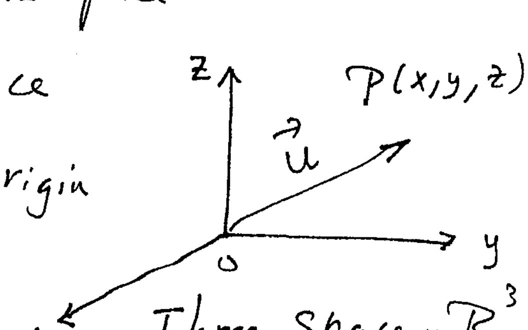
REVIEW OF VECTORS

ESSENTIALS FROM MATH 211

Review of Vectors

Definition 1: position Vectors in Three-Space



Three-Space "$\mathbb{R}^3$"

Let $P(x, y, z)$ be a point in 3-Space as shown. A directed line from the origin O to the point P is Called: A position Vector and may be denoted by $\vec{OP}$. You may also use a Single letter with an arrow on top such as $\vec{u}$.

The point "O" is Called: The Initial point, where as the point "P" is Called: The Terminal point.

The position Vector $\vec{u} = \vec{OP}$ will be represented by its Terminal point P and we write

$$\vec{u} = \vec{OP} = (x, y, z)$$

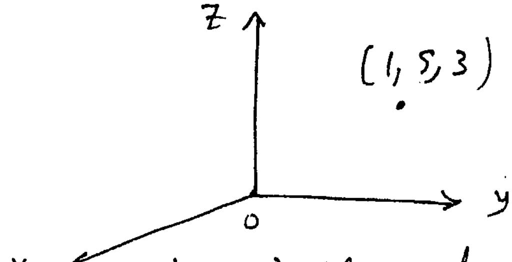
x, y, ad z are referred to as: Component of vector $\vec{u}$.

Remark: Given an ordered Triple (x, y, z) in $\mathbb{R}^3$:

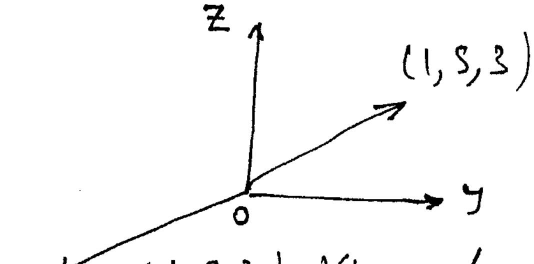
The ordered Triple may be <u>Viewed as a point</u> Q, and in which Case: x, y, z are the Coordinates of Q.

OR: It may be Viewed as a position Vector $\vec{v} = \vec{OQ}$, and in which Case: x, y, z are the Components of $\vec{v}$.

For Example: The ordered Triple $(1, 5, 3)$ may be Viewed as either a point or a position vector as shown in figures below:



(1, 5, 3) Viewed as a point



(1, 5, 3) Viewed as a position vector

Equality of position Vectors

let $\vec{u} = (x_1, y_1, z_1)$, and $\vec{v} = (x_2, y_2, z_2)$ be position vectors in $\mathbb{R}^3$. We shall say $\vec{u}$, and $\vec{v}$ are Equal and write $\vec{u} = \vec{v}$

$$\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$$

if and only if : Corresponding Components are Equal, namely : $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$

The Zero Vector :

The Special Vector $\vec{u} = (0, 0, 0)$ is Called : The Zero Vector and will be denoted by $\vec{0}$.

Arithmetics of Vectors

Let $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$ be position vectors in $\mathbb{R}^3$, and let $K \in \mathbb{R}$ be a scalar.

$\boxed{1}$ Sum of Vectors

The sum of $\vec{u}, \vec{v}$ in that order is denoted and defined by :

$$\vec{u} + \vec{v} = (x_1, y_1, z_1) + (x_2, y_2, z_2)$$
$$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

We Simply <u>added</u> Corresponding Components!

Note: Obviously: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$.

$\boxed{2}$ Difference of Vectors

The Difference of $\vec{u}$ and $\vec{v}$ in the order is denoted and defined by

$$\vec{u}-\vec{v} = (x_1, y_1, z_1) - (x_2, y_2, z_2)$$
$$= (x_1 - x_2, y_1 - y_2, z_1 - z_2)$$

We Simply subtrated Corresponding Components!

$\boxed{3}$ Scalar Multiplication of Vectors

The product of $\vec{u} = (x_1, y_1, z_1)$ by the real number K "Called: A scalar" is denoted and defined by

$$K\vec{u} = K(x_1, y_1, z_1)$$
$$= (Kx_1, Ky_1, Kz_1)$$

We Simply multiplied each Component of $\vec{u}$ by K.

Note: If $K=1$, we write $1\vec{u}$ as $\vec{u}$,
If $K=-1$, we write $(-1)\vec{u}$ as $-\vec{u}$

$\boxed{4}$ Ordinary Product / Quotient of Vectors

There are No ordinary products or Quotients of two Vectors!

In other words: The expressions $\vec{u}\vec{v}$ and $\dfrac{\vec{u}}{\vec{v}}$ are None Sense!

Two special products will be introduced later!

Geometric Interpretation of Sum and Scalar Multiplication

Sum: let $\vec{u}, \vec{v}$ be position vectors in $\mathbb{R}^3$.

Reposition the vector $\vec{v}$ so that the initial point of $\vec{v}$ coincides with the Terminal point of $\vec{u}$ and it remains parallel to $\vec{v}$. Refer to figures (1), (2), (3) below:

figure (1)

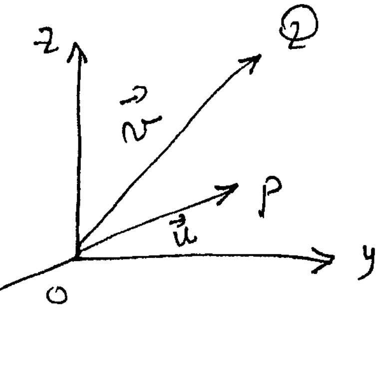
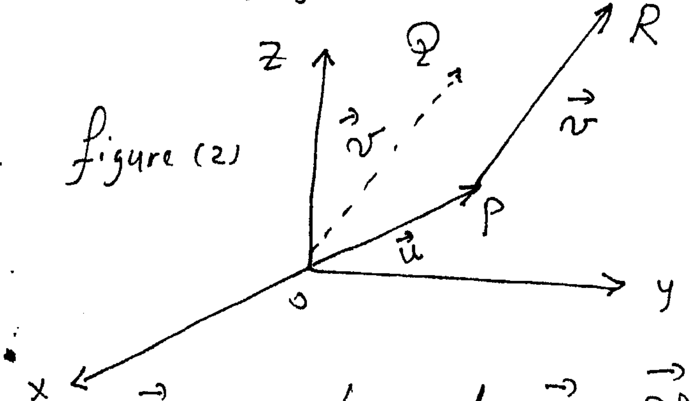
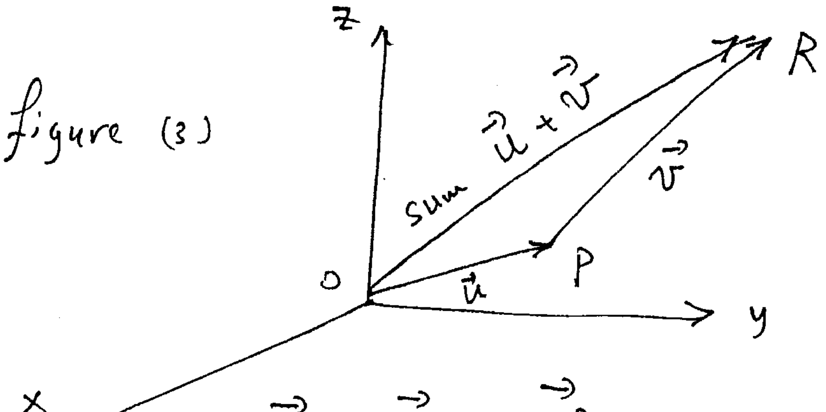


figure (2)



$\vec{v}$ repositioned: $\vec{v} = \vec{PR}$

figure (3)



$\vec{OP} + \vec{PR} = \vec{OR}$

You may think of the sum Law as follows: Take a Detour!
If you are at point O, and want to reach point R:
You have two choices:

Go directly from O to R: path $\vec{OR}$

or Go from O to P, then proceed from P to R
path: $\vec{OP} + \vec{PR}$. The Two are Equivalent!

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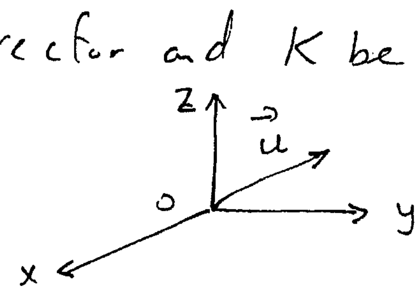
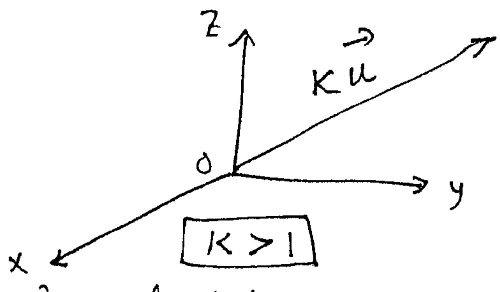
Scalar Multiplication:

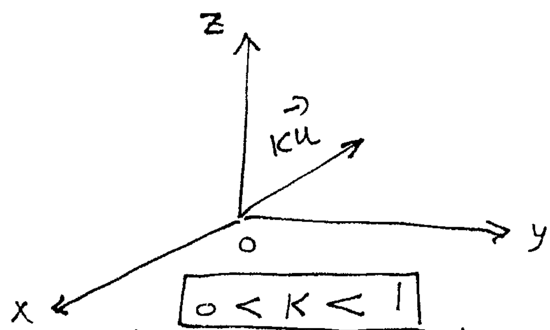
Let $\vec{u}$ be a non-zero position vector and K be a non-zero real number.



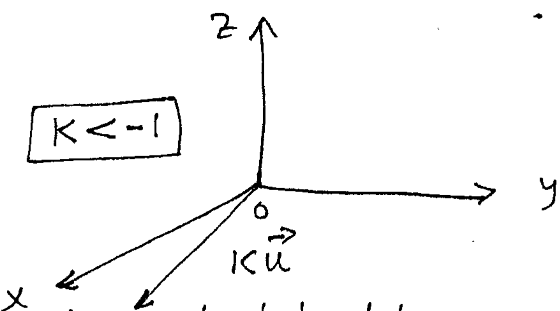
Refer to figures below:



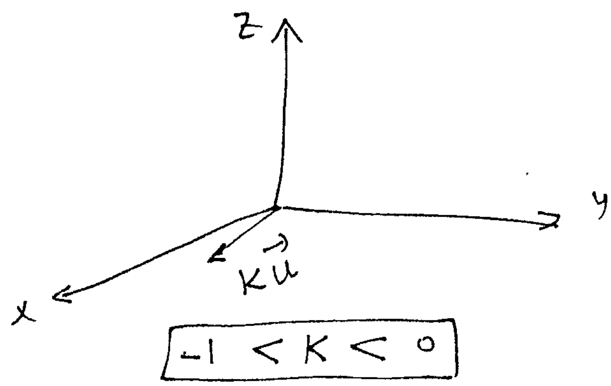
$\vec{u}$ is stretched by a factor of K in the <u>same</u> direction



$\vec{u}$ is compressed by a factor of K in the <u>same</u> direction



$\vec{u}$ is stretched by a factor of $|K|$ in the <u>opposite</u> direction



$\vec{u}$ is compressed by a factor of $|K|$ in the <u>opposite</u> direction

Important Conclusion

$K\vec{u}$ is a position vector parallel to $\vec{u}$ in the same direction if K is positive, and is parallel to $\vec{u}$ in the opposite direction if $\vec{u}$ is negative.

So far we have only introduced position vectors in $\mathbb{R}^3$.

Now, we shall define: General Vectors in $\mathbb{R}^3$.

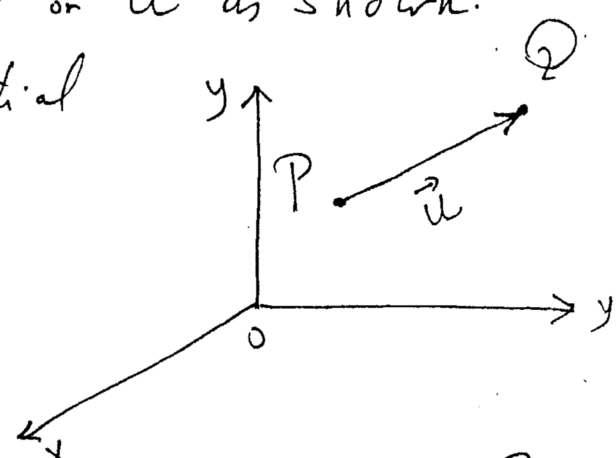
For simplicity sake, the general vectors in $\mathbb{R}^3$ will be called: "Vectors".

Hence, from now-on, there is no need for the word "position" any more!

Definition (2): General vectors in $\mathbb{R}^3$

Let $P(x_1, y_1, z_1)$, and $Q(x_2, y_2, z_2)$ be arbitrary points in $\mathbb{R}^3$. A directed line from P to Q is called: A general vector (or simply a vector) in $\mathbb{R}^3$ and may be denoted by $\overrightarrow{PQ}$ or $\vec{u}$ as shown.

Again: The point P is the Initial point, where as the point Q is the Terminal point of $\vec{u} = \overrightarrow{PQ}$.



A formula for $\overrightarrow{PQ}$:

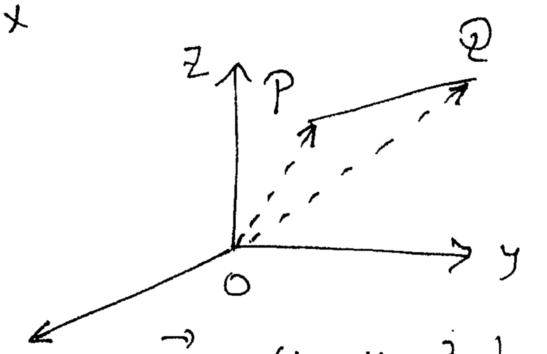
From the Sum Law:

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (x_2, y_2, z_2) - (x_1, y_1, z_1)$$

$$= \text{Terminal point } Q - \text{Initial } P$$



$\overrightarrow{OP} = (x_1, y_1, z_1)$

$\overrightarrow{OQ} = (x_2, y_2, z_2)$

Definition (3): Norm of a Vector

Let $\vec{u} = (x, y, z)$ be a vector in $\mathbb{R}^3$. The norm of $\vec{u}$ is denoted and defined by

$$\|\vec{u}\| = \sqrt{x^2 + y^2 + z^2}$$

Geometrically: $\|\vec{u}\|$ is the Length or Magnitude of the vector $\vec{u}$.

Clearly $\|\vec{u}\|$ is a Real Number and that

$$\|\vec{u}\| \geq 0$$

Note that: The only vector of zero length is the zero vector!

Therefore $\|\vec{u}\| = 0$ if and only if $\vec{u} = \vec{0}$.

EX: Let $\vec{u} = (3, -2, 6)$,

$$\therefore \|\vec{u}\| = \sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$$

EX: let $\vec{v} = \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$

$$\therefore \|\vec{v}\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{-2}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$$

Definition (4): A unit vector

A vector of length one unit is called: A unit vector and may be denoted by $\vec{n}$.

Therefore $\|\vec{n}\| = 1$

Construction of unit Vectors

Let $\vec{u}$ be a non-zero vector in $\mathbb{R}^3$. Then

(i) $\vec{n}_1 = + \dfrac{\vec{u}}{\|\vec{u}\|}$ is a unit vector in the direction of $\vec{u}$.

(ii) $\vec{n}_2 = - \dfrac{\vec{u}}{\|\vec{u}\|}$ is a unit vector in the opposite direction of $\vec{u}$.

Ex: Let $\vec{u} = (-4, 0, 3)$. Find a unit vector in the opposite direction of $\vec{u}$.

Indeed: A unit vector in the opposite direction of $\vec{u}$ is given by

$$n = - \frac{\vec{u}}{\|\vec{u}\|} \overset{\text{or}}{=} - \frac{1}{\|\vec{u}\|} \vec{u}$$

$$= - \frac{1}{\sqrt{(-4)^2 + 0^2 + 3^2}} (-4, 0, 3)$$

$$= - \frac{1}{5} (-4, 0, 3)$$

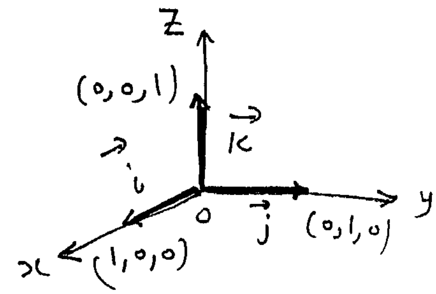
$$\overset{\text{or}}{=} \left(\frac{4}{5}, 0, \frac{-3}{5} \right)$$

Standard unit vectors in $\mathbb{R}^3$:

The standard unit vectors in $\mathbb{R}^3$ are:

$$\vec{i} = (1,0,0) \quad , \quad \vec{j} = (0,1,0) \quad , \text{ and } \quad \vec{k} = (0,0,1)$$

Note that if $\vec{u} = (x,y,z)$ is a vector in $\mathbb{R}^3$, then we can write



$$\vec{u} = (x,0,0) + (0,y,0) + (0,0,z)$$
$$= x(1,0,0) + y(0,1,0) + z(0,0,1)$$
$$\therefore \quad \vec{u} = x\vec{i} + y\vec{j} + z\vec{k}$$

Ex: $\vec{u} = (1,-3,7) \overset{\text{or}}{=} 1\vec{i} - 3\vec{j} + 7\vec{k}$.

$\vec{v} = 2\vec{i} + 3\vec{j} - 5\vec{k}$ is Equivalent to $\vec{v} = (2,3,-5)$

Special products: Two:

(A) The Dot Product:

Let $\vec{u} = (x_1, y_1, z_1)$, and $\vec{v} = (x_2, y_2, z_2)$ be vectors in $\mathbb{R}^3$. The Dot product of $\vec{u}$, and $\vec{v}$ in that order is denoted and defined by

$$\vec{u}\cdot\vec{v} = (x_1, y_1, z_1)\cdot(x_2, y_2, z_2)$$
$$= x_1x_2 + y_1y_2 + z_1z_2$$
$$= \text{A scalar quantity (A real Number!)}$$

We simply multiplied corresponding components and added up!

Ex Let $\vec{a} = (5,1,3)$, $\vec{b} = (-1,3,1)$.

$$\vec{a}\cdot\vec{b} = (5,1,3)\cdot(-1,3,1) = (5)(-1) + (1)(3) + (3)(1)$$
$$= -5 + 3 + 3 = 1$$

(B) The Cross Product:

Let $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$ be vectors in $\mathbb{R}^3$. The cross product of $\vec{u}, \vec{v}$ in that order is denoted and defined by

$$\vec{u} \times \vec{v} = \left(+ \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)$$

$= $ A vector quantity!

Note: The 2×2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Note also that if

$$A = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

then the first determinant $\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}$ is obtained from "A" by deleting first column. Similarly the remaining determinant may be obtained from "A" by deleting the 2nd. and 3rd. column respectively.

Ex: Let $\vec{u} = (2, -1, 7)$, $\vec{v} = (3, -4, 2)$. Find $\vec{u} \times \vec{v}$

Solution:

$$\vec{u} \times \vec{v} = \left(+ \begin{vmatrix} -1 & 7 \\ -4 & 2 \end{vmatrix}, - \begin{vmatrix} 2 & 7 \\ 3 & 2 \end{vmatrix}, + \begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix} \right)$$

$$= (26, 17, -5)$$

$$\begin{array}{c} \vec{u} \\ \vec{v} \end{array} \begin{pmatrix} 2 & -1 & 7 \\ 3 & -4 & 2 \end{pmatrix}$$

Properties of dot product: Similar to properties of real numbers

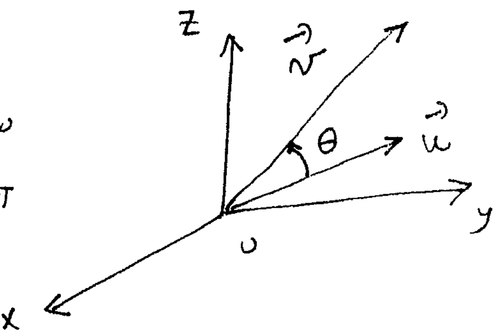
Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$, and $k \in \mathbb{R}$ be a scalar.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$, $\quad \vec{w} \cdot (\vec{u} + \vec{v}) = \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v}$.
3. $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$, $\quad \vec{u} \cdot (k\vec{v}) = k(\vec{u} \cdot \vec{v})$
4. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$.

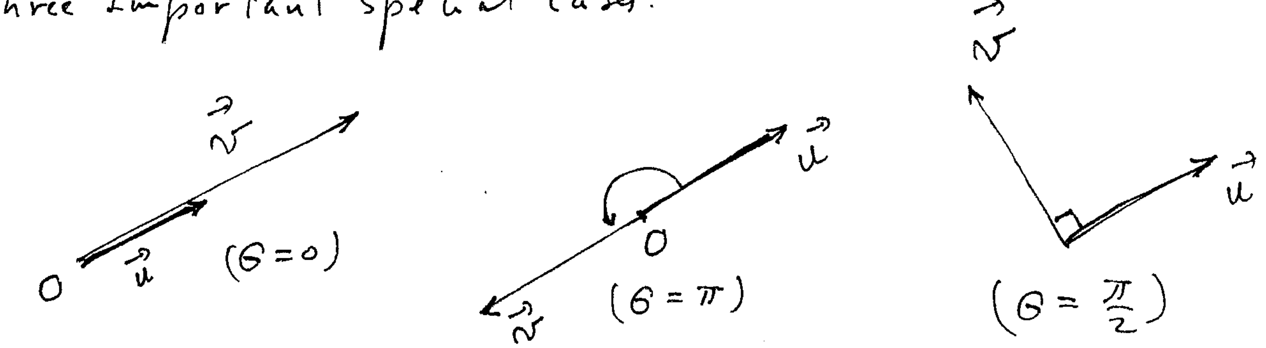
Application To dot product:

Let $\vec{u}, \vec{v}$ be non-zero vectors in $\mathbb{R}^3$, and let "θ" be the angle between $\vec{u}, \vec{v}$. Then

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}, \qquad 0 \leq \theta \leq 180^\circ$$
$$\text{or } 0 \leq \theta \leq \pi$$



Three Important special cases:



(i) If $\theta = 0$, then $\vec{u}$ is <u>parallel</u> to $\vec{v}$ in the <u>Same</u> direction and we write $\vec{u} \,/\!/\, \vec{v}$.

(ii) If $\theta = \pi$, then $\vec{u}$ is parallel to $\vec{v}$ in the opposite direction.

(iii) If $\theta = \frac{\pi}{2}$, then $\vec{u}$ is perpendicular <u>or</u> Orthogonal to $\vec{v}$ written $\vec{u} \perp \vec{v}$.

Note also that if $\theta = \frac{\pi}{2}$, we have

$$\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow \quad \vec{u} \cdot \vec{v} = 0$$

It follows that: $\vec{u} \perp \vec{v}$ if and only if $\vec{u} \cdot \vec{v} = 0$.

EX: let $\vec{a} = (5, -7, 1)$, $\vec{b} = (2, -3, -31)$

$$\therefore \quad \vec{a} \cdot \vec{b} = (5)(2) + (-7)(-3) + (1)(-31)$$

$$= 10 + 21 - 31 = 0$$

$$\therefore \quad \vec{a} \perp \vec{b}$$

EX: Find the angle θ between $\vec{x} = (-1, 0, 1)$, and $\vec{y} = (4, -1, 1)$.

<u>solution</u>: $\cos(\theta) = \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \dfrac{(-1, 0, 1) \cdot (4, -1, 1)}{\sqrt{(-1)^2 + 0^2 + 1^2} \cdot \sqrt{4^2 + (-1)^2 + 1^2}}$

$$= \frac{-4 + 0 + 1}{\sqrt{2}\sqrt{18}} = \frac{-3}{\sqrt{(2)(18)}} = -\frac{3}{\sqrt{36}}$$

$$= -\frac{3}{6} = -\frac{1}{2}$$

$$\therefore \quad \cos(\theta) = -\frac{1}{2}, \qquad 0 \leq \theta \leq \pi$$

$$\therefore \quad \theta = \frac{2\pi}{3} \qquad (\text{or } 120^\circ)$$

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Properties of Cross product:

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$ and $k \in \mathbb{R}$ be a scalar.

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

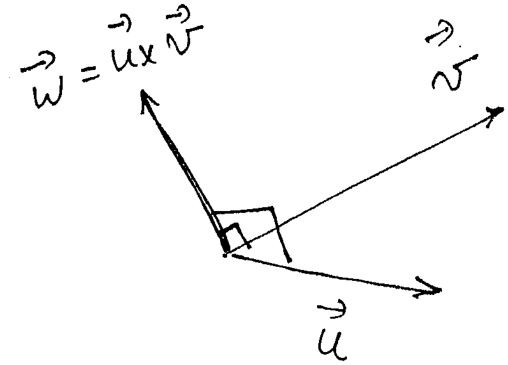
   $\vec{w} \times (\vec{u} + \vec{v}) = \vec{w} \times \vec{u} + \vec{w} \times \vec{v}$
3. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v})$, $\vec{u} \times (k\vec{v}) = k(\vec{u} \times \vec{v})$
4. $\vec{u} \times \vec{u} = \vec{0}$

Geometric Interpretation of the cross product:

Let $\vec{u}$, and $\vec{v}$ be non-zero vectors in $\mathbb{R}^3$. The cross product of $\vec{u}$, and $\vec{v}$ is a vector $\vec{w}$ in $\mathbb{R}^3$ orthogonal to both $\vec{u}$, and $\vec{v}$. That is if

$$\vec{w} = \vec{u} \times \vec{v} \text{ , then } \vec{w} \perp \vec{u} \text{ and } \vec{w} \perp \vec{v}$$

Refer to figure.



Ex: Find a unit vector orthogonal to the vectors $\vec{u} = (4, 2, -9)$, and $\vec{v} = (0, 2, 3)$.

Solution: First a vector $\perp$ $\vec{u}, \vec{v}$ is given by

$$\vec{w} = \vec{u} \times \vec{v}$$

$$= \left(\begin{vmatrix} 2 & -9 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 4 & -9 \\ 0 & 3 \end{vmatrix}, \begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} \right)$$

$$= (24, -12, 8) = 4(6, -3, 2)$$

There are two unit vector orthogonal to $\vec{u}$, $\vec{v}$, namely

$$\vec{n} = \pm \cdot \frac{\vec{w}}{\|\vec{w}\|}$$

Now, $\|\vec{w}\| = \|4(6,-3,2)\| = 4\,\|(6,-3,2)\|$

$$= 4\sqrt{6^2 + (-3)^2 + 2^2}$$

$$= 4\sqrt{36+9+4} = 4\sqrt{49}$$

$$= (4)(7) = 28$$

$$\therefore \vec{n} = \pm \frac{4(6,-3,2)}{28} = \pm \frac{1}{7}(6,-3,2)$$

Ex: Let $\vec{u} = \vec{i} + \vec{j} + \vec{k}$, $\vec{v} = -\vec{i} - \vec{j} - \vec{k}$, and $\vec{w} = \vec{i}$ be vectors in $\mathbb{R}^3$.

Verify that $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$

Answer: $(\vec{u} \times \vec{v}) \times \vec{w} = (0, 0, 0)$

$\vec{u} \times (\vec{v} \times \vec{w}) = (2, -1, -1)$

Important Remark: All definitions, properties and discussions for vectors in $\mathbb{R}^3$ (with the exception of the cross product) hold true for vectors in $\mathbb{R}^2$!

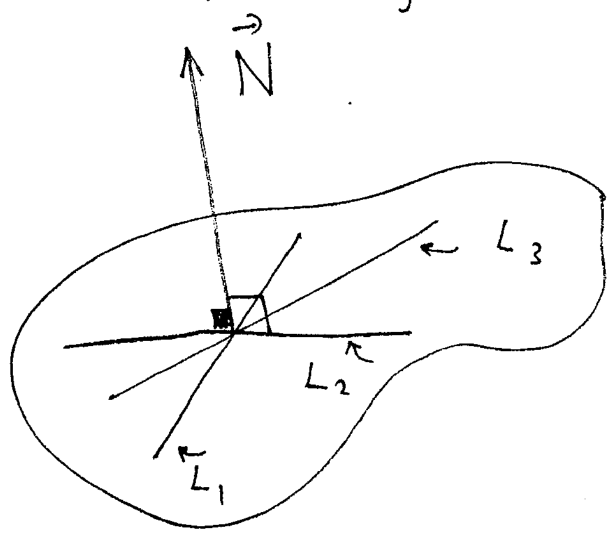
Applications to Vectors : plane and straight line in $\mathbb{R}^3$.

(A) The plane :

An infinite sheet in 3-space is called : a plane. Geometrically, a plane may be represented by an arbitrary closed figure as shown below.

Normal Vector to a plane:

A vector perpendicular to every straight line in the plane is referred to as a Normal Vector, and may be denoted by $\vec{N} = (a, b, c)$.



Note : $\vec{N}$ is perpendicular to $L_1, L_2, L_3, \ldots$ etc.

Equation of a plane: The point-normal form

The equation of a plane passing through the point $P(x_0, y_0, z_0)$ and has a normal vector $\vec{N} = (a, b, c)$ is given by

$$\vec{r} \cdot \vec{N} = \vec{r}_0 \cdot \vec{N} \quad \text{----} \quad (*)$$

where $\vec{r} = (x, y, z)$, and $\vec{r}_0 = (x_0, y_0, z_0)$.

Equation $(*)$ is the well known: point-normal form.

Remark: If we expand equation (*), we get an equation of the form:

$$ax + by + cz + d = 0.$$

This is the standard or general Equation of a plane. Observe that the Coefficients of x, y, and z, namely a, b, and c are the Components of a normal vector $\vec{N}$.

Ex: A vector normal to the plane

$$2x - 3y + 16z - 35 = 0$$

is given by $\vec{N} = (2, -3, 16)$.

Ex: Find an equation of the plane passing through the point $P(4, 2, -1)$ and has a normal vector $\vec{N} = (2, -3, 5)$.

Solution: Recall: point-normal form

$$\vec{r} \cdot \vec{N} = \vec{r_0} \cdot \vec{N} \quad ; \quad \vec{r} = (x, y, z)$$

Here $\vec{r_0} = (4, 2, -1)$, $\vec{N} = (2, -3, 5)$

$$\therefore \quad (x, y, z) \cdot (2, -3, 5) = (4, 2, -1) \cdot (2, -3, 5)$$

$$2x - 3y + 5z = 8 - 6 - 5$$

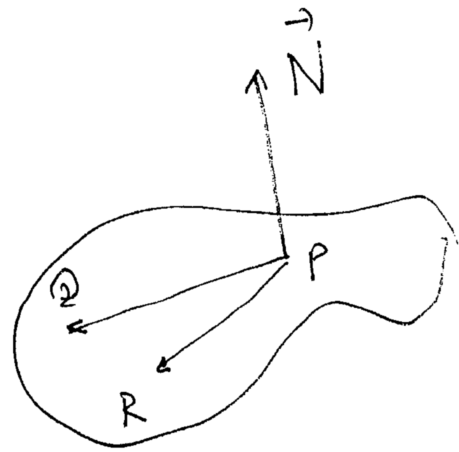
$$\Rightarrow \quad 2x - 3y + 5z + 3 = 0$$

Ex: Find an equation of a plane passing through the three points $P(1, 2, -1)$, $Q(3, 1, -2)$, and $R(2, 1, 4)$.

Solution: For an equation of a plane, one needs:

(i) A point : choose say point $P(1,2,-1)$. Hence $\vec{r_0} = (1,2,-1)$.

(ii) A normal vector $\vec{N}$:



Construct vectors $\vec{u}, \vec{v}$ as follows

$$\vec{u} = \vec{PQ} = (3,1,-2)-(1,2,-1)$$
$$= (2,-1,-1)$$
$$\vec{v} = \vec{PR} = (2,1,4)-(1,2,-1)$$
$$= (1,-1,5)$$

Clearly $\vec{N} \perp \vec{u}$, and $\vec{v}$. Hence

$$\vec{N} = \vec{u} \times \vec{v} = \left(\begin{vmatrix} -1 & -1 \\ -1 & 5 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} \right)$$
$$= (-6,-11,-1)$$

$$\begin{array}{c} \vec{u} \\ \vec{v} \end{array} \begin{pmatrix} 2 & -1 & -1 \\ 1 & -1 & 5 \end{pmatrix}$$

Equation of plane is thus given by

$$\vec{r} \cdot \vec{N} = \vec{r_0} \cdot \vec{N} \qquad ; \vec{r} = (x,y,z)$$

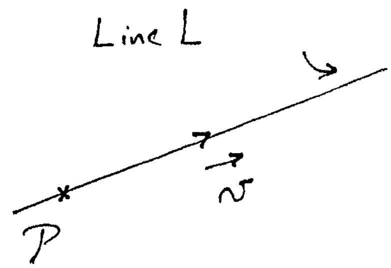
$$(x,y,z) \cdot (-6,-11,-1) = (1,2,-1) \cdot (-6,-11,-1)$$

$$\therefore \quad -6x - 11y - z = -6 - 22 + 1$$

$$\Rightarrow \quad -6x - 11y - z = -27$$

$$\text{or} \quad 6x + 11y + z = 27$$

(B) The straight Line:



Let L be the straight Line in $\mathbb{R}^3$ passing through the point $P(x_0, y_0, z_0)$ and has a direction vector $\vec{v} = (a,b,c)$.

The Vector equation of Line L is given by

$$\vec{r} = \vec{r_0} + t\vec{v} \quad , \quad t \in \mathbb{R}$$

where $\vec{r} = (x, y, z)$, and $\vec{r_0} = (x_0, y_0, z_0)$

$$\therefore \quad \boxed{(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)} \quad (*)$$

Equating Corresponding Components of (*) we get:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \qquad t \in \mathbb{R} \\ z &= z_0 + ct \end{aligned}$$

These are the well Known: Parametric Equations of Line L. They are the <u>most</u> useful!

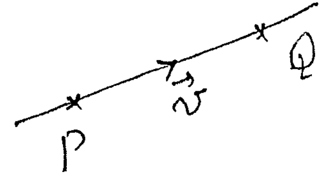
<u>EX</u>: Find parametric equations of the straight Line passing through the two points $P(1, -1, 4)$, and $Q(5, 2, -1)$.

<u>Solution</u>: For straight line, one needs:

(i) <u>A point</u>: choose say $P(1, -1, 4)$. Hence

$$\vec{r_0} = (1, -1, 4)$$

(ii) <u>A direction vector $\vec{v}$</u>:



Indeed: A vector in the direction of the Line is $\vec{v} = \vec{PQ}$

$$\begin{aligned} &= (5, 2, -1) - (1, -1, 4) \\ &= (4, 3, -5) \end{aligned}$$

Vector equation of line is thus given by

$$\vec{r} = \vec{r_0} + t\,\vec{v}, \quad t \in \mathbb{R}$$

$$\therefore \ (x, y, z) = (1, -1, 4) + t(4, 3, -5), \quad t \in \mathbb{R}$$

parametric equations are thus given by

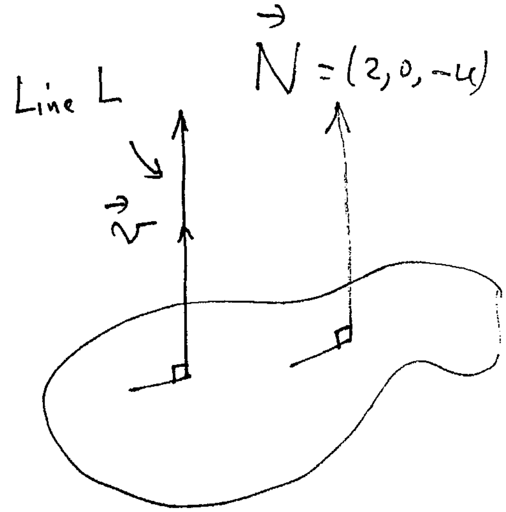
$$\begin{aligned} x &= 1 + 4t \\ y &= -1 + 3t \qquad t \in \mathbb{R} \\ z &= 4 - 5t \end{aligned}$$

Ex: Find parametric equations of the straight line passing through the origin and is perpendicular to the plane $2x - 4z + 91 = 0$.

Solution: For a straight line, one needs:

(i) A point: Given as $(0,0,0) \Rightarrow \vec{r_0} = (0,0,0)$

(ii) A direction vector $\vec{v}$?



Refer to figure:

Since line $\perp$ to plane,

$\therefore \ \vec{v} \parallel \vec{N}$

Take $\vec{v} = \vec{N} = (2, 0, -4)$.

parametric equations are:

$$\begin{aligned} x &= 0 + 2t \\ y &= 0 + 0t \quad , \ t \in \mathbb{R} \\ z &= 0 - 4t \end{aligned}$$

For the plane:

$2x + 0 \cdot y - 4z + 91 = 0$

normal vector $\vec{N} = (2, 0, -4)$

EX: Find an equation of the plane passing through the point $P(4,0,-3)$ and is perpendicular to the line
$$\vec{r} = (2+3t)\vec{i} + (1-7t)\vec{j} + 20\vec{k}$$

Solution: First: let us find parametric equation of the line: $\vec{r} = (2+3t)\vec{i} + (1-7t)\vec{j} + 20\vec{k}$

$$\vec{r} \overset{or}{=} (2+3t, 1-7t, 20)$$

$$\therefore (x,y,z) = (2+3t, 1-7t, 20)$$

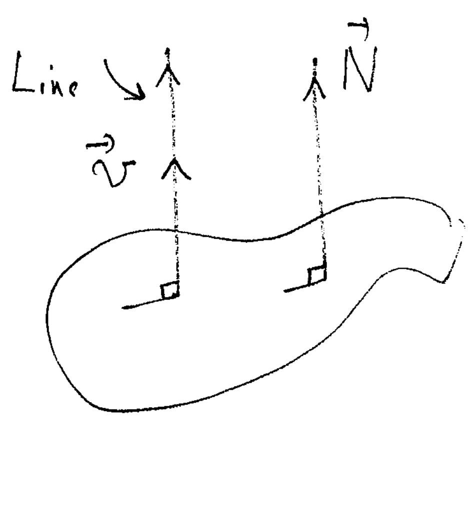
$$\therefore \quad x = 2 + 3t$$
$$y = 1 - 7t$$
$$z = 20 + 0t$$

So: A direction vector of line is $\vec{v} = (3, -7, 0)$.

For equation of a plane, one needs:

(i) A point: Given as $P(4,0,-3) \Rightarrow \vec{r}_0 = (4,0,-3)$

(ii) A normal vector $\vec{N}$?



From figure: $\vec{v} \parallel \vec{N}$

$\therefore$ Take $\vec{N} = \vec{v} = (3,-7,0)$

Recall: point-normal form

$$\vec{r}\cdot\vec{N} = \vec{r}_0\cdot\vec{N}$$

$$(x,y,z)\cdot(3,-7,0) = (4,0,-3)\cdot(3,-7,0)$$

$$\therefore \quad 3x - 7y = 12 + 0 + 0$$

$$\Rightarrow \quad 3x - 7y - 12 = 0$$