when other than z, centre is changed not last

MATH 277

Problem Set # 5 for Labs

Note : Problems marked with $(*)$ are left for students to do at home.

1. Find the radius of curvature of each of the following plane curves :

   $(i)\ \ x^2 + y^2 = 4$ 2 $\qquad (ii)\ \ x^2 + y^2 - 2x + 4y - 20 = 0$ (x-1)^2-1+(y+2)^2-4=20 radius of 5

   $(iii)\ \ (12x-1)^2 + (12y+3)^2 = 1.$ 144x^2-24x+1+144y+72y+9=1 x^2-24/144x+1/144+y^2+y/2+9/144=1/144

   Hint : **First identify the curve**. radius of 1/12

2. A frictionless highway turn lies along a circle of radius 230 metres. Find the angle at which turn should be banked if the maximum posted speed is $80\ \ km/hr$

   v^2 = pgtantheta

   tan^-(80000^2/(9.8*230*3600^2)) = theta

3. Identify each of the following surfaces:

   (i) $z = 1 - 3\sqrt{4x^2 + y^2}$ (z-1)^2/9 = 4x^2+y^2 cone $\qquad$ (ii) $y = 2 - x^2 - z^2$ spherical paraboloid (0,-2,0) down $\qquad$ (iii) $2 - 5x^2 - 3y^2 - 2z^2 = 0$ ellipsoid 5/2x^2+3/2y^2+z^2=1

   (iv) $\dfrac{x^2}{49} - \dfrac{y^2}{81} - \dfrac{z^2}{36} = 1$ hyperboloid 2 $\qquad$ (v) $x = z^2$ cylinder $\qquad$ (vi) $3x + 2y + 41 = 0$ plane

   (vii) $x^2 + y^2 + z^2 + 8y = 0$ sphere radius 4 $\qquad$ (viii) $z^2 + y^2 - x^2 - 4x = 5$ -(x+2)^2 = 9 hyperboloid one sheet

4. (a) If $z = \ln(xy)^{\tan(xy)}$, $x > 0$, $y > 0$, find $\dfrac{\partial z}{\partial x}$. Hint : First , simplify Logarithm.

   (b) Let $f(x,y) = y^{\ln(x)} + \sinh^{-1}(x^2)$, find $f_{yx}(x,y)$.

5. If $z = \sqrt{x^2 - y^2}$, $where\ x = r\cos(\theta)$, $and\ \ y = r\sin(\theta)$, find $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial \theta}$ at $(r,\theta) = (1, \frac{\pi}{6})$

   both directly and by the chain rule.

6. Let $z = x^4 + 2xy$, where $x = 1 - \sin(2t)$, and $y = t\ln(1+t)$.

   Use the Chain Rule to find $\dfrac{dz}{dt}$ at $t = 0$.

7. Let $W = x^2 + 2xyz$, where $x(t) = e^t$, $y(t) = \tan(3t) + 1$, and $z(t) = \cos^{-1}(t)$. Find $\dfrac{dW}{dt}$ at $t = 0$.

8. Let $z = u(s,t)$, where $s = x^2 - y^2$, and $t = 2xy$.

   Determine $\dfrac{\partial z}{\partial x}$ at $x = 2$, $y = -1$ given that $\dfrac{\partial u}{\partial s}(3,-4) = 7$, and $\dfrac{\partial u}{\partial t}(3,-4) = -5.$

9. Let $z = \ln(x^3 + 2y)$, where $x = x(r,s)$, and $y = y(r,s)$. Find $\frac{\partial z}{\partial s}$ at $r = 1$, $s = 3$

given that $x(1,3) = 0$, $y(1,3) = \frac{1}{2}$, $\frac{\partial x}{\partial s}(1,3) = -1$, and $\frac{\partial y}{\partial s}(1,3) = 2$.

10* Let $u = f(x,y,z)$, where $x = x(s,t)$, $y = y(s,t)$, and $z = z(s,t)$.

Determine $\frac{\partial u}{\partial t}$ at $s = -1$, and $t = 3$ given that $x(-1,3) = 4$, $y(-1,3) = 1$, $z(-1,3) = 9$,

$\frac{\partial x}{\partial t}(-1,3) = 7$, $\frac{\partial y}{\partial t}(-1,3) = 1$, $\frac{\partial z}{\partial t}(-1,3) = -6$, $\frac{\partial f}{\partial x}(4,1,9) = -2$, $\frac{\partial f}{\partial y}(4,1,9) = 1$,

and $\frac{\partial f}{\partial z}(4,1,9) = -5$.

11. Let $g(t)$ be a twice continuously differentiable function and let $z = g(x - 3y)$.

Find $\frac{\partial^2 z}{\partial y \partial x}$ at $x = 3$, $y = 1$ given that $g''(0) = 5$.

12* Let $f(u)$ and $g(v)$ be arbitrary functions having continuous second order derivatives on some real interval $\mathbf{I}$. Show that $w(x,t) = f(x + ct) + g(x - ct)$, (where c is a positive constant real number) satisfies the one – dimensional wave equation $c^2 w_{xx} = w_{tt}$ on $\mathbf{I}$.

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Solutions to Problem Set # 5

1. (i) $x^2+y^2=4$ is an equation of a circle of radius $a=\sqrt{4}=2$. Hence the radius of curvature is given by $\rho=a$; that is $\rho=2$.

(ii) $x^2+y^2-2x+4y-20=0$ is an equation of a circle.

Let us first complete the square in both x, and y-terms

$$(x^2-2x)+(y^2+4y)=20$$

$$\left[x^2-2x+\left(\tfrac{-2}{2}\right)^2\right]+\left[y^2+4y+\left(\tfrac{4}{2}\right)^2\right]=20+\left(\tfrac{-2}{2}\right)^2+\left(\tfrac{4}{2}\right)^2$$

$$\left(x^2-2x+(-1)^2\right)+\left(y^2+4y+2^2\right)=20+1+4$$

$$(x-1)^2+(y+2)^2=25$$

$\therefore$ Radius $=\sqrt{25}=5$, hence $\rho=5$ as well.

(iii) $(12x-1)^2+(12y+3)^2=1$

Rewrite equation in the form

$$\left[12\left(x-\tfrac{1}{12}\right)\right]^2+\left[12\left(y+\tfrac{3}{12}\right)\right]^2=1$$

$$144\left(x-\tfrac{1}{12}\right)^2+144\left(y+\tfrac{1}{4}\right)^2=1 \qquad (\div 144)$$

$$\left(x-\tfrac{1}{12}\right)^2+\left(y+\tfrac{1}{4}\right)^2=\frac{1}{144}$$

This is an equation of circle of radius $\sqrt{\frac{1}{144}}=\frac{1}{12}$, hence $\rho=\frac{1}{12}$.

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2. Recall

Banking angle

$$\theta = \tan^{-1}\left(\frac{v^2}{\rho g}\right)$$

Here $\rho = 230$, $g = 9.8\ m/s^2$, and

$v = 80\ Km/h$; that is $\frac{80}{3.6}\ m/s$

$$\therefore \theta = \tan^{-1}\left(\frac{\left(\frac{80}{3.6}\right)^2}{(230)(9.8)}\right)$$

$$\approx 12^\circ$$

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3. (i) $z = 1 - 3\sqrt{4x^2 + y^2}$

$$\Rightarrow \quad z - 1 = -3\sqrt{4x^2 + y^2}$$

This is an Equation of the lower nappe of a Cone with vertex at $(0, 0, -1)$, and axis of symmetry being the z-axis.

(ii) $y = 2 - x^2 - z^2$

$$\Rightarrow \quad y - 2 = -(x^2 + z^2)$$

This is an Equation of a paraboloid with vertex at $(0, 2, 0)$, axis of symmetry being the y-axis and which opens to the left.

(iii) $2 - 5x^2 - 3y^2 - 2z^2 = 0$

$$\Rightarrow \quad 5x^2 + 3y^2 + 2z^2 = 2$$

This is an Equation of an Ellipsoid with Centre at the origin.

(iv) $\dfrac{x^2}{49} - \dfrac{y^2}{81} - \dfrac{z^2}{36} = 1 \quad \Rightarrow$

$$-\frac{x^2}{49} + \frac{y^2}{81} + \frac{z^2}{36} = -1$$

This is an Equation of a Hyperboloid of two sheets with Centre at $(0, 0, 0)$ and axis of symmetry is the x-axis.

(v) $x = z^2$ is an Equation of a Cylinder with generators parallel to the y-axis.

(vi) $3x + 2y + 41 = 0$ is an Equation of a plane (perpendicular to xy-plane).

(vii) $x^2 + y^2 + z^2 + 8y = 0$

$\Rightarrow x^2 + (y^2 + 8y) + z^2 = 0$

↳ Complete the square!

Add $(\frac{8}{2})^2 = 4^2$ to both sides of Equation

$x^2 + (y^2 + 8y + 4^2) + z^2 = 4^2$

$\Rightarrow x^2 + (y+4)^2 + z^2 = 16$

This is an Equation of a sphere, centred at $(0,-4,0)$ and has radius 4 units

(viii) $z^2 + y^2 - x^2 - 4x = 5$

$\Rightarrow z^2 + y^2 - (x^2 + 4x) = 5$

↳ Complete the square!

Add $-(\frac{4}{2})^2$ to both sides

$z^2 + y^2 - (x^2 + 4x + 2^2) = 5 - 2^2$

$z^2 + y^2 - (x+2)^2 = 1$

$\Rightarrow (x+2)^2 + y^2 + z^2 = +1$

This is an Eq. of Hyperboloid of one sheet centred at $(-2, 0, 0)$ and axis of symmetry is the x-axis.

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4\. (a) $Z = \ln(xy)^{\tan(xy)}$

First, Simplify: $Z = \tan(xy)\ln(xy)$

$$= \tan(xy)\left[\ln(x) + \ln(y)\right]$$

$$\therefore \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\tan(xy)\right).\left[\ln(x) + \ln(y)\right] + \tan(xy).\frac{\partial}{\partial y}\left[\ln(x) + \ln(y)\right]$$

$$= \sec^2(xy).\, x\left[\ln(x) + \ln(y)\right] + \tan(xy).\left[0 + \frac{1}{y}\right]$$

$$= x\sec^2(xy).\ln(xy) + \frac{\tan(xy)}{y}$$

(b) $f(x,y) = y^{\ln(x)} + \sinh^{-1}(x^2)$

$$\frac{\partial f}{\partial y} \overset{\text{or}}{=} f_y = \ln(x)\, y^{\ln(x)-1} + 0 = \ln(x)\, y^{\ln(x)-1}$$

$$\therefore f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\ln(x)\, y^{\ln(x)-1}\right)$$

$$= \frac{1}{x}\, y^{\ln(x)-1} + \ln(x).\, y^{\ln(x)-1}\ln(y).\frac{1}{x}$$

$$= \frac{1}{x}\, y^{\ln(x)-1}\left[1 + \ln(x)\ln(y)\right]$$

Note: We have used:

$$\frac{d}{dt}\left(a^{u(t)}\right) = a^{u(t)}\ln(a).\, u'(t), \text{ where}$$

$a > 0,\ a \neq 1$ is a constant real number

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5. Let $z = f(x,y) = \sqrt{x^2 - y^2}$,

$x = r\cos(\theta)$, $y = r\sin(\theta)$.

<u>Method 1</u>: Calculate $\frac{\partial z}{\partial r}$, and $\frac{\partial z}{\partial \theta}$ <u>directly</u>:

$z = \sqrt{x^2 - y^2}$. substituting for $x = r\cos(\theta)$, and $y = r\sin(\theta)$, we obtain:

$$z = \sqrt{(r\cos(\theta))^2 - (r\sin(\theta))^2}$$

$$= \sqrt{r^2(\cos^2(\theta) - \sin^2(\theta))} = r\sqrt{\cos^2(\theta) - \sin^2(\theta)}$$

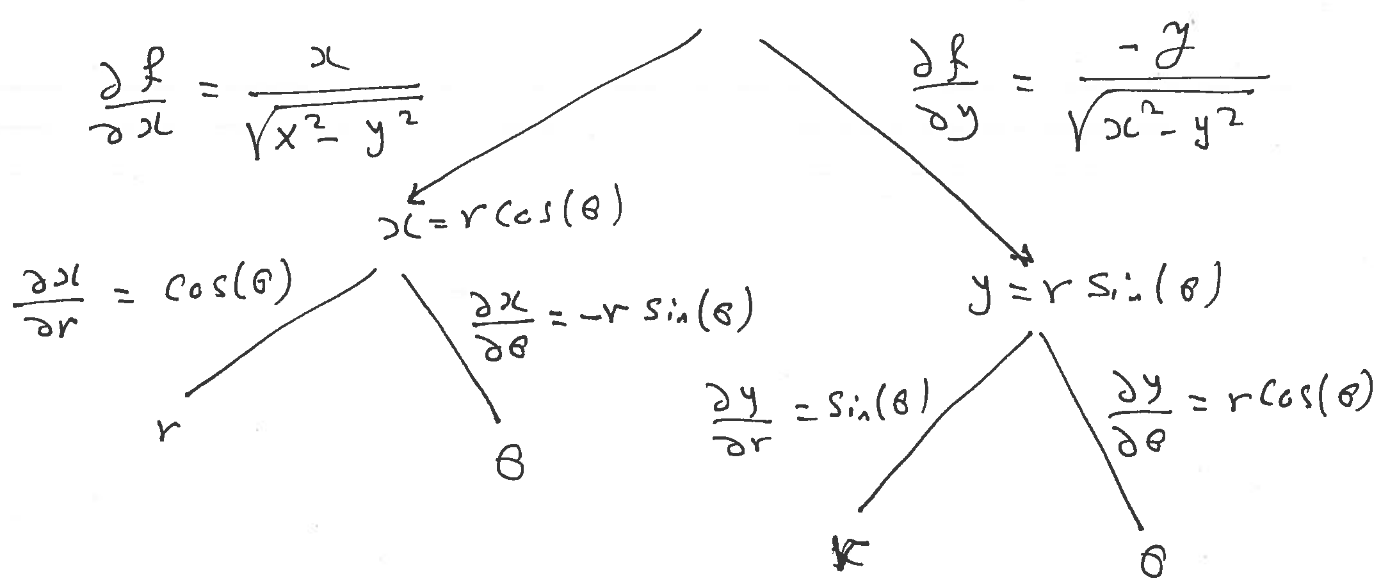
$$\therefore \quad z = r\sqrt{\cos(2\theta)}\Big|_{\substack{r=1\\ \theta=\pi/6}} = \frac{1}{\sqrt{2}}$$

$$\frac{\partial z}{\partial r} = \sqrt{\cos(2\theta)}, \quad \frac{\partial z}{\partial \theta} = r.\frac{1}{2}(\cos(2\theta))^{-\frac{1}{2}}(-\sin(2\theta).2)$$

$$\therefore \quad \frac{\partial z}{\partial \theta} = -\frac{r\sin(2\theta)}{\sqrt{\cos(2\theta)}}\Big|_{\substack{r=1\\ \theta=\pi/6}} = -\sqrt{\frac{3}{2}}$$

<u>Method 2</u>: Using the <u>chain rule</u> to calculate $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$:

$$z = f(x,y) = \sqrt{x^2 - y^2} \overset{\text{or}}{=} (x^2 - y^2)^{\frac{1}{2}}$$



$$\frac{\partial z}{\partial r} = \frac{x}{\sqrt{x^2 - y^2}} \cos(\theta) + \frac{-y}{\sqrt{x^2 - y^2}} \sin(\theta)$$

and $$\frac{\partial z}{\partial \theta} = \frac{x}{\sqrt{x^2 - y^2}} (-r\sin(\theta)) + \frac{-y}{\sqrt{x^2 - y^2}} (r\cos(\theta)).$$

At $r = 1$, $\theta = \pi/6$, we have

$x = r\cos(\theta) = 1\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$,

$y = r\sin(\theta) = 1\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\therefore \frac{\partial z}{\partial r} = \frac{x}{\sqrt{x^2 - y^2}} \cos(\theta) - \frac{y}{\sqrt{x^2 - y^2}} \sin(\theta) \Bigg|_{\substack{x = \frac{\sqrt{3}}{2},\, y = \frac{1}{2},\\ r = 1,\, \theta = \pi/6}}$$

$$= \frac{\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2}}{\sqrt{\frac{3}{4} - \frac{1}{4}}} = \frac{\frac{3}{4} - \frac{1}{4}}{\sqrt{\frac{1}{2}}} = \frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}}.$$

and

$$\frac{\partial z}{\partial \theta} = \frac{-\frac{\sqrt{3}}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{\sqrt{\frac{3}{4} - \frac{1}{4}}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = -\sqrt{\frac{3}{2}}$$

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6. $z = f(x,y) = x^4 + 2xy$, where

$x = 1 - \sin(2t)$, $y = t\ln(1+t)$

Clearly $z = z(t)$.

$$\therefore \quad \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

(or use diagram below).

$$\begin{tikzcd}[column sep=small]
& z = f(x,y) = x^4 + 2xy \arrow[dl, "\frac{\partial f}{\partial x} = 4x^3 + 2y"'] \arrow[dr, "\frac{\partial f}{\partial y} = 2x"] & \\
x = 1 - \sin(2t) \arrow[d, "\frac{dx}{dt} = -2\cos(2t)"'] & & y = t\ln(1+t) \arrow[d, "\frac{dy}{dt} = \ln(1+t) + \frac{t}{1+t}"] \\
t & & t
\end{tikzcd}$$

Note: At $t = 0$, $x = 1 - \sin(0) = 1$, $y = 0\ln(1+0) = 0$

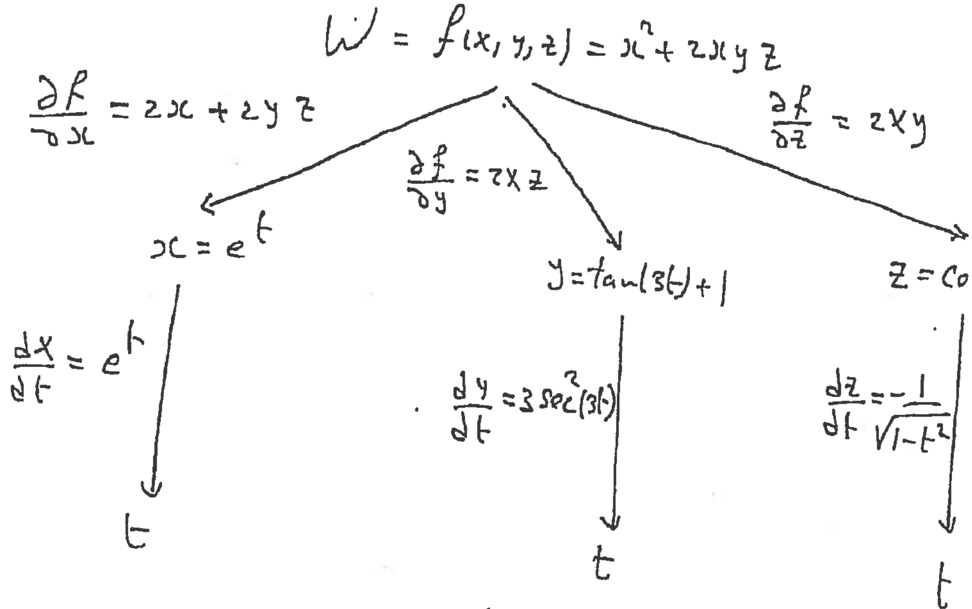
$$\left.\frac{dz}{dt}\right|_{t=0} = \left.(4x^3+2y)(-2\cos(2t)) + 2x\left[\ln(1+t) + \frac{t}{1+t}\right]\right|_{\substack{t=0\\x=1\\y=0}}$$

$$= (4+0)(-2) + 2(0+0) = -8$$

7. Let $f(x, y, z) = x^2 + 2xyz$

$\therefore\ W = f(x, y, z) = x^2 + 2xyz$

We shall use the chain rule illustrated by the Tree diagram below:



$$\therefore\ \frac{dW}{dt} = (2x + 2yz)e^t + 2xz \cdot 3\sec^2(3t) + (2xy)\left(-\frac{1}{\sqrt{1-t^2}}\right)$$

But at $t = 0$, we have $x = e^0 = 1$, $y = \tan(0) + 1 = 1$, and $z = \cos^{-1}(0) = \frac{\pi}{2}$.

$$\therefore\ \left.\frac{dW}{dt}\right|_{t=0} = \left.(2x + 2yz)e^t + 6xz\sec^2(3t) - \frac{2xy}{\sqrt{1-t^2}}\right|_{t=0,\ x=1,\ y=1,\ z=\frac{\pi}{2}}$$

$$= \left(2 + 2\cdot\frac{\pi}{2}\right)e^0 + 6(1)\cdot\frac{\pi}{2}\sec^2(0) - \frac{2(1)(1)}{\sqrt{1-0}}$$

$$= 2 + \pi + 3\pi - 2 = 4\pi$$

8. $$Z = U(s,t)$$

Since s, t are functions of x, and y, then clearly Z is a function of x, and y; that is

$$Z = Z(x,y)$$

We shall use the chain rule illustrated by the optional Tree Diagram below:

$$\begin{tikzcd}[column sep=small]
& Z(x,y) = U(s,t) \arrow[dl, "\frac{\partial U}{\partial s}(s,t)"'] \arrow[dr, "\frac{\partial U}{\partial t}(s,t)"] & \\
s = x^2 - y^2 \arrow[d, "\frac{\partial s}{\partial x} = 2x"'] & & t = 2xy \arrow[d, "\frac{\partial t}{\partial x} = 2y"] \\
x & & x
\end{tikzcd}$$

$$\therefore \quad \frac{\partial Z}{\partial x} = \frac{\partial U}{\partial s}(s,t) \cdot 2x + \frac{\partial U}{\partial t}(s,t) \cdot 2y$$

At $x = 2$, $y = -1$ we have $s = 2^2 - (-1)^2 = 3$, $t = 2(2)(-1) = -4$

$$\therefore \quad \left.\frac{\partial Z}{\partial x}\right|_{\substack{x=2 \\ y=-1}} = \left.\frac{\partial U}{\partial s}(s,t) \cdot 2x + \frac{\partial U}{\partial t}(s,t) \cdot 2y \right|_{\substack{x=2,\ y=-1 \\ s=3,\ t=-4}}$$

$$= \frac{\partial U}{\partial s}(3,-4) \cdot 2(2) + \frac{\partial U}{\partial t}(3,-4) \cdot 2(-1)$$

$$= 7 \cdot 4 + (-5)(-2) = 28 + 10 = 38$$

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9. $z = \ln(x^3 + 2y)$, where

$x = x(r, s)$, $y = y(r, s)$

$$\begin{tikzcd}[column sep=small]
& z = f(x,y) = \ln(x^3+2y) \arrow[dl, "\frac{\partial f}{\partial x} = \frac{3x^2}{x^3+2y}"'] \arrow[dr, "\frac{\partial f}{\partial y} = \frac{2}{x^3+2y}"] & \\
x = x(r,s) \arrow[d, "\frac{\partial x}{\partial s}(r,s)"'] & & y = y(r,s) \arrow[d, "\frac{\partial y}{\partial s}(r,s)"] \\
s & & s
\end{tikzcd}$$

By chain rule,

$$\frac{\partial z}{\partial s} = \frac{3x^2}{x^3+2y} \cdot \frac{\partial x}{\partial s}(r,s) + \frac{2}{x^3+2y} \frac{\partial y}{\partial s}(r,s)$$

At $r = 1$, $s = 3$, we have

$$\left.\frac{\partial z}{\partial s}\right|_{\substack{r=1\\ s=3}} = \left.\frac{3x^2}{x^3+2y} \frac{\partial x}{\partial s}(r,s) + \frac{2}{x^3+2y} \frac{\partial y}{\partial s}(r,s)\right|_{\substack{(r,s)=(1,3)\\ (x,y)=(0,\frac{1}{2})}}$$

$$= 0 + \frac{2}{0 + 2(\frac{1}{2})} \cdot 2 = \frac{2}{1} \cdot 2$$

$$= 4$$

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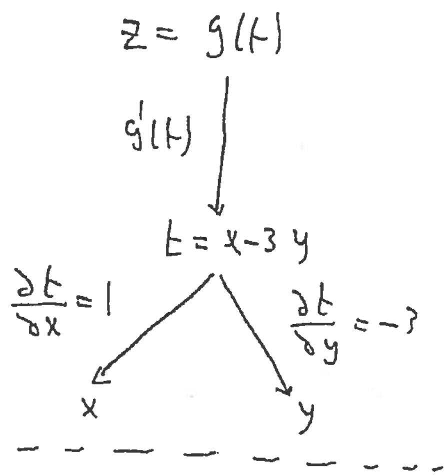
10. For students to do at home.

Answer : 17

11. $z = g(x-3y)$

let $t = x-3y$

$\therefore \; z = g(t)$, where $t = x-3y$



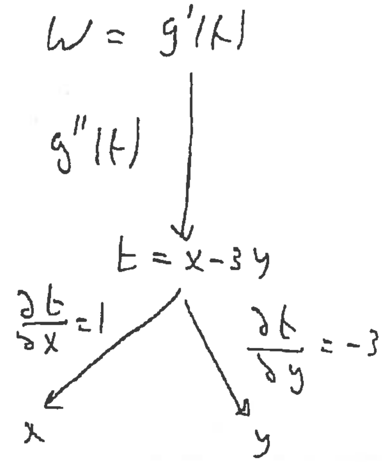
$\therefore \; \frac{\partial z}{\partial x} = g'(t) \cdot 1$

$= g'(t)$.

Next, let $w = g'(t)$

$\therefore \; \frac{\partial w}{\partial y} = g''(t) \cdot (-3)$

$= -3\, g''(t)$



Now,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(w)$$

$$= -3\, g''(t)$$

So $$\left.\frac{\partial^2 z}{\partial y \partial x}\right|_{\substack{x=3 \\ y=1}} = \left.-3\, g''(t)\right|_{\substack{t = 3-3(1) \\ = 0}}$$

$$= -3\, g''(0) = -3(5)$$

$$= -15$$