Department of Mathematics and Statistics
MATH 375
Information sheet # 6
Partial Differential Equations

Modeling of the three equations of mathematicsl physics

1. *The heat conduction (transfer) equation* for the temperature $u = u(x,t)$ at time t at point with the coordinate x in a rod has the form

$$k\,\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < a, \quad t > 0,$$

the positive constant k describes diffusivity (one-dimensional case).

Boundary conditions $u(0,t) = A$, $u(a,t) = B$ (the temperature at the ends is kept at A and B level, respectively), a particular case is homogeneous boundary conditions $u(0,t) = u(a,t) = 0$ (Dirichlet conditions), or $\dfrac{\partial u}{\partial x}(0,t) = A$, $\dfrac{\partial u}{\partial x}(a,t) = B$, where A and B describe the heat flux through the ends, a particular case is homogeneous $\dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(a,t) = 0$ (insulated ends, Neumann boundary conditions).

2. *The wave equation* for the small transverse vibration of an elastic string stretched between two fixed points on x-axis for the displacement $u = u(t,x)$ at time t at point with the coordinate x is

$$c\,\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < a, \quad t > 0,$$

c is a positive constant.

3. *The potential (Laplace) equation* for the field potential $u(x,y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Using the Laplace differential operator

$$\Delta u(x,y) = \nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

we can rewrite the potential equation as $\Delta u = 0$ or $\nabla^2 u = 0$.

Sturm-Liouville problems

The Sturm-Liouville problem is a boundary value problem for a second order ordinary differential equation. We are interested in the Sturm-Liouville equation

$$X''(x) + \lambda X(x) = 0,\ a < x < b.$$

This equation with the general homogeneous boundary conditions

$$\alpha_1 X(a) + \alpha_2 X'(a) = 0, \quad \beta_1 X(b) + \beta_2 X'(b) = 0$$

is called *the Sturm-Liouville problem*.

For example, the special cases are
$X''(x) + \lambda X(x) = 0, \quad a < x < b, \quad X(a) = X(b) = 0,$ $X''(x) + \lambda X(x) = 0, \quad a < x < b, \quad X'(a) = X'(b) = 0.$

The values of λ for which the Sturm-Liouville problem has a nontrivial solution, are called *eigenvalues*, and corresponding solutions $X(t)$ *eigenfunctions*.

Example 1. Find eigenvalues and eigenfunctions for the problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi, \quad X(0) = X(\pi) = 0.$$

Solution. The characteristic equation $r^2 + \lambda = 0$ has solutions $r_1 = r_2 = 0$ for $\lambda = 0$, $r = \pm\sqrt{\lambda}$ for $\lambda < 0$ and $r = \pm\sqrt{\lambda}i$ for $\lambda > 0$.

Case 1. $\lambda = 0$, then the general solution is $X(x) = A + Bx$. Substituting the boundary conditions we get $X(0) = A = 0$ and $X(\pi) = A + B\pi = 0$, so $A = B = 0$, there are no nontrivial solutions, $\lambda = 0$ is not an eigenvalue.

Case 2. $\lambda < 0$, for example, $\lambda = -\alpha^2$, then the general solution is

$$X(x) = A\cosh(\alpha x) + B\sinh(\alpha x)$$

(we recall $\cosh(y) = \dfrac{e^y + e^{-y}}{2}$, $\sinh(y) = \dfrac{e^y - e^{-y}}{2}$). Substituting the boundary conditions we get $X(0) = A = 0$ and $X(\pi) = A\cosh(\alpha\pi) + B\sinh(\alpha\pi) = B\sinh(\alpha\pi) = 0$, so $A = B = 0$, there are no nontrivial solutions, $\lambda < 0$ is not an eigenvalue.

Case 3. $\lambda > 0$, for example, $\lambda = \beta^2$, then the general solution is

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

Substituting the boundary conditions we get $X(0) = A = 0$ and $X(\pi) = B\sin(\beta\pi) = 0$ with $B \neq 0$ if $\beta\pi = n\pi$, or $\beta = n$, $\lambda = n^2$, $n \in \mathbb{N}$. The eigenvalues are $\lambda_n = n^2$, the corresponding eigenfunctions are $X_n = \sin(nx)$.

Generally, the Sturm-Liouville problem $X''(x) + \lambda X(x) = 0, X(0) = 0, \ X(a) = 0$ has eigenvalues $\lambda_n = \left(\dfrac{n\pi}{a}\right)^2$ with eigenfunctions $X_n = \sin\left(\dfrac{n\pi}{a}x\right)$, $n = 1, 2, \ldots$.

The Sturm-Liouville problem $X''(x) + \lambda X(x) = 0, X'(0) = 0, \ X'(a) = 0$ has eigenvalues $\lambda_0 = 0$, $\lambda_n = \left(\dfrac{n\pi}{a}\right)^2$ with eigenfunctions $X_n = \cos\left(\dfrac{n\pi}{a}x\right)$, $n = 1, 2, \ldots$.

Separation of variables

The idea. We look for the general solution in the form $u(x,t) = X(x)T(t)$. Using the boundary conditions only, we obtain collection of solutions $u_n = X_n T_n$. By the superposition principle, their sum $\sum_{n=1}^{\infty} C_n X_n(t) T_n(t)$ is also a solution. Substituting the initial conditions, we find the coefficients C_n.

Example 2. Find the solution of the initial-boundary value problem for the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \ t > 0, \quad u(0,t) = 0, \ u(\pi,t) = 0, \ t > 0, \ u(x,0) = \pi - x, \ 0 < x < \pi.$$

Solution. Let $u(x,t) = X(x)T(t)$, then $u_{xx} = X''T$, $u_t = XT'$. Separating variables

$$X''T = XT' \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{T},$$

we get that a function of x is identically equal to a function of t which is only possible when they are both constant, say, equal to $-\lambda$. Thus $X''(x) + \lambda X = 0$, $T' = -\lambda T$.

Substituting the initial conditions, we get $X(x)T(0) = \pi - x$. The boundary conditions lead to $X(0)T(t) = 0$ and $X(\pi)T(t) = 0$, which leads for nontrivial $T(t)$ to $X(0) = 0$, $X(\pi) = 0$.

The Sturm-Liouville problem $X''(x) + \lambda X(x) = 0, X(0) = 0, \ X(\pi) = 0$ has eigenvalues $\lambda_n = n^2$ with eigenfunctions $X_n(x) = \sin(nx)$, $n = 1, 2, \ldots$. For $\lambda_n = n^2$, the equation $T' = -\lambda T = -n^2 T$ has solutions $T_n(t) = e^{-n^2 t}$. Each function $u_n = \sin(nx)e^{-n^2 t}$. By the superposition principle, we are looking for the solution in the form

$$\sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}.$$

Substituting the initial condition, we get $u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = \pi - x$. We have to find the sine series for $f(x) = \pi - x$.

$$b_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin(nx) \ dx = \frac{2}{\pi} \left[-\frac{\pi - x}{n} \cos(nx) - \frac{\sin(nx)}{n^2} \right]_0^{\pi} = \frac{2}{\pi} \frac{\pi}{n} = \frac{2}{n}.$$

Finally, the solution to the heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx) e^{-n^2 t} = 2 \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-n^2 t}, \quad 0 \le x \le \pi.$$

Example 3. Find the potential $u(x,y)$ in a rectangular plate if it satisfies

$$\nabla^2 u(x,y) = \Delta u = 0, \quad 0 < x < 1, \ 0 < y < 2,$$

$$u(0,y) = 0, \ u(1,y) = 0, \quad u(x,0) = 0, \ u(x,2) = 4\sin(3\pi x)\sinh(6\pi).$$

Solution. We substitute $u(x,y) = X(x)Y(y)$, $u_{xx} = X''(x)Y(y)$, $u_{yy} = X(x)Y''(y)$ in the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. We separate variables and conclude that both sides are equal to a constant, say, $-\lambda$:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Substituting the boundary conditions we get $X(0)Y(y) = 0 = X(1)Y(y)$, so $X(0) = X(1) = 0$, $X(x)Y(0) = 0$, $X(x)Y(2) = \sin(3\pi x)\sinh(6\pi)$ gives $Y(0) = 0$.

The Sturm-Liouville problem $X''(x) + \lambda X(x) = 0, X(0) = 0,\ X(1) = 0$ has eigenvalues $\lambda_n = n^2\pi^2$ with eigenfunctions $X_n(x) = \sin(n\pi x)$, $n = 1, 2, \ldots$.

Consider the second Sturm-Liouville problem $Y''(y) - \lambda Y(y) = 0$ with $Y(0) = 0$ which has the same eigenvalues $\lambda_n = n^2\pi^2$, so $Y'' - n^2\pi^2 Y = 0$ leads to eigenfunctions $Y_n(y) = A_n\cosh(n\pi y) + B_n\sinh(n\pi y)$. The condition $Y(0) = 0$ gives $A_n = 0$, so the eigenfunctions are $Y_n(y) = \sinh(n\pi y)$, and

$$u_n = \sin(n\pi x)\sinh(n\pi y), \quad u(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh(n\pi y).$$

From the boundary condition $u(x,2) = 4\sin(3\pi x)\sinh(6\pi)$. Substituting $y = 2$ in the solution

$$u(x,2) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh(2n\pi) = 4\sin(3\pi x)\sinh(6\pi)$$

and comparing the terms we obtain $b_n = 0$, $n \neq 3$, $b_3 = 4$, so $u(x,y) = 4\sin(3\pi x)\sinh(3\pi y)$.