Summary

Integrating factor

$$y' + p(t)y = f(t), \qquad \text{integrating factor: } \mu(t) = e^{\int p(t)dt}$$

$$\Rightarrow (\mu(t)y)' = \mu(t)f(t)$$

Then: integrate both sides and divide by $\mu(t)$

Separation of variables

$y' = g(x)h(y)$, if your equation can be rewritten in this form,

separate variables: $\frac{1}{h(y)}y' = g(x)$

$$\Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x) \quad \Rightarrow \quad \frac{dy}{h(y)} = g(x)dx$$

Then: Integrate both sides.

Bernoulli equation

$$y' + p(t)y = q(t)y^m, \quad (m \neq 0, 1) \qquad \Rightarrow \quad y^{-m}y' + p(t)y^{1-m} = q(t)$$

substitute: $u = y^{1-m}$ and $u' = (1-m)y^{-m}y'$,

$$\Rightarrow \quad \frac{1}{1-m}u' + p(t)u = q(t)$$

Then: Solve for u (you can use integrating factor or, if possible, separation of variables) don't forget to come back to y.

Exact Differential Equations

$$M(x,y)dx + N(x,y)dy = 0, \qquad \text{if } \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad \Rightarrow \quad \text{Exact Differential Equation}$$

Find $G(x,y)$ such that

$$\frac{\partial G(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial G(x,y)}{\partial y} = N(x,y)$$

Solution: $G(x,y) = c$

Integrating factor for Non-Exact Differential Equation

If $M(x,y)dx + N(x,y)dy = 0$ is **not** an Exact Differential Equation

i.e. $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$ $\Rightarrow$ use integrating factor

$$\mu(x) = e^{\int u(x)dx}, \text{ where } u(x) = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)}$$

$$\mu(y) = e^{\int v(y)dy}, \text{ where } v(y) = \frac{N_x(x,y) - M_y(x,y)}{N(x,y)}$$

Then: $\big[\mu(x)M(x,y)\big]dx + \big[\mu(x)N(x,y)\big]dy = 0$ or

$\big[\mu(y)M(x,y)\big]dx + \big[\mu(y)N(x,y)\big]dy = 0$ is an exact diff. eq.

$\rightsquigarrow$ solve this exact differential equation

Homogeneous first order differential equations

$$y' = g\left(\frac{y}{x}\right), \qquad \text{substitute } u = \frac{y}{x} \text{ and } y' = u + xu' \quad \Rightarrow \quad u + xu' = g(u),$$

now solve for u. You can use separation of variables:

$$\Rightarrow \quad \frac{1}{g(u)-u}du = \frac{1}{x}dx$$

Don't forget the solutions that might come from $g(u) - u = 0$

Examples		
Equation	Method	Notes
$y' + \frac{1}{t}y = \frac{7}{t^2} + 3$	Integration factor	
$e^{t^2}y' + e^{t^2}\left(2t + \frac{1}{t}\right)y = t^2$	Integration factor	Divide by e^{t^2} to get the standard form
$xy' + 3y = \frac{e^x}{x}$	Integration factor	Divide by x to get the standard form
$\frac{y'}{2y} - \frac{1}{2}\cos(x) = \frac{\sin(x)\cos(x)}{y}$	Integration factor	Multiply by 2y to get the standard form
$ty' + y + y^2 = 0$	Separation of variables, Bernoulli equation	Divide by t
$y' + 2x(y^2 - 3y + 2) = 0$	Separation of variables	
$x^2yy' = (y^2 - 1)^{3/2}$	Separation of variables	
$y' + \frac{1}{t}y = -ty^3$	Bernoulli equation	m=3
$ty' + y = y^2\ln(t)$	Bernoulli equation	m=2
$(3x^2 + y^2 - 4xy - 3y)dx + (-2x^2 + 6y + 2xy - 3x)dy = 0$	Exact Differential equation	
$\big(x\ln(y) + y\ln(x)\big)dx + \left(\frac{x^2}{2y} + x\ln(x) - x\right)dy = 0$	Exact Differential equation	
$xy' = y + x\cos^2\left(\frac{y}{x}\right)$	Homogeneous 1^{st} order diff. eq.	Divide by x
$3xyy' = x^2 + 4y^2$	Homogeneous 1^{st} order diff. eq.	Divide by x^2