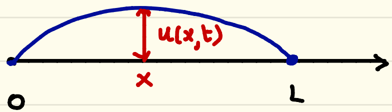
Wave Equation

The Initial-Boundary Value Problem

$$\begin{cases} c^2 u_{xx} = u_{tt}, & 0<x<L, \quad t>0 \\ u(0,t)=0, \quad u(L,t)=0, & t>0 \\ u(x,0)=f(x), \quad u_t(x,0)=g(x), & 0<x<L \end{cases}$$



models the vertical displacement $u(x,t)$ of a string with endpoints at $x=0$ and $x=L$. At equilibrium the string is a straight line segment. In response to being plucked or struck, it starts vertically vibrating. At any $t=t_0$, the graph of $u(x,t_0)$ gives the shape of the string. $u(x,0)=f(x)$ is the initial shape of the string and $u_t(x,0)=g(x)$ prescribes the initial velocity.

To solve the IBVP, we use the method of separation of variables and look for solutions of the BVP

$$\begin{cases} c^2 u_{xx} = u_{tt}, & 0<x<L, \ t>0 \\ u(0,t)=0, \quad u(L,t)=0, & t>0 \end{cases}$$

in the form $u(x,t) = X(x)\, T(t)$.

Substituting into $c^2 u_{xx} = u_{tt}$, we get $\dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{c^2 T(t)} = -\lambda$

Hence $X(x)$ and $T(t)$ are solutions of

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0)=0, \quad X(L)=0 \end{cases} \quad \text{and} \quad \left\{ T''(t) + \lambda c^2 T(t) = 0 \right.$$

Solving for non trivial solutions, we get

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad X(x) = \sin\left(n\frac{\pi}{L}x\right), \quad T(t) = \cos\left(n\frac{\pi}{L}ct\right), \ \sin\left(n\frac{\pi}{L}ct\right)$$

$$n = 1, 2, 3, \ldots$$

Next, we look for the solution of the IBVP

$$\begin{cases} c^2 u_{xx} = u_{tt}, & 0 < x < L, \quad t > 0 \\ u(0,t) = 0, \quad u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & 0 < x < L \end{cases}$$

in the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(n \frac{\pi}{L} x\right) \left(\alpha_n \cos\left(n \frac{\pi}{L} c t\right) + \beta_n \sin\left(n \frac{\pi}{L} c t\right) \right)$$

Notice that $u(x,t)$ satisfies the boundary conditions, regardless of what α_n & β_n are. To determine the coefficients α_n & β_n we require

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \iff \begin{cases} \sum_{n=1}^{\infty} \alpha_n \sin\left(n \frac{\pi}{L} x\right) = f(x) \\ \sum_{n=1}^{\infty} c n \frac{\pi}{L} \beta_n \sin\left(n \frac{\pi}{L} x\right) = g(x) \end{cases}$$

Hence $\displaystyle\sum_{n=1}^{\infty} \alpha_n \sin\left(n \frac{\pi}{L} x\right)$ must be the Fourier Sine series of

$f(x)$, $0 < x < L$

and

$\sum c n \frac{\pi}{L} \beta_n \sin\left(n \frac{\pi}{L} x\right)$ must be the Fourier Sine series of

$g(x)$, $0 < x < L$

Thus

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx$$

and $n = 1, 2, 3, \ldots$

$$n \frac{\pi}{L} c \cdot \beta_n = \frac{2}{L} \int_0^L g(x) \sin\left(n \frac{\pi}{L} x\right) dx$$

Summary

The solution of the Initial-Boundary value problem

$$\begin{cases} c^2 u_{xx} = u_{tt}, & 0 < x < L, \quad t > 0 \qquad (c > 0) \\ u(0,t) = 0, \quad u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & 0 < x < L \end{cases}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(n \frac{\pi}{L} x\right) \left[\alpha_n \cos\left(n \frac{\pi}{L} c t\right) + \beta_n \sin\left(n \frac{\pi}{L} c t\right) \right]$$

where

$$\sum_{n=1}^{\infty} \alpha_n \sin\left(n \frac{\pi}{L} x\right) = f(x) \implies \alpha_n = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx$$

$$\sum_{n=1}^{\infty} n \frac{\pi}{L} c \cdot \beta_n \sin\left(n \frac{\pi}{L} x\right) = g(x) \implies n \frac{\pi}{L} c \, \beta_n = \frac{2}{L} \int_0^L g(x) \sin\left(n \frac{\pi}{L} x\right) dx$$

Example

Solve the IBVP
$$\begin{cases} 4u_{xx} = u_{tt} & 0<x<1,\ t>0 \\ u(0,t) = 0, \quad u(1,t)=0, & t>0 \\ u(x,0) = x(1-x), \quad u_t(x,0)=0 & 0<x<1 \end{cases}$$

Here $c=2$, $L=1$, $f(x) = x(1-x)$, $g(x) = 0$

The solution is given by
$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(n\frac{\pi}{L}x\right)\left(\alpha_n \cos\left(n\frac{\pi}{L}ct\right) + \beta_n \sin\left(n\frac{\pi}{L}ct\right)\right)$$
$$= \sum_{n=1}^{\infty} \sin(n\pi x)\left(\alpha_n \cos(2n\pi t) + \beta_n \sin(2n\pi t)\right)$$

To compute the coefficients α_n and β_n, we use the initial conditions

$$\begin{cases} u(x,0) = x(1-x) \\ u_t(x,0) = 0 \end{cases} \iff \begin{cases} \displaystyle\sum_{n=1}^{\infty} \alpha_n \sin(n\pi x) = x(1-x), \quad 0<x<1 \\ \displaystyle\sum_{n=1}^{\infty} 2n\pi\beta_n \sin(n\pi x) = 0 \end{cases}$$

Hence

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(n\frac{\pi}{L}x\right)dx = \frac{2}{1}\int_0^1 x(1-x)\,\sin(n\pi x)\,dx$$

$$= 2\Big[-\frac{1}{n\pi}x(1-x)\cos(n\pi x) + \frac{1}{n^2\pi^2}(1-2x)\sin(n\pi x)$$

$$-\frac{2}{n^3\pi^3}\cos(n\pi x)\Big|_0^1\Big]$$

$$= 2\left[\frac{-2}{n^3\pi^3}\cos(n\pi) + \frac{2}{n^3\pi^3}\right] = \frac{4}{\pi^3}\,\frac{1-\cos(n\pi)}{n^3}$$

$x(1-x)$		$\sin(n\pi x)$
$1-2x$	$+$	$-\frac{1}{n\pi}\cos(n\pi x)$
-2	$-$	$-\frac{1}{n^2\pi^2}\sin(n\pi x)$
0	$+$	$\frac{1}{n^3\pi^3}\cos(n\pi x)$
	$-\int$	

$$\beta_n = 0$$

It follows

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4}{\pi^3}\,\frac{1-\cos(n\pi)}{n^3}\sin(n\pi x)\cdot\cos(2n\pi t)$$

Example

Find the solution of the IBVP

$$\begin{cases} 9\, u_{xx} = u_{tt} & 0<x<3, \quad t>0 \\ u(0,t) = 0, \quad u(3,t) = 0, & t>0 \\ u(x,0) = 3\sin(\frac{\pi}{3}x) - \frac{1}{2}\sin(2\frac{\pi}{3}x) & \\ u_t(x,0) = 0 & 0<x<3 \end{cases}$$

The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\tfrac{\pi}{3}x)\left(\alpha_n \cos(n\tfrac{\pi}{3}3t) + \beta_n \sin(n\tfrac{\pi}{3}3t)\right)$$

$$= \sum_{n=1}^{\infty} \sin(n\tfrac{\pi}{3}x)\left(\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t)\right)$$

To compute the coefficients α_n and β_n, we use the initial conditions

$$\begin{cases} u(x,0) = 3\sin(\frac{\pi}{3}x) - \frac{1}{2}\sin(2\frac{\pi}{3}x) \\ u_t(x,0) = 0 \end{cases} \iff$$

$$\begin{cases} \sum_{n=1}^{\infty} \alpha_n \sin\left(n \frac{\pi}{3} x\right) = 3 \sin\left(\frac{\pi}{3} x\right) - \frac{1}{2} \sin\left(2 \frac{\pi}{3} x\right) \\ \sum_{n=1}^{\infty} n\pi \beta_n \sin\left(n \frac{\pi}{3} x\right) = 0 \end{cases} \iff$$

$$\begin{cases} \alpha_1 = 3, \quad \alpha_2 = -\frac{1}{2}, \quad \alpha_3 = \alpha_4 = \cdots = 0 \\ \beta_n = 0, \quad n = 1, 2, 3, \ldots \end{cases}$$

Hence the solution

$$u(x,t) = \sin\left(\frac{\pi}{3} x\right) \left(3 \cos(\pi t) \right) + \sin\left(2 \frac{\pi}{3} x\right) \left(-\frac{1}{2} \cos(2\pi t) \right)$$

$$u(x,t) = 3 \sin\left(\frac{\pi}{3} x\right) \cos(\pi t) - \frac{1}{2} \sin\left(2 \frac{\pi}{3} x\right) \cos(2\pi t)$$

Laplace Equation

The solution of the boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < L, \quad 0 < y < H \\ u(x,0) = g(x), \quad u(x,H) = 0 & 0 < x < L \\ u(0,y) = 0, \quad u(L,y) = 0 & 0 < y < H \end{cases}$$

is given by

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n \sin\left(n \frac{\pi}{L} x\right) \sinh\left(n \frac{\pi}{L} (H-y)\right)$$

where

$$\alpha_n \cdot \sinh\left(n \frac{\pi}{L} H\right) = \frac{2}{L} \int_0^L g(x) \sin\left(n \frac{\pi}{L} x\right) dx, \quad n = 1, 2, 3, \ldots$$

Example

Solve the boundary value problem

$$u_{xx} + u_{yy} = 0 \qquad 0<x<2, \quad 0<y<2$$

$$u(0,y) = 0, \quad u(2,y)=0 \qquad 0<y<2$$

$$u(x,0) = x, \quad u(x,2) = 0 \qquad 0<x<2$$

The solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n \sin\left(n \frac{\pi}{2} x\right) \quad \sinh n \frac{\pi}{2}(2-y)$$

To determine the coefficients α_n, we use the boundary condition

$$u(x,0)=1 \iff \sum_{n=1}^{\infty} \alpha_n \sin\left(n\frac{\pi}{2}x\right) \sinh(n\pi) = x, \qquad 0<x<2$$

$$\iff \sum_{n=1} \alpha_n \sinh(n\pi) \cdot \sin\left(n \frac{\pi}{2}x\right) = x, \quad 0<x<2$$

Hence $\alpha_n \sinh(n\pi)$ are the coefficients of the Fourier Sine series of the function $f(x) = x, \quad 0<x<2$

Thus

$$\alpha_n \sinh(n\pi) = \frac{2}{2} \int_0^2 x \sin\left(n \frac{\pi}{2} x\right) dx$$

$$= -\frac{4}{n\pi} \cos(n\pi) \quad \Longrightarrow \quad \alpha_n = -\frac{4}{n\pi} \cos(n\pi) \frac{1}{\sinh(n\pi)}$$

and the solution is

$$u(x,y) = \sum_{n=1}^{\infty} -\frac{4}{n\pi} \cos(n\pi) \sin\left(n \frac{\pi}{2} x\right) \cdot \frac{\sinh\left(n \frac{\pi}{2}(2-y)\right)}{\sinh(n\pi)}$$