MATH 375
Handout # 7 - Answers, Hints, Solutions
Fourier Series

1. Find the Fourier series of each of the following functions
a) $f(x) = x$, $x \in [-\pi, \pi]$
b) $f(x) = 3\pi^2 + 5x - 12x^2$, $-\pi < x < \pi$
c) $f(x) = 3x^2 + 1$, $x \in [-\pi, \pi]$
d) $f(x) = \begin{cases} 1, & -\frac{1}{2} < x \le 0 \\ -1, & 0 < x < \frac{1}{2} \end{cases}$

Solution. a) For the segment $[-\pi, \pi]$, we have $\omega = 1$, and we find the Fourier coefficients as

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\ dx = \frac{x^2}{2}\Big|_{-\pi}^{\pi} = 0.$$

We could guess from the beginning that $a_0 = a_n = 0$, $n = 1, 2, \ldots$, as f is an odd function. Applying integration by parts with $u = x$, $dv = \sin(nx)\ dx$, $v = -\cos(nx)/n$, we obtain

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\ dx = -\frac{2}{n\pi}x\cos(nx)\Big|_0^{\pi} + \int_0^{\pi} \frac{2}{n\pi}\cos(nx)\ dx$$

$$= \frac{2}{n\pi}(-\pi\cos(n\pi) + 0) + \frac{1}{n^2}(\sin(n\pi) - \sin\ 0) = \frac{2}{n}(-1)(-1)^n + 0 = \frac{2(-1)^{n+1}}{n}.$$

The Fourier series are $x \sim \displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)$.

b) The Fourier coefficient a_0 is

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(3\pi^2 + 5x - 12x^2)\ dx = \frac{1}{\pi}\left[3\pi^2 x + \frac{5}{2}x^2 - 4x^3\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(3\pi^3 + \frac{5}{2}\pi^2 - 4\pi^3 - (-3\pi^3) - \frac{5}{2}(-\pi)^2 + 4(-\pi)^3\right) = \frac{1}{\pi}(6\pi^3 - 8\pi^3) = -2\pi^2,$$

$a_0/2 = -\pi^2$. Further, applying integration by parts, we get

$$\int x\sin(nx)\ dx = -\frac{1}{n}x\cos(nx) + \frac{1}{n^2}\sin(nx) + C,$$

$$\int x^2\cos(nx)\ dx = \frac{1}{n}x^2\sin(nx) + \frac{2}{n^2}x\cos(nx) - \frac{2}{n^3}\sin(nx) + C,$$

while $x^2\sin x$, $\sin x$ and $x\cos(x)$ are odd functions, and their integrals over $[-\pi, \pi]$ are equal to zero, thus

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(3\pi^2+5x-12x^2)\cos(nx)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(3\pi^2-12x^2)\cos(nx)\ dx$$

$$= \frac{1}{\pi}\left[\frac{3\pi^2}{n}\sin(nx) - \frac{12}{n}x^2\sin(nx) - \frac{24}{n^2}x\cos(nx) + \frac{24}{n^3}\sin(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - 0 - \frac{24}{n^2}\pi\cos(\pi n) - \frac{2}{n^2}\pi\cos(-\pi n)\right] = -\frac{48}{n^2}\cos(\pi n) = -\frac{48(-1)^n}{n^2},$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(3\pi^2 + 5x - 12x^2)\sin(nx)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 5x\sin(nx)\ dx \\ &= \frac{5}{\pi}\left[-\frac{1}{n}x\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{\pi} \\ &= \frac{5}{\pi}\left[-\frac{1}{n}\pi\cos(\pi n) + \frac{1}{n^2}\sin(\pi n) - \frac{1}{n}\pi\cos(-\pi n) - \frac{1}{n^2}\sin(-\pi n)\right] \\ &= \frac{5}{\pi}\left[-\frac{\pi}{n}(-1)^n + 0 - \frac{\pi}{n}(-1)^n - 0\right] = -\frac{10(-1)^n}{n}, \end{aligned}$$

the Fourier series are $3\pi^2 + 5x - 12x^2 \sim -\pi^2 - \sum_{n=1}^{\infty}\left[\frac{48(-1)^n}{n^2}\cos(nx) + \frac{10(-1)^n}{n}\sin(nx)\right]$.

c) The Fourier (in fact, the cosine series, it is an even function) series are

$$3x^2 + 1 \sim 1 + \pi^2 + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

d) $f(x)$ is an odd function, so the Fourier series are sine series, where $1/\ell = 1/0.5 = 2$, $\omega = \pi/0.5 = 2\pi$ and

$$b_n = \frac{2}{\ell}\int_0^{1/2} f(x)\sin(\omega nx)\ dx = -4\int_0^{1/2}\sin(2\pi nx)\ dx$$

$$= \frac{4}{2\pi n}\cos(2\pi nx)\Big|_0^{1/2} = \frac{2}{\pi n}(\cos(\pi n) - \cos(0)) = \frac{2}{\pi n}[(-1)^n - 1],$$

so $f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\sin(2\pi nx)$. It is easy to see that $(-1)^n - 1$ equals zero for even n and -2 for odd n, so the series can be rewritten as

$$f(x) \sim -\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin(2\pi(2k+1)x).$$

2. Find the Fourier sine series of $f(x) = \begin{cases} 1, & 0 < x \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x \le \pi \end{cases}$

Solution. We have to compute b_n only for the sine series assuming f is odd:

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\ dx = \frac{2}{\pi}\int_0^{\pi/2}\sin(nx)\ dx = -\frac{2}{\pi n}\cos(nx)\ dx\Big|_0^{\pi/2}$$

$= \frac{2}{\pi n}\left[-\cos\left(\frac{\pi n}{2}\right) + \cos 0\right] = \frac{2}{\pi n}\left[1 - \cos\left(\frac{\pi n}{2}\right)\right]$. Thus the sine series for f are $f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1 - \cos\left(\frac{\pi n}{2}\right)}{n}\sin(nx)$.

3. Find the Fourier cosine series of $f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & 1 < x \le \pi \end{cases}$

Solution. We have to compute a_n and a_n only for the cosine series assuming f is even:

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\ dx = \frac{2}{\pi}\int_0^1 1\ dx = \frac{2}{\pi},$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\ dx = \frac{2}{\pi}\int_0^1 \cos(nx)\ dx = \frac{2}{\pi n}\sin(nx)\ dx\Big|_0^1$$

$= \dfrac{2}{\pi n}[\sin(n) - \sin(0)] = \dfrac{2}{\pi n}\sin(n)$. Thus the cosine series are

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos(nx) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{\sin n}{n}\cos(nx).$$

4. Find the Fourier sine series and the Fourier cosine series for each of the following functions

a) $f(x) = x$, $0 < x < 1$

b) $f(x) = 1$, $0 < x < \pi$.

Solution. a) Here $\ell = 1$ $\omega = \frac{\pi}{\ell} = \pi$, the coefficients of the sine series are computed with integration by parts:

$$b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin(\omega n x)\ dx = 2\int_0^1 f(x)\sin(n\pi x)\ dx = 2\int_0^1 x\sin(n\pi x)\ dx$$

$$= -\frac{2x}{n\pi}\cos(n\pi x)\Big|_0^1 + \int_0^1 \frac{2}{n\pi}\cos(n\pi x)\ dx = -\frac{2}{n\pi}\cos(n\pi) + \left[\frac{2}{\pi^2 n^2}\sin(n\pi x)\right]_0^1$$

$= -\dfrac{2(-1)^n}{n\pi} + 0 - 0 = \dfrac{2(-1)^{n+1}}{\pi n}$, thus the sine series for f are $f(x) \sim \dfrac{2}{\pi}\displaystyle\sum_{n=1}^\infty \dfrac{(-1)^{n+1}}{n}\sin(n\pi x)$.

The coefficients of the cosine series are $a_0 = 2\displaystyle\int_0^1 f(x)\ dx = 2\int_0^1 x\ dx = x^2\Big|_0^1 = 1$,

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\ dx = 2\int_0^1 x\cos(n\pi x)\ dx = \frac{2}{\pi n}\sin(n\pi x)\Big|_0^1 - \frac{2}{\pi n}\int_0^1 \sin(n\pi x)\ dx$$

$= \dfrac{2}{\pi n}[\sin(n\pi) - \sin(0)] + \dfrac{2}{\pi^2 n^2}\Big|_0^1 = 0 - 0 + \dfrac{2}{\pi^2 n^2}(\cos(\pi n) - \cos(0)) = \dfrac{2}{\pi^2 n^2}[(-1)^n - 1]$ and

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi^2 n^2}\sum_{n=1}^\infty \frac{(-1)^n - 1}{n^2}\cos(n\pi x).$$

b) The coefficients of the sine series are

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\ dx = \frac{2}{\pi}\int_0^\pi \sin(nx)\ dx = -\frac{2}{\pi n}\cos(nx)\Big|_0^1 = -\frac{2}{\pi n}(\cos(n\pi) - \cos(0))$$

$= \dfrac{2}{\pi n}(1-(-1)^n)$, and the sine series for f are $f(x) \sim \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1-(-1)^n}{n}\sin(nx)$.

For the cosine series $a_0 = \dfrac{1}{\pi}\int_0^{\pi} f(x)\ dx = \dfrac{1}{\pi}\int_0^{\pi} 1\ dx = \dfrac{\pi}{\pi} = 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\ dx = \frac{2}{\pi}\int_0^{\pi}\cos(nx)\ dx = \frac{2}{\pi n}\sin(nx)\Big|_0^1 = \frac{2}{\pi n}[\sin(n\pi) - \sin(0)] = 0,$$

the cosine series for f are $f(x) \sim \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = 1$.

5. Define and sketch the even and the odd extensions of f if
   a) $f(x) = x$, $0 < x < 1$
   b) $f(x) = \sin(x)$, $0 < x < \pi$
   c) $f(x) = 1 - x$, $0 < x < 1$
   d) $f(x) = x^2$, $0 < x < 1$

   Solution. a) The even extension is $f_e(x) = |x|$, $-1 < x < 1$, the odd extension is $f_o(x) = x$, $-1 < x < 1$ (the function $f(x) = x$ is odd).
   b) The even extension is $f_e(x) = |\sin(x)|$, $-\pi < x < \pi$, the odd extension is $f_o(x) = \sin(x)$, $-\pi < x < \pi$ (the function $f(x) = \sin(x)$ is odd).
   c) The even extension is $f_e(x) = \begin{cases} x+1 & -1 < x < 0 \\ 1-x & 0 < x < 1 \end{cases}$, the odd extension is $f_o(x) = \begin{cases} -x-1 & -1 < x < 0 \\ 1-x & 0 < x < 1 \end{cases}$
   d) The even extension is $f_e(x) = x^2$, $-1 < x < 1$ (the function $f(x) = x^2$ is even), the odd extension is $f_o(x) = \begin{cases} -x^2 & -1 < x < 0 \\ x^2 & 0 < x < 1 \end{cases}$.

6. Consider the function
$$f(x) = \begin{cases} x^2 - 1 & 0 \le x < 1 \\ x & 1 \le x < 2 \\ -1 & 2 \le x < 4 \end{cases}$$

   Determine the values to which the Fourier series of f converges at $x = \frac{1}{2}$, $x = 1$, $x = 2$, and $x = 4$.
   Solution. The Fourier series converges to $f(x)$ at all points of $(0, 4)$ where f is continuous, to the average $\dfrac{f(x^-) + f(x^+)}{2}$ at all the points $x \in (0, 4)$, and to the average of the values at endpoints at $x = 0$ and $x = 4$ (the sum of the Fourier series is a periodic function). As $f(\dfrac{1}{2}) = 0.5^2 - 1 = -0.75$, $(f(1^-) + f(1^+))/2 = (1^2 - 1 + 1)/2 = \dfrac{1}{2}$, $f(2^-) + f(2^+))/2 = (2-1)/2 = \dfrac{1}{2}$, $(f(4^-) + f(0^+))/2 = (-1 + 0^2 - 1)/2 = -1$, the Fourier series converges to $-\frac{3}{4}$ at $x = \frac{1}{2}$, to $\frac{1}{2}$ at $x = 1$, to $\frac{1}{2}$ at $x = 2$, and to -1 at $x = 4$.

7. For the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x+1 & 0 \le x < \frac{\pi}{2} \\ 2x-1 & \frac{\pi}{2} < x < \pi \end{cases}$$

determine the values to which the Fourier series of f converges at $x = 0$, $x = 1$, $x = \frac{\pi}{2}$, and $x = \pi$.

Answer. At $x = 0$ the Fourier series of f converges to $\frac{1}{2}$, at $x = 1$ to 2, at $x = \frac{\pi}{2}$ to $\frac{3\pi}{4}$, and at $x = \pi$ to $\pi - \frac{1}{2}$.

8. If $f(x) = \begin{cases} x^2 + c^2 & 0 < x < 2 \\ 3c + 2x & 2 < x < 3 \end{cases}$, determine all possible values of the constant real number c such that the Fourier series of $f(x)$ converges to 6 at $x = 2$.

Solution. The Fourier series converges to the average of $x^2 + c^2$ and $3c + 2x$ at $x = 2$, thus

$$\frac{2^2 + c^2 + 2c + 4}{2} = 6 \;\Rightarrow\; c^2 + 3c - 4 = 0 \;\Rightarrow\; c = -4 \;\text{ or }\; 1.$$