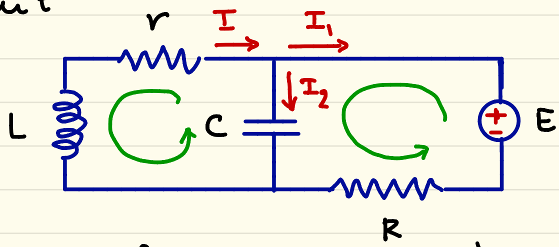
Linear Systems of Differential Equations

Consider the two-loops electrical circuit



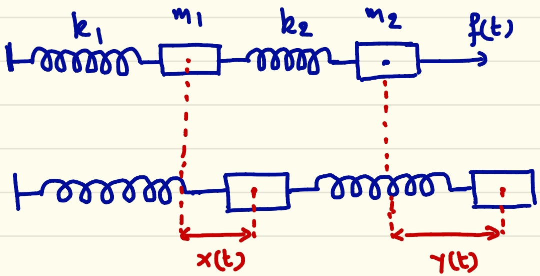
Using Kirchoff's voltage and current laws leads to the system of differential equations

$$\begin{cases} Q' = -\dfrac{1}{RC}\,Q + I + \dfrac{E}{R} \\[2mm] I' = -\dfrac{1}{LC}\,Q - \dfrac{r}{L}\,I \end{cases}$$

$Q = Q(t)$ is the charge in the capacitor

$$I = I_1 + I_2$$

Consider the mechanical system



Using Newton's second Law of motion and Hooke's Law, lead to the displacements x and y of the two objects satisfy the system of differential equations.

$$\begin{cases} m_1 x'' = -k_1 x + k_2(y-x) \\ m_2 y'' = -k_2(y-x) \end{cases}$$

Given $a_{11}(t), a_{12}(t), \dots, a_{1n}(t), a_{21}(t), a_{22}(t), \dots, a_{2n}(t), \dots,$
$a_{n1}(t), a_{n2}(t), \dots, a_{nn}(t), f_1(t), f_2(t), \dots, f_n(t)$
functions defined in an interval (a,b)

$$(S_n)\begin{cases} y_1' = a_{11}(t)\,y_1 + a_{12}(t)\,y_2 + \cdots + a_{1n}(t)\,y_n + f_1(t) \\ y_2' = a_{21}(t)\,y_1 + a_{22}(t)\,y_2 + \cdots + a_{2n}(t)\,y_n + f_2(t) \\ \quad\vdots \\ y_n' = a_{n1}(t)\,y_1 + a_{n2}(t)\,y_2 + \cdots + a_{nn}(t)\,y_n + f_n(t) \end{cases}$$

is called a first order linear system of differential equations.

$y_1 = y_1(t), y_2 = y_2(t), \dots, y_n = y_n(t)$ are the unknown functions to be determined.

If we let $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$, $\vec{F}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$,

$$Q(t) = \begin{bmatrix} q_{11}(t) & q_{12}(t) & \cdots & q_{1n}(t) \\ q_{21}(t) & q_{22}(t) & \cdots & q_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}(t) & q_{n2}(t) & \cdots & q_{nn}(t) \end{bmatrix}$$

and if we define

$\vec{y}\,'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{bmatrix}$, then (S_n) can be written as

$$\vec{y}\,' = Q(t)\,\vec{y} + \vec{F}(t)$$

Example consider the system of differential equations

$$\begin{cases} y_1' = t y_1 + y_2 + e^t \\ y_2' = 3y_1 - t y_2 - \sin(t) \end{cases}$$

To rewrite the system in matrix form we proceed as follows

$$\begin{cases} y_1' = t y_1 + y_2 + e^t \\ y_2' = 3y_1 - t y_2 - \sin t \end{cases} \iff \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} t y_1 + y_2 + e^t \\ 3y_1 - t y_2 - \sin t \end{bmatrix}$$

$$\iff \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} t y_1 + y_2 \\ 3y_1 - t y_2 \end{bmatrix} + \begin{bmatrix} e^t \\ -\sin(t) \end{bmatrix}$$

$$\iff \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} t & 1 \\ 3 & -t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} e^t \\ -\sin(t) \end{bmatrix}$$

or else $\vec{y}\,' = Q(t)\vec{y} + \vec{F}(t)$

with $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $Q(t) = \begin{bmatrix} t & 1 \\ 3 & -t \end{bmatrix}$, and $\vec{F}(t) = \begin{bmatrix} e^t \\ -\sin(t) \end{bmatrix}$

Example

Rewrite the system $\vec{y}\,' = \begin{bmatrix} t & -2t \\ 5 & \ln(t) \end{bmatrix} \vec{y} + \begin{bmatrix} \sin(t) \\ 2 \end{bmatrix}$ in equations form.

Letting y_1 and y_2 be the components of $\vec{y}$, we have

$$\begin{cases} y_1' = t\,y_1 - 2t\,y_2 + \sin(t) \\ y_2' = 5t_1 + \ln(t)\,y_2 + 2 \end{cases}$$

Example consider the 2^{nd} order diff. eq.

$$a(t)\, y'' + b(t)\, y' + c(t)\, y = f(t)$$

we want to rewrite the equation as a first order linear system. To do that, we start by solving for the highest derivative of y in the equation. We get

$$y'' = -\frac{c(t)}{a(t)}\, y - \frac{b(t)}{a(t)}\, y' + \frac{f(t)}{a(t)}$$

Next we set $y_1 = y$ & $y_2 = y'$. It follows

$$\begin{cases} y_1' = y_2 \\ y_2' = -\dfrac{c(t)}{a(t)}\, y_1 - \dfrac{b(t)}{a(t)}\, y_2 + \dfrac{f(t)}{a(t)} \end{cases}$$

In matrix form, the system is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\dfrac{c(t)}{a(t)} & -\dfrac{b(t)}{a(t)} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{f(t)}{a(t)} \end{bmatrix}$$

Example Rewrite $ty'' + 5y' + e^t y = \frac{1}{t}$ as a system of 1^{st} order diff. eqs.

start by solving for y'' to get $y'' = -\frac{e^t}{t}y - \frac{5}{t}y' + \frac{1}{t^2}$

Then set $y_1 = y$ & $y_2 = y'$. It follows

$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{e^t}{t}y_1 - \frac{5}{t}y_2 + \frac{1}{t^2} \end{cases} \iff \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{e^t}{t} & -\frac{5}{t} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{t^2} \end{bmatrix}$$

Definition

Consider the 1^{st} order linear system $\vec{Y}' = Q(t)\vec{Y} + \vec{F}(t)$

$$\vec{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad Q(t) = \begin{bmatrix} q_{11}(t) & \cdots & q_{1n}(t) \\ \vdots & \ddots & \vdots \\ q_{n1}(t) & \cdots & q_{nn}(t) \end{bmatrix}, \quad \vec{F}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$\vec{U}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$ is called a solution of the system in

an interval (a,b), if $\vec{U}(t)$ is differentiable in (a,b)

and $\vec{U}'(t) = Q(t)\vec{U}(t) + \vec{F}(t)$, for any t in (a,b)

The general solution of the system, is a vector function

$\vec{Y}(t, c_1, c_2, \dots, c_n)$ that depends on n constants such that every

solution $\vec{U}(t)$ is a special instance of $\vec{Y}(t, c_1, c_2, \dots, c_n)$

By definition solving the system $\vec{Y}' = Q(t)\vec{Y} + \vec{F}(t)$ is finding its general solution $\vec{Y}(t, c_1, c_2, \dots, c_n)$

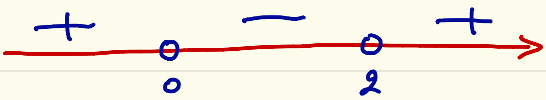
Result (Existence & Uniqueness)

consider the initial value problem $\begin{cases} \vec{Y}' = Q(t)\vec{Y} + \vec{F}(t) \\ \vec{Y}(t_0) = \vec{Y}_0 \end{cases}$

If $Q(t)$ and $\vec{F}(t)$ are continuous at and around $t = t_0$, then the ivp has a unique solution that is defined in the largest open interval (a, b) that contains t_0 and where both $Q(t)$ and $\vec{F}(t)$ are continuous.

This result is almost identical to the existence and uniqueness result we have stated for the scalar ivp $\begin{cases} y' + p(t)y = f(t) \\ y(t_0) = y_0 \end{cases}$

$t^2-2t>0 \iff t(t-2)>0$



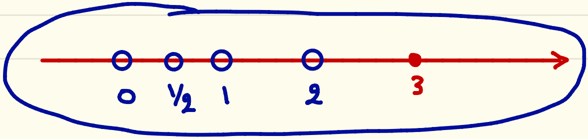
Example

Find the largest interval (a,b) where the solution of the ivp

$$\begin{cases} y_1' = \dfrac{1}{t-1}\,y_1 + \dfrac{t-1}{2t-1}\,y_2 + \ln(t^2-2t) \\ y_2' = 2t\,y_1 + \cos(t)\,y_2 + |t+1| \\ y_1(3) = -2 \;\;\&\;\; y_2(3) = 5 \end{cases}$$

is defined.

Solution The coefficient matrix $Q(t) = \begin{bmatrix} \frac{1}{t-1} & \frac{t-1}{2t-1} \\ 2t & \cos(t) \end{bmatrix}$ is continuous in $(-\infty, 1/2) \cup (1/2, 1) \cup (1, +\infty)$. The "forcing term" $\vec{F}(t) = \begin{bmatrix} \ln(t^2-2t) \\ |t+1| \end{bmatrix}$ is continuous in $(-\infty, 0) \cup (2, +\infty)$. The largest open interval that contains $t=3$, where both $Q(t)$ and $\vec{F}(t)$ are continuous is $(2, +\infty)$. By the existence and uniqueness theorem, the ivp has a unique solution defined in $(2, +\infty)$.



Main Result

The general solution of the homogeneous system

$$\vec{Y}' = Q(t)\,\vec{Y}, \quad a < t < b$$

is given by $\vec{Y}(t) = c_1 \vec{Y}_1(t) + c_2 \vec{Y}_2(t) + \cdots + c_n \vec{Y}_n(t)$ where $\vec{Y}_1(t), \vec{Y}_2(t), \ldots, \vec{Y}_n(t)$ are n solutions of the system that are linearly independent. By linearly independent, we mean none is equal to a linear combination of the others, i.e., we can't have $\vec{Y}_1(t) = k_2 \vec{Y}_2(t) + k_3 \vec{Y}_3(t) + \cdots + k_n \vec{Y}_n(t)$, or

$\vec{Y}_2(t) = k_1 \vec{Y}_1(t) + k_3 \vec{Y}_3(t) + \cdots + k_n \vec{Y}_n(t)$, or ..., or

$\vec{Y}_n(t) = k_1 \vec{Y}_1(t) + k_2 \vec{Y}_2(t) + \cdots + k_{n-1} \vec{Y}_{n-1}(t)$

An easy way to check that n solutions $\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_n(t)$ are linearly independent is to compute

$$W(t) = \det\left(\left[\vec{y}_1(t) \;\; \vec{y}_2(t) \;\; \cdots \;\; \vec{y}_n(t) \right] \right)$$

and make sure it is never equal to zero in the interval (a,b). $W(t)$ is called the wronskian of the solutions

$$\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_n(t)$$

Definition

If $\vec{y}_1(t), \vec{y}_2(t), \ldots, \vec{y}_n(t)$ are n solutions of the homogeneous system $\vec{y}\,' = Q(t)\vec{y}$ and are linearly independent, we say that they form a fundamental set of solutions of the system.

As a consequence of the main result and the definition, to solve $\vec{Y}' = Q(t)\vec{Y}$, we need to find a fundamental set of solutions $\{\vec{Y}_1(t), \vec{Y}_2(t), \dots, \vec{Y}_n(t)\}$. If we are successful then the general solution of the system is

$$\vec{Y}(t, c_1, c_2, \dots, c_n) = c_1 \vec{Y}_1(t) + c_2 \vec{Y}_2(t) + \cdots + c_n \vec{Y}_n(t)$$

In the case where the coefficient matrix $Q(t)$ is a function of t, it is quite a challenge to find a fundamental set of solutions. However if $Q(t) = A$, where A is a constant matrix, it is relatively easy

We want to find n linearly independent solutions of the system $\vec{Y}' = A\vec{Y}$, where A is an $n\times n$ constant matrix

To do that, we look for solutions in the form

$$\vec{Y} = e^{\lambda t}\vec{V} = e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix}$$

where λ and $\vec{V}$ are constant to be determined.

We have

$$\vec{Y}' = A\vec{Y} \iff \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix}' = A\left(e^{\lambda t}\vec{V}\right) \iff \begin{bmatrix} v_1 \lambda e^{\lambda t} \\ v_2 \lambda e^{\lambda t} \\ \vdots \\ v_n \lambda e^{\lambda t} \end{bmatrix} = e^{\lambda t} A.\vec{V} \iff$$

$$\lambda e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = e^{\lambda t} A.\vec{V} \iff \lambda e^{\lambda t}\vec{V} = e^{\lambda t} A\vec{V} \iff A\vec{V} = \lambda\vec{V}$$

This shows that for $\vec{y} = e^{\lambda t}\vec{V}$ to be a solution of the system $\vec{y}\,' = A\vec{y}$, we need λ to be an eigenvalue of A and $\vec{V}$ to be an eigenvector of A associated with the eigenvalue λ.

From Linear Algebra, the eigenvalues of the matrix A are solutions of

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} a_{11}-\lambda & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{vmatrix} = 0$$

$\det(A - \lambda I) = 0$ is called the <u>characteristic equation</u> of of the matrix A.

If λ is an eigenvalue of A, then the associated eigenvectors $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ are the nonzero solution of the system $(A - \lambda I)\vec{V} = \vec{0}$

That's

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \iff \begin{cases} (a_{11}-\lambda)v_1 + a_{12}v_2 + \cdots + a_{1n}v_n = 0 \\ a_{21}v_1 + (a_{22}-\lambda)v_2 + \cdots + a_{2n}v_n = 0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + (a_{nn}-\lambda)v_n = 0 \end{cases}$$

Example Solve $\begin{cases} v_1 - 2v_2 + 3v_3 = 0 \\ -3v_1 + 6v_2 + v_3 = 0 \\ -2v_1 + 4v_2 + 4v_3 = 0 \end{cases}$

Start by writing the augmented matrix of the system and finding its reduced row-echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & 0 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2+3R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right] \xrightarrow{R_2/10} \left[\begin{array}{cccc} 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right] \xrightarrow[R_1-3R_2]{R_3-10R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Next, convert back to a system, and solve for the leading variables v_1 & v_2 in terms of the nonleading variable v_3, to get

$$\begin{cases} v_1 - 2v_2 = 0 \\ v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = 2v_2 \\ v_3 = 0 \end{cases}$$

Hence the solution of the system is $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_2 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

The constant solution $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is called a basic solution of the system.

Example

Find the basic solutions of the system $\begin{cases} 2v_1 - 3v_2 + 5v_3 = 0 \\ -4v_1 + 6v_2 - 10v_3 = 0 \\ -2v_1 + 3v_2 - 5v_3 = 0 \end{cases}$

We start with the augmented matrix and convert it to its reduced row-echelon form :

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 0 \\ -4 & 6 & -10 & 0 \\ -2 & 3 & -5 & 0 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+2R_1} \left[\begin{array}{ccc|c} 2 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Next, we convert back to a system and solve for the leading variable v_1 in terms of the nonleading variables v_2 & v_3 to get $2v_1 - 3v_2 + 5v_3 = 0 \Rightarrow 2v_1 = 3v_2 - 5v_3 \Rightarrow v_1 = \frac{3}{2}v_2 - \frac{5}{2}v_3$

Hence the solution of the system is

$$\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}v_2 - \frac{5}{2}v_3 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}v_2 \\ v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{5}{2}v_3 \\ 0 \\ v_3 \end{bmatrix} = \frac{v_2}{2}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + \frac{v_3}{2}\begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix}$$

The basic solutions of the system are $\vec{V}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ & $\vec{V}_3 = \begin{bmatrix} -5 \\ 0 \\ 2 \end{bmatrix}$

Here are the steps to follow when solving the homogeneous, linear, constant coefficient matrix, system

$$\vec{Y}' = A\vec{Y}$$

- Find all the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A.

These are the roots of the characteristic equation

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

We will assume that all the eigenvalues are simple

Find the associated eigenvectors

For each eigenvalue $\lambda = r$, determine the associated eigenvectors $\vec{V} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ by solving the system $(A - rI)\vec{V} = \vec{0}$

The result should be $\vec{V} = \alpha \vec{V}_1$, with α an arbitrary constant and $\vec{V}_1$ a fixed solution of the system.

As an eigenvector associated with the eigenvalue $\lambda = r$, you may take $\vec{V}_1$ or, to simplify the algebra, any multiple thereof.

Keep in mind that if $\lambda = a + bi$ is complex and if $\vec{V}$ is an associated eigenvector, there is no need to find an eigenvector associated with the conjugate eigenvalue $\bar{\lambda} = a - bi$

Build a Fundamental Set of Solutions

- To every real eigenvalue $\lambda = r$ with corresponding eigenvector $\vec{V}$, associate the solution

$$e^{rt}\,\vec{V}$$

- To every complex eigenvalue $\lambda = a + bi$ with corresponding eigenvector $\vec{V}$ and its complex conjugate eigenvalue $\bar{\lambda} = a - bi$, associate the two solutions

$$\vec{Y}_1(t) = \text{Real}\left(e^{(a+bi)t}\,\vec{V} \right)$$

$$\vec{Y}_2(t) = \text{Imaginary}\left(e^{(a+bi)t}\,\vec{V} \right)$$

When done with the last step, you should get n solutions $\vec{Y}_1(t), \vec{Y}_2(t), \dots, \vec{Y}_n(t)$ of $\vec{Y}' = A\vec{Y}$

They will form a fundamental set of solutions, and the general solution of the system is

$$\vec{Y}(t) = c_1 \vec{Y}_1(t) + c_2 \vec{Y}_2(t) + \dots + c_n \vec{Y}_n(t)$$

Example Find the general solution of the system

$$\begin{cases} y_1' = 2y_1 + 4y_2 \\ y_2' = 3y_1 + 3y_2 \end{cases} \iff \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \iff \vec{y}\,' = \underbrace{\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}}_{A} \vec{y}$$

- Find all eigenvalues. They are solutions of

$\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - (4)(3) = 6 - 2\lambda - 3\lambda + \lambda^2 - 12$$

$$= \lambda^2 - 5\lambda - 6 = (\lambda+1)(\lambda-6)$$

$$\left(\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} \overset{C_1+C_2}{=} \begin{vmatrix} 6-\lambda & 4 \\ 6-\lambda & 3-\lambda \end{vmatrix} \overset{R_2-R_1}{=} \begin{vmatrix} 6-\lambda & 4 \\ 0 & -1-\lambda \end{vmatrix} = (6-\lambda)(-1-\lambda) \right)$$

$$\det(A - \lambda I) = 0 \iff (\lambda+1)(\lambda-6) = 0 \iff \lambda_1 = -1, \quad \lambda_2 = 6$$

- Find the corresponding eigenvectors

Eigenvalue $\lambda_1 = -1$

If $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = -1$, then

$$(A - \lambda_1 I)\vec{V} = \vec{0} \iff (A + I)\vec{V} = \vec{0} \iff \begin{bmatrix} 2+1 & 4 \\ 3 & 3+1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} 3v_1 + 4v_2 = 0 \\ 3v_1 + 4v_2 = 0 \end{cases} \iff \{ 3v_1 + 4v_2 = 0 \implies v_1 = -\frac{4}{3}v_2$$

so $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}v_2 \\ v_2 \end{bmatrix}$. setting $v_2 = 3$, we get

the eigen vector $\vec{V_1} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Eigenvalue $\lambda_2 = 6$

If $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector associated with $\lambda_2 = 6$, then

$$(A - \lambda_2 I)\vec{V} = \vec{0} \iff (A - 6I)\vec{V} = \vec{0} \iff \begin{bmatrix} 2-6 & 4 \\ 3 & 3-6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{cases} -4v_1 + 4v_2 = 0 \\ 3v_1 - 3v_2 = 0 \end{cases} \iff \begin{cases} v_1 - v_2 = 0 \\ v_1 - v_2 = 0 \end{cases} \iff \{ v_1 - v_2 = 0 \iff v_1 = v_2$$

so $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}$. setting $v_2 = 1$, leads to the eigenvector $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Build a Fundamental set of solutions

To $\lambda_1 = -1$ and $\vec{V}_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$, we associate the solution

$$\vec{Y}_1(t) = e^{\lambda_1 t}\,\vec{V}_1 = e^{-t}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4e^{-t} \\ 3e^{-t} \end{bmatrix}$$

To $\lambda_2 = 6$ and $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we associate the solution

$$\vec{Y}_2(t) = e^{\lambda_2 t}\,\vec{V}_2 = e^{6t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}$$

The general solution is

$$\vec{Y}(t) = c_1\vec{Y}_1(t) + c_2\vec{Y}_2(t) = c_1\begin{bmatrix} -4e^{-t} \\ 3e^{-t} \end{bmatrix} + c_2\begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix} = \begin{bmatrix} -4c_1e^{-t} + c_2e^{6t} \\ 3c_1e^{-t} + c_2e^{6t} \end{bmatrix}$$

Example Solve the system $\begin{cases} y_1' = 2y_1 + 3y_2 \\ y_2' = -3y_1 + 2y_2 \end{cases}$

In matrix form the system is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Find all the eigenvalues. They are solutions of

$\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - (3)(-3) = (2-\lambda)^2 + 9$$

$$\det(A - \lambda I) = 0 \iff (2-\lambda)^2 + 9 = 0 \iff (\lambda-2)^2 = (3i)^2$$

$$\iff \lambda - 2 = \pm 3i \iff \lambda = 2 \pm 3i$$

Example Solve the system $\begin{cases} y_1' = 2y_1 + 4y_2 \\ y_2' = 3y_1 + 3y_2 \end{cases}$

In matrix form the system is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \iff \vec{y}\,' = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix} \vec{y}$$

- Find all the eigenvalue of the matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$.

  They are solutions of $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - (4)(3) = \lambda^2 - 5\lambda - 6$$

$$= (\lambda+1)(\lambda-6)$$

$$\det(A - \lambda I) = 0 \iff (\lambda+1)(\lambda-6) = 0 \iff \lambda = -1,\ 6$$

Hence the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 6$

(Another way of computing $\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} \underset{=}{\overset{C_1+C_2}{=}} \begin{vmatrix} 6-\lambda & 4 \\ 6-\lambda & 3-\lambda \end{vmatrix} \underset{=}{\overset{R_2-R_1}{=}} \begin{vmatrix} 6-\lambda & 4 \\ 0 & -1-\lambda \end{vmatrix} = (6-\lambda)(-1-\lambda)$$)

- Find the associated eigenvectors

- Eigenvector associated with the eigenvalue $\lambda_1 = -1$

If $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = -1$

then $(A - \lambda_1 I)\vec{V} = \vec{0} \iff (A + I)\vec{V} = \vec{0} \iff$

$$\begin{bmatrix} 2+1 & 4 \\ 3 & 3+1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} 3v_1 + 4v_2 = 0 \\ 3v_1 + 4v_2 = 0 \end{cases} \iff \{ 3v_1 + 4v_2 = 0$$

$\Rightarrow \quad v_1 = -\frac{4}{3}v_2 \ , \quad v_2$ any real number $\neq 0$

$$\Rightarrow \quad \vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}v_2 \\ v_2 \end{bmatrix}$$

selecting $v_2 = 3$, leads to the eigenvector $\vec{V_1} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

- Eigenvector associated with the eigenvalue $\lambda_2 = 6$

If $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector associated with $\lambda_2 = 6$

then $(A-\lambda_2 I)\vec{V}=\vec{0} \iff (A-6I)\vec{V}=\vec{0} \iff \begin{bmatrix} 2-6 & 4 \\ 3 & 3-6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\iff \begin{cases} -4v_1 + 4v_2 = 0 \\ 3v_1 - 3v_2 = 0 \end{cases} \iff \begin{cases} v_1 - v_2 = 0 \\ v_1 - v_2 = 0 \end{cases} \iff \{ v_1 = v_2$$

$\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}$ $\quad v_2$ any real number $\neq 0$

Select $v_2 = 1$, to get $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Build a Fundamental set of solutions

For $\lambda_1 = -1$ & $\vec{V}_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ associate the solution

$$\vec{Y}_1(t) = e^{\lambda_1 t}\vec{V}_1 = e^{-t}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4e^{-t} \\ 3e^{-t} \end{bmatrix}$$

For $\lambda_2 = 6$ & $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ associate the solution

$$\vec{Y}_2(t) = e^{\lambda_2 t}\vec{V}_2 = e^{6t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}$$

The general solution is

$$\vec{Y}(t) = c_1 \vec{Y}_1(t) + c_2 \vec{Y}_2(t)$$

$$= c_1 \begin{bmatrix} -4e^{-t} \\ 3e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}$$

$$= \begin{bmatrix} -4c_1 e^{-t} + c_2 e^{6t} \\ 3c_1 e^{-t} + c_2 e^{6t} \end{bmatrix} = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}$$

Example Find the general solution of $\begin{cases} Y_1' = 2Y_1 + 3Y_2 \\ Y_2' = -3Y_1 + 2Y_2 \end{cases}$

In matrix form the system is

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \iff \vec{Y}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \vec{Y}$$

- Find all the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

They are solutions of $\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - (3)(-3) = (2-\lambda)^2 + 9$$

$$\det(A-\lambda I) = 0 \iff (2-\lambda)^2 + 9 = 0 \iff (\lambda-2)^2 = -9 = (3i)^2$$

$$\iff \lambda - 2 = \pm 3i \iff \lambda = 2 \pm 3i$$

- Eigenvectors associated with $\lambda_1 = 2+3i$

If $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = 2+3i$,

then it is a solution of $(A-\lambda_1 I)\vec{V} = \vec{0} \iff \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} (-3i)v_1 + 3v_2 = 0 \\ -3v_1 - 3i\,v_2 = 0 \end{cases} \overset{(i)\,Eq_1}{\iff} \begin{cases} 3v_1 + 3i\,v_2 = 0 \\ -3v_1 - 3i\,v_2 = 0 \end{cases} \iff 3v_1 + 3i\,v_2 = 0$$

solve for v_1 to get $v_1 = -i\,v_2$, v_2 any number

$$\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -i\,v_2 \\ v_2 \end{bmatrix}$$

If we select $v_2 = i$, we get the eigenvector $\vec{V_1} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

- ## Fundamental set of solutions

A fundamental set of real solutions is

$$\left\{ \vec{Y}_1(t) = \mathrm{Re}\left(e^{\lambda_1 t} \vec{V}_1 \right), \quad \vec{Y}_2(t) = \mathrm{Im}\left(e^{\lambda_1 t} \vec{V}_1 \right) \right\}$$

$$e^{\lambda_1 t} \vec{V}_1 = e^{(2+3i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{2t} \cdot e^{(3t)i} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{2t} \begin{bmatrix} e^{(3t)i} \\ e^{(3t)i} \cdot i \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} \cos(3t) + i \sin(3t) \\ (\cos(3t) + i \sin(3t))\, i \end{bmatrix}$$

Recall Euler Formula

$$e^{\theta i} = \cos(\theta) + i \sin(\theta)$$

$$= e^{2t} \begin{bmatrix} \cos(3t) + i \cdot \sin(3t) \\ -\sin(3t) + i \cdot \cos(3t) \end{bmatrix}$$

$$= e^{2t} \left(\begin{bmatrix} \cos(3t) \\ -\sin(3t) \end{bmatrix} + i \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} \right)$$

$$= \left(e^{2t} \begin{bmatrix} \cos(3t) \\ -\sin(3t) \end{bmatrix} \right) + i \left(e^{2t} \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} \right) = \vec{Y}_1(t) + i\, \vec{Y}_2(t)$$

$$\vec{Y}_1(t) = \begin{bmatrix} e^{2t}\cos(3t) \\ -e^{2t}\sin(3t) \end{bmatrix} , \quad \vec{Y}_2(t) = \begin{bmatrix} e^{2t}\sin(3t) \\ e^{2t}\cos(3t) \end{bmatrix}$$

The general solution is

$$\vec{Y}(t) = c_1\vec{Y}_1(t) + c_2\vec{Y}_2(t) = c_1\begin{bmatrix} e^{2t}\cos(3t) \\ -e^{2t}\sin(3t) \end{bmatrix} + c_2\begin{bmatrix} e^{2t}\sin(3t) \\ e^{2t}\cos(3t) \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t}\left(c_1\cos(3t) + c_2\sin(3t)\right) \\ e^{2t}\left(-c_1\sin(3t) + c_2\cos(3t)\right) \end{bmatrix} \begin{matrix} \longleftarrow \text{this is } Y_1(t) \\ \longleftarrow \text{this is } Y_2(t) \end{matrix}$$

Example

Consider the system

$$\begin{cases} y_1' = y_1 + y_2 + y_3 \\ y_2' = 2y_1 + y_2 - y_3 \\ y_3' = -8y_1 - 5y_2 - 3y_3 \end{cases}$$

and suppose its coefficient matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = 2$,

and corresponding eigenvectors $\vec{V}_1 = \begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}$, $\vec{V}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$,

respectively. Solve the system.

Solution

All that is left is building a fundament set of solutions. That is

$$\left\{ \vec{y}_1(t) = e^{\lambda_1 t}\vec{V}_1 = e^{-2t}\begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix},\ \vec{y}_2(t) = e^{\lambda_2 t}\vec{V}_2 = e^{-t}\begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix},\ \vec{y}_3(t) = e^{\lambda_3 t}\vec{V} = e^{2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

The general solution is then

$$\vec{y}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) = c_1 e^{-2t}\begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4c_1 e^{-2t} + 3c_2 e^{-t} \\ -5c_1 e^{-2t} - 4c_2 e^{-t} + c_3 e^{2t} \\ -7c_1 e^{-2t} - 2c_2 e^{-t} - c_3 e^{2t} \end{bmatrix}$$

Example Use Laplace transform to solve the system

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = 5y_1 - 3y_2 \end{cases}$$ with initial conditions $$\begin{matrix} y_1(0) = 2 \\ y_2(0) = 3 \end{matrix}$$

solution Take the Laplace transform of both equations

$$\begin{cases} \mathcal{L}\{y_1'\}(s) = \mathcal{L}\{y_1\}(s) - \mathcal{L}\{y_2\}(s) \\ \mathcal{L}\{y_2'\}(s) = 5\mathcal{L}\{y_1\}(s) - 3\mathcal{L}\{y_2\}(s) \end{cases} \iff$$

$$\begin{cases} s\mathcal{L}\{y_1\}(s) - y_1(0) = \mathcal{L}\{y_1\}(s) - \mathcal{L}\{y_2\}(s) \\ s\mathcal{L}\{y_2\}(s) - y_2(0) = 5\mathcal{L}\{y_1\}(s) - 3\mathcal{L}\{y_2\}(s) \end{cases} \iff$$

$$\begin{cases} (s-1)\mathcal{L}\{y_1\}(s) + \mathcal{L}\{y_2\}(s) = y_1(0) = 2 \\ -5\mathcal{L}\{y_1\}(s) + (s+3)\mathcal{L}\{y_2\}(s) = y_2(0) = 3 \end{cases}$$

Using Cramer's rule to solve the system in $\mathcal{L}\{y_1\}(s)$ & $\mathcal{L}\{y_2\}(s)$

we get

$$\mathcal{L}\{y_1\}(s) = \frac{\begin{vmatrix} 2 & 1 \\ 3 & s+3 \end{vmatrix}}{\begin{vmatrix} s-1 & 1 \\ -5 & s+3 \end{vmatrix}} = \frac{2(s+3)-3}{(s-1)(s+3)+5} = \frac{2s+3}{s^2+2s+2}$$

$$\mathcal{L}\{y_2\}(s) = \frac{\begin{vmatrix} s-1 & 2 \\ -5 & 3 \end{vmatrix}}{\begin{vmatrix} s-1 & 1 \\ -5 & s+3 \end{vmatrix}} = \frac{3(s-1)+10}{s^2+2s+2} = \frac{3s+7}{s^2+2s+2}$$

$$y_1(t) = \mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+2s+2}\right\}(t) \qquad y_2(t) = \mathcal{L}^{-1}\left\{\frac{3s+7}{s^2+2s+2}\right\}(t)$$

$$y_1(t) = \mathcal{L}^{-1}\left\{\frac{2(s+1)+1}{(s+1)^2+1}\right\}(t) = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1}\right\}(t)$$

$$= e^{-t}\,\mathcal{L}^{-1}\left\{2\frac{s}{s^2+1} + \frac{1}{s^2+1}\right\}(t) = e^{-t}\left(2\cos(t) + \sin(t)\right)$$

$$y_2(t) = \mathcal{L}^{-1}\left\{\frac{3(s+1)+4}{(s+1)^2+1}\right\}(t) = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+1}\right\}(t)$$

$$= e^{-t}\cdot\mathcal{L}^{-1}\left\{3\frac{s}{s^2+1} + 4\cdot\frac{1}{s^2+1}\right\}(t) = e^{-t}\left(3\cos(t) + 4\sin(t)\right)$$