Department of Mathematics and Statistics
MATH 375
Information sheet # 4
Linear Systems of Differential Equations

A linear homogeneous system of n equations with n unknown functions $x_1(t), x_2(t), \dots, x_n(t)$ can be written as

$$X'(t) = AX(t),$$

where X is a column of $x_1(t), x_2(t), \dots, x_n(t)$ and A is a square matrix.
Method of solution: find eigenvalues λ_i and corresponding eigenvectors v_i of the matrix A. If A has n linearly independent eigenvectors $v_1, v_2, \dots, v_n$ then the general solution is

$$X = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + \ldots + C_n e^{\lambda_n t} v_n.$$

Remark. If $\lambda_{1,2} = a \pm bi$ and v_1 is an eigenvector corresponding to λ_1 then the basic solutions are real and imaginary parts of $e^{\lambda_1 t} v_1$, respectively.

Example 1. Find the general solution of the system of differential equations

$$\begin{aligned} x_1' &= 4x_1 + x_2 + x_3 \\ x_2' &= x_1 + 4x_2 + x_3 \\ x_3' &= x_1 + 2x_2 + 3x_3 \end{aligned}$$

This system can be written in the matrix form $X' = AX$, where $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$. Let us find the eigenvalues of A. Then after subtracting the second row from the first row, taking away the common factor of the first row $3-\lambda$ and then adding the first column to the second column we have

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & \lambda-3 & 0 \\ 1 & 4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 1 & -1 & 0 \\ 1 & 4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 5-\lambda & 1 \\ 1 & 3 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 5-\lambda & 1 \\ 3 & 3-\lambda \end{vmatrix}$$

$= (3-\lambda)[(5-\lambda)(3-\lambda) - 1 \cdot 3] = (3-\lambda)(15 - 8\lambda + \lambda^2 - 3) = (3-\lambda)(12 - 8\lambda + \lambda^2) = 0$, thus $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 6$.
Let us find eigenvectors corresponding to $\lambda_1 = 2$.

$$A - 2I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix},$$

thus we have the system $2r+s+t=0$, $r+2s+t=0$. Subtracting the second equation from the first one, we have $r-s=0$, or $r=s$, then $t=-3s$, the general solution is $(s,s,-3s)$, we can take $v_1 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$. Similarly, for $\lambda_2 = 3$, we have $r+s+t=0$, $r+2s=0$, we choose $v_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, for $\lambda_3 = 6$, we can take $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The general solution is

$$X(t) = C_1 e^{2t} v_1 + C_2 e^{3t} v_2 + C_3 e^{6t} v_3 = C_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{6t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Example 2. A real 2×2 matrix A has an eigenvalue $3+2i$ and a corresponding eigenvector $(-i, 1)^T$. Find the general solution of the system $X'(t) = AX$.
Solution. We have a complex solution of the system

$$e^{3t} \begin{bmatrix} -i \\ 1 \end{bmatrix} (\cos 2t + i \sin 2t) = e^{3t} \begin{bmatrix} -i\cos 2t - i^2 \sin 2t \\ \cos 2t + i \sin 2t \end{bmatrix} = e^{3t} \begin{bmatrix} \sin 2t - i \cos 2t \\ \cos 2t + i \sin 2t \end{bmatrix}$$

$$= e^{3t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} + i e^{3t} \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}.$$

Both the real and the imaginary parts of this solution are (linearly independent) solutions of the system. Thus the general solution is

$$X(t) = C_1 e^{3t} \begin{bmatrix} \sin 2x \\ \cos 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}.$$