Math 375		Fall 2016
	First Order Differential Equations	
Worksheet # 1	**Part 3**	**October 03 - 07**

The problems marked with $(*)$ are to be attempted during the tutorial time. Students are strongly encouraged to attempt the remaining problems on their own. Solutions to all the problems will be available on the course's D2L website Friday, October 07. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Mixing Problems

01. A tank initially contains 40 gallons of pure water. A solution with 1 gram of salt per gallon of water is added to the tank at 3 gallons per minute, and the resulting well mixed solution leaves the tank at the same rate. Find the quantity $Q(t)$ of salt in the tank at time t, and $\lim_{t\to+\infty} Q(t)$

02*. A 500-litre tank initially contains 100 litres of a salt solution with concentration of 0.1 gram per litre. A salt solution with concentration of 0.3 gram per litre, is added to the tank at a rate of 5 litres per second. The resulting mixture is drained out at a rate of 3 litres per second. Find the concentration $c(t)$ of salt in the tank at any given time t. How much salt will the tank contain when it is on the point of overflowing.

Radioactive Decay

03. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of that substance to decay to $\dfrac{1}{10}$ of its original mass.

04*. Initially, 100 grams of a radioactive material is present. After 3 days, only 75 grams remains. How much additional time will it take for radioactive decay to reduce the amount present to 30 grams?

Heating and Cooling

05*. A body of temperature $80^\circ F$ is placed at time $t = 0$ in a medium the temperature of which is maintained at $50^\circ F$. At the end of 5 minutes, the body has cooled to a temperature of $70^\circ F$.

- **a.** What is the temperature of the body at the end of 10 minutes?
- **b.** When will the temperature of the body be $60^\circ F$?

06. A student performs the following experiment using two identical cups of water. One cup is removed from a refrigerator at $34^\circ F$ and allowed to warm in its surroundings to room temperature $\left(72^\circ F\right)$. A second cup is simultaneously taken from room temperature surroundings and placed in the refregirator to cool. The time at which each cup of water reached a temperature of $53^\circ F$ is recorded. Are the two recorded times the same?

Electrical Circuits

07. A generator having an emf of $100\,V$ is connected in series with a $10\,\Omega$ resistor and a $2\,H$ inductor. Find the current $I(t)$, if $I(0)=0$.

08. Let $p \neq 0,\ q,\ \omega,\ y_0$ be real constants. Show that the solution of the initial value problem $\begin{cases} y' + p\,y = q\,\cos(\omega\,t) \\ y(0) = y_0 \end{cases}$ is

$$y(t) = \frac{q}{p^2+\omega^2}\Big(p\,\cos(\omega\,t) + \omega\,\sin(\omega\,t)\Big) + \left(y_0 - \frac{p\,q}{p^2+\omega^2}\right)\,\mathrm{e}^{-p\,t}$$

Note: the solution above remains valid when $q=0$ or $\omega=0$.

09*. A generator having an emf $E(t) = 20\,\cos(5\,t)$, is connected in series with a $10\,\Omega$ resistor and a $2\,H$ inductor. Find the current $I(t)$, if $I(0)=0$.

10. A decaying emf $E(t) = 200\,\mathrm{e}^{-5\,t}$ is connected in series with a $20\,\Omega$ resistor and a $\dfrac{1}{100}\,F$ capacitor.

a. Find the charge $Q(t)$ and the current $I(t)$ at any time, if $Q(0)=0$

b. Find the maximum charge of the capacitor and the time it takes to reach that maximum.

Answers and Solutions

01. Let $Q(t)$ be the amount of salt (in grams) in the tank at time t (in minutes), and let $V(t)$ be the volume (in gallons) of the solution in the tank at time t. Making use of the Balance Law

net rate = rate in - rate out

for both $V=V(t)$ and $Q=Q(t)$, we have

$$\begin{cases} V' = 3\left[\dfrac{\text{gal}}{\text{min}}\right] - 3\left[\dfrac{\text{gal}}{\text{min}}\right] \\ V(0) = 40\ [\text{gal}] \end{cases} \iff V(t) = 40$$

and

$$\begin{cases} Q' = 3\left[\dfrac{\text{gal}}{\text{min}}\right]\times 1\left[\dfrac{\text{gr}}{\text{gal}}\right] - 3\left[\dfrac{\text{gal}}{\text{min}}\right]\times\dfrac{Q}{V}\left[\dfrac{\text{gr}}{\text{gal}}\right] \\ Q(0) = 0\ [\text{gr}] \end{cases} \iff \begin{cases} Q' + \dfrac{3}{40}\,Q = 3 \\ Q(0) = 0 \end{cases}$$

Solving we get $\Big($ an integrating factor is $\mathrm{e}^{3\,t/40}$ $\Big)$

$$Q(t) = 40 - 40\,\mathrm{e}^{-3t/40} \quad \text{and} \quad \lim_{t\to+\infty} Q(t) = 40\ [\text{gr}]$$

02. Let $Q(t)$ be the amount of salt (in grams) in the tank at time t (in seconds), and let $V(t)$ be the volume (in litres) of the solution in the tank at time t. Making use of the Balance Law

net rate = rate in - rate out

for both $V=V(t)$ and $Q=Q(t)$, we have

$$\begin{cases} V' = 5\left[\dfrac{\mathrm{L}}{\mathrm{s}}\right] - 3\left[\dfrac{\mathrm{L}}{\mathrm{s}}\right] \\ \\ V(0) = 100\ [\mathrm{L}] \end{cases} \iff \begin{cases} V' = 2 \\ \\ V(0) = 100 \end{cases} \iff V(t) = 2\,t + 100$$

and

$$\begin{cases} Q' = 5\left[\dfrac{\mathrm{L}}{\mathrm{s}}\right]\times 0.3\left[\dfrac{\mathrm{gr}}{\mathrm{L}}\right] - 3\left[\dfrac{\mathrm{L}}{\mathrm{s}}\right]\times\dfrac{Q}{V}\left[\dfrac{\mathrm{gr}}{\mathrm{L}}\right] \\ \\ Q(0) = 100\ [\mathrm{L}]\times 0.1\left[\dfrac{\mathrm{gr}}{L}\right] \end{cases} \iff \begin{cases} Q' + \dfrac{3}{2\,t+100}\,Q = \dfrac{3}{2} \\ \\ Q(0) = 10 \end{cases}$$

Solving we obtain $\Big($ an integrating factor is $(t+50)^{3/2}\Big)$ $Q(t) = \dfrac{3}{5}\,(t+50) - \dfrac{20\,(50)^{3/2}}{(t+50)^{3/2}}$.
Hence the concentration

$$c(t) = \frac{Q(t)}{V(t)} = \frac{3}{10} - \frac{10\,(50)^{3/2}}{(t+50)^{5/2}}$$

The tank starts to overflow when $V(t) = 500 \iff 2\,t + 100 = 500 \iff t = 200[\mathrm{s}]$.
The amount of salt present in the tank at that time is $Q(200) = 150 - \dfrac{4}{\sqrt{5}} \approx 148.21\ [\mathrm{gr}]$

03. Letting $Q(t)$ be the amount (in unit of mass) of the substance at time t (in days) and $Q_0 = Q(0)$ be the initial mass at $t = 0$ [days], then

$$\begin{cases} Q' = -k\,Q \\ Q(0) = Q_0 \end{cases} \iff \begin{cases} Q' + k\,Q = 0 \\ Q(0) = Q_0 \end{cases} \qquad \text{solving we obtain} \qquad Q(t) = Q_0\,\mathrm{e}^{-kt}$$

Making use of the fact that the half life is $\tau = 2$ [days], we have

$$Q(2) = \frac{Q_0}{2} \iff Q_0\,\mathrm{e}^{-2\,k} = \frac{Q_0}{2} \iff -2\,k = -\ln\big(2\big) \iff k = \frac{\ln\big(2\big)}{2}$$

Thus $\qquad Q(t) = Q_0\,\mathrm{e}^{-\frac{\ln(2)}{2}\,t}$

To find the time t required for the substance to decay to $\dfrac{Q_0}{10}$, solve

$$Q(t) = \frac{Q_0}{10} \iff Q_0\,\mathrm{e}^{-\frac{\ln(2)}{2}\,t} = \frac{Q_0}{10} \iff -\frac{\ln(2)}{2}\,t = -\ln(10) \iff t = \frac{2\ln(10)}{\ln(2)} \approx 6.64\ [\mathrm{days}]$$

That's about 6 days and 15 hours.

04. Letting $Q(t)$ be the amount (in grams) of the material at time t (in days), then

$$\begin{cases} Q' = -k\,Q \\ Q(0) = 100\ [\mathrm{grams}] \end{cases} \iff \begin{cases} Q' + k\,Q = 0 \\ Q(0) = 100 \end{cases} \qquad \text{solving we obtain} \qquad Q(t) = 100\,\mathrm{e}^{-kt}$$

Making use of the fact that $Q(3) = 75$ [grams], we can solve for the decay constant

$$Q(3) = 75 \iff 100\,\mathrm{e}^{-3\,k} = 75 \iff k = \frac{\ln(4/3)}{3}$$

Hence $Q(t) = 100\,\mathrm{e}^{-\ln\left(4/3\right)\,t/3}$. To find the total time it takes the material to decay from 100 grams to 30 grams, we solve

$$Q(t) = 30 \iff 100\,\mathrm{e}^{-\ln\left(4/3\right)\,t/3} = 30 \iff t = 3\,\frac{\ln\left(10/3\right)}{\ln\left(4/3\right)}\ [\mathrm{days}]$$

Hence the additional time is $3\,\dfrac{\ln\left(10/3\right)}{\ln\left(4/3\right)} - 3 \approx 9.55$ [days]. That's about 9 days and 13 hours.

05. Letting $T(t)$ be the temperature (in degree fahrenheit) of the body at time t (in minutes), and making use of Newton's Law of Cooling, $T = T(t)$ satisfies the initial value problem

$$\begin{cases} T' = k\big(M_0 - T\big) \\ T(0) = T_0 \end{cases} \iff \begin{cases} T' + k\,T = M_0\,k \\ T(0) = T_0 \end{cases} \iff \begin{cases} T' + k\,T = 50\,k \\ T(0) = 80 \end{cases}$$

Solving we obtain $T(t) = 50 + 30\,\mathrm{e}^{-kt}$. To find k, we use

$$T(5) = 70 \iff 50 + 30\,\mathrm{e}^{-5k} = 70 \iff k = \frac{\ln\big(3/2\big)}{5}$$

Hence $T(t) = 50 + 30\,\mathrm{e}^{-\ln(3/2)\,t/5}$.

a. At the end of 10 minutes the temperature of the body is

$$T(10) = 50 + 30\,\mathrm{e}^{-\ln(3/2)\;10/5} = 50 + 30\,\mathrm{e}^{-2\ln(3/2)} = 50 + 30\,\mathrm{e}^{\ln(4/9)} = 50 + 30\,\frac{4}{9} = \frac{190}{3}$$

That's about $63^\circ F$

b. To find the time when the temperature reaches $60^\circ F$, we solve

$$T(t) = 60 \iff 50 + 30\,\mathrm{e}^{-\ln(3/2)\,t/5} = 60 \iff \mathrm{e}^{-\ln(3/2)\,t/5} = \frac{1}{3} \iff -\ln(3/2)\,t/5 = \ln(1/3)$$

$$\iff t = \frac{5\,\ln(3)}{\ln(3/2)} \approx 13\,\text{minutes}\;33\,\text{seconds}$$

06. For both experiments the temperature $T(t)$ of the cup of water satisfies

$$\begin{cases} T' = k\,\big(M_0 - T\big) \\ T(0) = T_0 \end{cases} \iff \begin{cases} T' + k\,T = k\,M_0 \\ T(0) = T_0 \end{cases} \iff T(t) = M_0 + \big(T_0 - M_0\big)\,\mathrm{e}^{-k\,t}$$

a. In the 1st experiment, $T_0 = 34^\circ F$, $M_0 = 72^\circ F$. Thus $T(t) = 72 + \big(34 - 72\big)\,\mathrm{e}^{-k\,t} = 72 - 38\,\mathrm{e}^{-k\,t}$. Letting t_1 be the time where the water in the cup reaches $53^\circ F$, we have $53 = 72 - 38\,\mathrm{e}^{-k\,t_1} \iff \mathrm{e}^{-k\,t_1} = \dfrac{1}{2}$

b. In the 2nd experiment, $T_0 = 72^\circ F$, $M_0 = 34^\circ F$. Thus $T(t) = 34 + \big(72 - 34\big)\,\mathrm{e}^{-k\,t} = 34 + 38\,\mathrm{e}^{-k\,t}$. Letting t_2 be the time where the water in the cup reaches $53^\circ F$, we have $53 = 34 + 38\,\mathrm{e}^{-k\,t_2} \iff \mathrm{e}^{-k\,t_2} = \dfrac{1}{2}$

Thus $\mathrm{e}^{-k\,t_1} = \mathrm{e}^{-k\,t_2} \iff -k\,t_1 = -k\,t_2 \iff t_1 = t_2$.

07. The current $I(t)$ satisfies the differential equation

$$R\,I + L\,\frac{\mathrm{d}I}{\mathrm{d}t} = E \iff \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{R}{L}\,I = \frac{E}{L} \iff \frac{\mathrm{d}I}{\mathrm{d}t} + \frac{10}{2}\,I = \frac{100}{2} \iff \frac{\mathrm{d}I}{\mathrm{d}t} + 5\,I = 50$$

Solving leads to $I(t) = 10 + C\,\mathrm{e}^{-5\,t}$. Setting $t = 0$ and solving the equation $I(0) = 0$ for the constant C, we get $C = -10$. Hence

$$I(t) = 10 - 10\,\mathrm{e}^{-5\,t}$$

08. You may use

$$\int \mathrm{e}^{p\,t}\,\cos(\omega\,t)\,\mathrm{d}t = \frac{\mathrm{e}^{p\,t}}{p^2 + \omega^2}\,\Big(p\,\cos(\omega\,t) + \omega\,\sin(\omega\,t)\Big)$$

which can be established using two successive integrations by parts.

09. The current $I(t)$ satisfies the differential equation

$$R\,I+L\,\frac{\mathrm{d}I}{\mathrm{d}t}=E \iff \frac{\mathrm{d}I}{\mathrm{d}t}+\frac{R}{L}\,I=\frac{E}{L} \iff \frac{\mathrm{d}I}{\mathrm{d}t}+\frac{10}{2}\,I=\frac{20\,\cos(5\,t)}{2} \iff \frac{\mathrm{d}I}{\mathrm{d}t}+5\,I=10\,\cos(5\,t)$$

Solving (use problem (08) above) leads to $I(t)=\cos(5\,t)+\sin(5\,t)-\mathrm{e}^{-5\,t}$.

10. The charge $Q(t)$ satisfies the equation $R\,I+\dfrac{1}{C}\,Q=E(t)$. But $I=\dfrac{\mathrm{d}Q}{\mathrm{d}t}$. It follows

$$R\,I+\frac{1}{C}\,Q=E(t) \iff R\,\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{1}{C}\,Q=E(t) \iff \frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{1}{R\,C}\,Q=\frac{1}{R}\,E(t) \iff \frac{\mathrm{d}Q}{\mathrm{d}t}+5\,Q=10\,\mathrm{e}^{-5\,t}$$

An integrating factor is $\mu=\mathrm{e}^{\int 5\,\mathrm{d}t}=\mathrm{e}^{5\,t}$. Multiplying the differential equation by $\mu=\mathrm{e}^{5\,t}$, leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{5\,t}\,Q\right)=10 \implies \mathrm{e}^{5\,t}\,Q=10\,t+C \implies Q(t)=(10\,t+C)\,\mathrm{e}^{-5\,t}$$

Solving for C the equation $Q(0)=0$ gives $C=0$. Hence $Q(t)=10\,t\,\mathrm{e}^{-5\,t}$

The current is $I(t)=\dfrac{\mathrm{d}Q}{\mathrm{d}t}(t)=10\,\mathrm{e}^{-5\,t}-50\,t\,\mathrm{e}^{-5\,t}=10\,\big(1-5\,t\big)\,\mathrm{e}^{-5\,t}$

It is not hard to verify that $Q(t)$ has an absolute maximum at $t=\dfrac{1}{5}$ with value

$Q(1/5)=2\,\mathrm{e}^{-1}=\dfrac{2}{\mathrm{e}}\approx 0.736\,\mathrm{Coulomb}$