The Undetermined Coefficients Methods (UCM)

Consider the differential eq.

$$y'' + 3y' + 2y = f(t) \quad \cdots \quad (*)$$

The associated homogeneous eq. $y'' + 3y' + 2y = 0$

has c. Eq. $\lambda^2 + 3\lambda + 2 = 0$, roots $-2, -1$,

a fundamental set $\{ e^{-2t}, e^{-t} \}$ and a general sol.

$c_1 e^{-2t} + c_2 e^{-t}$.

The general solution of $(*)$ is

$$y(t) = c_1 e^{-2t} + c_2 e^{-t} + y_{\text{Part.}}(t)$$

Find $y_{\text{Part.}}(t)$ in the following cases:

Case 1 $f(t) = 5$. The equation becomes $y'' + 3y' + 2y = 5$.

we try $y_{\text{Part.}}(t) = A$, A constant. Substituting into the equation leads to $0 + 0 + 2A = 5 \Rightarrow A = 5/2$. Hence $y_{\text{Part.}}(t) = \frac{5}{2}$ and the general solution is $y(t) = c_1 e^{-2t} + c_2 e^{-t} + \frac{5}{2}$.

Case 2 $f(t) = t^2 + 2$. The equation becomes $y'' + 3y' + 2y = t^2 + 2$.

we try $y_{\text{Part.}}(t) = At^2 + Bt + C$. Substituting into the equation leads to

$$(2A) + 3(2At + B) + 2(At^2 + Bt + C) = t^2 + 2 \iff$$

$$2At^2 + (6A + 2B)t + (2A + 3B + 2C) = t^2 + 0.t + 2 \Rightarrow \begin{cases} 2A = 1 \\ 6A + 2B = 0 \\ 2A + 3B + 2C = 2 \end{cases}$$

solving the system, we get

$$A = 1/2$$
$$B = -3A = -3/2$$
$$C = (2 - 2A - 3B)/2 = (1 + 9/2)/2 = 11/4$$

Hence $y_{\text{Part.}}(t) = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{11}{4}$ and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{11}{4}$$

Case 3 $f(t) = 3e^{2t}$. The equation becomes $y'' + 3y' + 2y = 3e^{2t}$

We try $y_{Part.}(t) = Ae^{2t}$. Substitute into the equation to get

$$4Ae^{2t} + 3(2Ae^{2t}) + 2(Ae^{2t}) = 3e^{2t} \iff 12Ae^{2t} = 3e^{2t} \Rightarrow 12A = 3$$
$$\Rightarrow A = 1/4$$

Hence $y_{Part.}(t) = \frac{1}{4}e^{2t}$ and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{4}e^{2t}$$

Case 4 $f(t) = 2e^{-t}$. The equation becomes $y'' + 3y' + 2y = 2e^{-t}$

As in the previous, we try $y_{Part.}(t) = Ae^{-t}$. Substitute into the equation to get

$$Ae^{-t} + 3(-Ae^{-t}) + 2(Ae^{-t}) = 2e^{-t} \iff (A - 3A + 2A)e^{-t} = 2e^{-t} \iff 0 = 2e^{-t}$$

Question why is it that when we substituted Ae^{-t} into $y'' + 3y' + 2y = 2e^{-t}$ the left side vanished?

Answer Because Ae^{-t} is a solution of the associated homogeneous diff. equation $y''+3y'+2y=0$

Remedy Instead of looking for $y_{Part.}(t)$ in the form $y_{Part.}(t) = Ae^{-t}$, we look for it in the form $y_{Part.}(t) = t.(Ae^{-t})$

when we substitute this time, the left side won't vanish.

Let us try it.

$$(Ate^{-t})' = Ae^{-t} - Ate^{-t} = (A-At)e^{-t}$$

$$(Ate^{-t})'' = \left((A-At)e^{-t}\right)' = -Ae^{-t} - (A-At)e^{-t} = (-2A+At)e^{-t}$$

It follows $(-2A+At)e^{-t} + 3\left((A-At)e^{-t}\right) + 2(Ate^{-t}) = 2e^{-t} \iff$

$(-2A+At+3A-3At+2At)e^{-t} = 2e^{-t} \iff Ae^{-t} = 2e^{-t} \implies A = 2$

Hence $y_{Part.}(t) = 2te^{-t}$ and the general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{-t} + 2te^{-t}$$

Case 5 $\quad f(t) = t\cos(2t)$. The equation becomes $y'' + 3y' + 2y = t\cos(2t)$

We try $y_{Part.}(t) = (At+B)\cos(2t) + (Ct+D)\sin(2t)$. Differentiating

we get: $y'_{Part.}(t) = (2Ct + A + 2D)\cos(2t) + (-2At - 2B + C)\sin(2t)$

$$y''_{Part.}(t) = (-4At - 4B + 4C)\cos(2t) + (-4Ct - 4A - 4D)\sin(2t)$$

Substituting into the diff. eq. we get

$$(-4At - 4B + 4C)\cos(2t) + (-4Ct - 4A - 4D)\sin(2t)$$

$$+ \ (6Ct + 3A + 6D)\cos(2t) + (-6At - 6B + 3C)\sin(2t)$$

$$+ \ (2At + 2B)\cos(2t) + (2Ct + 2D)\sin(2t) = t\cos(2t)$$

$$\left[(-2A + 6C)t + (3A - 2B + 4C + 6D)\right]\cos(2t) + \left[(-6A - 2C)t + (-4A - 6B + 3C - 2D)\right]\sin(2t)$$

$$= t\cos(2t) + 0\sin(2t)$$

Hence the system
$$\begin{cases} -2A+6C=1 \\ -6A-2C=0 \\ 3A-2B+4C+6D=0 \\ -4A-6B+3C-2D=0 \end{cases}$$

Let us solve the system in A & C

$$\begin{cases} -2A+6C=1 \\ -6A-2C=0 \end{cases} \iff \begin{cases} -2A-18A=1 \\ C=-3A \end{cases} \iff \begin{cases} A=-1/20 \\ C=3/20 \end{cases}$$

The system in B & D becomes

$$\begin{cases} -2B+6D=-3A-4C \\ -6B-2D=4A-3C \end{cases} \iff \begin{cases} -2B+6D=-\frac{9}{20} \\ -6B-2D=-\frac{13}{20} \end{cases} \iff \begin{cases} B=3/25 \\ D=-7/200 \end{cases}$$

Hence $y_{\text{Part.}}(t) = \left(-\frac{1}{20}t+\frac{3}{25}\right)\cos(2t) + \left(\frac{3}{20}t-\frac{7}{200}\right)\sin(2t)$

General Case (UCM)

Consider $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$ ... (NH)

$a_n \neq 0, a_{n-1}, \ldots, a_1, a_0$ are real constants

Suppose

$$f(t) = e^{at}\left(P(t)\cos(bt) + Q(t)\sin(bt)\right)$$

a, b are real numbers and $P(t), Q(t)$ are polynomials of degree p, q

Then (NH) has a particular solution in the form

$$y_{Part.}(t) = t^k \cdot e^{at}\left(R(t)\cos(bt) + S(t)\sin(bt)\right)$$

where $R(t), S(t)$ are polynomials of degree $\max\{p, q\}$

and $k=0$, if $a+bi$ is not a root of the characteristic equation

and $k =$ multiplicity of $a+bi$, if it is a root of the C. Equation

Remarks

- Except for the t^k term, $f(t)$ and $y_{Part.}(t)$ have the same form with $y_{Part.}(t)$ depending on arbitrary constants to be determined.

- If $a=0$, $b=0$, $f(t) = P(t)$ and $y_{Part.}(t) = t^k \cdot R(t)$

- If $a\neq 0$, $b=0$, $f(t) = e^{at} P(t)$ and $y_{Part.}(t) = t^k e^{at} R(t)$

- If $a=0$, $b\neq 0$, $f(t) = P(t)\cos(bt) + Q(t)\sin(bt)$ and

$$y_{Part.}(t) = t^k \big(R(t)\cos(bt) + S(t)\sin(bt)\big)$$

- $y_{Part.}(t)$ doesn't have t^k if $a+bi$ is not a root of the characteristic equation.

Example Write down the form of the particular solution solution of $y'' + 2y' - 8 = f(t)$ in the following cases

- $f(t) = t^3$
- $f(t) = t\cos(2t)$
- $f(t) = te^{2t}\sin(3t)$
- $f(t) = (2t-1)e^{-4t}$

The characteristic equation of the associated homogeneous equation is $\lambda^2 + 2\lambda - 8 = 0 \iff (\lambda+4)(\lambda-2) = 0$ It has roots $\lambda_1 = -4$ and $\lambda_2 = 2$. Both simple.

Case $f(t) = t^3$

Comparing t^3 with $e^{at}(P(t)\cos(bt) + Q(t)\sin(bt))$, we conclude that $a+bi = 0+0i = 0$. This is not a root of the c. Eq. Hence $k=0$ and $y_{\text{Part.}}(t) = At^3 + Bt^2 + Ct + D$

Case $f(t) = t\cos(2t)$

comparing $t\cos(2t)$ to $e^{at}\left(P(t)\cos(bt) + Q(t)\sin(bt)\right)$, we conclude that $a+bi = 0+2i = 2i$. This is not a root of the C. Eq. Hence $k=0$, and

$$Y_{\text{Part.}}(t) = (At+B)\cos(2t) + (Ct+D)\sin(2t)$$

Case $f(t) = t\,e^{2t}\sin(3t)$

Comparing $t\,e^{2t}\sin(3t)$ to $e^{at}\left(P(t)\cos(bt) + Q(t)\sin(bt)\right)$, we conclude that $a+bi = 2+3i$. This is not a root of the C. Eq. Hence $k=0$, and

$$Y_{\text{Part.}}(t) = e^{2t}\left((At+B)\cos(3t) + (Ct+D)\sin(3t)\right)$$

Case $f(t) = (2t-1)e^{-4t}$

Comparing $(2t-1)e^{-4t}$ to $e^{at}\left(P(t)\cos(bt) + Q(t)\sin(bt)\right)$, we conclude that $a+bi = -4+0i = -4$. This is a root of the characteristic equation with multiplicity 1.

Hence $k=1$, and

$$Y_{\text{part.}}(t) = t^1 \cdot (At+B)e^{-4t}$$

Example Consider $y''' - 2y'' - 4y' + 8y = 2t+1 + te^{-2t} + (t^2+1)e^{2t}$

write down the form of the particular solution.

Solution

First, the characteristic Equation of the associated homogeneous equation is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0 \iff (\lambda+2)(\lambda-2)^2 = 0$

Hence the roots $\lambda_1 = -2$ (simple) and $\lambda_2 = 2$ (double)

- comparing $2t+1$ to $e^{at}(P(t)\cos(bt) + Q(t)\sin(bt))$, we conclude that $a+bi = 0+0i = 0$. This not a root of the C. Eq. Hence $k=0$, and the form of the particular solution associated with $2t+1$ is $y_{1,p}(t) = At+B$

- comparing te^{-2t} to $e^{at}(P(t)\cos(bt)+Q(t)\sin(bt))$, we conclude that $a+bi=-2+0.i=-2$. This is a root of the C. Eq. with multiplicity 1. Therefore $k=1$, and the form of the particular solution associated with te^{-2t} is

$$Y_{2,p}(t)=t^1.(Ct+D)e^{-2t}$$

- Comparing $(t^2+1)e^{2t}$ to $e^{at}(P(t)\cos(bt)+Q(t)\sin(bt))$, we see that $a+bi=2+0i=2$. This is a root of the C. Eq. with multiplicity 2. Hence $k=2$, and the form of the particular solution associated with $(t^2+1)e^{2t}$ is

$$Y_{3,p}(t)=t^2.(Et^2+Ft+G)e^{2t}$$

By the superposition principle, the form of the particular solution of

$$y''' - 2y'' - 4y' + 8y = 2t+1 + te^{-2t} + (t^2+1)e^{2t}$$

is

$$y_{part.}(t) = y_{1,p}(t) + y_{2,p}(t) + y_{3,p}(t)$$

$$= (At+B) + t(Ct+D)e^{-2t} + t^2(Et^2+Ft+G)e^{2t}$$