SPECIAL PLANE CURVES

ESSENTIALS FROM HIGH SCHOOL MATH

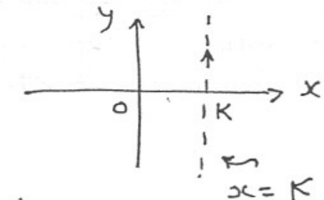
MATH 249/251

SHEET (VII) — Standard Curves

1 The Straight line

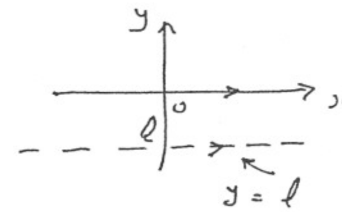
(a) Vertical line: A vertical line is a line parallel to the y-axis. It equation is of the form $\boxed{x = K}$, $K \in \mathbb{R}$. The line is K-units apart from y-axis.

In particular $\boxed{x = 0}$ is the equation of the y-axis.



(b) Horizontal line: A horizontal line is a line parallel to the x-axis. It equation is of the form $\boxed{y = \ell}$, $\ell \in \mathbb{R}$. The line is ℓ-units apart from x-axis

In particular $\boxed{y = 0}$ is the equation of the x-axis.

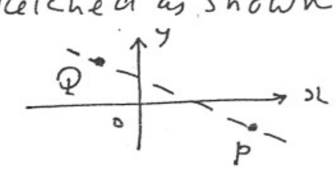


(c) General equation of a straight line:

An equation of the form $ax + by + c = 0$ where $a, b,$ and c are real numbers and a, b are not both zeros, represents an equation of a straight line (whose slope $m = -\frac{a}{b}$, $b \neq 0$).

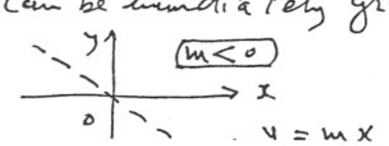
To sketch the straight line, one needs to determine only two points on it say P and Q, then join and extend beyond!

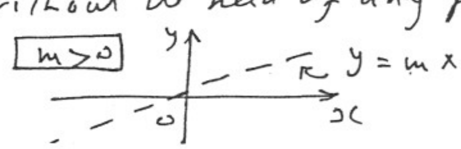
Remark: To determine a point on the line we assign an arbitrary value to one of the variables x or y and determine the other from equation of line. However this does not apply to horizontal or vertical lines which must be identified and sketched as shown in (a), and (b) above.



(d) Line through origin:

An equation of the form $y = mx$ represents an equation of a straight line which has slope "m" and passes through origin. This line can be immediately graph without the need of any points as follows



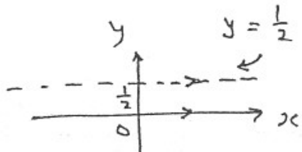


Ex: sketch (a) $2y-1=0$ (b) $x=-7$

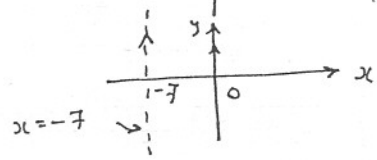
(c) $y=3x$ (d) $2y+5x=0$

(e) $2x-3y+6=0$

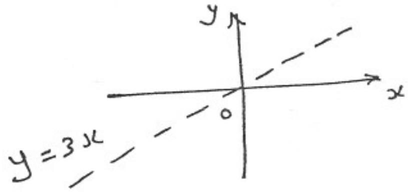
Soln. (a) $2y-1=0 \Rightarrow y=\frac{1}{2}$ (a horizontal line)



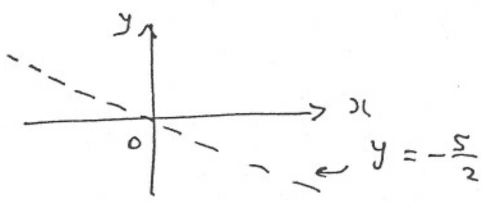
(b) $x=-7$ (a vertical line)



(c) $y=3x$, a line through "0" with positive slope ($m=3>0$)



(d) $2y+5x=0 \Rightarrow y=-\frac{5}{2}x$, a line through "0" with negative slope $m=-\frac{5}{2}$



(e) $2x-3y+6=0$, a general line. Need "2" points to sketch.

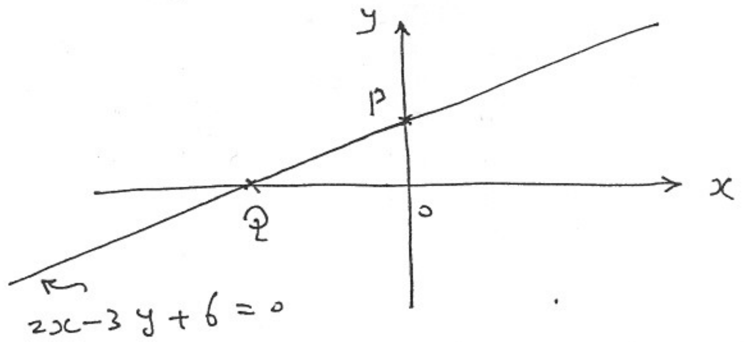
1st. point : let say $x=0$, hence $2(0)-3y+6=0$, $y=2$

$\therefore$ $P(0,2)$

2nd point : let say $y=0$, hence $2x-3(0)+6=0$, $x=-3$

$\therefore$ $Q(-3,0)$

Note : P, and Q are the y, and x - intercepts of the straight line respectively. These are the most recommended pair!



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2 The Circle:

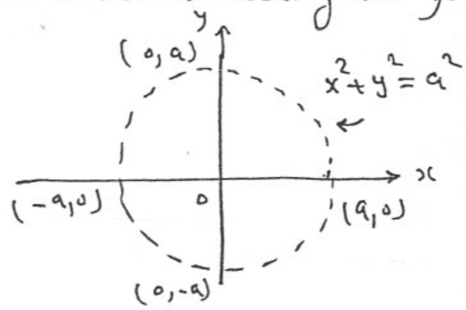
(a) The standard Equation:

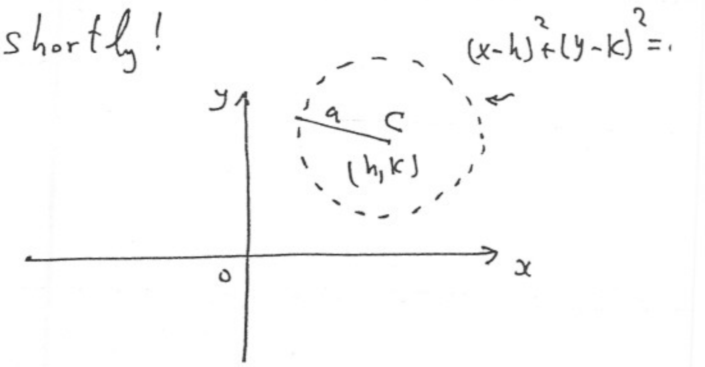
An equation of the form $(x-h)^2+(y-k)^2=a^2$, $a>0$ represents a circle centred at $C(h,k)$ and is of radius "a".

In particular if circle centred at the origin, its equation becomes

$$\boxed{x^2+y^2=a^2}$$

which we shall investigate further shortly!





(b) The General Equation of a circle

An equation of the form $x^2+y^2+ax+by+c=0$ represents an equation of a circle with centre at $(h,k)=\left(-\frac{a}{2},-\frac{b}{2}\right)$, and radius $R=\sqrt{h^2+k^2-c}$ provided $h^2+k^2-c>0$

(Note: If $h^2+k^2-c=0$, the circle degenerates to a point, and if $h^2+k^2-c<$ the circle is imaginary (or from outerspace!).

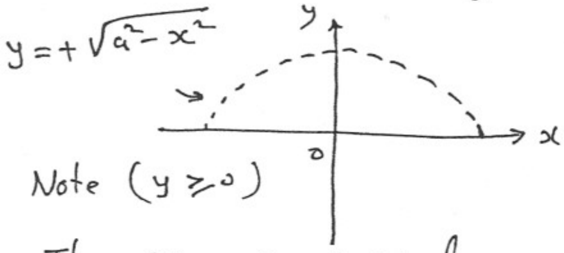
(c) The Special standard form:

Consider the special circle: $\boxed{x^2+y^2=a^2}$

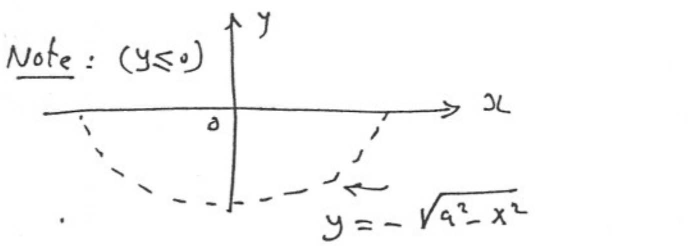
(i) If we solve for y, we get: $y^2=a^2-x^2 \Rightarrow y=\pm\sqrt{a^2-x^2}$

Therefore either $y=+\sqrt{a^2-x^2}$ " The upper semi-circle "

or $y=-\sqrt{a^2-x^2}$ " The lower semi-circle "



The upper semi-circle.



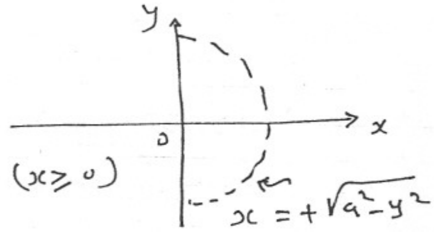
The lower semi-circle.

(ii) If we solve for x, we get: $x^2 = a^2 - y^2 \Rightarrow x = \pm\sqrt{a^2 - y^2}$.

Therefore Either $x = +\sqrt{a^2 - x^2}$ (The Right semi-circle)

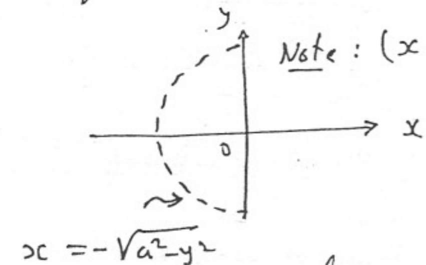
or $x = -\sqrt{a^2 - x^2}$ (The left semi-circle)

Note ($x \geq 0$)



The right semi-circle.

Note: ($x \leq 0$)



The left semi-circle.

Note: Same discussions apply to standard eq. $(x-h)^2 + (y-k)^2 = a^2$.

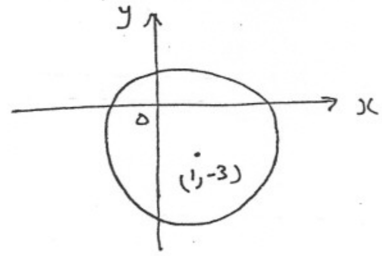
EX: sketch (a) $(x-1)^2 + (y+3)^2 = 16$

(b) $x^2 + y^2 - 4x + 6y - 23 = 0$

(c) $x = -\sqrt{9 - y^2}$

(d) $y = +\sqrt{25 - x^2}$

Soln. (a) $(x-1)^2 + (y+3)^2 = 16$ .. A circle centred at $(h, k) = (1, -3)$, and radius 4 units.

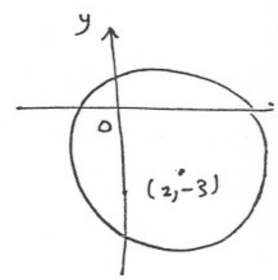


(b) $x^2 + y^2 - 4x + 6y - 23 = 0$

Here $a = -4$, $b = 6$, and $c = -23$

$\therefore$ Centre $(h, k) = \left(-\frac{a}{2}, -\frac{b}{2}\right) = (2, -3)$

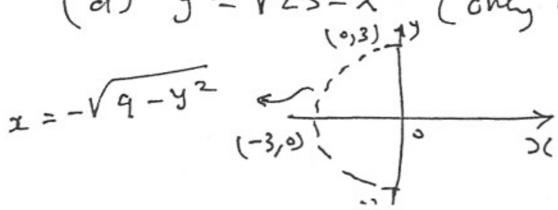
radius $R = \sqrt{h^2 + k^2 - c} = \sqrt{2^2 + (-3)^2 - (-23)} = \sqrt{36} = 6$

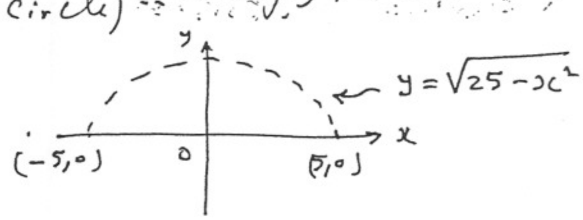


(c) $x = -\sqrt{9 - y^2}$ .. 1st. note that $x \leq 0$

Now, $x^2 = 9 - y^2 \Rightarrow x^2 + y^2 = 9$ which is a circle centred at $(0, 0)$ and is of radius 3. But since $x \leq 0$, we get only the left semi-circle (Not the whole circle!)

(d) $y = \sqrt{25 - x^2}$ (only upper semi circle) .. Verify!



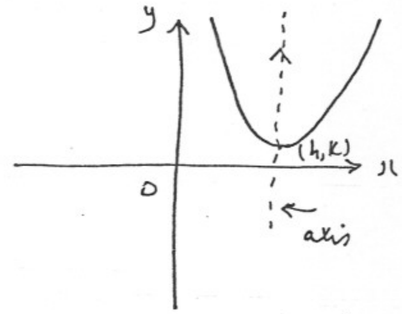


3) The parabola:

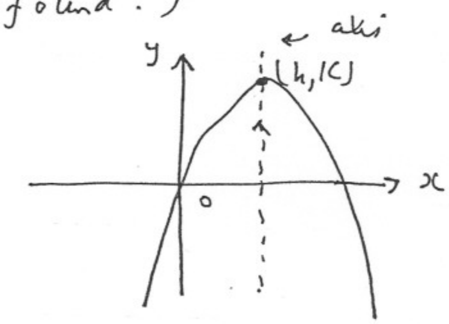
(a) Parabola with axis parallel to y-axis:

An equation of the form $y = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ with $a \neq 0$ is an equation of a parabola whose axis (or axis of symmetry) is parallel to y-axis. The parabola opens upward if $\boxed{a > 0}$ and opens downwards if $\boxed{a < 0}$.

To graph the parabola you may either determine its x, and y intercepts (if any!) and which way it opens or if more precise graphing is required, determine its vertex (h, k) where $h = -\frac{b}{2a}$ (and hence the y-coordinate k can be easily found!)



$(a > 0)$: opens upward



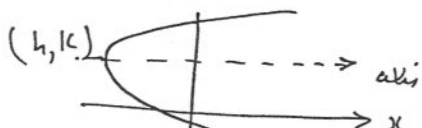
$(a < 0)$: opens downwards

(b) Parabola with axis parallel to x-axis:

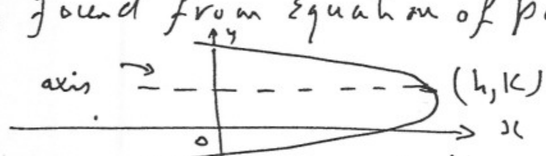
An equation of the form $x = ay^2 + by + c$, where $a, b, c \in \mathbb{R}$ with $a \neq 0$ is an equation of a parabola whose axis (or axis of symmetry) is parallel to x-axis. The parabola opens to the right if $\boxed{a > 0}$, and opens to the left if $\boxed{a < 0}$.

To graph the parabola you may either determine its x, and y-intercepts (if any!) and which way it opens or if more accurate graphing is desired, determine also its vertex (h, k) where $k = -\frac{b}{2a}$ (and hence the x-coordinate h can be easily found from equation of parabola!)

$\boxed{a > 0}$



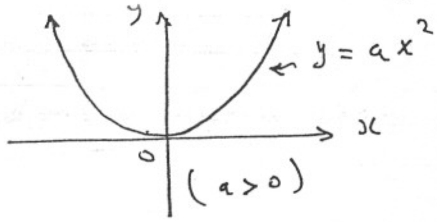
opens to the Right

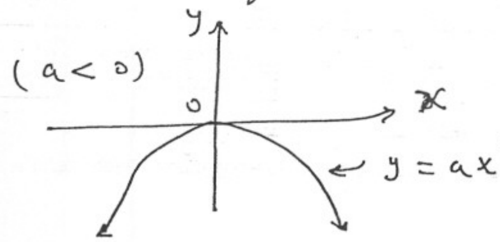


$(a < 0)$. opens to the left

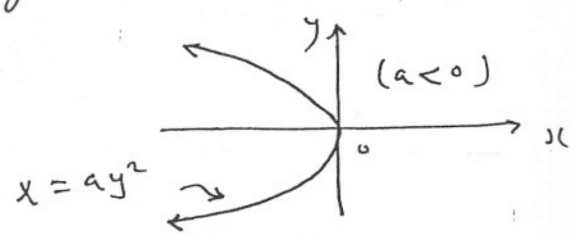
(c) Special parabolas: Two:

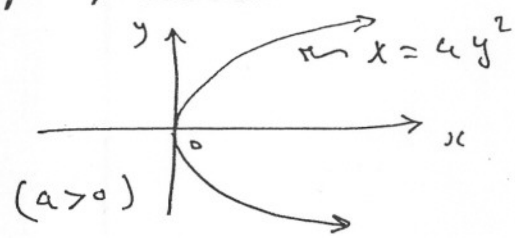
(i) The equation $y = ax^2$, $a \neq 0$ is an equation of a parabola with vertex at the origin and axis along the y-axis. It opens upwards if $\boxed{a > 0}$ and it opens downwards if $\boxed{a < 0}$.





(ii) The equation $x = ay^2$, $a \neq 0$ is an equation of a parabola with vertex at the origin and axis along x-axis. It opens to the right if $\boxed{a > 0}$ and it opens to the left if $\boxed{a < 0}$.





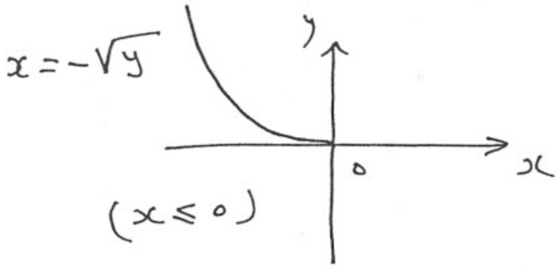
An Important remark:

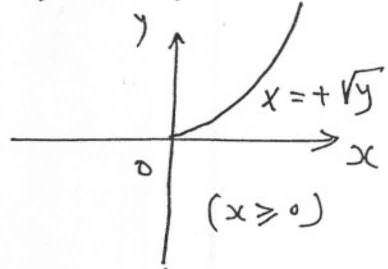
Sometimes an equation of a parabola represents only one of its two branches: upper, lower, right, or left branch but not the entire parabola. For instance:

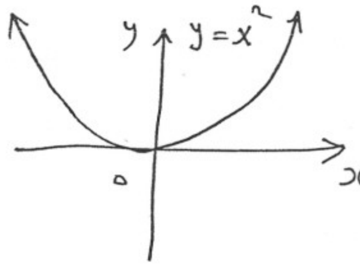
If $y = x^2$, then $x = \pm\sqrt{y}$

$\therefore$ $x = \sqrt{y}$ represents only the right branch (Since $x \geq 0$)

and $x = -\sqrt{y}$ represents only the left branch (Since $x \leq 0$)







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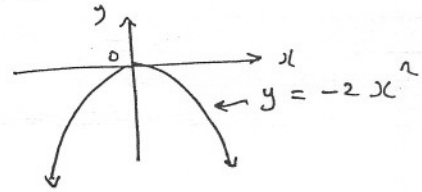
Example: sketch

(a) $y = -2x^2$ (b) $3x = y^2$

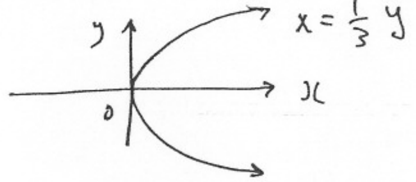
(c) $5x^2 - y = 0$ (d) $3x + 7y^2 = 0$

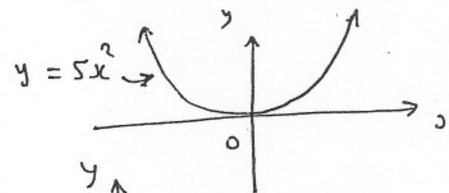
soln. (a) $y = -2x^2$ is a special parabola with vertex at "O", axis along the y-axis and which opens downwards.



(b) $3x = y^2 \Rightarrow x = \frac{1}{3} y^2$

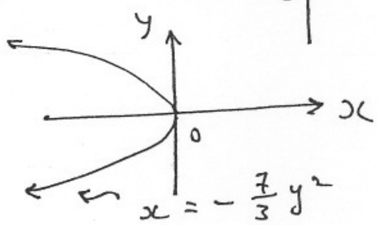
Again: A special parabola with vertex at $(0,0)$ and axis along x-axis. It opens to the right





(c) $5x^2 - y = 0 \Rightarrow y = 5x^2$

(d) $3x + 7y^2 = 0 \Rightarrow x = -\frac{7}{3} y^2$



EX: sketch (a) $y = x^2 - x - 6$

(b) $x = -y^2 - 4y + 5$

solution: (a) For rough sketch, let us determine intercepts:

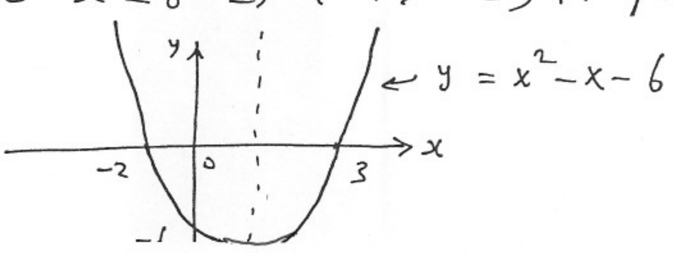
If $x = 0$, $y = -6$; $(0, -6)$ is the y-intercept

If $y = 0$, $0 = x^2 - x - 6 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = 3, -2$

$\therefore$ $(3,0)$, $(-2,0)$ are the x-intercepts.

Further $y = 1x^2 - x - 6 \Rightarrow a = 1 > 0 \Rightarrow$ it opens upwards!

Note: axis // to y-axis



(b) $x = -y^2 - 4y + 5$ is an equation of a parabola with axis parallel to x-axis and which opens to the left.

To graph this parabola more precisely than what we have done in part (a), we have to determine its vertex (h, K).

Indeed $x = -1 \cdot y^2 - 4y + 5 \Rightarrow a = -1, \ b = -4$

$\therefore$ vertex is given by $K = -\frac{b}{2a} = -\frac{(-4)}{2(-1)} = -2 = y$-coordinate

$\therefore$ If we put $y = -2$ into equation of parabola, we get

$$x = -(-2)^2 - 4(-2) + 5 = 9 = h$$

$$\therefore (h, K) = (9, -2)$$

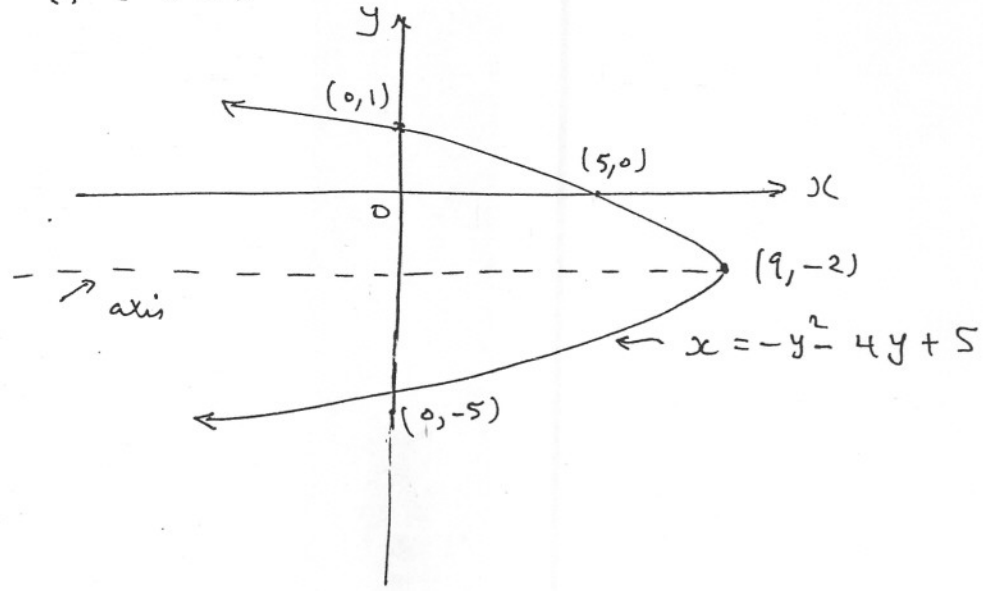
Next: The Intercepts:

let $y = 0$, we get $x = 5$; $(5, 0)$ is the x-intercept

let $x = 0$, we get, $0 = -y^2 - 4y + 5$ or $y^2 + 4y - 5 = 0$

$$\Rightarrow (y-1)(y+5) = 0 \Rightarrow y = 1, -5$$

$\therefore (0, 1)$, and $(0, -5)$ are the y-intercepts.



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EX: sketch

(a) $y = \sqrt{-2x}$ (b) $x = \sqrt{3y}$

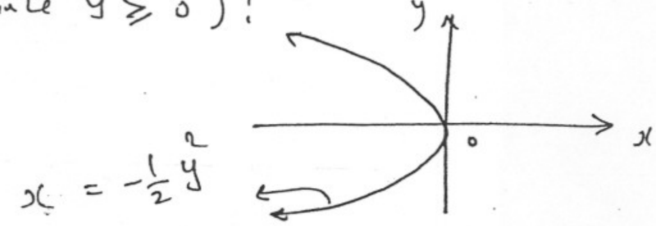
(c) $x = -\sqrt{y}$ (d) $y = -\sqrt{-3x}$

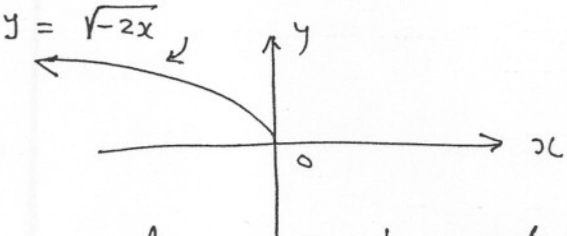
Soln. (a) $y = \sqrt{-2x}$. To identify curve, it helps if we get rid of the square root by squaring each side. We get

$$y^2 = -2x \quad \text{or} \quad \boxed{x = -\tfrac{1}{2}y^2}$$

This is an equation of the "special" parabola with vertex at $(0,0)$, axis along x-axis and which opens to the left.

However $y = \sqrt{-2x}$ is the equation of only its upper branch (since $y \geq 0$)!

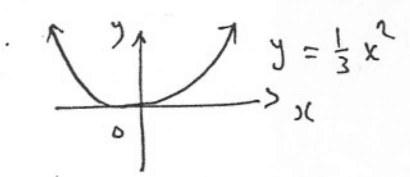


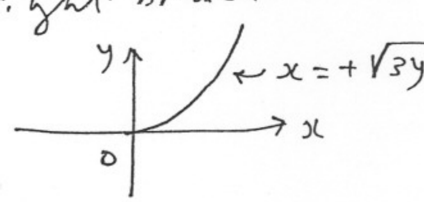


(only upper branch is needed!)

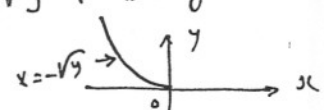
(b) $x = \sqrt{3y} \implies x^2 = 3y$ or $y = \frac{1}{3}x^2$

But since $x \geq 0$, we need only sketch right branch!





(c) $x = -\sqrt{y}$. Do yourself!

Ans: 

only left branch required!

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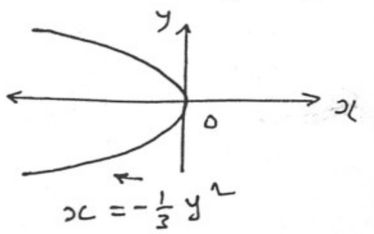
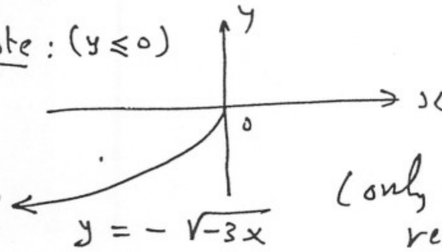
(d) $y = -\sqrt{-3x} \implies y^2 = -3x$ .. parabola opens to the left



Note: $(y \leq 0)$

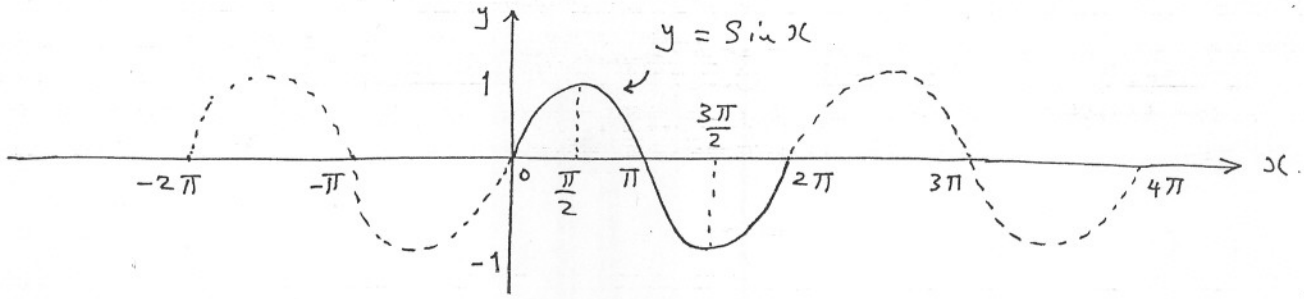


(only lower branch required!)

4 The Sine and Cosine functions (standard):

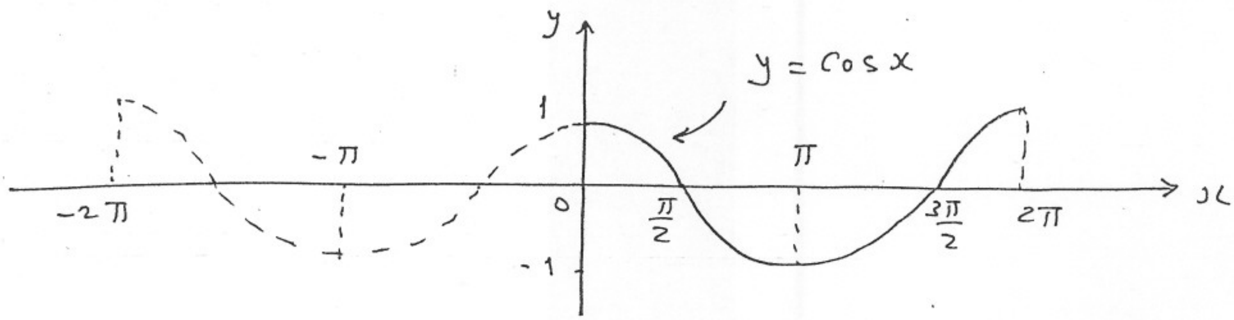
(a) The Sine function: $y = \sin x$, $x \in \mathbb{R}$

The function is periodic with period 2π meaning its graph repeats itself over every interval of length 2π.



(b) The Cosine function: $y = \cos x$, $x \in \mathbb{R}$

The function is periodic with period 2π (Explained above!)



(c) More general Sine/Cosine functions: Let A, b be positive real numbers

(i) $y = A \sin bx$ is periodic with period $p = \frac{2\pi}{b}$ (instead of 2π !!).
The constant real number "A" is called "The amplitude". If $A > 1$, the graph of Sine function is stretched vertically by a factor of "A" whereas if $0 < A < 1$, the Sine function is Compressed vertically by a factor of A.
Note: If A were negative, the graph is further reflected in the x-axis.

(ii) $y = A \cos bx$. Similar discussion holds!

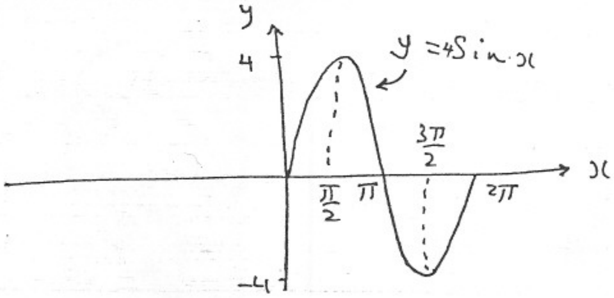
Ex: sketch (1) $y = 4 \sin x$ (2) $y = \cos\left(\frac{x}{3}\right)$

(3) $y = -2\cos(4x)$ (4) $y = \frac{1}{4}\cos(2x)$

Solution : (1) $y = 4 \sin x$

period $p = \frac{2\pi}{1} = 2\pi$ (no change)

amplitude $A = 4 > 1$ (vertical stretch by a factor of 4)



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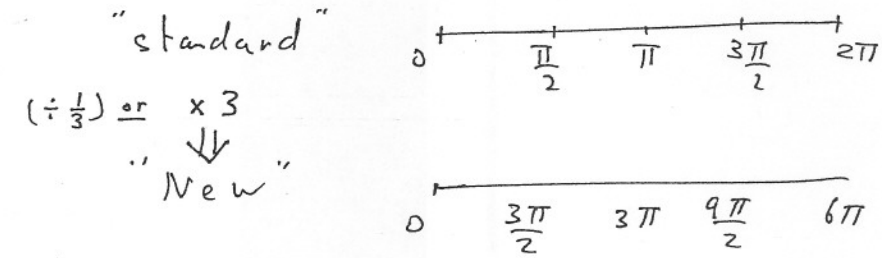
(2) $y = \cos\left(\frac{x}{3}\right)$

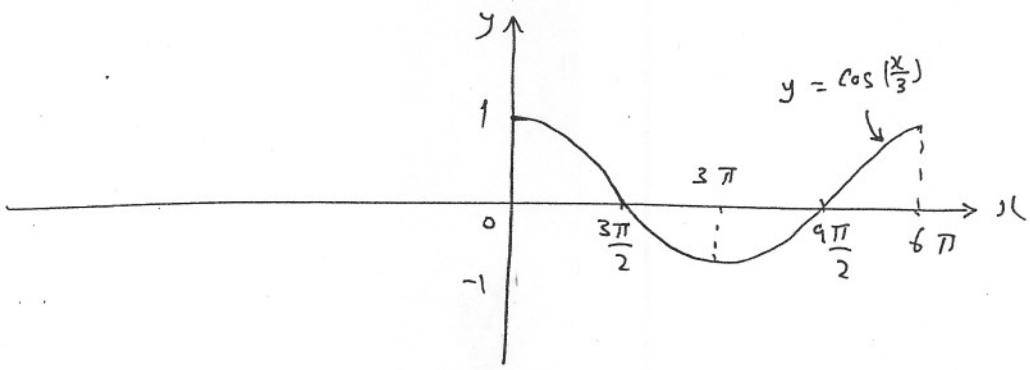
period $p = \frac{2\pi}{\frac{1}{3}} = 6\pi$

amplitude $A = 1$ (no change)

Note : The new x-intercepts are obtained from the standard ones by dividing each by $\frac{1}{3}$ (or equivalently multiplying each by $\frac{3}{1} = 3$).

Note $\boxed{b = \frac{1}{3}}$ $\Rightarrow$ $(\div \frac{1}{3})$ or $\times 3$





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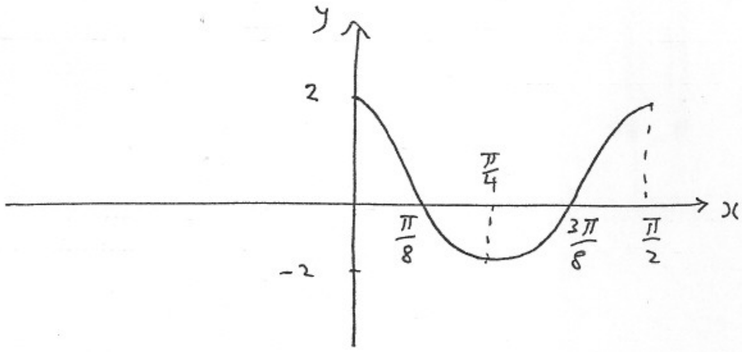
(3) $y = -2\cos(4x)$

Amplitude $A = +2$ (vertical stretch by a factor of 2), followed by a reflection in the x-axis

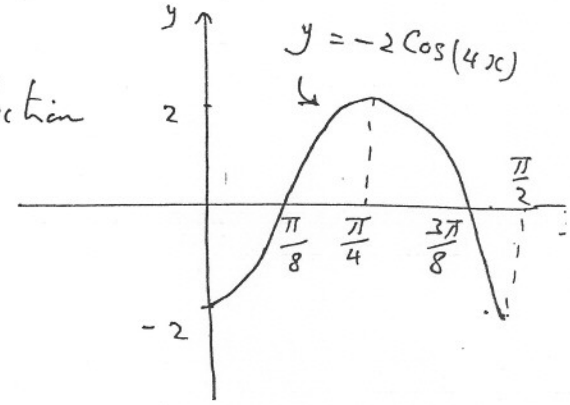
standard: $0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

Note $\boxed{b=4}$ $\Rightarrow$ $\div 4$ $\Downarrow$

new: $0 \quad \frac{\pi}{8} \quad \frac{\pi}{4} \quad \frac{3\pi}{8} \quad \frac{2\pi}{4} = \frac{\pi}{2}$



reflection $\Longrightarrow$



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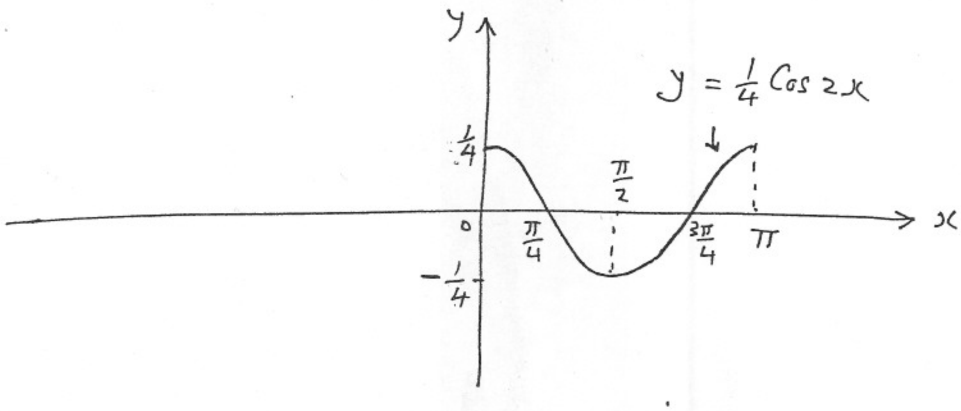
(4) $y = \frac{1}{4}\cos(2x)$

Amplitude $A = \frac{1}{4} < 1$ (vertical compression by a factor of $\frac{1}{4}$)

standard: $0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

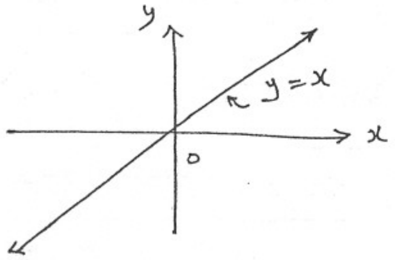
Note: $\boxed{b=2}$ $\Rightarrow$ $\div 2$ $\Downarrow$

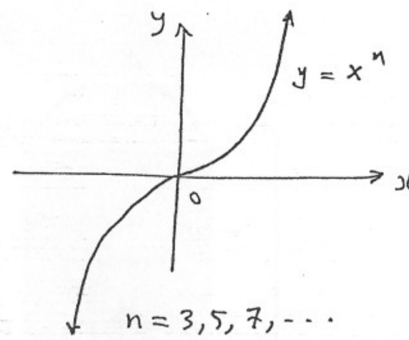
"New": $0 \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4} \quad \frac{2\pi}{2} = \pi$



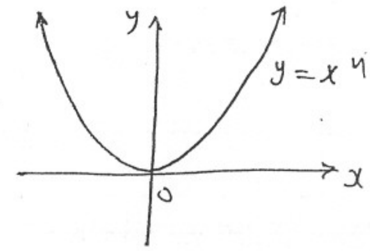
$\boxed{5}$ The power function:

Consider the power function $y = x^n$ for $n = 1, 2, 3, \dots$





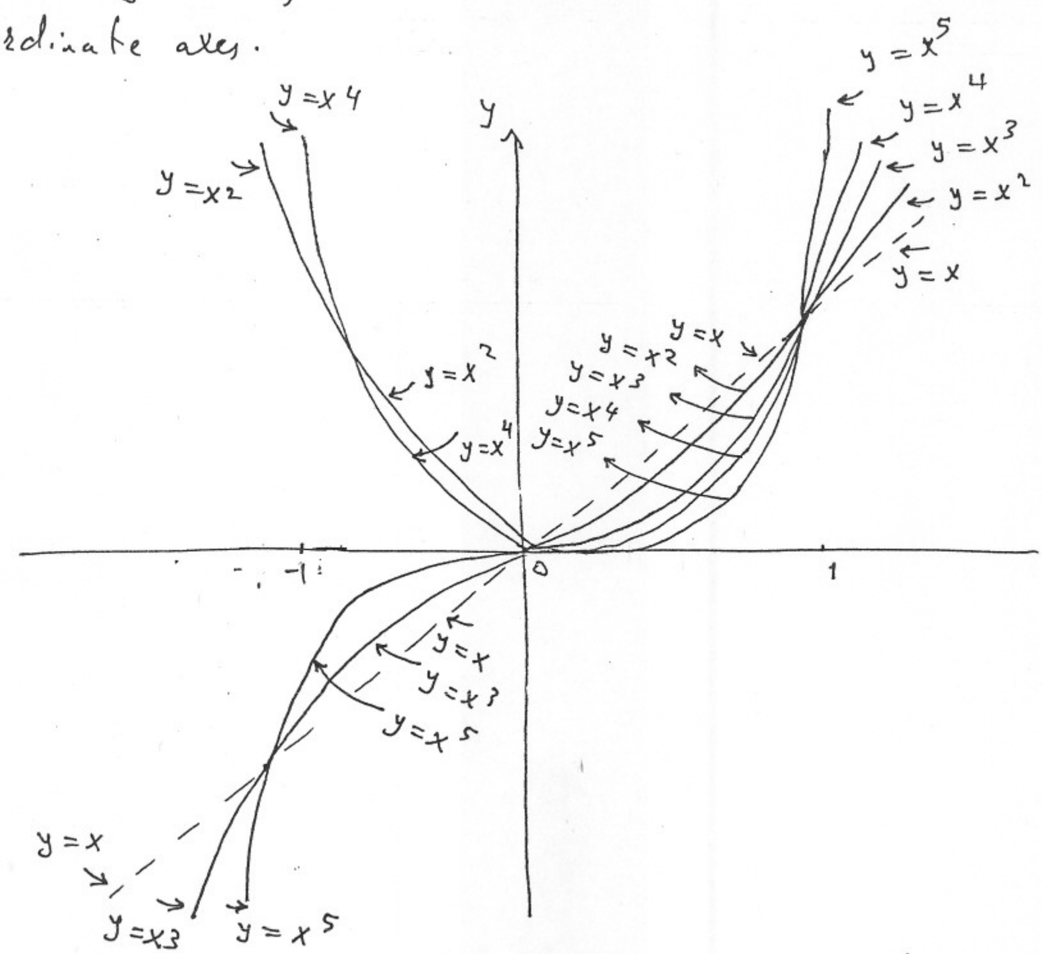
$n = 3, 5, 7, \dots$
(odd)



$n = 2, 4, 6, \dots$
(Even)

Note: The graphs of two power functions may intersect at either $x = 0, 1, -1$ or at $x = 0, 1$ depending on powers involved!

EX: sketch $y = x^n$ for $n = 1, 2, 3, 4,$ and 5 on the same set of coordinate axes.



Note: The larger the power "n", the narrower the graph becomes!

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$\boxed{\text{END}}$