Math 375 **Fall 2016**

First Order Differential Equations

Worksheet # 1 **Part 1** **September 19 - 23**

The problems marked with $(*)$ are to be attempted during the tutorial time. Students are strongly encouraged to attempt the remaining problems on their own. Solutions to all the problems will be available on the course's D2L website Friday, September 23. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Basics

01*. For each of the differential equations, determine its order and whether it is linear or nonlinear.

a. $\left(t\,y''+y\right)\left(t^2+\mathrm{e}^t\right)=1$ **b.** $\left((t^2+1)\,y'\right)'+y^3=t$ **c.** $t\,y'+|y|=\tan(t)$

02. Verify that $u(t)=\dfrac{t^2}{3}+\dfrac{C}{t}$, is a solution of the differential equation $t\,y'+y=t^2$

03*. Verify that $y^2-2\,x\,y-x^2-8\,y+4\,x=C$, is an implicit solution of the differential equation

$$y'=\frac{y+x-2}{y-x-4}$$

04. Show that $u(t)=\begin{cases}1-\mathrm{e}^{-t} & \text{if} \quad t<0\\ \mathrm{e}^t-1 & \text{if} \quad t\geq 0\end{cases}$ is a solution of the differential equation

$$y'-|y|=1 \ \text{ in } \ (-\infty,+\infty)$$

Linear First Order

05. Find the general solution of the given differential equation.

a. $y'+\dfrac{1}{t}\,y=\dfrac{7}{t^2}+3$ **b*.** $t\,y'+\left(2\,t^2+1\right)y=t^3\,\mathrm{e}^{-t^2}$

c. $x^2y'+3\,x\,y=\mathrm{e}^x$ **d*.** $y'=y\,\cos(x)+\sin(2\,x)$

06. Solve the initial value problems.

a. $\begin{cases}y'+2\,x\,y=x\\ y(0)=3\end{cases}$ **b*.** $\begin{cases}x\,y'+3\,y=\dfrac{2}{x(x^2+1)}\\ y(-1)=0\end{cases}$

c. $\begin{cases}(t^2-5)\,y'-2\,t\,y=-2\,t\,(t^2-5)\\ y(2)=7\end{cases}$

07. Find the largest interval $(a\,,\,b)$ on which the solution of the given initial value problem is guaranteed to exist.

a*. $\begin{cases}(t+5)\,y'+t\,y=\ln\left(t-\dfrac{4}{t}\right)\\ y(-1)=1\end{cases}$ **b*.** $\begin{cases}(t+5)\,y'+t\,y=\ln\left(t-\dfrac{4}{t}\right)\\ y(3)=1\end{cases}$

c. $\begin{cases}y'+\dfrac{t}{t^2-4}\,y=\sqrt{5-t}\\ y(3)=0\end{cases}$ **d.** $\begin{cases}y'+\dfrac{t}{t^2-4}\,y=\sqrt{5-t}\\ y(1)=0\end{cases}$

Separable Equations

08. Solve the following differential equations.

a*. $t\,y' + y + y^2 = 0$ **b.** $(x-1)\,y\,\mathrm{d}y + x\,(y-3)\,\mathrm{d}x = 0$

c. $y' + 2\,x\big(y^2 - 3\,y + 2\big) = 0$ **d.** $x^2\,y\,y' = \big(y^2-1\big)^{3/2}$

09. Solve the initial value problem $\begin{cases} y' + 2\,t\,\big(y^2 - 3\,y + 2\big) = 0 \\ y(0) = 3 \end{cases}$ and find the largest interval of validity of the solution.

10. Solve the initial value problem, and find the largest interval of validity of the solution.

a. $\begin{cases} (1+2\,y)y' - 2\,x = 0 \\ y(2) = 3 \end{cases}$ **b*.** $\begin{cases} y' - t\,y^2 = 0 \\ y(0) = 1 \end{cases}$

Answers and Solutions

01a. Rewriting the differential equation as $t\,y'' + y = \dfrac{1}{t^2 + \mathrm{e}^t}$ shows that it is a second order linear differential equation.

01b. Rewrite the differential equation as $\big(t^2+1\big)\,y'' + 2\,t\,y' + y^3 = t$, to see that it is second order. The y^3-term makes it nonlinear.

01c. The differential equation is first order. The $|y|$-term makes it nonlinear.

02. For $t \neq 0$, we have

$$t\,u'(t) + u(t) = t\left(\frac{2\,t}{3} - \frac{C}{t^2}\right) + \left(\frac{t^2}{3} + \frac{C}{t}\right) = \frac{2\,t^2}{3} - \frac{C}{t} + \frac{t^2}{3} + \frac{C}{t} = t^2$$

Hence $u(t) = \dfrac{t^2}{3} + \dfrac{C}{t}$ is a solution of the differential equation in both $(-\infty\,,\,0)$ and $(0\,,\,\infty)$

03. Recall the implicit differentiation formula

If $F(x,y) = C$ defines y as a differentiable function of x, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{\dfrac{\partial F}{\partial x}(x,y)}{\dfrac{\partial F}{\partial y}(x,y)}$

Here $F(x,y) = y^2 - 2\,x\,y - x^2 - 8\,y + 4\,x$. Therefore

$$y' = -\frac{\dfrac{\partial}{\partial x}\Big(y^2 - 2\,x\,y - x^2 - 8\,y + 4\,x\Big)}{\dfrac{\partial}{\partial y}\Big(y^2 - 2\,x\,y - x^2 - 8\,y + 4\,x\Big)} = -\frac{-2\,y - 2\,x + 4}{2\,y - 2\,x - 8} = -\frac{-2\,\big(y + x - 2\big)}{2\,\big(y - x - 4\big)} = \frac{y + x - 2}{y - x - 4}$$

04. We have $u'(t) = \begin{cases} e^{-t} & \text{if} \quad t < 0 \\ e^{t} & \text{if} \quad 0 < t \end{cases}$ Furthermore, the left and right derivatives at $t = 0$ are

$$u'_{-}(0) = \lim_{h \to 0-} \frac{u(0+h) - u(0)}{h} = \lim_{h \to 0-} \frac{1 - e^{-h} - 0}{h} \overset{\text{By L'Hôpital's}}{\underset{\text{rule}}{=\!=\!=\!=\!=}} \lim_{h \to 0-} \frac{e^{-h}}{1} = 1$$

and

$$u'_{+}(0) = \lim_{h \to 0+} \frac{u(0+h) - u(0)}{h} = \lim_{h \to 0+} \frac{e^{h} - 1 - 0}{h} \overset{\text{By L'Hôpital's}}{\underset{\text{rule}}{=\!=\!=\!=\!=}} \lim_{h \to 0+} \frac{e^{h}}{1} = 1$$

Since equal, $u(t)$ is differentiable at $t = 0$, and we have $u'(0) = 1$.

Hence $u'(t) = \begin{cases} e^{-t} & \text{if} \quad t < 0 \\ e^{t} & \text{if} \quad 0 \le t \end{cases}$ It follows

$$\begin{aligned}
u'(t) - |u(t)| &= \begin{cases} e^{-t} & \text{if} \quad t < 0 \\ e^{t} & \text{if} \quad 0 \le t \end{cases} - \begin{cases} \left|1 - e^{-t}\right| & \text{if} \quad t < 0 \\ \left|e^{t} - 1\right| & \text{if} \quad 0 \le t \end{cases} \\
&= \begin{cases} e^{-t} & \text{if} \quad t < 0 \\ e^{t} & \text{if} \quad 0 \le t \end{cases} - \begin{cases} -\left(1 - e^{-t}\right) & \text{if} \quad t < 0 \\ \left(e^{t} - 1\right) & \text{if} \quad 0 \le t \end{cases} \\
&= \begin{cases} e^{-t} + 1 - e^{-t} & \text{if} \quad t < 0 \\ e^{t} - e^{t} + 1 & \text{if} \quad 0 \le t \end{cases} = 1
\end{aligned}$$

Recall: To solve the first order linear equation $y' + p(t)\, y = f(t)$, you need an integrating factor, i.e., a function $\mu(t)$, such that $\mu(t)\left(y' + p(t)\, y\right) = \left(\mu(t)\, y\right)'$. The formula $\mu(t) = e^{\int p(t)\, dt}$, provides all the integrating factors. Go for the simplest

05a. The differential equation $y' + \frac{1}{t}\, y = \frac{7}{t^2} + 3$, is already in normal form. An integrating factor is given by

$$\mu(t) = e^{\int 1/t\, dt} = e^{\ln|t|} = |t| = \pm t$$

$\mu(t) = t$ will do. Multiplying both sides of the equation by $\mu(t)$, leads to

$$t\, y' + y = \frac{7}{t} + 3\, t \iff \left(t\, y\right)' = \frac{7}{t} + 3\, t$$

Integrating both sides and combining the two constants of integration, we get $t\, y = 7 \ln|t| + \frac{3}{2}\, t^2 + C$, and the general solution is $y = \frac{7 \ln|t|}{t} + \frac{3}{2}\, t + \frac{C}{t}$

05b. Rewrite the differential equation in normal form as $y' + \left(2\, t + \frac{1}{t}\right) y = t^2\, e^{-t^2}$. An integrating factor is given by

$$\mu(t) = e^{\int (2\, t + 1/t)\, dt} = e^{t^2 + \ln|t|} = e^{t^2}\, e^{\ln|t|} = e^{t^2}\, |t| = \pm t\, e^{t^2}$$

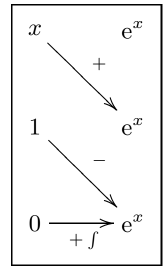
$\mu(t) = t\, e^{t^2}$ will do. Multiplying both sides of the equation (in normal form) by $\mu(t)$, leads to

$$\left(t\, e^{t^2}\, y\right)' = t^3$$

Integrating both sides and combining the two constants of integration, we get $t\, e^{t^2}\, y = \frac{1}{4}\, t^4 + C$, and the general solution is $y = \frac{1}{4}\, t^3\, e^{-t^2} + C\, \frac{e^{-t^2}}{t}$

05c. Rewrite the differential equation in normal form as
$y' + \dfrac{3}{x}\, y = \dfrac{e^x}{x^2}$. An integrating factor is given by

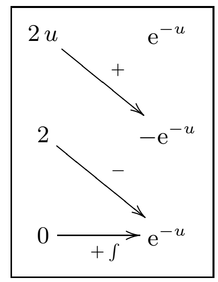
$$\mu(x) = e^{\int (3/x)\, dx} = e^{3\, \ln|x|} = e^{\ln|x|^3} = |x|^3 = \pm x^3$$



$\mu(x) = x^3$ will do. Multiplying both sides of the equation by $\mu(x)$, leads to $\left(x^3\, y\right)' = x\, e^x$. Integrating both sides, the right one by parts and combining the two constants of integration, we get
$x^3\, y = (x-1)\, e^x + C$ Hence the general solution

$$y = \frac{(x-1)\, e^x}{x^3} + \frac{C}{x^3}$$

05d. Rewrite the differential equation in normal form and use the identity $\sin(2\,x) = 2\, \sin(x)\, \cos(x)$, to get $y' - \big(\cos(x)\big)\, y = 2\, \sin(x)\, \cos(x)$. An integrating factor is



$$\mu(x) = e^{\int -\cos(x)\ dx} = e^{-\sin(x)}$$

Multiplying both sides of the equation by $\mu(x)$, leads to

$$\left(e^{-\sin(x)}\ y\right)' = 2\, \sin(x)\ \cos(x)\ e^{-\sin(x)}$$

Integrating both sides, we have

$$e^{-\sin(x)}\ y = \int 2\, \sin(x) \cos(x)\, e^{-\sin(x)}\ dx \ \underset{u = \sin(x)}{\overset{\text{Set}}{=\!=\!=}} \ \int 2\, u\, e^{-u}\, du$$
$$= (-2\,u - 2)\, e^{-u} + C = (-2\,u - 2)\, e^{-u} + C = (-2\, \sin(x) - 2)\, e^{-\sin(x)} + C$$

Hence the general solution is

$$y = -2\, \sin(x)\ - 2 + C\, e^{\sin(x)}$$

06a. <u>First</u> find the general solution of the differential equation. The equation is already in normal form. An integrating factor is $\mu(x) = e^{\int 2\,x\ dx} = e^{x^2}$. Multiplying the equation by $\mu(x)$, leads to $\left(e^{x^2}\, y\right)' = x\, e^{x^2}$ Integrating both sides (use the substitution $u = x^2$ to integrate the right side), we get $e^{x^2}\, y = \dfrac{1}{2}\, e^{x^2} + C$. Hence the general solution $y = \dfrac{1}{2} + C\, e^{-x^2}$
<u>Next</u> select the constant of integration C so that the initial condition $y(0) = 3$, is satisfied.

$$y(0) = 3 \iff \frac{1}{2} + C\, e^0 = 3 \iff C = 3 - \frac{1}{2} = \frac{5}{2}$$

Hence $y = \dfrac{1}{2} + \dfrac{5}{2}\, e^{-x^2}$

06b. <u>First</u> find the general solution of the differential equation. To do that, rewrite the equation in normal form as $y' + \frac{3}{x}\, y = \frac{2}{x^2\left(x^2+1\right)}$. An integrating factor is given by

$$\mu(x) = \mathrm{e}^{\int (3/x)\, \mathrm{d}x} = \mathrm{e}^{3\, \ln|x|} = \mathrm{e}^{\ln|x|^3} = |x|^3 = \pm x^3$$

Clearly $\mu(x) = x^3$ will do. Multiplying both sides of the equation by $\mu(x)$, we get $\left(x^3\, y\right)' = \frac{2\,x}{x^2+1}$. Integrating both sides, we obtain $x^3\, y = \ln\left(x^2+1\right) + C$. (the integral on the right was computed using the substitution $u = x^2+1$).

Hence the general solution is $y = \frac{\ln\left(x^2+1\right)}{x^3} + \frac{C}{x^3}$

<u>Next</u> select the constant of integration C so that the initial condition $y(-1) = 0$, is satisfied. Thus $y(-1) = 0 \iff \frac{\ln(2)}{-1} + \frac{C}{-1} = 0 \iff C = -\ln(2)$. Hence the solution $y = \frac{\ln\left(x^2+1\right) - \ln(2)}{x^3}$

06c. <u>First</u> we find the general solution of the differential equation. To do that, we rewrite the equation in normal form as $y' - \frac{2\,t}{t^2-5}\, y = -2\,t$. An integrating factor is given by

$$\mu(t) = \mathrm{e}^{\int -2\,t/\left(t^2-5\right)\, \mathrm{d}t} = \mathrm{e}^{-\ln\left|t^2-5\right|} = \mathrm{e}^{\ln\left(\left|t^2-5\right|\right)^{-1}} = \frac{1}{\left|t^2-5\right|} = \pm\frac{1}{t^2-5}$$

$\mu(t) = \frac{1}{t^2-5}$ will do. Multiplying both sides of the equation by $\mu(t)$, we get $\left(\frac{1}{t^2-5}\, y\right)' = -\frac{2\,t}{t^2-5}$. Integrating both sides, we obtain $\frac{1}{t^2-5}\, y = -\ln\left|t^2-5\right| + C$. Hence the general solution

$$y = -\left(t^2-5\right)\, \ln\left|t^2-5\right| + C\left(t^2-5\right)$$

<u>Next</u> we select the constant of integration C so that the initial condition $y(2) = 7$, is satisfied. Thus $y(2) = 7 \iff -(4-5)\, \ln|4-5| + C\,(4-5) = 7 \iff C = -7$. Hence

$$y = -\left(t^2-5\right)\, \ln\left|t^2-5\right| - 7\left(t^2-5\right)$$

Because of the initial condition $y(2) = 7$, the solution $y = y(t)$ has to be defined at and around $t = 2$. Thus $t^2 - 5 < 0 \implies \left|t^2-5\right| = -\left(t^2-5\right) = 5 - t^2$. It follows

$$y = \left(5-t^2\right)\left(7 + \ln\left(5-t^2\right)\right)$$

> To determine the largest interval I, the solution of the ivp $\begin{cases} y' + p(t)\, y = g(t) \\ y(t_0) = y_0 \end{cases}$ is guaranteed to be defined in, start by finding the endpoints $t_1 < t_2 < \cdots < t_k$ of the largest intervals where $p(t)$ and $f(t)$ are continuous. Then I is the interval amongst $\left(-\infty\,,\, t_1\right)$, $\left(t_1\,,\, t_2\right)$, $\cdots$, $\left(t_k\,,\, +\infty\right)$ that contains the initial time t_0.

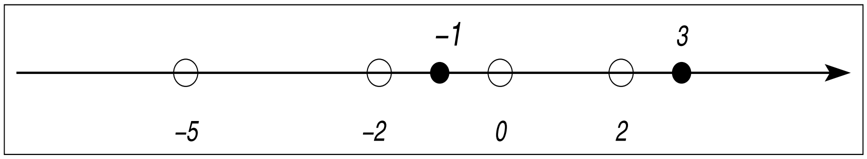
07. Rewrite the equation in normal form as $y' + \frac{t}{t+5}\, y = \frac{1}{t+5}\, \ln\left(\frac{t^2-4}{t}\right)$

$p(t) = \frac{t}{t+5}$ is continuous in $\left(-\infty\,,\, -5\right)$ and $\left(-5\,,\, +\infty\right)$

$g(t) = \frac{1}{t+5}\, \ln\left(\frac{t^2-4}{t}\right)$ is continuous in $\left(-2\,,\, 0\right)$ and $\left(2\,,\, +\infty\right)$

a. The largest interval that contains the initial time $t = -1$ where both $p(t)$ and $g(t)$ are continuous is $\big(-2\,,\,0\big)$. By the existence and uniqueness theorem, the solution of the I.V.P. is at least defined in $\big(-2\,,\,0\big)$

b. The largest interval that contains the initial time $t = 3$ where both $p(t)$ and $g(t)$ are continuous is $\big(2\,,\,+\infty\big)$. By the existence and uniqueness theorem, the solution of the I.V.P. is at least defined in $\big(2\,,\,+\infty\big)$.



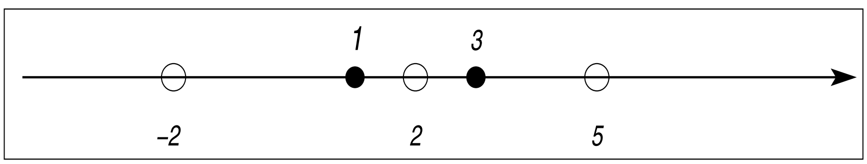
07. The equation is already in normal form.

$p(t) = \dfrac{t}{t^2 - 4}$ is continuous in $\big(-\infty\,,\,-2\big)$, $\big(-2\,,\,2\big)$ and $\big(2\,,\,+\infty\big)$

$g(t) = \sqrt{5 - t}$ is continuous in $\big(-\infty\,,\,5\big)$

c. The largest interval that contains the initial time $t = 3$ where both $p(t)$ and $g(t)$ are continuous is $\big(2\,,\,5\big)$. By the existence and uniqueness theorem, the solution of the I.V.P. is at least defined in $\big(2\,,\,5\big)$

d. The largest interval that contains the initial time $t = 1$ where both $p(t)$ and $g(t)$ are continuous is $\big(-2\,,\,2\big)$. By the existence and uniqueness theorem, the solution of the I.V.P. is at least defined in $\big(-2\,,\,2\big)$.



When separating variables in $F(t, y, y') = 0$, start by substituting $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ for y', and multiply both sides of the equation by $\mathrm{d}t$. If the result can be written as $p(t)\,q(y)\;\mathrm{d}t + u(t)\,v(y)\;\mathrm{d}y = 0$, then the equation is separable, since it can be rewritten as

$$u(t)\,v(y)\;\mathrm{d}y = -p(t)\,q(y)\;\mathrm{d}t \quad\Longrightarrow\quad \frac{v(y)}{q(y)}\;\mathrm{d}y = -\frac{p(t)}{u(t)}\;\mathrm{d}t$$

08a. Separating variables we successively write

$$t\,\frac{\mathrm{d}y}{\mathrm{d}t} = -\big(y^2 + y\big) \quad\Longrightarrow\quad \frac{1}{y^2 + y}\,\mathrm{d}y = -\frac{1}{t}\,\mathrm{d}t \;\Longleftrightarrow\; \left(\frac{1}{y} - \frac{1}{y+1}\right)\,\mathrm{d}y = -\frac{1}{t}\,\mathrm{d}t$$

Integrating both sides of the last equation, we obtain

$$\ln|y| - \ln|y+1| = -\ln|t| + C \implies \ln|y| - \ln|y+1| + \ln|t| = C \implies \ln\left|\frac{y\,t}{y+1}\right| = C \implies$$

$$\left|\frac{y\,t}{y+1}\right| = \mathrm{e}^C \implies \frac{y\,t}{y+1} = \pm\mathrm{e}^C \implies \frac{y}{y+1} = \frac{\pm\mathrm{e}^C}{t} \implies \frac{y+1}{y} = \frac{t}{\pm\mathrm{e}^C} \implies$$

$$1 + \frac{1}{y} = \pm\mathrm{e}^{-C}\,t \implies \frac{1}{y} = \pm\mathrm{e}^{-C}\,t - 1 \implies y = \frac{1}{\pm\mathrm{e}^{-C}\,t - 1}$$

Hence the general solution $y = \dfrac{1}{C\,t - 1}$, where we have renamed $\pm\mathrm{e}^{-C}$ as C.
Recall that in the process of separating the variables, we had to divide through by $y^2 + y = y(y+1)$. So we implicitly assumed that $y \neq 0$ and $y \neq -1$. Notice that both $y = 0$ and $y = -1$, are solutions of the differential equation.

08b. Separating variables we successively write

$$(x-1)\,y\;\mathrm{d}y + x\,(y-3)\;\mathrm{d}x = 0 \implies (x-1)\,y\;\mathrm{d}y = -x\,(y-3)\;\mathrm{d}x \implies \frac{y}{y-3}\,\mathrm{d}y = -\frac{x}{x-1}\;\mathrm{d}x$$

$$\implies \left(1 + \frac{3}{y-3}\right)\;\mathrm{d}y = \left(-1 - \frac{1}{x-1}\right)\;\mathrm{d}x$$

Integrating both sides of the last equation, we obtain $y + 3\,\ln|y-3| = -x - \ln|x-1| + C$.
Hence the general solution in implicit form is

$$y + 3\,\ln|y-3| + x + \ln|x-1| = C$$

In the process of separating the variables, we had to divide through by $y - 3$. That means, we implicitly assumed that $y \neq 3$. It is easy to check that $y = 3$, is a solution of the differential equation as well.

08c. Separating variables we successively write

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2\,x\left(y^2 - 3\,y + 2\right) = 0 \implies \mathrm{d}y + 2\,x\left(y^2 - 3\,y + 2\right)\;\mathrm{d}x = 0 \implies$$

$$\mathrm{d}y = -2\,x\left(y^2 - 3\,y + 2\right)\;\mathrm{d}x \implies \frac{1}{y^2 - 3\,y + 2}\;\mathrm{d}y = -2\,x\;\mathrm{d}x \implies$$

$$\frac{1}{(y-2)\,(y-1)}\;\mathrm{d}y = -2\,x\;\mathrm{d}x \implies \left(\frac{1}{y-2} - \frac{1}{y-1}\right)\;\mathrm{d}y = -2\,x\;\mathrm{d}x$$

Integrating both sides of the last equation, we obtain

$$\ln|y-2| - \ln|y-1| = -x^2 + C \implies \ln\left|\frac{y-2}{y-1}\right| = -x^2 + C \implies \left|\frac{y-2}{y-1}\right| = \mathrm{e}^{-x^2+C}$$

$$\implies \frac{y-2}{y-1} = \pm\,\mathrm{e}^C\;\mathrm{e}^{-x^2} \implies 1 - \frac{1}{y-1} = \pm\mathrm{e}^C\;\mathrm{e}^{-x^2}$$

$$\implies \frac{1}{y-1} = \pm\mathrm{e}^C\;\mathrm{e}^{-x^2} + 1 \implies y - 1 = \frac{1}{\pm\mathrm{e}^C\;\mathrm{e}^{-x^2} + 1}$$

$$\implies y = 1 + \frac{1}{\pm\mathrm{e}^C\;\mathrm{e}^{-x^2} + 1}$$

Hence the general solution $y = 1 + \dfrac{1}{C\,\mathrm{e}^{-x^2} + 1}$, where $\pm\mathrm{e}^C$ has been renamed as C.
Recall that in the process of separating the variables, we had to divide through by $y^2 - 3\,y + 2 = (y-1)\,(y-2)$. So we implicitly assumed that $y \neq 1$ and $y \neq 2$. Both $y = 1$ and $y = 2$, are solutions of our differential equation.

08d. Separating variables we successively write

$$x^2\, y\, \frac{\mathrm{d}y}{\mathrm{d}x} = \left(y^2 - 1\right)^{3/2} \implies y\,\left(y^2 - 1\right)^{-3/2}\,\mathrm{d}y = x^{-2}\,\mathrm{d}x$$

Integrating both sides of the last equation (use the subtitution $u = y^2 - 1$ to integrate the left side), we obtain

$$-\left(y^2 - 1\right)^{-1/2} = -x^{-1} + C \iff \frac{1}{\left(y^2-1\right)^{1/2}} = \frac{1}{x} - C \implies \left(y^2 - 1\right)^{1/2} = \frac{x}{1 - C\,x}$$
$$\implies y^2 - 1 = \frac{x^2}{\left(1 - C\,x\right)^2}$$

Hence the general solution in implicit form is

$$y^2 = 1 + \frac{x^2}{\left(1 - C\,x\right)^2}$$

In the process of separating the variables, we had to divide through by $\left(y^2 - 1\right)^{3/2}$. Consequently, we implicitly assumed that $y \neq \pm 1$. It is easy to check that $y = -1$ and $y = 1$, are both solutions of our differential equation.

09. According to problem (8c.), the general solution of $y' + 2\,t\left(y^2 - 3\,y + 2\right) = 0$ is $y = 1 + \dfrac{1}{C\,\mathrm{e}^{-t^2} + 1}$. To solve the initial value problem, we need to select the constant C so that the initial condition $y(0) = 3$ is satisfied. Thus

$$1 + \frac{1}{C\,\mathrm{e}^0 + 1} = 3 \iff \frac{1}{C+1} = 2 \iff C + 1 = \frac{1}{2} \iff C = -\frac{1}{2}$$

Hence the solution

$$y = 1 + \frac{1}{-\frac{1}{2}\,\mathrm{e}^{-t^2} + 1} = 1 + \frac{2}{2 - \mathrm{e}^{-t^2}}$$

Since the denominator is never zero, the solution is defined in $\mathbb{R}$.

10a. We start by finding the general solution of $(1 + 2\,y)\,y' - 2\,x = 0$. This is a separable equation. We separate the variables by successively writing

$$(1 + 2\,y)\,\frac{\mathrm{d}y}{\mathrm{d}x} - 2\,x = 0 \iff (1 + 2\,y)\,\mathrm{d}y - 2\,x\,\mathrm{d}x = 0 \iff (1 + 2\,y)\,\mathrm{d}y = 2\,x\,\mathrm{d}x$$

Integrating both sides, gives the general solution $y + y^2 = x^2 + C$. Now we select the constant C so that $y(2) = 3$. Setting $x = 2$ and $y = 3$ into $y + y^2 = x^2 + C$ leads to $3 + 9 = 4 + C \iff C = 8$. Hence the solution $y^2 + y = x^2 + 8$

Completing the square and solving for y, we get $y = -\frac{1}{2} + \frac{1}{2}\sqrt{4\,x^2 + 33}$ Clearly, the largest interval that contains the initial time where the solution is defined is $\left(-\infty, +\infty\right)$

10b. We start by finding the general solution of $y' - t\,y^2 = 0$. This is a separable equation. We separate the variables by successively writing

$$\frac{\mathrm{d}y}{\mathrm{d}t} - t\,y^2 = 0 \iff \mathrm{d}y - t\,y^2\,\mathrm{d}t = 0 \iff \mathrm{d}y = t\,y^2\,\mathrm{d}t \implies \frac{\mathrm{d}y}{y^2} = t\,\mathrm{d}t$$

Integrating both sides, gives the general solution $-\dfrac{1}{y} = \dfrac{t^2}{2} + C$. Now we select the constant C so that $y(0) = 1$. Setting $t = 0$ and $y = 1$ into $-\dfrac{1}{y} = \dfrac{t^2}{2} + C$, leads to $-1 = C \iff C = 8$.

Hence the solution

$$-\frac{1}{y} = \frac{t^2}{2} - 1 = \frac{t^2 - 2}{2} \implies -y = \frac{2}{t^2 - 2} \iff y = \frac{2}{2 - t^2}$$

The largest interval that contains the initial time where the solution is defined is $\left(-\sqrt{2}\,,\, \sqrt{2} \right)$