MATH 375
Handout # 9: Answers, Hints, Solutions
The Laplace Transform

1. Find the Laplace Transform of the following functions

a) $f(t) = 2\sin t - \cos t$ b) $f(t) = t^3 e^{-t}$ c) $f(t) = \sin(mt)\cos(nt)$

d) $f(t) = (t+1)\sin 2t$ e) $f(t) = \displaystyle\int_0^t \cos^2(\omega u)\, du$ f) $f(t) = e^{3t}\sin^2 t$

g) $f(t) = \sin(t-b)u_b(t)$

2. Find the Laplace Transform of the following functions:

a) $f(t) = \begin{cases} e^{5t}, & 0 \le t < 3, \\ e^{5t} + 5t^2 - 2t + 7, & t \ge 3. \end{cases}$ b) $f(t) = \begin{cases} t-3, & 0 \le t < 1, \\ 2t^2 + t - 4, & 1 \le t < 2, \\ t^2 + 2t - 4, & t \ge 2. \end{cases}$

3. Find the inverse Laplace Transform of the following functions

a) $F(s) = \dfrac{1}{s^2+4s+5}$ b) $F(s) = \dfrac{s+2}{(s+1)(s-2)(s^2+4)}$ c) $F(s) = \dfrac{s}{(s^2+1)^2}$

d) $F(s) = \dfrac{e^{-s}}{s(s-1)}$ e) $F(s) = \dfrac{1}{s^2+1}\left(e^{-2s} + 2e^{-3s} + 3e^{-4s}\right)$ f) $F(s) = \dfrac{e^{-s/3}}{s(s^2+1)}$

4. Using the Laplace Transform solve the initial value problems

a) $x'' + 3x' = e^t,\ x(0) = 0,\ x'(0) = -1$

b) $x'' + 2x' - 3x = e^{-t},\ x(0) = 0,\ x'(0) = 1$

c) $x'' + 2x' + x = \sin t,\ x(0) = 0,\ x'(0) = -1$

d) $x''' + 2x'' + 5x' = 0,\ x(0) = -1,\ x'(0) = 2,\ x''(0) = 0$

e) $x'' - 2x' + x = t - \sin t,\ x(0) = 0,\ x'(0) = 0$

f) $x''' + x'' = \cos t,\ x(0) = -2,\ x'(0) = 0,\ x''(0) = 0$

g) $x''' + x' = e^{2t},\ x(0) = 0,\ x'(0) = 0,\ x''(0) = 0$

h) $x'' + 4x = f(t),\ x(0) = 0,\ x'(0) = -1,\ f(t) = \begin{cases} 4t, & 0 \le t < 2 \\ 5t-2, & t \ge 2 \end{cases}$

Answers. 1. a) $F(s) = \dfrac{2-s}{s^2+1}$ b) $F(s) = \dfrac{6}{(s+1)^4}$ c) $F(s) = \dfrac{m(s^2+m^2-n^2)}{(s^2+m^2+n^2)^2 - 4m^2n^2}$

d) $F(s) = \dfrac{2s^2+4s+8}{(s^2+4)^2}$ e) $F(s) = \dfrac{s^2+2\omega^2}{s^2(s^2+4\omega^2)}$ f) $F(s) = \dfrac{1}{2(s-3)} - \dfrac{1}{2}\dfrac{s-3}{(s-3)^2+4}$

g) $F(s) = \dfrac{e^{-bs}}{s^2+1}$.

2. a) $F(s) = \dfrac{1}{s-5} + e^{-3s}\left(\dfrac{10}{s^3} + \dfrac{28}{s^2} + \dfrac{46}{s}\right)$.

b) $F(s) = \dfrac{1}{s^2} - \dfrac{3}{s} + e^{-s}\left(\dfrac{4}{s^3} + \dfrac{4}{s^2} + \dfrac{1}{s}\right) - e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{3}{s^2} + \dfrac{2}{s}\right)$.

3. a) $f(t) = e^{-2t}\sin t$ b) $f(t) = \frac{1}{6}e^{2t} - \frac{1}{15}e^{-t} - \frac{1}{10}\cos(2t) - \frac{1}{5}\sin(2t)$ c) $f(t) = \frac{1}{2}t\sin t$
d) $f(t) = (e^{t-1} - 1)u_1(t)$ e) $f(t) = \sin(t-2)u_2(t) + 2\sin(t-3)u_3(t) + 3\sin(t-4)u_4(t)$
f) $f(t) = u_{1/3}(t)(1 - \cos(t - \frac{1}{3}))$

4. a) $x(t) = \frac{1}{4}e^t + \frac{5}{12}e^{-3t} - \frac{2}{3}$ b) $x(t) = \frac{1}{8}(3e^t - e^{-3t} - 2e^{-t})$
c) $x(t) = \frac{1}{2}(e^{-t} - te^{-t} - \cos t)$ d) $x(t) = \frac{3}{5}e^{-t}\sin 2t - \frac{4}{5}e^{-t}\cos 2t - \frac{1}{5}$
e) $x(t) = 2 + t - \frac{1}{2}\cos t + \frac{1}{2}te^t - \frac{3}{2}e^t$ f) $x(t) = -1 - \frac{1}{2}(\cos t + \sin t + e^{-t})$
g) $x(t) = \frac{1}{10}e^{2t} - \frac{1}{2} + \frac{2}{5}\cos t - \frac{1}{5}\sin t$ h) $x(t) = t - \sin(2t) + \dfrac{1}{4}u_2(t)\left(t - 2 - \dfrac{1}{2}\sin(2t-4)\right)$

Solutions. 1. a) $F(s) = 2\mathcal{L}[\sin t] - \mathcal{L}[\cos t] = \dfrac{2-s}{s^2+1}$

b) $F(s) = -\dfrac{d^3}{ds^3}\mathcal{L}[e^{-t}] = -\left(\dfrac{1}{s+1}\right)''' = -\left(-\dfrac{1}{(s+1)^2}\right)'' = -\left(\dfrac{2}{(s+1)^3}\right)' = \dfrac{6}{(s+1)^4}$

c) $F(s) = \mathcal{L}\left[\dfrac{1}{2}(\sin((m+n)t) + \sin((m-n)t))\right] = \dfrac{1}{2}\left(\dfrac{m+n}{s^2+(m+n)^2} + \dfrac{m-n}{s^2+(m-n)^2}\right)$

$= \dfrac{1}{2}\dfrac{(m+n)(s^2+m^2+n^2-2mn) + (m-n)(s^2+m^2+n^2+2mn)}{(s^2+m^2+n^2+2mn)(s^2+m^2+n^2-2mn)}$

$= \dfrac{1}{2}\dfrac{2m(s^2+m^2+n^2) - 4mn^2}{(s^2+m^2+n^2)^2 - (2mn)^2} = \dfrac{m(s^2+m^2-n^2)}{(s^2+m^2+n^2)^2 - 4m^2n^2}$

d) $F(s) = \mathcal{L}[t\sin 2t] + \mathcal{L}[\sin 2t] = -\dfrac{d}{ds}\mathcal{L}[\sin 2t] + \dfrac{2}{s^2+4}$

$= -\left(\dfrac{2}{s^2+4}\right)' + \dfrac{2}{s^2+4} = \dfrac{4s}{(s^2+4)^2} + \dfrac{2}{s^2+4} = \dfrac{2s^2+4s+8}{(s^2+4)^2}$

e) $F(s) = \dfrac{1}{s}\mathcal{L}[\cos^2(\omega t)] = \dfrac{1}{s}\mathcal{L}\left[\dfrac{1+\cos(2\omega t)}{2}\right]$

$= \dfrac{1}{s}\left[\dfrac{1}{2}\dfrac{1}{s} + \dfrac{1}{2}\dfrac{s}{s^2+4\omega^2}\right] = \dfrac{s^2+4\omega^2+s^2}{2s^2(s^2+4\omega^2)} = \dfrac{s^2+2\omega^2}{s^2(s^2+4\omega^2)}$

f) $F(s) = \mathcal{L}[\sin^2 t](s-3) = \mathcal{L}\left[\dfrac{1-\cos(2t)}{2}\right](s-3) = \dfrac{1}{2}\dfrac{1}{s-3} - \dfrac{1}{2}\dfrac{s-3}{4+(s-3)^2}$

$= \dfrac{1}{2(s-3)} - \dfrac{1}{2}\dfrac{s-3}{(s-3)^2+4}$

g) $F(s) = e^{-bs}\mathcal{L}[\sin t] = \dfrac{e^{-bs}}{s^2+1}.$

2. a) $F(s) = \mathcal{L}[e^{5t} + (5t^2 - 2t + 7)u_3(t)] = \dfrac{1}{s-5} + e^{-3s}\mathcal{L}[[5(t+3)^2 - 2(t+3) + 7]$

$= \dfrac{1}{s-5} + e^{-3s}\mathcal{L}[5t^2 + 28t + 46] = \dfrac{1}{s-5} + e^{-3s}\left(\dfrac{10}{s^3} + \dfrac{28}{s^2} + \dfrac{46}{s}\right).$

b) $F(s) = \mathcal{L}\left[t - 3 + (2t^2-1)u_1(t) + (t-t^2)u_2(t)\right] = \dfrac{1}{s^2} - \dfrac{3}{s} + e^{-s}\mathcal{L}[2(t+1)^2 - 1]$

$+e^{-2s}\mathcal{L}[t+2-(t+2)^2] = \dfrac{1}{s^2} - \dfrac{3}{s} + e^{-s}\mathcal{L}[2t^2+4t+1] + e^{-2s}\mathcal{L}[-t^2-3t-2]$

$= \dfrac{1}{s^2} - \dfrac{3}{s} + e^{-s}\left(\dfrac{4}{s^3} + \dfrac{4}{s^2} + \dfrac{1}{s}\right) - e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{3}{s^2} + \dfrac{2}{s}\right).$

3. a) $F(s)=\dfrac{1}{(s+2)^2+1}$, so $f(t)=e^{-2t}\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right]=e^{-2t}\sin t$

b) Let us present $F(s)$ as a sum of partial fractions:

$F(s)=\dfrac{s+2}{(s+1)(s-2)(s^2+4)}=\dfrac{A}{s+1}+\dfrac{B}{s-2}+\dfrac{Cs+D}{s^2+4}$, bringing to the common denominator, we find that the numerators are equal:

$A(s-2)(s^2+4)+B(s+1)(s^2+4)+(Cs+D)(s+1)(s-2)=s+2$. Substituting $s=2$, we have $24B=4$, or $B=1/6$. Substituting $s=-1$, we obtain $-15A=1$, or $A=-1/15$, substituting $s=0$, we have $-8A+4B-2D=8/15+2/3-2D=2$, or $2D=-4/5$, $d=-2/5$. Substituting $s=1$, we obtain $-5A+10B-2(C+D)=3$, which implies $C+D=-1/2$, so $C=-1/2-D=-1/10$. Thus

$\mathcal{L}^{-1}[F]=-\dfrac{1}{15}\mathcal{L}^{-1}\left[\dfrac{1}{s+1}\right]+\dfrac{1}{6}\mathcal{L}^{-1}\left[\dfrac{1}{s-2}\right]-\dfrac{1}{10}\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4}\right]-\dfrac{2}{5}\mathcal{L}^{-1}\left[\dfrac{1}{s^2+4}\right]$

$=-\dfrac{1}{15}e^{-t}+\dfrac{1}{6}e^{2t}-\dfrac{1}{10}\cos(2t)-\dfrac{1}{5}\sin(2t)$

c) Since $F(s)=\dfrac{s}{(s^2+1)^2}=\dfrac{1}{2}\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right)$, we have

$f(t)=\dfrac{1}{2}t\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right)]=\dfrac{1}{2}t\sin t.$

d) $F(s)=e^{-s}\left(\dfrac{1}{s-1}-\dfrac{1}{s}\right)$, so

$f(t)=u_1(t)\mathcal{L}^{-1}\left[\dfrac{1}{s-1}-\dfrac{1}{s}\right](t-1)=(e^{t-1}-1)u_1(t)$

e) $f(t)=u_2(t)\left(\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right]\right)(t-2)+2u_3(t)\left(\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right]\right)(t-3)$

$+3u_4(t)\left(\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right]\right)(t-4)=\sin(t-2)u_2(t)+2\sin(t-3)u_3(t)+3\sin(t-4)u_4(t)$

f) $F(s)=e^{-s/3}\left[\dfrac{1}{s}-\dfrac{s}{s^2+1}\right]$, so

$f(t)=u_{1/3}(t)\left(\mathcal{L}^{-1}\left[\dfrac{1}{s}-\dfrac{s}{s^2+1}\right]\right)\left(t-\dfrac{1}{3}\right)=u_{1/3}(t)(1-\cos(t-\dfrac{1}{3}).$

4 a) Taking the Laplace Transform of both sides of $x''+3x'=e^t$, we have

$s^2\mathcal{L}[x]-s\cdot 0+1+3(s\mathcal{L}[x]-0)=\dfrac{1}{s-1}$, so $\mathcal{L}[x](s^2+3s)=\dfrac{1}{s-1}-1=\dfrac{2-s}{s-1}$ and

$\mathcal{L}[x]=\dfrac{2-s}{s(s+3)(s-1)}$. Presenting the right-hand side as a sum of partial fractions

$\dfrac{2-s}{s(s+3)(s^2+1)}=\dfrac{A}{s}+\dfrac{B}{s+3}+\dfrac{C}{s-1}$ we obtain $A(s-1)(s-3)+Bs(s+3)+Cs(s-1)=2-s$.

Substituting $s=1$, $s=0$ and $s=-3$ we have $4B=1$, $-3A=2$ and $12C=5$, so $A=-2/3$, $B=1/4$, $C=5/12$. The solution of the differential equation is

$\mathcal{L}^{-1}\left[-\dfrac{2}{3}\dfrac{1}{s}+\dfrac{1}{4}\dfrac{1}{s-1}+\dfrac{5}{12}\dfrac{1}{s+3}\right]=\dfrac{1}{4}e^t+\dfrac{5}{12}e^{-3t}-\dfrac{2}{3}.$

b) Taking the Laplace Transform of both sides of $x''+2x'-3x=e^{-t}$, we have

$s^2\mathcal{L}[x]-s\cdot 0-1+2(s\mathcal{L}[x]-0)-3\mathcal{L}[x]=\dfrac{1}{s+1}$, so $\mathcal{L}[x](s^2+2s-3)=\dfrac{1}{s+1}+1=\dfrac{s+2}{s+1}$

and
$\mathcal{L}[x] = \dfrac{s+2}{(s+1)(s^2+2s-3)} = \dfrac{s+2}{(s+1)(s-1)(s+3)}$. Presenting the right-hand side as a sum of partial fractions, we obtain $\dfrac{s+2}{(s+1)(s-1)(s+3)} = -\dfrac{1}{4}\dfrac{1}{s+1} + \dfrac{3}{8}\dfrac{1}{s-1} - \dfrac{1}{8}\dfrac{1}{s+3}$. Thus the solution is $\mathcal{L}^{-1}\left[-\dfrac{1}{4}\dfrac{1}{s+1} + \dfrac{3}{8}\dfrac{1}{s-1} - \dfrac{1}{8}\dfrac{1}{s+3}\right] = -\dfrac{1}{4}e^{-t} + \dfrac{3}{8}e^{t} - \dfrac{1}{8}e^{-3t}$.
c) Taking the Laplace Transform of both sides of $x'' + 2x' + x = \sin t$, we have $s^2\mathcal{L}[x] + 1 + 2s\mathcal{L}[x] + \mathcal{L}[x] = \dfrac{1}{s^2+1}$, so $\mathcal{L}[x](s^2+2s+1) = \dfrac{1}{s^2+1} - 1 = -\dfrac{s^2}{s^2+1}$ and $\mathcal{L}[x] = \dfrac{-s^2}{(s^2+1)(s+1)^2}$. Presenting the right-hand side as a sum of partial fractions $\dfrac{-s^2}{(s^2+1)(s+1)^2} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{Cs+D}{s^2+1}$ we obtain $A(s+1)(s^2+1) + B * s^2 + 1) + (Cs+D)(s+1)^2 = -s^2$. Comparing coefficients, we get $A = 1/2$, $B = -1/2$, $C = -1/2$, $D = 0$. Thus the solution is $\mathcal{L}^{-1}\left[\dfrac{1}{2}\dfrac{1}{s+1} - \dfrac{1}{2}\dfrac{1}{(s+1)^2} - \dfrac{1}{2}\dfrac{s}{s^2+1}\right] = \dfrac{1}{2}\left(e^{-t} - te^{-t} - \cos t\right)$.

h) The function $f(x)$ can be expressed in terms of the unit step function as

$$f(t) = 4t + (5t - 2 - 4t)u_2(t) = 4t + (t-2)u_2(t).$$

First, we apply the Laplace Transform to both sides of $x'' + 4x = 4t + (t-2)u_2(t)$ using the first differentiation formula $\mathcal{L}[x''] + 4\mathcal{L}[x] = \mathcal{L}[4t] + \mathcal{L}[(t-2)u_2(t)]$, thus $s^2\mathcal{L}[x] - sx(0) - x'(0) = s^2\mathcal{L}[x] + 1 = \frac{1}{s^2} + \mathcal{L}[(t-2)u_2(t)]$. By the second shift formula, $\mathcal{L}[(t-2)u_2(t)] = e^{-2s}\mathcal{L}[t] = e^{-2s}\frac{1}{s^2}$. From $s^2\mathcal{L}[x] + 1 + 4\mathcal{L}[x] = \dfrac{1}{s^2} + e^{-2s}\dfrac{1}{s^2}$ we find

$$\mathcal{L}[x] = \frac{4}{s^2(s^2+4)} + \frac{e^{-2s}}{s^2(s^2+4)} - \frac{1}{s^2+4}.$$

As $\dfrac{4}{s^2(s^2+4)} = \dfrac{1}{s^2} - \dfrac{1}{s^2+4}$,

$$\mathcal{L}[x] = \frac{1}{s^2} - \frac{1}{s^2+4} + \frac{1}{4}e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right) - \frac{1}{s^2+4} = \frac{1}{s^2} - \frac{2}{s^2+4} + \frac{1}{4}e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right).$$

Finally,

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)\right].$$

Using the shift formula, we obtain

$$x(t) = t - \sin(2t) + \frac{1}{4}u_2(t)\left(t - 2 - \frac{1}{2}\sin(2(t-2))\right).$$