Math 375 **Fall 2016**

Linear Systems of First Order Differential Equations

Worksheet # 4 **November 21-25**

The problems on this worksheet refer to material from sections §§7.1, and, 7.4 of our text.
Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Matrix Form

01. Express each of the following system in matrix form $\overrightarrow{Y}' = \boldsymbol{Q}(t)\,\overrightarrow{Y} + \overrightarrow{F}(t)$

a. $\begin{cases} y_1' = y_1 + (2\,t+1)\,y_2 + \dfrac{1}{t^2+1} \\ y_2' = t\,y_1 + \tan(t)\,y_2 + \cosh(t) \end{cases}$ **b*.** $\begin{cases} y_1' = 2\,y_1 + t\,y_2 - 3\,y_3 + t \\ y_2' = -y_1 + \cos(t)\,y_2 + \sec(t) \\ y_3' = t\,y_1 + 4\,y_3 + \ln(t) \end{cases}$

02. Rewrite each of the differential equations as a first order linear system

a. $t\,y'' - 2\,y' + (1 - \mathrm{e}^t)\,y = \sin(t)$ **b*.** $y''' - t\,y'' - \mathrm{e}^t\,y' + y = \ln(t)$

Existence and Uniqueness

03*. Consider the initial value problem $\begin{cases} y_1' = t\,y_1 + 2\,y_2 + \ln(5-t) \\ y_2' = 3\,y_1 - \dfrac{t}{t-1}\,y_2 + \csc(t) \\ y_1(t_0) = 1 \text{ and } y_2(t_0) = -1 \end{cases}$

For each of the following cases, find the largest open interval where the solution to the initial value problem is garanteed to be defined.

a. $t_0 = -1$ **b.** $t_0 = 2$ **c.** $t_0 = 4$

04. Find the largest interval (a, b) such that a unique solution to the initial value problem $\begin{cases} (t+1)^2\,y_1' &=& \cos(t)\,y_1 + y_2 + 2 \\ \sin(t)\,y_2' &=& \cos(t)\,y_1 + y_2 + \sec t \\ y_1(1) = 3 &\&& y_2(1) = 2 \end{cases}$, is guaranteed to exist.

Simple real eigenvalues

05. The coefficient matrix of the system $\begin{cases} y_1' = -2\,y_1 + y_2 \\ y_2' = \ y_1 - 2\,y_2 \end{cases}$ has eigenvalues $\lambda_1 = -3$, $\lambda_2 = -1$, and corresponding eigenvectors $\overrightarrow{V}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\overrightarrow{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Write down the general solution of the system, then find the solution $\overrightarrow{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ that satisfies $\overrightarrow{Y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

06. Solve the initial value problem $\begin{cases} \overrightarrow{Y}' = \boldsymbol{A}\,\overrightarrow{Y} \\ \overrightarrow{Y}(0) = \overrightarrow{Y}_0 \end{cases}$ in each of the following cases.

a. $\boldsymbol{A} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$, $\overrightarrow{Y}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ **b*.** $\boldsymbol{A} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, $\overrightarrow{Y}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

07*. Consider the system $\begin{cases} y_1' = y_1 + y_2 + y_3 \\ y_2' = 2\,y_1 + y_2 - y_3 \\ y_3' = -8\,y_1 - 5\,y_2 - 3\,y_3 \end{cases}$ and let $\boldsymbol{A}$ be its coefficient matrix. If you know that $\boldsymbol{A}$ has eigenvalues $\lambda_1 = -2,\ \lambda_2 = -1,\ \lambda_3 = 2,$ and corresponding eigenvectors $\vec{V}_1 = \begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}$, $\vec{V}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, write down the general solution of the system, then find the solution $\vec{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$ that satisfies $\vec{Y}(0) = \begin{bmatrix} 1 \\ -2 \\ 8 \end{bmatrix}$

08. Solve the initial value problem $\begin{cases} \vec{Y}' = \boldsymbol{A}\,\vec{Y} \\ \vec{Y}(0) = \vec{Y}_0 \end{cases}$ where $\boldsymbol{A} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}$, and $\vec{Y}_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Simple Complex Eigenvalues

09. Given that the coefficient matrix of the system $\begin{cases} y_1' = y_1 - y_2 \\ y_2' = 5\,y_1 - 3\,y_2 \end{cases}$ has eigenvalue $\lambda_1 = -1 + i$, and corresponding eigenvector $\vec{V}_1 = \begin{bmatrix} 1 \\ 2 - i \end{bmatrix}$, find the general solution of the system.

10. Solve the initial value problem $\begin{cases} \vec{Y}' = \boldsymbol{A}\,\vec{Y} \\ \vec{Y}(0) = \vec{Y}_0 \end{cases}$ in each of the following cases.

a. $\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, $\vec{Y}(0) = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ **b*.** $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$, $\vec{Y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Answers and Solutions

01a.

$$\vec{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \boldsymbol{Q}(t) = \begin{bmatrix} 1 & 2\,t+1 \\ t & \tan(t) \end{bmatrix}, \quad \vec{F}(t) = \begin{bmatrix} \dfrac{1}{t^2+1} \\ \cosh(t) \end{bmatrix}$$

01b.

$$\vec{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \boldsymbol{Q}(t) = \begin{bmatrix} 2 & t & -3 \\ -1 & \cos(t) & 0 \\ t & 0 & 4 \end{bmatrix}, \quad \vec{F}(t) = \begin{bmatrix} t \\ \sec(t) \\ \ln(t) \end{bmatrix}$$

02a. Setting $y_1 = y$ and $y_2 = y'$, it follows

$$y_1' = y_2$$

and

$$y_2' = y'' = \frac{1}{t}\Big(-(1-\mathrm{e}^t)\,y + 2\,y' + \sin(t) \Big) = \frac{\mathrm{e}^t - 1}{t}\,y_1 + \frac{2}{t}\,y_2 + \frac{\sin(t)}{t}$$

Hence the system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{\mathrm{e}^t - 1}{t} & \dfrac{2}{t} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\sin(t)}{t} \end{bmatrix}$$

02b. Setting $y_1 = y$, $y_2 = y'$, and $y_3 = y''$, it follows

$$y_1' = y_2 \qquad\qquad y_2' = y_3$$

and

$$y_3' = y''' = -y + \mathrm{e}^t\,y' + t\,y'' + \ln(t) = -y_1 + \mathrm{e}^t\,y_2 + t\,y_3 + \ln(t)$$

Hence the system

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \mathrm{e}^t & t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ln(t) \end{bmatrix}$$

03. In matrix form the system is $\vec{Y}\,' = \mathbf{Q}(t)\,\vec{Y} + \vec{F}(t)$, where $\mathbf{Q}(t) = \begin{bmatrix} t & 2 \\ 3 & -t/(t-1) \end{bmatrix}$, and $\vec{F}(t) = \begin{bmatrix} \ln(5-t) \\ \csc(t) \end{bmatrix}$. Based on their entries,

$\mathbf{Q}(t)$ is defined and continuous in $\big(-\infty\,,\,1\big) \cup \big(1\,,\,+\infty\big)$, while $\vec{F}(t)$ is defined and continuous in $\cdots\big(-4\,\pi\,,\,-3\,\pi\big) \cup \big(-3\,\pi\,,\,-2\,\pi\big) \cup \big(-2\pi\,,\,-\pi\big) \cup \big(-\pi\,,\,0\big) \cup \big(0\,,\,\pi\big) \cup \big(\pi\,,\,5\big)$

a. By the existence and uniqueness theorem, the IVP is guaranteed to have a unique solution defined on the largest open interval that contains $t_0 = -1$, where both $\mathbf{Q}(t)$, and $\vec{F}(t)$ are continuous. That's the interval $\big(-\pi\,,\,0\big)$

b. By the existence and uniqueness theorem, the IVP is guaranteed to have a unique solution defined on the largest open interval that contains $t_0 = 2$, where both $\mathbf{Q}(t)$, and $\vec{F}(t)$ are continuous. That's the interval $\big(1\,,\,\pi\big)$

c. By the existence and uniqueness theorem, the IVP is guaranteed to have a unique solution defined on the largest open interval that contains $t_0 = 4$, where both $\mathbf{Q}(t)$, and $\vec{F}(t)$ are continuous. That's the interval $\big(\pi\,,\,5\big)$

04. Rewrite the system in standard matrix form as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} \dfrac{\cos(t)}{(t+1)^2} & \dfrac{1}{(t+1)^2} \\ \dfrac{\cos(t)}{\sin(t)} & \dfrac{1}{\sin(t)} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \dfrac{2}{(t+1)^2} \\ \dfrac{1}{\cos(t)\,\sin(t)} \end{bmatrix}$$

The largest interval that contains $t = 1$, where the coefficient matrix and the right side are continuous is $\big(0\,,\,\pi/2\big)$

05. A fundamental set of solutions is $\left\{ \mathrm{e}^{-3\,t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \,,\, \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. It follows that the general solution is

$$\vec{Y}(t) = C_1\,\mathrm{e}^{-3\,t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\,\mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1\,\mathrm{e}^{-3\,t} + C_2\,\mathrm{e}^{-t} \\ -C_1\,\mathrm{e}^{-3\,t} + C_2\,\mathrm{e}^{-t} \end{bmatrix}$$

The solution of the initial value problem is obtained from the general solution by selecting the constants C_1, and C_2 so that

$$\vec{Y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \iff \begin{bmatrix} C_1 + C_2 \\ -C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \iff \begin{cases} C_1 + C_2 = 3 \\ -C_1 + C_2 = 1 \end{cases} \iff \begin{cases} C_1 = 1 \\ C_2 = 2 \end{cases}$$

Hence the solution $\qquad \vec{Y}(t) = \begin{bmatrix} e^{-3t} + 2e^{-t} \\ -e^{-3t} + 2e^{-t} \end{bmatrix}$

06a. The eigenvalues are solutions of $\det\big(\boldsymbol{A} - \lambda \boldsymbol{I}\big) = 0$. We have

$$\det\big(\boldsymbol{A} - \lambda \boldsymbol{I}\big) = \begin{vmatrix} 2-\lambda & -2 \\ -1 & 1-\lambda \end{vmatrix} = \lambda\,(\lambda - 3)$$

Hence the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 3$.

If $\vec{V} = \begin{bmatrix} r \\ s \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1 = 0$, then

$$\big(\boldsymbol{A} - \lambda_1 \boldsymbol{I}\big)\,\vec{V} = \vec{O} \iff \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $r - s = 0 \iff r = s$, and $\vec{V} = \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Selecting $s = 1$, we get the eigenvector $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If $\vec{V} = \begin{bmatrix} r \\ s \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2 = 3$, then

$$\big(\boldsymbol{A} - \lambda_2 \boldsymbol{I}\big)\,\vec{V} = \vec{O} \iff \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ -1 & -2 & 0 \end{array}\right] \xrightarrow{(-1)\,R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $r + 2s = 0 \iff r = -2s$, and $\vec{V} = \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ Selecting $s = -1$, we get the eigenvector $\vec{V}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ It follows that a fundamental set of solutions is

$$\left\{ e^{\lambda_1 t}\,\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ e^{\lambda_2 t}\,\vec{V}_2 = e^{3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

The general solution is then $\vec{Y}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\, e^{3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

To solve the initial value problem, we set

$$\vec{Y}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \iff C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\, e^0 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \iff \begin{cases} C_1 + 2C_2 = -1 \\ C_1 - C_2 = 1 \end{cases}$$

$$\iff \begin{cases} C_1 = 1/3 \\ C_2 = -2/3 \end{cases}$$

Hence the solution $\qquad \vec{Y}(t) = \dfrac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \dfrac{2}{3}\, e^{3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3} - \dfrac{4}{3}\, e^{3t} \\[2ex] \dfrac{1}{3} + \dfrac{2}{3}\, e^{3t} \end{bmatrix}$

06b. The eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 4$

Associated eigenvectors are $\vec{V}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},\ \vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

A fundamental set of solutions is $\left\{ \vec{Y}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad , \quad \vec{Y}_2(t) = e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Solution $\vec{Y}(t) = e^{2t} \begin{bmatrix} -3/2 \\ -9/2 \end{bmatrix} + e^{4t} \begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$

07. A fundamental set of solutions is $\left\{ e^{-2t} \begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix} \ , \ e^{-t} \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} \ , \ e^{2t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

Hence the general solution is

$$\vec{Y}(t) = C_1 e^{-2t} \begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} + C_3 e^{2t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4C_1 e^{-2t} + 3C_2 e^{-t} \\ -5C_1 e^{-2t} - 4C_2 e^{-t} + C_3 e^{2t} \\ -7C_1 e^{-2t} - 2C_2 e^{-t} - C_3 e^{2t} \end{bmatrix}$$

The solution of the initial value problem is obtained from the general solution by selecting the constants C_1, C_2, and C_3 so that

$$\vec{Y}(0) = \begin{bmatrix} 1 \\ -2 \\ 8 \end{bmatrix} \iff \begin{cases} 4C_1 + 3C_2 = 1 \\ -5C_1 - 4C_2 + C_3 = -2 \\ -7C_1 - 2C_2 - C_3 = 8 \end{cases} \iff \begin{cases} C_1 = -2 \\ C_2 = 3 \\ C_3 = 0 \end{cases}$$

Thus the solution of the IVP is $\vec{Y}(t) = \begin{bmatrix} -8e^{-2t} + 9e^{-t} \\ 10e^{-2t} - 12e^{-t} \\ 14e^{-2t} - 6e^{-t} \end{bmatrix}$

08. The eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 5$

Associated eigenvectors are $\vec{V}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ \vec{V}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},\ \vec{V}_3 = \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix}$

A fundamental set of solutions is $\left\{ e^{2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \ , \ e^{3t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \ , \ e^{5t} \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix} \right\}$

Solution $\vec{Y}(t) = -4e^{2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 11e^{3t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - 3e^{5t} \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix}$

09. A fundamental set of solutions is $\left\{ \vec{Y}_1(t) = \mathcal{R}e\left(e^{\lambda_1 t} \vec{V}_1\right) \ , \ \vec{Y}_2(t) = \mathcal{I}m\left(e^{\lambda_1 t} \vec{V}_1\right) \right\}$
with

$$\begin{aligned} e^{\lambda_1 t} \vec{V}_1 &= e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t} \begin{bmatrix} e^{it} \\ (2-i)e^{it} \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} \cos(t) + i \sin(t) \\ 2\cos(t) + \sin(t) + i\left(2\sin(t) - \cos(t)\right) \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + i\, e^{-t} \begin{bmatrix} \sin(t) \\ \left(2\sin(t) - \cos(t)\right) \end{bmatrix} \end{aligned}$$

Hence $\vec{Y}_1(t) = e^{-t} \begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix}$ and $\vec{Y}_2(t) = e^{-t} \begin{bmatrix} \sin(t) \\ \left(2\sin(t) - \cos(t)\right) \end{bmatrix}$

The general solution is then

$$\vec{Y}(t) = C_1 e^{-t} \begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix}$$

10a. The eigenvalues are solutions of $\det\big(A-\lambda I\big)=0$. We have

$$\det\big(A-\lambda I\big)=\begin{vmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}=(3-\lambda)(1-\lambda)+2=\big(\lambda-2\big)^2+1$$

Hence the eigenvalues $\lambda_1=2+i$ and $\lambda_2=2-i$.

If $\overrightarrow{V}=\begin{bmatrix} r \\ s \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1=2+i$, then

$$\big(A-\lambda_1 I\big)\overrightarrow{V}=\overrightarrow{O}\iff\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]\xrightarrow{(1+i)\,R_1}\left[\begin{array}{cc|c} 2 & 1+i & 0 \\ -2 & -1-i & 0 \end{array}\right]\xrightarrow{R_2+R_1}\left[\begin{array}{cc|c} 2 & 1+i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $2\,r+(1+i)\,s=0\iff r=-\dfrac{1+i}{2}\,s\implies\overrightarrow{V}=\begin{bmatrix} r \\ s \end{bmatrix}=\begin{bmatrix} -(1+i)\,s/2 \\ s \end{bmatrix}$ Selecting s so that $-\dfrac{(1+i)\,s}{2}=1\iff s=-\dfrac{2}{1+i}=-\dfrac{2\,(1-i)}{(1+i)\,(1-i)}=-\dfrac{2(1-i)}{2}=-1+i$, leads to the eigenvector $\overrightarrow{V}_1=\begin{bmatrix} 1 \\ -1+i \end{bmatrix}$ It follows that a fundamental set of solutions is

$$\left\{\mathcal{R}\mathrm{e}\left(\mathrm{e}^{\lambda_1 t}\,\overrightarrow{V}_1\right)=\mathcal{R}\mathrm{e}\left(\mathrm{e}^{(2+i)\,t}\begin{bmatrix} 1 \\ -1+i \end{bmatrix}\right)\ ,\ \mathcal{I}\mathrm{m}\left(\mathrm{e}^{\lambda_1 t}\,\overrightarrow{V}_1\right)=\mathcal{I}\mathrm{m}\left(\mathrm{e}^{(2+i)\,t}\begin{bmatrix} 1 \\ -1+i \end{bmatrix}\right)\right\}$$

But

$$\begin{aligned}\mathrm{e}^{(2+i)\,t}\begin{bmatrix} 1 \\ -1+i \end{bmatrix}&=\mathrm{e}^{2\,t}\big(\cos(t)+i\,\sin(t)\big)\begin{bmatrix} 1 \\ -1+i \end{bmatrix}=\mathrm{e}^{2\,t}\begin{bmatrix} \cos(t)+i\,\sin(t) \\ \big(\cos(t)+i\,\sin(t)\big)\,(-1+i) \end{bmatrix}\\ &=\mathrm{e}^{2\,t}\begin{bmatrix} \cos(t)+i\,\sin(t) \\ \big(-\cos(t)-\sin(t)\big)+i\,\big(\cos(t)-\sin(t)\big) \end{bmatrix}\\ &=\mathrm{e}^{2\,t}\begin{bmatrix} \cos(t) \\ -\cos(t)-\sin(t) \end{bmatrix}+i\,\mathrm{e}^{2\,t}\begin{bmatrix} \sin(t) \\ \cos(t)-\sin(t) \end{bmatrix}\end{aligned}$$

Hence the fundamental set of solutions is

$$\left\{\mathrm{e}^{2\,t}\begin{bmatrix} \cos(t) \\ -\cos(t)-\sin(t) \end{bmatrix}\ ,\ \mathrm{e}^{2\,t}\begin{bmatrix} \sin(t) \\ \cos(t)-\sin(t) \end{bmatrix}\right\}$$

and the general solution is

$$\overrightarrow{Y}(t)=C_1\,\mathrm{e}^{2\,t}\begin{bmatrix} \cos(t) \\ -\cos(t)-\sin(t) \end{bmatrix}+C_2\,\mathrm{e}^{2\,t}\begin{bmatrix} \sin(t) \\ \cos(t)-\sin(t) \end{bmatrix}$$

To solve the initial value problem, we set

$$\overrightarrow{Y}(0)=\begin{bmatrix} 8 \\ 6 \end{bmatrix}\iff C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}+C_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 8 \\ 6 \end{bmatrix}\iff\left\{\begin{array}{l} C_1=8 \\ -C_1+C_2=6 \end{array}\right.\iff\left\{\begin{array}{l} C_1=8 \\ C_2=14 \end{array}\right.$$

Hence the solution

$$\overrightarrow{Y}(t)=8\,\mathrm{e}^{2\,t}\begin{bmatrix} \cos(t) \\ -\cos(t)-\sin(t) \end{bmatrix}+14\,\mathrm{e}^{2\,t}\begin{bmatrix} \sin(t) \\ \cos(t)-\sin(t) \end{bmatrix}=\begin{bmatrix} \mathrm{e}^{2\,t}\,\big(8\,\cos(t)+14\,\sin(t)\big) \\ \mathrm{e}^{2\,t}\,\big(6\,\cos(t)-22\,\sin(t)\big) \end{bmatrix}$$

10b. The eigenvalues are $\lambda_1 = 2+3i,\ \ \lambda_2 = 2-3i$

Associated eigenvector is $\vec{V}_1 = \begin{bmatrix} 2 \\ -1+3i \end{bmatrix}$

A fundamental set of solutions is $\left\{ e^{2t} \begin{bmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{bmatrix},\ e^{2t} \begin{bmatrix} 2\sin(3t) \\ 3\cos(3t) - \sin(3t) \end{bmatrix} \right\}$

Solution $\vec{Y}(t) = 2e^{2t} \begin{bmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{bmatrix} + e^{2t} \begin{bmatrix} 2\sin(3t) \\ 3\cos(3t) - \sin(3t) \end{bmatrix}$