Math 375		**Fall 2016**
	Higher Order Linear Differential Equations	
Worksheet # 2	**Part 3**	**October 24 - 28**

The problems on this worksheet refer to material from sections §4.2, and §3.5 of your text. Solutions to all problems are included. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Higher Order Homogeneous With Constant Coefficients

01. Find the general solution of the following differential equations

a*. $y''' + 3\,y'' - 16\,y' - 48\,y = 0$ **b*.** $y''' - 5\,y'' + 4\,y' - 20\,y = 0$

c*. $y^{(4)} - 4\,y''' + 14\,y'' - 20\,y' + 25\,y = 0$ **d*.** $y^{(4)} - 8\,y'' + 16\,y = 0$

02. Find a fundamental set of solutions for the homogeneous linear differential equations

a. $y''' + y'' + y' + y = 0$ **b.** $y''' - y'' - y' + y = 0$

c. $y^{(4)} + 6\,y'' + 9\,y = 0$ **d.** $y''' + 6\,y'' + 12\,y' + 8\,y = 0$

03. Write down, in normal form, the linear homogeneous constant coefficients differential equation associated with the given fundamental solution set.

a. $\{\,1\,,\ t\,,\ e^{2\,t}\,\}$ **b.** $\{\,e^t\,,\ e^t\cos(t)\,,\ e^t\sin(t)\,\}$

c. $\{\,\cos(t)\,,\ \sin(t)\,,\ t\cos(t)\,,\ t\sin(t)\,\}$ **d.** $\{\,\cosh(t)\,,\ \sinh(t)\,\}$

04. Solve the following initial value problems

a*. $\begin{cases} y''' + y'' - 9\,y' - 9\,y = 0 \\ y(0) = y'(0) = y''(0) = -3 \end{cases}$ **b.** $\begin{cases} y^{(4)} + 8\,y'' + 16\,y = 0 \\ y(0) = 3,\ y'(0) = 2 \\ y''(0) = 1,\ y'''(0) = 2 \end{cases}$

05. A sixth order linear homogeneous differential equation with constant coefficients has $t\,\big(2 + 3\,\cos(3\,t)\big)$ as a solution. Find the normal form of the equation.

The Undetermined Coefficients Method

06*. Suppose the method of undetermined coefficients is used to find a particular solution $y_p(t)$. Write down the form of $y_p(t)$ in each of the following cases. Do not compute $y_p(t)$.

a. $4\,y'' - 3\,y' = t\,e^{3\,t/4}$ **b.** $y'' + y = \sin(t) - \cos(t)$

c. $y'' - 3\,y' + 2\,y = \big(t + e^t\big)^2$ **d.** $y'' + 16\,y = \sin\big(4\,t + \pi/3\big)$

07. Find the general solution of the differential equations.

a*. $y'' - 4\,y' + 4\,y = t^2$ **b.** $y'' + 4\,y' + 4\,y = 8\,e^{-2\,t}$

c*. $y'' + y = 4\,t\,\cos(t)$ **d.** $y'' - 3\,y' + 2\,y = \big(t^2 + t\big)\,e^{3\,t}$

08. Solve the following initial value problems.

a. $\begin{cases} y'' + y' = e^{-t} \\ y(0) = 1, \ \ y'(0) = -1 \end{cases}$ **b.** $\begin{cases} y^{(4)} - y = 8\,e^{t} \\ y(0) = -1, \ \ y'(0) = y'''(0) = 0, \ \ y''(0) = 1 \end{cases}$

Answers and Solutions

01a. The characteristic equation is $\lambda^3 + 3\lambda^2 - 16\lambda - 48 = 0$. Rewriting it as

$$\lambda^2\,(\lambda + 3) - 16\,(\lambda + 3) = 0 \iff (\lambda^2 - 16)\,(\lambda + 3) = 0$$

we conclude that the roots are $\lambda_1 = -4$, $\lambda_2 = -3$, and $\lambda_3 = 4$.
Hence a fundamental set of solutions is $\left\{ e^{-4t}, \ e^{-3t}, \ e^{4t} \right\}$, and the general solution is

$$y(t) = C_1\,e^{-4t} + C_2\,e^{-3t} + C_3\,e^{4t}$$

01b. The characteristic equation is $\lambda^3 - 5\lambda^2 + 4\lambda - 20 = 0$. Rewriting it as

$$\lambda^2\,(\lambda - 5) + 4\,(\lambda - 5) = 0 \iff (\lambda^2 + 4)\,(\lambda - 5) = 0$$

we conclude that the roots are $\lambda_1 = 5$, $\lambda_2 = 2i$, and $\lambda_3 = -2i$
Hence a fundamental set of solutions is $\left\{ e^{5t}, \ \cos(2t), \ \sin(2t) \right\}$, and the general solution is

$$y(t) = C_1\,e^{5t} + C_2\,\cos(2t) + C_3\,\sin(2t)$$

01c. The characteristic equation is $\lambda^4 - 4\lambda^3 + 14\lambda^2 - 20\lambda + 25 = 0$. Rewriting it as

$$(\lambda^2 + a\lambda + 5)\,(\lambda^2 + b\lambda + 5) = 0 \iff \lambda^4 + (a+b)\,\lambda^3 + (a\,b + 10)\,\lambda^2 + 5\,(a+b)\,\lambda + 25 = 0$$

leads to $a = b = -2$. Thus the characteristic equation takes the form $(\lambda^2 - 2\lambda + 5)^2 = 0$. It follows that the roots are $\lambda_1 = \lambda_2 = 1 + 2i$, $\lambda_3 = \lambda_4 = 1 - 2i$. Hence a fundamental set of solutions is

$$\left\{ e^{t}\cos(2t), \ e^{t}\sin(2t), \ t\,e^{t}\cos(2t), \ t\,e^{t}\sin(2t) \right\}$$

and the general solution is

$$y(t) = C_1\,e^{t}\cos(2t) + C_2\,e^{t}\sin(2t) + C_3\,t\,e^{t}\cos(2t) + C_3\,t\,e^{t}\sin(2t)$$

01d. The characteristic equation is $\lambda^4 - 8\lambda^2 + 16 = 0$. Rewriting it as

$$\lambda^4 - 8\lambda^2 + 16 = 0 \iff (\lambda^2 - 4)^2 = 0 \iff (\lambda + 2)^2\,(\lambda - 2)^2 = 0$$

leads to the roots $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = \lambda_4 = 2$. Hence a fundamental set of solutions is

$$\left\{ e^{-2t}, \ t\,e^{-2t}, \ e^{2t}, \ t\,e^{2t} \right\}$$

and the general solution is

$$y(t) = C_1\,e^{-2t} + C_2\,t\,e^{-2t} + C_3\,e^{2t} + C_4\,t\,e^{2t}$$

02a. The characteristic equation is $\lambda^3 + \lambda^2 + \lambda + 1 = 0$. Rewriting it as

$$\lambda^2\,(\lambda + 1) + (\lambda + 1) = 0 \iff (\lambda^2 + 1)\,(\lambda + 1) = 0$$

leads to the roots $\lambda_1 = -1$, $\lambda_2 = i$, and $\lambda_3 = -i$ Hence a fundamental set of solutions is

$$\left\{ e^{-t}, \ \cos(t), \ \sin(t) \right\}$$

02b. The characteristic equation is $\lambda^3 - \lambda^2 - \lambda + 1 = 0$. Rewriting it as

$$\lambda^2\left(\lambda - 1\right) - \left(\lambda - 1\right) = 0 \iff \left(\lambda^2 - 1\right)\left(\lambda - 1\right) = 0 \iff \left(\lambda + 1\right)\left(\lambda - 1\right)^2 = 0$$

leads to the roots $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 1$ Hence a fundamental set of solutions is

$$\left\{ e^{-t} , e^{t} , t\, e^{t} \right\}$$

02c. The characteristic equation is $\lambda^4 + 6\lambda^2 + 9 = 0$. Rewriting it as $\left(\lambda^2 + 3\right)^2 = 0$, leads to the roots $\lambda_1 = \lambda_2 = \sqrt{3}\, i$, and $\lambda_3 = \lambda_4 = -\sqrt{3}\, i$. Hence a fundamental set of solutions is

$$\left\{ \cos\left(\sqrt{3}\, t\right) , \sin\left(\sqrt{3}\, t\right) , t \cos\left(\sqrt{3}\, t\right) , t \sin\left(\sqrt{3}\, t\right) \right\}$$

02d. The characteristic equation is $\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0 \iff \left(\lambda + 2\right)^3 = 0$. Hence the roots $\lambda_1 = \lambda_2 = \lambda_3 = -2$, and a fundamental set of solutions is

$$\left\{ e^{-2t} , t\, e^{-2t} , t^2 e^{-2t} \right\}$$

03a. The solutions $y_1(t) = 1 = 1\, e^{0\,t}$, $y_2(t) = t = t\, e^{0\,t}$, $y_3(t) = e^{2\,t}$, show that the roots of the characteristic equation are $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = 2$. Consequently the characteristic equation is

$$\left(\lambda - 0\right)^2\left(\lambda - 2\right) = 0 \iff \lambda^2\left(\lambda - 2\right) = 0 \iff \lambda^3 - 2\lambda^2 = 0$$

Hence the differential equation (in normal form), is

$$y''' - 2\, y'' = 0$$

03b. The fact that $y_1(t) = e^{t}$, $y_2(t) = e^{t} \cos(t)$, and $y_3(t) = e^{t} \sin(t)$, are solutions shows that the roots of the characteristic equation are $\lambda_1 = 1$, $\lambda_2 = 1 + i$, and $\lambda_3 = 1 - i$. Consequently the characteristic equation is

$$\left(\lambda - 1\right)\left(\lambda - (1+i)\right)\left(\lambda - (1-i)\right) = 0 \iff \left(\lambda - 1\right)\left(\lambda^2 - 2\lambda + 2\right) = 0 \iff \lambda^3 - 3\lambda^2 + 4\lambda - 2 = 0$$

Hence the differential equation (in normal form) is

$$y''' - 3\, y'' + 4\, y' - 2\, y = 0$$

03c. $y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$, $y_3(t) = t \cos(t)$, and $y_4(t) = t \sin(t)$, being solutions implies that the roots of the characteristic equation are $\lambda_1 = \lambda_2 = i$, and $\lambda_3 = \lambda_4 = -i$. Consequently the characteristic equation is

$$\left(\lambda - i\right)^2\left(\lambda + i\right)^2 = 0 \iff \Big(\left(\lambda - i\right)\left(\lambda + i\right)\Big)^2 = 0 \iff \left(\lambda^2 + 1\right)^2 = 0 \iff \lambda^4 + 2\lambda^2 + 1 = 0$$

Hence the differential equation (in normal form) is

$$y^{(4)} + 2\, y'' + y = 0$$

03d. Recalling that $\cosh(t) = \dfrac{1}{2} e^{t} + \dfrac{1}{2} e^{-t}$, and $\sinh(t) = \dfrac{1}{2} e^{t} - \dfrac{1}{2} e^{-t}$, it follows from the superposition principle for homogeneous equations, that $y_1(t) = e^{-t} = \cosh(t) - \sinh(t)$, and $y_2(t) = e^{t} = \cosh(t) + \sinh(t)$ are solutions as well. This shows that the roots of the characteristic equation are $\lambda_1 = -1$, and $\lambda_2 = 1$. Consequently the characteristic equation is

$$\left(\lambda - (-1)\right)\left(\lambda - 1\right) = 0 \iff \left(\lambda + 1\right)\left(\lambda - 1\right) = 0 \iff \lambda^2 - 1 = 0$$

Hence the differential equation (in normal form) is

$$y'' - y = 0$$

04a. The characteristic equation is

$$\lambda^3+\lambda^2-9\,\lambda-9=0 \iff \lambda^2\big(\lambda+1\big)-9\,\big(\lambda+1\big)=0 \iff \big(\lambda^2-9\big)\,\big(\lambda+1\big)=0$$

Its roots are $\lambda_1=-3$, $\lambda_2=-1$, and $\lambda_3=3$. Hence the general solution is

$$y(t)=C_1\,\mathrm{e}^{-3\,t}+C_2\,\mathrm{e}^{-t}+C_3\,\mathrm{e}^{3\,t}$$

To solve the initial value problem, the arbitrary constants C_1, C_2, C_3, should be selected so that the initial conditions are satisfied. From $y'(t)=-3\,C_1\,\mathrm{e}^{-3\,t}-C_2\,\mathrm{e}^{-t}+3\,C_3\,\mathrm{e}^{3\,t}$ and $y''(t)=9\,C_1\,\mathrm{e}^{-3\,t}+C_2\,\mathrm{e}^{-t}+9\,C_3\,\mathrm{e}^{3\,t}$, it follows

$$\begin{cases} y(0)=-3 \\ y'(0)=-3 \\ y''(0)=-3 \end{cases} \iff \begin{cases} C_1+C_2+C_3=-3 \\ -3\,C_1-C_2+3\,C_3=-3 \\ 9\,C_1+C_2+9\,C_3=-3 \end{cases}$$

Solving by Gauss elimination or by Cramer's rule, we find $C_1=1$, $C_2=-3$, and $C_3=-1$. Hence the solution of the initial value problem is

$$y(t)=\mathrm{e}^{-3\,t}-3\,\mathrm{e}^{-t}-\mathrm{e}^{3\,t}$$

04b. The characteristic equation is $\lambda^4+8\,\lambda^2+16=0 \iff \big(\lambda^2+4\big)^2=0$. Hence the roots $\lambda_1=\lambda_2=2\,i$, and $\lambda_3=\lambda_4=-2\,i$, and the general solution is

$$y(t)=\big(C_1+C_2\,t\big)\,\cos(2\,t)+\big(C_3+C_4\,t\big)\,\sin(2\,t)$$

To solve the initial value problem, we select the arbitrary constants C_1, C_2, C_3, C_4, so that the initial conditions are satisfied. From

$$y'(t)=\big(C_2+2\,C_3+2\,C_4\,t\big)\,\cos(2\,t)+\big(-2\,C_1+C_4-2\,C_2\,t\big)\,\sin(2\,t)$$
$$y''(t)=\big(-4\,C_1+4\,C_4-4\,C_2\,t\big)\,\cos(2\,t)+\big(-4\,C_2-4\,C_3-4\,C_4\,t\big)\,\sin(2\,t)$$
$$y'''(t)=\big(-12\,C_2-8\,C_3-8\,C_4\,t\big)\,\cos(2\,t)+\big(8\,C_1-12\,C_4+8\,C_2\,t\big)\,\sin(2\,t)$$

it follows

$$\begin{cases} y(0)=3 \\ y'(0)=2 \\ y''(0)=1 \\ y'''(0)=2 \end{cases} \iff \begin{cases} C_1=3 \\ C_2+2\,C_3=2 \\ -4\,C_1+4\,C_4=1 \\ -12\,C_2-8\,C_3=2 \end{cases} \iff \begin{cases} C_1=3 \\ C_2=-5/4 \\ C_3=13/8 \\ C_4=13/4 \end{cases}$$

The solution of the initial value problem is then

$$y(t)=\left(3-\frac{5}{4}t\right)\,\cos(2\,t)+\left(\frac{13}{8}+\frac{13}{4}t\right)\,\sin(2\,t)$$

05. The fact that $t\,\big(2+3\,\cos(3\,t)\big)=2\,t\,\mathrm{e}^{0\,t}+3\,t\,\mathrm{e}^{0\,t}\,\cos(3\,t)$, is a solution of the differential equation, implies that $\lambda=0$ is a double root, and $\lambda=0+3\,i=3\,i$ is a double root as well, which in turn implies that $\lambda=0-3\,i=-3\,i$ is a double root as well. Hence the characteristic equation is

$$\begin{aligned}\big(\lambda-0\big)^2\,\big(\lambda-3\,i\big)^2\,\big(\lambda-(-3\,i)\big)^2=0 &\iff \lambda^2\,\Big(\big(\lambda-3\,i\big)\,\big(\lambda+3\,i\big)\Big)^2=0 \\ &\iff \lambda^2\,\big(\lambda^2+9\big)=0 \iff \lambda^6+18\,\lambda^4+81\,\lambda^2=0\end{aligned}$$

and the differential equation in normal form is

$$y^{(6)}+18\,y^{(4)}+81\,y''=0$$

06a. The characteristic equation of the homogeneous equation is $4\lambda^2 - 3\lambda = 0$. It has roots $\lambda_1 = 0$ and $\lambda_2 = \dfrac{3}{4}$. The right hand side $f(t) = t\,\mathrm{e}^{(3/4)\,t}$ has the exponential-polynomial-cosine-sine form $\mathrm{e}^{\alpha\,t}\Big(M(t)\,\cos(\beta\,t) + N(t)\,\sin(\beta\,t)\Big)$, with $\alpha = \dfrac{3}{4}$ and $\beta = 0$. It follows that the Undetermined Coefficients Method (UCM), is applicable, and we look for a particular solution in the form

$$y_P(t) = t^k\,\mathrm{e}^{(3/4)\,t}\Big(A\,t + B\Big)$$

where k is the multiplicity of $\alpha + \beta\,i$ as a root of the characteristic equation. Here

$$\alpha + \beta\,i = \frac{3}{4} + 0\,i = \frac{3}{4}$$

is clearly a root of the characteristic equation, with multiplicity 1. Hence $k = 1$, and

$$y_P(t) = t\,\Big(A\,t + B\Big)\,\mathrm{e}^{(3/4)\,t}$$

Finding the constants is not part of the problem, but if you substitute and solve, you will get

$$y_p(t) = \left(\frac{1}{6}\,t^2 - \frac{4}{9}\,t\right)\,\mathrm{e}^{3\,t/4}$$

06b. The characteristic equation of the homogeneous equation is $\lambda^2 + 1 = 0$. It has roots $\lambda_1 = i$ and $\lambda_2 = -i$. The right hand side $f(t) = \sin(t) - \cos(t)$ has the exponential-polynomial-cosine-sine form $\mathrm{e}^{\alpha\,t}\Big(M(t)\,\cos(\beta\,t) + N(t)\,\sin(\beta\,t)\Big)$, with $\alpha = 0$ and $\beta = 1$. It follows that the Undetermined Coefficients Method (UCM), is applicable, and we look for a particular solution in the form

$$y_P(t) = t^k\,\Big(A\,\cos(t) + B\,\sin(t)\Big)$$

where k is the multiplicity of $\alpha + \beta\,i$ as a root of the characteristic equation. Here

$$\alpha + \beta\,i = 0 + 1\,i = i$$

is clearly a root of the characteristic equation, with multiplicity 1. Hence $k = 1$, and

$$y_p(t) = t\,\big(A\,\cos(t) + B\sin(t)\big)$$

Finding the constants is not part of the problem, but if you substitute and solve, you will get

$$y_p(t) = -\frac{1}{2}\,t\,\cos(t) - \frac{1}{2}\,t\,\sin(t)$$

06c. The characteristic equation of the homogeneous equation is $\lambda^2 - 3\lambda + 2 = 0 \iff (\lambda-1)(\lambda-2) = 0$ Its roots are $\lambda_1 = 1$ and $\lambda_2 = 2$. Expanding the right side, the equation becomes

$$y'' - 3y' + 2y = t^2 + 2t\,e^t + e^{2t}$$

According to the superposition principle for nonhomogeneous equations, the particular solution is

$$y_p(t) = y_{p,1}(t) + y_{p,2}(t) + y_{p,3}(t)$$

where

$$y_{p,1}(t) \quad \text{is a particular solution of} \quad y'' - 3y' + 2y = t^2$$
$$y_{p,2}(t) \quad \text{is a particular solution of} \quad y'' - 3y' + 2y = 2t\,e^t$$
$$y_{p,3}(t) \quad \text{is a particular solution of} \quad y'' - 3y' + 2y = e^{2t}$$

According to the undetermined coefficients method, the form of each particular solution, is

$$y_{p,1}(t) = t^k \left(A t^2 + B t + C\right)$$

with $k = 0$, since $\alpha + \beta i = 0 + 0i = 0$, is not a root of the characteric equation.

$$y_{p,2}(t) = t^k \left(D t + E\right) e^t$$

with $k = 1$, since $\alpha + \beta i = 1 + 0i = 1$ is a simple root of the characteristic equation.

$$y_{p,3}(t) = t^k F e^{2t}$$

with $k = 1$, since $\alpha + \beta i = 2 + 0i = 2$, is a simple root of the characteristic equation.
Hence the form of $y_p(t)$, is

$$y_p(t) = y_{p,1}(t) + y_{p,2}(t) + y_{p,3}(t) = A t^2 + B t + C + \left(D t^2 + E t\right) e^t + F t\,e^{2t}$$

Finding the constants is not part of the problem, but if you substitute and solve, you will get

$$y_p(t) = \frac{1}{2} t^2 + \frac{3}{2} t + \frac{7}{4} - \left(t^2 + 2t\right) e^t + t\,e^{2t}$$

06d. The characteristic equation of the homogeneous equation is $\lambda^2 + 16 = 0$. It has roots $\lambda_1 = 4i$ and $\lambda_2 = -4i$. Rewriting the right side as

$$f(t) = \sin\left(4t + \frac{\pi}{3}\right) = \sin(4t)\,\cos\left(\frac{\pi}{3}\right) + \cos(4t)\,\sin\left(\frac{\pi}{3}\right) = \frac{1}{2}\sin(4t) + \frac{\sqrt{3}}{2}\cos(4t)$$

shows that it has the exponential-polynomial-cosine-sine form $e^{\alpha t}\Big(M(t)\cos(\beta t) + N(t)\sin(\beta t)\Big)$, with $\alpha = 0$ and $\beta = 4$. It follows that the (UCM) is applicable, and we look for a particular solution in the form

$$y_p(t) = t^k \left(A\cos(4t) + B\sin(4t)\right)$$

Here $\alpha + \beta i = 0 + 4i = 4i$, is a root of the characteristic equation, with multiplicity 1. Thus $k = 1$, and

$$y_p(t) = t\left(A\cos(4t) + B\sin(4t)\right) = A t\cos(4t) + B t\sin(4t)$$

Finding the constants is not part of the problem, but if you substitute and solve, you will get

$$y_p(t) = -\frac{1}{16} t\cos(4t) + \frac{\sqrt{3}}{16} t\sin(4t)$$

07a. The characteristic equation of the associated homogeneous equation is $\lambda^2 - 4\lambda + 4 = 0$. Its roots are $\lambda_1 = \lambda_2 = 2$, and a fundamental set of solutions is $\left\{ e^{2t}, \ t e^{2t} \right\}$. Thus the general solution of the homogeneous equation is

$$y_h(t) = C_1 e^{2t} + C_2 t e^{2t}$$

A particular solution $y_p(t)$ can be found by using the undetermined coefficients method. It should have the form $y_p(t) = A t^2 + B t + C$. Substituting into the equation and solving for the constants, we get

$$y_p(t) = \frac{1}{4} t^2 + \frac{1}{2} t + \frac{3}{8}$$

Hence the general solution is

$$y(t) = y_h(t) + y_p(t) = C_1 e^{2t} + C_2 t e^{2t} + \frac{1}{4} t^2 + \frac{1}{2} t + \frac{3}{8}$$

07b. Characteristic equation: $\lambda^2 + 4\lambda + 4 = 0$
The roots: $\lambda_1 = \lambda_2 = -2$
Fundamental set of solutions: $\left\{ e^{-2t}, \ t e^{-2t} \right\}$
General solution of the homogeneous: $y_h(t) = C_1 e^{-2t} + C_2 t e^{-2t}$
Particular solution: $y_p(t) = t^2 A e^{-2t} = 4 t^2 e^{-2t}$
General solution of the nonhomogeneous equation

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-2t} + C_2 t e^{-2t} + 4 t^2 e^{-2t}$$

07c. Characteristic equation: $\lambda^2 + 1 = 0$
The roots: $\lambda_1 = i$ and $\lambda_2 = -i$
Fundamental set of solutions: $\left\{ \cos(t), \ \sin(t) \right\}$
General solution of the homogeneous: $y_h(t) = C_1 \cos(t) + C_2 \sin(t)$
Particular solution: $y_p(t) = t \Big(\big(A t + B\big) \cos(t) + \big(C t + D\big) \sin(t) \Big) = t \cos(t) + t^2 \sin(t)$
General solution of the nonhomogeneous equation

$$y(t) = y_h(t) + y_p(t) = C_1 \cos(t) + C_2 \sin(t) + t \cos(t) + t^2 \sin(t)$$

07d. Characteristic equation: $\lambda^2 - 3\lambda + 2 = 0$
The roots: $\lambda_1 = 1$, $\lambda_2 = 2$
Fundamental set of solutions: $\left\{ e^{t}, \ e^{2t} \right\}$
General solution of the homogeneous: $y_h(t) = C_1 e^{t} + C_2 e^{2t}$
Particular solution: $y_p(t) = \big(A t^2 + B t + C\big) e^{3t} = \left(\frac{1}{2} t^2 - t + 1 \right) e^{3t}$
General solution of the nonhomogeneous equation

$$y(t) = y_h(t) + y_p(t) = C_1 e^{t} + C_2 e^{2t} + \left(\frac{1}{2} t^2 - t + 1 \right) e^{3t}$$

08a. Characteristic equation: $\lambda^2 + \lambda = 0$
The roots: $\lambda_1 = 0,\ \lambda_2 = -1$
Fundamental set of solutions: $\left\{ 1\ ,\ e^{-t} \right\}$
General solution of the homogeneous: $y_h(t) = C_1 + C_2\, e^{-t}$
Particular solution: $y_p(t) = t\, e^{-t} A = -t\, e^{-t}$
General solution of the nonhomogeneous equation

$$y(t) = y_h(t) + y_p(t) = C_1 + C_2\, e^{-t} - t\, e^{-t}$$

Solution of the IVP: $y(t) = 1 - t\, e^{-t}$

08b. Characteristic equation: $\lambda^4 - 1 = 0$
The roots: $\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = i,\ \lambda_4 = -i$
Fundamental set of solutions: $\left\{ e^{-t}\ ,\ e^{t}\ ,\ \cos(t)\ ,\ \sin(t) \right\}$
General solution of the homogeneous: $y_h(t) = C_1\, e^{-t} + C_2\, e^{t} + C_3\, \cos(t) + C_4\, \sin(t)$
Particular solution: $y_p(t) = t\, e^{t} A = 2\, t\, e^{t}$
General solution of the nonhomogeneous equation

$$y(t) = y_h(t) + y_p(t) = C_1\, e^{-t} + C_2\, e^{t} + C_3\, \cos(t) + C_4\, \sin(t) + 2\, t\, e^{t}$$

Solution of the IVP: $y(t) = e^{-t} + (2\, t - 3)\, e^{t} + \cos(t) + 2\, \sin(t)$