Math 375 **Fall 2016**

Laplace Transform

Worksheet # 3 **Part 1** **November 07-11**

The problems on this worksheet refer to material from sections §6.1, §6.2, 6.3, and §6.4 of your text. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

The Multiplication by t Formula

01. Evaluate

a. $\mathcal{L}\{t\cos(3t)\}(s)$ **b.** $\mathcal{L}\left\{\big(t+\sin(t)\big)^2\right\}(s)$

c*. $\mathcal{L}\left\{(t-2)\cos(3t)\,e^{2t}\right\}(s)$ **d.** $\mathcal{L}\left\{t^2\sin(a\,t)\right\}(s)$

02. Evaluate the integrals

a. $\displaystyle\int_0^{+\infty} t\,e^{-2t}\cos(t)\,dt$ **b.** $\displaystyle\int_0^{+\infty} t^2\sin(2t)\,e^{-t}\,dt$

Think of each integral as a Laplace transform evaluated at a specific value s.

03. Compute $\mathcal{L}\left\{t\,e^{2t}\cos(3t)\sin(4t)\right\}(s)$

Division by t Formula

04. Compute the following Laplace transforms:

a*. $\mathcal{L}\left\{\dfrac{\sinh(t)}{t}\right\}(s)$ **b.** $\mathcal{L}\left\{\dfrac{e^{3t}-1}{t}\right\}(s)$ **c.** $\mathcal{L}\left\{\dfrac{1-\cos(2t)}{t}\right\}(s)$

Hint: Use the division by t formula: $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\}(s)=\displaystyle\int_s^{+\infty}\mathcal{L}\{f(t)\}(r)\,dr$, where $f(t)$ is piecewise continuous and of exponential order in $[0,+\infty)$, and $\displaystyle\lim_{t\to 0^+}\frac{f(t)}{t}$ is a real number.

The Transform of Periodic Functions

05. Compute the Laplace transform of the following periodic functions.

a*. $f(t)=\begin{cases}1 & \text{if } 0<t<1\\ -1 & \text{if } 1<t<2\end{cases}$ and $f(t+2)=f(t)$, for any t

b. $f(t)=t,\ 0<t<1$, and $f(t+1)=f(t)$ for any t

c. $f(t)=|\sin(t)|$

Piecewise Functions and their Laplace Transforms

06. Express each function $f(t)$ in terms of unit step functions, then compute $\mathcal{L}\{f(t)\}(s)$.

a*. $f(t) = \begin{cases} 2t-1 & \text{if } 0 \le t < 2 \\ t & \text{if } 2 \le t \end{cases}$

b*. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \\ 6 & \text{if } 2 \le t \end{cases}$

c. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 2 \\ e^{-2t} & \text{if } 2 \le t \end{cases}$

d. $f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ t\sin(t) & \text{if } \pi \le t \end{cases}$

Transforms of Derivatives

07. Use the formula $\mathcal{L}\{f''(t)\}(s) = s^2\mathcal{L}\{f(t)\}(s) - s\,f(0) - f'(0)$ to compute

a. $\mathcal{L}\{\cos(bt)\}(s)$

b. $\mathcal{L}\{\sin(bt)\}(s)$

c. $\mathcal{L}\{\cosh(bt)\}(s)$

d. $\mathcal{L}\{\sinh(bt)\}(s)$

08. Compute $\mathcal{L}\{y\}(s)$, given that $y = y(t)$ is the solution of the initial value problem.

a. $\begin{cases} y'' + y = t \\ y(0) = 1, \quad y'(0) = -2 \end{cases}$

b*. $\begin{cases} y'' + 4y = f(t) \\ y(0) = 0, \quad y'(0) = 0 \end{cases}$ where $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ -1 & \text{if } 1 \le t < 2 \\ 0 & \text{if } 2 \le t \end{cases}$

> Multiplication by t Formula
>
> If $f(t)$ is piecewise continuous and of exponential order in $[0\,,\,+\infty)$, then
> $$\mathcal{L}\left\{t\,f(t)\right\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{f(t)\right\}(s)\Big)$$

01a. Using the multiplication by t formula, we have

$$\mathcal{L}\left\{t\,\cos(3\,t)\right\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{\cos(3\,t)\right\}(s)\Big) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+9}\right) = -\frac{1\cdot\left(s^2+9\right)-s\cdot(2\,s)}{\left(s^2+9\right)^2} = \frac{s^2-9}{\left(s^2+9\right)^2}$$

01b. Expanding and making use of the double angle formula $\sin^2(t) = \dfrac{1}{2} - \dfrac{1}{2}\cos(2\,t)$, we have

$$\begin{aligned}\mathcal{L}\left\{\left(t+\sin(t)\right)^2\right\}(s) &= \mathcal{L}\left\{t^2+2\,t\,\sin(t)+\sin^2(t)\right\}(s) = \mathcal{L}\left\{t^2+2\,t\,\sin(t)+\frac{1}{2}-\frac{1}{2}\,\cos(2\,t)\right\}(s)\\ &= \mathcal{L}\left\{t^2\right\}(s)+2\,\mathcal{L}\left\{t\,\sin(t)\right\}(s)+\frac{1}{2}\,\mathcal{L}\left\{1\right\}(s)-\frac{1}{2}\,\mathcal{L}\left\{\cos(2\,t)\right\}(s)\\ &= \frac{2}{s^3}+2\,\mathcal{L}\left\{t\,\sin(t)\right\}(s)+\frac{1}{2}\,\frac{1}{s}-\frac{1}{2}\,\frac{s}{s^2+4}\end{aligned}$$

Using the multiplication by t formula, we have

$$\mathcal{L}\left\{t\,\sin(t)\right\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{\sin(t)\right\}(s)\Big) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = \frac{2\,s}{\left(s^2+1\right)^2}$$

Hence

$$\mathcal{L}\left\{\left(t+\sin(t)\right)^2\right\}(s) = \frac{2}{s^3}+\frac{4\,s}{\left(s^2+1\right)^2}+\frac{1}{2\,s}-\frac{s}{2\left(s^2+4\right)}$$

01c. Making use of the first shift formula $\mathcal{L}\left\{f(t)\,\mathrm{e}^{a\,t}\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s-a)$, we have

$$\mathcal{L}\left\{(t-2)\,\cos(3\,t)\,\mathrm{e}^{2\,t}\right\}(s) = \mathcal{L}\left\{(t-2)\,\cos(3\,t)\right\}(s-2)$$

Next we compute $\mathcal{L}\left\{(t-2)\,\cos(3\,t)\right\}(s)$. We have

$$\begin{aligned}\mathcal{L}\left\{(t-2)\,\cos(3\,t)\right\}(s) &= \mathcal{L}\left\{t\,\cos(3\,t)\right\}(s)-2\,\mathcal{L}\left\{\cos(3\,t)\right\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{\cos(3\,t)\right\}(s)\Big)-2\,\frac{s}{s^2+9}\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+9}\right)-\frac{2\,s}{s^2+9} = \frac{s^2-9}{\left(s^2+9\right)^2}-\frac{2\,s}{s^2+9}\end{aligned}$$

Thus

$$\mathcal{L}\left\{(t-2)\,\cos(3\,t)\,\mathrm{e}^{2\,t}\right\}(s) = \mathcal{L}\left\{(t-2)\,\cos(3\,t)\right\}(s-2) = \frac{(s-2)^2-9}{\left((s-2)^2+9\right)^2}-\frac{2\,(s-2)}{(s-2)^2+9}$$

01d. Using the multiplication by t formula twice, we have

$$\begin{aligned}\mathcal{L}\left\{t^2\,\sin(a\,t)\right\}(s) &= -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{t\sin(a\,t)\right\}(s)\Big) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{\sin(a\,t)\right\}(s)\Big)\right)\\ &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big(\mathcal{L}\left\{\sin(a\,t)\right\}(s)\Big) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{a}{s^2+a^2}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{2\,a\,s}{\left(s^2+a^2\right)^2}\right)\\ &= \frac{2\,a\left(3\,s^2-a^2\right)}{\left(s^2+a^2\right)^3}\end{aligned}$$

02a. $\displaystyle\int_0^{+\infty} t\,\mathrm{e}^{-2\,t}\,\cos(t)\,\mathrm{d}t = \left(\int_0^{+\infty} t\,\cos(t)\,\mathrm{e}^{-s\,t}\,\mathrm{d}t\right)\Big|_{s=2} = \mathcal{L}\left\{t\,\cos(t)\right\}(2)$

Now

$$\mathcal{L}\left\{t\,\cos(t)\right\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{\cos(t)\right\}(s)\Big) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{\left(s^2+1\right)^2}$$

Hence

$$\int_0^{+\infty} t\,\mathrm{e}^{-2\,t}\,\cos(t)\,\mathrm{d}t = \frac{s^2-1}{\left(s^2+1\right)^2}\Bigg|_{s=2} = \frac{4-1}{(4+1)^2} = \frac{3}{25}$$

02b. $\displaystyle\int_0^{+\infty} t^2\,\sin(2\,t)\,\mathrm{e}^{-t}\,\mathrm{d}t = \left(\int_0^{+\infty} t^2\,\sin(2\,t)\,\mathrm{e}^{-s\,t}\,\mathrm{d}t\right)\Big|_{s=1} = \mathcal{L}\left\{t^2\,\sin(2\,t)\right\}(1)$

To compute $\mathcal{L}\left\{t^2\,\sin(2\,t)\right\}(s)$, we apply the multiplication by t formula twice, to get

$$\begin{aligned}\mathcal{L}\left\{t^2\,\sin(2\,t)\right\}(s) &= -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{t\,\sin(2\,t)\right\}(s)\Big) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{\sin(2\,t)\right\}(s)\Big)\right)\\ &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big(\mathcal{L}\left\{\sin(2\,t)\right\}(s)\Big) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{2}{s^2+4}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{-4\,s}{\left(s^2+4\right)^2}\right)\\ &= \frac{4\left(3\,s^2-4\right)}{\left(s^2+4\right)^3}\end{aligned}$$

It follows

$$\int_0^{+\infty} t^2\,\sin(2\,t)\,\mathrm{e}^{-t}\,\mathrm{d}t = \mathcal{L}\left\{t^2\,\sin(2\,t)\right\}(1) = \frac{4\,(3-4)}{(1+4)^3} = -\frac{4}{125}$$

03. Using the multiplication by t formula followed by the first shift formula, we have

$$\begin{aligned}\mathcal{L}\left\{t\,\mathrm{e}^{2\,t}\,\cos(3\,t)\,\sin(4\,t)\right\}(s) &= -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{\mathrm{e}^{2\,t}\,\cos(3\,t)\,\sin(4\,t)\right\}(s)\Big)\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\left\{\cos(3\,t)\,\sin(4\,t)\right\}(s-2)\Big)\end{aligned}$$

To compute $\mathcal{L}\left\{\cos(3\,t)\,\sin(4\,t)\right\}(s)$, we start by linearizing $\cos(3\,t)\,\sin(4\,t)$

$$\begin{aligned}\cos(3\,t)\,\sin(4\,t) &= \frac{\mathrm{e}^{3\,t\,i}+\mathrm{e}^{-3\,t\,i}}{2}\;\frac{\mathrm{e}^{4\,t\,i}-\mathrm{e}^{-4\,t\,i}}{2\,i} = \frac{\mathrm{e}^{7\,t\,i}-\mathrm{e}^{-t\,i}+\mathrm{e}^{t\,i}-\mathrm{e}^{-7\,t\,i}}{4\,i}\\ &= \frac{\left(\mathrm{e}^{7\,t\,i}-\mathrm{e}^{-7\,t\,i}\right)+\left(\mathrm{e}^{t\,i}-\mathrm{e}^{-t\,i}\right)}{4\,i} = \frac{2\,i\,\sin(7\,t)+2\,i\,\sin(t)}{4\,i}\\ &= \frac{1}{2}\,\sin(7\,t)+\frac{1}{2}\,\sin(t)\end{aligned}$$

Hence

$$\mathcal{L}\left\{\cos(3\,t)\,\sin(4\,t)\right\}(s)=\mathcal{L}\left\{\frac{1}{2}\,\sin(7\,t)+\frac{1}{2}\,\sin(t)\right\}(s)=\frac{1}{2}\,\frac{7}{s^2+49}+\frac{1}{2}\,\frac{1}{s^2+1}$$

and

$$\mathcal{L}\left\{\cos(3\,t)\,\sin(4\,t)\right\}(s-2)=\frac{1}{2}\,\frac{7}{(s-2)^2+49}+\frac{1}{2}\,\frac{1}{(s-2)^2+1}$$

Thus

$$\begin{aligned}\mathcal{L}\left\{t\,\mathrm{e}^{2\,t}\,\cos(3\,t)\,\sin(4\,t)\right\}(s)&=-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{7}{2}\,\frac{1}{(s-2)^2+49}+\frac{1}{2}\,\frac{1}{(s-2)^2+1}\right)\\&=-\left(-\frac{7}{2}\,\frac{2\,(s-2)}{\left((s-2)^2+49\right)^2}-\frac{1}{2}\,\frac{2\,(s-2)}{\left((s-2)^2+1\right)^2}\right)\\&=\frac{7\,(s-2)}{\left((s-2)^2+49\right)^2}+\frac{(s-2)}{\left((s-2)^2+1\right)^2}\end{aligned}$$

Division by t Formula

If $f(t)$ is piecewise continuous and of exponential order in $[0\,,\,+\infty)$, and if $\lim\limits_{t\to 0^+}\dfrac{f(t)}{t}=L$, where L is a real number, then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)=\int_s^{+\infty}\mathcal{L}\left\{f(t)\right\}(r)\,\mathrm{d}r$$

04a. Clearly $\dfrac{\sinh(t)}{t}$ satisfies the conditions of the division by t formula. It follows

$$\begin{aligned}\mathcal{L}\left\{\frac{\sinh(t)}{t}\right\}(s)&=\int_s^{+\infty}\mathcal{L}\left\{\sinh(t)\right\}(r)\,\mathrm{d}r=\int_s^{+\infty}\frac{1}{r^2-1}\,\mathrm{d}r\\&=\int_s^{+\infty}\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)\,\mathrm{d}r=\frac{1}{2}\left(\ln(r-1)-\ln(r+1)\right)\Big|_s^{+\infty}\\&=\frac{1}{2}\,\ln\frac{r-1}{r+1}\Big|_s^{+\infty}=\frac{1}{2}\left(\ln(1)-\ln\frac{s-1}{s+1}\right)=-\frac{1}{2}\,\ln\frac{s-1}{s+1}\\&=\frac{1}{2}\,\ln\frac{s+1}{s-1}\end{aligned}$$

04b. Clearly $\dfrac{\mathrm{e}^{3\,t}-1}{t}$ satisfies the conditions of the division by t formula. It follows

$$\begin{aligned}\mathcal{L}\left\{\frac{\mathrm{e}^{3\,t}-1}{t}\right\}(s)&=\int_s^{+\infty}\mathcal{L}\left\{\mathrm{e}^{3\,t}-1\right\}(r)\,\mathrm{d}r=\int_s^{+\infty}\left(\frac{1}{r-3}-\frac{1}{r}\right)\,\mathrm{d}r\\&=\Big(\ln(r-3)-\ln(r)\Big)\Big|_s^{+\infty}=\ln\frac{r-3}{r}\Big|_s^{+\infty}\\&=\ln(1)-\ln\frac{s-3}{s}=-\ln\frac{s-3}{s}=\ln\frac{s}{s-3}\end{aligned}$$

04c. Clearly $\dfrac{1-\cos(3\,t)}{t}$ satisfies the conditions of the division by t formula. It follows

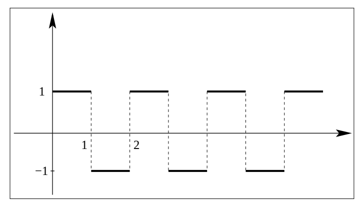
$$\begin{aligned}\mathcal{L}\left\{\frac{1-\cos(3\,t)}{t}\right\}(s) &= \int_s^{+\infty} \mathcal{L}\{1-\cos(3\,t)\}(r)\,\mathrm{d}r = \int_s^{+\infty}\left(\frac{1}{r}-\frac{r}{r^2+9}\right)\mathrm{d}r\\ &= \ln(r)-\frac{1}{2}\ln\left(r^2+9\right)\Big|_s^{+\infty} = \frac{1}{2}\ln\frac{r^2}{r^2+9}\Big|_s^{+\infty}\\ &= \frac{1}{2}\left(\ln(1)-\ln\frac{s^2}{s^2+9}\right) = -\frac{1}{2}\ln\frac{s^2}{s^2+9} = \frac{1}{2}\ln\frac{s^2+9}{s^2}\end{aligned}$$

Laplace Transform of a Periodic Function

If $f(t)$ is periodic with period T, and piecewise continuous in $[0\,,T]$, then

$$\mathcal{L}\{f(t)\}(s) = \frac{\int_0^T f(t)\,\mathrm{e}^{-s\,t}\,\mathrm{d}t}{1-\mathrm{e}^{-T\,s}}$$

05a. The graph of the function is shown in the figure to the right. The function is 2-periodic, and is known as the square wave function. We have



$$\mathcal{L}\{f(t)\}(s) = \frac{\int_0^2 f(t)\,\mathrm{e}^{-s\,t}\,\mathrm{d}t}{1-\mathrm{e}^{-2\,s}}$$

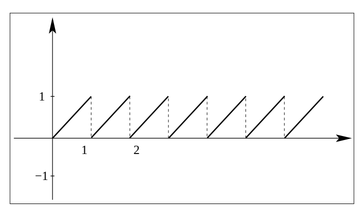
We have

$$\begin{aligned}\int_0^2 f(t)\,\mathrm{e}^{-s\,t}\,\mathrm{d}t &= \int_0^1 \mathrm{e}^{-s\,t}\,\mathrm{d}t + \int_1^2 -\mathrm{e}^{-s\,t}\,\mathrm{d}t = -\frac{1}{s}\,\mathrm{e}^{-s\,t}\Big|_{t=0}^{t=1} + \frac{1}{s}\,\mathrm{e}^{-s\,t}\Big|_{t=1}^{t=2}\\ &= -\frac{\mathrm{e}^{-s}}{s}+\frac{1}{s}+\frac{\mathrm{e}^{-2\,s}}{s}-\frac{\mathrm{e}^{-s}}{s} = \frac{\left(1-\mathrm{e}^{-s}\right)^2}{s}\end{aligned}$$

It follows

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \frac{\left(1-\mathrm{e}^{-s}\right)^2}{s\left(1-\mathrm{e}^{-2\,s}\right)} = \frac{\left(1-\mathrm{e}^{-s}\right)^2}{s\left(1-\mathrm{e}^{-s}\right)\left(1+\mathrm{e}^{-s}\right)} = \frac{1-\mathrm{e}^{-s}}{s\left(1+\mathrm{e}^{-s}\right)}\\ &= \frac{\mathrm{e}^{s/2}-\mathrm{e}^{-s/2}}{s\left(\mathrm{e}^{s/2}+\mathrm{e}^{-s/2}\right)} = \frac{2\,\sinh(s/2)}{s\,2\,\cosh(s/2)} = \frac{1}{s}\,\tanh(s/2)\end{aligned}$$

05b. The graph of the function is shown in the figure to the right. The function is 1-periodic, and is known as the saw tooth wave function. We have



$$\mathcal{L}\{f(t)\}(s) = \frac{\int_0^1 f(t)\,\mathrm{e}^{-s\,t}\,\mathrm{d}t}{1-\mathrm{e}^{-s}}$$

Using integration by parts, we have

$$\begin{aligned}\int_0^1 f(t)\,\mathrm{e}^{-s\,t}\,\mathrm{d}t &= \int_0^1 t\,\mathrm{e}^{-s\,t}\,\mathrm{d}t = -\frac{1}{s^2}\left(s\,t+1\right)\mathrm{e}^{-s\,t}\Big|_{t=0}^{t=1}\\ &= -\frac{1}{s^2}\left(s+1\right)\mathrm{e}^{-s}+\frac{1}{s^2} = \frac{-s\,\mathrm{e}^{-s}-\mathrm{e}^{-s}+1}{s^2} = \frac{1-\mathrm{e}^{-s}}{s^2}-\frac{\mathrm{e}^{-s}}{s}\end{aligned}$$

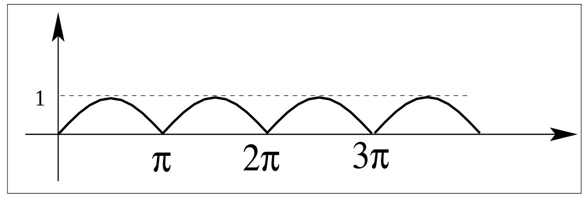
It follows

$$\mathcal{L}\{f(t)\}(s) = \frac{\dfrac{1-e^{-s}}{s^2} - \dfrac{e^{-s}}{s}}{\left(1-e^{-s}\right)} = \frac{1}{s^2} - \frac{e^{-s}}{s\left(1-e^{-s}\right)}$$

05c. The graph of the function

$$f(t) = |\sin(t)|$$

is shown in the figure to the right. The function is π-periodic, since $|\sin(t+\pi)| = |-\sin(t)| = |\sin(t)|$. It is known as the rectified sine wave function.



We have

$$\mathcal{L}\{|\sin(t)|\}(s) = \frac{\int_0^{\pi} |\sin(t)|\, e^{-s\,t}\, dt}{1-e^{-\pi\,s}} = \frac{\int_0^{\pi} \sin(t)\, e^{-s\,t}\, dt}{1-e^{-\pi\,s}}$$

Using integration by parts, we have

$$\int \sin(t)\, e^{-s\,t}\, dt = -\frac{\cos(t) + s\ \sin(t)}{s^2+1}\ e^{-s\,t}$$

Consequently

$$\int_0^{\pi} |\sin(t)|\ e^{-s\,t}\, dt = \int_0^{\pi} \sin(t)\, e^{-s\,t}\, dt = -\frac{\cos(t) + s\ \sin(t)}{s^2+1}\ e^{-s\,t}\Bigg|_{t=0}^{t=\pi}$$
$$= \frac{e^{-\pi\,s}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi\,s}}{s^2+1}$$

It follows

$$\mathcal{L}\{|\sin(t)|\}(s) = \frac{\dfrac{1+e^{-\pi\,s}}{s^2+1}}{1-e^{-\pi\,s}} = \frac{1}{s^2+1}\,\frac{1+e^{-\pi\,s}}{1-e^{-\pi\,s}}\,\frac{1}{s^2+1}\,\frac{e^{\pi\,s/2}+e^{-\pi\,s/2}}{e^{\pi\,s/2}-e^{-\pi\,s/2}} = \frac{1}{s^2+1}\ \coth\left(\pi\,s/2\right)$$

Recall that to express the piecewise defined function $f(t) = \begin{cases} f_1(t) & \text{if} \quad 0 \le t < a \\ f_2(t) & \text{if} \quad a \le t < b \\ f_3(t) & \text{if} \quad b \le t \end{cases}$ in terms of unit functions write

$$f(t) = \underbrace{f_1(t)}_{f_1(t)\text{ up to } t=a} + \underbrace{f_2(t)\ u_a(t) - f_1(t)\ u_a(t)}_{\text{at } t=a \text{ switch on } f_2(t) \text{ and switch off } f_1(t)} + \underbrace{f_3(t)\ u_b(t) - f_2(t)\big)\ u_b(t)}_{\text{at } t=b \text{ switch on } f_3(t) \text{ and switch off } f_2(t)}$$

or else

$$f(t) = \underbrace{f_1(t)}_{f_1(t)\text{ up to } t=a} + \underbrace{\Big(f_2(t) - f_1(t)\Big)}_{\text{Jump of } f(t) \text{ at } t=a}\ u_a(t) + \underbrace{\Big(f_3(t) - f_2(t)\Big)}_{\text{Jump of } f(t) \text{ at } t=b}\ u_b(t)$$

06a. Rewrite $f(t)$ in terms of unit step functions as

$$f(t) = (2\,t-1) - (2\,t-1)\,u_2(t) + t\,u_2(t) = 2\,t - 1 - (t-1)\,u_2(t)$$

Using the linearity of the Laplace transform and the second shift formula

$$\mathcal{L}\left\{g(t)\,u_a(t)\right\}(s) = \mathcal{L}\left\{g(t+a)\right\}(s)\ \mathrm{e}^{-a\,s}$$

we successively write

$$\begin{aligned}\mathcal{L}\left\{f(t)\right\}(s) &= 2\,\mathcal{L}\left\{t\right\}(s) - \mathcal{L}\left\{1\right\}(s) - \mathcal{L}\left\{(t-1)\,u_2(t)\right\}(s) = 2\,\frac{1}{s^2} - \frac{1}{s} - \mathcal{L}\left\{(t+2)-1\right\}(s)\ \mathrm{e}^{-2\,s} \\ &= \frac{2}{s^2} - \frac{1}{s} - \mathcal{L}\left\{t+1\right\}(s)\ \mathrm{e}^{-2\,s} = \frac{2}{s^2} - \frac{1}{s} - \left(\frac{1}{s^2} + \frac{1}{s}\right)\ \mathrm{e}^{-2\,s}\end{aligned}$$

06b. Rewrite $f(t)$ in terms of unit step functions as

$$f(t) = t - t\,u_1(t) + (2-t)\,u_1(t) - (2-t)\,u_2(t) + 6\,u_2(t) = t - (2\,t-2)\,u_1(t) + (t+4)\,u_2(t)$$

Using the linearity of the Laplace transform and the second shift formula, we successively write

$$\begin{aligned}\mathcal{L}\left\{f(t)\right\}(s) &= \mathcal{L}\left\{t\right\}(s) - \mathcal{L}\left\{(2\,t-2)\,u_1(t)\right\}(s) + \mathcal{L}\left\{(t+4)\,u_2(t)\right\}(s) \\ &= \frac{1}{s^2} - \mathcal{L}\left\{2\,(t+1)-2\right\}(s)\ \mathrm{e}^{-s} + \mathcal{L}\left\{(t+2)+4\right\}(s)\ \mathrm{e}^{-2\,s} \\ &= \frac{1}{s^2} - \mathcal{L}\left\{2\,t\right\}(s)\ \mathrm{e}^{-s} + \mathcal{L}\left\{t+6\right\}(s)\ \mathrm{e}^{-2\,s} \\ &= \frac{1}{s^2} - \frac{2}{s^2}\ \mathrm{e}^{-s} + \left(\frac{1}{s^2} + \frac{6}{s}\right)\ \mathrm{e}^{-2\,s}\end{aligned}$$

06c. Rewrite $f(t)$ in terms of unit step functions as

$$f(t) = t - t\,u_1(t) + 0\,u_1(t) - 0\,u_2(t) + \mathrm{e}^{-2\,t}\,u_2(t) = t - t\,u_1(t) + \mathrm{e}^{-2\,t}\,u_2(t)$$

Using the linearity of the Laplace transform, the second and first shift formula, we successively write

$$\begin{aligned}\mathcal{L}\left\{f(t)\right\}(s) &= \mathcal{L}\left\{t\right\}(s) - \mathcal{L}\left\{t\,u_1(t)\right\}(s) + \mathcal{L}\left\{\mathrm{e}^{-2\,t}\,u_2(t)\right\}(s) \\ &= \frac{1}{s^2} - \mathcal{L}\left\{(t+1)\right\}(s)\ \mathrm{e}^{-s} + \mathcal{L}\left\{u_2(t)\right\}\big(s-(-2)\big) \\ &= \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{s}\right)\ \mathrm{e}^{-s} + \mathcal{L}\left\{u_2(t)\right\}(s+2) \\ &= \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{s}\right)\ \mathrm{e}^{-s} + \frac{\mathrm{e}^{-2\,(s+2)}}{s+2}\end{aligned}$$

Note: We could have computed $\mathcal{L}\left\{\mathrm{e}^{-2\,t}\,u_2(t)\right\}(s)$ by using the second shift formula followed by the first. Doing that would have given

$$\begin{aligned}\mathcal{L}\left\{\mathrm{e}^{-2\,t}\,u_2(t)\right\}(s) &= \mathcal{L}\left\{\mathrm{e}^{-2\,(t+2)}\right\}(s)\ \mathrm{e}^{-2\,s} = \mathcal{L}\left\{\mathrm{e}^{-4}\ \mathrm{e}^{-2\,t}\right\}(s)\ \mathrm{e}^{-2\,s} \\ &= \mathrm{e}^{-4}\ \frac{1}{s-(-2)}\ \mathrm{e}^{-2\,s} = \frac{\mathrm{e}^{-2\,(s+2)}}{s+2}\end{aligned}$$

06d. Rewrite $f(t)$ in terms of unit step functions as

$$f(t) = 1 - 1\,u_\pi(t) + t\,\sin(t)\,u_\pi(t) = 1 + \big(t\,\sin(t) - 1\big)\,u_\pi(t)$$

Using the linearity of the Laplace transform and the second shift formula, we successively write

$$\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \mathcal{L}\{1\}(s) + \mathcal{L}\left\{\big(t\,\sin(t) - 1\big)\,u_\pi(t)\right\}(s) = \frac{1}{s} + \mathcal{L}\{(t+\pi)\,\sin(t+\pi) - 1\}(s)\,\mathrm{e}^{-\pi s} \\
&= \frac{1}{s} + \mathcal{L}\{-(t+\pi)\,\sin(t) - 1\}(s)\,\mathrm{e}^{-\pi s} \\
&= \frac{1}{s} - \Big(\mathcal{L}\{t\,\sin(t)\}(s) + \pi\,\mathcal{L}\{\sin(t)\}(s) + \mathcal{L}\{1\}(s)\Big)\,\mathrm{e}^{-\pi s} \\
&= \frac{1}{s} - \left(\mathcal{L}\{t\,\sin(t)\}(s) + \pi\,\frac{1}{s^2+1} + \frac{1}{s}\right)\,\mathrm{e}^{-\pi s}
\end{aligned}$$

To compute $\mathcal{L}\{t\,\sin(t)\}(s)$, we use the multiplication by t formula to get

$$\mathcal{L}\{t\,\sin(t)\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathcal{L}\{\sin(t)\}(s)\Big) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = \frac{2\,s}{\left(s^2+1\right)^2}$$

Hence

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{s} - \left(\frac{2\,s}{\left(s^2+1\right)^2} + \frac{\pi}{s^2+1} + \frac{1}{s}\right)\,\mathrm{e}^{-\pi s}$$

If $f(t)$ is continuous and of exponential order in $[0\,,+\infty)$, and if $f'(t)$ is piecewise continuous in $[0\,,+\infty)$, then

$$\mathcal{L}\left\{f'(t)\right\}(s) = s\,\mathcal{L}\{f(t)\}(s) - f(0)$$

A similar formula applies to the second derivative.

If $f(t)$, $f'(t)$ are continuous and of exponential order in $[0\,,+\infty)$, and if $f''(t)$ is piecewise continuous in $[0\,,+\infty)$, then

$$\mathcal{L}\left\{f''(t)\right\}(s) = s^2\,\mathcal{L}\{f(t)\}(s) - s\,f(0) - f'(0)$$

More generally

If $f(t), f'(t), \cdots, f^{(n-1)}(t)$ are continuous and of exponential order in $[0\,,+\infty)$, and if $f^{(n)}(t)$ is piecewise continuous in $[0\,,+\infty)$, then

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n\,\mathcal{L}\{f(t)\}(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - f^{(n-2)}(0)$$

07a. Substituting $\cos(b\,t)$ for $f(t)$ in the formula $\mathcal{L}\left\{f''(t)\right\}(s) = s^2\,\mathcal{L}\{f(t)\}(s) - s\,f(0) - f'(0)$, leads to

$$\begin{aligned}
\mathcal{L}\left\{-b^2\,\cos(b\,t)\right\}(s) = s^2\,\mathcal{L}\{\cos(b\,t)\}(s) - s \quad &\Longrightarrow\quad -b^2\,\mathcal{L}\{\cos(b\,t)\}(s) = s^2\,\mathcal{L}\{\cos(b\,t)\}(s) - s \\
&\Longrightarrow\quad s = \left(s^2 + b^2\right)\mathcal{L}\{\cos(b\,t)\}(s) \\
&\Longrightarrow\quad \mathcal{L}\{\cos(b\,t)\}(s) = \frac{s}{s^2+b^2}
\end{aligned}$$

07b. Substituting $\sin(b\,t)$ for $f(t)$ in the formula $\mathcal{L}\left\{f''(t)\right\}(s) = s^2\,\mathcal{L}\left\{f(t)\right\}(s) - s\,f(0) - f'(0)$, leads to

$$\begin{aligned}
\mathcal{L}\left\{-b^2\,\sin(b\,t)\right\}(s) = s^2\,\mathcal{L}\left\{\sin(b\,t)\right\}(s) - b \;&\Longrightarrow\; -b^2\,\mathcal{L}\left\{\sin(b\,t)\right\}(s) = s^2\,\mathcal{L}\left\{\sin(b\,t)\right\}(s) - b\\
&\Longrightarrow\; b = \left(s^2+b^2\right)\mathcal{L}\left\{\sin(b\,t)\right\}(s)\\
&\Longrightarrow\; \mathcal{L}\left\{\sin(b\,t)\right\}(s) = \frac{b}{s^2+b^2}
\end{aligned}$$

07c. Substituting $\cosh(b\,t)$ for $f(t)$ in the formula $\mathcal{L}\left\{f''(t)\right\}(s) = s^2\,\mathcal{L}\left\{f(t)\right\}(s) - s\,f(0) - f'(0)$, leads to

$$\begin{aligned}
\mathcal{L}\left\{b^2\,\cosh(b\,t)\right\}(s) = s^2\,\mathcal{L}\left\{\cosh(b\,t)\right\}(s) - s \;&\Longrightarrow\; b^2\,\mathcal{L}\left\{\cosh(b\,t)\right\}(s) = s^2\,\mathcal{L}\left\{\cosh(b\,t)\right\}(s) - s\\
&\Longrightarrow\; s = \left(s^2-b^2\right)\mathcal{L}\left\{\cos(b\,t)\right\}(s)\\
&\Longrightarrow\; \mathcal{L}\left\{\cosh(b\,t)\right\}(s) = \frac{s}{s^2-b^2}
\end{aligned}$$

07d. Substituting $\sinh(b\,t)$ for $f(t)$ in the formula $\mathcal{L}\left\{f''(t)\right\}(s) = s^2\,\mathcal{L}\left\{f(t)\right\}(s) - s\,f(0) - f'(0)$, leads to

$$\begin{aligned}
\mathcal{L}\left\{b^2\,\sinh(b\,t)\right\}(s) = s^2\,\mathcal{L}\left\{\sinh(b\,t)\right\}(s) - b \;&\Longrightarrow\; b^2\,\mathcal{L}\left\{\sinh(b\,t)\right\}(s) = s^2\,\mathcal{L}\left\{\sinh(b\,t)\right\}(s) - b\\
&\Longrightarrow\; b = \left(s^2-b^2\right)\mathcal{L}\left\{\sinh(b\,t)\right\}(s)\\
&\Longrightarrow\; \mathcal{L}\left\{\sinh(b\,t)\right\}(s) = \frac{b}{s^2-b^2}
\end{aligned}$$

08a. Taking the Laplace Transform of $y'' + y = t$, leads to

$$\begin{aligned}
\mathcal{L}\left\{y''(t)\right\}(s) + \mathcal{L}\left\{y(t)\right\}(s) = \mathcal{L}\left\{t\right\}(s) \;&\Longleftrightarrow\; s^2\,\mathcal{L}\left\{y(t)\right\}(s) - s\,y(0) - y'(0) \;+\; \mathcal{L}\left\{y(t)\right\}(s) = \frac{1}{s^2}\\
&\Longleftrightarrow\; \left(s^2+1\right)\mathcal{L}\left\{y(t)\right\}(s) - s + 2 = \frac{1}{s^2}\\
&\Longleftrightarrow\; \left(s^2+1\right)\mathcal{L}\left\{y(t)\right\}(s) = \frac{s^3-2\,s^2+1}{s^2}\\
&\Longleftrightarrow\; \mathcal{L}\left\{y(t)\right\}(s) = \frac{s^3-2\,s^2+1}{s^2\left(s^2+1\right)}
\end{aligned}$$

08b. Taking the Laplace Transform of $y'' + 4\,y = f(t)$, leads to
$\mathcal{L}\left\{y''(t)\right\}(s) + 4\,\mathcal{L}\left\{y(t)\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s)$, or else
$s^2\,\mathcal{L}\left\{y(t)\right\}(s) - s\,y(0) - y'(0) \;+\; 4\,\mathcal{L}\left\{y(t)\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s)$, which is equivalent to

$$\left(s^2+4\right)\mathcal{L}\left\{y(t)\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s) \;\Longleftrightarrow\; \mathcal{L}\left\{y(t)\right\}(s) = \frac{\mathcal{L}\left\{f(t)\right\}(s)}{s^2+4}$$

To compute $\mathcal{L}\left\{f(t)\right\}(s)$, we start by expressing $f(t)$ in terms of unit step functions as

$$f(t) = 1 - u_1(t) - u_1(t) + u_2(t) + 0\,u_2(t) = 1 - 2\,u_1(t) + u_2(t)$$

It follows

$$\mathcal{L}\left\{f(t)\right\}(s) = \mathcal{L}\left\{1\right\}(s) - 2\,\mathcal{L}\left\{u_1(t)\right\}(s) + \mathcal{L}\left\{u_2(t)\right\}(s) = \frac{1}{s} - 2\,\frac{\mathrm{e}^{-s}}{s} + \frac{\mathrm{e}^{-2\,s}}{s}$$

Hence

$$\mathcal{L}\left\{y(t)\right\}(s) = \frac{1}{s\left(s^2+4\right)} - \frac{2\,\mathrm{e}^{-s}}{s\left(s^2+4\right)} + \frac{\mathrm{e}^{-2\,s}}{s\left(s^2+4\right)} = \frac{\left(1-\mathrm{e}^{-s}\right)^2}{s\left(s^2+4\right)}$$