Math 375	**Fourier Series and Boundary Value Problems**	**Fall 2016**
Worksheet # 5	**Part 1**	**Nov. 28 - Dec. 02**

The problems on this worksheet refer to material from sections §§10.2, 10.3, and, 10.4 of our text. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Fourier Series

01. Find the Fourier series of each of the following functions.

a. $f(x) = x, \;\; 0 < x < 2\pi, \;$ and $\; f(x + 2\pi) = f(x)$

b. $f(x) = 3\pi^2 + 5x - 12x^2, \;\; -\pi < x < \pi, \;$ and $\; f(x + 2\pi) = f(x)$

c*. $f(x) = 3x^2 + 1, \;\; -\pi < x < \pi, \;$ and $\; f(x + 2\pi) = f(x)$

d. $f(x) = \begin{cases} 1 & \text{if } -\dfrac{1}{2} < x \leq 0 \\ -1 & \text{if } \;\; 0 < x < \dfrac{1}{2} \end{cases}$ and $\; f(x + 1) = f(x)$

Even and Odd Extensions

02. Define and sketch the even and odd extensions of the function $f(x)$.

a. $f(x) = x, \quad 0 \leq x \leq 1$ **b.** $f(x) = \sin(x), \quad 0 \leq x \leq \pi$

c. $f(x) = 1 - x, \quad 0 \leq x \leq 1$ **d.** $f(x) = x^2, \quad 0 \leq x \leq 1$

03. Sketch the even and odd extensions of the given function $f(x)$

a. $f(x) = \begin{cases} 1 - x & \text{if } \; 0 \leq x < 1 \\ 0 & \text{if } \; 1 \leq x \leq 2 \end{cases}$ **b.** $f(x) = \begin{cases} -\sin(x) & \text{if } \; 0 \leq x < \pi/2 \\ -1 & \text{if } \; \pi/2 \leq x \leq \pi \end{cases}$

Fourier Sine and Fourier Cosine Series

04*. Find the Fourier sine series of $f(x) = \begin{cases} 1 & \text{if } \; 0 < x \leq \dfrac{\pi}{2} \\ 0 & \text{if } \; \dfrac{\pi}{2} < x < \pi \end{cases}$

05* Find the Fourier cosine series of $f(x) = \begin{cases} 1 & \text{if } \; 0 \leq x \leq 1 \\ 0 & \text{if } \; 1 < x \leq \pi \end{cases}$

06. Find the Fourier sine series and the Fourier cosine series of each of the following functions

a. $f(x) = x, \quad 0 < x < 1$ **b.** $f(x) = 1, \quad 0 < x < \pi$

07*. Consider the function $f(x) = \begin{cases} x^2-1 & \text{if } 0 \le x < 1 \\ x & \text{if } 1 \le x < 2 \\ -1 & \text{if } 2 \le x < 4 \end{cases}$ Determine the values to which the Fourier series of $f(x)$ converges at $x = \dfrac{1}{2}$, $x = 2$, and $x = 4$.

08*. Consider the function $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x+1 & \text{if } 0 \le x < \dfrac{\pi}{2} \\ 2\,x-1 & \text{if } \dfrac{\pi}{2} < x < \pi \end{cases}$ Determine the values to which the Fourier series of $f(x)$ converges at $x = 0$, $x = 1$, $x = \dfrac{\pi}{2}$, and $x = \pi$.

09. Determine all values of the constant c so that the Fourier series of the function

$f(x) = \begin{cases} x^2+c^2 & \text{if } 0 < x < 2 \\ 2\,x+3\,c & \text{if } 2 < x < 3 \end{cases}$ converges to 6 at $x = 2$.

10. Let $f(x) = \begin{cases} x & \text{if } 0 \le x < \pi \\ 2\,\pi - x & \text{if } \pi \le x < 2\,\pi \end{cases}$ and $f(x+2\,\pi) = f(x)$.

a. Find the Fourier series of $f(x)$

b. Prove the following identities

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8} \qquad \text{and} \qquad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

If $f(x)$ is piecewise continuous and $2\,L$-periodic, i.e., periodic with period $2\,L$, then its Fourier series is $\dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right) + b_n \sin\left(n\frac{\pi}{L}x\right)$, where

$$a_n = \frac{1}{L}\int_I f(x)\cos\left(n\frac{\pi}{L}x\right)\,\mathrm{d}x \qquad \text{and} \qquad b_n = \frac{1}{L}\int_I f(x)\sin\left(n\frac{\pi}{L}x\right)\,\mathrm{d}x$$

I is any interval of length $2\,L$. In practice it is the interval where the formula that defines $f(x)$ is given. An important simplification:

If $f(x)$ is even, then $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)\,\mathrm{d}x$ and $b_n = 0$

If $f(x)$ is odd, then $a_n = 0$, and $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)\,\mathrm{d}x$

01a. The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,\mathrm{d}x = \frac{1}{2\pi}\,x^2\Big|_0^{2\pi} = 2\,\pi$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} x\cos(n\,x)\,\mathrm{d}x = \frac{x\sin(n\,x)}{\pi\,n} + \frac{\cos(n\,x)}{\pi\,n^2}\Big|_0^{2\pi} \\
&= \frac{1}{\pi\,n^2} - \frac{1}{\pi\,n^2} = 0 \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} x\sin(n\,x)\,\mathrm{d}x = -\frac{x\cos(n\,x)}{\pi\,n} + \frac{\sin(n\,x)}{\pi\,n^2}\Big|_0^{2\pi} \\
&= -\frac{2}{n}
\end{aligned}$$

$$\begin{array}{lcl}
x & & \cos(n\,x) \\
& \searrow^{+} & \\
1 & & \dfrac{1}{n}\sin(n\,x) \\
& \searrow^{-} & \\
0 & \xrightarrow[+\int]{} & -\dfrac{1}{n^2}\cos(n\,x)
\end{array}$$

Hence

$$\begin{aligned}
f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}\Big(a_n\cos(n\,x) + b_n\sin(n\,x)\Big) \\
&\sim \pi + \sum_{n=1}^{+\infty}\frac{-2}{n}\sin(n\,x) \\
&\sim \pi - 2\sum_{n=1}^{+\infty}\frac{1}{n}\sin(n\,x)
\end{aligned}$$

$$\begin{array}{lcl}
x & & \sin(n\,x) \\
& \searrow^{+} & \\
1 & & -\dfrac{1}{n}\cos(n\,x) \\
& \searrow^{-} & \\
0 & \xrightarrow[+\int]{} & -\dfrac{1}{n^2}\sin(n\,x)
\end{array}$$

01b. The Fourier coefficients are

$$\begin{array}{lcl} 3\,\pi^2-12\,x^2 & & \cos(n\,x)\\ & \searrow^{+} & \\ -24\,x & & \dfrac{1}{n}\,\sin(n\,x)\\ & \searrow^{-} & \\ -24 & & -\dfrac{1}{n^2}\,\cos(n\,x) \\ & \searrow^{+} & \\ 0 & \xrightarrow[-\int]{} & -\dfrac{1}{n^3}\,\sin(n\,x)\end{array}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(3\,\pi^2+5\,x-12\,x^2\right)\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi}\left(3\,\pi^2-12\,x^2\right)\mathrm{d}x = \frac{2}{\pi}\left(3\,\pi^2\,x-4\,x^3\right)\Big|_0^{\pi} \\ &= \frac{2}{\pi}\left(3\,\pi^3-4\,\pi^3\right) = -2\,\pi^2 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(3\,\pi^2+5\,x-12\,x^2\right)\,\cos(n\,x)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi}\left(3\,\pi^2-12\,x^2\right)\,\cos(n\,x)\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\left(\frac{\left(3\,\pi^2-12\,x^2\right)\,\sin(n\,x)}{n}-\frac{24\,x\,\cos(n\,x)}{n^2}+\frac{24\,\sin(n\,x)}{n^3}\right)\Bigg|_0^{\pi} \\ &= \frac{2}{\pi}\left(-\frac{24\,x\,\cos(n\,x)}{n^2}\right)\Bigg|_0^{\pi} = \frac{2}{\pi}\,\frac{-24\,\pi\,\cos(n\,\pi)}{n^2} = -\frac{48\,\cos(n\,\pi)}{n^2} \\ &= -48\,\frac{(-1)^n}{n^2} \end{aligned}$$

and

$$\begin{array}{lcl} 5\,x & & \sin(n\,x)\\ & \searrow^{+} & \\ 5 & & -\dfrac{1}{n}\,\cos(n\,x)\\ & \searrow^{-} & \\ 0 & \xrightarrow[+\int]{} & -\dfrac{1}{n^2}\,\sin(n\,x)\end{array}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(3\,\pi^2+5\,x-12\,x^2\right)\,\sin(n\,x)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi}\left(5\,x\right)\,\sin(n\,x)\,\mathrm{d}x \\ &= \frac{2}{\pi}\left(-\frac{5\,x\,\cos(n\,x)}{n}+\frac{5\,\sin(n\,x)}{n^2}\right)\Bigg|_0^{\pi} \\ &= -\frac{10\,\cos(n\,\pi)}{n} = -10\,\frac{(-1)^n}{n} \end{aligned}$$

Hence

$$3\,\pi^2+5\,x-12\,x^2 \sim -\pi^2-\sum_{n=1}^{+\infty}\left(48\,\frac{(-1)^n}{n^2}\,\cos(n\,x)+10\,\frac{(-1)^n}{n}\,\sin(n\,x)\right)$$

01c. Notice that the function is even, as a result $b_n = 0$, and

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(3\,x^2+1\right)\mathrm{d}x = \frac{2}{\pi}\left(x^3+x\right)\Bigg|_0^{\pi} = \frac{2}{\pi}\left(\pi^3+\pi\right) = 2\,\pi^2+2$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}\left(3\,x^2+1\right)\,\cos(n\,x)\,\mathrm{d}x = \frac{2}{\pi}\left(\frac{\left(3\,x^2+1\right)\,\sin(n\,x)}{n}+\frac{6\,x\,\cos(n\,x)}{n^2}-\frac{6\,\sin(n\,x)}{n^3}\right)\Bigg|_0^{\pi} \\ &= \frac{2}{\pi}\left(\frac{6\,x\,\cos(n\,x)}{n^2}\right)\Bigg|_0^{\pi} = \frac{2}{\pi}\,\frac{6\,\pi\,\cos(n\,x)}{n^2} = 12\,\frac{(-1)^n}{n^2} \end{aligned}$$

Hence

$$3\,x^2+1 \sim \frac{a_0}{2}+\sum_{n=1}^{+\infty} a_n\,\cos(n\,x) = \pi^2+1+12\,\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\,\cos(n\,x)$$

01d. Notice that the function is odd, as a result $a_n=0$.
The period is $2\,L=1 \implies L=\frac{1}{2}$. It follows

$$b_n=\frac{2}{1/2}\int_0^{1/2} f(x)\,\sin\left(n\,\frac{\pi}{1/2}\,x\right)\,\mathrm{d}x = 4\int_0^{1/2} -\sin(n\,2\,\pi\,x)\,\mathrm{d}x = 4\,\frac{\cos(2\,\pi\,n\,x)}{2\,\pi\,n}\Bigg|_0^{1/2}$$

$$=2\,\frac{\cos(2\,\pi\,n\,x)}{\pi\,n}\Bigg|_0^{1/2} = 2\,\frac{\cos(n\,\pi)}{n\,\pi}-2\,\frac{1}{n\,\pi}=\frac{2}{\pi}\,\frac{(-1)^n-1}{n}$$

$$f(x)\sim\frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n-1}{n}\,\sin(2\,\pi\,n\,x)$$

If $f(x)$ is a function defined in an interval $(0\,,\,b)$, then its <u>odd extension</u>, is the function $f_{\text{odd}}(x)$, that

► is defined in $(-b\,,\,0)\cup(0\,,\,b)$, ► is odd, ► is equal to $f(x)$ in $(0\,,\,b)$, i.e.,

$$f_{\text{odd}}(x)=\begin{cases} -f(-x) & \text{if} \quad -b<x<0 \\ f(x) & \text{if} \quad 0<x<b \end{cases}$$

Notice that if you want $f_{\text{odd}}(x)$ to be defined at $x=0$, then you must have $f_{\text{odd}}(0)=0$, regardless whether $f(0)=0$ or not. Similarly, if $f(b)$ is defined, then $f_{\text{odd}}(b)=f(b)$ and $f_{\text{odd}}(-b)=-f(b)$

If $f(x)$ is a function defined in an interval $(0\,,\,b)$, then its <u>even extension</u>, is the function $f_{\text{even}}(x)$, that

► is defined in $(-b\,,\,0)\cup(0\,,\,b)$, ► is even, ► is equal to $f(x)$ in $(0\,,\,b)$, i.e.,

$$f_{\text{even}}(x)=\begin{cases} f(-x) & \text{if} \quad -b<x<0 \\ f(x) & \text{if} \quad 0<x<b \end{cases}$$

Notice that if $f(0)$ is defined, then $f_{\text{even}}(0)=f(0)$. Similarly, if $f(b)$ is defined, then $f_{\text{even}}(b)=f(b)$, and $f_{\text{even}}(-b)=f(b)$.

02a. $f_{\text{even}}(x)=\begin{cases} -x & \text{if} \quad -1\le x<0 \\ x & \text{if} \quad 0\le x<\le 1 \end{cases} = |x|$ <u>and</u> $f_{\text{odd}}(x)=x, \quad -1\le x\le 1$

02b. $f_{\text{even}}(x)=\begin{cases} -\sin(x) & \text{if} \quad -\pi\le x<0 \\ \sin(x) & \text{if} \quad 0\le x\le\pi \end{cases} = |\sin(x)|\,, \qquad f_{\text{odd}}(x)=\sin(x), \quad -\pi\le x\le\pi$

02c. $f_{\text{even}}(x)=\begin{cases} 1+x & \text{if} \quad -1\le x<0 \\ 1-x & \text{if} \quad 0\le x\le 1 \end{cases}\,, \quad f_{\text{odd}}(x)=\begin{cases} -1-x & \text{if} \quad -1\le x<0 \\ 0 & \text{if} \quad x=0 \\ 1-x & \text{if} \quad 0<x\le 1 \end{cases}$

02d. $f_{\text{even}}(x)=x^2, \quad -1\le x\le 1, \quad f_{\text{odd}}(x)=\begin{cases} -x^2 & \text{if} \quad -1\le x<0 \\ x^2 & \text{if} \quad 0\le x\le 1 \end{cases} = x\,|x|$

03a. $f_{\text{even}}(x) = \begin{cases} 0 & \text{if} \quad -2 \le x < -1 \\ x+1 & \text{if} \quad -1 \le x < 0 \\ 1-x & \text{if} \quad 0 \le x < 1 \\ 0 & \text{if} \quad 1 \le x \le 2 \end{cases}$ and $f_{\text{odd}}(x) = \begin{cases} 0 & \text{if} \quad -2 \le x < -1 \\ -x-1 & \text{if} \quad -1 \le x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1-x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad 1 \le x \le 2 \end{cases}$

03b. $f_{\text{even}}(x) = \begin{cases} -1 & \text{if} \quad -\pi \le x < -\pi/2 \\ \sin(x) & \text{if} \quad -\pi/2 \le x < 0 \\ -\sin(x) & \text{if} \quad 0 \le x < \pi/2 \\ -1 & \text{if} \quad \pi/2 \le x \le \pi \end{cases}$

and $\quad f_{\text{odd}}(x) = \begin{cases} 1 & \text{if} \quad -\pi \le x < -\pi/2 \\ -\sin(x) & \text{if} \quad -\pi/2 \le x < -\pi/2 \\ -1 & \text{if} \quad \pi/2 \le x \le \pi \end{cases}$

04. Here $L = \pi$, and the Fourier sine series is $\displaystyle\sum_{n=1}^{+\infty} b_n \sin(n\,x)$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin(n\,x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi/2} \sin(n\,x)\,\mathrm{d}x = \frac{2}{\pi}\,\frac{-\cos(n\,x)}{n}\,\Bigg|_0^{\pi/2}$$
$$= \frac{2}{\pi}\,\frac{1-\cos(n\,\pi/2)}{n}$$

Hence

$$f(x) \sim \frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{1-\cos\big(n\,\pi/2\big)}{n}\,\sin(n\,x)$$

05. Here $L = \pi$, and the Fourier cosine series is $\displaystyle\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(n\,x)$, where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{1} \mathrm{d}x = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\cos(n\,x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{1} \cos(n\,x)\,\mathrm{d}x = \frac{2}{\pi}\,\frac{\sin(n\,x)}{n}\,\Bigg|_0^{1}$$
$$= \frac{2}{\pi}\,\frac{\sin(n)}{n}$$

Hence

$$f(x) \sim \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{\sin(n)}{n}\,\cos(n\,x)$$

06a. Fourier sine series: $\displaystyle x \sim -\frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}\,\sin(n\,\pi\,x)$

Fourier cosine series: $\displaystyle x \sim \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{+\infty} \frac{(-1)^n - 1}{n^2}\,\cos(n\,\pi\,x)$

06b. Fourier sine series: $\displaystyle 1 \sim -\frac{2}{\pi}\sum_{n=1}^{+\infty} \frac{(-1)^n - 1}{n}\,\sin(n\,x)$

Fourier cosine series: $1 \sim 1$

07. Both $f(x)$ and $f'(x)$ are piecewise continuous in $\big[0,4\big]$, the Fourier series associated with $f(x)$ converges at every x in $\big[1,4\big]$ and has value $\dfrac{f(x+0)+f(x-0)}{2}$.

a. At $x=\dfrac{1}{2}$, the function is continuous, and the Fourier series has value

$$\frac{f(1/2+0)+f(1/2-0)}{2} = f(1/2) = \left(\frac{1}{2}\right)^2 - 1 = -\frac{3}{4}$$

b. At $x=2$, the Fourier series has value

$$\frac{f(2+0)+f(2-0)}{2} = \frac{-1+2}{2} = \frac{1}{2}$$

c. At $x=4$, the Fourier series has value

$$\frac{f(4+0)+f(4-0)}{2} = \frac{f(0+0)+f(4-0)}{2} = \frac{-1-1}{2} = -1$$

08. Both $f(x)$ and $f'(x)$ are piecewise continuous in $\big[-\pi,\pi\big]$, the Fourier series associated with $f(x)$ converges at every x in $\big[-\pi,\pi\big]$ and has value $\dfrac{f(x+0)+f(x-0)}{2}$.

a. At $x=0$, the Fourier series has value

$$\frac{f(0+0)+f(0-0)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

b. At $x=\pi/2$, the Fourier series has value

$$\frac{f(\pi/2+0)+f(\pi/2-0)}{2} = \frac{(\pi-1)+(\pi/2+1)}{2} = \frac{3\,\pi}{4}$$

08c. At $x=\pi$, the Fourier series has value

$$\frac{f(\pi+0)+f(\pi-0)}{2} = \frac{f(0+0)+f(\pi-0)}{2} = \frac{0+2\,\pi-1}{2} = \frac{2\,\pi-1}{2}$$

09. Both $f(x)$ and $f'(x)$ are piecewise continuous in $\big[-\pi,\pi\big]$, the Fourier series associated with $f(x)$ converges at every x in $\big[0,3\big]$ and has value $\dfrac{f(x+0)+f(x-0)}{2}$. In particular

$$\begin{aligned}\frac{f(2+0)+f(2-0)}{2} = 6 \iff \frac{4+3\,c+4+c^2}{2} = 6 &\iff c^2+3\,c-4=0\\ \iff (c+4)\,(c-1)=0 &\iff c=-4,\ 1\end{aligned}$$

10a. First notice that $f(x)$ is an even function. It follows $b_n = 0,\ n = 1, 2, \cdots$

$$a_0 = \frac{2}{L} \int_0^L f(x)\,\mathrm{d}x = \frac{2}{\pi} \int_0^\pi x\,\mathrm{d}x = \left.\frac{x^2}{\pi}\right|_0^\pi = \pi$$

and

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos\left(n\,\frac{\pi}{L}\,x\right) \mathrm{d}x = \frac{2}{\pi} \int_0^\pi x \cos(n\,x)\,\mathrm{d}x \\ &= \frac{2}{\pi} \left(\frac{x \sin(n\,x)}{n} + \frac{\cos(n\,x)}{n^2}\right) \Bigg|_0^\pi = \frac{2\left((-1)^n - 1\right)}{\pi\,n^2} \end{aligned}$$

Hence

$$f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{+\infty} \frac{2\left((-1)^n - 1\right)}{\pi\,n^2} \cos(n\,x)$$

But $(-1)^n - 1 = \begin{cases} -2 & \text{if} \quad n = 1, 3, 5, \cdots \\ 0 & \text{if} \quad n = 2, 4, 6, \cdots \end{cases}$ It follows

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{\cos\left((2\,n+1)\,x\right)}{(2\,n+1)^2}$$

10b. Because $f(x)$ is continuous in its domain, we have

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{\cos\left((2\,n+1)\,x\right)}{(2\,n+1)^2} \qquad \text{for any } x$$

Setting $x = 0$, leads to

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{1}{(2\,n+1)^2} \implies \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{1}{(2\,n+1)^2} = \frac{\pi}{2} \implies \sum_{n=0}^{+\infty} \frac{1}{(2\,n+1)^2} = \frac{\pi^2}{8}$$

To establish the second identity, write

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{(2\,n)^2} + \sum_{n=0}^{+\infty} \frac{1}{(2\,n+1)^2} = \frac{1}{4} \sum_{n=1}^{+\infty} \frac{1}{n^2} + \sum_{n=0}^{+\infty} \frac{1}{(2\,n+1)^2}$$

Solving for $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$, we get

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=0}^{+\infty} \frac{1}{(2\,n+1)^2} = \frac{4}{3}\,\frac{\pi^2}{8} = \frac{\pi^2}{6}$$