Heating and Cooling

These problems are based on Newton's law of cooling:

The rate of change of the temperature $T = T(t)$ of an object, is proportional to the temperature difference between the object and the medium surrounding it.

Hence, if M is the temperature of the medium surrounding the object, then $T' = k(M-T)$, where $k>0$ is a constant.

If $T(0) = T_0$, then $T = T(t)$ is solution of the ivp

$$\begin{cases} T' = k(M-T) \\ T(0) = T_0 \end{cases} \iff \begin{cases} T' + kT = kM \\ T(0) = T_0 \end{cases}$$

If we assume that $M = M_0$ is constant, then the solution is

$$T(t) = M_0 + (T_0 - M_0)\, e^{-kt}$$

Usually both T_0 and M_0 are given. To determine the constant k, we need to know the temperature of the object at another time t_1. For instance, if $T(t_1) = T_1$, then

$$T(t_1) = T_1 \iff M_0 + (T_0 - M_0)\, e^{-kt_1} = T_1 \iff e^{-kt_1} = \frac{T_1 - M_0}{T_0 - M_0}$$

It follows:

$$T(t) = M_0 + (T_0 - M_0)\, e^{-kt} = M_0 + (T_0 - M_0)\, e^{-kt_1 \cdot \frac{t}{t_1}}$$

$$= M_0 + (T_0 - M_0) \left(e^{-kt_1}\right)^{\frac{t}{t_1}} = M_0 + (T_0 - M_0) \left(\frac{T_1 - M_0}{T_0 - M_0}\right)^{\frac{t}{t_1}}$$

Example

A dish is baked in an oven at $325°C$ and cooled in a room, the temperature of which is $25°C$. After 4 min., the temperature of the dish drops to $225°C$

a. what is the temperature of the dish 8 min. after it was removed from the oven?

b. When will the temperature of the dish be $75^\circ C$?

Solution

a. The temperature of the dish at any time t is given by

$$T(t) = M_0 + (T_0 - M_0)\left(\frac{T_1 - M_0}{T_0 - M_0}\right)^{\frac{t}{t_1}}$$

Here $M_0 = 25^\circ c$, $T_0 = 325^\circ c$, $t_1 = 4$ min., $T_1 = 225^\circ c$

Substituting, we get $T(t) = 25 + 300\left(\frac{200}{300}\right)^{\frac{t}{4}} = 25 + 300\left(\frac{2}{3}\right)^{t/4}$

The temperature of the dish 8 min. after it has been removed is

$$T(8) = 25 + 300\left(\frac{2}{3}\right)^{8/4} = 25 + 300\left(\frac{2}{3}\right)^2 = 25 + \frac{400}{3} = \frac{475}{3} \approx 158^\circ C$$

b. The time t when the temperature of the dish reaches $75^\circ c$, satisfies

$$T(t) = 75 \iff 25 + 300\left(\frac{2}{3}\right)^{t/4} = 75 \iff \left(\frac{2}{3}\right)^{t/4} = \frac{1}{6} \iff t = \frac{4\ln(6)}{\ln(3) - \ln(2)} \approx 18 \text{ min.}$$