# MATH 375
# Handout # 8: Answers, Hints, Solutions
# Partial Differential Equations

1. Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = 0,\ X'(\pi/2) = 0.$$

**Solution.** The characteristic equation $r^2 + \lambda = 0$ has solutions $r_1 = r_2 = 0$ for $\lambda = 0$, $r = \pm\sqrt{\lambda}$ for $\lambda < 0$ and $r = \pm\sqrt{\lambda}i$ for $\lambda > 0$.

*Case 1.* $\lambda = 0$, then the general solution is $X(x) = A + Bx$. Substituting the boundary conditions we get $X'(0) = B = 0$ and $X'(\pi/2) = 0$ for any $A$, so $\lambda = 0$ is an eigenvalue with any constant as an eigenfunction, say, $X_0 = 1$.

*Case 2.* $\lambda < 0$, for example, $\lambda = -\alpha^2$, then the general solution is $X(x) = A\cosh(\alpha x) + B\sinh(\alpha x)$. Substituting the boundary conditions we get $X'(0) = \alpha B = 0$ and $X'(\pi/2) = A\alpha\sinh(\alpha\pi/2) + B\alpha\cosh(\alpha\pi/2) = B\alpha\cosh(\alpha\pi/2) = 0$, so $A = B = 0$, there are no nontrivial solutions, $\lambda < 0$ is not an eigenvalue.

*Case 3.* $\lambda > 0$, for example, $\lambda = \beta^2$, then the general solution is $X(x) = A\cos(\beta x) + B\sin(\beta x)$. Substituting the boundary conditions we get $X'(0) = B = 0$ and $X'(\pi/2) = A\sin(\beta\pi) = 0$ with $A \neq 0$ if $\beta\pi/2 = n\pi$, or $\beta = 2n$, $\lambda = 4n^2$, $n \in \mathbb{N}$.

The eigenvalues are $\lambda_0 = 0$, $\lambda_n = 4n^2$, $n = 1, 2, \ldots$, the corresponding eigenfunctions are $X_0 = 1$, $X_n = \cos(4n^2 x)$.

2. a) Formulate the boundary value problem for heat transfer (conduction) in a slab of length $\pi$ with $k = 4$ and the initial temperature distribution $u(x, 0) = x^2 - \pi x$ and the temperature at the ends kept at zero.
b) Use the method of separation of variables to determine the temperature $u(t, x)$.
**Solution.** a) The problem is

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi,\ t > 0,$$

$$u(0, t) = 0,\ u(\pi, t) = 0,\ t > 0,\ u(x, 0) = x^2 - \pi x,\ 0 < x < \pi.$$

Let $u(x, t) = X(x)T(t)$, then $u_{xx} = X''T$, $u_t = XT'$. Separating variables

$$4X''T = XT' \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{4T},$$

we get that a function of $x$ is identically equal to a function of $t$ which is only possible when they are both constant, say, equal to $-\lambda$. Thus $4X''(x) + \lambda X = 0$, $T' = -\lambda T$.

Substituting the initial conditions, we get $X(x)T(0) = x^2 - \pi x$. The boundary conditions lead to $X(0)T(t) = 0$ and $X(\pi)T(t) = 0$, which leads for nontrivial $T(t)$ to $X(0) = 0$, $X(\pi) = 0$.

The Sturm-Liouville problem $X''(x) + \lambda X(x) = 0, X(0) = 0,\ X(\pi) = 0$ has eigenvalues $\lambda_n = n^2$ with eigenfunctions $X_n(x) = \sin(nx),\ n = 1, 2, \ldots$. For $\lambda_n = n^2$, the equation $T' = -4\lambda T = -4n^2 T$ has solutions $T_n(t) = e^{-4n^2 t}$. Each function $u_n(x,t) = \sin(nx)e^{-n^2 t}$. By the superposition principle, we are looking for the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-4n^2 t}.$$

Substituting the initial condition, we get $u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = x^2 - \pi x$. We have to find the sine series for $f(x) = x^2 - \pi$. Applying twice integration by parts, we find

$$\int (x^2 - \pi x)\sin(nx)\ dx = \frac{(\pi x - x^2)\cos(nx)}{n} + \frac{(2x - \pi)\sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3} + C.$$

Using it in the computation of the Fourier coefficients, we obtain

$$b_n = \frac{2}{\pi}\int_0^{\pi} (x^2 - \pi x)\sin(nx)\ dx$$

$$= \frac{2}{\pi}\left[\frac{(\pi x - x^2)\cos(nx)}{n} + \frac{(2x - \pi)\sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - 0 + 0 - 0 + \frac{2}{n^3}(\cos(n\pi) - \cos(0)\right] = \frac{4}{\pi}\frac{(-1)^n - 1}{n^3}.$$

Finally, the solution to the heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi}\frac{(-1)^n - 1}{n^3}\sin(nx)e^{-4n^2 t} \quad 0 \le x \le \pi.$$

3. Use the method of separation of variables to find the solution to the heat conduction problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi,\ t > 0,$$
$$u(0,t) = u(\pi,t) = 0,\ t > 0, \quad u(x,0) = \sin(2x),\ 0 < x < \pi.$$

**Solution.** Similar to the previous problem,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}.$$

Substituting the initial condition, we have

$$\sum_{n=1}^{\infty} b_n \sin(nx) = \sin(2x),$$

thus (comparing coefficients) $b_n = 0$, $n \neq 2$, $b_2 = 1$ and

$$u(t,x) = e^{-4t}\sin(2x).$$

4. Use the method of separation of variables to find the solution to the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < 2$$

$$u(0, y) = 0, \; u(\pi, y) = 0, \quad 0 < y < 2, \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 2) = 6 \sin(3x), \; 0 < x < \pi$$

**Solution.** We substitute $u(x, y) = X(x)Y(y)$, $u_{xx} = X''(x)Y(y)$, $u_{yy} = X(x)Y''(y)$ in the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. We separate variables and conclude that both sides are equal to a constant, say, $-\lambda$:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Substituting the boundary conditions we get $X(0)Y(y) = 0 = X(\pi)Y(y)$, so $X(0) = X(\pi) = 0$, $X(x)Y(0) = 0$, $X(x)Y'(2) = 6 \sin(3x)$ gives $Y'(0) = 0$.

The Sturm-Liouville problem $X''(x)+\lambda X(x) = 0, X(0) = 0, \; X(pi) = 0$ has eigenvalues $\lambda_n = n^2$ with eigenfunctions $X_n(x) = \sin{(nx)}$, $n = 1, 2, \ldots$.

Consider the second Sturm-Liouville problem $Y''(y) - \lambda Y(y) = 0$ with $Y(0) = 0$ which has the same eigenvalues $\lambda_n = n^2$, so $Y'' - n^2 Y = 0$ leads to eigenfunctions $Y_n(y) = A_n \cosh(ny) + B_n \sinh{(ny)}$. The condition $Y'(0) = 0$ gives $B_n = 0$, so the eigenfunctions are $Y_n(y) = \cosh{(ny)}$, and

$$u_n(x, y) = \sin{(nx)} \cosh{(ny)},$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin{(nx)} \cosh{(ny)},$$

and

$$u_y(x, y) = \sum_{n=1}^{\infty} n b_n \sin{(nx)} \sinh{(ny)}$$

From the boundary condition $u_y(x, 2) = 6 \sin(3x)$, substituting $y = 2$ in the solution

$$u_y(x, 2) = \sum_{n=1}^{\infty} n b_n \sin{(nx)} \sinh{(2n)} = 7 \sin(3x)$$

and comparing the terms we obtain $b_n = 0$, $n \neq 3$, $b_3 = 6/(3 \sinh(6))$, so
$u(x, y) = \dfrac{2}{\sinh(6)} \sin{(3x)} \cosh{(3y)}$.

5. Use the method of separation of variables to find the solution to the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < 1$$

$$u(0,y)=0,\ u(\pi,y)=0,\quad 0<y<1,\quad \frac{\partial u}{\partial y}(x,0)=0,\quad \frac{\partial u}{\partial y}(x,1)=x,\ 0<x<\pi$$

**Solution.** We substitute $u(x,y)=X(x)Y(y)$, $u_{xx}=X''(x)Y(y)$, $u_{yy}=X(x)Y''(y)$ in the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. We separate variables and conclude that both sides are equal to a constant, say, $-\lambda$:

$$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda.$$

Substituting the boundary conditions we get $X(0)Y(y)=0=X(\pi)Y(y)$, so $X(0)=X(\pi)=0$, $X(x)Y(0)=0$, $X(x)Y(1)=x$ gives $Y(0)=0$.

The Sturm-Liouville problem $X''(x)+\lambda X(x)=0, X(0)=0,\ X(1)=0$ has eigenvalues $\lambda_n=n^2$ with eigenfunctions $X_n(x)=\sin(nx)$, $n=1,2,\ldots$.

Consider the second Sturm-Liouville problem $Y''(y)-n^2Y(y)=0$ with $Y(0)=0$, the eigenfunctions are $Y_n(y)=A_n\cosh(ny)+B_n\sinh(ny)$. The condition $Y'(0)=0$ gives $B_n=0$, so the eigenfunctions are $Y_n(y)=\cosh(ny)$, and

$$u_n=\sin(nx)\cosh(ny)\,,\quad u(x,y)=\sum_{n=1}^{\infty}B_n\sin(nx)\cosh(ny)\,.$$

From the boundary condition $u_y(x,1)=x$. Substituting $y=1$ in the solution

$$\frac{\partial u}{\partial y}(x,1)=\sum_{n=1}^{\infty}nB_n\sin(nx)\sinh(n)=x.$$

The sine series for $f(x)=x$ can be found as

$$b_n=\frac{2}{\pi}\int_0^{\pi}x\sin(nx)=\frac{2}{\pi}\left[-\frac{x\cos(nx)}{n}+\frac{\sin(nx)}{n^2}\right]$$

$$=\frac{2}{\pi}\frac{\pi}{n}\cos(n\pi)=\frac{2}{n}(-1)^{n+1}.$$

Comparing the terms we obtain

$$nB_n\sinh(n)=\frac{2}{n}(-1)^{n+1}\quad\Rightarrow\quad B_n=\frac{2}{n^2\sinh(n)}(-1)^{n+1}$$

and

$$u(x,y)=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\sinh(n)}\sin(nx)\cosh(ny)\,.$$

6. Use the method of separation of variables to find the solution $u(t,x)$ of a vibrating string problem:

$$4\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2},\quad 0<x<\pi,\ t>0,$$

$$u(0,t)=0,\ u(\pi,t)=0,\ t>0,\quad u(x,0)=0,\quad \frac{\partial u}{\partial t}(x,0)=8\sin(3x),\ 0<x<\pi.$$

**Solution.** We substitute $u(x,y)=X(x)T(t)$, $u_{xx}=X''(x)T(t)$, $u_{tt}=X(x)T''(t)$ in the equation, separate variables and conclude that both sides are equal to a constant, say, $-\lambda$:

$$\frac{X''}{X}=\frac{1}{4}\frac{T''}{T}=-\lambda.$$

Substituting the boundary conditions we get $X(0)T(t)=0=X(\pi)T(t)$, so $X(0)=X(\pi)=0$, $X(x)T(0)=0$ gives $T(0)=0$.

The Sturm-Liouville problem $X''(x)+\lambda X(x)=0, X(0)=0,\ X(\pi)=0$ has eigenvalues $\lambda_n=n^2$ with eigenfunctions $X_n(x)=\sin{(nx)}$, $n=1,2,\ldots$.

Consider the second Sturm-Liouville problem $T''(y)+4\lambda T(t)=T''+4n^2T=0$ with $T(0)=0$, the eigenfunctions are $T_n(t)=A_n\cos(2nt)+B_n\sin{(2nt)}$. The condition $T(0)=0$ gives $A_n\cos(0)+0=0$, so $A_n=0$ and the eigenfunctions are $Y_n(y)=\sin{(2nt)}$, and

$$u_n=\sin{(nx)}\sin{(2nt)}\,,\quad u(x,y)=\sum_{n=1}^{\infty}B_n\sin{(nx)}\sin{(2nt)}\,.$$

From the boundary condition $u_t(x,0)=8\sin(3x)$. Substituting $t=0$ in the derivative

$$u_t(x,t)=\sum_{n=1}^{\infty}2nB_n\sin{(nx)}\cos{(2nt)}\,,$$

we obtain

$$u_t(x,0)=\sum_{n=1}^{\infty}2nB_n\sin{(nx)}\cos(0)=8\sin(3x).$$

The comparison gives $B_n=0$, $n\neq 3$, and $2\cdot 3B_3=8$, or $B_3=4/3$. The solution is

$$u(t,x)=\frac{4}{3}\sin(6t)\sin(3x).$$

7. Use the method of separation of variables to find the displacement $u(t,x)$ of a vibrating elastic string which satisfies the conditions:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},\quad 0<x<2,\ t>0,$$

$$u(t,0)=0,\ u(t,2)=0,\ t>0,\quad u(0,x)=f(x)=\begin{cases} x, & 0\le x\le 1,\\ 2-x, & 1<x\le 2,\end{cases}$$

$\frac{\partial u}{\partial t}(0,x)=0,\ 0<x<2.$

**Solution.** We substitute $u(x,y)=X(x)T(t)$, $u_{xx}=X''(x)T(t)$, $u_{tt}=X(x)T''(t)$ in the equation, separate variables and conclude that both sides are equal to a constant, say, $-\lambda$:

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda.$$

Substituting the boundary conditions we get $X(0)T(t) = 0 = X(2)T(t)$, so $X(0) = X(2) = 0$, $X(x)T'(0) = 0$ gives $T'(0) = 0$.

The Sturm-Liouville problem $X''(x) + \lambda X(x) = 0, X(0) = 0,\ X(2) = 0$ has eigenvalues $\lambda_n = \left(\frac{\pi n}{2}\right)^2$ with eigenfunctions $X_n(x) = \sin\left(\frac{\pi n}{2}x\right)$, $n = 1, 2, \ldots$.

Consider the second Sturm-Liouville problem $T''(y) + \lambda T(t) = T'' + \left(\frac{\pi n}{2}\right)^2 T = 0$ with $T'(0) = 0$, the eigenfunctions are $T_n(t) = A_n \cos\left(\frac{\pi n}{2}t\right) + B_n \sin\left(\frac{\pi n}{2}t\right)$. The condition $T'(0) = 0$ gives $\frac{\pi n}{2} B_n \cos(0) + 0 = 0$, so $B_n = 0$ and the eigenfunctions are $Y_n(y) = \cos\left(\frac{\pi n}{2}t\right)$, and

$$u_n = \sin\left(\frac{\pi n}{2}x\right)\cos\left(\frac{\pi n}{2}t\right), \quad u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{2}x\right)\cos\left(\frac{\pi n}{2}t\right).$$

From the boundary condition $u_t(x,0) = f(x)$. Substituting $t = 0$, we get

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{2}x\right),$$

so we need to find the sine series for $f(x)$ on $[0,2]$. The coefficients of the sine series are

$$b_n = \int_0^2 f(x)\sin\left(\frac{\pi n}{2}x\right)\, dx = \int_0^1 x\sin\left(\frac{\pi n}{2}x\right)\, dx + \int_1^2 (2-x)\sin\left(\frac{\pi n}{2}x\right)\, dx$$

$$= -\frac{2}{n\pi}x\cos\left(\frac{\pi n}{2}x\right)\Big|_0^1 + \frac{4}{n^2\pi^2}\sin\left(\frac{\pi n}{2}x\right)\Big|_0^1 - \frac{4}{n\pi}\cos\left(\frac{\pi n}{2}x\right)\Big|_1^2 + \frac{2}{n\pi}x\cos\left(\frac{\pi n}{2}x\right)\Big|_1^2$$

$$-\frac{4}{n^2\pi^2}\sin\left(\frac{\pi n}{2}x\right)\Big|_1^2 = -\frac{4}{n\pi}\cos\left(\frac{\pi n}{2}\right) + \frac{4}{n\pi}\cos(n\pi) + \frac{8}{n^2\pi^2}\sin\left(\frac{\pi n}{2}\right)$$

$$-\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\cos\left(\frac{\pi n}{2}\right) = \frac{8}{n^2\pi^2}\sin\left(\frac{\pi n}{2}\right),$$

thus

$$u(t,x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}\sin\left(\frac{n\pi}{2}x\right)\cos\left(\frac{n\pi}{2}x\right).$$