Math 375	**Fourier Series and Boundary Value Problems**	**Spring 2016**
Worksheet # 5	**Part 2**	**December 05-09**

The problems on this worksheet refer to material from sections §§10.2, 10.3, and, 10.4 of our text. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Heat Equation

01. Solve the initial-boundary value problems for the heat conduction in a wire.

a.

$$\begin{cases} 4\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0, \quad t > 0 \\ u(x,0) = x - x^2, \quad 0 < x < 1 \end{cases}$$

b.

$$\begin{cases} 2\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0, \quad t > 0 \\ u(x,0) = 2\,\sin(\pi\,x) - \dfrac{1}{3}\,\sin(3\,\pi\,x) \quad 0 < x < 1 \end{cases}$$

02. Solve the initial-boundary value problem for the heat conduction in a wire.

a.

$$\begin{cases} 2\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0, \quad t > 0 \\ u(x,0) = 4\,\sin(\pi\,x)\,\cos^3(\pi\,x), \quad 0 < x < 1 \end{cases}$$

b.

$$\begin{cases} 2\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < 3, \quad t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(3,t) = 0, \quad t > 0 \\ u(x,0) = 4\,\cos\left(\dfrac{2}{3}\pi\,x\right)\,\cos^3\left(\dfrac{4}{3}\pi\,x\right), \quad 0 < x < 3 \end{cases}$$

Wave Equation

03. Solve the initial-boundary value problem for the vibrating string

$$\begin{cases} 4\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = u(2,t) = 0, & t > 0 \\ u(x,0) = f(x) \quad \dfrac{\partial u}{\partial t}(x,0) = 0, 0 < x < 2 \end{cases}$$

where $f(x) = \begin{cases} x & \text{if } \; 0 \le x \le 1 \\ 2 - x & \text{if } \; 1 < x \le 2 \end{cases}$

04. Solve the initial-boundary value problem for the vibrating string

$$\begin{cases} 4\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = u(2,t) = 0, & t > 0 \\ u(x,0) = 0 \quad \dfrac{\partial u}{\partial t}(x,0) = 8\,\sin(3\,\pi\,x), 0 < x < 2 \end{cases}$$

Laplace Equation

05. Solve

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 \le y \le H \\ u(x,0) = 0 \quad u(x,H) = 0, \quad 0 < x < L \\ u(0,y) = y\,(H - y) \quad u(L,y) = 0, \quad 0 < y < H \end{cases}$$

The solution of the initial-boundary value problem for the heat conduction in a wire

$$\begin{cases} k\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0 \\ u(0,t) = u(1,t) = 0, \quad t > 0 \\ u(x,0) = f(x), \quad 0 < x < L \end{cases}$$

is $u(x,t) = \sum_{n=1}^{+\infty} b_n \sin\left(n\,\frac{\pi}{L}\,x\right) \mathrm{e}^{-\left(n\,\frac{\pi}{L}\right)^2 k\,t}$, with

$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(n\,\frac{\pi}{L}\,x\right)\,\mathrm{d}x$ A convenient way to remember what the coefficients b_n are, is to set $t = 0$ in the formula defining $u(x,t)$ to get $f(x) = \sum_{n=1}^{+\infty} b_n \sin\left(n\,\frac{\pi}{L}\,x\right)$. In other words, $\sum_{n=1}^{+\infty} b_n \sin\left(n\,\frac{\pi}{L}\,x\right)$ is the Fourier sine series of $f(x),\ 0 < x < L$.

Similarly, the solution of the initial-boundary value problem for the heat equation

$$\begin{cases} k\,\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0 \\ u_x(0,t) = u_x(1,t) = 0, \quad t > 0 \\ u(x,0) = f(x), \quad 0 < x < L \end{cases}$$

is $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\left(n\,\frac{\pi}{L}\,x\right) \mathrm{e}^{-\left(n\,\frac{\pi}{L}\right)^2 k\,t}$, with

$a_n = \frac{2}{L}\int_0^L f(x)\,\cos\left(n\,\frac{\pi}{L}\,x\right)\,\mathrm{d}x$

01a. Here $k = 4,\ L = 1,\ \frac{\pi}{L} = \pi$. It follows

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{+\infty} \frac{4\left(1-\cos(n\,\pi)\right)}{n^3\,\pi^3} \sin\left(n\,\pi\,x\right) \mathrm{e}^{-4\,\pi^2\,n^2\,t} \\ &= \sum_{n=1}^{+\infty} \frac{8}{(2\,n-1)^3\,\pi^3} \sin\left((2\,n-1)\,\pi\,x\right) \mathrm{e}^{-4\,\pi^2\,(2\,n-1)^2\,t} \end{aligned}$$

01b. Here $k = 2,\ L = 1,\ \frac{\pi}{L} = \pi$. It follows

$$u(x,t) = 2\,\sin(\pi\,x)\,\mathrm{e}^{-2\,\pi^2\,t} - \frac{1}{3}\,\sin(3\,\pi\,x)\,\mathrm{e}^{-18\,\pi^2\,t}$$

Notice that in this problem, there is no need to integrate to find the Fourier coefficients b_n. Indeed $u(x,0) = 2\,\sin(\pi\,x) - \frac{1}{3}\,\sin(3\,\pi\,x)$ leads to

$$\sum_{n=1}^{+\infty} b_n \sin\left(n\,\pi\,x\right) = 2\,\sin(\pi\,x) - \frac{1}{3}\,\sin(3\,\pi\,x) \iff \begin{cases} b_1 = 2 \\ b_3 = -1/3 \\ b_2 = b_4 = b_5 = \cdots = 0 \end{cases}$$

02a. First linearize $4\,\sin(\pi\,x)\,\cos^3(\pi\,x)$ as $\sin(2\,\pi\,x) + \frac{1}{2}\,\sin(4\,\pi\,x)$

The solution is

$$u(x,t) = \sin(2\,\pi\,x)\,\mathrm{e}^{-8\,\pi^2\,t} + \frac{1}{2}\,\sin(4\,\pi\,x)\,\mathrm{e}^{-32\,\pi^2\,t}$$

02b. Linearizing, we get

$$4\cos\left(\frac{2\pi}{3}x\right)\cos^3\left(\frac{4\pi}{3}x\right)=\frac{3}{2}\cos\left(2\frac{\pi}{3}x\right)+\frac{3}{2}\cos\left(6\frac{\pi}{3}x\right)+\frac{1}{2}\cos\left(10\frac{\pi}{3}x\right)+\frac{1}{2}\cos\left(14\frac{\pi}{3}x\right)$$

The solution is

$$u(x,t)=\frac{3}{2}\cos\left(2\frac{\pi}{3}x\right)\mathrm{e}^{-2\left(2\pi/3\right)^2t}+\frac{3}{2}\cos\left(6\frac{\pi}{3}x\right)\mathrm{e}^{-2\left(6\pi/3\right)^2t}+\frac{3}{2}\cos\left(10\frac{\pi}{3}x\right)\mathrm{e}^{-2\left(10\pi/3\right)^2t}$$
$$+\frac{3}{2}\cos\left(14\frac{\pi}{3}x\right)\mathrm{e}^{-2\left(14\pi/3\right)^2t}$$

The solution of the initial-boundary value problem for the wave equation of a vibrating string

$$\begin{cases} c^2\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}, \quad 0<x<L,\quad t>0\\ u(0,t)=u(L,t)=0,\quad t>0\\ u(x,0)=f(x),\quad \dfrac{\partial u}{\partial t}(x,0)=g(x),\quad 0<x<L\end{cases}$$

is

$$u(x,t)=\sum_{n=1}^{+\infty}\sin\left(n\frac{\pi}{L}x\right)\left(b_n\cos\left(n\frac{\pi}{L}ct\right)+b_n^*\sin\left(n\frac{\pi}{L}ct\right)\right)$$

where

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)\mathrm{d}x \qquad \text{and} \qquad b_n^*=\frac{2}{n\pi c}\int_0^L g(x)\sin\left(n\frac{\pi}{L}x\right)\mathrm{d}x$$

A convenient way to remember what the coefficients b_n and b_n^* are, is to set $t=0$ in the formulas defining $u(x,t)$ and $\dfrac{\partial u}{\partial t}(x,t)$ to get

$$f(x)=\sum_{n=1}^{+\infty}b_n\sin\left(n\frac{\pi}{L}x\right) \qquad \text{and} \qquad g(x)=\sum_{n=1}^{+\infty}\frac{n\pi c}{L}b_n^*\cos\left(n\frac{\pi}{L}x\right)$$

which suggests that,

$$\sum_{n=1}^{+\infty}b_n\sin\left(n\frac{\pi}{L}x\right) \quad \text{is the \underline{Fourier sine series} of } f(x),\ 0<x<L$$

and

$$\sum_{n=1}^{+\infty}\frac{n\pi c}{L}b_n^*\cos\left(n\frac{\pi}{L}x\right) \quad \text{is the \underline{Fourier cosine series} of } g(x),\ 0<x<L$$

03. Here $c = 2,\ L = 2,\ \dfrac{\pi}{L} = \dfrac{\pi}{2},\ \dfrac{\pi}{L}\, c = \pi$. It follows

$$u(x,t) = \sum_{n=1}^{+\infty} \sin\left(n\, \frac{\pi}{2}\, x\right) \left(b_n\, \cos(n\, \pi\, t) + b_n^*\, \sin(n\, \pi\, t) \right)$$

Setting $t = 0$, leads to

$$u(x,0) = \sum_{n=1}^{+\infty} b_n\, \sin\left(n\, \frac{\pi}{2}\, x\right) \iff f(x) = \sum_{n=1}^{+\infty} b_n\, \sin\left(n\, \frac{\pi}{2}\, x\right)$$

Hence

$$\begin{aligned} b_n &= \frac{2}{2} \int_0^2 f(x)\, \sin\left(n\, \frac{\pi}{2}\, x\right)\, \mathrm{d}x = \int_0^1 x\, \sin\left(n\, \frac{\pi}{2}\, x\right)\, \mathrm{d}x + \int_1^2 (2-x)\, \sin\left(n\, \frac{\pi}{2}\, x\right)\, \mathrm{d}x \\ &= \frac{8\, \sin(n\, \pi/2)}{n^2\, \pi^2} \end{aligned}$$

Differentiating with respect to t and setting $t = 0$, leads to

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{+\infty} n\, \pi\, b_n^*\, \sin\left(n\, \frac{\pi}{2}\, x\right) \iff 0 = \sum_{n=1}^{+\infty} n\, \pi\, b_n^*\, \sin\left(n\, \frac{\pi}{2}\, x\right) \implies b_n^* = 0,\ n = 1, 2, \cdots$$

Hence the solution is

$$u(x,t) = \sum_{n=1}^{+\infty} \frac{8\, \sin(n\, \pi/2)}{n^2\, \pi^2}\, \sin\left(n\, \frac{\pi}{2}\, x\right) \cos\left(n\, \pi\, t\right)$$

04. Here $c = 2,\ L = 2,\ \dfrac{\pi}{L} = \dfrac{\pi}{2},\ \dfrac{\pi}{L}\, c = \pi$. It follows

$$u(x,t) = \sum_{n=1}^{+\infty} \sin\left(n\, \frac{\pi}{2}\, x\right) \left(b_n\, \cos(n\, \pi\, t) + b_n^*\, \sin(n\, \pi\, t) \right)$$

Setting $t = 0$, leads to

$$u(x,0) = \sum_{n=1}^{+\infty} b_n\, \sin\left(n\, \frac{\pi}{2}\, x\right) \iff 0 = \sum_{n=1}^{+\infty} b_n\, \sin\left(n\, \frac{\pi}{2}\, x\right) \implies b_n = 0,\ n = 1, 2, \cdots$$

Differentiating with respect to t and setting $t = 0$, leads to

$$\begin{aligned} \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{+\infty} n\, \pi\, b_n^*\, \sin\left(n\, \frac{\pi}{2}\, x\right) &\iff 8\, \sin(3\, \pi\, x) = \sum_{n=1}^{+\infty} n\, \pi\, b_n^*\, \sin\left(n\, \frac{\pi}{2}\, x\right) \\ &\implies \begin{cases} b_6 = 8 \\ b_n^* = 0,\ n \neq 6 \end{cases} \end{aligned}$$

Hence the solution

$$u(x,t) = \frac{4}{3\, \pi}\, \sin(3\, \pi\, x)\, \sin(6\, \pi\, t)$$

The solution of the boundary value problem for Laplace equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < H \\ u(x,0) = 0, \quad u(x,H) = 0, \quad 0 < x < L \\ u(0,y) = g(y), \quad u(L,y) = 0, \quad 0 < y < H \end{cases}$$

is

$$u(x,y) = \sum_{n=1}^{+\infty} c_n \ \sinh\left(n\,\frac{\pi}{H}\,(L-x)\right)\ \sin\left(n\,\frac{\pi}{H}\,y\right)$$

where the coefficients c_n are selected so that

$$u(0,y) = g(y) \iff \sum_{n=1}^{+\infty} c_n \ \sinh\left(n\,\frac{\pi}{H}\,L\right)\ \sin\left(n\,\frac{\pi}{H}\,y\right) = g(y)$$

Hence $c_n \ \sinh\left(n\,\frac{\pi}{H}\,L\right)$ are the coefficients of the Fourier sine series of $g(y)$.

$$c_n \ \sinh\left(n\,\frac{\pi}{H}\,L\right) = \frac{2}{H}\int_0^H g(y)\,\sin\left(n\frac{\pi}{H}y\right)\,\mathrm{d}y$$

05. The solution is given by

$$u(x,y) = \sum_{n=1}^{+\infty} c_n \sinh\left(n\frac{\pi}{H}(L-x)\right)\ \sin\left(n\frac{\pi}{H}y\right)$$

Setting $x=0$ and equating the result to $y\,(H-y)$, leads to

$$u(0,y) = y\,(H-y) \iff \sum_{n=1}^{+\infty} c_n \sinh\left(n\frac{\pi}{H}L\right)\ \sin\left(n\frac{\pi}{H}y\right) = y\,(H-y)$$

It follows that

$$c_n \ \sinh\left(n\frac{\pi}{H}L\right) = \frac{2}{H}\int_0^H y\,(H-y)\,\sin\left(n\frac{\pi}{H}\,y\right)\,\mathrm{d}y = \frac{4\,H^2}{n^3\,\pi^3}\,(1-\cos(n\,\pi))\ \frac{1}{\sinh\left(n\frac{\pi}{H}\,L\right)}$$

Hence

$$u(x,y) = \frac{4\,H^2}{\pi^3}\sum_{n=1}^{+\infty}\frac{1-\cos(n\,\pi)}{n^3}\ \frac{\sinh\left(n\frac{\pi}{H}(L-x)\right)}{\sinh\left(n\frac{\pi}{H}L\right)}\ \sin\left(n\frac{\pi}{H}y\right)$$