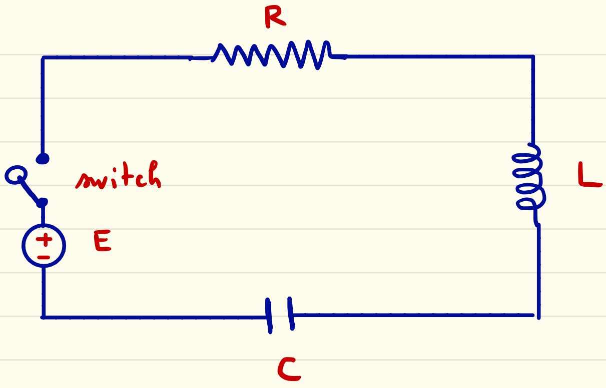
Electrical Circuits

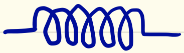


A typical electrical circuit is shown in the figure on the right. It consists of the following element :

- An Electromotive Force, usually a generator with symbol  or a battery with symbol  that produces an electrical current by "pushing" electrons from one pole, around the circuit, to the other pole. The "force" pushing the electrons around the circuit is called voltage. It is measured in volts and will be denoted by either E or V

- A Resistor, with symbol . This is a device such as a light bulb, a heater, a TV, ... that resists the flow of the electrical current, thereby causing the voltage to drop between the two ends of the resistor.

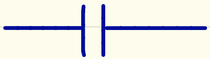
  By Ohm's law, the voltage drop ΔV_{res} is proportional to the current I passing through the resistor, i.e., $\Delta V_{res} = R\, I$. The constant of proportionality R is called the resistance of the resistor and has units ohm (Ω)

- An Inductor, which usually consists of a coil of wire with the property of opposing sudden changes in the current flowing through it. The symbol of an inductor is 

An inductor also causes the voltage between its two ends to drop. That voltage drop is proportional to the rate of change of the current I passing through it i.e.,

$$\Delta V_{ind} = L \frac{dI}{dt}.$$

The constant L is called the inductance of the inductor. It has units Henry.

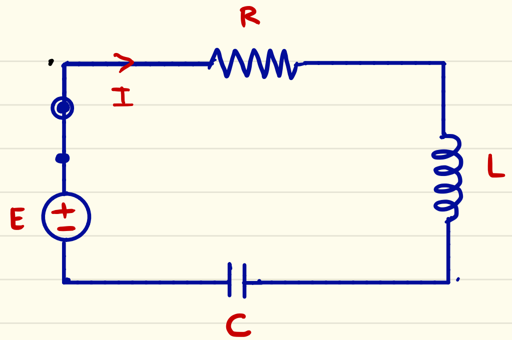
- A capacitor. Typically it consists of two plates separated by a gap across which no current flows. The symbol for a capacitor is . One of its many uses is to store an electric charge with the purpose of releasing it very quickly when needed. Think of the flash in a camera.

The voltage drop accross a capacitor is proportional to the charge Q on the capacitor, i.e., $\Delta V_{cap} = \frac{1}{C} Q$

The constant C is called the capacitance of the capacitor. It is measured in farad. Eventhough no current flows accross the gap between the plates of the capacitor, a current will flow through a closed circuit to which the two plates are attached. If I is the intensity of that current, then $I = \frac{dQ}{dt}$

- A switch, used to close and open the circuit.

Turning on the switch in an LRC circuit closes the circuit which results in a current with intensity $I = I(t)$ amperes to flow accross the circuit. To determine $I(t)$, we apply Kirchoff's voltage law:



> Around any closed loop, the sum of all the emfs and all the potential drops accross resistors, inductors, and capacitors must be equal to zero.

That leads to the equation

$$E - RI - L\frac{dI}{dt} - \frac{1}{C}Q = 0 \iff L\frac{dI}{dt} + RI + \frac{1}{C}Q = E$$

$$I = \frac{dQ}{dt}$$

Case 1 **No capacitance**

In this case, the equation $L\frac{dI}{dt}+RI+\frac{1}{C}Q=E$, becomes $\frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L}$

This is a 1^{st} order linear differential equation in $I=I(t)$.

Case 2 **No inductance**

In this case, the equation $L\frac{dI}{dt}+RI+\frac{1}{C}Q=E$ becomes

$RI+\frac{1}{C}Q=E$. Using the relation $I=\frac{dQ}{dt}$, leads to

$\frac{dQ}{dt}+\frac{1}{RC}Q=\frac{E}{R}$, which is a 1^{st} order linear differential equation in

$Q=Q(t)$. Solve to get Q, then differentiate to get I.

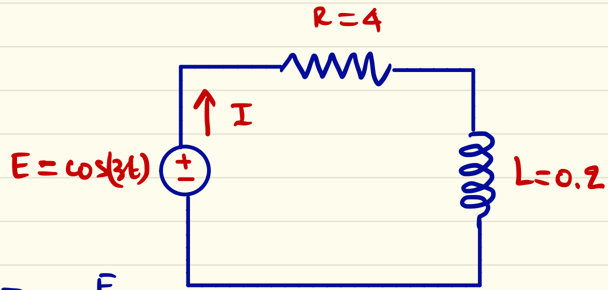
Case 3 **General case**

Using the relation $I=\frac{dQ}{dt}$, the equation $L\frac{dI}{dt}+RI+\frac{1}{C}Q=E$,

becomes $L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E$. This is a 2^{nd} order linear differential equation, that will be studied in the next chapter.

Example Consider an RL-circuit, with $R = 4$ ohms, $L = 0.2$ henry, $E = \cos(3t)$. Find $I(t)$ given that $I(0) = 0$

Solution



By Kirchof voltage Law:

$$E - RI - L\frac{dI}{dt} = 0$$

$$L\frac{dI}{dt} + RI = E \iff \frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

$$\frac{dI}{dt} + \frac{4}{0.2}I = \frac{\cos(3t)}{0.2} \iff \begin{cases} \frac{dI}{dt} + 20I = 5\cos(3t) \\ I(0) = 0 \end{cases}$$

An integrating factor is

$$\mu = e^{\int 20\,dt} = e^{20t}$$

Multiply the diff. eq. by e^{20t}

$$\left(e^{20t} I\right)' = 5e^{20t}\cos(3t)$$

$$e^{20t} I = \int 5e^{20t}\cos(3t)\,dt$$

To compute the integral, we use integration by parts to get

$$e^{20t} I = \frac{5}{9}\left(3\sin(3t) - 20\cos(3t)\right) e^{20t} + C$$

It follows

$$I(t) = \frac{5}{9}\left(3\sin(3t) - 20\cos(3t)\right) + c\, e^{-20t}$$

To find C, we solve

$$I(0) = 0 \iff \frac{5}{9}(0 - 20) + c = 0 \iff c = \frac{100}{9}$$

Hence

$$I(t) = \frac{5}{9}\left(3\sin(3t) - 20\cos(3t)\right) + \frac{100}{9} e^{-20t}$$

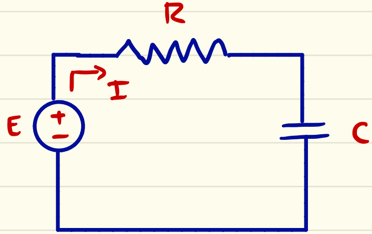
$$\begin{array}{ccc}
5e^{20t} & & \cos(3t) \\
 & + & \\
100e^{20t} & & \frac{1}{3}\sin(3t) \\
 & - & \\
2000e^{20t} & +\int & -\frac{1}{9}\cos(3t)
\end{array}$$

$$\int 5e^{20t}\cos(3t) = \frac{5}{3} e^{20t}\sin(3t) - \frac{100}{9} e^{20t}\cos(3t) - \int \frac{2000}{9} e^{20t}\cos(3t)\, dt$$

$$\Longrightarrow \left(5 + \frac{2000}{9}\right)\int e^{20t}\cos(3t)\, dt = \left(\frac{5}{3}\sin(3t) - \frac{100}{9}\cos(3t)\right)e^{20t} + C$$

Example Consider an RC- circuit, with $R = 10$ ohms, $C = 0.02$ farad, and $E = 10e^{-2t}$. Find $Q(t)$ and $I(t)$, given that $Q(0) = 0$.

What is the maximum charge on the capacitor and when does it occur?



Solution

By Kirchoff's law, we have

$$E - RI - \frac{1}{C}Q = 0 \iff RI + \frac{1}{C}Q = E$$

Using the relation $\frac{dQ}{dt} = I$, we get $R\frac{dQ}{dt} + \frac{1}{C}Q = E \iff \frac{dQ}{dt} + \frac{1}{RC}Q = \frac{E}{R}$

substituting R, C, E by their respective values, leads to

$$\begin{cases} Q' + 5Q = e^{-2t} \\ Q(0) = 0 \end{cases}$$

An integrating factor is $\mu(t) = e^{\int 5\,dt} = e^{5t}$

Multiplying the differential equation by $\mu(t) = e^{5t}$

we get $(e^{5t}Q)' = e^{3t} \Rightarrow e^{5t}Q = \frac{1}{3}e^{3t} + c \Rightarrow Q = \frac{1}{3}e^{-2t} + c\,e^{-5t}$

To determine the constant, we solve $Q(0)=0 \iff \frac{1}{3}e^{0} + ce^{0} = 0 \iff c = -\frac{1}{3}$

Hence $Q(t) = \frac{1}{3}e^{-2t} - \frac{1}{3}e^{-5t} = \frac{1}{3}\left(e^{-2t} - e^{-5t}\right)$

To find the maximum value of $Q(t)$, we compte $Q'(t) = \frac{1}{3}\left(-2e^{-2t} + 5e^{-5t}\right)$

$$Q'(t) = 0 \iff -2e^{-2t} + 5e^{-5t} = 0 \iff -2e^{3t} + 5 = 0 \iff e^{3t} = \frac{5}{2} \iff 3t = \ln\left(\frac{5}{2}\right)$$

$$\iff t = \frac{1}{3}\ln\left(\frac{5}{2}\right)$$

The table of variations of $Q(t)$ in $[0, +\infty)$ is

x	0		$\frac{1}{3}\ln\left(\frac{5}{2}\right)$		$+\infty$
$Q'(x)$		$+$	0	$-$	
$Q(x)$	0	$\nearrow$	$\frac{1}{5}\left(\frac{4}{25}\right)^{1/3}$	$\searrow$	0

Hence the maximum charge is $\frac{1}{5}\left(\frac{4}{25}\right)^{1/3} \approx 0.109$

and it occurs at time $\frac{1}{3}\ln\left(\frac{5}{2}\right) \approx 0.305$

Useful Solutions

The general solution of $y' + ay = b\cos(\omega t)$ is

$$\boxed{y(t) = C e^{-at} + \frac{ab}{a^2+\omega^2}\cos(\omega t) + \frac{b\omega}{a^2+\omega^2}\sin(\omega t)}$$

The general solution of $y' + ay = b\sin(\omega t)$ is

$$\boxed{y(t) = C e^{-at} + \frac{ab}{a^2+\omega^2}\sin(\omega t) - \frac{b\omega}{a^2+\omega^2}\cos(\omega t)}$$

The general solution of $y' + ay = b e^{\omega t}$ is

$$\boxed{y(t) = C e^{-at} + \begin{cases} \dfrac{b}{\omega+a} e^{\omega t} & \text{if} \quad \omega + a \neq 0 \\ b t e^{-at} & \text{if} \quad \omega + a = 0 \end{cases}}$$