Department of Mathematics and Statistics
MATH 375 - Outline # 2
Second Order Differential Equations
General Theory

Consider the linear equation

$$y'' + p(x)y' + q(x)y = 0. \qquad (1)$$

1. (existence and uniqueness) If p and q are continuous in an open interval I and $x_0 \in I$ then the initial value problem (1), $y(x_0) = y_0$, $y'(x_0) = y_0'$ has a unique solution for any y_0,y_0'.
2. (superposition) If y_1 and y_2 are solutions of (1), then $y = C_1y_1 + C_2y_2$ is also a solution of (1) for any constants C_1 and C_2.
3. (general solution) If y_1 and y_2 are linearly independent (fundamental) solutions of (1) then $y = C_1y_1 + C_2y_2$ is the general solution of (1). The functions y_1 and y_2 are linearly independent if the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \neq 0.$$

4. (Abel's Theorem) If $p(x)$ and $q(x)$ are continuous, then

$$W(x) = Ce^{-\int p(x)\,dx}.$$

Thus the Wronskian either identically equal to zero or never vanishes.
5. The difference of any two solutions of the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \qquad (2)$$

satisfies (1).
If y_1 and y_2 are linearly independent solutions of (1) and $Y(t)$ is a particular solution of (2) then the general solution of (2) is

$$y = C_1y_1 + C_2y_2 + Y.$$

Homogeneous Equations with Constant Coefficients $\mathbf{ay'' + by' + cy = 0}$ (3)

Method of solution: Let us write the characreristic equation $\mathbf{ar^2 + br + c = 0}$. **(4)**

Case 1. $\Delta = b^2 - 4ac > 0$, the characteristic equation (4) has two real roots:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Then the fundamental solutions of (3) are $y_1 = e^{r_1x}$, $y_2 = e^{r_2x}$; the general solution of (1) is $y = C_1e^{r_1x} + C_2e^{r_2x}$.

Case 2. $\Delta = b^2 - 4ac < 0$, the characreristic equation (4) has two complex roots:

$$r_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-b}{2a} + \frac{\sqrt{-\Delta}}{2a}i = \alpha + \beta i,\ r_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-b}{2a} - \frac{\sqrt{-\Delta}}{2a}i = \alpha - \beta i.$$

Then the basic solutions of (1) are $y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$; the general solution of (1) is $y = e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right)$.

Case 3. $\Delta = b^2 - 4ac = 0$, the characreristic equation (4) has two equal real roots $r_1 = r_2 = r = -\frac{b}{2a}$. Then the basic solutions of (1) are $y_1 = e^{rx}$, $y_2 = xe^{rx}$; the general solution of (1) is $y = C_1 e^{rx} + C_2 x e^{rx}$.

Higher order equations

For a higher order equation with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = 0$$

we find roots of the characteristic equation

$$a_0 r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0.$$

Then distinct real roots r_1, r_2, r_3 correspond to basic solutions $y_1 = e^{r_1 x}, y_2 = e^{r_2 x}, y_3 = e^{r_3 x}$, equal real roots $r_1 = r_2 = r_3 = r$ correspond to basic solutions $y_1 = e^{rx}, y_2 = xe^{rx}, y_3 = x^2 e^{rx}$, complex roots $r_{1,2} = \alpha \pm \beta i$ - to $y_1 = e^{\alpha x}\cos(\beta x), y_2 = e^{\alpha x}\sin(\beta x)$,
equal complex roots $r_1 = r_2 = \alpha + \beta i, r_3 = r_4 = \alpha - \beta i$ correspond to
$y_1 = e^{\alpha x}\cos(\beta x), y_2 = e^{\alpha x}\sin(\beta x), y_3 = xe^{\alpha x}\cos(\beta x), y_4 = xe^{\alpha x}\sin(\beta x)$.

Nonhomogeneous Equations with Constant Coefficients $\mathbf{ay'' + by' + cy = f(x)}$ (5)

The general solution of (5) is a sum of a general solution of the corresponding homogeneous equation (5) y_{gen} and a particular solution y_{part} of (5).

The method of undetermined coefficients

If the right hand side is a polynomial function

$$P_n(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots b_1 x + b_0$$

multiplied by one of the functions $e^{kx}, \cos(kx), \sin(kx)$, then the particular solution of (3) has a similar form (see the table below), only the polynomial has unknown coefficients $Q_n(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots A_1 x + A_0$, where A_i are determined by substitution in (3).

$f(x)$ in (3)	y_{part}	Comment
$P_n(x)$	$Q_n(x)$	$r = 0$ is not a root of (4)
$P_n(x)e^{kx}$	$Q_n(x)e^{kx}$	$r = k$ is not a root of (4)
$P_n(x)e^{kx}\cos(bx)$	$e^{kx}[Q_n(x)\cos(bx) + R_n(x)\sin(bx)]$	$r = k + bi$ is not a root of (4)
$P_n(x)e^{kx}\sin(bx)$	$e^{kx}[Q_n(x)\cos(bx) + R_n(x)\sin(bx)]$	$r = k + bi$ is not a root of (4)
$P_n(x)$	$x^s Q_n(x)$	$r = 0$ is a root, multiplicity s
$P_n(x)e^{kx}$	$x^s Q_n(x)e^{kx}$	$r = k$ is a root, multiplicity s
$P_n(x)e^{kx}\cos(bx)$	$x^s e^{kx}[Q_n(x)\cos(bx) + R_n(x)\sin(bx)]$	$r = k + bi$ is a root, multiplicity s
$P_n(x)e^{kx}\sin(bx)$	$x^s e^{kx}[Q_n(x)\cos(bx) + R_n(x)\sin(bx)]$	$r = k + bi$ is a root, multiplicity s

The method of the variation of parameters

If the general solution of (3) is $y = C_1 y_1 + C_2 y_2$, then the general solution of (5) is $y = C_1(x) y_1 + C_2(x) y_2$, where $C_1(x), C_2(x)$ are found from the following system of equations:

$$\begin{aligned} C_1' y_1 \quad &+ C_2' y_2 &&= 0, \\ a(C_1' y_1' \quad &+ C_2' y_2') &&= f(x). \end{aligned}$$