# 1 January 11, 2016

## 1.1 Vector Notation

Let $P$ be a point in $\mathbb{R}^3$.

**Definition.** A directed line from the origin to $P$ is a **position vector**, and may be denoted by $\overrightarrow{OP}$, or as a single letter $\vec{u}$.

The origin $O$ is referred to as the initial point, and the point $P$ as the terminal point of $\vec{u}$. Vector $\vec{u}$ is represented by the terminal point $P$, and is written as $\vec{u} = (x, y, z)$.

*Remark.* Given an ordered tuple $(x, y, z)$ in $\mathbb{R}^3$, we represent the point as the set of coordinates, and represent the position vector by its components. Whereas the $x, y, z$ coordinates indicate a point, the $x, y, z$ components indicate a directed line from the origin to those coordinates.

**Definition.** A **zero vector** is the vector with all components equal to 0. Ie: $\overrightarrow{0} = (0, 0, 0 \dots)$.

When each component of a vector corresponds to the components of another vector, then these two vectors are equal.

## 1.2 Vector Arithmetic

1. The sum of vectors is the sum of the individual corresponding components.
2. The difference of vectors is the difference between the components of the first vector with the components of the second vector.
3. Scalar multiplication is the multiplication of each component of a vector with a scalar real number $k$.

*Remark.* We write multiplication by a scalar as $k\vec{u}$. $\vec{u}\vec{v}$ and $\frac{\vec{u}}{\vec{v}}$ do not exist (There is no ordinary piecewise multiplication and division between vectors.

## 1.3 Geometric Interpretation of Vectors

- Addition: Place all vectors from head to tail. The sum is the resultant vector.
- Scalar Multiplication: Place $k$ number of the vectors from head to tail.

# 2 January 13, 2016

## 2.1 Interpretation of Vectors

Let $\vec{u}$ be a non-zero vector and $k$ be a non-zero scalar. $k\vec{u}$ is a vector parallel to $\vec{u}$. If $k > 0$, then the resultant vector is in the same direction. If $k < 0$, then the resultant vector is in the opposite direction.

**Definition.** Let $P$ and $Q$ be points $(x, y, z)$ and $(x', y', z')$ respectively. The directed line from $P$ to $Q$ is a **general vector** and denoted by $\overrightarrow{PQ}$ or $\vec{u}$.

We note that,

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$

rearranging, we get:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

## 2.2 Normal Vector

**Definition.** Let $\vec{u}$ be the vector $(x, y, z)$ in $\mathbb{R}^3$. The norm (or magnitude) of $\vec{u}$ is denoted and defined by:

$$\|\vec{u}\| = \sqrt{x^2 + y^2 + z^2}$$

Geometrically, this indicates the length of the vector.

*Remark.* It can be noted that $\|\vec{u}\| \in \mathbb{R}^3$, and $\|\vec{u}\| \geq 0$ (with $\|\vec{u}\| = 0$ when $\vec{u}$ is $\overrightarrow{0}$). It can also be found that for any scalar $k$, $\|k\vec{u}\| = |k|\|\vec{u}\|$.

Ie: Let $\vec{u} = (4, -2, 4)$. Find $\|\vec{u}\|$. $\|\vec{u}\| = 6$

**Definition.** A **unit vector** is a vector with a magnitude of length 1. It may be denoted by $\vec{n}$ to distinguish from other vectors. Since it has a magnitude of 1, then $\|\vec{n}\| = 1$.

Ie: Show whether or not $\left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ is a unit vector. This is a unit vector, since the magnitude is 1.

## 2.3 Properties of the Unit Vector

Given a non-zero vector $\vec{u}$, we note the following properties:

1. $\vec{n_1} = \frac{\vec{u}}{\|\vec{u}\|}$, where $\vec{n_1}$ is the unit vector in the direction of $\vec{u}$.

2. $\vec{n_2} = -\frac{\vec{u}}{\|\vec{u}\|}$, where $\vec{n_2}$ is the unit vector in the opposite direction of $\vec{u}$.

Ie: Let $\vec{a}$ be the vector $(4, 0, -3)$. Then the unit vector in the opposite direction of $\vec{a}$ is $(-\frac{4}{5}, 0, \frac{3}{5})$.

**Definition.** The **standard unit vectors** are denoted and defined by $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$. Note that we can write the vector $\vec{u} = (x, y, z)$ as $x\vec{i} + y\vec{j} + z\vec{k}$ since $\vec{u} = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$.

For instance, the vector $(2, -3, 5)$ is the same as the vector expressed as $2\vec{i}-3\vec{j}+\vec{5}$. Likewise, the vector $2\vec{i} - 7\vec{k}$ is the same as $(2, 0, -7)$.

# 3 January 15, 2016

## 3.1 Vector Operations

**Definition.** Let $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$. The **dot product**, denoted and defined by $\vec{u} \cdot \vec{v}$, is the real scalar given by summing the product of corresponding components of the two vectors, and is equal to $x_1x_2 + y_1y_2 + z_1z_2$.

The following are properties related to the dot product:

1. *$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. *$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$
3. *$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = \vec{w} \cdot (\vec{u} + \vec{v})$
4. $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$

*Note that the starred properties are the same for real numbers.

*Remark.* Let $\vec{u}$ and $\vec{v}$ be non-zero vectors, and let $\theta$ be the angle between $\vec{u}$ and $\vec{v}$. It can be shown that,

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

1. If $\theta = 0$, then $\vec{u}$ and $\vec{v}$ are parallel and in the same direction.
2. If $\theta = \pi$, then $\vec{u}$ and $\vec{v}$ are parallel and in the opposite direction.
3. If $\theta = \frac{\pi}{2}$, then $\vec{u}$ and $\vec{v}$ are **orthogonal** (perpendicular).

*Remark.* Two vectors are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$.

**Definition.** Let $\vec{u}$ and $\vec{v}$ be vectors, specifically in $\mathbb{R}^3$. The **cross product** of the two vectors is denoted by $\vec{u} \times \vec{v}$, and defined as

$$\vec{u} \times \vec{v} = \left( + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)$$

*Remark.* Note that this does not need to be memorized, as it simply derives from the determinant of the two vectors.

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
3. $\vec{w} \times (\vec{u} + \vec{v}) = \vec{w} \times \vec{u} + \vec{w} \times \vec{v}$
4. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$
5. $\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = 0$

*Remark.* let $\vec{u}$ and $\vec{v}$ be non-zero vectors in $\mathbb{R}^3$. The cross product of $\vec{u}$ and $\vec{v}$ is the vector $\vec{w}$ which is orthogonal to both $\vec{u}$ and $\vec{v}$.

Ie: Find the vector that is orthogonal to the vectors $(4, 2, -9)$ and $(0, 2, 3)$. Recall that a vector orthogonal to $\vec{u}$ and $\vec{v}$ is the cross product of both vectors. The result is $(24, -12, 8)$. Note that the dot product of this vector with $\vec{u}$ or $\vec{v}$ is 0 (This can be used to check for accuracy).

### 3.2 Vector Function

**Definition.** A vector function of a single variable. a vector function $\vec{v}$ is a rule that assigns to each permissible real number $t$ one and only one ordered tuple, which we write as

$$\begin{aligned}\vec{v}(t) &= ((x(t), (y(t), (z(t)) \\ &= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\end{aligned}$$

Let $\vec{v} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $t \in I$. Assuming that $x(t)$, $y(t)$, and $z(t)$ are continuous for all $t$ in the interval $I$, then geometrically, $\vec{v}(t)$ is the position of a moving object at time $t$.

## 4 January 18, 2016

### 4.1 Vector Function

Given the vector function $\vec{r}(t) = x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$, and assuming that the function is continuous, then $\vec{r}(t)$ may be thought of as the position of a particle $P$ moving in three space at time $t$.

Note that as time $t$ varies, the terminal point $P$ traces a curve $C$, which is the path of the particle. Curve $C$ is said to be given parametrically by the vector function $\vec{r}(t)$, where $t \in \mathbb{Z}$. Conversely, $C$ is given parametrically by three scalar equations:

$$C = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad t \in \mathbb{Z}$$

**Definition.** Let $C$ be a space curve given by $\vec{r}(t)$, where $t \in [a, b]$. Define the initial and terminal points of curve $C$ respectively by $P = \vec{r}(a)$, and $Q = \vec{r}(b)$. The **orientation** of $C$ is the direction from $P$ to $Q$, indicated by 1 or 2 arrowheads.

### 4.2 Properties

1. Position: By definition, position is the vector function $\vec{r}(t)$.

2. Velocity: Denoted by $\vec{v}(t)$. By definition, average velocity is $\frac{\vec{d}}{t}$.

   Let $P$ and $Q$ denote the position of an object at time $t$ and $t + \Delta t$ respectively. Then, average $\vec{v} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$. We note that $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ implies that $\vec{r}(t) + \overrightarrow{PQ} = \vec{r}(t + \Delta t)$. Rearranging this gives $\overrightarrow{PQ} = \vec{r}(t + \Delta t) - \vec{r}(t)$.

   Instantaneous velocity is found when $\Delta t$ approaches 0. So, $\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$, which is $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$. Thus, velocity can be found by taking the derivative of position.

   *Remark.* As $\Delta t$ approaches 0, point $Q$ approaches $P$, so $\overrightarrow{PQ}$ approaches the tangent line of curve $C$ at $P$. In other words, the velocity vector $\vec{v}(t)$ is tangent to curve $C$ at point $P$.

3. Speed: $v$ or $\|\vec{v}\|$ is the magnitude of the vector.

4. Acceleration: By definition, acceleration is the rate of change of velocity, so $\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$.

5. Distance (or Arc Length): Distance is denoted by $L$. Let $P$ and $Q$ be positions of objects moving along a space curve at time $t = a$ and $t = b$ respectively. To determine the distance from $P$ to $Q$, consider that speed is equal to the rate of change in distance.

$$\begin{aligned}
\|\vec{v}\| &= \frac{\mathrm{d}L}{\mathrm{d}t} \\
\mathrm{d}L &= \|\vec{v}\|\mathrm{d}t \\
L &= \int\limits_{t=a}^{t=b} \|\vec{v}\|\mathrm{d}t \\
&= \int\limits_{a}^{b} (speed)(time)
\end{aligned}$$

*Remark.* Note that $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\right) = \frac{\mathrm{d}x}{\mathrm{d}t}\vec{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\vec{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\vec{k}$.

### 4.3 Summary

1. Position: $\vec{r}(t)$
2. Velocity: $\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$
3. Acceleration: $\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$
4. Speed: $v(t) = \|\vec{v}\|$
5. Distance (Arc Length): $L = \int\limits_{a}^{b} \|\vec{v}\|\mathrm{d}t$

**Example.** *The position of a particle moving in three space is given by the vector equation $\vec{r}(t) = (1-2t)\vec{i} + \frac{1}{3}t^3\vec{j} + t^2\vec{k}$, where $t \geq 0$. Find the velocity, acceleration, and speed at time $t = 3$. Find the distance from point $P = (1,0,0)$ to the point $Q = (-5,9,9)$. Find the vector equation of a straight line tangent to the curve $C$ at $Q = (-5,9,9)$.*

## 5 January 20, 2016

### 5.1 Solution to Vector Problems

For the first question, we take the derivative of to get $\vec{v} = \left(-2, t^2, 2t\right)$, and $\vec{a} = (0, 2t, 2)$. Substituting $t = 3$, we get $\vec{v} = (-2, 9, 6)$ and $\vec{a} = (0, 6, 2)$. To get speed, we note $v = \sqrt{(-2)^2 + (t^2)^2 + (2t)^2}$, which can be reduced to $v = t^2 + 2$. Substituting in $t = 3$, we get 11.

To solve the second question, we note that $P$ occurs at time 0, and $Q$ at time 3. Using these values for $t$, we get,

$$
\begin{aligned}
L &= \int_0^3 \|\vec{v}\| \mathrm{d}t \\
&= \int_0^3 (t^2+2)\mathrm{d}t \\
&= \frac{1}{3}t^3 + 2t\Big|_0^3 \\
&= \left(\frac{1}{3}(3)^3 + 2(3)\right) - \left(\frac{1}{3}(0)^3 + 2(0)\right) \\
&= (9+6) - (0+0) \\
&= 15
\end{aligned}
$$

For the third question, we recall that the tangent line to the curve is in the direction of the velocity $\vec{v} = (-2, t^2, 2t)$. At $t = 3$, this means that,

$$
\begin{aligned}
\vec{r}(t) &= \vec{r_0} + s\vec{v} \\
&= (-5, 9, 9) + s(-2, 9, 6), \quad s \in \mathbb{R}
\end{aligned}
$$

**Example.** *Find the arc length of the space curve given by* $\vec{r}(t) = \left(6t^2+4, 9t^4-5, 9t^6\right)$, *where* $0 \leq t \leq 1$. *Recall that arc length is equal to distance.*

To determine $\vec{v}$, we take the derivative of $\vec{r}$ to get:

$$
\begin{aligned}
\vec{v}(t) &= \vec{r}(t)' \\
&= \left(\left(6t^2+4\right)', \left(9t^4-5\right)', (9t^6)'\right) \\
&= \left(12t, 36t^3, 54t^5\right)
\end{aligned}
$$

Let us simplify $\vec{v}$ by factoring out $6t$:

$$
\begin{aligned}
\vec{v}(t) &= \left(12t, 36t^3, 54t^5\right) \\
&= 6t\left(2, 6t^2, 9t^4\right)
\end{aligned}
$$

Thus,

$$\begin{aligned}\|\vec{v}(t)\| &= |6t|\sqrt{(2)^2 + (6t^2)^2 + (9t^4)^2)} \\ &= |6t|\sqrt{4 + 36t^4 + 81t^8} \\ &= |6t|\sqrt{(2 + 9t^4)^2} \\ &= |6t|\left(2 + 9t^4\right) \\ &= 12t + 54t^5\end{aligned}$$

Now to determine the arc length, we use the formula given for distance,

$$\int\limits_a^b \|\vec{v}(t)\|\mathrm{d}t$$

Substituting into the equation, we get:

$$\begin{aligned}\int\limits_{t=0}^{t=1} (12t + 54t^5)\mathrm{d}t =& \ 6t^2 + 9t^6\Big|_0^1 \\ &= \left(6(1)^2 + 9(1)^6\right) - \left(6(0)^2 + 9(0)^6\right) \\ &= (6 + 9) - (0 + 0) \\ &= 15\end{aligned}$$

Thus, the arc length is 15.

# 6 January 22, 2016

## 6.1 Additional Exercises

**Example.** *The acceleration of a moving particle in three space is given by*

$$\vec{a}(t) = -10e^{-2t}\vec{i} + \sin(t)\vec{j} + \frac{1}{1+t}\vec{k} \quad ,t \leq 0$$

*Find an expression for velocity given that the initial velocity is* $2\vec{i} + 3\vec{j} - 7\vec{k}$.

Recall that $\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$. To find velocity, we integrate acceleration.

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)\mathrm{d}t \\ &= \int \left(-10e^{-2t}, \sin(t), \frac{1}{1+t}\right)\mathrm{d}t \\ &= \left(5e^{-2t} + C, -\cos(t) + C_2, \ln(1+t) + C_3\right)\end{aligned}$$

Now, we set $t = 0$, since we are given the initial velocity. At $t = 0$, $\vec{v}(0) = (5 + C_1, -1 + C_2, 0 + C_3)$. Therefore, $C_1 = -3$, $C_2 = 4$, and $C_3 = -7$. Substituting these values back into the equation for $\vec{v}$, we get $\vec{v}(t) = \left(5e^{-2t} - 3, 4 - \cos(t), \ln(1+t) - 7\right)$.

**Example.** *The position of a moving particle in three space is given by*

$$\vec{r}(t) = (t-12)^2\vec{i} + \frac{4\sqrt{7}}{3}(t-12)^{\frac{3}{2}}\vec{j} + 7\sqrt{6}(t+9)\vec{k}$$

*When will the speed of the particle be 21 units?*

First, observe that $\vec{r}(t)$ contains the quantity $(t-12)^{\frac{3}{2}} = \sqrt{(t-12)}^3$. Clearly, $\vec{r}(t)$ is defined only if $t \geq 12$. The speed $v = \|\vec{v}\|$, so now we need to determine the velocity.

$$\begin{aligned}
\vec{v}(t) &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left((t-12)^2, \frac{4\sqrt{7}}{3}(t-12)^{\frac{3}{2}}, 7\sqrt{6}(t+9)\right) \\
&= \left(2(t-12), \frac{4\sqrt{7}}{3} * \frac{3}{2}(t-12)^{\frac{1}{2}}, 7\sqrt{6}\right) \\
&= \left(2(t-12), 2\sqrt{7}(t-12)^{\frac{1}{2}}, 7\sqrt{6}\right)
\end{aligned}$$

Now, let $u = t - 12$, so the equation becomes $\vec{v}(t) = \left(2u, 2\sqrt{7}\sqrt{u}, 7\sqrt{6}\right)$. Therefore,

$$\begin{aligned}
v &= \|\vec{v}\| \\
&= \sqrt{(2u)^2 + \left(2\sqrt{7}\sqrt{u}\right)^2 + \left(7\sqrt{6}\right)^2} \\
&= \sqrt{4u^2 + 28u + 49(6)}
\end{aligned}$$

We then equate this with 21, which is the speed given by the problem description. Now, we square both sides, and utilize the quadratic formula to determine $u = \frac{7}{2}$ and $-\frac{21}{2}$. By converting back from $u$ to $t$, we get $t = \frac{31}{2}$ and $t = \frac{3}{2}$. However, because we have restricted $t \geq 12$, we reject $\frac{3}{2}$ as a possible solution.

## 6.2 Special Parametric Curves in Two and Three Space

Let $C$ be a curve in $\mathbb{R}^2$ given parametrically by the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, where $t \in \mathbb{Z}$.

**Definition.** By **cartesian equation** of curve $C$, we mean a direct relationship between $x$ and $y$. This relationship can be easily obtained by simply eliminating $t$ among $x(t)$ and $y(t)$.

**Example.** *Let $C$ be the parametric curve given by $\vec{r}(t) = (t-1)\vec{i} + (t^2+2)\vec{j}$, where $t \in \mathbb{R}$. Find the cartesian equation of $C$. Identify and sketch.*

$\vec{r}(t) = (t-1, t^2+2) = (x(t), y(t))$. So $x = t-1$, and $y = t^2+2$. Eliminate $t$ among the first result. Now substitute $t = x+1$ into second equation to get $y = (x+1)^2+2$. This is the same as $y - 2 = (x+1)^2$, which can be re-written as $y = x^2+2x+3$. This is a parabola with vertex at point $(-1, 2)$ and which open upwards.

### 6.3 Standard Parametric Curves in $\mathbb{R}^2$

The vector equation of a **straight line** segment joining the points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is given by $\vec{r}(t) = \vec{r_0} + t\vec{v}$.

$$
\begin{aligned}
PQ &= Q - P \\
&= (x_2, y_2) - (x_1, y_1) \\
&= (x_2 - x_1, y_2 - y_1).
\end{aligned}
$$

Therefore, $\vec{r}(t) = (x_1, y_1) + t(x_2 - x_1, y_2 - y_1)$, where $0 \leq t \leq 1$. This is because when $t$ is between 0 and 1, the line segment is between P and Q.

## 7 January 25, 2016

### 7.1 Standard Parametric Curves in $\mathbb{R}^2$ Cont'd

The vector equation for an **ellipse** with center at $(\alpha, \beta)$ and with semi-axes of length $a, b$ is given by

$$\vec{r}(t) = (\alpha + a\cos(t))\,\vec{i} + (\beta + b\sin(t))\,\vec{j} \quad , t \in [0, 2\pi]$$

Let us justify, that $x = \alpha + a\cos(t)$, and $y = \beta + b\sin(t)$. To find the cartesian equation, we simply eliminate $t$ among the two above equations. We know that $\frac{x-\alpha}{a} = \cos(t)$, and $\frac{y-\beta}{b} = \sin(t)$. But $\cos^2 + \sin^2 = 1$, so substituting presents the equation of the ellipse centered at $(\alpha, \beta)$, with a semi-axes of length $(a, b)$

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$$

The vector equation of a **circle** centered at $(\alpha, \beta)$ that has a radius of $a$ is given by

$$\vec{r}(t) = (\alpha + a\cos(t))\vec{i} + (\beta + a\sin(t))\vec{j} \quad ,0 \leq t \leq 2\pi$$

Note that this is a special case of the ellipse where $a = b$. Additionally, the cartesian equation becomes

$$(x-\alpha)^2 + (y-\beta)^2 = a^2$$

The vector equation of the right hand branch of the **hyperbola** centered at $(\alpha, \beta)$ that has a semi-axes of length $a, b$ is given by

$$\vec{r}(t) = (\alpha + a\cosh(t))\vec{i} + (\beta + b\sinh(t))\vec{j} \quad ,t \in \mathbb{R}$$

Let us justify, by using $\cosh^2 - \sinh^2 = 1$, that the cartesian equation becomes

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1$$

Note that this is only a representation of the right hand branch, since $x = \alpha + a\cosh(t)$ where $x \geq \alpha + a$ is only the right hand side.

## 7.2 Standard Parametric Curves in $\mathbb{R}^3$

The **straight line** segment from point $P = (x_{,1}, y_1, z_1)$ towards the direction of point $Q = (x_2, y_2, z_2)$, is given by the equation $\vec{r}(t) = \vec{r_0} + t\vec{v}$. This becomes

$$\vec{r}(t) = (x_1, y_1, z_1) + t(x_2 - x_1, y_2 - y_1, z_2 - z_1) \quad ,0 \leq t \leq 1$$

Note that at $t = 0$, we obtain the initial point, and when $t = 1$ we obtain the terminal point. Also remember to record the range of $t$ in calculations.

The **helix** is a wire wrapped around a cylinder. The vector equation of a helix is given by

$$\vec{r}(t) = a\cos(t)\vec{i} + a\sin(t)\vec{j} + bt\vec{k} \quad ,t \in \mathbb{R}$$

The Helix plays an important role in DNA analysis.

**Definition.** A **surface** is any equation that consists of one, two, or three variables in $\mathbb{R}^3$.

**Example.** *$x = 3$ is the equation of a plane in $\mathbb{R}^3$. $x^2 + y^2 = 4$ is a cylinder in $\mathbb{R}^3$. $z = x^2 + y^2$ is a surface called a paraboloid in $\mathbb{R}^3$.*

*Remark.* The intersection of any two surfaces $S_1$ and $S_2$ results in a space curve $C$.

**Example.** *In each case, identify the plane curve.* $\vec{r}(t) = (2 + 4\cos(t))\vec{i} + (-7 + 4\sin(t))\vec{j}$*, where* $0 \leq t \leq 1$*.* $\vec{r}(t) = (-1 + 3\cos(t), 5 + 8\sin(t))$*, where* $0 \leq t \leq 2\pi$*.* $\vec{r}(t) = 2\cosh^2(t)\vec{i} + \sinh(t)\vec{j}$*, where* $t \in \mathbb{R}$*.*

The first equation denotes the equation of a circle centered at (2,-7) and has a radius of 4. The second equation is an ellipse centered at (-1, 5) and has a semi-axis of $a = 3$, and $b = 8$. The third equation is not a standard parametric curve. Here, $x = 2\cosh^2(t)$, and $y = \sinh(t)$. Therefore, $\frac{x}{2} = \cosh^2(t)$, and $y = \sinh(t)$. But $\cosh^2(t) - \sinh^2(t) = 1$. So, $y^2 = \frac{x}{2} - 1$. This is the equation of a parabola with a vertex at (2, 0) which opens to the right.

# 8 January 27, 2016

## 8.1 Additional Exercises

**Example.** *Find a vector equation for the curve of intersection of the two surfaces,* $4x^2 + y^2 = 16$*, and* $3x - 2y + z = 7$*.*

To find a parametrization of a curve of intersection between two surfaces, we begin with the equation containing only two variables. We then view that equation in $\mathbb{R}^2$.

$$\frac{4x^2}{16} + \frac{y^2}{16} = \frac{16}{16}$$

By dividing both sides by 16, we get

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

This is the equation of an ellipse centered at $(\alpha, \beta) = (0, 0)$, with semi-axes of length is 2, 4.

Its parametric equation is thus given by $x = \alpha + a\cos(t)$, and $y = \beta + b\sin(t)$, where $t \in [0, 2\pi]$. Substituting values, this becomes $x = 2\cos(t)$, and $y = 4\sin(t)$. To find the value of z, consider the second equation to substitute for z. The vector equation of curve is thus given by

$$\begin{aligned}\vec{r}(t) &= x\vec{i} + y\vec{j} + z\vec{k}\\ &= 2\cos(t)\vec{i} + 4\sin(t)\vec{j} + (7 - 6\cos(t) + 8\sin(t))\vec{k} \quad , t \in [0, 2\pi]\end{aligned}$$

**Example.** *Find the parametric equation of the curve of intersection of the surface* $z = 4x^2 + y^2$ *with the plane of* $8x + 2y + z = 31$

Let us first create an equation with two variables by removing the $z$. The equation then becomes $(4x^2+8x)+(y^2+2y)=31$. We need to view this equation in $\mathbb{R}^2$. We then complete the square. Thus, $4x^2+8x=4(x^2+2x)=4(x+1)^2-4$, and for $y$, it becomes $(y+1)^2-1$. The entire equation becomes:

$$\begin{aligned} 4(x+1)^2-4+(y+1)^2-1&=31\\ 4(x+1)^2+(y+1)^2&=36\\ \frac{4(x+1)^2}{36}+\frac{(y+1)^2}{36}&=1\\ \frac{(x+1)^2}{9}+\frac{(y+1)^2}{36}&=1 \end{aligned}$$

This is the equation of an ellipse with centre (-1,-1) and with semi-axes lengths of 3 and 6. The parametric equations are therefore: $x=-1+3\cos(t)$, $y=-1+6\sin(t)$, and $z=41-24\cos(t)-12\sin(t)$, $0\le t\le 2\pi$. The parametric equation is therefore

$$\begin{cases} x=-1+3\cos(t) \\ y=-1+6\sin(t) & \qquad 0\le t\le 2\pi \\ z=41-24\cos(t)-12\sin(t) \end{cases}$$

## 8.2 Motion Involving a Varying Mass

*Application to Rockets*

A rocket moves forward by the backwards expulsion of the fuel.

- $M$: The initial mass.
- $m=m(t)$: The mass of the rocket at time $t$.
- $m+\Delta m$: The mass of the rocket at time $t+\Delta t$.
  Note that $\Delta m<0$, hence $-\Delta m>0$.
- $\vec{v}(t)$: The velocity of the rocket at time $t$
- $\vec{v}+\Delta v$: The velocity of the rocket at time $t+\Delta t$.
- $\vec{v_e}$: The velocity of the ejected fuel in relation to the rocket (assume this is constant). Hence, the velocity of the gas relative to the Earth is $\vec{v}+\vec{v_e}$.
- $P(t)$: The momentum, which is equal to $(mass)(velocity)$.

Recall Newton's Second Law of Motion, which states that the rate of change of momentum = The net force acting on an object. That is

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\vec{F}$$

Or, in other words

$$\begin{aligned}\vec{F} &= \frac{\mathrm{d}P}{\mathrm{d}t} \\ &= \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t}\end{aligned}$$

# 9 January 29, 2016

## 9.1 Motion Involving Varying Mass Cont'd

$$\begin{aligned}P(t) &= \text{Initial Momentum} \\ &= m\vec{v}\end{aligned}$$

$$\begin{aligned}P(t+\Delta t) &= \text{Momentum of Rocket} + \text{Momentum of Gas} \\ &= (m+\Delta m)(\vec{v}+\Delta\vec{v}) + (-\Delta m)(\vec{v_e}) \\ &= m\vec{v} + m\Delta\vec{v} + \vec{v}\Delta m + \Delta m\Delta\vec{v} - \Delta m\vec{v_e}\end{aligned}$$

$$\begin{aligned}\Delta P &= P(t+\Delta t) - P(t) \\ &= m\Delta\vec{v} - \vec{v_e}\Delta m\end{aligned}$$

$$lim_{\Delta t\to 0}\frac{\mathrm{d}P}{\mathrm{d}t} = lim_{\Delta t\to 0} m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - \vec{v_e}\frac{\mathrm{d}m}{\mathrm{d}t} = \vec{F}$$

Assume that there are no forces acting on the rocket. That is, $\vec{F} = 0$. Therefore

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - \vec{v_e}\frac{\mathrm{d}m}{\mathrm{d}t} = 0$$

For motion in a straight line, we write $\vec{0} = 0\vec{i}$. Where $v_1$ and $v_2$ are the speeds of the rocket and ejected gases respectively. We get

$$\begin{aligned}m\frac{\mathrm{d}v}{\mathrm{d}t} - v_e\frac{\mathrm{d}m}{\mathrm{d}t} &= 0 \\ v = -v_e \ln m + C\end{aligned}$$

Assume that the rocket starts at rest. It follows that

$$v = -v_e\ln(m) + v_e\ln(M) = v_e\ln\left(\frac{M}{m}\right)$$

Assume $v_e$ is constant. Assume that the gas is ejected at a constant rate, where $\alpha > 0$ per unit of time. Therefore, $m(t) = M - \alpha t$.

### 9.2 Summary

In conclusion

$$v = v_e \ln\left(\frac{M}{m}\right), \quad m(t) = M - \alpha t$$

Where $v$ is the speed of the rocket at time $t$, $v_e$ is the speed of the ejected gases (constant), $\alpha$ is the rate of ejected gases (constant), $M$ is the total initial mass of the rocket, and $m(t)$ is the mass of the rocket at time $t$.

*Remark.* Note that we assume that no force acts on the rocket, and the rocket starts from rest.

**Example.** *A rocket with a mass of 52000 kg, which includes 39000 of a fuel mixture, is fired vertically upwards in a vacuum (that is, free space where the gravitational field is negligible). Assume that gases are ejected at a constant rate of 1300 kg/s, and with a constant velocity of magnitude 500 m/s. If the rocket starts from rest, find its speed at times 15, 20, 30, and 35 seconds.*

Recall that $v = v_0 \ln\left(\frac{M}{m(t)}\right) = v_0 \ln\left(\frac{M}{M-\alpha t}\right)$. We note that $M = 52000kg$, $v_0 = 500m/s$, and $\alpha = 1300kg/s$. We get 235m/s, 347 m/s, and 693 m/s. Observe that all of the fuel is used up when 39000 = 1300t = 30 seconds. Therefore, the rocket will maintain a constant speed of 693 m/s for all $t \geq 30$

**Example.** *A rocket moves forward in a straight line only under the forces of the ejected gas. Assume that the gases are ejected at a constant rate of 1000 kg/s, and at a constant velocity with magnitude 400 m/s. Assume that the rocket starts from rest. Let M denote the total initial mass of the rocket, and assume that it starts from rest. What percentage of total initial mass M would the rocket have to burn in order to accelerate to a speed of 800 m/s? What is the speed of the rocket if 40% of its initial mass is ejected as fuel in the burning process.*

$v_e = 400m/s$, $\alpha = 1000kg/s$, and $M = 1000t$. The mass burned is the difference in the initial mass and the mass at time $t$.

Let $P$ be the rate of amount of fuel burnt. This is equal to $\frac{M-m}{M}$, or $1 - \frac{m}{M}$. Therefore, $\ln\left(\frac{M}{m}\right) = e$. $P = 1 - \frac{m}{M} = 1 - \frac{1}{e^2}$. Multiply by 100%. The answer is therefore 86.5%. To answer the second question, we determine the speed. We know that $m = M - 0.4M$. We get $v = 400\ln\left(\frac{1}{.6}\right) = 204m/s$.

## 10 February 1, 2016

### 10.1 Derivative Rules for Vector Functions

Let $\vec{u}(t)$ and $\vec{w}(t)$ be vector functions in two or three space.

1. The Sum / Difference Rule: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{u}(t) \pm \vec{w}(t)\right) = \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\mathrm{d}\vec{w}}{\mathrm{d}t}$.
2. Constant Multiple Rule: $\frac{\mathrm{d}}{\mathrm{d}t}(k\vec{u}) = k\frac{\mathrm{d}\vec{u}}{\mathrm{d}t}$
3. Scalar Product Rule: Let $f(t)$ be a scalar function. Then $(f(t)\vec{u}(t))' = f(t)'\vec{u}(t) + f(t)\vec{u}(t)'$
4. Dot Product Rule: $(\vec{u}(t) \cdot \vec{w}(t))' = \vec{u}(t)' \cdot \vec{w}(t) + \vec{u}(t) \cdot \vec{w}(t)'$
5. Cross Product Rule: $(\vec{u}(t) \times \vec{w}(t))' = \vec{u}(t)' \times \vec{w}(t) + \vec{u}(t) \times \vec{w}(t)'$.
6. Constant Vector Rule: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{C}\right) = \vec{0}$, where $\vec{C}$ is a constant vector.

**Definition.** Let $C$ be a plane or a space curve given parametrically by the vector function $\vec{r}(t), \quad t \in \mathbb{Z}$.

- The Unit Tangent Vector $\vec{T}(t)$: The unit tangent vector of curve $C$ is $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ divided by $\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\|$. Or
$$\vec{T}(t) = \frac{\vec{r}(t)'}{\|\vec{r}(t)'\|}$$
This is assuming that $\vec{r}(t)'$ exists, and is not equal to 0. Note that it points towards the orientation of curve $C$ as shown.

- The Principal Unit Vector $\vec{N}(t)$: Recall that since $\vec{T}(t)$ is a unit vector, then $\|\vec{T}(t)\| = 1$, so $\|\vec{T}(t)\|^2 = 1$. Applying the dot product rule, $\vec{T}' \cdot \vec{T} + \vec{T} \cdot \vec{T}' = 0$. This implies that $2\vec{T}' \cdot \vec{T} = 0$, so $\vec{T}' \cdot \vec{T} = 0$. It follows that $\vec{T}(t)'$ is a vector perpendicular to $\vec{T}(t)$. A unit vector in the direction of $\vec{T}(t)'$ is thus given by
$$\vec{N}(t) = \frac{\vec{T}(t)'}{\|\vec{T}(t)'\|}$$
This vector is called the principle unit normal (unit normal for short), and will be denoted by $\vec{N}(t)$. Observe that the unit normal is always perpendicular to the tangent.

- The Unit Binormal Vector $\vec{B}(t)$: For a space curve $C$, the cross product of $\vec{T}$ and $\vec{N}$ is a unit vector orthogonal to both $\vec{T}$ and $\vec{N}$. We justify that $\vec{B}(t)$ is indeed a unit vector. This vector is called the unit binormal vector (binormal vector for short), and is denoted by $\vec{B}(t)$, where
$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$
Thus,
$$\|\vec{B}(t)\| = \|\vec{T} \times \vec{N}\| = \|\vec{T}\|\|\vec{N}\| = 1$$

- The Curvature $\kappa$: The curvature of a plane or space curve $C$ is denoted and defined by

$$\kappa = \frac{\|\mathrm{d}\vec{T}\|}{\|\mathrm{d}s\|}$$

$s$ denotes the arc length of a curve $C$ measured from a fixed point $P_0$ to an arbitrary point $P$. Geometrically, the curvature provides a measure of how rapidly the unit tangent vector is turning as the arc length $s$ is traversed. If $\vec{T}(t)$ is a function of $\vec{T}$, that is $\vec{T}(t) = \vec{T}$, then by the chain rule, $\frac{\mathrm{d}\vec{T}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{T}}{\mathrm{d}s} * v$. Therefore

$$\kappa = \frac{1}{v}\frac{\|\mathrm{d}\vec{T}\|}{\|\mathrm{d}t\|}$$

- The Radius of Curvature $\rho$: At a point $P$ on the curve $C$ where $\kappa = / = 0$, we define the radius of curvature by

$$\rho \; = \frac{1}{\kappa}$$

The circle of radius $\rho$ that is tangent to curve $C$ of $P$ on the concave side is called the circle of curvature. Geometrically, the circle of curvature is the circle that best fits the curve $C$ in the neighbourhood of point $P$.

- The Torsion $\tau$: The torsion for a space curve $C$ is denoted ad defined by

$$\tau = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}s} \cdot \vec{N}$$

Where $S$ is the length of curve C measured from a fixed point $P_0$ to arbitrary point $P$. Note that if $\vec{B}$ is a function of $r$, then by the chain rule $\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{B}}{\mathrm{d}s} * \frac{\mathrm{d}s}{\mathrm{d}t}$. We note that $\frac{\mathrm{d}s}{\mathrm{d}t} = v$. Substituting, we get

$$\tau = -\frac{1}{v}\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} \cdot \vec{N}$$

This is a scalar quantity. Geometrically, the torsion measures to some extent the amount by which a twisted curve lies outside the plane containing $\vec{T}$ and $\vec{N}$.

# 11 February 3, 2016

## 11.1 Alternative Formulas

Alternative formulas for $\vec{T}(t), \vec{N}(t), \vec{B}(t), \kappa, \rho, \tau$. Now, we shall derive an easy way to compute the formulas for the six quantities above using velocity, acceleration and speed.

- The Unit Tangent Vector $\vec{T}(t)$: We note that velocity is $\vec{r}(t)'$, and speed is $\|\vec{r}(t)'\|$. Therefore
$$\vec{T}(t) = \frac{\vec{v}(t)}{v}$$
- Curvature $\kappa$: It can be shown that
$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3}$$
- The Unit Binormal Vector $\vec{B}(t)$: This can be simplified to
$$\vec{B}(t) = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|}$$
- The Radius of Curvature $\rho$: Recall that $\rho = \frac{1}{\kappa}$. Therefore
$$\rho = \frac{v^3}{\|\vec{v} \times \vec{a}\|}$$
- The Principal Unit Vector $\vec{N}(t)$: Perhaps for most planes or space curves, the easiest formula is given by
$$\vec{N}(t) = \vec{B} \times \vec{T}$$
This follows from the fact that the triple $\vec{T}, \vec{N}, \vec{B}$ form a right handed set of mutually orthogonal unit vectors.
- The Torsion $\tau$: It can be shown using derivation that
$$\tau = \frac{(\vec{v} \times \vec{a})\vec{a}(t)'}{\|\vec{v} \times \vec{a}\|^2}$$

### 11.2 Tangential and Normal Components of Acceleration

We have shown earlier that
$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\vec{T} + \kappa v^2 \vec{N}$$
If we let $a_T = \frac{\mathrm{d}v}{\mathrm{d}t}$, and $a_N = \kappa v^2$, then we get
$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

We shall call $a_T$ and $a_N$ that Tangential and Normal Component.

*Remark.* Note that $a_T \vec{T}$ is the Tangential Acceleration, and $a_N \vec{N}$ is the Normal Acceleration.

By recalling from the definition of $a_T$ and $a_N$, we note that

$$a_n = \frac{\|\vec{v} \times \vec{a}\|}{v}$$

By simplifying by taking dot product of acceleration equation on left with $\vec{v}$, we get that $\vec{v} \cdot \vec{a} = \vec{v}\left(a_T\vec{T} + a_N\vec{N}\right)$. We note that $\vec{v} = v\vec{T}$, and that $\|\vec{T}\| = 1$, and $\vec{T}$ is perpendicular with $\vec{N}$, so the dot product between these is 0.

$$a_T = \frac{\vec{v} \cdot \vec{a}}{v}$$

*Remark.* We note that a curve in $\mathbb{R}^2$ may be viewed as a curve in $\mathbb{R}^3$ by simply inserting a 2-component of z-component of zero value. Hence, all the formulas may be applied to plane or space curves.

# 12 February 5, 2016

## 12.1 Summary

Let $\vec{r}(t)$ be the position of a moving particle in two or three space, and let $C$ be the curve given parametrically by $\vec{r}(t)$.

- $$\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$$
- $$\vec{a}(t) = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$
- $$\vec{T}(t) = \frac{\vec{v}(t)}{v}$$
- $$\vec{B}(t) = \frac{\vec{v} \times \vec{a}}{\|\vec{v} \times \vec{a}\|}$$
- $$\vec{N}(t) = \vec{B} \times \vec{T}$$
- $$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3}$$
- $$\rho = \frac{v^3}{\|\vec{v} \times \vec{a}\|}$$

- $$\tau = \frac{(\vec{v} \times \vec{a})\vec{a}(t)'}{\|\vec{v} \times \vec{a}\|^2}$$
- $$a_T = \frac{\vec{v} \cdot \vec{a}}{v}$$
- $$a_N = \frac{\|\vec{v} \times \vec{a}\|}{v}$$

*Remark.* For convenience, we may think of a plane curve $C$ as a space curve by simply inserting a $z$-component of zero value. hence, all formulas above may be applied.

**Example.** *The position of a moving object in space is given by*

$$\vec{r}(t) = t^2\vec{i} + t\vec{j} + \frac{1}{2}t^2\vec{k}$$

*Find the tangential and normal component of acceleration at time $t = 4$.*

We note that $\vec{r}$ can be written as $\vec{r} = \left(t^2, t, \frac{1}{2}t^2\right)$, $\vec{v} = (2t, 1, t)$, and $\vec{a} = (2, 0, 1)$. We substitute $t = 4$ to determine speed. Then, we substitute into the formulas. Thus we get $a_T = \frac{20}{9}$, and $a_N = \frac{\sqrt{5}}{9}$.

**Example.** *The curve $C$ in three space is given by*

$$\vec{r}(t) = \cosh(t)\vec{i} - \sinh(t)\vec{j} + t\vec{k}$$

*Find all of the six quantities at $t = 0$.*

We note that $\vec{r} = (\cosh(t), -\sinh(t), t)$. Thus, $\vec{v} = (\sinh(t), -\cosh(t), 1)$ and $\vec{a} = (\cosh(t), -\sinh(t), 0)$. We substitute for $t = 0$. Then substitute into the formulas. Doing so results in

$$\vec{T} = \frac{1}{\sqrt{2}}(0, -1, 1) = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\vec{B} = \frac{1}{\sqrt{2}}(0, 1, 1) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\vec{N} = \frac{1}{2}(2, 0, 0) = (1, 0, 0)$$

$$\kappa = \frac{1}{2}$$

$$\rho = 2$$

$$\tau = -\frac{1}{2}$$

**Example.** *Find $\vec{T}, \vec{N}, \kappa, \rho$ for the plane curve given by the cartesian equation*

$$y = \ln(\cos(x))$$

*Let us first find the parametric equation of the curve. *Note that this curve is not one of the standard curves.*

Let us say $x = t$. Hence, $y = \ln(\cos(t))$. Therefore, $\vec{r}(t) = t\vec{i} + \ln(\cos(t))\vec{j}$. For convenience, let us view the curve as a space curve by inserting a z-component of zero. We get $\vec{r} = (t, \ln(\cos(t)), 0)$ where $t = \frac{\pi}{4}$.

Thus $\vec{v}(t) = \left(1, -\frac{\sin(t)}{\cos(t)}, 0\right)$ and $\vec{a}(t) = \left(0, \sec^2(t), 0\right)$. At $t = \frac{\pi}{4}$, we get $\vec{v}(t) = (1, -1, 0)$ and $\vec{a}(t) = (0, -2, 0)$. Solving for speed, $v = \sqrt{2}$. By calculating the remaining values using the provided formulas, we get

$$\vec{T} = \frac{1}{\sqrt{2}}(1, -1, 0)$$

$$\vec{B} = (0, 0, -1)$$

$$\vec{N} = \frac{1}{\sqrt{2}}(-1, 1, 0)$$

$$\kappa = \frac{1}{\sqrt{2}}$$

$$\rho = \sqrt{2}$$

# 13 February 8, 2016

## 13.1 Functions of Two and Three Variables

**Definition.** A function $f$ of two independent variables is a rule that assigns to each ordered pair $(x, y)$ one and only one real number $z$ which we write $f(x, y) = z$. Let $z = f(x, y)$ be a function of two independent variables.

The **domain** of $f$ is the set of all ordered pairs $(x, y)$ such that $f$ is defined and real. It is denoted by $dmf$ or $\mathcal{D}$.

Recall that the graph of a function of a single variable $y = f(x)$ is the set of all ordered pairs $(x, y)$ such that $y = f(x)$. $y = f(x)$ is referred to as a **curve** in $\mathbb{R}^2$. Likewise, the graph of a function of two independent variables is the set of all ordered triples $(x, y, z)$ such that $z = f(x, y)$. This is referred to as a **surface** in $\mathbb{R}^3$.

**Level curves** are the horizontal cross sections of a particular surface. It is given by $z = c$, where $c$ is a real number. In other words, $f(x, y) = c$.

*Remark.* The graph of a function of three or more variables is referred to as a hypersurface. For a function $w = f(x, y, z)$, the level surface is given by $f(x, y, z) = c$.

### 13.2 Summary of Standard Plane Curves

1. The Circle: The equation $(x-h)^2 + (y-k)^2 = r^2$ is the equation of a circle centered at $(h, k)$ with a radius of $r$.

2. The Ellipse: The equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is the equation of an ellipse centered at $(h, k)$ with a semi-axis of $a, b > 0$.

3. The Hyperbola: The equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \pm 1$ is the equation of an hyperbola centered at $(h, k)$ with a semi-axis of $a, b > 0$. If the right hand side is $+1$, it opens left and right. If the right hand side is $-1$, it opens from top to bottom.

4. The Parabola: The equations $y - k = a(x-h)^2$ and $x - h = a(y-k)^2$ are equations of parabolas with a vertex of $(h, k)$. Note that the equations open upwards and towards the right respectively if $a > 0$, and open downwards and towards the left respectively if $a < 0$.

5. The Straight Line: The equation $ax + by + c = 0$, where $a, b, c$ not all 0 is an equation of a straight line with a slope of $-\frac{a}{b}$. The equation of a vertical line is $x = k$ and the equation of a horizontal line is $y = l$ where $k, l \in \mathbb{R}$.

## 14 February 10, 2016

### 14.1 Summary of Standard Plane Curves Examples

**Example.** *In each case, find the domain of the given function.*

$$f(x) = \frac{1}{x^2 + y - 1}$$

$$f(x, y) = \left(3x - 7y^2 + 32\right)^{1/3}$$

$$f(x, y) = \sqrt{(\ln(17 - x^2 - y^2))}$$

$$f(x, y) = \ln(x + y)$$

We note that the domain of $f(x, y)$ consists of all ordered pairs $(x, y)$ such that $f$ is defined and is real. For the first question, we note $f$ is defined and real provided that the denominator is not 0. In other words, $\mathcal{D} = \{(x, y) | x^2 + y - 1 \neq 0\}$. We note that $x^2 + y - 1 = 0$ is the equation of a parabola with vertex at $(0, 1)$ and which opens downwards. So f is defined for everything other than the curve.

Consider the second equation. Clearly, the cube root is defined and real for all $(x, y) \in \mathbb{R}^2$. Thus, the domain is the entire xy-plane. $\mathcal{D} = \mathbb{R}^2$.

Recall that the domain of $f$ is the set of all $(x, y)$ such that $f$ is defined and real. For logarithms, the domain is $t > 0$, and $\ln(t) \geq 0$ for all $t \geq 1$. $f$ is defined and real provided that the quantity under the square root is greater than or equal to 0. So, we require that $\ln\left(17 - x^2 - y^2\right) \geq 0$. We consider $17 - x^2 - y^2$ as $t$. Thus, we require that $17 - x^2 - y^2 \geq 1$. Therefore, $\mathcal{D} = \{(x,y)|x^2 + y^2 \leq 16\}$.

For the fourth equation, we note that $f$ is defined and real only when $x + y > 0$. Thus, $\mathcal{D} = \{(x,y)|x + y > 0\}$.

*Remark.* A dashed line is used for all quantities that are $>, <$ while a solid line is for all quantities $\geq, \leq$.

# 15 February 12, 2016

## 15.1 Continued Examples

**Example.** *Let $f(x,y) = y^2 + 4x^2 + 4$. Sketch the level curve of the function corresponding to $c = 0, 4, 8$ on the same set of coordinate axes. Recall that level curves are given by $f(x,y) = c$. That is*

$$y^2 - 4x^2 + 4 = c$$

For $c = 0$, we obtain $4x^2 - y^2 = 4$, which reduces to $x^2 - \frac{y^2}{4} = 1$. This is the equation of a hyperbola with centre at $(0,0)$, a semi-axis of $a = 1$, $b = 2$, and which opens to the left and right. The vertices are $(\pm a, 0) = (\pm 1, 0)$.

For $c = 4$, the equation becomes $y^2 - 4x^2 = 0$. Factoring, we get $(y-2x)(y+2x)$. So $y = 2x$, or $y = -2x$. This is a pair of lines through the origin.

For $c = 8$, we obtain $y^2 - 4x^2 + 4 = 8$. This reduces to $x^2 - \frac{y^2}{4} = -1$. This is the equation of a hyperbola with a centre at $(0,0)$, a semi-axis of $a = 1$, $b = 2$, and which opens up and down. The vertices are $(0, \pm b) = (0, \pm 2)$.

**Example.** *Given $f(x,y) = \frac{4x^2+10y^2-64}{12x^2+6y^2}$, sketch the level curves of $f$ corresponding to $c = 0, 1, -1$. That is*

$$\frac{4x^2 + 10y^2 - 64}{12x^2 + 6y^2} = c$$

For $c = -1$, cross multiply. After simplifying, we get that $x^2 + y^2 = 4$. This is the equation of a circle centered at $(0,0)$ with a radius of 2.

For $c = 1$, cross multiply. We get $4x^2 + 10y^2 - 64 = 12x^2 + 6y^2$. By combining like terms, $8x^2 - 4y^2 = -64$. We can further reduce this to $\frac{x^2}{8} - \frac{y^2}{16} = -1$. This is the equation of a hyperbola centered at $(0,0)$, with semi-axis $a = \sqrt{8}$, $b = 4$, and which opens up and down with vertices $(0, \pm b) = (0, \pm 4)$.

For $c = 0$, we cross multiply and reduce to get $\frac{x^2}{16} + \frac{y^2}{6.4} = 1$, which is the equation of an ellipse with a centre at $(0,0)$, semi-axis of $a = 4$, $b = \sqrt{6.4}$, with vertices at $(\pm 4, 0)$ and $(0, \pm\sqrt{6.4})$.

## 15.2 Quadric Surfaces

A second degree equation in three variables is called a **quadric surface**. The following are instances of quadric surfaces, which fit into three families.

Let $a, b, c$ be positive real numbers.

1. The Ellipsoid Family: It is composed of two members.
   The **ellipsoid** centered at $(0,0,0)$ and with semi axes of $a, b, c$ is given
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
   The **sphere** centered at $(0,0,0)$ with radius $a$ is given
   $$x^2 + y^2 + z^2 = a^2$$

2. The Paraboloid Family: It is composed of three members.
   The **elliptic paraboloid** has vertices at the origin with the z-axis as its axis of symmetry. It opens upwards if $z > 0$, and downwards otherwise. It is given by
   $$z = \pm\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right), \quad a \neq b$$
   The **circular paraboloid** has vertices at the origin with the z-axis as its axis of symmetry. It opens upwards if $z > 0$, and downwards otherwise. (Looks like an upside down bowl). It is given by
   $$z = \pm\left(\frac{x^2}{a^2} + \frac{y^2}{a^2}\right)$$
   The **hyperbolic paraboloid** has vertices at the origin with the z-axis as its axis of symmetry. It is given by
   $$z = \pm\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

3. The Hyperboloid Family: It is composed of three members.
   The **hyperboloid of one sheet** has a centre at the origin with the z-axis as its axis of symmetry (vertical wormhole). It is given by
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
   The **hyperboloid of two sheets** has a centre at the origin with the z-axis as its axis of symmetry (vertical bowls). It is given by
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

The **cone** has a vertex at the origin with the z-axis as its axis of symmetry. If the cross section of a cone by a horizontal plane is a circle, parabola, ellipse, or hyperbola, then the cone is referred to as circular, parabolic, elliptic, or hyperbolic respectively (vertical cones). It is given by

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

# 16 February 22, 2016

## 16.1 Quadric Surfaces Cont'd

*Remark.* All eight quadric surface equations can have $x, y, z$ replaced with $(x - h), (y - k), (z - l)$ respectively to obtain a translated quadric surface. An equation of a quadric surface with axes of symmetry in the x or y axes (or parallel to the x or y-axes) is similar to the ones with the axes of symmetry being the z-axis.

**Example.**

$$\frac{(x-1)^2}{a^2} - \frac{(y+3)^2}{b^2} + \frac{(z-2)^2}{c^2} = 1$$

*is an equation of a hyperboloid of one sheet centered at* $(1, -3, 2)$ *that is parallel to the y-axis.*

## 16.2 Two Special Surfaces

1. The **plane** is given by the following equation, where $a, b, c$ are not all zero. If $d$ is 0, then the plane passes through the origin. We note that when $z = 0$, then this refers to the $xy$ plane, and when $z = c$ where $c$ is a constant, then this is a plane parallel to the $xy$ plane. This is also true of $x, y$.

$$ax + by + cz + d = 0$$

2. The **special cylinder** is an equation in $\mathbb{R}^3$ which contains only two of the three variables of $x, y, z$. The equation of a cylinder is generated by a straight line parallel to the axis determined by the missing variable in the equation. For instance, in $\mathbb{R}^3$, a cylinder parallel to the z-axis is given by

$$y = x^2$$

**Example.** *In each case, identify the surface and sketch.*

$$x = y^2 + z^2$$

$$z = \sqrt{9 - x^2 - y^2}$$

$$x^2 + y^2 + z^2 = 9$$
$$x + 2y + 4z = 12$$
$$-225x^2 + 100y^2 - 36z^2 + 900 = 0$$
$$z = -\sqrt{x^2 + y^2}$$
$$z = 2 - 4x^2 - 4y^2$$
$$x^2 + y^2 + z^2 - 8z = 0$$
$$x^2 + z^2 = 1$$
$$y = \sin(x), \quad 0 \leq x \leq \pi$$

$x = y^2 + z^2$. This is a paraboloid with a vertex at the origin and an axis of symmetry in the x-axis. It opens towards the front (appears like a single bowl facing outwards).

$z = \sqrt{9 - x^2 - y^2}$. We first square both sides to get $x^2 + y^2 + z^2 = 9$. This is the equation of a sphere centered at the origin with a radius of 3. Note however that $z > 0$ from the square root. Therefore, the equation represents only the upper hemisphere.

$x + 2y + 4z = 12$. We note that this is the equation of a plane. To sketch this, we simply find the $x, y, z$ intercepts.

$-225x^2 + 100y^2 - 36z^2 + 900 = 0$. We note that this is equivalent to $225x^2 - 100y + 36z^2 = 900$. We then divide both sides by 900. The equation becomes $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{25} = 1$. This is therefore a hyperboloid of one sheet. To sketch, we note that it has a centre at the origin, and an axis of symmetry in the y-axis,

$z = -\sqrt{x^2 + y^2}$. We first square both sides to get $z^2 = x^2 + y^2$. This is the equation of a cone with a vertex at the origin and with the z-axis as the axis of symmetry. However, $z \leq 0$ from initial equation. Therefore, the equation represents only the bottom cone.

$z = 2 - 4x^2 - 4y^2$. This can be reduced to $z - 2 = -\left(4x^2 + 4y^2\right)$. This is the equation of a paraboloid (circular) with a vertex at $(0, 0, 2)$ and an axis of symmetry in the z-axis which opens downwards.

$x^2 + y^2 + z^2 - 8z = 0$. We will need to complete the square. The equation then becomes $x^2 + y^2 + (z - 4)^2 = 16$. This is the equation of a sphere centered at $(0, 0, 4)$ with a radius of 4.

$x^2 + z^2 = 1$. This is the equation of a cylinder since there are only two variables. It is centered at the origin, has a radius of 1 and a generator parallel to the y-axis.

$y = \sin(x)$ where $0 \leq x \leq \pi$. This is an equation of a cylinder with a generator parallel to the z-axis. Simply sketch the graph of $y = \sin(x)$, then pile up in the z direction.

# 17 February 24, 2016

## 17.1 Partial Derivatives of a Function of Several Variables

**Definition.** The partial derivative of a function of two independent variables. Let $z = f(x, y)$ be a function of the two independent variabeles $x, y$. Provided that the limit exists, the partial derivative of $z$ with respect to $x$ is denoted and defined by

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Provided that the limit exists, the partial derivative of $z$ with respect to $y$ is denoted and defined by

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

*Remark.* The partial derivative $\frac{\partial z}{\partial x}$ is the derivative of $z$ with respect to $x$ by treating $y$ as a constant. Similarly, the partial derivative $\frac{\partial z}{\partial y}$ is the derivative of $z$ with respect to $y$ by treating $x$ as a constant.

## 17.2 Other Notations for Partial Derivatives

*Note that in the following, "1" indicates the position of the first variable and "2" indicates the position of the second variable. We do not use prime notation for partial derivatives.

- $\frac{\mathrm{d}f}{\mathrm{d}x}$ and $\frac{\mathrm{d}f}{\mathrm{d}y}$
- $f_x(x, y)$ and $f_y(x, y)$
- $f_1(x, y)$ and $f_2(x, y)$

# 18 February 26, 2016

## 18.1 Partial Derivatives Cont'd

*Remark.* The partial derivatives of a function of three or more variables are computed similarly. For instance, let $w = f(x, y, z)$. Then $\frac{\partial w}{\partial y}$ is the derivative of $w$ with respect to $y$ but holding both $x, z$ to be constant.

**Example.** *Let* $f(x, y) = x^3 y + \sin(xy) + \tan^{-1}(x)$. *Find* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$.

We know that $f(x, y) = x^3 y + \sin(xy) + \tan^{-1}(x)$ can be re-written as $f(x, y) = yx^3 + \sin(yx) + \tan^{-1}(x)$. We now solve for the partial derivatives of $f$ with respect to $x$ and $y$. We get $\frac{\partial f}{\partial x} = 3x^2 y + y\cos(xy) + \frac{1}{1+x^2}$ and $\frac{\partial f}{\partial y} = x^3 + x\cos(xy)$.

**Example.** $f(x, y, z) = xy + xz + yz$. *Find* $\frac{\partial f}{\partial z}$.

$f_z(x, y, z) = 0 + x + y = x + y$.

### 18.2 Higher Order Partial Derivatives

Let $z = f(x,y)$. From now on, we may call the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ the **first order partial derivatives** of $f$. There are four **second order partial derivatives** for the function $z = f(x,y)$.They are denoted and defined as follows:

1. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$
2. $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$
3. $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$
4. $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

*Remark.* Note that the last two are **mixed partials**. The mixed partials are equal under certain conditions (mainly continuity of partial derivatives). That is, in certain conditions, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

### 18.3 Other Notations for Second Order Partials

- $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$. (Mixed partials evaluated right to left).
- $f_{xx}(x,y), f_{yy}(x,y), f_{yx}(x,y), f_{xy}(x,y)$. (Mixed partials evaluated left to right).
- $f_{11}(x,y), f_{22}(x,y), f_{21}(x,y), f_{12}(x,y)$. (Mixed partials evaluated left to right).

*Remark.* Second order partial derivatives can be generalized for functions of three or more variables. For instance, if we let $w = f(x,y,z)$, then the second order partials are given by $f_{xx}(x,y,z), f_{xy}(x,y,z), f_{xz}(x,y,z), f_{yx}(x,y,z), f_{yy}(x,y,z), f_{yz}(x,y,z), f_{zx}(x,y,z), f_{zy}(x,y,z), f_{zz}(x,y,z)$.

*Remark.* Partial derivatives of order three or higher are defined similarly. For instance, it we let $w = f(x,y,z)$, then $\frac{\partial^3 w}{\partial x \partial^2 y}$ is the third order partial derivative of $w$ with respect to $y, y, z$. Likewise, $\frac{\partial^3 w}{\partial z \partial x \partial y}$ is the third order partial derivative of $w$ with respect to $y, x, z$.

**Example.** *Let* $z = x^{\sin(y)} + y$. *Find* $\frac{\partial^2 z}{\partial x \partial y}$.

In order to compute this, we compute from right to left. That is, we evaluate $\frac{\partial z}{\partial y}$ before evaluating $\frac{\partial}{\partial x}$. We use the fact that $a^b = e^{b\ln(a)}$. Thus, $z = e^{\sin(y)\ln(x)} + y$. So $\frac{\partial z}{\partial y} = e^{\sin(y)\ln(x)}\ln(x)\cos(y) + 1$. Therefore

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= \frac{\cos(y)e^{\sin(y)\ln(x)}}{x} + \frac{\sin(y)e^{\sin(y)\ln(x)}\ln(x)\cos(y)}{x} \\ &= \frac{\cos(y)e^{\sin(y)\ln(x)}\left(1+\ln(x)\sin(y)\right)}{x} \\ &= \frac{\cos(y)x^{\sin(y)}\left(1+\ln(x)\sin(y)\right)}{x} \\ &= \cos(y)x^{\sin(y)-1}\left(1+\ln(x)\sin(y)\right)\end{aligned}$$

**Example.** *Let $f(x,y) = xe^{\frac{y}{x}}$. Find all second order partials.*

We note that $f(x,y) = xe^{yx^{-1}}$. Additionally, $\frac{\partial f}{\partial x} = \left(1 - yx^{-1}\right)e^{yx^{-1}}$ and $\frac{\partial f}{\partial y} = e^{yx^{-1}}$.

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial}{\partial y}\left(e^{yx^{-1}}\right) \\ &= e^{\frac{y}{x}}\left(\frac{1}{x}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(1 - yx^{-1}\right)e^{yx^{-1}} \\ &= yx^{-2}e^{yx^{-1}} + \left(1 - yx^{-1}\right)e^{yx^{-1}}\left(-yx^{-2}\right) \\ &= e^{\frac{y}{x}}\left(\frac{y^2}{x^3}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ &= \frac{\partial}{\partial y}\left(1 - yx^{-1}\right)e^{yx^{-1}} \\ &= \left(-x^{-1}\right)e^{yx^{-1}} + \left(1 - yx^{-1}\right)e^{yx^{-1}}\left(x^{-1}\right) \\ &= e^{\frac{y}{x}}\left(-\frac{y}{x^2}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(e^{yx^{-1}}\right) \\ &= e^{\frac{y}{x}}\left(-\frac{y}{x^2}\right)\end{aligned}$$

*Remark.* Note that it is much easier to computer $\partial x \partial y$ in that order first.

# 19 February 29, 2016

## 19.1 Chain Rule

The chain rule for functions of several variables. Let $y = f(x)$, and let $x$ be itself a function of $t$. Then $y$ is indirectly a function of $t$. We want $\frac{\mathrm{d}y}{\mathrm{d}t}$, so we interpret the functions as a chained function. Thus, by the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

Likewise, Let $z = f(x, y)$, where $x = x(t)$, and $y = y(t)$. Hence, $z$ itself is a function of $t$. Thus, $\frac{\mathrm{d}z}{\mathrm{d}t}$ is the sum of two chains of single variables (one for $x$, and one for $y$).

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

## 19.2 Other Versions of Chain Rule

Let $w = f(x, y, z)$ where $x = x(t)$, $y = y(t)$, and $z = z(t)$. Therefore, $w = w(t)$. Thus

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

*Remark.* Note that the partial symbol is used since the function $f$ is a function of many variables.

Let $z = f(x, y)$ where $x, y$ are functions of $u, v$ such that $x = x(u, v)$ and $y = y(u, v)$. Clearly, $z = z(u, v)$. Indeed, we note that

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Similarly for $v$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

### 19.3 Tree Diagram

The Tree Diagram is a powerful tool which eliminates the need to memorize several versions of the chain rule for functions of several variables.

Given $z = f(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$. We want to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$. We shall call $x, y$ original variables, and $u, v$ new variables. We construct the tree diagram by starting at the tree top with $z = f(x, y)$. Then draw a first generation branch for each original variable. Following this, draw a second generation branch for each new variable. Finally, compute partial derivatives or derivatives along each branch.

For instance, top level branches are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. The second level branches are $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$. Therefore, $\frac{\partial z}{\partial u}$ is the sum of product of a partial derivative along a first generation and a partial derivative along a second generation starting from $z$ and towards $u$.

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

Likewise for $v$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

## 20 March 2, 2016

### 20.1 Tree Diagram Examples

**Example.** *Let $w = f(x, y, z) = e^{2x+5y+z}$ where $x = t + \sin(2t - 2)$, $y = 3 - 4t$, and $z = 3e^{t^2-1}$. Use the chain rule to find $\frac{\mathrm{d}w}{\mathrm{d}t}$ at $t = 1$.*

We note that $w$ is a function of $x, y, z$, where $x, y, z$ are themselves functions of $t$. Therefore, $w$ is indirectly a function of $t$. That is, $w = w(t)$. We want to find $\frac{\mathrm{d}w}{\mathrm{d}t}$. Here, the **original variables** are $x, y, z$. We will draw 3 first generation branches. We only have 1 **new variable** $t$. We will draw 1 second generation branch from each first generation branch. Now, compute the partial derivatives along the branches.

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\
&= 2e^{2x+5y+z}\left(1 + 2\cos(2t - 2)\right) + 5e^{2x+5y+z}(-4) + e^{2x+5y+z}\left(3e^{t^2-1} * 2t\right) \\
&= e^{2x+5y+z}\left(2(1 + 2\cos(2t - 2)) - 20 + 6te^{t^2-1}\right)
\end{aligned}$$

We note that at $t = 1$, $x = 1$, $y = -1$, and $z = 3$ by substitution into their formulas. Substituting these values, we get

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -8$$

**Example.** *Let $z = f(x, y)$ where $x = uv^2$, and $y = \frac{u}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at $(u, v) = (2, -1)$ given that $f_x(2, -2) = 3$, $f_y(2, -2) = -2$, $f_x(2, -1) = -7$, and $f_y(2, -1) = 16$.*

We note that we have 2 original variables $x, y$ and 2 new variables $u, v$. We first calculate the partial derivatives to get $\frac{\partial x}{\partial u} = v^2$, $\frac{\partial x}{\partial v} = 2uv$, $\frac{\partial y}{\partial u} = \frac{1}{v}$, and $\frac{\partial y}{\partial v} = -\frac{u}{v^2}$. Furthermore, at $(u, v) = (2, -1)$, we get that $x = 2$ and $y = -2$ by solving for $x$ and $y$ respectively. We note that by substituting into the formula, we get that

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\
&= f_x(x, y)\left(v^2\right) + f_y(x, y)\left(\frac{1}{v}\right) \\
&= f_x(2, -2)\left((-1)^2\right) + f_y(2-, 2)\left(-1\right) \\
&= 5
\end{aligned}$$

**Example.** *Let $z = f(x, y) = 4\cosh(x) + \cos(y)$ where $x = t + 2s$ and $y = \frac{\pi}{2}e^t + s$. Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ at $t = \ln(2)$ and $s = 0$.*

At $t = \ln(2)$ and $s = 0$, we get that $x = \ln(2)$ and $y = \pi$. We also note that $\frac{\partial f}{\partial x} = 4\sinh(x)$, and $\frac{\partial f}{\partial y} = -\sin(y)$. Furthermore, after calculation, we can show that $\frac{\partial x}{\partial t} = 1$, $\frac{\partial x}{\partial s} = 2$, $\frac{\partial y}{\partial t} = \frac{\pi}{2}e^t$, and $\frac{\partial y}{\partial s} = 1$. Substituting into the equation, we get

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\
&= 4\sinh(x)(2) - \sin(y)(1) \\
&= 8\sinh(\ln(2)) - \sin(\pi) \\
&= 8\left(\frac{1}{2}\left(e^{\ln(2)} - e^{-\ln(2)}\right)\right) - 0 \\
&= 8\left(\frac{1}{2}\left(e^{\ln(2)} - e^{\ln(2^{-1})}\right)\right) \\
&= 8\left(\frac{1}{2}\left(2 - 2^{-1}\right)\right) \\
&= 8\left(\frac{1}{2}\left(\frac{3}{2}\right)\right) \\
&= 8\left(\frac{3}{4}\right) \\
&= 6
\end{aligned}$$

# 21 March 4, 2016

## 21.1 Gradient

Let $F(x, y, z)$ be a function of three independent variables $x, y, z$.

**Definition.** The **gradient** of $F$ at the point $P(x_0, y_0, z_0)$ is denoted by either by $gradF(P)$, or we use $\vec{\nabla}F(P)$.

$$\vec{\nabla}F(P) = F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0))$$

Geometrically, the gradient is a vector orthogonal to surface $S : F(x, y, z) = 0$ at $P$. That is, the gradient is a vector orthogonal to tangent $P$ to the surface $S$ at $P$.

**Example.** *Let $F(x, y, z) = x^3y^2z^4 - 35$. Find the gradient for point $P = (1, 1, -2)$.*

Finding the partial derivatives, then substituting the point in, we get that $\vec{\nabla}F(P) = (48, 32, -32)$.

## 21.2 Equation of a Plane

The point-normal form. The equation of a plane passing through the point $P$ that has a normal vector of $\vec{N}$ is given by the following equation given that $\vec{r} = (x, y, z)$.

$$\vec{r} \cdot \vec{N} = \vec{r_0} \cdot \vec{N}$$

1. The Equation of the Tangent Plane.

$$\vec{r} \cdot \vec{N} = \vec{r_0} \cdot \vec{N}$$

   where $\vec{N} = gradF(P) = \vec{\nabla} F(P)$

2. The Vector Equation of Normal Line.

$$\vec{r} = \vec{r_0} + t\vec{v}, \quad t \in R$$

   where $\vec{v}$ is the direction vector $= \vec{\nabla} F(P)$

**Example.** *Find the equation of the Tangent Plane and the Normal Line to the surface* $xz + yz + xy = -3$ *at the point* $P = (1, 1, -2)$ *on the surface.*

The equation can be re-written so that $f(x, y, z) = xz + yz + xy + 3 = 0$. We need to compute $\vec{\nabla} F(P)$. Finding the partial derivatives, we get that $\vec{\nabla} F(P) = (z + y, z + x, x + y)$. By substituting the initial point $P$, we find the gradient is $(-1, -1, 2)$. This is our normal vector $\vec{N}$.

The equation of the tangent plane is thus given by $\vec{r} \cdot \vec{N} = \vec{r_0} \cdot \vec{N}$. So, $(x, y, z) \cdot (-1, -1, 2) = (1, 1, -2) \cdot (-1, -1, 2)$. Thus

$$x + y - 2z = 6$$

Vector equation of normal line requires initial point and direction vector, which is $(-1, -1, 2)$. The equation of normal line is thus given by

$$(x, y, z) = (1, 1, -2) + t(-1, -1, 2), \quad t \in \mathbb{R}$$

### 21.3 Linearization of a Function of Two or Three Variables

Let $z = f(x, y)$ be a function of two independent variables $x, y$. Let us find the equation of the tangent plane to the surface $S : z = f(x, y)$. That is, $f(x, y) - z = 0$. at the point $P = (x_0, y_0, f(x_0, y_0))$. We need a point and a normal vector (where $\vec{N} = \vec{\nabla} F(P)$). We get that $\vec{N} = (f_x(x_0, y_0), f_y(x_0, y_0), -1)$. Substituting into the equation for tangent, we get that

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We call this z-coordinate of the equation of the tangent plane above the linearization of $f(x, y)$ at point $P$. It is denoted by $L(x, y)$. It follows that

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$