MATH 375 - Fall 2016
Lectures Follow-up

Week 1

Various types of equations: linear, quadratic, trigonometric, logarithmic. In all of them the unknown is a number. In a differential equation the unknown is a function! Differential equations as a main motivation for developing calculus.

The notion of an ordinary differential equation as an equation connecting an unknown function of a single independent variable with some of its derivatives in this variable. Dependent and independent variables, order of differential equations is the order of the highest derivative present. A solution is a function (with sufficient number of derivatives) such that its substitution turns the equation into an identity on a given interval.

Example 1: the Malthusian model $N'(t) = rN(t)$ is a first order equation. If $r = 8$, for which k will $N(t) = e^{kt}$ be a solution of this equation?

Example 2: for a falling ball, the differential equation $\ddot{y} = -g$ is a second order equation. How can we solve an ordinary differential equation? After integrating $y''(t) = -g$ twice, we have *the general solution* $y = -\frac{gt^2}{2} + C_1 t + C_2$. Usually a differential equation has an infinite number of solutions. A collection of all solutions is called the general solution. To pick up a specific (particular) solution, we need an initial condition. For example, if the initial location is 1.5 m, and the initial velocity is 0.5 mps, the particular solution is $y = -\frac{gt^2}{2} + 0.5t + 1.5$.

An equation is linear if the function and its derivatives are included linearly (all functions of an independent variable are treated as constants). Example 3: the equation $e^s \frac{d^3u}{ds^3} = \sin(s^5)\frac{du}{ds} - s^5 \ln(s^4 + s^2 + 1)$ is linear, while $\left(\frac{du}{ds}\right)^2 + 5u = s + 7$ is not. Linear equations play an important role in this course.

An initial value problem is a differential equation with an initial condition. For n-th order equations, we have n initial conditions:

$$y^{(n)}(t) = f\left(t, y, y', y'', \dots, y^{(n-1)}\right), \quad y(t_0) = y_0, y'(t_0) = y_0', \dots, y^{(n-1)}(t_0) = y_0^{n-1}.$$

Partial differential equations connect an unknown function of several variables with its derivatives. Example 4. The equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ (heat transfer equation) is a linear second order partial differential equation.

Example 5. Another example of a linear equation is the Newton law of cooling (heating) $T'(t) = -k\left(T - T_{medium}\right)$. Let us remark that equations that do not include an independent variable explicitly are called autonomous equations. There is also a differential form of first order ordinary differential equations: $dT = -k\left(T - T_{medium}\right)\, dt$.

Linear first order differential equations. Their general form is

$$y'(t) + p(t)y = q(t),$$

where the left-hand side is not a derivative but after multiplication by the integrating factor $\mu(t) = e^{\int p(t)\, dt}$ we have

$$e^{\int p(t)\, dt} y'(t) + p(t) e^{\int p(t)\, dt} y(t) = q(t) e^{\int p(t)\, dt},$$

which can be rewritten as $\frac{d}{dt}\left(e^{\int p(t)\, dt} y\right)' = q(t) e^{\int p(t)\, dt}$ and integrated.

Example 1. For $y' + \frac{3}{x} y - x = 0$, the integrating factor is $\mu(x) = x^3$ (do not introduce a constant of integration and mind operations with exponentials), and the general solution is $y = \frac{x^2}{5} + \frac{C}{x^3}$.

Example 2. Consider the initial value problem $\cos(t) y' = y \sin(t) + \cos^2(t)$, $y(0) = 4$. The general solution is $y = \frac{C}{\cos(t)} + \frac{t}{2\cos(t)} + \frac{\sin(t)}{2}$, substituting $t = 0$, we get $C = 4$, and $y(t) = \frac{4}{\cos(t)} + \frac{t}{2\cos(t)} + \frac{\sin(t)}{2}$.

Theorem. The solution of an initial value problem for a linear equation exists and is unique on the interval where coefficients p and q are continuous.

Example 3. Find the interval on which a solution of the initial value problem for the equation $(t^2 - 9)y' + \ln(t + 7)y = \cos(3t)$ is guaranteed to have a solution if the initial condition is: $y(0) = 5$ (the interval is $(-3, 3)$), $y(-5) = 2$ (the interval is $(-7, -3)$).

For nonlinear equations, we cannot easily find where the solution exists by inspecting the equation.

Separable equations have the form $\frac{dy}{dt} = f(t)g(y)$. The method of solution consists of writing the equation in a differential form, separating y and t in different parts of the equation and integrating.

Example 1. Solve the initial value problems $y' = -\frac{x}{y}$, $y(1) = 1$ and $y' = -\frac{x}{y}$, $y(1) = -2$ and find the interval where the solution exists. In the first case $y(x) = \sqrt{2 - x^2}$ exists on $(-\sqrt{2}, \sqrt{2})$, in the second case $y(x) = -\sqrt{5 - x^2}$ exists on $(-\sqrt{5}, \sqrt{5})$.

Week 2

Let us note that it is not always easy to express $y(x)$ or $y(t)$ explicitly. Sometimes we have solutions that are not parts of the general solution (singular solutions).

Example 2. The general solution of the equation $y'(1 + x^2) = \frac{1}{2}(y^2 - 1)$ is $\frac{y-1}{y+1} = Ce^{\arctan(x)}$, or $y = \frac{1 + Ce^{\arctan(x)}}{1 - Ce^{\arctan(x)}}$, for any $C \in \mathbb{R}$. A *singular solution* is $y = -1$.

We can solve a differential equation if we present it in the form $dF(x, y) = 0$.

Exact equations in the differential form can be written as $M(x, y)\, dx + N(x, y)\, dy = 0$

if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. We are looking for the potential function $F(x, y)$ such that

$$M(x, y) = \frac{\partial F}{\partial x}, \quad N(x, y) = \frac{\partial F}{\partial y}.$$

The general solution is $F(x, y) = C$.

Example 1. The equation $(2xy + 3y^2)y' + y^2 + \cos(x) = 0$ has the general solution $xy^2 + y^3 + \sin(x) = C$.

Example 2. The equation $(y\cos(x) + 2xe^y)\ dx + (\sin(x) + x^2e^y - 1)\ dy = 0$ has the general solution $y\sin(x) + x^2e^y - y = C$.

Example 3. The equation $ax^3y^2 + 6x^byy' = 0$ is exact for $a = 12$, $b = 4$ only.

Bernoulli equations $y' + p(x)y = q(x)y^n$ are solved with the substitution $z = y^{1-n}$, $n \neq 0, 1$, we obtain a linear equation. Example 1. The equation $y' + y + xy^2 = 0$ has the general solution $y = (Ce^x - x - 1)^{-1}$.

Example 2. For the equation $ty' + y = (10t^2 + 3)y^{-2}$, the general solution is $y = \left(6t^2 + 3 + \dfrac{C}{t^3}\right)^{1/3}$.

First order "homogeneous" equations: if we have $y' = f(y/x)$, the substitution $u = \frac{y}{x}$ leads to a separable equation for $u = u(x)$.

Example 1. The equation $y' = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$ has the general solution $\sin\left(\dfrac{y}{x}\right) = Cx$.

Example 2. The initial value problem $(x - y)y' + x + y = 0$, $y(1) = 0$ has the solution $x^2 + 2xy - y^2 = 1$.

An integrating factor: Sometimes $M(x, y)\ dx + N(x, y)\ dy = 0$ is not an exact equation but we can multiply by $\mu(x)$, or $\mu(y)$, or $\mu(x, y)$ to get an exact equation. For example, if there is $\mu(x)$, then $\dfrac{\mu'}{\mu} = \dfrac{M_y - N_x}{N}$ should depend on x only.

Example 1. The integrating factor $\mu(x)$ for $(x + y^2)dx - 2xy\ dy = 0$ is $\mu(x) = \dfrac{1}{x^2}$, the general solution is $\ln|x| - \dfrac{y^2}{x} = C$.

Example 2. The integrating factor $\mu(y)$ for $y(1 + xy)dx - x\ dy = 0$ is $\mu(x) = \dfrac{1}{y^2}$, the general solution is $\dfrac{x}{y} + \dfrac{x^2}{2} = C$.

Example 3. For what values of m and n will $\mu(x, y) = x^ny^m$ be an integrating factor for the differential equation $(6y + 14x)dx + (4x + 6x^2y^{-1})dy = 0$? The values are $n = 5$, $m = 3$, and the solution is $-x^6y^4 + 2x^7y^3 = C$.

Week 3

Applications of first order equations. **Exponential growth and decay** $y' = ky$, where k is either positive (growth) or negative (decay).

Example 1: if a capital is doubled in 8 years, how much time it will take (for the same constant interest rate) for it to triple? ($8\ln(3)/\ln(2) \approx 12.7$ years).

Newton's law of cooling $T'(t) = k(S(t) - T(t))$. Example 2: a metal object is heated to 200°C and then placed in a large room with a constant temperature of 20°C to cool. After 10 min, the temperature of the object is 100°C. How long will it take the object to cool to 25°C? (≈ 44.19 min). When was the metal object 140°C? (5 min after brought to the room).

Mixing problems $Q'(t) = c_1 r_1 - c_2 r_2$. Example 3. A tank contains 1000 litres of water with 2 kg of salt. A valve is opened and water containing 0.02 kg of salt per litre flows into the tank at a rate of 5 litres/min. The mixture is well sturred and drains from the tank at a rate of 5 litres/min. Find $Q(t)$ $(20 - 18e^{-0.005t})$, $Q_L = \lim_{t\to\infty} Q(t)$ (20) . When will the limit be 99% of Q_L? (≈ 15 hours).

Electric circuits: definitions of an emf, a resistor, an inductor, a capacitor and a key (switch). The relation $Q'(t) = I(t)$ between the charge and the current, the voltage drop across the resistor (Ohm's law) $E_R = RI$ (R is the resistance), the inductor $E_L = LI'(t)$ (L is the inductance), and the capacitor $E_C = \frac{Q}{C}$ (C is the capacitance). The Kirchhoff law. Model 1: the "LR" circuit leads to

$$E = L\frac{dI}{dt} + RI.$$

Model 2: the "CR" circuit leads to

$$E = R\frac{dQ}{dt} + \frac{Q}{C}.$$

Example 4: an energy source with emf 100 volts is connected in a series with a 10 ohm resistor and an inductor of 2 henries. If the switch is closed at time $t = 0$, what is the current $I(t)$? ($I(t) = 10(1 - e^{-5t})$, $t > 0$).

Example 5: a variable energy source with emf $E = 200e^{-50t}$ volts is connected in a series with a 2 ohm resistor and a 0.01 farad capacitor. If the initial charge on the capacitor is zero, find the current and the charge at any time $t > 0$ ($Q(t) = 100te^{-50t}$, $I(t) = 100e^{-50t}(1 - 50t)$), and the maximal charge stored (at $t = \frac{1}{50}$, $Q(0.02) = \frac{2}{e} \approx 0.73576$ coulomb.

General review of first order equations: determine the type of a differential equation.

Differential equations of the second order: the importance in applications (the equations of motion, the equation of a spring) and the general form. Homogeneous equations. A combination of two solutions is a solution. The notion of linearly independent functions. Example 1: are the following functions linearly dependent: e^x and e^{2x} on $(-\infty, \infty)$ (no), $\ln(x)$ and $\ln(x^5)$ on $(0, \infty)$ (yes)?

Week 4

If we have two fundamental (linearly independent) solutions of a homogeneous equation, their linear combination is a general solution. The notion of a Wronskian. If the Wronskian is different from zero at some point t_0, the functions are linearly independent at any open interval including t_0. Example 2: check that the following sets of functions are linearly independent: e^x and e^{2x}, $\cos(x)$ and $\sin(x)$.

The general theory of linear differential equations (with continuous coefficients). Theorem 1: existence and uniqueness. Theorem 2 (superposition principle): a combination of solutions of a homogeneous equation is a solution. The definition of the fundamental set of solutions. Theorem 3: if the Wronskian is different from zero at some point then functions are linearly independent.

Theorem 4 (Abel's theorem for the second order equation $y'' + p(t)y' + q(t)y = 0$): the Wronskian is $W(t) = W(t_0)\exp\left\{-\int_{t_0}^{t} p(s)\, ds\right\}$. Remark: the Wronskian is either identically equal to zero or never vanishes. Example 3. The Wronskian of the Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ for $y_1(1) = 3$, $y_1'(1) = 1$, $y_2(1) = 2$, $y_2'(1) = 2$ is $W(x) = 4/x$.

Theorem 5: for a non-homogeneous equation, the difference of two solutions is a solution of the homogeneous equation, so the general solution of a non-homogeneous equation is a general solution of the homogeneous equation plus a (particular) solution of a non-homogeneous equation.

Second order equations: if the equation includes y', y'' only, after denoting $u = y'$ we have the first orderequation.

Reduction of order: if a solution y_1 is known, we are looking for the general solution as $y(t) = y_1(t)z(t)$. Example1. Find the solution of $t^2y'' + 3ty' + y = 0$ knowing that $y = t^{-1}$ is a solution.($y = t^{-1}(C_1 + C_2 \ln|t|)$)

Linear equations of the second order with constant coefficients: we look for solutions in the form e^{rt}, the notion of the characteristic equation. In the case of two real roots, e^{r_1t} and e^{r_2t} are fundamental solutions. Example 2: the initial value problem $y'' - 2y' - 8y = 0$, $y(0) = 7$, $y'(0) = -8$ has the solution $y = 6e^{-2t} + e^{4t}$.

In case of two equal roots of the characteristic equation, e^{rt} and te^{rt} are fundamental solutions. Example 3. Find the general solution of $y'' - 6y' + 9y = 0$.

In the case of the complex roots $r = \alpha \pm \beta i$, $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are fundamental solutions. Example 4. Find the general solution of $y'' - 6y' + 10y = 0$.

The notion of a boundary value problem. Example 5. Solve $y'' + 4y = 0$ if $y(0) = 0$ and either $y(\pi/2) = 1$ or $y(\pi/2) = 0$.

Week 5

The notion of a boundary value problem. Example 6. It is convenient to present the solution of $y'' - 4y = 0$ as a combination of $\cosh(x)$ and $\sinh(x)$, then it is easier to solve it for $y(0) = 0$, $y(\pi/2) = 1$.

The theory of higher order ordinary differential equations. The superposition principle, fundamental solutions, the Wronskian. Example 1. Find three linearly independent solutions of $y''' + y' = 0$ and prove that they are linearly independent.

Higher order equations with constant coefficients. Example 2. The general solution of $y''' - 27y = 0$ is $y = C_1e^{3t} + e^{-3t/2}\left[C_1\cos\left(\frac{3\sqrt{3}}{2}\right) + C_2\sin\left(\frac{3\sqrt{3}}{2}\right)\right]$.

Example 3. Does there exist a homogeneous linear differential equation with constant coefficients which has a solution $y = t^2e^{3t} + te^{-t}$ a) of the fourth order? b) of the fifth order?

Example 4. The general solution of $y^{(4)} - 3y'' - 4y = 0$ is $y = C_1e^{2t} + C_2e^{-2t} + C_3\cos(t) + C_4\sin(t)$.

The method of undetermined coefficients. Example 1. The general solution of $y'' + y' + 2y = 2x^2$ is $y = e^{-x/2}\left[C_1\cos\left(\frac{\sqrt{7}}{2}x\right) + C_2\sin\left(\frac{\sqrt{7}}{2}x\right)\right] + x^2 - x - \frac{1}{2}$. Example 2. The general solution of $y'' - 4y' = 3\cos(t)$ is $y = C_1 + C_2e^{4t} - \frac{3}{17}\cos(t) - \frac{12}{17}\sin(t)$. The general rules of the method of undetermined coefficients (non-resonance case). Example 3. The general solution of $y'' - 4y' = t + 2$ is $y = C_1 + C_2e^{4t} - \frac{1}{8}t^2 - \frac{9}{16}t$.

Week 6

The rules for the resonance case. An example of the falling ball $y'' = -g$.

Example 4. The general solution of $y'' + 9y = -18\cos(3t)$ is $y = C_1\cos(3t) + C_2\sin(3t) - 3t\sin(3t)$.

Example 5. Write down the form of the particular solution only for $y^{(5)} - y' = t\cos(t)$, $y^{(5)} - y' = e^{2x} + e^{-x} + xe^{x}$, $y^{(5)} - y' = x^3e^{5x}$.

The method of variation of parameters. In order to solve $y'' + p(t)y' + q(t)y = f(t)$, once we have two fundamental solutions y_1 and y_2 of the equation $y'' + p(t)y' + q(t)y = 0$, we solve the system $C_1'y_1 + C_2'y_2 = 0$, $C_1'y_1' + C_2'y_2' = f(t)$.

Example 1. The solution of $y'' + y = \frac{1}{\sin(t)}$ is $y = C_1\cos(t) + C_2\sin(t) - t\cos(t) + \sin(t)\ln|\sin(t)|$.

The review of methods of solution for non-homogeneous equations.

Example 2 (UC). The general solution of $y^{(5)} - y' = 8e^{-x}$ is $y = C_1 + C_2e^{x} + C_3e^{-x} + C_4\cos(x) + C_5\sin(x) + 2xe^{-x}$.

Example 3 (UC). Find the third order equation with constant coefficients with Ae^{3x} in the right-hand side which has a solution $e^x + \cos(3x) + 7\sin(3x) - 2e^{3x}$ ($y''' - y'' + 9y' - 9y = -72e^{3x}$).

Example 4 (variation of parameters). The solution of $y'' + y = \cot(t)$ is $y = C_1\cos(t) + C_2\sin(t) + \ln\left|\tan\left(\frac{t}{2}\right)\right|$.

The Laplace Transform. Definition: $\mathcal{L}[f](s) = \int_0^\infty e^{-st}f(t)\,dt$, $s > 0$, where f is piecewise continuous and satisfies $|f(t)| \leq Me^{bt}$ for some fixed $M > 0$, $b > 0$.

Example 1: $\mathcal{L}[1] = \frac{1}{s}$.

Example 2: $\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$.

Using integration by parts, it is possible to get

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2},\ s > 0,\ \mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2},\ s > 0.$$

Week 7

Properties of the Laplace Transform.
Property 1 (linearity): $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f] + b\mathcal{L}[g]$ (proof).

Example 3: $\mathcal{L}\left[3\cos(at) + 5e^{-4t}\right] = \dfrac{3s}{s^2+4} + \dfrac{5}{s+4}$.

Example 4: $\mathcal{L}\left[\sin^2(t)\right] = \dfrac{1}{2s} - \dfrac{s}{2(s^2+4)}$.

Property 2: if $|f(t)| \le Me^{bt}$ then $\mathcal{L}[f](s)$ is defined for $s > b$.

Property 3 (differentiation): $\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$ (proof).
Generally, (the first differentiation rule),

$$\mathcal{L}\left[\frac{d^n}{dt^n}f\right] = s^n\mathcal{L}[f] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$$

Property 4 (multiplication of the original by t): $\mathcal{L}[tf(t)] = -\dfrac{d}{ds}\mathcal{L}[f](s)$.

Generally, $\mathcal{L}[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}\mathcal{L}[f](s)$ (the second differentiation formula).

Example 5: $\mathcal{L}[te^t] = \dfrac{1}{(s-1)^2}$.

Example 6: $\mathcal{L}[t^3] = \dfrac{6}{s^4}$. Generally, we can find the transform of t^n:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}.$$

Property 5 (the first shift formula): $\mathcal{L}\left[e^{at}f(t)\right](s) = \mathcal{L}[f](s-a)$.

Example 7: $\mathcal{L}\left[e^{at}\cos(bt)\right] = \dfrac{s-a}{(s-a)^2+b^2}$,

$\mathcal{L}\left[e^{at}\sin(bt)\right] = \dfrac{b}{(s-a)^2+b^2}$.

Example 8: $\mathcal{L}\left[(t^2-5t+6)e^{2t}\right] = \dfrac{2}{(s-2)^3} - \dfrac{5}{(s-2)^2} + \dfrac{6}{s-2}$.

Property 6 (the transform of the integral): $\mathcal{L}\left[\displaystyle\int_0^t f(u)\,du\right] = \dfrac{1}{s}\mathcal{L}[f](s)$.

Example 9: $\mathcal{L}\left[\displaystyle\int_0^t e^{8u}\cos(6u)\,du\right] = \dfrac{s-8}{s((s-8)^2+36)}$.
One of the advantages of the Laplace Transform is that it can deal with discontinuous functions.

The step function $u_a(t) = \begin{cases} 0, & 0 \le t < a, \\ 1, & t \ge a \end{cases}$

Property 7: $\mathcal{L}\left[u_a(t)\right] = \dfrac{e^{-as}}{s}$.

Example 10: If $f(t) = \begin{cases} 3, & 5 \le t < 8, \\ 0, & \text{otherwise} \end{cases}$ then $\mathcal{L}[f] = \dfrac{3}{s}\left(e^{-5s} - e^{-8s}\right)$.

Property 8 (the second shift formula): $\mathcal{L}[u_a(t)g(t)] = e^{-as}\mathcal{L}[g(t+a)]$.

Example 11: the Laplace transform of

$$f(t) = \begin{cases} t^2 + t, & 0 \le t < 2, \\ 2t + 1, & 2 \le t < 7, \\ t - 3, & 7 \le t \end{cases}$$

is

$$\frac{2}{s^3} + \frac{1}{s^2} + e^{-2s}\left(-\frac{2}{s^3} - \frac{3}{s^2} - \frac{1}{s}\right) - e^{-7s}\left(\frac{1}{s^2} + \frac{11}{s}\right).$$

Week 8

Remark: the Laplace Transform can be used for computing definite integrals, for example,

$$\int_0^\infty e^{-3t}\cos(5t)\,dt = \frac{3}{34}.$$

The Laplace Transform of periodic functions: the Laplace Transform of $f(t) = 2 - e^{-3t}$, $0 \le t < 3$ satisfying $f(t+3) = f(t)$ is

$$F(s) = \frac{1}{1 - e^{-3s}}\left[\frac{2}{s} - \frac{2}{s}e^{-3s} + \frac{1}{s+3}e^{3s-9} - \frac{1}{s+3}\right].$$

The inverse Laplace Transform.

Example 1: a) $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+16}\right] = \cos(4t)$,

b) $\mathcal{L}^{-1}\left[\dfrac{s+7}{s^2+6s+13}\right] = e^{-3t}\cos(2t) + 2e^{-3t}\sin(2t)$.

Computation with partial fractions.

Example 2: $\mathcal{L}^{-1}\left[\dfrac{s-2}{s^2-s-6}\right] = \dfrac{1}{5}e^{3t} + \dfrac{4}{5}e^{-2t}$.

The case of the exponentials (using the second shift formula).

Example 3: $\mathcal{L}^{-1}\left[\dfrac{1-e^{-2s}}{s^2}\right] = t-(t-2)u_2(t)$.

Example 4: $\mathcal{L}^{-1}\left[e^{-5s}\dfrac{1}{(s+4)^3}\right] = \dfrac{1}{2}e^{-4(t-5)}(t-5)^2u_5(t)$.

Applications of the Laplace Transform include solution of ordinary differential equations and analysis of electric circuits.

Solution of differential equations using the Laplace Transform.

Example 1: the solution of the initial value problem $y''(t)-y'(t)-6y=0$, $t>0$, $y(0)=1$, $y'(0)=-1$ is $y(t)=\dfrac{1}{5}e^{3t}+\dfrac{4}{5}e^{-2t}$.

Example 2: the solution of the initial value problem $y''(t)+y=\sin(2t)$, $y(0)=2$, $y'(0)=1$ is $y(t)=2\cos(t)+\dfrac{5}{3}\sin(t)-\dfrac{1}{3}\sin(2t)$.

Example 3: the solution of the initial value problem

$$y'(t)+y=\begin{cases} -8, & 0\le t<4, \\ -1, & t\ge 4 \end{cases},\ y(0)=-7$$

is

$$\begin{cases} e^{-t}-8, & 0\le t<4, \\ e^{-t}-1-7e^{4-t} & t\ge 4 \end{cases}.$$

Week 9

Systems of differential equations. The connection between systems and higher order equations. Example 1: we can write the second order equation $y''(t)-5y'(t)+6y(t)=0$ as a system $X'(t)=AX(t)$, with $y=x_1$, $y'=x_2$, $A=\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

Method of solution: we assume that $X(t)=e^{\lambda t}v$ and substitute. We have non-trivial solutions if λ is an eigenvalue and v is an associated eigenvector of matrix A. For a 2×2 matrix with 2 real distinct eigenvalues $\lambda_1\neq\lambda_2$ and associated eigenvectors v_1 and v_2, the general solution is $X=C_1e^{\lambda_1 t}v_1+C_2e^{\lambda_2 t}v_2$. Note that in Example 1 these eigenvalues coincide with the roots of the characteristic equation, and the general solution is $X(t)=C_1e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}+C_2e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Example 2: the general solution of $X'(t)=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}X(t)$ is $X(t)=C_1\begin{bmatrix} -2 \\ 1 \end{bmatrix}+C_2e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Elements of the general theory of systems of linear homogeneous differential equations. Properties of eigenvaluers (the sum of eigenvalues is a sum of the diagonal entries of the matrix, the product of eigenvalues equals to the determinant of the matrix, if a real matrix A has a complex eigenvalue $\bar{\lambda}$ and an eigenvector $\mathbf{v}$ The case of complex eigenvalues. We only need to find one complex solution: its real and its imaginary parts are fundamental solutions.

Example 3: find the solution of the initial value problem $X'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X(t)$, $X(0) = (2,3)^T$. The general solution is $X(t) = C_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$, the solution of the initial value problem is $X(t) = 2 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + 3 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$.

Example 4: the general solution of the system of differential equations

$$\begin{aligned} x_1' &= 4x_1 + x_2 + x_3 \\ x_2' &= x_1 + 4x_2 + x_3 \\ x_3' &= x_1 + 2x_2 + 3x_3 \end{aligned}$$

is $X(t) = C_1 e^{2t} v_1 + C_2 e^{3t} v_2 + C_3 e^{6t} v_3 = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{6t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Example 5: If a real 2×2 matrix A has an eigenvalue $3+2i$ and a corresponding eigenvector $(-i,1)^T$, the general solution of the system $X'(t) = AX$ is $X(t) = C_1 e^{3t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}$.

Week 10

Example 6: the solution of the problem $Y'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} Y(t)$, $Y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ satisfies (choose the correct answer)

a) $Y(2\pi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad Y(t)$ is periodic $\quad$ c) $Y(t+2\pi) = e^{2\pi} Y(t)$ for any t

d) $Y(t+2\pi) - e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a periodic function.

Example 7: for which real a, b, all solutions of the system

$$X'(t) = \begin{bmatrix} -a & b \\ -b & a \end{bmatrix} X(t)$$

are periodic?

Fourier series.

Definition of periodic functions, examples of sine and cosine functions, $f(x) = \{x\}$. $\cos x$ and $\sin x$ are periodic with the period 2π, while $\cos(n\omega x)$ and $\sin(n\omega x)$ are periodic with the period $2\pi/\omega$ for any n. A constant function is periodic with any period.

Let f be defined on $[-\ell, \ell]$. Odd an even functions. The graph of an even function is symmetric with respect to the y-axis. The graph of an odd function is symmetric with respect to the origin. Constants, $f(x) = x^{2n}$, $n \in \mathbb{N}$, $\cos(\omega x)$ are examples of even functions. For an even function $\displaystyle\int_{-\ell}^{\ell} f(x)\, dx = 2\int_0^{\ell} f(x)\, dx$.

$f(x) = x^{2n+1}$, $n \in \mathbb{N}$, $\sin(\omega x)$, $\tan(\omega x)$ are examples of even functions. A sum of odd (even) functions is an odd (even) function. A product of even functions is even, a product of odd functions is even, a product of an odd function and an even function is an odd function (is similar to operations with positive and negative signs). An integral of an odd function over a symmetric interval is equal to zero.

Example 1. The function under the integral can be presented as a sum of an odd and an even function: $\displaystyle\int_{-\pi}^{\pi} \left(x^{16} + 5x\right) \sin(nx)\, dx = \frac{(-1)^{n+1} 10\pi}{n}$, as $\sin(n\pi) = 0$, $\cos(n\pi) = (-1)^n$.

Even and odd extensions of functions

$$f_e(x) = \begin{cases} f(-x), & x \in [-l, 0] \\ f(x), & x \in [0, l] \end{cases} \qquad \text{and} \qquad f_o(x) = \begin{cases} -f(-x), & x \in [-l, 0] \\ f(x), & x \in [0, l], \end{cases}$$

The notion of the periodic extension of a function. The periodic extension of $f(x) = x$, $0 < x < 1$, to the real line is $\{x\}$.

Fourier series approximation of a function on $[-\pi, \pi]$, assuming that f is periodic with a period 2π is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right],$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{-\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

These formulas can be extended to any interval of the length 2π, with the bounds of integration updated accordingly.

Example 2. The Fourier series of $f(x) = x^2$, $0 < x < 2\pi$ are

$$f(x) \sim \frac{4}{3}\pi^2 + 4 \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \cos(nx) - \frac{\pi}{n} \sin(nx)\right].$$

So far we wrote $f(x) \sim$ its Fourier series. What does the Fourier series converge to? Let f be a piecewise continuous function on (α, β), so that the left-hand limit $f(c^-) = \lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c, x < c} f(x)$ and the right-hand limit $f(c^+) = \lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c, x > c} f(x)$ exist and is finite.

Dirichlet's theorem describes the convergence.

1. If $f(x)$ is continuous at $x \in (\alpha, \beta)$ $(f(x^-) = f(x^+) = f(x))$ then the Fourier series at x converges to $f(x)$, and $\sim$ can be replaced with the equality sign:

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega x) + b_n \sin(n\omega x)\right], \quad \omega = 2\pi/(\beta - \alpha),$$

where

$$a_0 = a_n = \frac{1}{\ell}\int_\alpha^\beta f(x)\cos(n\omega x)\ dx, \quad b_n = \frac{1}{\ell}\int_\alpha^\beta f(x)\sin(n\omega x)\ dx, \quad \ell = \frac{\beta-\alpha}{2}.$$

2. If $f(x)$ has a finite jump discontinuity at $x \in (\alpha, \beta)$ ($f(x^-) \neq f(x^+)$ exist and are finite) then the Fourier series at x converges to the average of the left-hand and the right-hand limits

$$\frac{1}{2}\left[f(x^-) + f(x^+)\right] = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega x) + b_n\sin(n\omega x)\right], \quad \omega = 2\pi/(\beta-\alpha).$$

3. At the endpoints α and β the Fourier series converges to $\frac{1}{2}\left[f(\alpha^+) + f(\beta^-)\right]$. We recall that the function is extended continuously with a period of $\beta-\alpha$, so, if $f(\alpha^+) = f(\beta^-)$, the Fourier series converges to this value, otherwise to their average (in this case, we have the Gibbs phenomenon).

Example 3. For the function $f(x) = \begin{cases} x^2+1 & 0 < x \leq 2 \\ \frac{12}{x} & 2 < x \leq 3 \\ 3x-1 & 3 < x < 7 \end{cases}$ determine the values to which the Fourier series of f converges at each of the points x=0,2,3,4 and 7.

Week 11

Fourier series for a function $f(x)$ on the segment $[\alpha, \beta]$ of length $\beta - \alpha = 2\ell$ are

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega x) + b_n\sin(n\omega x)\right], \quad \omega = \frac{\pi}{\ell},$$

where

$$a_0 = \frac{1}{2\ell}\int_\alpha^\beta f(x)\ dx, \quad a_n = \frac{1}{\ell}\int_\alpha^\beta f(x)\cos(n\omega x)\ dx, \quad b_n = \frac{1}{\ell}\int_\alpha^\beta f(x)\sin(n\omega x)\ dx.$$

Here we assume that f is periodic with the period 2ℓ.

If f is an even function on $[-\ell, \ell]$ then we have the Fourier cosine series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega x), \quad \omega = \frac{\pi}{\ell},$$

where

$$a_0 = \frac{1}{\ell}\int_0^\ell f(x)\ dx, \quad a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos(n\omega x)\ dx, \ n \in \mathbb{N}.$$

If f is an odd function on $[-\ell, \ell]$ then we have the Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\omega x), \quad \omega = \frac{\pi}{\ell},$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin(n\omega x)\ dx, n \in \mathbb{N}.$$

In exercises, it is usually required to expand a function given on an interval into either Fourier series (sines and cosine), or in either cosine or sine series (assuming even and odd extensions, respectively).

Example 4. The sine series of $f(x) = 1 - x$, $0 < x < 1$ is

$$1 - x \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x).$$

Remark. If a function on $[-\ell, \ell]$ is a combination of a constant, $\cos(n\omega x)$ and $\sin(n\omega x)$ for integer n, then its Fourier series coincides with the function.

Partial differential equations of mathematical physics. The heat conduction (transfer) equation for the temperature $u = u(x, t)$ at time t at point with the coordinate x in a rod. Boundary conditions. Dirichlet and Neumann boundary conditions. Initial conditions.

The wave equation.

The Laplace operator. The potential equation.

We are interested in the equation

$$X''(x) + \lambda X(x) = 0,\ a < x < b.$$

This equation with the general homogeneous boundary conditions

$$\alpha_1 X(a) + \alpha_2 X'(a) = 0, \quad \beta_1 X(b) + \beta_2 X'(b) = 0$$

is called *the Sturm-Liouville problem.*

For example, the special cases are
$X''(x) + \lambda X(x) = 0, \quad a < x < b, \quad X(a) = X(b) = 0,$ $X''(x) + \lambda X(x) = 0, \quad a < x < b, \quad X'(a) = X'(b) = 0.$

The values of λ for which the Sturm-Liouville problem has a nontrivial solution, are called *eigenvalues*, and corresponding solutions $X(t)$ *eigenfunctions.*

Example 1. The eigenvalues and eigenfunctions for the problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi, \quad X(0) = X(\pi) = 0$$

are $\lambda_n = n^2$, the corresponding eigenfunctions are $X_n = \sin(nx)$.

The idea of separation of variables: we look for the general solution in the form $u(x, t) = X(x)T(t)$. Using the boundary conditions only, we obtain collection of solutions $u_n = X_n T_n$.

By the superposition principle, their sum $\sum_{n=1}^{\infty} C_n X_n(t) T_n(t)$ is also a solution. Substituting the initial conditions, we find the coefficients C_n.

Example 2. The solution of the initial-boundary value problem for the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi,\ t > 0, \quad u(0,t) = 0,\ u(\pi,t) = 0,\ t > 0,\ u(x,0) = \pi - x,\ 0 < x < \pi$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx) e^{-n^2 t} = 2 \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-n^2 t}, \quad 0 \leq x \leq \pi.$$

Week 12

Review: the idea of separation of variables.

Example 3. The solution of the initial-boundary value problem for the heat conduction equation

$$5 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi,\ t > 0,$$

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = 0, \quad \left. \frac{\partial u}{\partial x} \right|_{x=\pi} = 0,\ t > 0,\ u(x,0) = \cos(3x),\ 0 < x < \pi$$

is

$$u(x,t) = \cos(3x) e^{-45t}$$

Example 4. The solution $u(t,x)$ of a vibrating string problem

$$4 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < \pi,\ t > 0,$$

$$u(0,t) = 0,\ u(\pi,t) = 0,\ t > 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 12 \sin(3x),\ 0 < x < \pi$$

is $u(t,x) = 2 \sin(6t) \sin(3x)$.

Example 5. The potential $u(x,y)$ in a rectangular plate which satisfies

$$\nabla^2 u(x,y) = \Delta u = 0, \quad 0 < x < 1,\ 0 < y < 2,$$

$$u(0,y) = 0,\ u(1,y) = 0, \quad u(x,0) = 0,\ u(x,2) = 4 \sin(3\pi x) \sinh(6\pi)$$

is $u(x,y) = 4 \sin(3\pi x) \sinh(3\pi y)$.

Week 13

Review of the material of the course.