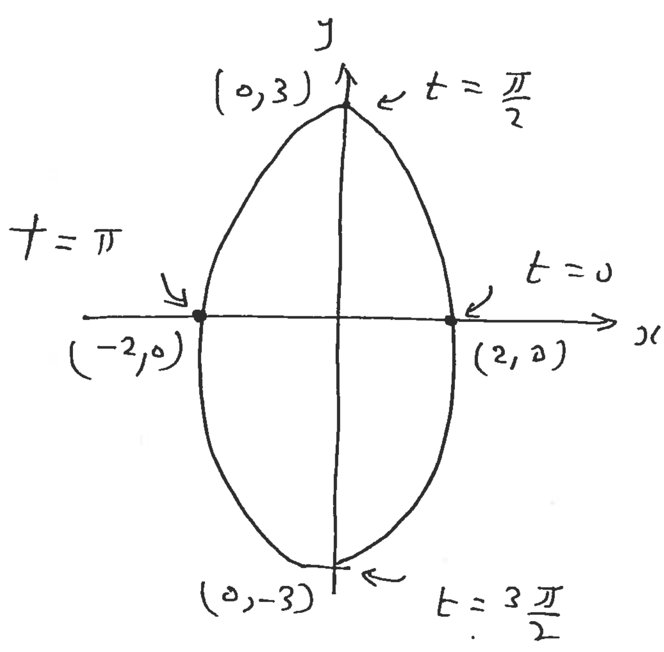
$$K = \frac{6}{(4+0)^{\frac{3}{2}}} = \frac{6}{4^{\frac{3}{2}}} = \frac{6}{4\sqrt{4}} = \frac{3}{4}$$

Next, K attains its minimum value when $\cos^2(t) = 1$ which occurs at $t = 0$, $t = \pi$.

The Minimum Value is

$$K = \frac{6}{(4+5)^{\frac{3}{2}}} = \frac{6}{9^{\frac{3}{2}}} = \frac{6}{9\sqrt{9}} = \frac{2}{9}$$

Refer to figure for the location of points of Maximum and Minimum Curvature.



Maximum Curvature occur at the points $(0,-3)$, $(0,3)$

Minimum Curvature occur at the points $(-2,0)$, $(2,0)$

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8. For students to do at home.

Answer : Maximum Curvature $K = \frac{1}{2}$, occurs at the point $(0, 2)$.

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9. Recall: Parametric equations of st. line passing through the point (x_0, y_0, z_0) and in the direction of the vector (a, b, c) are given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad \text{and} \quad z = z_0 + ct$$

$$\therefore \vec{r}(t) = (x_0 + at,\ y_0 + bt,\ z_0 + ct), \quad t \in \mathbb{R}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (a,\ b,\ c) \quad \Rightarrow v = \|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (0, 0, 0)$$

$$\therefore \vec{v} \times \vec{a} = (a, b, c) \times (0, 0, 0) = \vec{0}$$

$$\|\vec{v} \times \vec{a}\| = 0$$

$$\therefore K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{0}{\sqrt{a^2 + b^2 + c^2}} = 0$$

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10. For students to do at home.

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