AMAT 375
Handout # 6 - Answers, Hints, Solutions
Systems of linear differential equations

1. Find the general solution of the following systems of differential equations

$$a)\ \ X'(t)=\begin{bmatrix} 2 & 1\\ 3 & 4\end{bmatrix}X(t) \qquad b)\ X'(t)=\begin{bmatrix} 1 & -3\\ 3 & 1\end{bmatrix}X(t)$$

$$c)\ X'(t)=\begin{bmatrix} 3 & 1\\ -5 & -3\end{bmatrix}X \qquad d)\ X'=\begin{bmatrix} 1 & -1 & 1\\ 1 & 1 & -1\\ 2 & -1 & 0\end{bmatrix}X(t)$$

$$e)\ X'=\begin{bmatrix} 1 & -2 & -1\\ -1 & 1 & 1\\ 1 & 0 & -1\end{bmatrix}X \qquad f)\ \ X'=\begin{bmatrix} 3 & -1 & 1\\ 1 & 1 & 1\\ 4 & -1 & 4\end{bmatrix}X$$

$$g)\ X'=\begin{bmatrix} -1 & -5\\ 1 & 1\end{bmatrix}X(t)$$

Solutions and answers. a) First we find eigenvalues and eigenvectors of A.

$$det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 1\\ 3 & 4-\lambda\end{vmatrix}=(2-\lambda)(4-\lambda)-3=\lambda^2-6\lambda+5=(\lambda-1)(\lambda-5)=0.$$

The eigenvalues are $\lambda_1=1$ and $\lambda_2=5$, and eigenvectors are solutions of $(A-I)v_1=0$ and $(A-5I)v_2$, respectively. Solving the systems, we get

$$A-I=\begin{bmatrix} 1 & 1\\ 3 & 3\end{bmatrix}\Rightarrow v_1=\begin{bmatrix} 1\\ -1\end{bmatrix},\quad A-5I=\begin{bmatrix} -3 & 1\\ 3 & -1\end{bmatrix}\Rightarrow v_2=\begin{bmatrix} 1\\ 3\end{bmatrix}.$$

Thus the general solution is

$$X=C_1v_1e^{\lambda_1 t}+C_2v_2e^{\lambda_2 t}=C_1\begin{bmatrix} 1\\ -1\end{bmatrix}e^t+C_2\begin{bmatrix} 1\\ 3\end{bmatrix}e^{5t}=\begin{bmatrix} C_1e^t+C_2e^{5t}\\ -C_1e^t+3C_2e^{5t}\end{bmatrix}.$$

b) As $det(A-\lambda I)=(\lambda-1)^2-(-3)\cdot 3=\lambda^2-2\lambda+10$, the eigenvalues are $\lambda=\dfrac{2\pm\sqrt{2^2-40}}{2}=1\pm 3i$, the eigenvectors are $v_1=(i,1)^T, v_2=(-i,1)^T$ (it is enough to find only one, the second one is conjugate). One of the solutions is

$$X(t)=v_1e^t(\cos(3t)+i\sin(3t))=e^tv_1e^t(\cos(3t)+i\sin(3t))$$

$$=e^t\begin{bmatrix} i(\cos(3t)+i\sin(3t))\\ \cos(3t)+i\sin(3t)\end{bmatrix}=e^t\begin{bmatrix} -\sin(3t)\\ \cos(3t)\end{bmatrix}+ie^t\begin{bmatrix} \cos(3t)\\ \sin(3t)\end{bmatrix},$$

its real and imaginary parts are fundamental solutions. Thus the general solution is their combination

$$X(t) = e^t \left(C_1 \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} C_2 \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} + \right).$$

c) $X = C_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}.$

d) $X = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} e^{-t}.$

e) $X = C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} e^{-t}.$

f) $X = C_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{5t}.$

g) The eigenvalues are the roots of the equation $(x-1)(x+1)+5 = x^2-1+5 = x^2+4 = 0$, so $\lambda = \pm 2i$. Solving the system of two proportional equations $(-1-2i)r - 5s = 0$, $r + (1-2i)s = 0$, we have, for example, an eigenvector $v_1 = (1-2i, -1)^T$ corresponding to $\lambda_1 = 2i$. Thus the real and the imaginary part of this solution can be found from

$$e^{2it} \begin{bmatrix} 1-2i \\ 1 \end{bmatrix} = (\cos(2t) + i\sin(2t)) \begin{bmatrix} 1-2i \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2t) + 2\sin(2t) \\ -\cos(2t) \end{bmatrix} + i \begin{bmatrix} \sin(2t) - 2\cos(2t) \\ -\sin(2t) \end{bmatrix},$$

and the general solution is $X = C_1 \begin{bmatrix} \cos(2t) + 2\sin(2t) \\ -\cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) - 2\cos(2t) \\ -\sin(2t) \end{bmatrix}.$

2. Find the solution of the initial value problem

$$X' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} X, X(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Answer. The general solution is

$$X(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t, \quad X(0) = \begin{bmatrix} C_1 - C_2 \\ C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

thus $C_1 - C_2 = 1$, $C_1 + C_2 = 3$. Thus $2C_1 = 4$, or $C_1 = 2$, $C_2 = 1$, and
$X = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t == \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t.$

3. Find the general solution of the system $X'(t) = AX(t)$, where for the matrix $A = 2\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}$ we have that the determinant of $A - \lambda I$ is $(7-\lambda)(\lambda^2 - 4\lambda + 5$.

Answer. The eigenvalues are $7, 2 \pm i$. The general solution is

$$X(t) = C_1 e^{7t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 5\sin(t) - \cos(t) \\ \sin(t) + 5\cos(t) \\ -\cos(t) \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\sin(t) - 5\cos(t) \\ 5\sin(t) - \cos(t) \\ -\sin(t) \end{bmatrix}$$

4. For a system of two equations $X'(t) = AX(t)$ with a real constant matrix A which has an eigenvalue $\lambda = 6 + i$ and a corresponding eigenvector $X = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$, find the general solution.

Solution. The real and the imaginary part of this solution can be found from

$$e^{(6+i)t} \begin{bmatrix} 1-i \\ 1 \end{bmatrix} = e^{6t}(\cos(t) + i\sin(t)) \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

$$= e^{6t} \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix} + ie^{6t} \begin{bmatrix} \sin(t) - \cos(t) \\ \sin(t) \end{bmatrix},$$

and the general solution is $X = C_1 e^{6t} \begin{bmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} \sin(t) - \cos(t) \\ \sin(t) \end{bmatrix}$.

5. For a homogeneous linear system of order three $X'(t) = AX(t)$, where a real constant matrix A has eigenvalues $3, -5, 0$ and corresponding eigenvectors $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}$, find the general solution.

Answer. $X(t) = C_1 e^{3t} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} + C_3 \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}$

6. Find the general solution of the system $x_1'(t) = 2x_1 + 3x_3$, $x_2'(t) = 2x_2$, $x_1'(t) = 3x_1 + 2x_3$.

Answer. $X(t) = C_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, or $x_1(t) = C_2 e^{5t} + C_3 e^{-t}$, $x_2(t) = C_1 e^{2t}$, $x_1(t) = C_2 e^{5t} - C_3 e^{-t}$.