Math 375		**Fall 2016**
	Variation of Parameters and Laplace Transform	
Worksheet # 2	**Part 4**	**October 31-November 04**

The problems on this worksheet refer to material from sections §3.6, §4.4, and §6.1, of your text. Solutions to all problems will be available on the course's D2L website Friday, November 4. Please report any typos, omissions and errors to aiffam@ucalgary.ca

The Variation of Parameters Method

01. Find the general solution of the following equations.

a. $y'' + y = \sec^3(t)$ **b.** $y'' - 2\,y' + y = \dfrac{e^t}{t^2+1}$

Laplace Transform of Basic Functions

02. Compute $\mathcal{L}\{f(t)\}(s)$, if

a. $f(t) = (t^2+1)^2$ **b.** $f(t) = 3\,\cos(5\,t) - 2\,\sin(2\,t)$

c. $f(t) = \cos^4(t) - \sin^4(t)$ **d*.** $f(t) = \cosh(2\,t)\,\sinh(3\,t)$

e*. $f(t) = \cos^3(t)$ **f.** $f(t) = \sin^3(t)$

03. Use the substitution $x = \sqrt{s\,t}$ to show

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}(s) = \frac{2}{\sqrt{s}}\int_0^{+\infty} e^{-x^2}\,dx, \quad \text{for any} \quad s > 0$$

Given that $\displaystyle\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$, compute $\mathcal{L}\left\{\dfrac{1}{\sqrt{t}}\right\}(s)$

04. Use integration by parts to show that $\mathcal{L}\left\{\sqrt{t}\right\}(s) = \dfrac{1}{2\,s}\,\mathcal{L}\left\{\dfrac{1}{\sqrt{t}}\right\}(s)$, $s > 0$, then compute $\mathcal{L}\left\{\sqrt{t}\right\}(s)$

First Shift Formula

05. Find $\mathcal{L}\{f(t)\}(s)$, if

a. $f(t) = t^2\,e^{-3\,t}$ **b*.** $f(t) = (t+1)^2\,e^t$ **c*.** $f(t) = \big(\sin(2\,t) + \cos(3\,t)\big)\,e^{-t}$

d. $f(t) = \dfrac{e^{-2\,t}}{\sqrt{t}}$ **e.** $f(t) = \sqrt{t}\,e^{2\,t}$

Answers and Solutions

01a. The characteristic equation of the homogeneous equation is $\lambda^2+1=0$. It has roots $\lambda_1=i$ and $\lambda_2=-i$. Hence a fundamental set of solutions for the homogeneous equation is

$$\left\{ y_1(t)=\cos(t)\ ,\ y_2(t)=\sin(t) \right\}$$

According to the variation of parameters method, we look for a particular solution in the form

$$y_p(t)=y_1(t)\,u_1(t)+y_2(t)\,u_2(t)$$

with $u_1'(t)$ and $u_2'(t)$ solutions of the system

$$\left\{\begin{array}{l} y_1(t)\,u_1'(t)+y_2(t)\,u_2'(t)=0 \\ \\ y_1'(t)\,u_1'(t)+y_2'(t)\,u_2'(t)=f(t) \end{array}\right. \iff \left\{\begin{array}{l} u_1'(t)=\dfrac{1}{W(t)}\begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t)\end{vmatrix} \\ \\ u_2'(t)=\dfrac{1}{W(t)}\begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t)\end{vmatrix} \end{array}\right.$$

where $W(t)=\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t)\end{vmatrix}$, is the wronskian of the fundamental set of solutions. We have

$$W(t)=\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t)\end{vmatrix}=1$$

$$u_1'(t)=\frac{1}{1}\begin{vmatrix} 0 & \sin(t) \\ \sec^3(t) & \cos(t)\end{vmatrix}=-\sin(t)\ \sec^3(t)=-\tan(t)\ \sec^2(t) \implies u_1(t)=-\frac{1}{2}\ \tan^2(t)$$

and

$$u_2'(t)=\frac{1}{1}\begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \sec^3(t)\end{vmatrix}=\cos(t)\ \sec^3(t)=\sec^2(t) \implies u_1(t)=\tan(t)$$

Hence the particular solution

$$\begin{aligned} y_p(t)&=u_1(t)\,y_1(t)+u_2(t)\,y_2(t)=\Big(-\frac{1}{2}\ \tan^2(t)\Big)\ \cos(t)+\Big(\tan(t)\Big)\ \sin(t) \\ &=\Big(-\frac{1}{2}\ \tan(t)\ \cos(t)+\sin(t)\Big)\ \tan(t)=\Big(-\frac{1}{2}\ \sin(t)+\sin(t)\Big)\ \tan(t)=\frac{1}{2}\ \sin(t)\ \tan(t) \\ &=\frac{\sin^2(t)}{2\ \cos(t)}=\frac{1-\cos^2(t)}{2\ \cos(t)}=\frac{1}{2}\ \sec(t)-\frac{1}{2}\ \cos(t) \end{aligned}$$

Now because $\frac{1}{2}\cos(t)$, is a solution of the associated homogeneous equation (remember $\cos(t)$ is part of the fundamental set of solutions), we can drop off $-\frac{1}{2}\cos(t)$ from $y_p(t)$ to simply get

$$y_p(t)=\frac{1}{2}\ \sec(t)$$

The general solution of the equation is then $y(t)=C_1\ \cos(t)+C_2\ \sin(t)+\frac{1}{2}\ \sec(t) \quad \cdots \quad (*)$

Notice that if we haven't dropped off $-\frac{1}{2}\cos(t)$ from the particular solution, the general solution would have been

$$y(t)=C_1\ \cos(t)+C_2\ \sin(t)+\frac{1}{2}\ \sec(t)-\frac{1}{2}\ \cos(t)=\Big(C_1-\frac{1}{2}\Big)\ \cos(t)+C_2\ \sin(t)+\frac{1}{2}\ \sec(t)$$

Now because C_1 is an arbitrary constant, that's the same solution as $(*)$

01b. The characteristic equation of the homogeneous equation is $\lambda^2 - 2\,\lambda + 1 = 0 \iff (\lambda - 1)^2 = 0$. It has roots $\lambda_1 = \lambda_2 = 1$. Hence a fundamental set of solutions for the associated homogeneous equation is

$$\left\{ y_1(t) = \mathrm{e}^t \ , \ y_2(t) = t\,\mathrm{e}^t \right\}$$

According to the variation of parameters method, we look for a particular solution in the form

$$y_p(t) = y_1(t)\,u_1(t) + y_2(t)\,u_2(t)$$

with $u_1'(t)$ and $u_2'(t)$ solutions of the system

$$\left\{ \begin{array}{l} y_1(t)\,u_1'(t) + y_2(t)\,u_2'(t) = 0 \\[2ex] y_1'(t)\,u_1'(t) + y_2'(t)\,u_2'(t) = f(t) \end{array} \right. \iff \left\{ \begin{array}{l} u_1'(t) = \dfrac{1}{W(t)} \begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t) \end{vmatrix} \\[3ex] u_2'(t) = \dfrac{1}{W(t)} \begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t) \end{vmatrix} \end{array} \right.$$

where $W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$, is the wronskian of the fundamental set of solutions. We have

$$W(t) = \begin{vmatrix} \mathrm{e}^t & t\,\mathrm{e}^t \\ \mathrm{e}^t & (t+1)\,\mathrm{e}^t \end{vmatrix} = (t+1)\,\mathrm{e}^{2\,t} - t\,\mathrm{e}^{2\,t} = \mathrm{e}^{2\,t}$$

$$u_1'(t) = \frac{1}{\mathrm{e}^{2\,t}} \begin{vmatrix} 0 & t\,\mathrm{e}^t \\ \mathrm{e}^t/(t^2+1) & (t+1)\,\mathrm{e}^t \end{vmatrix} = \frac{-t\,\mathrm{e}^{2\,t}}{\mathrm{e}^{2\,t}\left(t^2+1\right)} = -\frac{t}{t^2+1} \implies u_1(t) = -\frac{1}{2}\,\ln\left(t^2+1\right)$$

and

$$u_2'(t) = \frac{1}{\mathrm{e}^{2\,t}} \begin{vmatrix} \mathrm{e}^t & 0 \\ \mathrm{e}^t & \mathrm{e}^t/(t^2+1) \end{vmatrix} = \frac{\mathrm{e}^{2\,t}}{\mathrm{e}^{2\,t}\left(t^2+1\right)} = \frac{1}{t^2+1} \implies u_2(t) = \tan^{-1}(t)$$

Hence the particular solution

$$y_p(t) = \mathrm{e}^t\left(-\frac{1}{2}\,\ln\left(t^2+1\right)\right) + t\,\mathrm{e}^t\left(\tan^{-1}(t)\right)$$

and the general solution is

$$y(t) = C_1\,\mathrm{e}^t + C_2\,t\,\mathrm{e}^t - \frac{1}{2}\,\mathrm{e}^t\,\ln\left(t^2+1\right) + t\,\mathrm{e}^t\,\tan^{-1}(t)$$

02a. Expand and use the formula $\mathcal{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$ to get

$$\begin{aligned} \mathcal{L}\left\{(t^2+1)^2\right\}(s) &= \mathcal{L}\left\{t^4 + 2\,t^2 + 1\right\}(s) = \mathcal{L}\left\{t^4\right\}(s) + 2\,\mathcal{L}\left\{t^2\right\}(s) + \mathcal{L}\{1\}(s) \\ &= \frac{4!}{s^5} + 2\,\frac{2!}{s^3} + \frac{1}{s} = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s} \end{aligned}$$

02b. Use the linearity of Laplace transform and the formulas $\mathcal{L}\{\cos(b\,t)\}(s) = \dfrac{s}{s^2+b^2}$ and $\mathcal{L}\{\sin(b\,t)\}(s) = \dfrac{b}{s^2+b^2}$, to get

$$\begin{aligned} \mathcal{L}\{3\,\cos(5\,t) - 2\,\sin(2\,t)\}(s) &= 3\,\mathcal{L}\{\cos(5\,t)\}(s) - 2\,\mathcal{L}\{\sin(2\,t)\}(s) \\ &= 3\,\frac{s}{s^2+25} - 2\,\frac{2}{s^2+4} = \frac{3\,s}{s^2+25} - \frac{4}{s^2+4} \end{aligned}$$

02c. First rewrite $\cos^4(t) - \sin^4(t)$ as

$$\cos^4(t) - \sin^4(t) = \big(\cos^2(t) + \sin^2(t)\big)\,\big(\cos^2(t) - \sin^2(t)\big) = \big(\cos^2(t) - \sin^2(t)\big) = \cos(2\,t)$$

It follows

$$\mathcal{L}\left\{\cos^4(t) - \sin^4(t)\right\}(s) = \mathcal{L}\left\{\cos(2\,t)\right\}(s) = \frac{s}{s^2+4}$$

02d. First rewrite $\cosh(2\,t)\,\sinh(3\,t)$ as

$$\cosh(2\,t)\,\sinh(3\,t) = \frac{e^{2\,t} + e^{-2\,t}}{2}\;\frac{e^{3\,t} - e^{-3\,t}}{2} = \frac{e^{5\,t} - e^{-t} + e^{t} - e^{-5\,t}}{4}$$

Now using the linearity of Laplace transform and the formula $\mathcal{L}\left\{e^{a\,t}\right\}(s) = \dfrac{1}{s-a}$, we have

$$\begin{aligned}
\mathcal{L}\left\{\cosh(2\,t)\,\sinh(3\,t)\right\} &= \frac{1}{4}\,\left(\mathcal{L}\left\{e^{5\,t}\right\}(s) - \mathcal{L}\left\{e^{-t}\right\}(s) + \mathcal{L}\left\{e^{t}\right\}(s) - \mathcal{L}\left\{e^{-5\,t}\right\}(s)\right)\\
&= \frac{1}{4}\,\left(\frac{1}{s-5} - \frac{1}{s+1} + \frac{1}{s-1} - \frac{1}{s+5}\right) = \frac{1}{4}\,\left(\frac{10}{s^2-25} + \frac{2}{s^2-1}\right)\\
&= \frac{5}{2\big(s^2-25\big)} + \frac{1}{2\,\big(s^2-1\big)}
\end{aligned}$$

02e. Using Euler formula

$$\cos(\theta) = \frac{e^{\theta\,i} + e^{-\theta\,i}}{2} \iff e^{\theta\,i} + e^{-\theta\,i} = 2\,\cos(\theta)$$

we can write

$$\begin{aligned}
\cos^3(t) &= \left(\frac{e^{t\,i} + e^{-t\,i}}{2}\right)^3 = \frac{e^{3\,t\,i} + 3\,e^{2\,t\,i}\,e^{-t\,i} + 3\,e^{t\,i}\,e^{-2\,t\,i} + e^{-3\,t\,i}}{8}\\
&= \frac{\left(e^{3\,t\,i} + e^{-3\,t\,i}\right) + 3\,\left(e^{t\,i} + e^{-t\,i}\right)}{8} = \frac{2\,\cos(3\,t) + 6\,\cos(t)}{8} = \frac{1}{4}\,\cos(3\,t) + \frac{3}{4}\,\cos(t)
\end{aligned}$$

It follows

$$\mathcal{L}\left\{\cos^3(t)\right\}(s) = \frac{1}{4}\,\mathcal{L}\left\{\cos(3\,t)\right\}(s) + \frac{3}{4}\,\mathcal{L}\left\{\cos(t)\right\}(s) = \frac{1}{4}\,\frac{s}{s^2+9} + \frac{3}{4}\,\frac{s}{s^2+1}$$

02f. Use Euler formula $\sin(\theta) = \dfrac{e^{\theta\,i} - e^{-\theta\,i}}{2\,i} \iff e^{\theta\,i} - e^{-\theta\,i} = 2\,i\,\sin(\theta)$, and proceed as in question (02e) above, to get $\sin^3(t) = -\dfrac{1}{4}\,\sin(3\,t) + \dfrac{3}{4}\,\sin(t)$ and

$$\mathcal{L}\left\{\sin^3(t)\right\}(s) = -\frac{1}{4}\,\mathcal{L}\left\{\sin(3\,t)\right\}(s) + \frac{3}{4}\,\mathcal{L}\left\{\sin(t)\right\}(s) = -\frac{1}{4}\,\frac{1}{s^2+9} + \frac{3}{4}\,\frac{1}{s^2+1}$$

03. Setting $x = \sqrt{s\,t} \implies t = \dfrac{x^2}{s} \implies \mathrm{d}t = \dfrac{2\,x}{s}\,\mathrm{d}x$. Substituting, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}(s) &= \int_0^{+\infty} \frac{1}{\sqrt{t}}\,\mathrm{e}^{-s\,t}\,\mathrm{d}t = \int_0^{+\infty} \frac{1}{\sqrt{t}}\,\mathrm{e}^{-x^2}\,\frac{2\,x}{s}\,\mathrm{d}x \\
&= \int_0^{+\infty} \frac{2}{\sqrt{s}}\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{2}{\sqrt{s}}\left(\int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right) \\
&= \frac{2}{\sqrt{s}}\,\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{\sqrt{s}}
\end{aligned}$$

The formula $\displaystyle\int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$ can be established as follows.

Letting $\displaystyle I = \int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x \implies I = \int_0^{+\infty} \mathrm{e}^{-y^2}\,\mathrm{d}x$, It follows

$$\begin{aligned}
I^2 &= I\left(\int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right) = \int_0^{+\infty} \mathrm{e}^{-x^2}\,I\,\mathrm{d}x = \int_0^{+\infty} \mathrm{e}^{-x^2}\left(\int_0^{+\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^{+\infty}\left(\int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{e}^{-y^2}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^{+\infty}\int_0^{+\infty} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \iint_Q \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

where Q is the first quadrant. Switching to polar coordinates

$$x = r\,\cos(\theta), \quad y = r\,\sin(\theta), \quad \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

we have

$$\begin{aligned}
I^2 &= \iint_Q \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \iint_Q \mathrm{e}^{-r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/2}\left(\int_0^{+\infty} \mathrm{e}^{-r^2}\,r\,\mathrm{d}r\right)\mathrm{d}\theta \\
&= \int_0^{\pi/2}\left(-\frac{1}{2}\mathrm{e}^{-r^2}\,\Big|_{r=0}^{r=+\infty}\right)\mathrm{d}\theta = \int_0^{\pi/2}\left(0 + \frac{1}{2}\right)\mathrm{d}\theta = \frac{1}{2}\,\theta\Big|_0^{\pi/2} = \frac{\pi}{4}
\end{aligned}$$

Taking the square root in $I^2 = \dfrac{\pi}{4}$, leads to $\displaystyle\int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$

04. Using integration by parts, we have

$\displaystyle\int \sqrt{t}\,\mathrm{e}^{-s\,t}\,\mathrm{d}t = -\frac{\sqrt{t}\,\mathrm{e}^{-s\,t}}{s} + \frac{1}{2\,s}\int \frac{1}{\sqrt{t}}\,\mathrm{e}^{-s\,t}\,\mathrm{d}t$. It follows

$$\int_0^{+\infty} \sqrt{t}\,\mathrm{e}^{-s\,t}\,\mathrm{d}t = -\frac{\sqrt{t}\,\mathrm{e}^{-s\,t}}{s}\,\Bigg|_{t=0}^{t=+\infty} + \frac{1}{2\,s}\int_0^{+\infty} \frac{1}{\sqrt{t}}\,\mathrm{e}^{-s\,t}\,\mathrm{d}t$$

$$\boxed{\begin{array}{ccc} \sqrt{t} & & \mathrm{e}^{-s\,t} \\ & \searrow^{+} & \\ \dfrac{1}{2\,\sqrt{t}} & \xrightarrow[-\int]{} & -\dfrac{\mathrm{e}^{-s\,t}}{s} \end{array}}$$

Thus

$$\mathcal{L}\left\{\sqrt{t}\right\}(s) = \frac{1}{2\,s}\,\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}(s) = \frac{1}{2\,s}\,\frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\sqrt{\pi}}{2\,s^{3/2}}$$

05a. Recall the first shift formula $\boxed{\mathcal{L}\left\{f(t)\,\mathrm{e}^{a\,t}\right\}(s) = \mathcal{L}\left\{f(t)\right\}(s-a)}$, where $\mathcal{L}\left\{f(t)\right\}(s-a)$, means compute $\mathcal{L}\left\{f(t)\right\}(s)$, then change s into $s-a$.
we have

$$\mathcal{L}\left\{t^2\,\mathrm{e}^{-3\,t}\right\}(s) = \mathcal{L}\left\{t^2\right\}\left((s-(-3)\right) = \mathcal{L}\left\{t^2\right\}(s+3)$$

Now, $\mathcal{L}\left\{t^2\right\}(s) = \dfrac{2!}{s^3} = \dfrac{2}{s^3}$. As a result

$$\mathcal{L}\left\{t^2\,\mathrm{e}^{-3\,t}\right\}(s) = \mathcal{L}\left\{t^2\right\}(s+3) = \frac{2}{(s+3)^3}$$

05b. By the first shift formula we have

$$\mathcal{L}\left\{(t+1)^2\,\mathrm{e}^t\right\}(s)=\mathcal{L}\left\{(t+1)^2\right\}(s-1)$$

Now,

$$\mathcal{L}\left\{(t+1)^2\right\}(s)=\mathcal{L}\left\{t^2+2\,t+1\right\}(s)=\mathcal{L}\left\{t^2\right\}(s)+2\,\mathcal{L}\left\{t\right\}(s)+\mathcal{L}\left\{1\right\}(s)=\frac{2}{s^3}+2\,\frac{1}{s^2}+\frac{1}{s}$$

Hence

$$\mathcal{L}\left\{(t+1)^2\,\mathrm{e}^t\right\}(s)=\frac{2}{(s-1)^3}+\frac{2}{(s-1)^2}+\frac{1}{s-1}$$

05c. By the first shift formula we have

$$\mathcal{L}\left\{\big(\sin(2\,t)+\cos(3\,t)\big)\,\mathrm{e}^{-t}\right\}(s)=\mathcal{L}\left\{\sin(2\,t)+\cos(3\,t)\right\}\big(s-(-1)\big)=\mathcal{L}\left\{\sin(2\,t)+\cos(3\,t)\right\}(s+1)$$

Now,

$$\mathcal{L}\left\{\sin(2\,t)+\cos(3\,t)\right\}(s)=\mathcal{L}\left\{\sin(2\,t)\right\}(s)+\mathcal{L}\left\{\cos(3\,t)\right\}(s)=\frac{2}{s^2+4}+\frac{s}{s^2+9}$$

Thus

$$\mathcal{L}\left\{\big(\sin(2\,t)+\cos(3\,t)\big)\,\mathrm{e}^{-t}\right\}(s)=\mathcal{L}\left\{\sin(2\,t)+\cos(3\,t)\right\}(s+1)=\frac{2}{(s+1)^2+4}+\frac{s+1}{(s+1)^2+9}$$

05d. Making use of the first shift formula, we have

$$\mathcal{L}\left\{\frac{\mathrm{e}^{-2\,t}}{\sqrt{t}}\right\}(s)=\mathcal{L}\left\{\frac{1}{\sqrt{t}}\,\mathrm{e}^{-2\,t}\right\}(s)=\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}\big(s-(-2)\big)=\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}\big(s+2\big)$$

Now, $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}(s)=\frac{\sqrt{\pi}}{\sqrt{s}}$. It follows

$$\mathcal{L}\left\{\frac{\mathrm{e}^{-2\,t}}{\sqrt{t}}\right\}(s)=\frac{\sqrt{\pi}}{\sqrt{s+2}}$$

05e. Making use of the first shift formula, we have

$$\mathcal{L}\left\{\sqrt{t}\,\mathrm{e}^{2\,t}\right\}(s)=\mathcal{L}\left\{\sqrt{t}\right\}\big(s-2\big)$$

But since $\mathcal{L}\left\{\sqrt{t}\right\}(s)=\frac{\sqrt{\pi}}{2\,s^{3/2}}$, it follows

$$\mathcal{L}\left\{\sqrt{t}\,\mathrm{e}^{2\,t}\right\}(s)=\frac{\sqrt{\pi}}{2\,(s-2)^{3/2}}$$