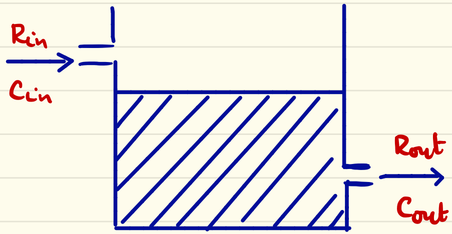
Mixing Problems



These involve a container that initially holds a volume V_0 that contains a mass m_0 of a substance (salt, sugar, drug, ...).

A brine that contains the substance with concentration C_{in} $\left(\frac{\text{mass}}{\text{volume}}\right)$ is poured into the container at a rate R_{in} $\left(\frac{\text{volume}}{\text{time}}\right)$

At the same time, the container is drained out at a rate R_{out} $\left(\frac{\text{volume}}{\text{time}}\right)$. The question is usually to determine the total mass $m = m(t)$ of the substance.

To translate this problem in mathematical terms, we use the balance law:

net rate = rate in − rate out

$$m'(t) = C_{in} \cdot R_{in} - C_{out} \cdot R_{out}$$

$$= C_{in} \cdot R_{in} - \frac{m(t)}{V(t)} \cdot R_{out}$$

Hence $m(t)$ is solution of the initial value problem

$$\begin{cases} m' + \dfrac{R_{out}}{V(t)} m = C_{in} \cdot R_{out} \\ m(0) = m_0 \end{cases}$$

The volume $V(t)$ is a solution of the ivp $\begin{cases} V' = R_{in} - R_{out} \\ V(0) = V_0 \end{cases}$

Notice that if $R_{in} = R_{out}$, then $V(t) = V_0$, while if $R_{in} \neq R_{out}$

then $V(t) = (R_{in} - R_{out})\, t + V_0$

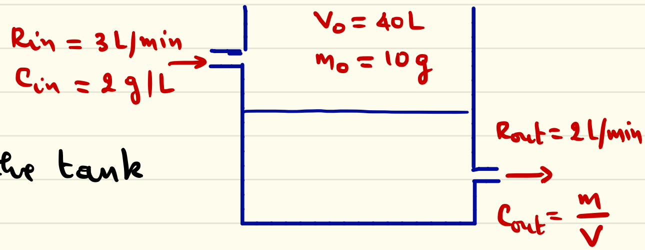
Example A tank initially contains $40\,L$ of water in which $10\,g$ of salt has been dissolved. Starting at $t = 0$, a brine containing $2\,g/L$ of dissolved salt, flows into the tank at the rate of $3\,L/min$. The mixture is kept uniform by continuous stirring, and the well stirred mixture simultaneously flows out of the tank at the slower rate of $2\,L/min$.

a. Find the amount of salt $m(t)$ at any given time t.

b. How much salt is in the tank at the end of 30 min. ?

Solution

Letting $m(t)$ and $V(t)$ be the mass of salt and the volume of the brine in the tank at time t, respectively, we have



$$m'(t) = R_{in} \cdot C_{in} - R_{out} \cdot C_{out} = \left(3\,\frac{L}{min}\right)\left(2\,\frac{g}{L}\right) - \left(2\,\frac{L}{min}\right)\frac{m(t)}{V(t)}$$

$$= 6 - \frac{2}{V(t)}\,m(t) \quad \cdots (*)$$

To determine $V(t)$, observe that $V'(t) = R_{in} - R_{out} = \left(3\,\frac{L}{min}\right) - \left(2\,\frac{L}{min}\right) = 1$

Integrating, we get $V(t) = t + C$. To determine the constant, we use the fact that $V(0) = 40\ L \iff 0 + C = 40 \iff C = 40$.

Hence $V(t) = t + 40$. Substituting into $(*)$, we get

$m'(t) = 6 - \dfrac{2}{t+40}\,m(t)$. Hence $m(t)$ is the solution of the ivp

$$\begin{cases} m' + \dfrac{2}{t+40}\,m = 6 \\ m(0) = 10 \end{cases}$$

An integrating factor is

$$\mu(t) = e^{\int 2/(t+40)\,dt} = (t+40)^2$$

Multiply by the integrating factor leads to $\left((t+40)^2 m\right)' = 6(t+40)^2$

Integrating, we get $m(t) = 2(t+40) + \dfrac{c}{(t+40)^2}$

To determine the constant, use the initial condition

$$m(0)=10 \iff 2(0+40)+\frac{c}{(0+40)^2}=10 \iff 80+\frac{c}{1600}=10 \iff c=-112000$$

Hence

$$m(t)=2(t+40)-\frac{112\,000}{(t+40)^2}$$

The amount of salt after 30 min is simply

$$m(30)=2(30+40)-\frac{112\,000}{(30+40)^2}=140-\frac{112\,000}{4900}=140-\frac{160}{7}$$

$$=\frac{820}{7}\approx 117{,}14\ g$$