Our goal in this second part of the chapter is to use Fourier series to solve some of the basic differential equations of Mathematical Physics.

We start with the

Heat Equation

It has the form $k\, u_{xx} = u_t$, where $u = u(x,t)$,

$u_{xx}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t)$ and $u_t(x,t) = \dfrac{\partial u}{\partial t}(x,t)$.

This partial differential equation models the temperature $u(x,t)$ at position x and time t in a wire or a rod that extends along the x-axis.

L is the length of the wire. We assume that the wire has a uniform density, that it has a constant cross-section, and that it is laterally insulated.

k is called the thermal diffusivity. It usually depends on the material of the wire or rod. Large value of k means high conduction of heat energy, and low value of k means poor conduction of heat energy. We will assume that k is independent of x and t.

In physical situations, the temperature distribution in the wire is usually known, i.e., $u(x,0) = f(x)$, $0 < x < L$. This is called an <u>initial condition</u>. To be able to predict what happens to the temperature forward in time, we need to know what happens at the end points of the wire for all times:

- if the temperature at the endpoints of the wire is maintained at 0°, then that is expressed by

$$u(0,t) = 0 \quad \& \quad u(L,t) = 0$$

- if the endpoints of the wire are insulated, then the flow of heat is zero at the end points. This is expressed by

$$u_x(0,t) = 0 \quad \& \quad u_x(L,t) = 0$$

- $u(0,t) = g(t)$, means the temperature at the left end of the wire is being maintained to be $g(t)$ at any $t > 0$. Similarly $u(L,t) = h(t)$, means the temperature at the right end of the wire is prescribed to be $h(t)$ at any time $t>0$.

- $u_x(0,t) = g(t)$, means the heat flux is prescribed at the left end of the wire.

  similarly $u_x(L,t) = h(t)$, means the heat flux is prescribed at the right end of the wire.

$u(0,t)=0$ & $u(L,t)=0$, $u(0,t)=g(t)$ & $u(L,t)=h(t)$,

$u_x(0,t)=0$ & $u_x(L,t)=0$, $u_x(0,t)=g(t)$ & $u_x(0,t)=h(t)$,

are all called boundary conditions.

Another possible boundary conditions is: $c_1\, u(x_0,t) + c_2\, u_x(x_0,t) = g(t)$ where x_0 is one of the end points of the wire.

The problems

$$\begin{cases} k\, u_{xx} = u_t & 0 < x < L, \quad t > 0 \\ u(x,0) = f(x) & 0 < x < L \\ u(0,t) = g(t) \ \& \ u(L,t) = h(t), & t > 0 \end{cases}$$

and

$$\begin{cases} k\, u_{xx} = u_t & 0 < x < L, \quad t > 0 \\ u(x,0) = f(x) & 0 < x < L \\ u(0,t) = g(t) \ \& \ u_x(0,t) = h(t) & \end{cases}$$

are called boundary value problems.

superposition principle

If $u_1(x,t)$ and $u_2(x,t)$ are solutions of $k u_{xx} = u_t$, then $c_1 u_1(x,t) + c_2 u_2(x,t)$ is also a solution of the same differential equation $k u_{xx} = u_t$.

In what follows, and to be specific, we will focus on solving the homogeneous boundary value problem

$$\begin{cases} k u_{xx} = u_t & 0 < x < L, \quad t > 0 \\ u(0,t) = 0 \quad \& \quad u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

A popular and simple method to solve the boundary value problem is known as the separation of variables method.

We assume the solution of $u(x,t)$ of $k\,u_{xx} = u_t$ has the form: $u(x,t) = X(x)\cdot Y(t)$, where $X(x)$ is a function of x only, and $Y(t)$ is a function of t only. Substituting into $k\,u_{xx} = u_t$, leads to

$$k\,X''(x)\,Y(t) = X(x)\cdot Y'(t) \Rightarrow \frac{X''(x)}{X(x)} = \frac{Y'(t)}{k\,Y(t)}$$

That is possible only if both fractions are equal to the same constant, say $-\lambda$. This leads to

$$\begin{cases} \dfrac{X''}{X} = -\lambda \\[2ex] \dfrac{Y'}{kY} = -\lambda \end{cases} \Rightarrow \begin{cases} X'' + \lambda X = 0 \\[2ex] Y' + \lambda k Y = 0 \end{cases}$$

The boundary conditions $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \Leftrightarrow \begin{cases} X(0)\,Y(t) = 0 \\ X(L)\,Y(t) = 0 \end{cases}$ $t > 0$

This implies that $X(0) = 0$ & $X(L) = 0$

Hence $X(x)$ is solution of the boundary value problem

$$(*)\begin{cases} X'' + \lambda X = 0, & 0<x<L \\ X(0)=0 \quad \& \quad X(L)=0 \end{cases}$$

$(*)$ is called a Sturm-Liouville problem.

The general solution of $(*)$ depends on the parameter λ.

More specifically, we have

$$X(x) = c_1 e^{-\sqrt{-\lambda}\, x} + c_2 e^{\sqrt{-\lambda}\, x}, \quad \text{if} \quad \lambda<0$$

$$X(x) = c_1 x + c_2, \quad \text{if} \quad \lambda = 0$$

$$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x), \quad \text{if} \quad \lambda>0$$

Imposing the boundary conditions $X(0)=0$ & $X(L)=0$, leads to:

- $X(x) \equiv 0$, if $\lambda \leq 0$. This is called the trivial solution. It leads to the solution $u(x,t) = X(x)\,T(t) \equiv 0$, which usually won't satisfy the initial condition $u(x,0) = f(x)$.

- $X(x) = \sin\left(n \frac{\pi}{L} x\right)$, if $\lambda = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$

The pair $\left(\left(\frac{n\pi}{L}\right)^2, \sin\left(n\frac{\pi}{L}x\right)\right)$, is called an eigenvalue-eigenfunction pair of the Sturm-Liouville problem

$$(*) \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

For every $\lambda = \left(\frac{n\pi}{L}\right)^2$, the differential equation satisfied by $T(t)$: $T' + \lambda k T = 0$ becomes $T' + k\left(\frac{n\pi}{L}\right)^2 T = 0$

Solving, we get $T(t) = c\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

Hence, we have established that the problem

$$\begin{cases} k u_{xx} = u_t & 0 < x < L,\quad t > 0 \\ u(0,t) = 0, \quad u(L,t) = 0 & t > 0 \end{cases}$$

has infinitely many solutions :

$$u_n(x,t) = X_n(x) \cdot T_n(t) = \sin\left(n\frac{\pi}{L}x\right) \cdot e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n = 1, 2, 3, \dots$$

None of these solutions can be made to satisfy the initial condition $u(x,0) = f(x)$, for a general $f(x)$.

so to build a solution of the boundary value problem

$$\begin{cases} k\, u_{xx} = u_t & 0<x<L, \quad t>0 \\ u(0,t)=0, \quad u(L,t)=0 & t>0 \\ u(x,0) = f(x) & 0<x<L \end{cases}$$

we use the superposition principle and look for $u(x,t)$ in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right) \cdot e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

and we select the coefficients b_n so that

$$u(x,0) = f(x) \iff \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right) = f(x)$$

From of study of Fourier series, we need the trigonometric series $\sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} x\right)$ to be the Fourier Sine series of the function $f(x)$, $0 < x < L$.

Hence

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx, \quad n = 1, 2, \ldots$$

Therefore

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx, \quad n = 1, 2, 3, \ldots \end{aligned}}$$

Summary

The solution of the boundary value problem

$$\begin{cases} k\, u_{xx} = u_t & 0 < x < L, \quad t > 0 \\ u(0,t) = 0, \quad u(L,t) = 0, & t > 0 \\ u(x,0) = f(x) & 0 < x < L \end{cases}$$

is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} x\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t}}$$

where $$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx, \quad n = 1, 2, 3, \ldots$$

are the coefficients of the Fourier sine series of the function $f(x)$, $0 < x < L$

Example

Solve the boundary value problem
$$\begin{cases} u_{xx} = u_t & 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = 1 & 0 < x < \pi \end{cases}$$

Here $k=1$, $L=1$, and $f(x) = 1$. The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} = \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-n^2 t}$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx = \frac{2}{1}\int_0^{\pi} 1 \cdot \sin(nx)\,dx = -\frac{2}{n}\cos(nx)\Big|_0^{\pi}$$

$$= \frac{2(1-\cos(n\pi))}{n}, \quad n = 1, 2, \dots$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n} \sin(nx)\, e^{-n^2 t}$$

Example

Solve the bvp
$$\begin{cases} 2\, u_{xx} = u_t, & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0, \quad u(3,t) = 0, & t > 0 \\ u(x,0) = 5\sin(x) + 2\sin(3x) & 0 < x < \pi \end{cases}$$

Here $k = 2$, $L = \pi$, and $f(x) = 5\sin(x) + 2\sin(3x)$

The solution of the bvp is
$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} = \sum_{n=1}^{\infty} b_n \sin(n x)\, e^{-2n^2 t}$$

with
$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(n x)\, dx$$

we could compute the integral to find b_n, for $n = 1, 2, 3, \dots$

A better way is to remember that
$$u(x,0) = f(x) \iff \sum_{n=1}^{\infty} b_n \sin(n x) = f(x) = 5\sin(x) + 2\sin(3x)$$

which we rewrite as

$$b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + b_4 \sin(4x) + \cdots = 5 \sin(x) + 2 \sin(3x)$$

to conclude that

$$b_1 = 5, \quad b_2 = 0, \quad b_3 = 2, \quad b_4 = b_5 = \cdots = 0$$

Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-2n^2 t} = 5 \sin(x)\, e^{-2t} + 2 \sin(2x)\, e^{-8t}$$

If the wire is insulated at both ends, the boundary value problem satisfied by the temperature $u(x,t)$ is

$$\begin{cases} k u_{xx} = u_t, & 0 < x < L, \quad t > 0 \\ u_x(0,t) = 0, \quad u_x(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

Proceeding as we did in the case $u(0,t) = u(L,t) = 0$, we look look for a solution of $\begin{cases} k u_{xx} = u_t, & 0<x<L, \quad t>0 \\ u_x(0,t) = u_x(L,t) = 0, & t>0 \end{cases}$

in the form $u(x,t) = X(x)\, Y(t)$. Substituting, leads to

$X(x)$ must be solution of $\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$

and $Y(t)$ must be solution of $Y' + \lambda k Y = 0$

Solving and retaining only the nontrivial solutions, we get the solutions

$$u_n(x,t) = \cos(n\tfrac{\pi}{L}x)\, e^{-k\left(\frac{n\pi}{L}\right)^2 x}, \quad n = 0, 1, 2, 3, \dots$$

These solutions don't always satify the initial condition $u(x,0) = f(x)$, when $f(x)$ is a general function.

Using the superposition principle, we look for the solution of the bvp

$$\begin{cases} k u_{xx} = u_t, & 0 < x < L, \quad t > 0 \\ u_x(0,t) = u_x(L,t) = 0, & t > 0 \\ u(x,0) = f(t), & 0 < x < L \end{cases}$$

in the form

$$u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t) = \sum_{n=0}^{\infty} c_n \cos(n\tfrac{\pi}{L}x)\, e^{-k\left(\frac{n\pi}{L}\right)^2 x}$$

and we select the coefficients $c_n,\ n = 0, 1, 2, \dots$ so that

$$u(x,0) = f(x) \iff \sum_{n=0}^{\infty} c_n \cos\left(n\frac{\pi}{L}x\right) = f(x)$$

$$\iff c_0 + \sum_{n=1}^{\infty} c_n \cos\left(n\frac{\pi}{L}x\right) = f(x)$$

From our study of Fourier series, we see that we need to select the trigonometric series to be the Fourier Cosine series of the function $f(x)$, $0 < x < L$.

Hence $c_0 = \frac{a_0}{2}$ and $c_n = a_n$, with $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{x}L\right)dx$

Summary

The solution of the boundary value problem

$$\begin{cases} k\, u_{xx} = u_t, & 0<x<L, \quad t>0 \\ u_x(0,t) = u_x(L,t) = 0, & t>0 \\ u(x,0) = f(x), & 0<x<L \end{cases}$$

is given by

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx, \qquad n = 0, 1, 2, \ldots$$

are the coefficients of the Fourier cosine series of $f(x)$, $0<x<L$.

Example

Solve the boundary value problem
$$\begin{cases} 3u_{xx} = u_t, & 0<x<\pi,\ t>0 \\ u_x(0,t) = u_x(\pi,t) = 0, & t>0 \\ u(x,0) = 3\sin^2(x), & 0<x<\pi \end{cases}$$

Here $k=3$, $L=\pi$, and $f(x) = 3\sin^2(x)$

The solution of the bvp is given by
$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\, e^{-3n^2 t}$$

with a_n defined by
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)\,dx, \quad n=0,1,2,\dots$$

Instead of computing a_n directly, we could rewrite
$$f(x) = 3\sin^2(x) = 3\,\frac{1-\cos(2x)}{2} = \frac{3}{2} - \frac{3}{2}\cos(2x).$$ It follows
$$u(x,0) = f(x) \iff \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = \frac{3}{2} - \frac{3}{2}\cos(2x)$$
$$\iff a_0 = 3,\ a_1 = 0,\ a_2 = -\frac{3}{2},\ a_3 = a_4 = \dots = 0$$

Hence the solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \cdot e^{-3n^2 t}$$

$$= \frac{3}{2} - \frac{3}{2}\cos(2x)\, e^{-12t}$$