# Department of Mathematics and Statistics
# MATH 375
# Information sheet # 5
## Fourier Series

Functions are **periodic with period** $T$ if $f(x+T) = f(x)$ for any $x \in \mathbb{R}$. For example, $\sin(\omega x)$ and $\cos(\omega x)$ are periodic with period $T = \dfrac{2\pi}{\omega}$. The functions $\sin(n\omega x)$ and $\cos(n\omega x)$ are periodic with the same period for any $n \in \mathbb{N}$.

If a function is defined on a segment $[\alpha, \beta]$ of length $\beta - \alpha = T$, we can construct a periodic extention of $f(x)$ to all $x \in \mathbb{R}$. For example, $f(x) = \{x\}$ (the fractional part of a real number, $f(x) = x - [x]$, where $[x]$ is the integer part) is a periodic extension of $f(x) = x$, $x = [0, 1]$.

Let $f$ be defined on $[-\ell, \ell]$. A function is **even** if $f(-x) = f(x)$ and is **odd** if $f(-x) = -f(x)$.

The graph of an even function is symmetric with respect to the $y$-axis. Constants, $f(x) = x^{2n}$, $n \in \mathbb{N}$, $\cos(\omega x)$ are examples of even functions. For an even function $\displaystyle\int_{-\ell}^{\ell} f(x)\ dx = 2\int_0^{\ell} f(x)\ dx$.

The graph of an odd function is symmetric with respect to the origin, $f(0) = 0$ for any odd function $f$. $f(x) = x^{2n+1}$, $n \in \mathbb{N}$, $\sin(\omega x)$, $\tan(\omega x)$ are examples of odd functions. A sum of odd (even) functions is an odd (even) function. A product of even functions is even, a product of odd functions is even, a product of an odd function and an even function is an odd function (is similar to operations with positive and negative signs). For an odd function $\displaystyle\int_{-\ell}^{\ell} f(x)\ dx = 0$.

**Example 1.** We compute $\displaystyle\int_{-\pi}^{\pi} \left(x^{16} + 5x\right) \sin(nx)\ dx$, presenting the integrand as a sum of an odd and an even function and later applying integration by parts:

$$\int_{-\pi}^{\pi} \left(x^{16} + 5x\right) \sin(nx)\ dx = \int_{-\pi}^{\pi} x^{16} \sin(nx)\ dx + \int_{-\pi}^{\pi} 5x \sin(nx)\ dx$$

$$= 0 + 10 \int_0^{\pi} x \sin(nx)\ dx = 10 \left[ \frac{-x}{n} \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_0^{\pi} = \frac{(-1)^{n+1} 10\pi}{n},$$

as $\sin(n\pi) = 0$, $\cos(n\pi) = (-1)^n$.

Even and odd extensions of a function given on $[0, l]$ are described by

$$f_e(x) = \begin{cases} f(-x), & x \in [-l, 0] \\ f(x), & x \in [0, l] \end{cases} \qquad \text{and} \qquad f_o(x) = \begin{cases} -f(-x), & x \in [-l, 0] \\ f(x), & x \in [0, l], \end{cases}$$

respectively.

Fourier series for a function $f(x)$ on the segment $[\alpha, \beta]$ of length $\beta - \alpha = 2\ell$ are

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega x) + b_n \sin(n\omega x)\right], \quad \omega = \frac{\pi}{l},$$

where

$$a_0 = \frac{1}{2\ell}\int_{\alpha}^{\beta} f(x)\ dx, \quad a_n = \frac{1}{\ell}\int_{\alpha}^{\beta} f(x)\cos(n\omega x)\ dx, \quad b_n = \frac{1}{\ell}\int_{\alpha}^{\beta} f(x)\sin(n\omega x)\ dx.$$

Here we assume that $f$ is periodic with period $2\ell$.

If $f$ is an even function on $[-\ell, \ell]$ then we have the Fourier cosine series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega x), \quad \omega = \frac{\pi}{\ell},$$

where

$$a_0 = \frac{1}{\ell}\int_{0}^{\ell} f(x)\ dx, \quad a_n = \frac{2}{\ell}\int_{0}^{\ell} f(x)\cos(n\omega x)\ dx, \ n \in \mathbb{N}.$$

If $f$ is an odd function on $[-\ell, \ell]$ then we have the Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\omega x), \quad \omega = \frac{\pi}{l},$$

where

$$b_n = \frac{2}{\ell}\int_{0}^{\ell} f(x)\sin(n\omega x)\ dx, n \in \mathbb{N}.$$

**Example 2.** Find the Fourier series of $f(x) = x^2$, $0 < x < 2\pi$.
**Solution.** We substitute $\alpha = 0, \beta = 2\pi, \ell = 2\pi/2 = \pi, \omega = \frac{\pi}{\ell} = \frac{\pi}{\pi} = 1,\ f(x) = x^2$ in the formulas,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{1}{\pi}\frac{x^3}{3}\Big|_0^{2\pi} = \frac{1}{3\pi}(2\pi)^3 = \frac{8}{3}\pi^2.$$

We compute $a_n$ and $b_n$ using integration by parts:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)\ dx = \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\frac{2(2\pi)}{n^2}\cos(2\pi n) = \frac{4}{n^2}(-1)^{2n} = \frac{4}{n^2},$$

since $\sin(\pi n) = 0$, $\cos(\pi n) = (-1)^n$, and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)\ dx = \frac{1}{\pi}\left[-\frac{x^2}{n}\cos(nx) + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3}(1-1)\right] = -\frac{4\pi}{n}.$$

We have $a_0/2 = \frac{4}{3}\pi^2$. Thus

$$f(x) \sim \frac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos(nx) - \frac{\pi}{n}\sin(nx)\right].$$

So far we wrote $f(x) \sim$ its Fourier series. What does the Fourier series converge to? Let $f$ be a piecewise continuous function on $(\alpha,\beta)$, so that the left-hand limit $f(c^-) = \lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c, x<c} f(x)$ and the right-hand limit $f(c^+) = \lim\limits_{x\to c^+} f(x) = \lim\limits_{x\to c, x>c} f(x)$ exist and is finite.

Dirichlet's theorem describes the convergence.

1. If $f(x)$ is continuous at $x \in (\alpha,\beta)$ $(f(x^-) = f(x^+) = f(x))$ then the Fourier series at $x$ converges to $f(x)$, and $\sim$ can be replaced with the equality sign:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega x) + b_n\sin(n\omega x)\right], \quad \omega = 2\pi/(\beta-\alpha).$$

2. If $f(x)$ has a finite jump discontinuity at $x \in (\alpha,\beta)$ $(f(x^-) \neq f(x^+)$ exist and are finite) then the Fourier series at $x$ converges to the average of the left-hand and the right-hand limits

$$\frac{1}{2}\left[f(x^-) + f(x^+)\right] = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega x) + b_n\sin(n\omega x)\right], \quad \omega = 2\pi/(\beta-\alpha).$$

3. At the endpoints $\alpha$ and $\beta$ the Fourier series converges to $\frac{1}{2}\left[f(\alpha^+) + f(\beta^-)\right]$. We recall that the function is extended continuously with a period of $\beta-\alpha$, so, if $f(\alpha^+) = f(\beta^-)$, the Fourier series converges to this value, otherwise to their average (in this case, we have the Gibbs phenomenon).

**Example 3.** For the function $f(x) = \begin{cases} x^2+1 & 0 < x \leq 2 \\ \frac{12}{x} & 2 < x \leq 3 \\ 3x-1 & 3 < x < 7 \end{cases}$ determine the values to which the Fourier series of $f$ converges at each of the points $x$=0,2,2.5,3 and 7.

**Solution.** At $x = 2$ we have

$f(2^-) = \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(x^2+1) = 2^2+1 = 5,$

$f(2^+) = \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}\frac{12}{x} = \frac{12}{2} = 6,$

so the Fourier series at $x = 2$ converges to $\dfrac{f(2^-)+f(2^+)}{2} = \dfrac{1}{2}(5+6) = \dfrac{11}{2} = 5.5$. Further, $f(3^-) = \frac{12}{3} = 4$, $f(3^+) = 3\cdot 3 - 1 = 8$, the Fourier series at $x = 2$ converges to 6. At the endpoints $x = 0$ and $x = 7$ the Fourier series converges to

$$\frac{f(7^-)+f(0^+)}{2} = \frac{1}{2}(3\cdot 7 - 1 + 0^2 + 1) = \frac{21}{2}.$$