**Math 375** **Fall 2016**

## First Order Differential Equations

**Worksheet # 1** **Part 2** **September 26 - 30**

The problems marked with $(*)$ are to be attempted during the tutorial time. Students are strongly encouraged to attempt the remaining problems on their own. Solutions to all the problems will be available on the course's D2L website Friday, September 30. Please report any typos, omissions and errors to aiffam@ucalgary.ca

### Bernoulli Equations

**01.** Solve the Bernoulli differential equations

**a.** $t\,y' + y = y^2\,\ln(t)$

**b.** $y' + \dfrac{1}{t}\,y + t\,y^3 = 0$

**c*.** $y' = \dfrac{t^3 + y^3}{t\,y^2}$

**d.** $y' - t\,y = t\,y^{3/2}$

**02.** Solve the initial value problems

**a.** $\begin{cases} t\,y' + y = t^4\,y^4 \\ y(1) = 1/2 \end{cases}$

**b*.** $\begin{cases} y' - 2y = 2\sqrt{y} \\ y(0) = 1 \end{cases}$

### Exact Differential Equations

**03.** Determine whether the given differential equation is exact or not in its domain.

**a.** $2\,x\,\sin(y)\ \mathrm{d}x + x^2\,\cos(y)\ \mathrm{d}y = 0$

**b.** $\left(\,3 + \mathrm{e}^x\,\cos(y)\,\right)\ \mathrm{d}x + \mathrm{e}^{-x}\,\sin(y)\ \mathrm{d}y = 0$

**c.** $\left(\,\mathrm{e}^{-y} - y\,\sin(x\,y)\,\right)\ \mathrm{d}x = \left(\,x\mathrm{e}^{-y} - x\,\sin(x\,y)\,\right)\ \mathrm{d}y$

**04.** Find all functions $M(x, y)$ such that the differential equation

$$M(x, y)\ \mathrm{d}x + \Big(x\,\cos(y) - 2\,y\,\cos(x)\Big)\ \mathrm{d}y = 0$$

is exact in $\mathbb{R}^2$.

**05.** Solve the following exact differential equations

**a.** $\left(\,3\,x^2 + y^2 - 4\,x\,y - 3\,y\,\right)\ \mathrm{d}x + \left(\,-2\,x^2 + 6\,y^2 + 2\,x\,y - 3\,x\,\right)\ \mathrm{d}y = 0$

**b.** $\left(\,2\,x\,\sin(y) + \mathrm{e}^x\,\cos(y)\,\right)\ \mathrm{d}x + \left(\,x^2\,\cos(y) - \mathrm{e}^x\,\sin(y)\,\right)\ \mathrm{d}y = 0$

**c*.** $\left(\,x\,\ln(y) + y\,\ln(x)\,\right)\ \mathrm{d}x + \left(\,\dfrac{x^2}{2\,y} + x\,\ln(x) - x\,\right)\ \mathrm{d}y = 0$

**06.** Solve the following initial value problems.

**a.** $\left\{\ (y^2 + 2\,x)\ \mathrm{d}x + \left(2\,x\,y + \dfrac{1}{y}\right)\ \mathrm{d}y = 0, \qquad y(1) = 1\right.$

**b*.** $\left\{\ \left(x\,y^2 + \cos(x)\right)\ \mathrm{d}x + \left(\mathrm{e}^{2\,y} + x^2\,y\right)\ \mathrm{d}y = 0, \qquad y(\pi/2) = 0\right.$

## Integrating Factors For Non Exact Equations

**07.** Each of the following differential equations has an integrating factor $\mu$ that depends on either $x$ alone or on $y$ alone. Find $\mu$, then find the general solution of the differential equation.

**a*.** $\big(5\,x\,y + 4\,y^2 + 1\big)\ \mathrm{d}x + \big(x^2 + 2\,x\,y\big)\ \mathrm{d}y = 0$

**b.** $x\,y^3\ \mathrm{d}x + \big(x^2\,y^2 + 1\big)\ \mathrm{d}y = 0$

**c*.** $\big(2\,x + \tan(y)\big)\ \mathrm{d}x + \big(x - x^2\ \tan(y)\big)\ \mathrm{d}y = 0$

**d.** $\big(y^2 - x\big)\ \mathrm{d}x + 4\,x\,y\ \mathrm{d}y = 0,\ \ x > 0$

**08.** Verify that the differential equation is not exact in $\mathbb{R}^2$, then find an integrating factor in the form $\mu(x,y) = x^m\,y^n$

**a*.** $\big(\,4\,x\,y^2 + 6\,y\,\big)\ \mathrm{d}x + \big(\,5\,x^2\,y + 8\,x\,\big)\ \mathrm{d}y = 0$

**b.** $\big(\,3\,x\,y - 2\,y^2\,\big)\ \mathrm{d}x + \big(\,2\,x^2 - 3\,x\,y\,\big)\ \mathrm{d}y = 0$

## Homogeneous Equations

**09.** Find the general solution of the differential equations

**a.** $x\,y' = y + x\cos^2\left(\dfrac{y}{x}\right)$ **b*.** $y' = \dfrac{x-y}{x+y}$

**10.** Solve the differential equations

**a.** $3x\,y\,y' = x^2 + 4y^2$ **b.** $xy' = \sqrt{x^2 - y^2} + y$

**11.** Find the general solution of $y' = -\dfrac{8\,x + 4\,y + 1}{4\,x + 2\,y + 1}$

Hint: Use the substitution $u = 4\,x + 2\,y$ to convert the equation into a separable equation.

## Answers and Solutions

To solve a Bernoulli equation $y' + p(t)\,y = q(t)\,y^m$, $m \neq 0, 1$, divide both sides of the equation by $y^m$, to get $y^{-m}\,y' + p(t)\,y^{1-m} = q(t)$. The substitution, $u = y^{1-m}$, converts the equation into a linear differential equation in $u$. Solve for $u$, then go back to $y$.

**01a.** Multiply both sides of the equation by $y^{-2}$ to eliminate $y^2$ from the right side, and obtain

$$y^{-2}\,y' + \frac{1}{t}\,y^{-1} = \frac{\ln(t)}{t} \qquad \cdots \quad (*)$$

Next make the substitution $u = y^{-1} \implies u' = -y^{-2}\,y' \implies y^{-2}\,y' = -u'$. Substituting into the equation $(*)$, we get $-u' + \dfrac{1}{t}\,u = \dfrac{\ln t}{t} \iff u' - \dfrac{1}{t}\,u = -\dfrac{\ln t}{t}$.

An integrating factor is $\mu = \dfrac{1}{t}$. Solving we get $u = C\,t + 1 + \ln t$. Finally going back to $y$, we obtain the general solution $y = u^{-1} = \dfrac{1}{C\,t + 1 + \ln t}$

Another solution is $y = 0$. It was overlooked when we multiplied the original equation by $y^{-2}$.

**01b.** Rewrite the equation as $y' + \frac{1}{t}\, y = -t\, y^3$, and multiply both sides by $y^{-3}$ to eliminate $y^3$ from the right side to obtain $\quad y^{-3}\, y' + \frac{1}{t}\, y^{-2} = -t \quad \cdots \quad (*)$

Next, make the substitution $u = y^{-2} \implies u' = -2\, y^{-3}\, y' \implies y^{-3}\, y' = -\frac{u'}{2}$

Substituting into $(*)$, we get $-\frac{u'}{2} + \frac{1}{t}\, u = -t \iff u' - \frac{2}{t}\, u = 2\, t$.

An integrating factor is $\mu = \frac{1}{t^2}$. Solving we get $u = C\, t^2 + 2\, t^2 \ln|t|$. Finally going back to $y$, we obtain the general solution ( in implicit form )

$$y^{-2} = C\, t^2 + 2\, t^2 \ln|t| \implies y^2 = \frac{1}{C\, t^2 + 2\, t^2 \ln|t|}$$

Another solution is $y = 0$. It was overlooked when we multiplied the original equation by $y^{-3}$.

**01c.** Rewrite the equation as $y' = \frac{t^3}{t\, y^2} + \frac{y^3}{t\, y^2} \iff y' - \frac{1}{t}\, y = t^2\, y^{-2}$, and multiply both sides by $y^2$ to eliminate $y^{-2}$ from the right side to obtain $\quad y^2\, y' - \frac{1}{t}\, y^3 = t^2 \quad \cdots \quad (*)$

Next, make the substitution $u = y^3 \implies u' = 3\, y^2\, y' \implies y^2\, y' = \frac{u'}{3}$

Substituting into $(*)$, we get $\frac{u'}{3} - \frac{1}{t}\, u = t^2 \iff u' - \frac{3}{t}\, u = 3\, t^2$.

An integrating factor is $\mu = \frac{1}{t^3}$. Solving we get $u = C\, t^3 + 3\, t^3 \ln|t|$. Finally going back to $y$, we obtain the general solution

$$y^3 = C\, t^3 + 3\, t^3 \ln|t| \iff y = \Big(C\, t^3 + 3\, t^3 \ln|t|\Big)^{1/3} \iff y = t\, \Big(C + 3 \ln|t|\Big)^{1/3}$$

**01d.** Multiply both sides of the equation by $y^{-3/2}$ to get $\quad y^{-3/2}\, y' - t\, y^{-1/2} = t \quad \cdots \quad (*)$

Making use of the substitution $u = y^{-1/2} \implies u' = -\frac{1}{2}\, y^{-3/2}\, y' \implies y^{-3/2}\, y' = -2\, u'$, and substituting into $(*)$, we get $-2\, u' - t\, u = t \iff u' + \frac{t}{2}\, u = -\frac{t}{2}$

An integrating factor is $\mu = e^{t^2/4}$. Solving we get $u = C\, e^{-t^2/4} - 1$. Finally going back to $y$, we obtain the general solution $\quad y = \frac{1}{\Big(C\, e^{-t^2/4} - 1\Big)^2}$

Another solution is $y = 0$. It was overlooked when we multiplied the original equation by $y^{-3/2}$.

**02a.** We start by finding the general solution. The differential equation is a Bernoulli's equation. To solve, we rewrite the equation as $y' + \frac{1}{t}\, y = t^3\, y^4$, then multiply both sides by $y^{-4}$ to eliminate $y^4$ from the right side and obtain $\quad y^{-4}\, y' + \frac{1}{t}\, y^{-3} = t^3 \quad \cdots \quad (*)$

This suggests the use of the substitution

$$u = y^{-3} \implies u' = -3\, y^{-4}\, y' \implies y^{-4}\, y' = -\frac{u'}{3}$$

Substituting into $(*)$, we get $-\frac{u'}{3} + \frac{1}{t}\, u = t^3 \iff u' - \frac{3}{t}\, u = -3\, t^3$

An integrating factor is $\mu = \frac{1}{t^3}$. Solving we get

$$u = C\, t^3 - 3\, t^4 \iff y^{-3} = C\, t^3 - 3\, t^4 \implies y = \Big(C\, t^3 - 3\, t^4\Big)^{-1/3} \iff y = \frac{1}{t\, (C - 3\, t)^{1/3}}$$

Now we select the constant $C$ so that the initial condition $y(1) = 1/2$ is satisfied.

$$y(1) = \frac{1}{2} \iff \frac{1}{(C-3)^{1/3}} = \frac{1}{2} \iff C = 11$$

Hence the solution $\quad y = \frac{1}{t\, (11 - 3\, t)^{1/3}}$

**02b.** We start by finding the general solution. The differential equation is a Bernoulli's equation. To solve, we multiply both sides of the equation by $y^{-1/2}$ to eliminate $\sqrt{y}$ from the right side and obtain $y^{-1/2}\,y' - 2\,y^{1/2} = 2 \quad \cdots \quad (*)$
This suggests the use of the substitution
$$u = y^{1/2} \implies u' = \frac{1}{2}\,y^{-1/2}\,y' \implies y^{-1/2}\,y' = 2\,u'$$
Substituting into $(*)$, we get $2\,u' - 2\,u = 2 \iff u' - u = 1$
An integrating factor is $\mu = \mathrm{e}^{-t}$. Solving we get $u = C\,\mathrm{e}^{t} - 1 \iff y^{1/2} = C\,\mathrm{e}^{t} - 1$.
Now we select the constant $C$ so that the initial condition $y(0) = 1$ is satisfied.
$$\big(y(0)\big)^{1/2} = C\,\mathrm{e}^{0} - 1 \iff 1 = C - 1 \iff C = 2$$
Hence $y^{1/2} = 2\,\mathrm{e}^{t} - 1 \implies y = \big(2\,\mathrm{e}^{t} - 1\big)^2$

**Exactness Test**
Consider the differential equation $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \quad \cdots(*)$, and assume that both $M(x,y)$ and $N(x,y)$ have continuous first partial derivatives in a one-piece region $D$, with no holes. Then the differential equation (*), is exact in $D$ if and only if
$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y), \text{ in } D$$

**03a.** We have $\dfrac{\partial}{\partial y}\Big(2\,x\,\sin(y)\Big) = 2\,x\,\cos(y)$ and $\dfrac{\partial}{\partial x}\Big(x^2\,\cos(y)\Big) = 2\,x\,\cos(y)$. The two partial derivatives are continuous and equal in the whole $xy$-plane, by the exactness test, the differential equation is exact in $\mathbb{R}^2$.

**03b.** We have $\dfrac{\partial}{\partial y}\Big(3+\mathrm{e}^{x}\,\cos(y)\Big) = -\mathrm{e}^{x}\,\sin(y)$ and $\dfrac{\partial}{\partial x}\Big(\mathrm{e}^{-x}\,\sin(y)\Big) = -\mathrm{e}^{-x}\,\sin(y)$. The two partial derivatives are continuous in $\mathbb{R}^2$, but cleary not equal. As a result of the exactness test, the equation is not exact in its domain.

**03c.** First rewrite the equation in the form $\big(\mathrm{e}^{-y} - y\,\sin(x\,y)\big)\,\mathrm{d}x + \big(-x\,\mathrm{e}^{-y} + x\,\sin(x\,y)\big)\,\mathrm{d}y = 0$. Then
$$\frac{\partial}{\partial y}\Big(\mathrm{e}^{-y} - y\,\sin(x\,y)\Big) = -\mathrm{e}^{-y} - \sin(x\,y) - x\,y\,\cos(x\,y)$$
and
$$\frac{\partial}{\partial x}\Big(-x\,\mathrm{e}^{-y} + x\,\sin(x\,y)\Big) = -\mathrm{e}^{-y} + \sin(x\,y) + x\,y\,\cos(x\,y)$$
The two partials being different, the equation is not exact in its domain $\mathbb{R}^2$.

**04.** Assuming that $M(x,y)$ has continuous first partials, by the exactness test, we have
$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial}{\partial x}\Big(x\,\cos(y) - 2\,y\,\cos(x)\Big) \iff \frac{\partial M}{\partial y}(x,y) = \cos(y) + 2\,y\,\sin(x)$$
Partially integrating both sides with respect to $y$, leads to
$$M(x,y) = \sin(y) + y^2\,\sin(x) + K(x)$$
where $K(x)$ is an arbitrary differentiable function of $x$.

Consider the differential equation $M(x,y)\ \mathrm{d}x + N(x,y)\ \mathrm{d}y = 0 \quad \cdots(*)$, and suppose it is exact in a region $D$ in the $xy$-plane. If $G(x,y)$ is a potential function of (*), then the general solution is given by $G(x,y) = C$. To find $G(x,y)$, solve the system $\begin{cases} \dfrac{\partial G}{\partial x}(x,y) = M(x,y) \\ \\ \dfrac{\partial G}{\partial y}(x,y) = N(x,y) \end{cases}$

Remember the simpler $G(x,y)$ is, the better.

**05a.** Given that the differential equation is exact, the general solution in implicit form is given by $G(x,y) = C$, where $G(x,y)$ is any potential function of the differential equation. To find $G(x,y)$, we solve the system

$$\begin{cases} \dfrac{\partial G}{\partial x}(x,y) = 3\,x^2 + y^2 - 4\,x\,y - 3\,y \\ \\ \dfrac{\partial G}{\partial y}(x,y) = -2\,x^2 + 6\,y^2 + 2\,x\,y - 3\,x \end{cases}$$

Partially integrating the first equation with respect to $x$, we get

$$G(x,y) = x^3 + x\,y^2 - 2\,x^2\,y - 3\,x\,y + K(y)$$

To determine $K(y)$, we substitute into the second equation, to get

$$2\,x\,y - 2\,x^2 - 3\,x + K'(y) = -2\,x^2 + 6\,y^2 + 2\,x\,y - 3\,x \iff K'(y) = 6\,y^2 \iff K(y) = 2\,y^3 + L$$

where $L$ is an arbitrary constant. Selecting $L = 0$, ( recall we want the simplest potential ), leads to $G(x,y) = x^3 + x\,y^2 - 2\,x^2\,y - 3\,x\,y + 2\,y^3$. Hence the general solution is

$$G(x,y) = C \iff x^3 - 2\,x^2\,y + x\,y^2 + 2\,y^3 - 3\,x\,y = C$$

<u>An alternate</u> way to find $G(x,y)$ is to partially integrate both equations. The first with respect to $x$, and the second with respect to $y$. Doing so we obtain

$$\begin{cases} G(x,y) = x^3 + x\,y^2 - 2\,x^2\,y - 3\,x\,y + K(y) \\ G(x,y) = -2\,x^2\,y + 2\,y^3 + x\,y^2 - 3\,x\,y + L(x) \end{cases}$$

Comparing the two expresions of $G(x,y)$, leads to

$$K(y) = 2\,y^3 + L(x) \implies K(y) = 2\,y^3 + K_0$$

where $K_0$ is an arbitrary constant. Ignoring the constant, a potential function is

$$G(x,y) = x^3 + x\,y^2 - 2\,x^2\,y - 3\,x\,y + 2\,y^3$$

**05b.** Given that the differential equation is exact, the general solution in implicit form is given by $G(x,y) = C$, where $G(x,y)$ is any potential function of the differential equation. To find $G(x,y)$, we solve the system $\begin{cases} \dfrac{\partial G}{\partial x}(x,y) = 2\,x\,\sin(y) + \mathrm{e}^x\,\cos(y) \\ \\ \dfrac{\partial G}{\partial y}(x,y) = x^2\,\cos(y) - \mathrm{e}^x\,\sin(y) \end{cases}$ Partially integrating the first equation with respect to $x$, and the second with respect to $y$, we get $\begin{cases} G(x,y) = x^2\,\sin(y) + \mathrm{e}^x\,\cos(y) + K(y) \\ \\ G(x,y) = x^2\,\sin(y) + \mathrm{e}^x\,\cos(y) + L(x) \end{cases}$ Comparing the two expresions of $G(x,y)$, we obtain $K(y) = L(x) \implies K(y) = K_0$, where $K_0$ is a constant. Ignoring the constant, we get the potential function $G(x,y) = x^2\,\sin(y) + \mathrm{e}^x\,\cos(y)$.

The general solution, in implicit form, is then

$$G(x,y) = C \iff x^2 \sin(y) + e^x \cos(y) = C$$

**05c.** The differential equation being exact, its general solution in implicit form is given by $G(x,y) = C$, where $G(x,y)$ is any potential function of the differential equation. To find $G(x,y)$, we solve the system $\begin{cases} \dfrac{\partial G}{\partial x}(x,y) = x \ln(y) + y \ln(x) \\[2ex] \dfrac{\partial G}{\partial y}(x,y) = \dfrac{x^2}{2\,y} + x \ln(x) - x \end{cases}$ Partially integrating the first equation with respect to $x$, and the second with respect to $y$, we get

$$\begin{cases} G(x,y) = \dfrac{1}{2}x^2 \ln(y) + y\,\big(x \ln(x) - x\big) + K(y) \\[2ex] G(x,y) = \dfrac{1}{2}x^2 \ln(y) + x \ln(x)\, y - x\,y + L(x) \end{cases} \implies K(y) = L(x) \implies K(y) = K_0$$

where $K_0$ is a constant. Hence the potential function $G(x,y) = \dfrac{1}{2}\,x^2 \ln(y) + x\,y \ln(x) - x\,y$, and the general solution, in implicit form is

$$G(x,y) = C \iff \frac{1}{2}\,x^2 \ln(y) + x\,y \ln(x) - x\,y = C$$

**06a.** We start by finding the general solution of the differential equation.
We have $\dfrac{\partial}{\partial y}\Big(y^2 + 2\,x\Big) = 2\,y$ and $\dfrac{\partial}{\partial x}\Big(2\,x\,y + \dfrac{1}{y}\Big) = 2\,y$. By the exactness test, the differential equation is exact in the upper half of the $xy$-plane. A potential function $G(x,y)$ can be found by solving the system $\begin{cases} \dfrac{\partial G}{\partial x}(x,y) = y^2 + 2\,x \\[2ex] \dfrac{\partial G}{\partial y}(x,y) = 2\,x\,y + \dfrac{1}{y} \end{cases}$ Partially integrating the first equation with respect to $x$, and the second with respect to $y$, we get $\begin{cases} G(x,y) = x\,y^2 + x^2 + K(y) \\ G(x,y) = x\,y^2 + \ln(y) + L(x) \end{cases}$ Comparing the two expressions of $G(x,y)$, leads to $K(y) = \ln(y) + L(x) \implies K(y) = \ln(y) + K_0$, where $K_0$ is a real constant. Hence the potential function $G(x,y) = x\,y^2 + x^2 + \ln(y)$, and the general solution is $G(x,y) = C \iff x\,y^2 + x^2 + \ln(y) = C$
Next, we select the constant $C$ so that the initial condition $y(1) = 1$ holds. Setting $x = 1$ and $y = 1$ into the general solution, we get $1 + 1 + \ln(1) = C \iff C = 2$. Hence the solution

$$x\,y^2 + x^2 + \ln(y) = 2$$

**06b.** We start by finding the general solution of the differential equation.
We have $\dfrac{\partial}{\partial y}\Big(x\,y^2 + \cos(x)\Big) = 2\,x\,y$ and $\dfrac{\partial}{\partial x}\Big(e^{2\,y} + x^2\,y\Big) = 2\,x\,y$. By the exactness test, the differential equation is exact in all of $\mathbb{R}^2$. A potential function $G(x,y)$ can be found by solving the system $\begin{cases} \dfrac{\partial G}{\partial x}(x,y) = x\,y^2 + \cos(x) \\[2ex] \dfrac{\partial G}{\partial y}(x,y) = e^{2\,y} + x^2\,y \end{cases}$ Partially integrating the first equation with respect to $x$, and the second with respect to $y$, we get

$$\left\{\begin{array}{l} G(x,y)=\dfrac{1}{2}\,x^2\,y^2+\sin(x)+K(y) \\ \\ G(x,y)=\dfrac{1}{2}\,e^{2\,y}+\frac{1}{2}\,x^2\,y^2+L(x) \end{array}\right. \implies K(y)=\frac{1}{2}\,e^{2\,y}+L(x) \implies K(y)=\frac{1}{2}\,e^{2\,y}+K_0,$$

where $K_0$ is a real constant. This leads to the potential function $G(x,y)=\dfrac{1}{2}\,x^2\,y^2+\sin(x)+\dfrac{1}{2}\,e^{2\,y}$. Hence the general solution is

$$\frac{1}{2}\,x^2\,y^2+\sin(x)+\frac{1}{2}\,e^{2\,y}=C$$

Next, we select the constant $C$ so that the initial condition $y\big(\pi/2\big)=0$ holds. Setting $x=\pi/2$ and $y=0$ into the formula for the general solution, we get $0+1+\dfrac{1}{2}=C \iff C=\dfrac{3}{2}$. Hence the solution

$$\frac{1}{2}\,x^2\,y^2+\sin(x)+\frac{1}{2}\,e^{2\,y}=\frac{3}{2} \iff x^2\,y^2+2\,\sin(x)+e^{2\,y}=3$$

Consider the differential equation $M(x,y)\;dx+N(x,y)\;dy=0 \quad \cdots(*)$, and suppose that it is not exact, i.e., $M_y(x,y)\neq N_x(x,y)$

If $\dfrac{N_x(x,y)-M_y(x,y)}{N(x,y)}=u(x)$, then $\mu(x)=e^{-\int u(x)\,dx}$ is an integrating factor of (*)

If $\dfrac{M_y(x,y)-N_x(x,y)}{M(x,y)}=v(y)$, then $\mu(y)=e^{-\int v(y)\,dy}$ is an integrating factor of (*)

**07a.** Here $\left\{\begin{array}{l} M(x,y)=5\,x\,y+4\,y^2+1 \\ N(x,y)=x^2+2\,x\,y \end{array}\right. \implies \left\{\begin{array}{l} M_y(x,y)=5\,x+8\,y \\ N_x(x,y)=2\,x+2\,y \end{array}\right.$ Clearly the two partials are different, consequently the differential equation is not exact in $\mathbb{R}^2$. Looking for an integrating factor $\mu$ that depends on one variable, we compute

$$M_y(x,y)-N_x(x,y)=\big(5\,x+8\,y\big)-\big(2\,x+2\,y\big)=3\,x+6\,y=3\,\big(x+2\,y\big)$$

Comparing with $M(x,y)=5\,x\,y+4\,y^2+1$ and $N(x,y)=x^2+2\,x\,y=x\,\big(x+2\,y\big)$, we see that $\dfrac{N_x(x,y)-M_y(x,y)}{N(x,y)}=\dfrac{-3\,\big(x+2\,y\big)}{x\,\big(x+2\,y\big)}=-\dfrac{3}{x}$, is a function of $x$ only. As a result, an integrating factor is given by $\mu(x)=e^{-\int -3/x\;dx}=e^{3\,\ln|x|}=|x|^3=\pm x^3$. Selecting $\mu(x)=x^3$, and multiplying the differential equation by $\mu(x)=x^3$, leads to the equation

$$\big(5\,x^4\,y+4\,x^3\,y^2+x^3\big)\;dx+\big(x^5+2\,x^4\,y\big)\;dy=0 \quad \cdots \quad (*)$$

This is an exact equation, since $\dfrac{\partial}{\partial y}\Big(5\,x^4\,y+4\,x^3\,y^2+x^3\Big)=\dfrac{\partial}{\partial x}\Big(x^5+2\,x^4\,y\Big)$, as you can easily verify. A potential function of the equation $(*)$ can be found by solving the system

$$\left\{\begin{array}{l} \dfrac{\partial G}{\partial x}(x,y)=5\,x^4\,y+4\,x^3\,y^2+x^3 \\ \\ \dfrac{\partial G}{\partial y}(x,y)=x^5+2\,x^4\,y \end{array}\right. \implies \left\{\begin{array}{l} G(x,y)=x^5\,y+x^4\,y^2+\dfrac{1}{4}\,x^4+K(y) \\ G(x,y)=x^5\,y+x^4\,y^2+L(x) \end{array}\right.$$

$$\implies K(y)=K_0$$

where $K_0$ is a real constant. Hence a potential function is $G(x,y)=x^5\,y+x^4\,y^2+\dfrac{1}{4}\,x^4$, and the general solution is $G(x,y)=C \iff x^5\,y+x^4\,y^2+\dfrac{1}{4}\,x^4=C$

**07b.** Here $\begin{cases} M(x,y) = x\,y^3 \\ N(x,y) = x^2\,y^2 + 1 \end{cases} \implies \begin{cases} M_y(x,y) = 3\,x\,y^2 \\ N_x(x,y) = 2\,x\,y^2 \end{cases}$ This shows that the differential equation is not exact in $\mathbb{R}^2$. To find an integrating factor $\mu$ that depends on one variable, we compute $M_y(x,y) - N_x(x,y) = \left(3\,x\,y^2\right) - \left(2\,x\,y^2\right) = x\,y^2$. Comparing with $M(x,y) = x\,y^3$ and $N(x,y) = x^2\,y^2 + 1$, we see that $\dfrac{M_y(x,y) - N_x(x,y)}{M(x,y)} = \dfrac{x\,y^2}{x\,y^3} = \dfrac{1}{y}$, is a function of $y$ only. As a result an integrating factor is given by

$$\mu(y) = \mathrm{e}^{-\int 1/y\ \mathrm{d}y} = \mathrm{e}^{-\ln|y|} = \frac{1}{|y|} = \pm\frac{1}{y}$$

Selecting $\mu(y) = \dfrac{1}{y}$, and multiplying the equation by $\mu(y) = \dfrac{1}{y}$, we obtain the equation

$$x\,y^2\ \mathrm{d}x + \left(x^2\,y + \frac{1}{y}\right)\ \mathrm{d}y = 0 \quad \cdots \quad (*)$$

This is an exact equation, since $\dfrac{\partial}{\partial y}\left(x\,y^2\right) = \dfrac{\partial}{\partial x}\left(x^2\,y + \dfrac{1}{y}\right)$, as you can easily verify.
A potential function of the equation $(*)$ can be found by solving the system

$$\begin{cases} \dfrac{\partial G}{\partial x}(x,y) = x\,y^2 \\[2ex] \dfrac{\partial G}{\partial y}(x,y) = x^2\,y + \dfrac{1}{y} \end{cases} \implies \begin{cases} G(x,y) = \dfrac{1}{2}\,x^2\,y^2 + K(y) \\[2ex] G(x,y) = \dfrac{1}{2}\,x^2\,y^2 + \ln|y| + L(x) \end{cases}$$

$$\implies K(y) = \ln|y| + L(x) \implies K(y) = \ln|y| + K_0$$

where $K_0$ is a real constant. Hence a potential function is $G(x,y) = \dfrac{1}{2}\,x^2\,y^2 + \ln|y|$, and the general solution is

$$G(x,y) = C \iff \frac{1}{2}\,x^2\,y^2 + \ln|y| = C \iff x^2\,y^2 + \ln\left(y^2\right) = C$$

The solution $y = 0$ has been overlooked when we multiplied the original equation by $\dfrac{1}{y}$.

**07c.** Here $\begin{cases} M(x,y) = 2\,x + \tan(y) \\ N(x,y) = x - x^2\,\tan(y) \end{cases} \implies \begin{cases} M_y(x,y) = \sec^2(y) \\ N_x(x,y) = 1 - 2\,x\,\tan(y) \end{cases}$ The first partials being different, the equation is not exact. To find an integrating factor that depends on one of the variables only, we compute

$$M_y(x,y) - N_x(x,y) = \sec^2(y) - \Big(1 - 2\,x\,\tan(y)\Big) = \tan^2(y) + 2\,x\,\tan(y) = \tan(y)\,\Big(\tan(y) + 2\,x\Big)$$

Comparing with $M(x,y) = 2\,x + \tan(y)$ and $N(x,y) = x\Big(1 - x\,\tan(y)\Big)$, we see that $\dfrac{M_y(x,y) - N_x(x,y)}{M(x,y)} = \dfrac{\tan(y)\,\Big(\tan(y) + 2\,x\Big)}{2\,x + \tan(y)} = \tan(y)$, is a function of $y$ only. As a result an integrating factor is given by

$$\mu(y) = \mathrm{e}^{-\int \tan(y)\ \mathrm{d}y} = \mathrm{e}^{\ln|\cos(y)|} = |\cos(y)| = \pm\cos(y)$$

Selecting $\mu(y) = \cos(y)$, and multiplying the equation by $\mu(y) = \cos(y)$, we obtain the equation

$$\Big(2\,x\,\cos(y) + \sin(y)\Big)\ \mathrm{d}x + \Big(x\,\cos(y) - x^2\,\sin(y)\Big)\ \mathrm{d}y = 0 \quad \cdots \quad (*)$$

This is an exact equation, since $\dfrac{\partial}{\partial y}\Big(2\,x\,\cos(y) + \sin(y)\Big) = \dfrac{\partial}{\partial x}\Big(x\,\cos(y) - x^2\,\sin(y)\Big)$, as you can easily verify. A potential function of $(*)$ can be found by solving the system

$$\left\{\begin{aligned} &\frac{\partial G}{\partial x}(x,y) = 2\,x\,\cos(y) + \sin(y) \\ &\frac{\partial G}{\partial y}(x,y) = x\,\cos(y) - x^2\,\sin(y) \end{aligned}\right. \implies \left\{\begin{aligned} &G(x,y) = x^2\,\cos(y) + x\,\sin(y) + K(y) \\ &G(x,y) = x\,\sin(y) + x^2\,\cos(y) + L(x) \end{aligned}\right.$$

$$\implies K(y) = K_0$$

where $K_0$ is a real constant. Hence the potential $G(x,y) = x^2\,\cos(y) + x\,\sin(y)$, and the general solution is

$$G(x,y) = C \iff x^2\,\cos(y) + x\,\sin(y) = C$$

**07d.** Here $\left\{\begin{aligned} M(x,y) &= y^2 - x \\ N(x,y) &= 4\,x\,y \end{aligned}\right. \implies \left\{\begin{aligned} M_y(x,y) &= 2\,y \\ N_x(x,y) &= 4\,y \end{aligned}\right.$ The two partials being different, the differential equation is not exact. Looking for an integrating factor $\mu$ that depends on one variable, we compute $M_y(x,y) - N_x(x,y) = \big(2\,y\big) - \big(4\,y\big) = -2\,y$ Comparing with $M(x,y) = y^2 - x$ and $N(x,y) = 4\,x\,y$, we see that $\dfrac{N_x(x,y) - M_y(x,y)}{N(x,y)} = \dfrac{-2\,y}{4\,x\,y} = -\dfrac{1}{2\,x}$, dependents on $x$ only. As a result, an integrating factor is given by

$$\mu(x) = \mathrm{e}^{-\int 1/(2\,x)\ \mathrm{d}x} = \mathrm{e}^{-(1/2)\ \ln|x|} = \mathrm{e}^{\ln\left(|x|^{-1/2}\right)} = |x|^{-1/2} = \frac{1}{\sqrt{|x|}} = \frac{1}{\sqrt{x}} \qquad \text{(recall } x > 0)$$

Multiplying the differential equation by $\mu(x) = \dfrac{1}{\sqrt{x}}$, leads to the equation

$$\left(\frac{y^2}{\sqrt{x}} - \sqrt{x}\right)\,\mathrm{d}x + 4\,\sqrt{x}\,y\ \mathrm{d}y = 0 \quad \cdots \quad (*)$$

This is an exact equation, since $\dfrac{\partial}{\partial y}\left(\dfrac{y^2}{\sqrt{x}} - \sqrt{x}\right) = \dfrac{\partial}{\partial x}\left(4\,\sqrt{x}\,y\right)$, as you can easily verify. A potential function of $(*)$ can be obtained by solving the system

$$\left\{\begin{aligned} &\frac{\partial G}{\partial x}(x,y) = \frac{y^2}{\sqrt{x}} - \sqrt{x} \\ &\frac{\partial G}{\partial y}(x,y) = 4\,\sqrt{x}\,y \end{aligned}\right. \implies \left\{\begin{aligned} &G(x,y) = 2\,\sqrt{x}\,y^2 - \frac{2}{3}\,x\,\sqrt{x} + K(y) \\ &G(x,y) = 2\,\sqrt{x}\,y^2 + L(x) \end{aligned}\right. \implies K(y) = K_0$$

where $K_0$ is a real constant. Hence a potential function is $G(x,y) = 2\,\sqrt{x}\,y^2 - \dfrac{2}{3}\,x\,\sqrt{x}$, and the general solution is

$$G(x,y) = C \iff 2\,\sqrt{x}\,y^2 - \frac{2}{3}x\,\sqrt{x} = C \iff 3\,\sqrt{x}\,y^2 - x\,\sqrt{x} = C$$

**08a.** From $\dfrac{\partial}{\partial y}\left(4\,x\,y^2 + 6\,y\right) = 8\,x\,y + 6 \neq \dfrac{\partial}{\partial x}\left(5\,x^2\,y + 8\,x\right) = 10\,x + 8$, it follows that the differential equation is not exact. Assuming that $\mu(x,y) = x^m\,y^n$, is an integrating factor, it follows that $x^m\,y^n\,\Big(\,\big(4\,x\,y^2 + 6\,y\big)\ \mathrm{d}x + \big(5\,x^2\,y + 8\,x\big)\ \mathrm{d}y\,\Big) = 0$, must be exact, i.e.,

$$\left(4\,x^{m+1}\,y^{n+2} + 6\,x^m\,y^{n+1}\right)\,\mathrm{d}x + \left(5\,x^{m+2}\,y^{n+1} + 8\,x^{m+1}\,y^n\right)\,\mathrm{d}y = 0$$

is exact. Hence $\dfrac{\partial}{\partial y}\left(4\,x^{m+1}\,y^{n+2} + 6\,x^m\,y^{n+1}\right) = \dfrac{\partial}{\partial x}\left(5\,x^{m+2}\,y^{n+1} + 8\,x^{m+1}\,y^n\right)$, or else

$$4\,(n+2)\,x^{m+1}\,y^{n+1} + 6\,(n+1)\,x^m\,y^n = 5\,(m+2)\,x^{m+1}\,y^{n+1} + 8\,(m+1)\,x^m\,y^n$$

Which we can rewrite as

$$x^m\,y^n\,\Big[(4\,n-5\,m-2)\,x\,y+(6\,n-8\,m-2)\Big]=0 \iff \left\{\begin{array}{l} 4\,n-5\,m-2=0\\ 6\,n-8\,m-2=0\end{array}\right. \iff \left\{\begin{array}{l} m=2\\ n=3\end{array}\right.$$

Thus $\mu(x,y)=x^2\,y^3$, is an integrating factor.

**08b.** From $\dfrac{\partial}{\partial y}\Big(3\,x\,y-2\,y^2\Big) = 3\,x-4\,y \neq \dfrac{\partial}{\partial x}\Big(2\,x^2-3\,x\,y\Big) = 4\,x-3\,y$, it follows that the differential equation is not exact. As in the previous question, assuming that $\mu(x,y)=x^m\,y^n$ is an integrating factor, it follows that $x^m\,y^n\,\Big(\,(3\,x\,y-2\,y^2)\;\mathrm{d}x+(2\,x^2-3\,x\,y)\;\mathrm{d}y\,\Big)=0$, must be exact., i.e.,

$$\big(3\,x^{m+1}\,y^{n+1}-2\,x^m\,y^{n+2}\big)\;\mathrm{d}x+\big(2\,x^{m+2}\,y^n-3\,x^{m+1}\,y^{n+1}\big)\;\mathrm{d}y=0$$

is exact. Hence $\dfrac{\partial}{\partial y}\Big(3\,x^{m+1}\,y^{n+1}-2\,x^m\,y^{n+2}\Big)=\dfrac{\partial}{\partial x}\Big(2\,x^{m+2}\,y^n-3\,x^{m+1}\,y^{n+1}\Big)$ or else

$$3\,(n+1)\,x^{m+1}\,y^n-2\,(n+2)\,x^m\,y^{n+1}=2\,(m+2)\,x^{m+1}\,y^n-3\,(m+1)\,x^m\,y^{n+1}$$

Which we can rewrite as

$$x^m\,y^n\,\Big[(3\,n-2\,m-1)\,x+(-2\,n+3\,m-1)\,y\Big]=0 \iff \left\{\begin{array}{l} 3\,n-2\,m-1=0\\ -2\,n+3\,m-1=0\end{array}\right. \iff \left\{\begin{array}{l} m=1\\ n=1\end{array}\right.$$

Thus $\mu(x,y)=x\,y$, is an integrating factor.

A first order differential equation that can be written in the form $y'=g\left(\dfrac{y}{x}\right)$, is called homogeneous. To solve, simply make the substitution $u=\dfrac{y}{x} \implies y=x\,u \implies y'=u+xu'$, to transform it into

$$u+x\,u'=g(u) \iff x\,\frac{\mathrm{d}u}{\mathrm{d}x}=g(u)-u \implies \frac{1}{g(u)-u}\;\mathrm{d}u=\frac{1}{x}\;\mathrm{d}x$$

which is a separable equation in $u$. Solve, then go back to $y$. Don't forget the solutions that may come from solving $g(u)-u=0$.

**09a.** Dividing by $x$, we get $y'=\dfrac{y}{x}+\cos^2\left(\dfrac{y}{x}\right)$. This is a homogeneous differential equation. To solve, we set $u=\dfrac{y}{x} \implies y=x\,u \implies y'=u+x\,u'$. Substituting, we get

$$u+x\,u'=u+\cos^2(u) \iff x\,\frac{\mathrm{d}u}{\mathrm{d}x}=\cos^2(u) \implies \frac{1}{\cos^2(u)}\;\mathrm{d}u=\frac{1}{x}\;\mathrm{d}x \iff \sec^2(u)\;\mathrm{d}u=\frac{1}{x}\;\mathrm{d}x$$

This is a separable equation. Integrating both sides, we get

$$\tan(u)=\ln|x|+C \iff u=\tan^{-1}\Big(\ln|x|+C\Big)$$

Going back to $y=x\,u$, leads to the general solution

$$y=x\,\tan^{-1}\Big(\ln|x|+C\Big)$$

The case $\cos^2(u)=0 \iff u=\dfrac{\pi}{2}+k\,\pi$, $k=0,\pm1,\pm2,\cdots$, leads to the solutions

$$\frac{y}{x}=\left(\frac{\pi}{2}+k\,\pi\right) \implies y=\left(\frac{\pi}{2}+k\,\pi\right)\,x, \qquad k=0,\pm1,\pm2,\cdots$$

**09b.** Dividing both numerator and denominator by $x$, we get $y' = \dfrac{1-\frac{y}{x}}{1+\frac{y}{x}}$, which shows that the differential equation is homogeneous. To solve, we set $u = \dfrac{y}{x} \implies y = x\,u \implies y' = u + x\,u'$. Substituting, we get

$$u + x\,u' = \frac{1-u}{1+u} \iff x\,\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1-2\,u-u^2}{1+u} \implies \frac{1+u}{1-2\,u-u^2}\,\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$$

Which we can rewrite as $\dfrac{2\,u+2}{u^2+2\,u-1}\,\mathrm{d}u = -\dfrac{2}{x}\,\mathrm{d}x$. Integrating both sides leads to

$$\ln\left|u^2+2\,u-1\right| = -2\,\ln|x| + K \iff \left|u^2+2\,u-1\right| = \frac{\mathrm{e}^K}{x^2} \implies u^2+2\,u-1 = \frac{C}{x^2}$$

Going back to $y$, using $u = \dfrac{y}{x}$, leads to $\dfrac{y^2}{x^2} + 2\,\dfrac{y}{x} - 1 = \dfrac{C}{x^2} \implies y^2 + 2\,x\,y - x^2 = C$. Hence the general solution $y^2 + 2\,x\,y - x^2 = C$.

The case $\dfrac{1-2\,u-u^2}{1+u} = 0 \iff u^2+2\,u-1 = 0 \iff u = -1 \pm \sqrt{2}$, leads to the solutions $y = -\left(1+\sqrt{2}\right)x$ and $y = \left(-1+\sqrt{2}\right)x$.

**10a.** Dividing both sides by $x^2$, converts the equation into $3\,\dfrac{y}{x}\,y' = 1 + 4\left(\dfrac{y}{x}\right)^2$. This shows that the differential equation is homogeneous. To solve, we set $u = \dfrac{y}{x} \implies y = x\,u \implies y' = u + x\,u'$. Substituting, we get $3\,u\left(u + x\,u'\right) = 1 + 4\,u^2$. Separating the variables leads to

$$3\,u\left(u + x\,u'\right) = 1 + 4\,u^2 \iff 3\,x\,u\,\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + u^2 \implies \frac{2\,u}{1+u^2}\,\mathrm{d}u = \frac{2}{3\,x}\,\mathrm{d}x$$

Integrating both sides, we get $\ln\left(1+u^2\right) = \dfrac{2}{3}\,\ln|x| + K \implies 1 + u^2 = C\,|x|^{2/3}, \quad C > 0$

Going back to $y$, leads to $1 + \dfrac{y^2}{x^2} = C\,|x|^{2/3} \implies x^2 + y^2 = C\,|x|^{8/3}$

Note: An alternate way of solving the equation is the to set $u = y^2$. That will convert it into a linear equation. Try it!

**10b.** Set $u = \dfrac{y}{x}$ and separate the variables to get $\dfrac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = \dfrac{1}{|x|}\,\mathrm{d}x$, $1 - u^2 \neq 0$. Thus

$$\sin^{-1}(u) = \begin{cases} -\ln(-x) + C & \text{if } x < 0 \\ \ln(x) + C & \text{if } x > 0 \end{cases} \iff \sin^{-1}\left(\frac{y}{|x|}\right) = \ln(|x|) + C$$

The case $1 - u^2$, leads to the singular solutions $y = \pm x$

**11.** Set $u = 4\,x + 2\,y$ to get $u' = 4 + 2\,y' \iff y' = \dfrac{1}{2}u' - 2$, and the equation becomes

$$\frac{1}{2}\,u' - 2 = -\frac{2\,u+1}{u+1} \iff \frac{1}{2}\,u' = \frac{1}{u+1} \iff u' = \frac{2}{u+1} \iff \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{u+1}$$
$$\implies (u+1)\,\mathrm{d}u = 2\,\mathrm{d}x$$

Which is a separable differential equation. Integrating we get $\dfrac{1}{2}\,(u+1)^2 = x^2 + C$. Returning back to $x$ and $y$, we obtain

$$(4\,x + 2\,y + 1)^2 = 4\,x + C$$