$$I = \frac{1}{3}\int_{t=1}^{t=9} \frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{3}\int_{1}^{9} t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{3}\left.\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right|_{1}^{9}$$

$$= \frac{2}{3}\sqrt{t}\,\Big|_{1}^{9} = \frac{2}{3}\left[\sqrt{9}-\sqrt{1}\right] = \frac{2}{3}(3-1) = \frac{4}{3}$$

New Limits

$t = 1 + x^3$

At $x = 0$, $t = 1 + 0 = 1$

At $x = 2$, $t = 1 + 2^3 = 9$

Ex4

Use Double Integrals to find the volume of the solid which lies below the surface $z = 1 + x^2$ and above the region in xy-plane enclosed by the lines $y = x$, $y = -x$, and $y = 1$

Solution:

Recall

$$V = \iint_{Base} \text{Height}\, dA$$

Base

Here: This is the Triangular region in xy-plane shown in figure below:

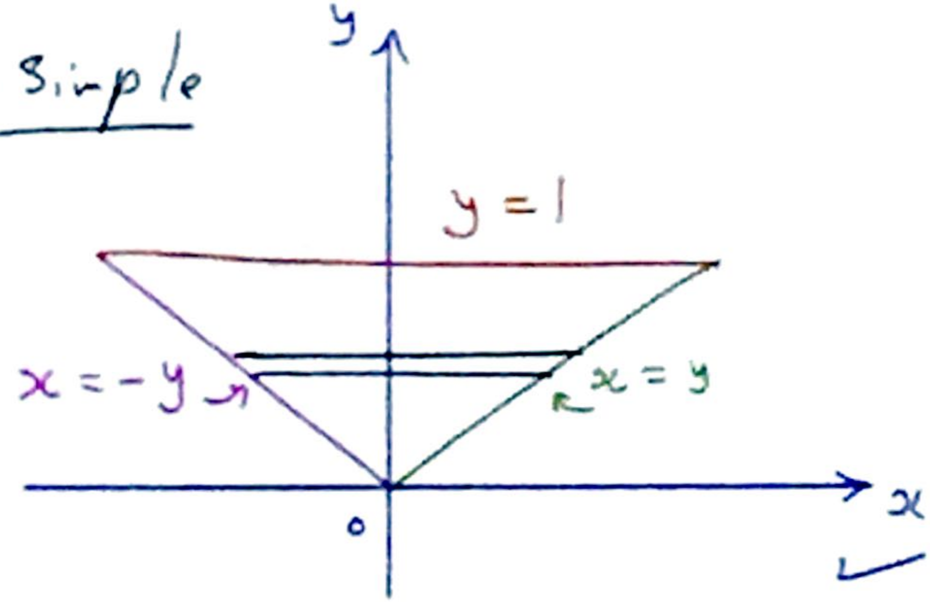
Height:

$$Z_{top} - Z_{bottom}$$

$Z_{top} = 1 + x^2$, and

$Z_{bottom} = xy\text{-plane } (z = 0)$
$= 0$

x-simple



Base: $\begin{cases} -y \le x \le y \\ 0 \le y \le 1 \end{cases}$

$$\text{Height} = (1 + x^2) - 0$$
$$= 1 + x^2 \checkmark$$

$$\therefore \text{ Volume } V = \int_{y=0}^{y=1} \int_{x=-y}^{y=x} (1 + x^2)\, dx\, dy$$

Volume $V = \int_0^1 x + \frac{1}{3}x^3 \Big|_{x=-y}^{x=y} dy$

$$= \int_0^1 \left[\left(y + \frac{1}{3}y^3\right) - \left(-y - \frac{1}{3}y^3\right)\right] dy$$

$$= \int_0^1 \left(2y + \frac{2}{3}y^3\right) dy$$

$$= y^2 + \frac{2}{3} \cdot \frac{1}{4}y^4 \Big|_0^1$$

$$= 1 + \frac{1}{6} = \frac{7}{6}$$

*__EX5__: Evaluate $\iint_R dA$

where R is the region in xy-plane enclosed by the curve $(2x-1)^2+(2y+3)^2=16$.

__Solution__:

First, observe that

$$(2x-1)^2+(2y+3)^2=16$$

$$\Rightarrow \quad 2^2\left(x-\frac{1}{2}\right)^2+2^2\left(y+\frac{3}{2}\right)^2=16 \qquad (\div 2^2=4)$$

$$\left(x-\frac{1}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=4$$

This is an equation of a circle centred at $\left(\frac{1}{2},-\frac{3}{2}\right)$ and has radius $a=\sqrt{4}=2$.

Observe that

$$\iint_R dA = A$$

$= \text{area of region } R$

$= \text{area of circle of radius } 2$

$= \pi a^2 = \pi(2)^2 = 4\pi$

Ex 6: Express the iterated integral

$$\int_0^1 \int_{z^2}^1 \int_0^{1-y} g(x,y,z)\,dx\,dy\,dz$$

as an equivalent integral in which the y-integration is performed first, z-integration second, and x-integration last.

Solution:

$$I = \int_{z=0}^{z=1} \int_{y=z^2}^{y=1} \int_{x=0}^{x=1-y} g(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_E g(x,y,z)\,dV$$

The requested order is

y, z, then x

There $dV = dy\, dz\, dx$

<u>Recall:</u>

(1) The inner-most limits ("y") can be functions of z, and x.

(2) The middle limits ("z") can be functions of only x

(3) The outer-most limits ("x") <u>must</u> be constant real numbers

$$I = \iiint_E g(x,y,z)\, dV$$

where E is the region described by

$$0 \le x \le 1-y \quad \text{----} \quad (1)$$

$$z^2 \le y \le 1 \quad \text{----} \quad (2)$$

$$0 \le z \le 1 \quad \text{------} \quad (3)$$

The idea is: To Rearrange inequalities (1), (2), (3).

First : Inner-most limit : Limits for "y"

From (1): $x \le 1-y$

$$\boxed{y \le 1-x} \quad \text{---} \quad (*)$$

From (2) : $\boxed{z^2 \leq y}$ ---- (**)

Combine (*), (**), we obtain

$$\boxed{z^2 \leq y \leq 1-x} \leftarrow \text{Inner-most}$$

Next : Middle limits : Limits for z

From (2) $z^2 \leq y$

But limits for z could be functions of "x" only

Indeed

$$z^2 \leq y \quad , \text{ but } \quad y \leq 1-x$$

$$\Rightarrow \quad z^2 \leq 1-x$$

$$\boxed{z \leq \sqrt{1-x}}$$

From (3): $\boxed{0 \leq z}$

Combing these inequalities, we obtain

$$\boxed{0 \leq z \leq \sqrt{1-x}} \quad \text{middle limits}$$

<u>Outer-most Limits</u>: Limits for "x"

From (1): $0 \leq x \leq 1 - y$

But limits for "x" must be Constant Real number. Hence,

$$0 \leq x \leq 1 - y \leq 1 \qquad (\text{because } y \geq 0)$$

$$\Rightarrow \quad \boxed{0 \leq x \leq 1} \quad \text{outer-most}$$

$$I = \int_{x=0}^{x=1} \int_{z=0}^{z=\sqrt{1-x}} \int_{y=z^2}^{y=1-x} g(x,y,z)\, dy\, dz\, dx$$

Polar, Cylindrical and Spherical Coordinates

1 Polar Coordinates

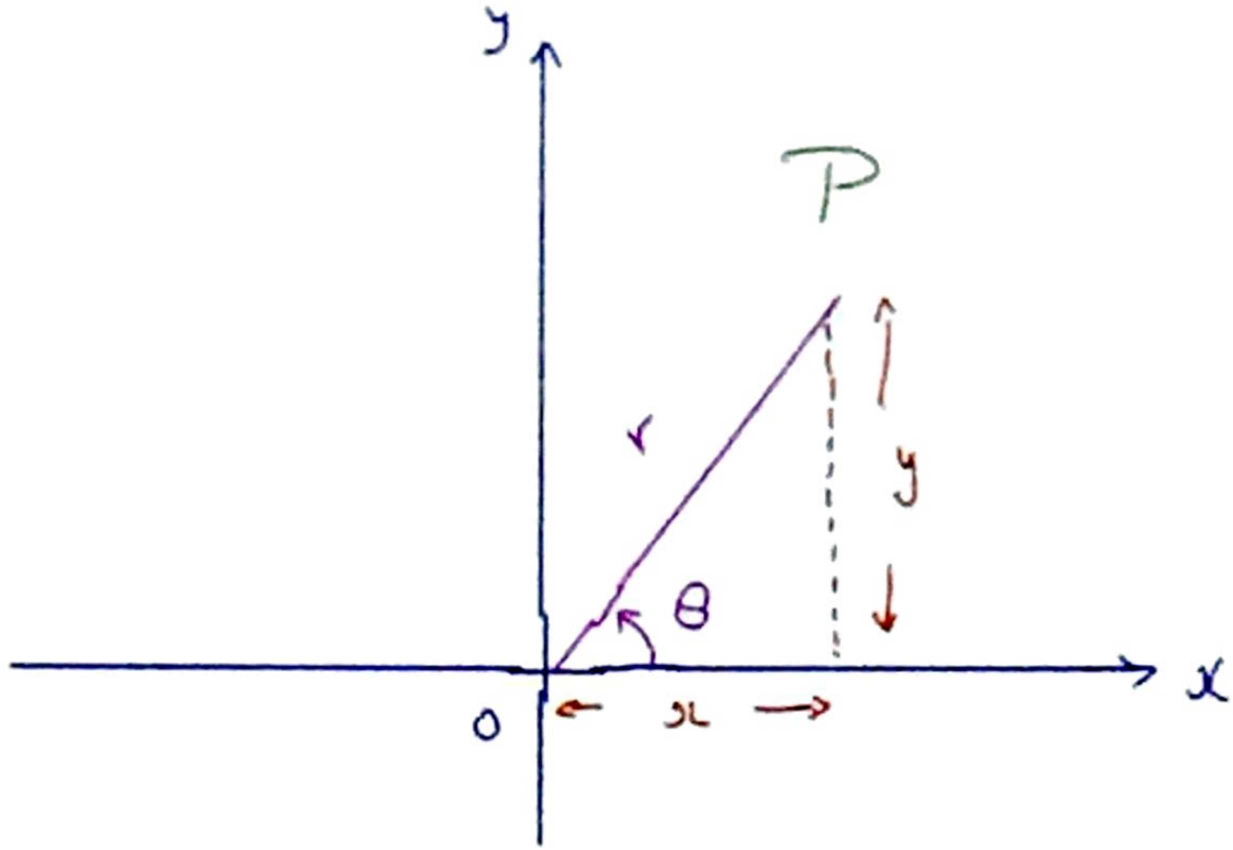
Let P be a point in xy-plane, say $P(x,y)$.

The polar coordinates of P are

r : The distance from origin to point P

θ : The angle made by $\vec{OP}$ and positive-half of x-axis (in radius).

Often polar and Cartesian coordinates are displayed on same set of coordinate axes as shown.



y
P
r
y
θ
x
o
x