MATH 375 MIDTERM - Answers, Hints, Solutions
Monday November 2, 2015, 18:00-19:30

1. The equation $y'' + ky'(y^2 - 1) + 3y''' = -2\cos(t)y^2$ is
   A) second order, linear B) second order, nonlinear
   C) third order, linear D) third order, nonlinear
   E) nonlinear, and the order depends on k
   Answer. The correct answer is D).

2. The general solution of the equation $tyy' = 1 - t^2$ for $t > 0$ can be written as
   A) $y = 2\ln(t) - t^2 + C$ B) $y = \pm\sqrt{C + 2\ln(t) - t^2}$
   C) $y = \pm\sqrt{2\ln(t) - t^2} + C$ D) $y = \sqrt{\ln(t^2)} + t + C$
   E) $y = \ln(t) - \dfrac{t^2}{2} + C$
   Answer. It is a separable equation, the correct answer is B).

3. The solution of the initial value problem $x' + \dfrac{5}{t}x = \dfrac{3}{t^3}$, $x(1) = 4$ is
   A) $x = 3e^{-5t} + 4 - 3e^{-5}$ B) $x = 3(t-1)e^{-5t} + 1$ C) $x = \dfrac{1}{t^2} + \dfrac{3}{t^5}$
   D) $x = \dfrac{2}{t} + \dfrac{2}{t^2}$ E) $x = t^3 + 3$
   Solution. This is a linear equation, after the multiplication by the integrating factor $\mu(t) = e^{\int (5/t)\, dt} = t^5$ we have $(t^5x)' = 3t^2$, or $t^5x = t^3 + C$, so $x(t) = \frac{1}{t^2} + \frac{C}{t^5}$, $x(1) = 1 + C = 4$. Thus $C = 3$ and $x(t) = \dfrac{1}{t^2} + \dfrac{3}{t^5}$, the correct answer is C).

4. The general solution of the equation $\dfrac{dy}{dt} + y = ty^2$ is
   A) $y = Ce^t + t + 1$ B) $y = Ce^{-t} + \dfrac{t^2}{2} + e^t$ C) $y = \left(Ce^{-2t} + \dfrac{t^2}{2}\right)^{-1}$
   D) $y = Ce^{-t} + \dfrac{1}{t} + 1$ E) $y = \left(Ce^t + t + 1\right)^{-1}$
   Answer. It is a Bernoulli equation, the correct answer is E).

5. The equation $e^{\alpha x + \beta y}\left(3 + \dfrac{dy}{dx}\right) = 0$ is exact
   A) for any $\alpha, \beta \in \mathbb{R}$ B) for $\alpha = 6$, $\beta = 2$ only
   C) for any $\alpha = 3\beta \in \mathbb{R}$ D) for $\alpha = 3$, $\beta = 1$ only
   E) never: there are no $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that the equation is exact
   Answer. We have $M(x,y) = 3e^{\alpha x + \beta y}$, $N(x,y) = e^{\alpha x + \beta y}$, $M_y = 3\beta e^{\alpha x + \beta y}$, $N_x = \alpha e^{\alpha x + \beta y}$, they are equal if $\alpha = 3\beta \in \mathbb{R}$, the correct answer is C).

6. A tank, containing 1 litre of liquid, has a brine solution entering at a constant rate of 1 litres per minute. The well-stirred solution leaves the tank at the same rate. The

concentration within the tank is monitored and found to be $c(t) = e^{-t/10}$ kg/L. Then the inflow concentration is
A) $c_{in}(t) = e^{-t/10}$ kg/L B) $c_{in}(t) = \dfrac{9}{10}e^{-t/10}$ kg/L
C) $c_{in}(t) = \dfrac{1}{10}$ kg/L D) $c_{in}(t) = 0$ kg/L
E) $c_{in}(t)$ cannot be found: not enough data
Solution. $Q(t) = c(t)V = e^{-t/10}$, $Q'(t) = -\dfrac{1}{10}e^{-t/10}$,

$$Q'(t) = c_{in}r_{in} - c_{out}r_{out} = c_{in} \cdot 1 - 1 \cdot e^{-t/10} = -\frac{1}{10}e^{-t/10},$$

thus $c_{in} = e^{-t/10} - \dfrac{1}{10}e^{-t/10} = \dfrac{9}{10}e^{-t/10}$ kg/L, the correct answer is B).

7. The solution of the initial value problem $xy' + 6y = -7x\cos(x^7)$, $y\left(\sqrt[7]{\dfrac{\pi}{2}}\right) = 0$ is
A) $y = \dfrac{1-\sin(x^7)}{x^6}$ B) $y = \dfrac{\sin(x^7)}{x^6} - 1$ C) $y = \cos(x^7)$
D) $y = \dfrac{\cos(x^7)}{x^6}$ E) $y = \dfrac{(x-\pi/2)\sin(x^7)}{x^6}$
Answer. The equation is linear, the correct answer is A).

8. According to the existence and uniqueness theorem, the largest open interval (a, b) on which the unique solution of the initial value problem $(t-4)y' + ty = \ln\left(t - \dfrac{1}{t}\right)$, $y(2) = 5$ is guaranteed to exist is
A) $(-\infty, \infty)$ B) $(-1, \infty)$ C) $(1, \infty)$ D) $(1, 4)$ E) $(1, 3)$
Solution. To bring to the standard form, we have to divide by $t - 4$. The function $\dfrac{t}{t-4}$ is continuous everywhere but at $t = 4$, while $\ln\left(t - \frac{1}{t}\right)/(t-4)$ is defined and continuous for $t - \frac{1}{t} > 0$, $t \neq 4$, or $(t+1)(t-1)/t > 0$, $t \in (-1, 0) \cup (1, 4) \cup (4, \infty)$. The largest interval containing 2 is $(1, 4)$, the correct answer is D).

9. We solve the initial value problem $y' = x^3 + \ln(y+1) + x\sin(y)$, $y(1) = 0$ using Euler's method with step size $h = 0.1$. Then
A) $y(1.1) \approx y_1 = 1$ B) $y(1.1) \approx y_1 = 0.1$ C) $y(1.1) \approx y_1 = 0$
D) $y(1.1) \approx y_1 = 0.2$ E) $y(1.1) \approx y_1 = 2$
Solution. $y_1 = y(1) + y'(1)h = 0 + (1^3 + \ln(0+1) + 1\sin(0))0.1 = 0 + (1+0+0)0.1 = 0.1$, the correct answer is B).

10. The general solution of the equation $x^2y'' + 11xy' + 25y = 0$, $x > 0$ is
A) $y = x^{-5}(C_1 + C_2\ln(x))$ B) $y = C_1e^{(-5.5+\sqrt{21}/2)x} + C_2e^{(-5.5+\sqrt{21}/2)x}$
C) $y = C_1x^{-5} + C_2x^{-5}$ D) $y = C_1e^{-5x} + C_2xe^{-5x}$ E) $y = C_1x^{-5} + C_2x^{-6}$

Solution. This is the Cauchy-Euler equation, after substitution $x = e^z$ we have the characteristic equation $r(r-1)+11r+25 = r^2+10r+25 = (r+5)^2 = 0$ for $y = y(z)$, the general solution is $y = x^{-5}\left(C_1 + C_2 \ln(x)\right)$, the correct answer is A).

11. A particular solution of the equation $y'' + y = 4\sin(t)$ is
A) $y = 4\sin(t)$ B) $y = -t\cos(t) + t\sin(t)$ C) $y = -2t\cos(t)$
D) $y = 2t\sin(t)$ E) $y = 4\cos(t) + 4\sin(t)$
Solution. Since $\sin(t)$ and $\cos(t)$ are solutions of the homogeneous equation, we are looking for a particular solution in the form $y = At\cos(t) + Bt\sin(t)$, $y' = A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t)$, $y'' = -2A\sin(t) - At\cos(t) + 2B\cos(t) - Bt\sin(t)$. Thus
$$y'' + y = -2A\sin(t) - At\cos(t) + 2B\cos(t) - Bt\sin(t) + At\cos(t) + Bt\sin(t)$$
$= -2A\sin(t) + 2B\cos(t) = 4\sin(t)$, so $A = -2$, $B = 0$, the particular solution is $-2t\cos(t)$, the correct answer is C).

12. A freshly poured coffee has a temperature of 92 degrees Celsius, and it is brought to the room where the temperature is kept at 22 degrees. Then, for a constant k, the coffee temperature is described by the initial value problem
A) $T'(t) = k(92 - T(t))$, $T(0) = 22$ B) $T'(t) = -kT(t)$, $T(0) = 92$ C) $T'(t) = k(22 + T(t))$, $T(0) = 92$
D) $T'(t) = k(T(t) - 22)$, $T(0) = 22$ E) $T'(t) = k(22 - T(t))$, $T(0) = 92$
Answer. The correct answer is E).

13. The general solution of the equation $\cos(x+y) + 3x^2 + \cos(x+y)\dfrac{dy}{dx} = 0$ is
A) $\sin(x+y) + x^3 = C$ B) $\cos(x+y) = C$ C) $\sin(x+y) = C$
D) $y = \sin(x+y) + x^3 + C$ E) $\cos(x+y) + x^3 = C$
Answer. The equation is exact, the correct answer is A).

14. A fundamental solution set of the equation $y''' - 3y'' + 3y' - y = 0$ is
A) $\left\{e^t, e^{-t}, e^{3t}, e^{-3t}\right\}$ B) $\left\{e^t, e^{-t}, e^{3t}\right\}$ C) $\left\{e^t, te^t, e^{-t}\right\}$
D) $\left\{e^t, e^{-t}, t\right\}$ E) $\left\{e^t, te^t, t^2e^t\right\}$
Solution. The characteristic equation is $r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$, so $r_1 = r_2 = r_3 = 1$, the fundamental solution set is e^t, te^t, t^2e^t, the correct answer is E).

15. The Wronskian for any two solutions y_1 and y_2 of the equation $y'' - \dfrac{1}{1+t}y'(t) + q(t)y(t) = 0$ can be
A) equal to $\dfrac{1}{1+t}$ B) equal to e^{1+t} C) equal to $(1+t)$
D) equal to $e^{t+t^2/2}$ E) none of the above
Solution. By Abel's theorem,
$$W[y_1, y_2](t) = Ce^{\int -p(t)\ dt} = Ce^{\ln(1+t)} = 1 + t,$$
the correct answer is C).

16. The **minimum order** m of the linear homogeneous differential equation with constant coefficients which can have a function $y(t) = t^2 e^{5t}\cos(2t)$ as a solution is
A) $m = 3$ B) $m = 4$ C) $m = 5$ D) $m = 6$ E) $m = 7$
Solution. The roots of the characteristic equation should be $r_{1,2} = 2 \pm 3i = r_{3,4} = r_{5,6}$ (t^2 indicates that the multiplicity is at least 3), so $m = 6$ (for the sixth order equation which has the characteristic equation $((r-5-2i)(r-5+2i))^3 = (r^2-10r+29)^3 = 0$), the correct answer is D).

17. According to **the method of undetermined coefficients**, the particular solution of the equation $y'' + 9y = 4\cos(3x) + e^{3x}$ has the form:
A) $C_1xe^{3x} + C_2\cos(3x)$ B) $C_1xe^{3x} + C_2\cos(3x) + C_3\sin(3x)$
C) $C_1e^{3x} + C_2\sin(3x)$ D) $C_1xe^{3x} + C_2x\cos(3x) + C_3x\sin(3x)$
E) $C_1e^{3x} + C_2x\cos(3x) + C_3x\sin(3x)$
Solution. The solutions of the homogeneous equation are $\cos(3x)$, $\sin(3x)$, the correct answer is E).

18. If y_1 and y_2 are solutions of the second order linear equation $y''+p(t)y'+q(t)y = t\ln(t)$, $t > 0$ then the function $y(t) = ay_1(t) + by_2(t)$ also satisfies this equation
A) for any $a, b \in \mathbb{R}$
B) for $a, b \in \mathbb{R}$ satisfying $a + b = 1$ only
C) for $a, b \in \mathbb{R}$ satisfying $a - b = 1$ only
D) for any $a = b \in \mathbb{R}$ only
E) never
Solution. Since $y'' + p(t)y' + q(t)y = a(y_1'' + p(t)y_1' + q(t)y_1) + b(y_2'' + p(t)y_2' + q(t)y_2)$ $= at\ln(t)+bt\ln(t) = (a+b)l\ln(t) = t\ln(t)$ for any t if and only if $a+b = 1$, the correct answer is B).

19. If $xe^{2x} + 3\cos(x)$ is a solution of the fourth-order differential equation $y^{(4)} + ay''' + by'' + cy' + dy = 0$ then the equation is
A) $y^{(4)} - 4y''' + 5y'' - 4y' + 4y = 0$
B) $y^{(4)} - 2y''' + y'' - 2y' = 0$
C) $y^{(4)} + 5y'' + 4y = 0$
D) $y^{(4)} - 5y''' + 7y'' - 5y' + 6y = 0$
E) cannot be found: not enough data
Solution. The characteristic equation has roots $r_{1,2} = 2$, $r_{3,4} = \pm i$, so it has the form (please note that the coefficient of the highest degree is one)

$$g(r) = (r-2)^2(r-i)(r+i) = (r^2-4r+4)(r^2+1) = r^4 - 4r^3 + 5r^2 - 4r + 4,$$

so the equation is $y^{(4)} - 4y''' + 5y'' - 4y' + 4y = 0$, the correct answer is A).

20. The linear equation $y'' + p(t)y' + q(t)y = 0$, $t > 0$ has solutions $y_1(t) = \dfrac{1}{t}$ and $y_2 = \dfrac{1}{t^2}$. Then a particular solution of the non-homogeneous equation $y'' + p(t)y' + q(t)y = \dfrac{1}{t}$ is

A) $\dfrac{1}{4t^3}$ B) $\dfrac{t^2}{2} - \dfrac{t^3}{3}$ C) $\dfrac{1}{6t^2}$ D) $\dfrac{t}{6}$

E) unknown without knowing $p(t)$ and $q(t)$

Solution. According to the method of variation of parameters, we are looking for solutions in the form $y = C_1(t)y_1 + C_2(t)y_2$, where

$$\frac{1}{t}C_1' + \frac{1}{t^2}C_2' = 0, \quad -\frac{1}{t^2}C_1' - \frac{2}{t^3}C_2' = \frac{1}{t}.$$

Dividing the first equation by t and adding the two equations, we obtain

$$-\frac{1}{t^3}C_2' = \frac{1}{t} \quad \Rightarrow \quad C_2'(t) = -t^2 \quad \Rightarrow \quad C_2(t) = C_2 - \frac{t^3}{3},$$

$$C_1' = -\frac{1}{t}C_2' = t \Rightarrow \quad C_1(t) = C_1 + \frac{t^2}{2}.$$

Thus the general solution is

$$y = C_1(t)y_1 + C_2(t)y_2 = \left(C_1 + \frac{t^2}{2}\right)\frac{1}{t} + \left(C_2 - \frac{t^3}{3}\right)\frac{1}{t^2}$$

$$= C_1\frac{1}{t} + C_2\frac{1}{t^2} + \frac{t}{2} - \frac{t}{3} = \frac{C_1}{t} + \frac{C_2}{t^2} + \frac{t}{6},$$

a particular solution is $\dfrac{t}{6}$, the correct answer is D).