Department of Mathematics and Statistics
MATH 375
Handout # 1 - ANSWERS, HINTS, SOLUTIONS
Differential equations in general; linear equations

1. For each of the following ordinary differential equations determine the order, the dependent variable (or the unknown function), and the independent variable. Determine whether the equation is linear or not.

   a) $\dfrac{dx}{dt} + 8x = e^{t+x}$ b) $\dfrac{d^2t}{dx^2} + xt = \sin(x^2)$

   c) $\left(\dfrac{dr}{ds}\right)^3 + \dfrac{d^2r}{ds^2} = 7$ d) $y^{(4)} - 2y''' + 5y' - 9y = \cos^2(t)$

   e) $\left(\dfrac{dp}{dq}\right)^6 = 2p$ f) $\dfrac{d^6p}{dq^6} = 2p$

   g) $t^2\ddot{s} - t\dot{s} + 4s = 1 - \sin(t)$

   Solution. a) $x = x(t)$, where t is an independent variable, x is a dependent variable, the equation is of the first order, it is nonlinear due to e^x in the right-hand side (if instead of e^{t+x} we had xe^t, it would be linear).
   b) $t = t(x)$, it is a second order linear equation.
   c) $r = r(s)$, it is a second order nonlinear equation (the cube of $r'(s)$ is included).
   d) $y = y(t)$, it is a fourth order linear equation.
   e) $p = p(q)$, it is a first order nonlinear equation (but later on we will study that this equation is separable).
   f) $p = p(q)$, it is a sixth order linear equation.
   g) $s = s(t)$ (recall that $\cdot$ above the variable always denotes the derivative in time t

2. For each of the following partial differential equations determine the order, the dependent variable (or the unknown function), and the independent variables. Determine whether the equation is linear or not.
   a) $\dfrac{\partial x}{\partial s} + 4\dfrac{\partial x}{\partial t} + x^2 = 5e^{s+t}$ b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \sin(x+y)$
   Solution. a) $x = x(s,t)$ is a dependent variable, s and t are independent, this is a first order nonlinear (includes x^2) equation. b) $u = u(x,y)$, this is a second order linear equation.

3. Which of the following are solutions to differential equations $\dfrac{dx}{dt} = \dfrac{x}{t}$, $t > 0$?
   a) $x = 0$ b) $x = 2$ c) $x = 2t$ d) $x = -3t$ e) $x = t^2$
   Solution. a) Substituting $x = 0$, we obtain for $t > 0$ the identity $0 = 0$, so $x = 0$ is a solution.
   b) $0 \neq 2/t$ for $t > 0$, so $x = 2$ is not a solution.
   c) $2 = (2t)/t$ for $t > 0$, so $x = 2t$ is a solution.
   d) $x = -3t$ is a solution. e) $x = t^2$ is not a solution.

4. Given the differential equation $t^2\dfrac{d^2x}{dt^2} - t\dfrac{dx}{dt} + x = 0,\ \ t > 0$
a) verify that $x(t) = t(C_1 + C_2\ln(t))$ for any $C_1, C_2 \in \mathbb{R}$ is a solution of the given differential equation; b) find the solution satisfying the initial conditions $x(1) = -3$, $x'(1) = 4$.
Solution. a) Substituting $x'(t) = C_1 + C_2 + C_2\ln(t))$, $x''(t) = C_2/t$ in the equation, we get an identity.
b) We have $x(1) = C_1 + C_2\ln(1) = C_1 = -3$, $x'(1) = C_1 + C_2 = 4$, so the solution is $C_1 = -3$, $C_2 = 7$, $x(t) = t(-3 + 7\ln(t))$.

5. Find the general solution of the differential equation $y' = -2x+2$. Sketch few members of the one-parameter family of solutions.
Solution. Integrating, we obtain $y(t) = \displaystyle\int(-2x+2)\,dx = C + 2x - x^2$, which are all parabolas opening downwards obtained by a vertical translation of a parabola with a vertex $(1,1)$ passing through $(0,0)$ and $(2,0)$.

6. For $t > 0$, find the equivalent forms of the differential equation $e^{x'+x} = t$ among the following:
a) $x' + x = \ln(t)$ b) $x' + x = e^{-t}$ c) $\dfrac{dx}{dt} = -x + \ln(t)$ d) $dx + (x - e^{-t})dt = 0$ e) $dx + (x - \ln(t))dt = 0$
Answer. Equations in a),c),e) are equivalent to $e^{x'+x} = t$, in b) and d) are not. d) and e) are differential equations in the differential form.

7. Find the general solution of the following differential equations
a) $y' - \dfrac{y}{x} = x$, $x > 0$
b) $(2x+1)y' = 4x + 2y$
c) $xy' + (x+1)y = 3x^2e^{-x}$
Answer. a) $y = x^2 + Cx$ b) $y = (2x+1)\left(\ln|2x+1| + C\right) + 1$ c) $y = x^2e^{-x} + C\frac{1}{x}e^{-x}$.
Solution. a) Since $\int -\frac{1}{x}dx = -\ln|x| + C$ then the integrating factor is $\mu(x) = e^{-\ln(x)} = 1/x$. Multiplying the equation by $1/x$, we obtain

$$\frac{1}{x}y' - \frac{1}{x^2}y = 1,$$

where the left-hand side is the derivative of the product of $1/x$ and y. Thus

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 1.$$

Integrating both sides, we obtain

$$\frac{1}{x}y = \int 1\ dx = x + C \ \Rightarrow\ y = x^2 + Cx.$$

b) The standard form of the linear equation is

$$y' - \frac{2}{2x+1}y = \frac{4x}{2x+1} = 2 - \frac{2}{2x+1}.$$

The integrating factor is $\mu(x) = e^{\int -2/(2x+1)\ dx} = e^{-\ln(2x+1)} = \dfrac{1}{2x+1}$. Dividing both sides by $2x+1$, we get

$$\frac{y'}{2x+1} - \frac{2y}{(2x+1)^2} = \frac{d}{dx}\left(\frac{y}{2x+1}\right) = \frac{2}{2x+1} - \frac{2}{(2x+1)^2}.$$

The left-hand side is the derivative of $y/(2x+1)$, so

$$\frac{y}{2x+1} = \int \left(\frac{2}{2x+1} - \frac{2}{(2x+1)^2}\right)\ dx = \ln|2x+1| + \frac{1}{2x+1} + C.$$

Multiplying both sides by $(2x+1)$, we obtain

$$y = (2x+1)\left(\ln|2x+1| + \frac{1}{2x+1} + C\right) \Rightarrow y = (2x+1)\left(\ln|2x+1| + C\right) + 1$$

c) The standard form of the linear equation is

$$y' + \frac{x+1}{x}y = y' + \left(1 + \frac{1}{x}\right)y = 3xe^{-x}.$$

The integrating factor is $\mu(x) = e^{\int (1+1/x)\ dx} = e^{x+\ln(x)} = xe^x$. Multiplying both sides by $\mu(x) = xe^x$, we have

$$xe^x y' + (x+1)e^x y = \frac{d}{dx}\left(xe^x y\right) = 3xe^{-x}xe^x = 3x^2.$$

After integration we have

$$xe^x y = \int 3x^2\ dx = x^3 + C \ \Rightarrow\ y = \frac{1}{x}e^{-x}\left(x^3 + C\right) = x^2e^{-x} + C\frac{1}{x}e^{-x}.$$

8. Find the general solution of the equation

$$xy' - 2y = x^3 + x;$$

find the solution satisfying $y(1) = 3$.
Answer. $y = x^3 - x + Cx^2$, $y = x^3 - x + 3x^2$.
Solution. The equation is linear $y' - 2\frac{1}{x}y = x^2 + 1$, the integrating factor is

$$\mu(x) = e^{-\int (2/x)\ dx} = e^{-2\ln(x)} = x^{-2} = x^2 + 1.$$

Dividing by x^2, we get

$$\frac{1}{x^2}y' - 2\frac{1}{x^3}y = \left(\frac{y}{x^2}\right)' = 1 + \frac{1}{x^2}.$$

Thus

$$\frac{y}{x^2} = \int \left(1 + \frac{1}{x^2}\right)\, dx = x - \frac{1}{x} + C \;\Rightarrow\; y = x^3 - x + Cx^2.$$

For the solution of the initial value problem we substitute $x = 1$, $y = 3$:
$3 = 1 - 1 + C \Rightarrow C = 3$, so $y = x^3 - x + 3x^2$.