Department of Mathematics and Statistics
MATH 375 - Outline # 1
First Order Differential Equations

Linear equations
Type: $y' + p(x)y = q(x)$
Method of solution (integrating factor): multiply both sides by the integrating factor

$$\mu(x) = e^{\int p(x)\ dx},$$

then we have the derivative of the product $\mu(x)y(x)$ in the left-hand side

$$e^{\int p(x)\ dx}y' + p(x)e^{\int p(x)\ dx}y = \frac{d}{dx}\left(e^{\int p(x)\ dx}y\right) = q(x)e^{\int p(x)\ dx},$$

thus

$$e^{\int p(x)\ dx}y(x) = C + \int e^{\int p(x)\ dx}\ dx,$$

or

$$y(x) = Ce^{-\int p(x)\ dx} + e^{-\int p(x)\ dx}\int e^{\int p(x)\ dx}\ dx.$$

There is no need to memorize this formula. It is sufficient to remember the formula for the integrating factor.

Example 1. Find the general solution of the equation $y' + xy - x = 0$.

Solution. The equation is linear $y' + xy = x$, so we first find the integrating factor

$$\mu(x) = e^{\int x\ dx} = e^{x^2/2}.$$

Note that $\mu' x = xe^{x^2/2}$. After multiplying by $\mu(x)$, we have

$$e^{x^2/2}y' + xe^{x^2/2}y = \left(e^{x^2/2}y\right)' = xe^{x^2/2},$$

integrating, we obtain

$$e^{x^2/2}y(x) = \int xe^{x^2/2}\ dx = C + e^{\frac{x^2}{2}} \ \Rightarrow\ y = Ce^{-\frac{x^2}{2}} + 1.$$

Separable equations
Type: $y' = f(x)g(y)$
Method of solution: separate factors with y in the left hand side, factors with x in the right hand side and integrate both sides.

Example 2. Find the general solution of the equation $y' = e^{x+y}$.
Solution. $e^{x+y} = e^x e^y$, so

$$\frac{dy}{dx} = e^x e^y \Rightarrow \frac{dy}{e^y} = e^{-y}\ dy = e^x\ dx \Rightarrow \int e^{-y}\ dy = \int e^x\ dx$$

$$C - e^{-y} = e^x \Rightarrow e^{-y} = C - e^x \Rightarrow -y = \ln\left(C - e^{-x}\right) \Rightarrow y = -\ln\left(C - e^{-x}\right) = \ln\left(\frac{1}{C - e^{-x}}\right).$$

Remark. We divided by e^y which does not vanish.

Exact equations
Type: $M(x,y)dx + N(x,y)dy = 0$, where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Method of solution: we check that the equation is exact (should be $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ for any x, y) and look for such a function that $\frac{\partial \phi}{\partial x} = M$, $\frac{\partial \phi}{\partial y} = N$. Then the general solution is $\phi(x,y) = C$.

Example 3. Find the general solution of the equation $2x + 3x^2 y + (x^3 - 3y^2)y' = 0$.
Solution. Here $M(x,y) = 2x + 3x^2 y, N(x,y) = x^3 - 3y^2$, $\frac{\partial M}{\partial y} = 3x^2$, $\frac{\partial N}{\partial x} = 3x^2$, so the equation is exact. Thus there exists $\phi(x,y)$ such that $\frac{\partial \phi}{\partial x} = M$, $\frac{\partial \phi}{\partial y} = N$.

$$\frac{\partial \phi}{\partial x} = 2x + 3x^2 y \Rightarrow \phi(x,y) = \int \left(2x + 3x^2 y\right) dx = x^2 + x^3 y + C(y), \quad \text{so}$$

$$\frac{\partial \phi}{\partial y} = x^3 + C'(y) = N(x,y) = x^3 - 3y^2 \Rightarrow C'(y) = -3y^2 \Rightarrow C(y) = \int (-3y^2) dy = -y^3.$$

Finally, $\phi(x,y) = x^2 + x^3 y + C(y) = x^2 + x^3 y - y^3$ and the general solution is

$$x^2 + x^3 y - y^3 = C.$$

Bernoulli's equations
Type: $y' + p(x)y = q(x)y^n$
Method of solution: the equation is divided by y^n; after the substitution $z = 1/y^{n-1}$, $z' = y'/(ny^n)$ we obtain a linear equation.

Example 4. Find the general solution of the equation $y' + y + xy^2 = 0$.
Solution. $y' + y + xy^2 = 0$, after dividing by y^2 we have

$$\frac{y'}{y^2} + \frac{1}{y} + x = 0, \quad \text{after substituting} \quad z = \frac{1}{y}, z' = -\frac{y'}{y^2} \quad \text{we obtain}$$

$-z' + z + x = 0$ which is a linear equation $z' - z = x$. The integrating factor is $\mu(x) = e^{\int -dx} = e^{-x}$, after multiplication we have

$$z'e^{-x} - ze^{-x} = \left(ze^{-x}\right)' = xe^{-x}.$$

Thus, applying integration by parts with $u_1 = x, dv = e^{-x}dx$, $v = -e^{-x}$, we have

$$z(x)e^{-x} = \int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = C - xe^{-x} - e^{-x}.$$

Thus $z = e^x(C - xe^{-x} - e^{-x}) = Ce^x - x - 1$, and
$y = \dfrac{1}{z} = \dfrac{1}{Ce^x - x - 1}$.

"Homogeneous" equations

Type: $y' = F(\frac{y}{x})$

Method of solution: the substitution $t = \frac{y}{x}$ leads to the separable equation with respect to $t = t(x)$; thus $y = tx$, $y' = t + xt'$.

Example 5. Find the general solution of the equation $2x^3y' = y(2x^2 - y^2)$.

Solution. The equation is homogeneous since $y' = \frac{y}{2x^3}(2x^2 - y^2) = \frac{y}{x} - \frac{1}{2}(\frac{y}{x})^3$. After the substitution $t = \frac{y}{x}$, $y' = t + xt'$ we have

$$t + xt' = t - \frac{1}{2}t^3 \Rightarrow xt' = x\frac{dt}{dx} = t - \frac{1}{2}t^3 - t = -\frac{1}{2}t^3$$

$$\Rightarrow -\frac{2dt}{t^3} = \frac{dx}{x} \Rightarrow -\int \frac{2dt}{t^3} = \int \frac{dx}{x} \Rightarrow \frac{1}{t^2} = \ln|x| + C_1 = \ln|x| + \ln C,\ C > 0$$

$$\frac{1}{t^2} = \ln(Cx),\ C \neq 0, \Rightarrow \frac{x^2}{y^2} = \ln(Cx) \Rightarrow y^2 = \frac{x^2}{\ln(Cx)} \Rightarrow y = \pm\frac{x}{\sqrt{\ln(Cx)}}$$

In the process of the solution we divided by x and $t = \frac{y}{x}$, let us check whether $x = 0$ and $y = 0$ are solutions. By substitution we get that $x = 0$ is not a solution while $y = 0$ (and thus $y' = 0$) turns the equation into an identity, so $y = 0$ is a solution.

Existence and Uniqueness Theorem. If $p(t)$ and $q(t)$ are continuous on an open interval (a, b) containing the point $t = t_0$ then there exists a unique solution of the initial value problem

$$y' + p(t)y = q(t), \quad y(t_0) = y_0$$

for any $y_0 \in \mathbb{R}$ on (a, b).

Example 6. Find an interval where the initial value problem

$$\ln(t^2 - 1)y' + (t^4 - 9)y = 5\cos(t),\ y(3) = 2$$

has a unique solution.

Solution. Rewriting the equation as $y' + \dfrac{t^4 - 9}{\ln(t^2 - 1)}y = \dfrac{5\cos(t)}{\ln(t^2 - 1)}$, we see that the coefficients are continuous whenever $\ln(t^2 - 1)$ is continuous and different from zero, $t \in (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, -1) \cup (1, \sqrt{2}) \cup (\sqrt{2}, \infty)$. The maximal interval containing 3 is $(\sqrt{2}, \infty)$.

WHENEVER DIVIDE, CHECK IF SOLUTION