THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
MIDTERM EXAMINATION - SAMPLE SOLUTIONS
MATH 375 Lec 01-04 - Version 11 FALL 2016
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Last Name	First Name

Student I.D. Number	Tutorial Number

This examination consists of 20 equally weighted questions. Please attempt all problems and record your answer by circling your choice in the exam booklet, and filling in the appropriate circle in the scantron sheet.

This is a closed book examination and calculators are not permitted.

01 One of the following statements is <u>incorrect</u>. Which one is it?

A. $y' + y = \dfrac{y^3 + 3\,y}{3\,t + t\,y^2}$ is a linear equation.

B. $2\,y' = \dfrac{(t+5)\,y^2 - \dfrac{1}{y}}{t\,y}$ is a Bernoulli equation.

C. $x^2 \ln(2\,x\,y)\,\mathrm{d}x + \dfrac{x^3 + \mathrm{e}^y}{3\,y}\,\mathrm{d}y = 0$ is an exact equation.

D. $y' + t^2\,y = 3\,t\,\mathrm{e}^y + t$ is a separable equation.

E. $x^2\,y' - x\,y = y^2 - x^2$ is a homogeneous equation.

<u>Solution</u>

The statement **(A)** is correct, since $\dfrac{y^3 + 3\,y}{3\,t + t\,y^2} = \dfrac{y\,(y^2+3)}{t\,(3+y^2)} = \dfrac{y}{t}$.

Hence the equation is equivalent to $y' + y = \dfrac{y}{t} \iff y' + \left(1 - \dfrac{1}{t}\right)\,y = 0.$

The statement **(B)** is also correct, since $\dfrac{(t+5)\,y^2 - \dfrac{1}{y}}{t\,y} = \dfrac{(t+5)\,y^2}{t\,y} - \dfrac{\dfrac{1}{y}}{t\,y} = \dfrac{t+5}{t}\,y - \dfrac{1}{t}\,y^{-2}$

As a result the equation becomes $2\,y' = \dfrac{t+5}{t}\,y - \dfrac{1}{t}\,y^{-2} \iff 2\,y' - \dfrac{t+5}{t}\,y = -\dfrac{1}{t}\,y^{-2}$

The statement **(C)** is correct. Indeed, we have $\dfrac{\partial}{\partial y}\Big(x^2\,\ln(2\,x\,y)\Big) = \dfrac{\partial}{\partial y}\Big(x^2\,\big[\ln(2\,x)+\ln(y)\big]\Big) = x^2\,\dfrac{1}{y} = \dfrac{x^2}{y}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x^3 + \mathrm{e}^y}{3\,y}\right) = \dfrac{3\,x^2}{3\,y} = \dfrac{x^2}{y}$

The two partial being equal, the differential equation is exact.

The statement **(D)** is the incorrect one. Since the equation can be rewritten as $y' = 3\,t\,\mathrm{e}^y + t - t^2\,y \iff y' = t\,\Big(3\,\mathrm{e}^y + 1 - t\,y\Big)$ It is not hard to convince yourself that the term between brackets can't be written as a produit of a function of t by a function of y.

The statement **(E)** is correct. Indeed, solving for y', we get

$$y' = \frac{y^2 - x^2 + x\,y}{x^2} = \frac{y^2}{x^2} - \frac{x^2}{x^2} + \frac{x\,y}{x^2} = \left(\frac{y}{x}\right)^2 - 1 + \frac{y}{x}$$

02. If $y(t)$ is the solution of $\begin{cases} (t+1)\,y' + 2\,y = t+1 \\ y(0) = 0 \end{cases}$ Then

A. $y(t) = \dfrac{1-(t+1)^3}{(t+1)^2}$ **B.** $y(t) = \dfrac{(t+1)^3-1}{(t+1)^2}$ **C.** $y(t) = \dfrac{1}{3}\,\dfrac{(t+1)^3-1}{(t+1)^2}$

D. $y(t) = \dfrac{1}{3}\left((t+1)^3 - 1\right)$ **E.** $y(t) = \dfrac{1}{4}\,\dfrac{(t+1)^4-1}{(t+1)^2}$

Solution

We start by rewriting the differential equation in normal form as $y' + \dfrac{2}{t+1}\,y = 1$. An integrating factor is $\mu(t) = \mathrm{e}^{\int 2/(t+1)\,\mathrm{d}t} = \mathrm{e}^{2\,\ln(t+1)} = \mathrm{e}^{\ln\left((t+1)^2\right)} = (t+1)^2$. Multiplying the differential equation by the integrating factor, leads to

$$\left((t+1)^2\,y\right)' = (t+1)^2 \implies (t+1)^2\,y = \frac{1}{3}\,(t+1)^3 + C \implies y = \frac{1}{3}\,(t+1) + \frac{C}{(t+1)^2}$$

Setting $t=0$ and using the condition $y(0)=0$, we get $0 = \dfrac{1}{3} + C \iff C = -\dfrac{1}{3}$. Hence

$$y(t) = \frac{1}{3}\,(t+1) - \frac{1}{3\,(t+1)^2} = \frac{1}{3}\,\frac{(t+1)^3-1}{(t+1)^2}$$

Hence the correct answer is **(C)**.

03. The initial value problem $\begin{cases} (t+6)\,y' + \dfrac{t+3}{(t-3)\,(t-5)}\,y = \ln(2-t) \\ y(-1) = 5 \end{cases}$ is guaranteed to have a unique solution in the open interval

A. $\big(-\infty\ ,\ 3\big)$ **B.** $\big(-6\ ,\ 2\big)$ **C.** $\big(-6\ ,\ 3\big)$

D. $\big(-6\ ,\ 5\big)$ **E.** $\big(-3\ ,\ 2\big)$

Solution

Rewriting the equation in normal form, we get $y' + \dfrac{t+3}{(t-3)\,(t-5)\,(t+6)}\,y = \dfrac{\ln(2-t)}{t+6}$

$\dfrac{t+3}{(t-3)\,(t-5)\,(t+6)}$ is defined and continuous in $(-\infty,-6)\cup(-6,3)\cup(3,5)\cup(5,+\infty)$, while $\dfrac{\ln(2-t)}{t+6}$ is defined and continuous in $(-\infty,-6)\cup(-6,2)$. It follows that the largest open interval that contains the initial time $t=-1$, and where both functions are continuous is $(-6,2)$. By the existence and uniqueness theorem, the solution of the initial value problem is guaranteed to be defined in $(-6,2)$. Hence the correct answer is **(B)**.

04. The general solution of $\big(t^2+1\big)\,y' + 2\,t\,y \,=\, t\,\big(t^2+1\big)\,y^3,$ in implicit form is given by

A. $y^2 = \big(C - t^2\big)\,\big(t^2+1\big)^2$ **B.** $y^2 = \dfrac{1}{\big(C-t^2\big)\,\big(t^2+1\big)^2}$

C. $y^2 = \dfrac{1}{t^2+1} + C\,\big(t^2+1\big)^2$ **D.** $y^2 = t^2+1+C\,\big(t^2+1\big)^2$

E. $y^2 = \dfrac{1}{t^2+1+C\,\big(t^2+1\big)^2}$

Solution

This is a Bernoulli equation. To solve, we multiply both sides by y^{-3} to get $\big(t^2+1\big)\,y'\,y^{-3} + 2\,t\,y^{-2} = t\,\big(t^2+1\big)$. Next we set $u = y^{-2} \implies u' = -2\,y^{-3}\,y' \implies y'\,y^{-3} = -\dfrac{1}{2}\,u'$. Substituting, leads to $-\dfrac{1}{2}\,\big(t^2+1\big)\,u' + 2\,t\,u = t\,\big(t^2+1\big) \implies u' - \dfrac{4\,t}{t^2+1}\,u = -2\,t$. An integrating factor for the differential equation in u, is

$$\mu(t) = \mathrm{e}^{\int -4\,t/(t^2+1)\,\mathrm{d}t} = \mathrm{e}^{-2\ln(t^2+1)} = \mathrm{e}^{\ln(t^2+1)^{-2}} = \frac{1}{\big(t^2+1\big)^2}$$

Multiplying the equation by the integrating factor, we get

$$\left(\frac{1}{\big(t^2+1\big)^2}\,u\right)' = -\frac{2\,t}{\big(t^2+1\big)^2} \implies \frac{1}{\big(t^2+1\big)^2}\,u = \frac{1}{t^2+1} + C \implies u = t^2+1+C\,\big(t^2+1\big)^2$$

Hence

$$y^{-2} = t^2+1+C\,\big(t^2+1\big)^2 \implies y^2 = \frac{1}{t^2+1+C\,\big(t^2+1\big)^2}$$

The correct answer is **(E)**

05. The solution of the initial value problem

$$\begin{cases} \left(3\,x^2 + y\,\mathrm{e}^{2\,x\,y}\right)\,\mathrm{d}x + \left(2\,\sin(2\,y) + x\,\mathrm{e}^{2\,x\,y}\right)\,\mathrm{d}y = 0, \\ y(0) = 0 \end{cases}$$ is given by

A. $6\,x + 4\,\cos(2\,y) + 2\,y^2\,\mathrm{e}^{2\,x\,y} = 4$

B. $x^3 + \dfrac{1}{2}\,\mathrm{e}^{2\,x\,y} - \cos(2\,y) = -\dfrac{1}{2}$

C. $x^3 + \dfrac{1}{2}\,\mathrm{e}^{2\,x\,y} = \dfrac{1}{2}$

D. $\dfrac{1}{2}\,\mathrm{e}^{2\,x\,y} - \cos(2\,y) = -\dfrac{1}{2}$

E. $x^3 + \dfrac{1}{2}\,\mathrm{e}^{2\,x\,y} + \cos(2\,y) = \dfrac{3}{2}$

Solution

The equation is exact in the whole xy-plane, since

$$\frac{\partial}{\partial y}\left(3\,x^2 + y\,\mathrm{e}^{2\,x\,y}\right) = \mathrm{e}^{2\,x\,y} + y\,\mathrm{e}^{2\,x\,y}\,2\,x = (2\,x\,y+1)\,\mathrm{e}^{2\,x\,y}$$

and

$$\frac{\partial}{\partial x}\left(2\,\sin(2\,y) + x\,\mathrm{e}^{2\,x\,y}\right) = \mathrm{e}^{2\,x\,y} + x\,\mathrm{e}^{2\,x\,y}\,2\,y = (2\,x\,y+1)\,\mathrm{e}^{2\,x\,y}$$

are equal. A potential function $G(x,y)$ can be obtained by solving

$$\begin{cases} G_x(x,y) = 3\,x^2 + y\,\mathrm{e}^{2\,x\,y} \\ G_y(x,y) = 2\,\sin(2\,y) + x\,\mathrm{e}^{2\,x\,y} \end{cases} \implies \begin{cases} G(x,y) = x^3 + \dfrac{1}{2}\,\mathrm{e}^{2\,x\,y} + K(y) \\ G(x,y) = -\cos(2\,y) + \dfrac{1}{2}\,\mathrm{e}^{2\,x\,y} + L(x) \end{cases}$$

$$\implies G(x,y) = x^3 + \frac{1}{2}\,\mathrm{e}^{2\,x\,y} - \cos(2\,y) + C$$

Hence the general solution is $x^3 + \dfrac{1}{2}\,\mathrm{e}^{2\,x\,y} - \cos(2\,y) = C$

Setting $t = 0$ and $y(0) = 0$, leads to $0 + \dfrac{1}{2} - 1 = C \iff C = -\dfrac{1}{2}$.

Hence the solution is $x^3 + \dfrac{1}{2}\,\mathrm{e}^{2\,x\,y} - \cos(2\,y) = -\dfrac{1}{2}$, and the correct answer is **(B)**.

06. An integrating factor for the non exact differential equation

$$\Big(\cos(x) + 2\,y^2\Big)\ \mathrm{d}x - x\,y\ \mathrm{d}y = 0$$

is given by

A. $\mu = x^5$ **B.** $\mu = x^3$ **C.** $\mu = y^5$ **D.** $\mu = x^{-5}$ **E.** $\mu = y^{-5}$

Solution

Letting $M(x, y) = \cos(x) + 2\,y^2$ and $N(x, y) = -x\,y$, we have

$$\frac{M_y(x, y) - N_x(x, y)}{M(x, y)} = \frac{4\,y - (-y)}{cos(x) + 2\,y^2} = \frac{5\,y}{\cos(x) + 2\,y^2}$$

and

$$\frac{N_x(x, y) - M_y(x, y)}{N(x, y)} = \frac{-y - 4\,y)}{-x\,y} = \frac{-5\,y}{-x\,y} = \frac{5}{x}$$

This shows that the differential equation has an integrating factor that depends on x only, given by

$$\mu(x) = \mathrm{e}^{-\int 5/x\,\mathrm{d}x} = \mathrm{e}^{-5\ \ln(x)} = \mathrm{e}^{\ln\left(x^{-5}\right)} = x^{-5}$$

Hence the correct answer is **(D)**.

07. The general solution of $(x+2)(y+1)\,\mathrm{d}x + (x+1)(y+3)\,\mathrm{d}y = 0$ is

A. $x+y+2\,\ln|y+1|+\ln|x+1| = C$ **B.** $-\dfrac{2}{(y+1)^2} + \dfrac{1}{(x+1)^2} = C$

C. $y-x+2\,\ln|y+1|-\ln|x+1| = C$ **D.** $\dfrac{y+6}{y+2} + \dfrac{x+4}{x+2} = C$

E. $x+y+\ln|(x+1)(y+1)| = C$

Solution
Rewriting the equation as

$$(x+1)(y+3)\,\mathrm{d}y = -(x+2)(y+1)\,\mathrm{d}x \implies \frac{y+3}{y+1}\,\mathrm{d}y = -\frac{x+2}{x+1}\,\mathrm{d}x \iff \left(1+\frac{2}{y+1}\right)\mathrm{d}y = -\left(1+\frac{1}{x+1}\right)\mathrm{d}x$$

Which shows that the equation is separable. Integrating leads to

$$y+2\,\ln|y+1| = -x-\ln|x+1|+C \iff x+y+2\,\ln|y+1|+\ln|x+1| = C$$

Hence the correct answer is **(A)**.

08. The solution of the initial value problem $\begin{cases} y' = \dfrac{y}{x} + 3\,\dfrac{y^2}{x^2} \\ y(1) = 1 \end{cases}$ is

A. $y = \dfrac{x}{1-3\,\ln(x)}$ **B.** $y = \dfrac{x}{1+3\,\ln(x)}$ **C.** $y = \dfrac{-x}{1-3\,\ln(x)}$

D. $y = \dfrac{1-3\,\ln(x)}{x}$ **E.** $y = \dfrac{1+3\,\ln(x)}{x}$

Solution
The equation is homogeneous. To solve, we set $u = \dfrac{y}{x} \implies y = x\,u \implies y' = u + x\,u'$ Substituting leads to

$$u + x\,u' = u + 3\,u^2 \iff x\,\frac{\mathrm{d}u}{\mathrm{d}x} = 3\,u^2 \implies \frac{1}{u^2}\,\mathrm{d}u = \frac{3}{x}\,\mathrm{d}x$$

Integrating, we get

$$-\frac{1}{u} = 3\,\ln|x| + C \implies u = -\frac{1}{3\,\ln|x|+C} \implies \frac{y}{x} = -\frac{1}{3\,\ln|x|+C} \implies y = -\frac{x}{3\,\ln|x|+C}$$

Setting $x=1,\ y=1,$ leads to

$$1 = -\frac{1}{C} \implies C = -1$$

Hence the solution is $y = -\dfrac{x}{3\,\ln|x|-1} = \dfrac{x}{1-3\,\ln(x)}$ and the correct answer is **(A)**.

09. What percentage is left of a sample of a radioactive material after two times its half-life has elapsed?

A. 0% **B.** 3.125% **C.** 6.25% **D.** 12.5% **E.** 25%

Solution
The mass at time t of the sample is given by $m(t) = m_0\, \mathrm{e}^{-k\,t}$, where m_0 is the initial mass and k is the decay constant. k is related to the half-life τ by the relation $k\,\tau = \ln(2)$. At $t = 2\,\tau$, the mass is

$$m(2\,\tau) = m_0\, \mathrm{e}^{-k\,(2\,\tau)} = m_0\, \mathrm{e}^{-2\,k\,\tau} = m_0\, \mathrm{e}^{-2\,\ln(2)} = m_0\, \mathrm{e}^{\ln(1/4)} = m_0\, \frac{1}{4}$$

It follows that the percentage of the mass left is

$$100\, \frac{m(2\,\tau)}{m_0} = 100\, \frac{1}{4} = 25$$

Hence the correct answer is **(E)**.

10. A tank with capacity 400 litres, initially contains 200 litres of pure water. Starting at $t = 0$, a brine solution with 0.1 kg of salt per litre, starts flowing into the tank at a rate of 4 litres per minute. The brine solution in the tank is kept well stirred and flows out of the tank at the rate of 2 litres per minute. What is the total mass of salt in the tank at the time the tank starts overflowing?

A. It can't be determined. **B.** 10 kg **C.** 20 kg **D.** 30 kg **E.** 40 kg

Solution
Applying the balance law to the mass $m(t)$ of salt, we get

$$m' = 4\left[\frac{L}{\min}\right]\,(0.1)\left[\frac{kg}{L}\right] - 2\left[\frac{L}{\min}\right]\,\frac{m}{V(t)}\left[\frac{kg}{L}\right] \iff m' + \frac{2}{V(t)}\,m = 0.4$$

The volume of the brine in the tank at time t is $V(t) = 2\,t + 200$.
Substituting into the differential equation, we obtain $m' + \dfrac{1}{t+100}\,m = \dfrac{2}{5}$.
An integrating factor is $\mu(t) = \mathrm{e}^{\int 1/(t+100)\,\mathrm{d}t} = \mathrm{e}^{\ln(t+100)} = t + 100$
Multiplying both sides of the equation by the integrating factor, we get

$$\Big((t+100)\,m\Big)' = \frac{2}{5}\,(t+100) \implies (t+100)\,m = \frac{1}{5}\,(t+100)^2 + C \implies m = \frac{1}{5}\,(t+100) + \frac{C}{t+100}$$

To find the constant C, we use the fact that $m(0) = 0$, to get

$$0 = \frac{1}{5}\,(100) + \frac{C}{100} \implies C = -2000 \implies m(t) = \frac{t+100}{5} - \frac{2000}{t+100}$$

The tank starts to overflow when $V(t) = 400 \iff 2t + 200 = 400 \iff t = 100$.
The mass of salt at that instant is $m(100) = \dfrac{100+100}{5} - \dfrac{2000}{100+100} = 40 - 10 = 30$
Hence the correct answer is **(D)**.

11. Given that $y_1 = t$, and $y_2 = \dfrac{1}{t}$ are solutions of $y'' + \dfrac{1}{t}\, y' - \dfrac{1}{t^2}\, y = 0$, the solution of the initial value problem $\begin{cases} y'' + \dfrac{1}{t}\, y' - \dfrac{1}{t^2}\, y = 0 \\ y(1) = 1 \quad y'(1) = -3 \end{cases}$, is given by

A. $y(t) = \dfrac{3 - 2\,t^2}{t}$ **B.** $y(t) = \dfrac{2\,t^2 - 1}{t}$ **C.** $y(t) = \dfrac{2 - t^2}{t}$

D. $y(t) = \dfrac{3\,t^2 - 2}{t}$ **E.** $y(t) = \dfrac{5\,t^2 - 4}{t}$

Solution

$y_1 = t$ and $y_2 = \dfrac{1}{t}$ being linearly independent, the general solution is $y(t) = C_1\, t + \dfrac{C_2}{t}$. To find the solution of the initial value problem, we select the constants C_1 and C_2 so that the initial conditions are satisfied. We have $y'(t) = C_1 - \dfrac{C_2}{t^2}$. It follows

$$\begin{cases} y(1) = 1 \\ y'(1) = -3 \end{cases} \iff \begin{cases} C_1 + C_2 = 1 \\ C_1 - C_2 = -3 \end{cases} \iff \begin{cases} C_1 + C_2 = 1 \\ 2\,C_1 = -2 \end{cases} \iff \begin{cases} C_2 = 1 - C_1 = 2 \\ C_1 = -1 \end{cases}$$

Hence the solution is $y(t) = -t + \dfrac{2}{t} = \dfrac{2 - t^2}{t}$, and the correct answer is **(C)**.

12. Suppose that e^t is a solution of the equation $t\,y'' - (2\,t + 1)\,y' + (t + 1)\,y = 0$. Using the reduction of order method, the general solution of the differential equation is given by

A. $y = \dfrac{C_1 + C_2\,t}{t}\,\mathrm{e}^t$ **B.** $y = \left(C_1\,t^2 + C_2\right)\mathrm{e}^{-t}$ **C.** $y = \left(C_1\, t^2 + C_2\right)\mathrm{e}^t$

D. $y = \left(C_1\,t + C_2\right)\mathrm{e}^t$ **E.** $y = C_1\,\mathrm{e}^{-t} + C_2\,\mathrm{e}^t$

Solution

Setting $y = \mathrm{e}^t\,u \implies y' = \mathrm{e}^t\left(u' + u\right) \implies y'' = \mathrm{e}^t\left(u'' + 2\,u' + u\right)$. Substituting into the differential equation, we get

$$t\,\mathrm{e}^t\left(u'' + 2\,u' + u\right) - (2\,t + 1)\,\mathrm{e}^t\left(u' + u\right) + (t + 1)\,\mathrm{e}^t\,u = 0 \iff \mathrm{e}^t\left(t\,u'' - u'\right) = 0 \iff u'' - \frac{1}{t}\,u' = 0$$

The last equation is equivalent to

$$\begin{cases} u' = v \\ v' - \dfrac{1}{t}\,v = 0 \end{cases}$$

An integrating factor for $v' - \dfrac{1}{t}\,v = 0$, is $\mu(t) = \mathrm{e}^{\int -1/t\,\mathrm{d}t} = \dfrac{1}{t}$. It follows

$$v' - \frac{1}{t}\,v = 0 \implies \left(\frac{1}{t}\,v\right)' = 0 \implies \frac{1}{t}\,v = C_1 \implies v = C_1\,t$$

and

$$u' = v \iff u' = C_1\,t \implies u = \frac{1}{2}\,C_1\,t^2 + C_2 \implies y = \mathrm{e}^t\left(\frac{1}{2}\,C_1\,t^2 + C_2\right)$$

Hence the general solution is $y = \left(C_1\,t^2 + C_2\right)\mathrm{e}^t$, and the correct answer is **(C)**.

13. The wronskian of two functions $y_1(t) = t\,e^{2t}$ and $y_2(t) = (2t+1)\,e^{2t}$ is given by

A. $W(t) = e^{4t}$ **B.** $W(t) = -e^{4t}$ **C.** $W(t) = \left(8t^2 + 8t + 1\right) e^{4t}$

D. $W(t) = -e^{2t}$ **E.** $W(t) = t\,e^{4t}$

Solution
We have

$$W(t) = \begin{vmatrix} t\,e^{2t} & (2t+1)\,e^{2t} \\ (2t+1)\,e^{2t} & (4t+4)\,e^{2t} \end{vmatrix} = e^{2t}\,e^{2t} \begin{vmatrix} t & (2t+1) \\ (2t+1) & (4t+4) \end{vmatrix}$$
$$= e^{4t}\left(t\,(4t+4) - (2t+1)^2\right) = -e^{4t}$$

Hence the correct answer is **(B)**.

14. The unique solution of the initial value problem

$$\begin{cases} (t-3)\,y'' + \dfrac{t}{t+4}\,y' + \dfrac{t-3}{2t+5}\,y = \sqrt{5-2t} \\ y(0) = 0 \quad \text{and} \quad y'(0) = 1 \end{cases}$$

is guaranteed to exist in the open interval

A. $\left(-4\,,\,3\right)$ **B.** $\left(-4\,,\,5/2\right)$ **C.** $\left(-5/2\,,\,3\right)$

D. $\left(-\infty\,,\,3\right)$ **E.** $\left(-5/2\,,\,5/2\right)$

Solution
Rewrite the equation in normal form as $y'' + \dfrac{t}{(t+4)\,(t+3)}\,y' + \dfrac{t-3}{(2t+5)\,(t+3)}\,y = \dfrac{\sqrt{5-2t}}{t+3}$

$p(t) = \dfrac{t}{(t+4)\,(t+3)}$, is continuous in $\left(-\infty\,,\,-4\right) \cup \left(-4\,,\,-3\right) \cup \left(-3\,,\,+\infty\right)$

$q(t) = \dfrac{t-3}{(2t+5)\,(t+3)}$ is continuous in $\left(-\infty\,,\,-3\right) \cup \left(-3\,,\,-5/2\right) \cup \left(-5/2\,,\,+\infty\right)$

$g(t) = \dfrac{\sqrt{5-2t}}{t+3}$ is continuous in $\left(-\infty\,,\,-3\right) \cup \left(-3\,,\,5/2\right)$

The largest open interval that contains the initial time $t = 0$ where $p(t)$, $q(t)$, and $g(t)$ are all continuous is $\left(-5/2\,,\,5/2\right)$. By the existence and uniqueness theorem, the solution of the I.V.P. is guaranteed to exist in $\left(-5/2\,,\,5/2\right)$ Hence the correct answer is **(E)**.

15. The general solution of $2\,y'' + 5\,y' + 2\,y = 0$, is given by

A. $y = C_1\,\mathrm{e}^{-2\,t} + C_2\,\mathrm{e}^{-t/2}$

B. $y = C_1\,\mathrm{e}^{t/2} + C_2\,\mathrm{e}^{2\,t}$

C. $y = C_1\,\mathrm{e}^{-2\,t} + C_2\,\mathrm{e}^{t/2}$

D. $y = C_1\,\mathrm{e}^{-t/2} + C_2\,\mathrm{e}^{2\,t}$

E. $y = C_1\,\mathrm{e}^{-3\,t/2} + C_2\,\mathrm{e}^{2\,t}$

Solution

The characteristic equation is $2\,\lambda^2 + 5\,\lambda + 2 = 0$. The roots are $\lambda_1 = -2$ and $\lambda_2 = -\dfrac{1}{2}$. It follows that a fundamental set of solutions is $\left\{\, \mathrm{e}^{-2\,t}\,,\ \mathrm{e}^{-t/2} \,\right\}$, and the general solution is $y = C_1\,\mathrm{e}^{-2\,t} + C_2\,\mathrm{e}^{-t/2}$. Hence the correct answer is **(A)**.

16. The solution of the initial value problem $\begin{cases} y'' + 2\,y' + 2\,y = 0 \\ y(0) = 4 \quad \text{and} \quad y'(0) = -2 \end{cases}$ is

A. $y = \Big(4\,\cos(t) - 6\,\sin(t)\Big)\,\mathrm{e}^{-t}$

B. $y = \Big(4\,\cos(t) - 2\,\sin(t)\Big)\,\mathrm{e}^{-t}$

C. $y = \Big(4\,\cos(t) + 2\,\sin(t)\Big)\,\mathrm{e}^{-t}$

D. $y = \Big(4\,\cos(t) + 6\,\sin(t)\Big)\,\mathrm{e}^{-t}$

E. $y = \Big(4\,\cos(t) + 2\,\sin(t)\Big)\,\mathrm{e}^{t}$

Solution

The characteristic equation is $\lambda^2 + 2\,\lambda + 2 = 0 \iff (\lambda + 1)^2 + 1 = 0$. The roots are $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Hence a fundamental set of solutions is $\left\{\, \mathrm{e}^{-t}\,\cos(t)\,,\ \mathrm{e}^{-t}\,\sin(t) \,\right\}$, and the general solution is

$$y = \mathrm{e}^{-t}\Big(C_1\,\cos(t) + C_2\,\sin(t)\Big) \implies y' = \mathrm{e}^{-t}\Big(-C_1\,\cos(t) - C_2\,\sin(t) - C_1\,\sin(t) + C_2\,\cos(t)\Big)$$

It follows $\begin{cases} y(0) = 4 \\ y'(0) = -2 \end{cases} \iff \begin{cases} C_1 = 4 \\ -C_1 + C_2 = -2 \end{cases} \iff \begin{cases} C_1 = 4 \\ C_2 = 2 \end{cases}$

Hence the solution of the initial value problem is $y(t) = \Big(4\,\cos(t) + 2\,\sin(t)\Big)\,\mathrm{e}^{-t}$, and the correct answer is **(C)**.

17. The general solution of $y'' + 6\,y' + 9\,y = 0$, is

A. $y = C_1\,e^{-3\,t} + C_2\,e^{3\,t}$ **B.** $y = \big(C_1 + C_2\,t\big)\,e^{3\,t}$ **C.** $y = \big(C_1 + C_2\,t\big)\,e^{-3\,t}$

D. $y = C_1\,t\,e^{-3\,t} + C_2\,t\,e^{3\,t}$ **E.** $y = C_1\,t\,e^{-3\,t} + C_2\,e^{3\,t}$

Solution

The characteristic equation is $\lambda^2 + 6\,\lambda + 9 = 0 \iff (\lambda+3)^2 = 0$. The roots are $\lambda_1 = \lambda_2 = -3$. It follows that a fundamental set of solutions is $\left\{\ e^{-3\,t}\ ,\ t\,e^{-3\,t}\ \right\}$, and the general solution is $y = C_1\,e^{-3\,t} + C_2\,t\,e^{-3\,t} = \Big(C_1 + C_2\,t\Big)\,e^{-3\,t}$. Hence the correct answer is **(C)**.

18. Which of the following differential equations has $3\,\cos(3\,t)\,e^{-2\,t}$ as a solution?

A. $y'' + 4\,y' + 13\,y = 0$ **B.** $y'' - 4\,y' + 13\,y = 0$ **C.** $y'' - 4\,y' - 13\,y = 0$

D. $y'' + 4\,y' - 13\,y = 0$ **E.** $y'' + 4\,y' + 9\,y = 0$

Solution

The fact that $3\,\cos(3\,t)\,e^{-2\,t}$, is a solution shows that the roots of the characteristic equation are $\lambda_1 = -2 + 3\,i$ and $\lambda_2 = -2 - 3\,i$. Consequently the characteristic equation is

$$\big(\lambda - (-2+3\,i)\big)\,\big(\lambda - (-2-3\,i)\big) = 0 \iff \big(\lambda + (2-3\,i)\big)\,\big(\lambda + (2+3\,i)\big) = 0 \iff \lambda^2 + 4\,\lambda + 13 = 0$$

Hence the differential equation (in normal form) is $y'' + 4\,y' + 13\,y = 0$, and the correct answer is **(A)**.

19. If the characteristic equation of a linear homogeneous differential equation with constant coefficients is $\lambda^2\left((\lambda-2)^2+1\right)=0$, then its general solution is given by

A. $y = C_1 + C_2\,t + \big(C_3\,\cos(2\,t) + C_4\,\sin(2\,t)\big)\,\mathrm{e}^{-t}$

B. $y = C_1\,t + C_2\,t^2 + \big(C_3\,\cos(t) + C_4\,\sin(t)\big)\,\mathrm{e}^{2\,t}$

C. $y = \Big(\big(C_1 + C_2\,t\big)\,\cos(t) + \big(C_3 + C_4\,t\big)\,\sin(t)\Big)\,\mathrm{e}^{2\,t}$

D. $y = C_1 + C_2\,t + \big(C_3\,\cos(t) + C_4\,\sin(t)\big)\,\mathrm{e}^{2\,t}$

E. $y = C_1\,\mathrm{e}^{t} + C_2\,t\,\mathrm{e}^{t} + \big(C_3\,\cos(t) + C_4\,\sin(t)\big)\,\mathrm{e}^{2\,t}$

Solution

The roots of the characteristic equation are $\lambda_1 = \lambda_2 = 0,\ \lambda_3 = 2+i,\ \lambda_4 = 2-i$. It follows that a fundamental set of solutions is $\left\{\ 1\ ,\ t\ ,\ \mathrm{e}^{2\,t}\,\cos(t)\ ,\ \mathrm{e}^{2\,t}\,\sin(t)\ \right\}$. Hence the general solution is $y(t) = C_1 + C_2\,t + C_3\mathrm{e}^{2\,t}\,\cos(t) + C_4\,\mathrm{e}^{2\,t}\,\sin(t) = C_1 + C_2\,t + \Big(C_3\,\cos(t) + C_4\,\sin(t)\Big)\,\mathrm{e}^{2\,t}$, and the correct answer is **(D)**.

20. A particular solution y_p of the differential equation $y'' + 2\,y' + 2\,y = t\,(t+1)$, is

A. $y_p = t\,(1-t)/2$ **B.** $y_p = t\,(t-1)/2$ **C.** $y_p = t\,(t+1)/2$

D. $y_p = t\,(t-2)/2$ **E.** $y_p = (t-1)^2/2$

Solution

The characteristic equation of the associated homogeneous equation is $\lambda^2 + 2\,\lambda + 2 = 0$. Its roots are $\lambda_1 = -1+i,\ \lambda_2 = -1-i$. A particular solution $y_p(t)$ can be found by using the undetermined coefficients method. It should have the form $y_p(t) = A\,t^2 + B\,t + C$. Substituting into the equation, we get

$$2\,A + 2\Big(2\,A\,t + B\Big) + 2\Big(A\,t^2 + B\,t + C\Big) = t\,(t+1) \iff$$

$$2\,A\,t^2 + \Big(4\,A + 2\,B\Big)\,t + \Big(2\,A + 2\,B + 2\,C\Big) = t^2 + t \implies \begin{cases} 2\,A = 1 \\ 4\,A + 2\,B = 1 \\ 2\,A + 2\,B + 2\,C = 0 \end{cases}$$

Solving for the constants, we get $A = 1/2,\ B = -1/2,\ C = 0$.
Hence $y_p(t) = \dfrac{1}{2}\,t^2 - \dfrac{1}{2}\,t = t(t-1)/2$, and the correct answer is **(B)**.