MATH 375
Handout # 5 - Answers, Hints, Solutions
Higher Order Linear Differential Equations

1. In each case , determine whether the set $\mathbf{S}$ is linearly independent on the interval $(-\infty, \infty)$. Use the Wronskian test or the definition.
a) $\mathbf{S} = \{x^2 + x, x^2 + 1, x^2 - 1\}$
b) $\mathbf{S} = \{\cos(2x), -3, 2\sin^2 x\}$
c) $\mathbf{S} = \{1, t, e^t\}$
Solution. a) The Wronskian can be simplified by subtracting the first column from the other columns:

$$\begin{vmatrix} x^2+x & x^2+1 & x^2-1 \\ 2x+1 & 2x & 2x \\ 2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} x^2+x & 1-x & -1-x \\ 2x+1 & -1 & -1 \\ 2 & 0 & 0 \end{vmatrix} = 2\begin{vmatrix} 1-x & -1-x \\ -1 & -1 \end{vmatrix}$$

$= 2(x-1-1-x) = -4 \neq 0$, so the functions are linearly independent.
b) The functions are linearly dependent, as $2\sin^2 x = 1 - \cos(2x)$ for any x, or $\cos(2x) + \frac{1}{3}(-3) + 2sin^2 x \equiv 0$.
c) The functions are linearly independent, as the Wronskian equals $e^t \neq 0$ for any t.

2. Find the general solution of the following differential equations
a) $y^{(4)} - 17y'' + 16y = 0$
b) $y^{(3)} - 64y = 0$
c) $y^{(4)} - 81y = 0$
d) $y^{(4)} + 12y'' + 11y = 0$
e) $y^{(5)} - 10y^{(3)} + 9y' = 0$
f) $y^{(3)} + 6y'' + 12y' + 8y = 0$
Solution. Here we assume that all the coefficients C_i are real numbers. a) The roots of the characteristic equation $r^4 - 17r^2 + 16 = 0$ correspond to $r^2 = 1, 16$, so $r = \pm 1, \pm 4$ and the general solution is

$$y = C_1 e^x + C_2 e^{-x} + C_3 e^{-4x} + C_4 e^{4x}.$$

b) Since $r^3 - 64 = (r-4)(r^2 + 4x + 16)$, we have $r = 4, -2 \pm 2\sqrt{3}i$ and

$$y = C_1 e^{4x} + e^{-2x}\left(C_2 \cos(2\sqrt{3}x) + C_3 \sin(2\sqrt{3}x)\right).$$

c) The roots of $r^4 - 81 = (r^2 - 9)(r^2 + 9)$ are $r = \pm 3, \pm 3i$, so the general solution is

$$y = C_1 e^{3x} + C_2 e^{-3x} + C_3 \cos(3x) + C_4 \sin(3x).$$

d) The roots are $r = \pm i, \pm\sqrt{11}i$, so

$$y = C_1 \cos x + C_2 \sin x + C_3 \cos(\sqrt{11}x) + C_4 \sin(\sqrt{11}x).$$

e) $r^5 - 10r^4 + 9r = r(r^2-1)(r^2-9) = r(r-1)(r+1)(r-3)(r+3)$, so

$$y = C_1 + C_2 e^x + C_3 e^{-x} + C_4 e^{3x} + C_5 e^{-3x}.$$

f) $r^3 + 6r^2 + 12r + 8 = (r+2)^3$, so $r_{1,2,3} = -2$ and the general solution is

$$y = e^{-2x}(C_1 x^2 + C_2 x + C_3).$$

3. Find the general solution to each of the following differential equations. The primes denote derivatives with respect to t, and $D \equiv \dfrac{d}{dt}$.

$a)\ \dfrac{d^3x}{dt^3} - 6\dfrac{d^2x}{dt^2} + 11\dfrac{dx}{dt} - 6x = 0$ $\qquad b)\ x''' - 6x'' + 2x' + 36x = 0$

$c)\ (D^4 + 8D^3 + 24D^2 + 32D + 16)x = 0$ $\qquad d)\ (D^5 - D^4 - 2D^3 + 2D^2 + D - 1)x = 0$

$e)\ D^3(D^2 + 6D + 25)(D^4 - 16)x = 0$ $\qquad f)\ (D^4 - 16)^3 x = 0$

Answers. a) $x = C_1e^t + C_2e^{2t} + C_3e^{3t}$ b) $x = C_1e^{-2t} + C_2e^{4t}\cos(\sqrt{2}t) + C_3e^{4t}\sin(\sqrt{2}t)$
c) $x = \left(C_1 + C_2t + C_3t^2 + C_4t^3\right)e^{-2t}$ d) $x = C_1e^t + C_2te^t + C_3t^2e^t + C_4e^{-t} + C_5e^{-t}$
e) $x = C_1 + C_2t + C_3t^2 + C_4e^{-3t}\cos(4t) + C_5e^{-3t}\sin(4t) + C_6e^{2t} + C_7e^{-2t} + C_8\cos(2t) + C_9\sin(2t)$
f) $x = (C_1 + C_2t + C_3t^2)e^{2t} + (C_4 + C_5t + C_6t^2)e^{-2t} + (C_7 + C_8t + C_9t^2)\cos(2t) + (C_{10} + C_{11}t + C_{12}t^2)$
$sin(2t)$.

4. According to the method of undetermined coefficients, find the form of the particular solution for the following equations. Do not compute the coefficients.

a) $y^{(4)} - 17y'' + 16y = x\cos x$
b) $y^{(3)} - 64y = 7e^{4x}$
c) $y^{(4)} - 81y = x^2e^{2x} + 6\cos(3x)$
d) $y^{(4)} + 12y'' + 11y = 3x^2 + e^x$
e) $y^{(5)} - 10y^{(3)} + 9y' = 3x^2 + e^x$
f) $y^{(3)} + 6y'' + 12y' + 8y = xe^{-2x}$

Answers. a) Since $r = i$ is not the root of the characteristic equation, the form of the particular solution is $y = (Ax + B)\cos x + (Cx + D)\sin x$.
b) $r = 4$ is a root (once), so $y = Axe^{4x}$.
c) $r = 2$ is not a root, $r = 3i$ is a root, so

$$y = (Ax^2 + Bx + C)e^{2x} + Dx\cos(3x) + Ex\sin(3x).$$

d) $r = 0, 1$ are not roots, so

$$y = Ax^2 + Bx + C + De^x.$$

e) $r = 0, 1$ are roots (once), so

$$y = x(Ax^2 + Bx + C) + Dxe^x.$$

f) $r = -2$ is a root of multiplicity 3, so

$$y = x^3(Ax + B)e^{-2x}.$$

5. Find a homogeneous linear differential equation with constant coefficients whose characteristic equation has the roots :
a) $\{1, i, -i\}$ b) $\{7, 7, 0, 0, 0, 2, 5i, 2, 5i, 2, 5i, 2, 5i\}$
Answers. a) A possible equation is $x''' + x'' + x' + x = 0$.
b) $D^3(D^2 - 49)(D^2 - 4D + 29)^4 x = 0$

6. Find the homogeneous linear differential equation with constant coefficients of the minimal order which has the following solution:
a) $x \cos x$ b) $x^2 e^{2x}$ c) $x^5 + e^{5x}$
Solution. a) This solution corresponds to the roots $r_{1,2} = i, r_{3,4} = -i$, the characteristic equation of the equation of the minimal order is $(r - i)^2(r + i)^2 = (r^2 + 1)^2 = r^4 + 2r^2 + 1 = 0$, the equation is

$$y^{(4)} + 2y'' + y = 0.$$

b) Since $r_1 = r_2 = r_3 = 2$ and $(r - 2)^3 = r^3 - 6r^2 + 12r - 8$, the equation is

$$y^{(30} - 6y'' + 12y' - 8y = 0.$$

c) The first term corresponds to $r_1 = \cdot = r_6 = 0$, the second term means $r_7 = 5$, $r^6(r - 5) = r^7 - 5r^6 = 0$, and the equation is

$$y^{(7)} - 5y^{(6)} = 0.$$