UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS & STATISTICS

FINAL EXAMINATION

Differential Equations for Engineers and Scientists – MATH 375
ALL SECTIONS (L01 - L04) – FALL 2015

VERSION 14

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Date: December 17, 2015 **Time: 2 hours**

SECT. (L01-L04)	INSTRUCTOR NAME	STUDENT I.D. NUMBER	FIRST NAME	LAST NAME

EXAMINATION RULES

1. This is a closed book examination, **the formula sheet is at the end of the exam**.
2. No aids are allowed for this examination.
3. Choose **and circle** a correct answer in the examination paper and fill the scantron sheet.
4. The scantron sheet must be filled out during the exam time period. No additional time will be granted to fill in the scantron form.
5. The use of personal electronic or communication devices is prohibited.
6. A University of Calgary Student ID card is required to write the Final Examination and could be requested for midterm examinations. If adequate ID isn't presented, the student must complete an Identification Form.
7. Students late in arriving will not be permitted after one-half hour of the examination time has passed.
8. No student will be permitted to leave the examination room during the first 30 minutes, nor during the last 15 minutes of the examination. Students must stop writing and hand in their exam immediately when time expires.
9. All inquiries and requests must be addressed to the exam supervisor.
10. Students are strictly cautioned against:
    (a) communicating to other students;
    (b) leaving answer papers exposed to view;
    (c) attempting to read other students' examination papers.
11. During the final examination, if a student becomes ill or receives word of domestic affliction, the student must report to the Invigilator, hand in the unfinished paper and request that it be cancelled. If ill, the student must report immediately to a physician/counselor for a medical note to support a deferred examination application.
12. Once the examination has been handed in for marking, a student cannot request that the examination be cancelled. Retroactive withdrawals from the course will be denied.
13. Failure to comply with these regulations will result in rejection of the examination paper.

1. The solution of the initial value problem $ty' + 4y = \frac{3}{t}$, $y(1) = 3$ is

A) $y = 2t^2 + t$

B) $y = \frac{2}{t^4} + \frac{1}{t}$

C) $y = \frac{1}{t^3} + \frac{2}{t}$

D) $y = \frac{2}{t^3} + \frac{1}{t}$

E) $y = \frac{1}{t^4} + \frac{2}{t}$

2. The solution of the initial value problem $\frac{dy}{dx} = -\frac{3x^2y^2 + 4x^3y^4}{2x^3y + 4x^4y^3}$, $y(1) = 1$ is

A) $y = 3x^3 - 2x^2$

B) $x^2y^2 + x^3y^4 + x^3y + x^4y^3 = 4$

C) $x + y = 2$

D) $3x^2y^2 + 4x^3y^4 + 2x^3y + 4x^4y^3 = 13$

E) $x^3y^2 + x^4y^4 = 2$

3. The largest open interval on which the unique solution of the initial value problem

$$(t^2 - 9)\, y'' + (\ln|t-1| + 4)\, y' + \frac{1}{t-7}\, y = 0, \quad y(2) = 5$$

is guaranteed to exist (according to the existence and uniqueness theorem) is
A) $(-3, \infty)$
B) $(1, 3)$
C) $(-3, 3)$
D) $(1, 7)$
E) $(-3, 5)$

4. A tank initially contains a solution with 80 kg of salt dissolved in 1000 litres of water. Pure water enters the tank at the rate of 6 litres/min. The solution is mixed and drains from the tank at the rate of 3 litres/min. Then the initial value problem describing the amount $Q(t)$ of salt in the tank at time t is

   A) $\dfrac{dQ}{dt} = -\dfrac{3Q}{1000+3t}, \quad Q(0) = 80$

   B) $\dfrac{dQ}{dt} = -\dfrac{3Q}{1000}, \quad Q(0) = 80$

   C) $\dfrac{dQ}{dt} = 80 - \dfrac{3Q}{1000-3t}, \quad Q(0) = 60$

   D) $\dfrac{dQ}{dt} = 6 - \dfrac{3Q}{1000}, \quad Q(0) = 80000$

   E) $\dfrac{dQ}{dt} = 6000 - 3000Q, \quad Q(0) = 80$

5. The first step of Euler's approximation for the solution of the initial value problem

   $\dfrac{dy}{dx} = 2\sin(x(y+1))$, $y(0) = 1$ with the step size $h = 0.1$ is

   A) $y(0.1) \approx 1 + 0.2\sin(0.1)$

   B) $y(0.1) \approx 1.1$

   C) $y(0.1) \approx 1$

   D) $y(0.1) \approx 1.2$

   E) $y(0.1) \approx 1 + 0.1\sin(0.2)$

6. A cake at 220° C is brought to a room at 20° C. If after 10 minutes the cake is 120° C, how long will it take to cool down from 120° C to 45° C?

   A) 45 minutes
   B) 30 minutes
   C) 20 minutes
   D) 10 minutes
   E) 7.5 minutes

7. The general solution of the equation $y^{(5)} + 2y^{(4)} + y^{(3)} = 0$ is

   A) $C_1e^t + C_2te^t + C_3e^{-2t} + C_4te^{-2t} + C_5e^{2t}$

   B) $C_1e^t + C_2te^t + C_3t^2e^t + C_4t^3e^t + C_5t^4e^t$

   C) $C_1e^t + C_2te^t + C_3 + C_4t + C_5t^2$

   D) $C_1e^{-t} + C_2te^{-t} + C_3 + C_4t + C_5t^2$

   E) $C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + C_4t^3e^{-t} + C_5t^4e^{-t}$

8. If $x(t) = 7\cos(3t) + \sin(t)$ is a solution of the fourth order differential equation $x^{(4)} + ax^{(3)} + bx'' + cx' + dx = 0$ then

   A) $a = 1, b = 3, c = 2, d = 6$

   B) $a = 0, b = 10, c = 0, d = 9$

   C) $a = 3, b = 1, c = 4, d = 9$

   D) $a = 0, b = -10, c = 0, d = -9$

   E) a, b, c, d cannot be found

9. According to Undetermined Coefficients method, a particular solution to the equation $y'' - 4y' + 4y = xe^{2x} + 3xe^{-2x} - 7$ should be sought in the form of

   A) $(Ax + B)e^{2x} + (Cx + D)e^{-2x} + E$

   B) $(Ax + B)e^{2x} + x^2(Cx + D)e^{-2x} + E$

   C) $Axe^{2x} + Bxe^{-2x} + C$

   D) $(Ax + B)e^{2x} + (Cx + D)e^{-2x} + Ex^2$

   E) $x^2(Ax + B)e^{2x} + (Cx + D)e^{-2x} + E$

10. A fundamental set of solutions for the equation $y^{(6)} + 9y^{(4)} = 0$ is

A) $\{1, t, t^2, t^3, \cos(3t), \sin(3t)\}$

B) $\{1, t, t^2, t^3, t^4, \cos(3t), \sin(3t)\}$

C) $\{1, \cos(3t), \sin(3t)\}$

D) $\{1, e^t \cos(3t), e^t \sin(3t)\}$

E) $\{1, t, t^2, t^3, e^{3t}, e^{-3t}\}$

11. The general solution of the equation $x^2y'' - 3xy' + 4y = 0, \quad x > 0$ is

A) $y = C_1x^2 + C_2x^2 \ln(x)$

B) $y = e^{\frac{3}{2}x} \left[C_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}x\right) \right]$

C) $y = C_1 + C_2e^{2x}$

D) $y = C_1e^{2x} + C_2xe^{2x}$

E) $y = x^2 \left[C_1 \cos\left(\frac{\sqrt{7}}{2}\ln(x)\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}\ln(x)\right) \right]$

12. If $y = C_1x + C_2x^2$ is the general solution of the homogeneous equation $x^2y'' + bxy' + cy = 0$ then a particular solution of the non-homogeneous equation $x^2y'' + bxy' + cy = x^3$ is

A) $\dfrac{x^4}{4}$

B) x^3

C) $\dfrac{x^5}{4} + \dfrac{x^4}{3}$

D) $\dfrac{x^3}{2}$

E) none of the above

13. The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 5$, $\lambda_2 = 1$ and $\lambda_3 = 0$ and eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ associated with $\lambda_1 = 5$ and $\lambda_2 = 1$, respectively.

Then the general solution of the system $X' = AX$ is

A) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^t \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

B) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^t \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

C) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^t \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

D) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^t \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

E) $C_1 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + C_2 e^t \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

14. A real 2×2 matrix A has an eigenvalue $\lambda_1 = 1 + i$ and an associated eigenvector $v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Then the general solution of the system $X' = AX$ is

A) $C_1 \begin{bmatrix} \cos(t) \\ -\cos(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}$

B) $C_1 e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}$

C) $C_1 e^t \begin{bmatrix} \cos(t) \\ -\cos(t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}$

D) $C_1 \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}$

E) unknown: insufficient data to find the general solution

15. The eigenvalues λ_n and eigenfunctions X_n of the Sturm-Liouville problem $X'' + \lambda X = 0$, $X'(0) = X'(\pi) = 0$ are
A) $\lambda_n = n$, $X_n = \sin(nx)$, $n = 1, 2, \ldots$ only
B) $\lambda_n = n^2$, $X_n = \sin(nx)$, $n = 1, 2, \ldots$ only
C) $\lambda_0 = 0$, $X_0 = 1$, $\lambda_n = \pi^2 n^2$, $X_n = \cos(n\pi x)$, $n = 1, 2, \ldots$ only
D) $\lambda_0 = 0$, $X_0 = 1$, $\lambda_n = n^2$, $X_n = \cos(nx)$, $n = 1, 2, \ldots$ only
E) none of the above

16. In the Fourier series $a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$ of period $2\ell = 4$ for the function

$$f(x) = \begin{cases} 3, & -2 < x \leq -1, \\ -3, & -1 < x \leq 1, \\ 3 & 1 < x < 2 \end{cases}$$

A) all $a_n \neq 0$, $b_n \neq 0$
B) all $b_n = 0$ but all $a_n \neq 0$, $n = 0, 1, 2, \ldots$
C) $a_0 = 0$, all $a_n \neq 0$, $b_n \neq 0$ for $n = 1, 2, \ldots$
D) all $a_n = 0$, $n = 0, 1, 2, \ldots$
E) $a_0 = 0$, $b_n = 0$, $n = 1, 2, \ldots$

17. The Fourier series $f(x) \sim a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)]$ of period $2\ell = 2$ for $f(x) = x^2 - x$, $x \in (-1, 1)$
A) at $x = -1$ converges to 2, at $x = 1$ converges to 0
B) at $x = -1$ converges to 0, at $x = 1$ converges to 2
C) at $x = -1$ converges to 1, at $x = 1$ converges to 1
D) at $x = -1$ converges to 0, at $x = 1$ converges to 0
E) at $x = -1$ converges to 2, at $x = 1$ converges to 2

18. The sine series of period 2π for $f(x) = x + 3$, $x \in [0, \pi]$ is

A) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n} \sin(nx)$

B) $\displaystyle 3 + \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$

C) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n} \sin(n\pi x)$

D) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{2}{n} - \frac{6}{\pi n}(-1)^n\right] \sin(nx)$

E) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n} \left[(-1)^{n+1} - \frac{3}{\pi}\left((-1)^n - 1\right)\right] \sin(nx)$

19. The solution $u(x,t)$ to the heat transfer (conduction) problem $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $0 < x < \pi$, $t > 0$, $u(0,t) = 0$, $u(\pi, t) = 0$, $t > 0$, $u(x, 0) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx)$, $0 < x < \pi$ is

A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx) e^{-n^6 t}$

B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx) e^{-n^2 t}$

C) $\displaystyle\sum_{n=1}^{\infty} \sin(nx) e^{-n^3 t}$

D) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(n\pi x) e^{-n^2 \pi^2 t}$

E) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx) \sinh(nt)$

20. A string of length L is secured at both ends. The string has no initial displacement, but has initial velocity $f(x)$ at any point x. This scenario is described by the partial differential equations and boundary and initial conditions

A) $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \;\; 0 < x < L, \; t > 0, \; u(0,t) = u(L,t) = 0, \; u(x,0) = 0, \;\; \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = f(x)$

B) $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \;\; 0 < x < L, \; t > 0, \;\; u(x,0) = u(x,L) = 0, \; u(x,0) = 0, \;\; \left.\dfrac{\partial u}{\partial x}\right|_{t=0} = f(x)$

C) $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \;\; 0 < x < L, \; t > 0, \; u(0,t) = u(L,t) = 0, \;\; \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = f(x)$

D) $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \;\; 0 < x < L, \; t > 0, \;\; u(x,0) = u(x,L) = 0, \;\; \left.\dfrac{\partial u}{\partial x}\right|_{t=0} = f(x)$

E) $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \;\; 0 < x < L, \; t > 0, \; u(x,L) = f(x), \; u(L,t) = 0, \;\; u(t,0) = 0$

21. The function $u(x,y) = a\sin(2\pi x)\sinh(by)$ is a solution of the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ satisfying $u(0,y) = 0, \; u(1,y) = 0, \; 0 \le y \le 2, \;\; u(x,0) = 0, \; u(x,2) = \sin(2\pi x), \;\; 0 \le x \le 1$ for

A) $a = 1$ and any $b \in \mathbb{R}$

B) $a = \dfrac{1}{\sinh(2\pi)}, \; b = \pi$

C) $a = \dfrac{1}{\sinh(4\pi)}, \; b = 2\pi$

D) $a = 1, \; b = 2$

E) $a = \dfrac{1}{\sinh(\pi)}, \; b = 2\pi$

22. Using the functions $u_c(t)$, the function $f(t) = \begin{cases} t, & 0 \le t < 1, \\ t^2 - 2, & 1 \le t < 3, \\ t - 2 & t \ge 3 \end{cases}$ can be written as

A) $f(t) = t + (t^2 - 2)u_1(t) + (t-2)u_3(t)$

B) $f(t) = t + (t^2 - 2 - t)u_1(t) - 2u_3(t)$

C) $f(t) = t + (t^2 - 2 - t)u_1(t) + (t - t^2)u_3(t)$

D) $f(t) = t - t^2 + 2 + (t^2 - t)u_1(t) - 2u_3(t)$

E) $f(t) = t + (t + 2 - t^2)u_1(t) - 2u_3(t)$

23. The Laplace Transform of the function $f(t) = \begin{cases} 1, & 0 \le t < 2, \\ 3t - 4, & t \ge 2 \end{cases}$ is

A) $\dfrac{1}{s} + \dfrac{3e^{-2s}}{s^2} + \dfrac{e^{-2s}}{s}$

B) $\dfrac{1}{s} + \dfrac{3e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s}$

C) $-\dfrac{3}{s} + \dfrac{3}{s^2}$

D) $\dfrac{3e^{-2s}}{s^2} - \dfrac{4e^{-2s}}{s}$

E) $\dfrac{1}{s} + \dfrac{3e^{-2s}}{s^2} - \dfrac{4e^{-2s}}{s}$

24. The inverse Laplace Transform of $F(s) = \dfrac{5 - s}{s^2 + 2s + 5}$ equals

A) $-\cos(t) + 5\sin(t)$

B) $-e^{2t}\cos(t) + 5e^{2t}\sin(t)$

C) $-e^{t}\cos(2t) + 2e^{t}\sin(2t)$

D) $-2e^{-t}\cos(2t) + 5e^{-t}\sin(2t)$

E) $-e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$

25. The inverse Laplace Transform of $F(s) = \dfrac{2e^{-2s}}{(s+1)(s+3)}$ is

A) $e^{-t} - e^{-3t}$

B) $u_2(t)\left(e^{-t} - e^{-3t}\right)$

C) $u_2(t)\left(\dfrac{1}{t^2} - \dfrac{1}{t^4}\right)$

D) $u_2(t)\left(e^{2-t} - e^{6-3t}\right)$

E) $u_2(t)\left(t - 2 + \dfrac{3}{t}\right)$

END OF THE TEST- THE FORMULA SHEET IS THE NEXT PAGE

TABLE OF LAPLACE TRANSFORM FORMULAS

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{1}{(n-1)!}t^{n-1}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \qquad \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$\mathcal{L}\left[\sin(at)\right] = \frac{a}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin(at)$$

$$\mathcal{L}\left[\cos(at)\right] = \frac{s}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$$

Differentiation and integration

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = s\mathcal{L}[f(t)] - f(0)$$

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

In the following formulas $F(s) = \mathcal{L}[f(t)]$, so $f(t) = \mathcal{L}^{-1}[F(s)]$.

$$\mathcal{L}\left[\int_0^t f(u)\ du\right] = \frac{1}{s}F(s) \qquad \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(u)\ du$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n\frac{d^n}{ds^n}(F(s)) \qquad \mathcal{L}^{-1}\left[\frac{d^nF(s)}{ds^n}\right] = (-1)^n t^n f(t)$$

Shift formulas

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a) \qquad \mathcal{L}^{-1}[F(s-a)] = e^{at}f(t)$$

$$\mathcal{L}\left[u_a(t)g(t)\right] = e^{-as}\mathcal{L}[g(t+a)] \qquad \mathcal{L}^{-1}\left[e^{-as}F(s)\right] = u_a(t)f(t-a)$$

Here $u_a(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$

Fourier Series

If $f(x)$ is defined on $(-\ell, \ell)$ then

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega x) + b_n\sin(n\omega x)\right], \quad \omega = \frac{\pi}{l},$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(n\omega x)\ dx,\ n = 0, 1, 2, \ldots, \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(n\omega x)\ dx, n = 1, 2, \ldots.$$