Fourier Series and Boundary Value Problems

Introduction

Let $f(x)$ be a function defined in an interval I

The goal is to express $f(x)$ as an infinite sum of cosine and sine functions i.e.,

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cos(n\omega x) + \beta_n \sin(n\omega x)$$

$$= \alpha_0 + (\alpha_1 \cos(\omega x) + \beta_1 \sin(\omega x)) + (\alpha_2 \cos(2\omega x) + \beta_2 \sin(2\omega x))$$

$$+ (\alpha_3 \cos(3\omega x) + \beta_3 \sin(3\omega x)) + \dots$$

$\omega > 0$ is a given real number

$\alpha_0, \alpha_1, \alpha_2, \dots$; $\beta_0, \beta_1, \beta_2, \dots$ are two sequences of real number.

writing $f(x)$ as an infinite sum of sine and cosine functions will be very useful when we try to solve the heat equation, the wave equation and Laplace's equation.

Definition

Given $\omega > 0$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, the infinite sum

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(n\omega x) + \beta_n \sin(n\omega x) \quad \cdots (*)$$

is called a trigonometric series

$F_0(x) = \alpha_0$, $\quad F_1(x) = \alpha_0 + \big(\alpha_1 \cos(\omega x) + \beta_1 \sin(\omega x)\big)$,

$F_2(x) = \alpha_0 + \big(\alpha_1 \cos(\omega x) + \beta_1 \sin(\omega x)\big) + \big(\alpha_2 \cos(2\omega x) + \beta_2 \sin(2\omega x)\big)$,

$\vdots$

are called the partial sums of the trigonometric series $(*)$

of order $0, 1, 2, 3, \ldots$ respectively

Notice that the bigger n is, the closer $F_n(x)$ is to the infinite sum (*)

Notice also that if $\alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos(n\omega x) + \beta_n \sin(n\omega x))$ adds up to a real number for every x, then this defines a function

$$g(x) = \alpha_0 + \sum_{n=1}^{\infty} \left(\alpha_n \cos(n\omega x) + \beta_n \sin(n\omega x)\right)$$

$\alpha_1 \cos(\omega x) + \beta_1 \sin(\omega x)$ is periodic with period $\frac{2\pi}{\omega}$

$\alpha_2 \cos(2\omega x) + \beta_2 \sin(2\omega x)$ is periodic with period $\frac{2\pi}{2\omega} = \frac{\pi}{\omega}$

$\alpha_3 \cos(3\omega x) + \beta_3 \sin(3\omega x)$ is periodic with period $\frac{2\pi}{3\omega}$

$\vdots$

This shows that $g(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(n\omega x) + \beta_n \sin(n\omega x)$ is periodic with period $\frac{2\pi}{\omega}$

Suppose we have a function $f(x)$ defined in an interval I, and suppose we are successful in representing $f(x)$ as trigonometric series

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(n\omega x) + \beta_n \sin(n\omega x)$$

Using integration by parts, we can show that

$$\alpha_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(x) \cos(n\omega x)\, dx \qquad n = 1, 2, 3, \dots$$

$$\beta_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(x) \sin(n\omega x)\, dx$$

$$\alpha_0 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} f(x)\, dx$$

Fourier Series

Definition

Let $f(x)$ be a function that is piecewise continuous and periodic with period $T = 2L$ $(L > 0)$.

The Fourier series of $f(x)$ is the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{L} x\right) + b_n \sin\left(n \frac{\pi}{L} x\right) \quad \dots\dots (*)$$

where $a_0, a_1, a_2, \dots ; \ b_1, b_2, \dots,$ are defined by

$$a_n = \frac{1}{L} \int_I f(x) \cos\left(n \frac{\pi}{L} x\right) dx \qquad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{1}{L} \int_I f(x) \sin\left(n \frac{\pi}{L} x\right) dx \qquad n = 1, 2, 3, \dots$$

I is an interval of length $2L$

a_n, b_n are called the Fourier coefficients of $f(x)$.

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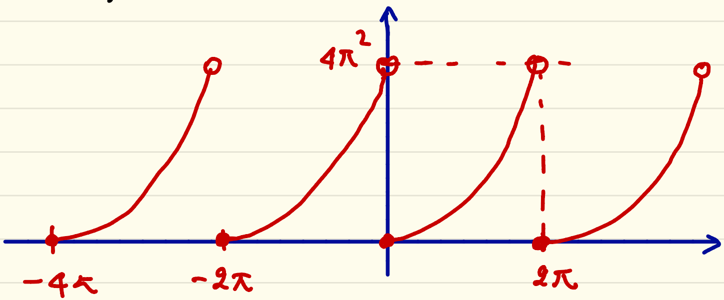
To express that the trigonometric series $(*)$ is the Fourier series of $f(x)$, we will write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{L} x\right) + b_n \sin\left(n \frac{\pi}{L} x\right)$$

Example

Compute the Fourier series of $f(x) = x^2$, $0 \leq x < 2\pi$

with $f(x + 2\pi) = f(x)$



The Fourier series associated with the function $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{\pi} x\right) + b_n \sin\left(n \frac{\pi}{\pi} x\right) =$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\,\frac{x^3}{3}\Big|_0^{2\pi} = \frac{8\pi^3}{3\pi} = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x).\cos\left(n\frac{\pi}{L}x\right)dx = \frac{1}{\pi}\int_0^{2\pi} x^2.\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\frac{1}{n}x^2\sin(nx) + \frac{2}{n^2}x\cos(nx) - \frac{2}{n^3}\sin(nx)\Big|_0^{2\pi}\right)$$

$$= \frac{1}{\pi}\left(\left(\frac{4\pi}{n^2}\right) - (0)\right) = \frac{4}{n^2}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x).\sin\left(n\frac{\pi}{L}x\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}x^2\cos(nx) + \frac{2}{n^2}x\sin(nx) + \frac{2}{n^3}\cos(nx)\Big|_0^{2\pi}\right)$$

$$= \frac{1}{\pi}\left(\left(-\frac{4\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{2}{n^3}\right)\right) = -\frac{4\pi}{n}$$

$$\begin{array}{lcl} x^2 & & \cos(nx) \\ & + & \\ 2x & & \frac{1}{n}\sin(nx) \\ & - & \\ 2 & & -\frac{1}{n^2}\cos(nx) \\ & + & \\ 0 & -\int & -\frac{1}{n^3}\sin(nx) \end{array}$$

$$\begin{array}{lcl} x^2 & & \sin(nx) \\ & + & \\ 2x & & -\frac{1}{n}\cos(nx) \\ & - & \\ 2 & & -\frac{1}{n^2}\sin(nx) \\ & + & \\ 0 & -\int & \frac{1}{n^3}\cos(nx) \end{array}$$

Hence the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$f(x) \sim \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos(nx) - \frac{4\pi}{n} \sin(nx) \right)$$

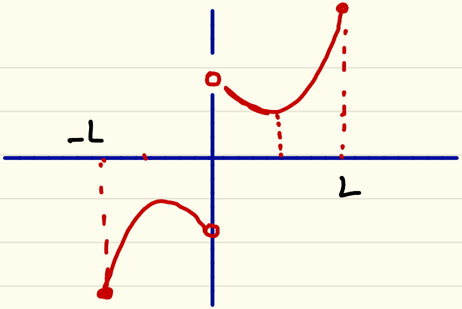
Definition

$f(x)$ is said to be an odd function if

$f(-x) = -f(x)$, for any x in the domain of $f(x)$

$f(x)$ is said to be an even function if

$f(-x) = f(x)$, for any x in the domain of $f(x)$



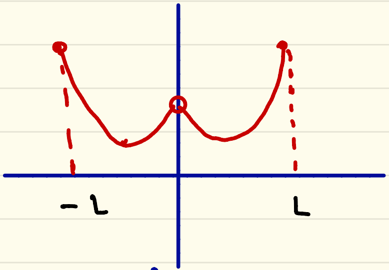
odd function

$f(x) = x$, $f(x) = x^3$,

$f(x) = x - \sin(x)$

$f(x) = x + 5x^7$

are odd functions



even function

$f(x) = 1$, $f(x) = x^2$

$f(x) = x^4$, $f(x) = \cos(x)$

$f(x) = x^2 + 3x^4$,

$f(x) = x^2 - \cos(x)$

are even functions

Result

If $f(x)$ is odd, then $\int_{-L}^{L} f(x)\,dx = 0$

If $f(x)$ is even, then $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$

Result

If $f(x)$ is an odd $2L$-periodic function, then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\left(n\frac{\pi}{L}x\right)dx = 0, \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\cdot\sin\left(n\frac{\pi}{L}x\right)dx, \qquad n = 1, 2, 3, \ldots$$

If $f(x)$ is an even, $2L$-periodic function, then

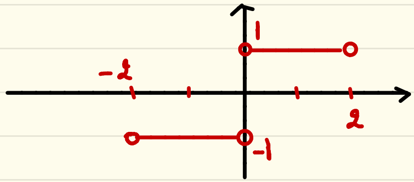
$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx \qquad n = 0, 1, 2, \ldots$$

$$b_n = 0, \qquad n = 1, 2, \ldots$$

Example

Find the Fourier series of $f(x) = \begin{cases} -1 & \text{if } -2 < x < 0 \\ 1 & \text{if } 0 < x < 2 \end{cases}$

and $f(x+4) = f(x)$.



Notice that $f(x)$ is odd, it follows

that $a_n = 0, \quad n = 0, 1, 2, \dots$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx = \frac{2}{2}\int_0^2 1.\sin\left(n\frac{\pi}{2}x\right)dx$$

$$= -\frac{2}{n\pi}\cos\left(n\frac{\pi}{2}x\right)\Big|_0^2 = -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{n\pi} = \frac{2(1-\cos(n\pi))}{n\pi}$$

Hence

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n\pi}\sin\left(n\frac{\pi}{2}x\right)$$

Theorem (Convergence – Dirichlet)

Let $f(x)$ be a $2L$-periodic function, and let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{L} x\right) + b_n \sin\left(n \frac{\pi}{L} x\right) \quad \cdots (*)$$

be its Fourier series.

If $f(x)$ and $f'(x)$ are piecewise continuous in $[-L, L]$ or $[0, 2L]$

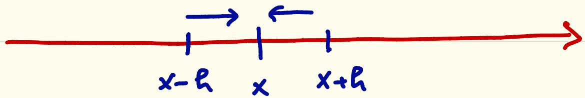
then the Fourier series $(*)$ converges for any x

Furthermore

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(n \frac{\pi}{L} x\right) + b_n \sin\left(n \frac{\pi}{L} x\right)\right) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \dfrac{f(x+0) + f(x-0)}{2} & \text{if } f \text{ is discont. at } x \end{cases}$$

where $f(x+0) = \lim_{h \to 0^+} f(x+h)$ and

$f(x-0) = \lim_{h \to 0^+} f(x-h)$



Example

Let $$f(x) = \begin{cases} x^2+1 & 0 < x \le 2 \\ \dfrac{12}{x} & 2 < x \le 3 \\ 3x-5 & 3 < x \le 7 \end{cases}, \qquad f(x+7) = f(x)$$

Evaluate the Fourier series of $f(x)$ at $x=2$, $x=5/2$, $x=3$, $x=7$

First notice that both $f(x)$ and $$f'(x) = \begin{cases} 2x & \text{if } 0<x<2 \\ -12/x^2 & \text{if } 2<x<3 \\ 3 & \text{if } 3<x<7 \end{cases}$$ are piecewise continuous in $[0, 7]$.

By the convergence theorem, the Fourier series of $f(x)$ converges at every x in $[0, 7]$.

Value at $x=2$

we have

$$f(2-0) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2+1) = 4+1 = 5$$

$$f(2+0) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{12}{x} = \frac{12}{2} = 6$$

Hence the value of the Fourier series at $x = 2$ is

$$\frac{f(2-0) + f(2+0)}{2} = \frac{5+6}{2} = \frac{11}{2}$$

Value at $x = 5/2$

$f(x)$ is continuous at $x = 5/2$. It follows that the value of the Fourier series at $x = 5/2$ is $f(5/2) = \dfrac{12}{5/2} = \dfrac{24}{5}$

Value at $x = 3$

we have $f(3-0) = \lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} \dfrac{12}{x} = \dfrac{12}{3} = 4$ and

$f(3+0) = \lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (3x-5) = 9-5 = 4$

Hence the value of the Fourier series at $x = 3$ is

$$\frac{f(3-0) + f(3+0)}{2} = \frac{4+4}{2} = 4$$

Value at $x=7$

we have $f(7-0) = \lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} (3x-5) = 21-5 = 16$

and $f(7+0) = \lim_{x \to 7^+} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2+1) = 0+1 = 1$

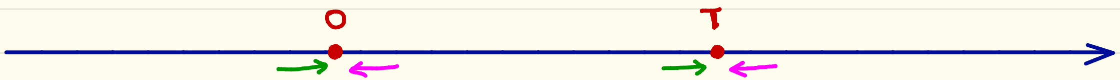
Hence the value of the Fourier series at $x=7$ is

$$\frac{f(7-0)+f(7+0)}{2} = \frac{16+1}{2} = \frac{17}{2}$$

Result

If $f(x)$ is periodic with period T, then

$$f(0-0) = f(T-0) = \lim_{x \to T^-} f(x) \quad \text{and} \quad f(T+0) = f(0+0) = \lim_{x \to 0^+} f(x)$$



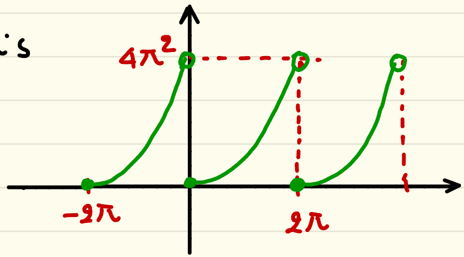
RH limits at $x=0$ and $x=T$ are equal, LH limits at $x=0$ and $x=T$ are equal

Example

In a previous example, we have showed that the Fourier series of

$f(x) = x^2$ for $0 \le x < 2\pi$, $f(x+2\pi) = f(x)$, is

$$\frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nx) - \frac{4\pi}{n}\sin(nx)$$



By the convergence theorem, we have

$$x^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nx) - \frac{4\pi}{n}\sin(nx), \qquad 0 < x < 2\pi$$

at $x=0$, we have $f(0-0) = f(2\pi - 0) = \lim_{x \to 2\pi^-} x^2 = 4\pi^2$,

$f(0+0) = \lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$. It follows

$$\frac{f(0-0)+f(0+0)}{2} = \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(0) - \frac{4\pi}{n}\sin(0) \Leftrightarrow \frac{4\pi^2+0}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

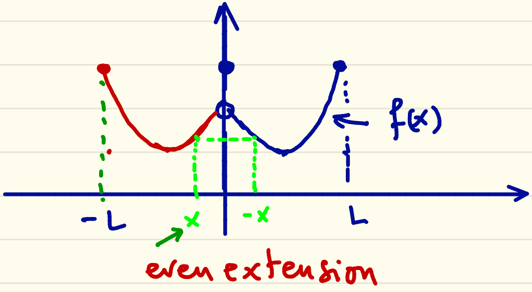
$$\Leftrightarrow 2\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2 - \frac{4\pi^2}{3}}{4} = \frac{\pi^2}{6}$$

Hence $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}$

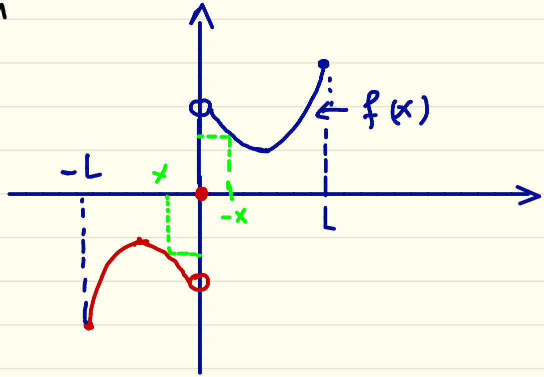
Definition

Let $f(x)$ be a function defined in $(0, L)$, $L>0$.

The even extension of $f(x)$ is the function $f_{even}(x)$ that is even and is equal to $f(x)$ in $(0,L)$.



The odd extension of $f(x)$ is the function $f_{odd}(x)$ that is odd and is equal to $f(x)$ in $(0,L)$.



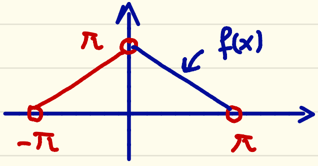
$$f_{even}(x) = \begin{cases} f(x) & \text{if } 0<x<L \\ f(-x) & \text{if } -L<x<0 \end{cases}$$

$$f_{odd}(x) = \begin{cases} f(x) & \text{if } 0<x<L \\ -f(-x) & \text{if } -L<x<0 \end{cases}$$

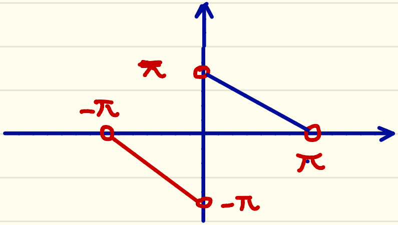
Example

Find the even and odd extensions of $f(x) = \pi - x$, $0 < x < \pi$

$$f_{even}(x) = \begin{cases} f(x) & \text{if } 0 < x < \pi \\ f(-x) & \text{if } -\pi < x < 0 \end{cases} = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ \pi + x & \text{if } -\pi < x < 0 \end{cases}$$

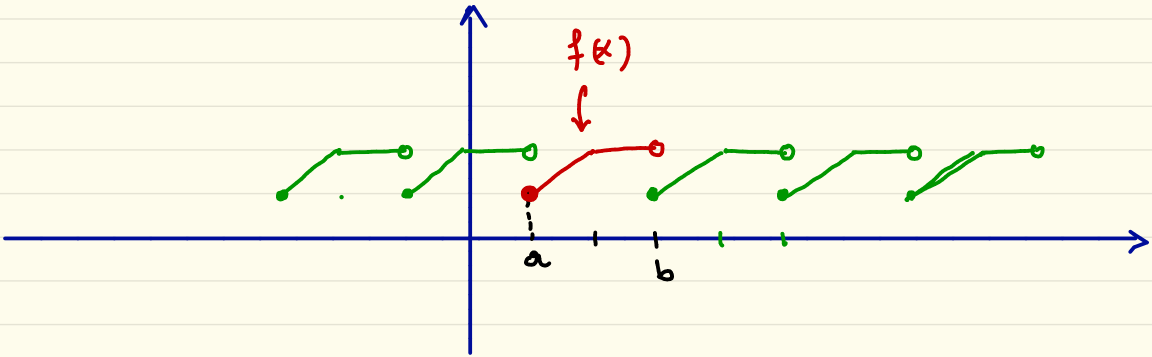


$$f_{odd}(x) = \begin{cases} f(x) & \text{if } 0 < x < \pi \\ -f(-x) & \text{if } -\pi < x < 0 \end{cases} = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ -(\pi + x) & \text{if } -\pi < x < 0 \end{cases}$$



Definition

Let $f(x)$ be a function defined in (a,b) and let $T=b-a$.
The periodic extension of $f(x)$, is the periodic function with period T and is equal to $f(x)$ in (a,b)



Definition (Fourier Sine and Cosine Series)

Let $f(x)$ be defined in $(0, L)$, $L > 0$.

The Fourier Sine Series of $f(x)$, is the Fourier series of the odd-periodic extension of $f(x)$

The Fourier Cosine Series of $f(x)$, is the Fourier series of the even-periodic extension of $f(x)$

Result

Let $f(x)$ be defined in $(0, L)$

The Fourier Sine series of $f(x)$ is $\sum_{n=1}^{\infty} b_n \sin(n \frac{\pi}{L} x)$

with $b_n = \frac{2}{L} \int_0^L f(x) \sin(n \frac{\pi}{L} x)\, dx$

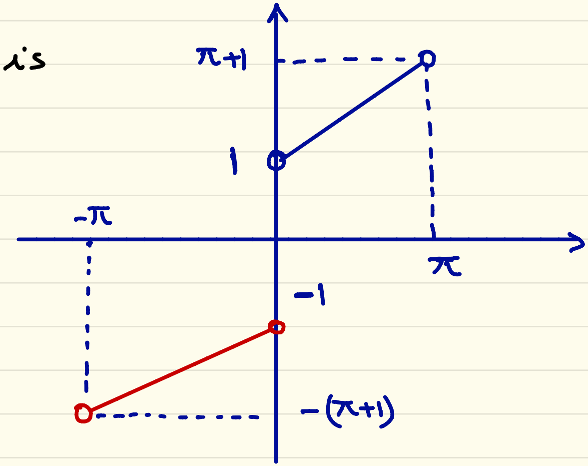
The Fourier Cosine series of $f(x)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n \frac{\pi}{L} x)$

with

$$a_n = \frac{2}{L} \int_0^L f(x) \cos(n \frac{\pi}{L} x)\, dx \qquad n = 0, 1, 2, \ldots$$

Example

Compute the Fourier Sine series of $f(x) = x+1,\ 0 < x < \pi$



The Fourier sine series of $f(x)$ is

$$\sum_{n=1}^{\infty} b_n \sin(n \frac{\pi}{\pi} x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

with

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x+1) \sin(nx)\, dx$$

Using integration by parts leads to

$$b_n = \frac{2}{\pi}\int_0^{\pi}(x+1)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}(x+1)\cos(nx) + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}(\pi+1)\cos(n\pi) + \frac{1}{n}\right]$$

$$= \frac{2}{n\pi}\left(1-(\pi+1)\cos(n\pi)\right)$$

$$\begin{array}{lcl} x+1 & & \sin(nx) \\ & \searrow^{+} & \\ 1 & & -\frac{1}{n}\cos(nx) \\ & \searrow^{-} & \\ 0 & \xrightarrow[+\int]{} & -\frac{1}{n^2}\sin(nx) \end{array}$$

Hence

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(\pi+1)\cos(n\pi)\right)\sin(nx)$$