FINAL EXAMINATION – Differential Equations – MATH 375

ALL SECTIONS (L01 - L04) – FALL 2015 - Answers, Hint, Solutions

1. The solution of the initial value problem $ty' + 4y = \dfrac{3}{t}$, $y(1) = 3$ is

   A) $y = 2t^2 + t$ B) $y = \dfrac{2}{t^4} + \dfrac{1}{t}$ C) $y = \dfrac{1}{t^3} + \dfrac{2}{t}$ D) $y = \dfrac{2}{t^3} + \dfrac{1}{t}$ E) $y = \dfrac{1}{t^4} + \dfrac{2}{t}$

   Solution. This is a linear equation $y' + \dfrac{4}{t}y = \dfrac{3}{t^2}$, after multiplication by the integrating factor $\mu(t) = e^{\int \frac{4}{t}dt} = e^{4\ln(t)} = t^4$ we have $(t^4y)' = 3t^2$, or $t^4y = C + t^3$, or $y = \dfrac{C}{t^4} + \dfrac{1}{t}$. As $y(1) = C + 1 = 2$, we have $C = 2$, the correct answer is B).

2. The solution of the initial value problem $\dfrac{dy}{dx} = -\dfrac{3x^2y^2 + 4x^3y^4}{2x^3y + 4x^4y^3}$, $y(1) = 1$ is

   A) $y = 3x^3 - 2x^2$ B) $x^2y^2 + x^3y^4 + x^3y + x^4y^3 = 4$ C) $x + y = 2$

   D) $3x^2y^2 + 4x^3y^4 + 2x^3y + 4x^4y^3 = 13$ E) $x^3y^2 + x^4y^4 = 2$

   Solution. The equation is exact $((3x^2y^2 + 4x^3y^4)dx + (2x^3y + 4x^4y^3)dy = 0)$ as $\dfrac{\partial}{\partial y}(3x^2y^2 + 4x^3y^4) = 6x^2y + 16x^3y^3 = \dfrac{\partial}{\partial x}(2x^3y + 4x^4y^3)$, with the potential $F(x, y) = x^3y^2 + x^4y^4 = C$. Since $y(1) = 1$, $F(1, 1) = 1 + 1 = 2$, $C = 2$, the correct answer is E).

3. The largest open interval on which the unique solution of the initial value problem $(t^2 - 9)y'' + (\ln|t - 1| + 4)y' + \frac{1}{t-7}y = 0$, $y(2) = 5$ is guaranteed to exist (according to the existence and uniqueness theorem) is

   A) $(-3, \infty)$ B) $(1, 3)$ C) $(-3, 3)$ D) $(1, 7)$ E) $(-3, 5)$

   Solution. The first coefficient vanishes at $t = \pm 3$, the second and the third are defined for $t \neq 1, 7$. The largest interval including 2 and none of $\pm 3, 1, 7$ is $(1, 3)$, the correct answer is B).

4. A tank initially contains a solution with 80 kg of salt dissolved in 1000 litres of water. Pure water enters the tank at the rate of 6 litres/min. The solution is mixed and drains from the tank at the rate of 3 litres/min. Then the initial value problem describing the amount $Q(t)$ of salt in the tank at time t is

   A) $\dfrac{dQ}{dt} = -\dfrac{3Q}{1000 + 3t}$, $Q(0) = 80$ B) $\dfrac{dQ}{dt} = -\dfrac{3Q}{1000}$, $Q(0) = 80$

   C) $\dfrac{dQ}{dt} = 80 - \dfrac{3Q}{1000 - 3t}$, $Q(0) = 60$ D) $\dfrac{dQ}{dt} = 6 - \dfrac{3Q}{1000}$, $Q(0) = 80000$

   E) $\dfrac{dQ}{dt} = 6000 - 3000Q$, $Q(0) = 80$

   Answer. The correct answer is A).

5. The first step of Euler's approximation for the solution of the initial value problem $\dfrac{dy}{dx} = 2\sin(x(y + 1))$, $y(0) = 1$ with the step size $h = 0.1$ is

   A) $y(0.1) \approx 1 + 0.2\sin(0.1)$ B) $y(0.1) \approx 1.1$ C) $y(0.1) \approx 1$

   D) $y(0.1) \approx 1.2$ E) $y(0.1) \approx 1 + 0.1\sin(0.2)$

   Solution. $\dfrac{dy}{dx}(0, 1) = 2\sin 0 = 0$, so $y(0.1) \approx y_1 = 1 + 0.1 \cdot 0 = 1$, the correct answer is C).

6. A cake at 220° C is brought to a room at 20° C. If after 10 minutes the cake is 120° C, how long will it take to cool down from 120° C to 45° C?

   A) 45 minutes B) 30 minutes C) 20 minutes D) 10 minutes E) 7.5 minutes

   Solution. The solution of the Newton's equation of cooling is $T(t) = 20 + 200e^{-kt}$, $T(10) = 20 + 200e^{-10k} =$

120, so $e^{-10k} = \frac{1}{2}$. We have $T(t) = 20 + 200e^{-kt} = 45$, or $e^{-kt} = \dfrac{1}{8} = (e^{-10k})^3 = e^{-30k}$, so $t = 30$ to cool down from 220 to 45° C, while only 20 minutes from 120° C to 45° C, the correct answer is C). The problem can also be solved using exponential decay: if the temperature drops twice after 10 minutes, it drops $8 = 2^3$ times in $3 \cdot 10 = 30$ minutes, or 20 minutes after it dropped to 120° C, the correct answer is C).

7. The general solution of the equation $y^{(5)} + 2y^{(4)} + y^{(3)} = 0$ is
A) $C_1e^t + C_2te^t + C_3e^{-2t} + C_4te^{-2t} + C_5e^{2t}$ B) $C_1e^t + C_2te^t + C_3t^2e^t + C_4t^3e^t + C_5t^4e^t$ C) $C_1e^t + C_2te^t + C_3 + C_4t + C_5t^2$

D) $C_1e^{-t} + C_2te^{-t} + C_3 + C_4t + C_5t^2$ E) $C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + C_4t^3e^{-t} + C_5t^4e^{-t}$
Solution. The roots of the characteristic equation $r^5 - 2r^4 + r^3 = r^3(r^2 - 2r + 1) = r^3(r-1)^2 = 0$ are $r_1 = r_2 = 1$, $r_3 = r_4 = r_5 = 0$, the correct answer is D).

8. If $x(t) = 7\cos(3t) + \sin(t)$ is a solution of the fourth order differential equation $x^{(4)} + ax^{(3)} + bx'' + cx' + dx = 0$ then
A) $a = 1, b = 3, c = 2, d = 6$ B) $a = 0, b = 10, c = 0, d = 9$ C) $a = 3, b = 1, c = 4, d = 9$
D) $a = 0, b = -10, c = 0, d = -9$ E) a, b, c, d cannot be found
Solution. The roots of the characteristic equation are $r = \pm 3i$, $r = \pm i$, so the characteristic equation is $(r^2+1)(r^2+9) = r^4 + 9r^2 + r^2 + 9 = r^4 + 0 \cdot r^3 + 10r^2 + 0 \cdot r + 9$, the correct answer is B).

9. According to Undetermined Coefficients method, a particular solution to the equation
$y'' - 4y' + 4y = xe^{2x} + 3xe^{-2x} - 7$ should be sought in the form of
A) $(Ax + B)e^{2x} + (Cx + D)e^{-2x} + E$ B) $(Ax + B)e^{2x} + x^2(Cx + D)e^{-2x} + E$

C) $Axe^{2x} + Bxe^{-2x} + C$ D) $(Ax + B)e^{2x} + (Cx + D)e^{-2x} + Ex^2$

E) $x^2(Ax + B)e^{2x} + (Cx + D)e^{-2x} + E$
Solution. The roots of the characteristic equation are $r_{1,2} = 2$, so both e^{2x} and xe^{2x} are solutions of the homogeneous equation (while a constant and e^{-2x} are not, the correct answer is E).

10. A fundamental set of solutions for the equation $y^{(6)} + 9y^{(4)} = 0$ is
A) $\{1, t, t^2, t^3, \cos(3t), \sin(3t)\}$ B) $\{1, t, t^2, t^3, t^4, \cos(3t), \sin(3t)\}$ C) $\{1, \cos(3t), \sin(3t)\}$

D) $\{1, e^t\cos(3t), e^t\sin(3t)\}$ E) $\{1, t, t^2, t^3, e^{3t}, e^{-3t}\}$

Solution. The roots of the characteristic equation are $r_1 = r_2 = r_3 = r_4 = 0$, $r_{5,6} = \pm 3i$ ($\cos(3t), \sin(3t)$ are solutions of the homogeneous equation). The correct answer is A).

11. The general solution of the equation $x^2y'' - 3xy' + 4y = 0, \quad x > 0$ is
A) $y = C_1x^2 + C_2x^2\ln(x)$ B) $y = e^{\frac{3}{2}x}\left[C_1\cos\left(\frac{\sqrt{7}}{2}x\right) + C_2\sin\left(\frac{\sqrt{7}}{2}x\right)\right]$

C) $y = C_1 + C_2e^{2x}$ D) $y = C_1e^{2x} + C_2xe^{2x}$ E) $y = x^2\left[C_1\cos\left(\frac{\sqrt{7}}{2}\ln(x)\right) + C_2\sin\left(\frac{\sqrt{7}}{2}\ln(x)\right)\right]$
Solution. This is the Cauchy-Euler equation, $r(r-1) - 3r + 4 = (r-2)^2 = 0$ has roots $r_1 = r_2 = 2$, the correct answer is A).

12. If $y = C_1x + C_2x^2$ is the general solution of the homogeneous equation $x^2y'' + bxy' + cy = 0$ then a particular solution of the non-homogeneous equation $x^2y'' + bxy' + cy = x^3$ is

A) $\dfrac{x^4}{4}$ B) x^3 C) $\dfrac{x^5}{4} + \dfrac{x^4}{3}$ D) $\dfrac{x^3}{2}$ E) none of the above
Solution. Applying the variation of parameters method, we obtain $C_1'x + C_2'x^2 = 0$, $C_1' + 2C_2'x = x^3/x^2 = x$. Multiplying the second equation by x and subtracting from it the first equation, we obtain $C_2'x^2 = x^2$, or $C_2(x) = x$ (plus some constant). Thus $C_1' = -C_2'x = -x$, or $C_1(x) = -\dfrac{x^2}{2}$ (plus some constant). Finally, a particular solution is $C_1x + C_2x^2 = -x\dfrac{x^2}{2} + xx^2 = \dfrac{x^3}{2}$, the correct answer is D).

13. The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 5$, $\lambda_2 = 1$ and $\lambda_3 = 0$ and eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ associated with $\lambda_1 = 5$ and $\lambda_2 = 1$, respectively. Then the general solution of the system $X' = AX$ is

A) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ B) $C_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

C) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ D) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

E) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

Solution. The eigenvector which is a solution of $Av_3 = 0$ is $(1, 1, -3)^T$, the correct answer is D).

14. A real 2×2 matrix A has an eigenvalue $\lambda_1 = 1 + i$ and an associated eigenvector $v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Then the general solution of the system $X' = AX$ is

A) $C_1 \begin{bmatrix} \cos(t) \\ -\cos(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}$ B) $C_1 e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}$

C) $C_1 e^t \begin{bmatrix} \cos(t) \\ -\cos(t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}$ D) $C_1 \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}$

E) unknown: insufficient data to find the general solution

Solution. The solution, by Euler's formula, is $e^t(\cos(t) + i\sin(t))(1, -i)^T$, its real and imaginary parts are $e^t(\cos(t), \sin(t))^T$ and $e^t(\sin(t), -\cos(t))$, respectively, the correct answer is B).

15. The eigenvalues λ_n and eigenfunctions X_n of the Sturm-Liouville problem $X'' + \lambda X = 0$, $X'(0) = X'(\pi) = 0$ are

A) $\lambda_n = n$, $X_n = \sin(nx)$, $n = 1, 2, \ldots$ only B) $\lambda_n = n^2$, $X_n = \sin(nx)$, $n = 1, 2, \ldots$ only

C) $\lambda_0 = 0$, $X_0 = 1$, $\lambda_n = \pi^2 n^2$, $X_n = \cos(n\pi x)$, $n = 1, 2, \ldots$ only

D) $\lambda_0 = 0$, $X_0 = 1$, $\lambda_n = n^2$, $X_n = \cos(nx)$, $n = 1, 2, \ldots$ only E) none of the above

Answer. The solution is a constant, for $\lambda = 0$, and $X_n = \cos(nx)$ for $\lambda_n = n^2$, the correct answer is D).

16. In the Fourier series $a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$ of $f(t) = \begin{cases} 3, & -2 < t \le -1, \\ -3, & -1 < t \le 1, \\ 3 & 1 < t < 2 \end{cases}$

A) all $a_n \neq 0$, $b_n \neq 0$ B) all $b_n = 0$ but all $a_n \neq 0$, $n = 0, 1, 2, \ldots$

C) $a_0 = 0$, all $a_n \neq 0$, $b_n \neq 0$ for $n = 1, 2, \ldots$ D) all $a_n = 0$, $n = 0, 1, 2, \ldots$

E) $a_0 = 0$, $b_n = 0$, $n = 1, 2, \ldots$

Solution. The function is even, so all $b_n = 0$, $n = 1, 2, \ldots$. Also, $a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx = 0$, so $a_0 = 0$ as well, the correct answer is E).

17. The Fourier series $f(x) \sim a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$ of period $2\ell = 2$ for $f(x) = x^2 - x$, $x \in (-1, 1)$

A) at $x = -1$ converges to 2, at $x = 1$ converges to 0 B) at $x = -1$ converges to 0, at $x = 1$ converges to 2

C) at $x = -1$ converges to 1, at $x = 1$ converges to 1 D) at $x = -1$ converges to 0, at $x = 1$ converges to 0

E) at $x = -1$ converges to 2, at $x = 1$ converges to 2

Solution. The Fourier series at (discontinuity or end) point converges to $(f(0^+) + f(2^-))/2 = (0 + 2)/2 = 1$ both at $x = -1$ and $x = 1$, the correct answer is C).

18. The sine series of period 2π for $f(x) = x + 3$, $x \in [0, \pi]$ is

A) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n} \sin(nx)$ B) $\displaystyle 3 + \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$ C) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n} \sin(n\pi x)$

D) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{2}{n} - \frac{6}{\pi n}(-1)^n\right] \sin(nx)$ E) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n} \left[(-1)^{n+1} - \frac{3}{\pi}\left((-1)^n - 1\right)\right] \sin(nx)$

Solution. Ax $\displaystyle\int x \ \sin(nx) \ dx = -\frac{x\cos(nx)}{n} + \int \frac{\cos(nx)}{n} dx = -\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$, we have

$$b_n = \frac{2}{\pi}\int_0^{\pi} (x+3)\sin(nx) \ dx = \frac{2}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2} - \frac{3\cos(nx)}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) - 0 + 0 - 0 - \frac{3}{n}\cos(n\pi) + \frac{3}{n}\right] = \frac{2}{n}\left[(-1)^{n+1} - \frac{3}{\pi}\left((-1)^n - 1\right)\right],$$

the correct answer is E).

19. The solution $u(x,t)$ to the heat transfer (conduction) problem $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $0 < x < \pi$, $t > 0$, $u(0,t) = 0$, $u(\pi,t) = 0$, $t > 0$, $u(x,0) = \sum_{n=1}^{\infty} \frac{1}{n^3}\sin(nx)$, $0 < x < \pi$ is

A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}\sin(nx)e^{-n^6 t}$ B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}\sin(nx)e^{-n^2 t}$

C) $\displaystyle\sum_{n=1}^{\infty} \sin(nx)e^{-n^3 t}$ D) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}\sin(n\pi x)e^{-n^2\pi^2 t}$ E) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}\sin(nx)\sinh(nt)$

Solution. Following the separation of variables method, we obtain for $u(x,0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin(nx)$:

$u(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2 t}$. Substituting the initial condition, we have

$b_n = \dfrac{1}{n^3}$, $u(t,x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}\sin(nx)e^{-n^2 t}$, the correct answer is B).

20. A string of length L is secured at both ends. The string has no initial displacement, but has initial velocity $f(x)$ at any point x. This scenario is described by the partial differential equations and boundary and initial conditions

A) $\alpha^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $0 < x < L$, $t > 0$, $u(0,t) = u(L,t) = 0$, $u(x,0) = 0$, $\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = f(x)$

B) $\alpha^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $0 < x < L$, $t > 0$, $u(x,0) = u(x,L) = 0$, $u(x,0) = 0$, $\left.\dfrac{\partial u}{\partial x}\right|_{t=0} = f(x)$

C) $\alpha^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $0 < x < L$, $t > 0$, $u(0,t) = u(L,t) = 0$, $\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = f(x)$

D) $\alpha^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $0 < x < L$, $t > 0$, $u(x,0) = u(x,L) = 0$, $\left.\dfrac{\partial u}{\partial x}\right|_{t=0} = f(x)$

E) $\alpha^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $0 < x < L$, $t > 0$, $u(x,L) = f(x)$, $u(L,t) = 0$, $u(t,0) = 0$

Answer. This is the wave equation, the correct answer is A).

21. The function $u(x,y) = a\sin(2\pi x)\sinh(by)$ is a solution of the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ satisfying $u(0,y) = 0$, $u(1,y) = 0$, $0 \le y \le 2$, $u(x,0) = 0$, $u(x,2) = \sin(2\pi x)$, $0 \le x \le 1$ for

A) $a = 1$ and any $b \in \mathbb{R}$ B) $a = \dfrac{1}{\sinh(2\pi)}$, $b = \pi$

C) $a = \dfrac{1}{\sinh(4\pi)}$, $b = 2\pi$ D) $a = 1$, $b = 2$ E) $a = \dfrac{1}{\sinh(\pi)}$, $b = 2\pi$

Solution. We have by direct computation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = -(2\pi)^2 u + b^2 u = 0$ for any (x, y) only if $b = 2\pi$. Then the first three boundary conditions are satisfied, the fourth condition gives $u(x, 2) = a\sin(2\pi x)\sinh(4\pi) = \sin(2\pi x)$ only if $a = 1/\sinh(4\pi)$, the correct answer is C).

22. Using the functions $u_c(t)$, the function $f(t) = \begin{cases} t, & 0 \le t < 1, \\ t^2 - 2, & 1 \le t < 3, \\ t - 2 & t \ge 3 \end{cases}$ can be written as

A) $f(t) = t + (t^2 - 2)u_1(t) + (t - 2)u_3(t)$ B) $f(t) = t + (t^2 - 2 - t)u_1(t) - 2u_3(t)$

C) $f(t) = t + (t^2 - 2 - t)u_1(t) + (t - t^2)u_3(t)$ D) $f(t) = t - t^2 + 2 + (t^2 - t)u_1(t) - 2u_3(t)$

E) $f(t) = t + (t + 2 - t^2)u_1(t) - 2u_3(t)$
Answer. The correct answer is C).

23. The Laplace Transform of the function $f(t) = \begin{cases} 1, & 0 \le t < 2, \\ 3t - 4, & t \ge 2 \end{cases}$ is

A) $\dfrac{1}{s} + \dfrac{3e^{-2s}}{s^2} + \dfrac{e^{-2s}}{s}$ B) $\dfrac{1}{s} + \dfrac{3e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s}$ C) $-\dfrac{3}{s} + \dfrac{3}{s^2}$

D) $\dfrac{3e^{-2s}}{s^2} - \dfrac{4e^{-2s}}{s}$ E) $\dfrac{1}{s} + \dfrac{3e^{-2s}}{s^2} - \dfrac{4e^{-2s}}{s}$
Solution. We have $f(t) = 1 + (3t - 5)u_2(t)$, so

$$\mathcal{L}[f] = \frac{1}{s} + \mathcal{L}\left[3(t-2)u_2(t) + u_2(t)\right] = \frac{1}{s} + \frac{3e^{-2s}}{s^2} + \frac{e^{-2s}}{s},$$

the correct answer is A).

24. The inverse Laplace Transform of $F(s) = \dfrac{5 - s}{s^2 + 2s + 5}$ equals

A) $-\cos(t) + 5\sin(t)$ B) $-e^{2t}\cos(t) + 5e^{2t}\sin(t)$ C) $-e^{t}\cos(2t) + 2e^{t}\sin(2t)$

D) $-2e^{-t}\cos(2t) + 5e^{-t}\sin(2t)$ E) $-e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$
Solution. By completing the square and using the table, we get

$$\frac{5 - s}{s^2 + 2s + 5} = \frac{-s + 5}{(s+1)^2 + 2^2} = \frac{-(s+1)}{(s+1)^2 + 2^2} + 3\frac{2}{(s+1)^2 + 2^2},$$

the inverse Laplace Transform is

$$\mathcal{L}^{-1}\left(\frac{5 - s}{s^2 + 2s + 5}\right) = -e^{-t}\cos(2t) + 3e^{-t}\sin(2t),$$

the correct answer is E).

25. The inverse Laplace Transform of $F(s) = \dfrac{2e^{-2s}}{(s+1)(s+3)}$ is

A) $e^{-t} - e^{-3t}$ B) $u_2(t)\left(e^{-t} - e^{-3t}\right)$

C) $u_2(t)\left(\dfrac{1}{t^2} - \dfrac{1}{t^4}\right)$ D) $u_2(t)\left(e^{2-t} - e^{6-3t}\right)$ E) $u_2(t)\left(t - 2 + \dfrac{3}{t}\right)$

Solution. We have $\dfrac{2}{(s+1)(s+3)} = \dfrac{1}{s+1} - \dfrac{1}{s+3}$, thus using the second shift formula we obtain

$$\mathcal{L}^{-1}\left[\frac{2e^{-2s}}{(s+1)(s+3)}\right] = u_2(t)\left(e^{-(t-2)} - e^{-3(t-2)}\right) = u_2(t)\left(e^{2-t} - e^{6-3t}\right),$$

the correct answer is D).