Department of Mathematics and Statistics
MATH 375
Handout # 3 - Answers, Hints, Solutions
Applications of First Order Equations

1. Radium decomposes at a rate proportional to the amount present. If the half-life of Radium is 1600 years , find percentage lost in 4800 years.
Solution. The radioactive decay is described by the initial value problem

$$\frac{dq}{dt} = -kq, \quad q(0) = q_0, \quad k > 0$$

which has a solution (as either a separable or a linear equation) $q(t) = q_0 e^{-kt}$, the half-life of radium gives

$$q(1600) = q_0 e^{-1600k} = \frac{1}{2} q_0 \ \Rightarrow \ e^{-1600k} = \frac{1}{2} \ \Rightarrow \ 1600k = \ln(2),$$

thus $k = \ln(2)/1600$, and

$$q(4800) = q_0 e^{-4800k} = q_0 e^{-4800 \ln(2)/1600} = q_0 e^{-4 \ln(2)} = q_0 2^{-4} = q_0/16 = 0.125 q_0,$$

or 12.5% of the original mass of radium. Thus. the loss is 100%-12.5%=87.5%.

Alternatively, we could notice that only half of the mass is left after 1600 years, and 4800/1600=3, so three times we get only half of the previous mass, so

$$q(4800) = q_0/2^3 = 0.125 q_0.$$

2. Iodine-131 decomposes at a rate proportional to the amount present. Find the half-life of Iodine-131 if after 40 days , 96. 875% of the original amount has disintegrated.
Answer. 8 days

3. A glass of boiling water (100°C) placed in a large room with constant temperature of 20°C cooled down to 60°C after 10 minutes. How much time will it take to cool water to 25°C?
Solution. The Newton law of cooling leads to the following initial value problem

$$T' = -k(T - 20), \quad T(0) = 100.$$

The equation is linear. The complementary equation $T' = -kT$ has a solution $y = Ce^{-kt}$, we look for the solution in the form $T = ue^{-kt}$, $T' = u'e^{-kt} - kue^{-kt}$. Thus

$$T' = u'e^{-kt} - kue^{-kt} = -kue^{-kt} + 20k \Rightarrow u' = 20ke^{kt} \Rightarrow u = 20e^{kt} + C,$$

and the solution is

$$T(t) = (C + 20e^{kt})e^{-kt} = 20 + Ce^{-kt}, \quad T(0) = 20 + C = 100,$$

so

$$T(t) = 20 + 80e^{-kt}, \quad T(10) = 20 + 80e^{-10k} = 60 \Rightarrow e^{-10k} = 0.5.$$

We are looking for t_1 such that

$$T(t_1) = 20 + 80e^{-kt_1} = 25, \quad \text{or} \quad e^{-kt_1} = 0.0625 = 1/16 = 0.5^4.$$

Thus

$$e^{-kt_1} = (e^{-10k})^{t_1/10} = (0.5)^{t_1/10} = (0.5)^4,$$

so $t_1/10 = 4$ and $t_1 = 40$.
Answer. After 40 minutes water will be 25°C.

4. If when the temperature of the air is 290 K , a certain substance cools from 370 K to 330 K in 10 minutes , find the temperature after 40 minutes.
   Answer. 295 K (the temperature is measured in Kelvin scale).

   Note : the zero in Kelvin scale is equivalent to -273.16° C (459. 67° F).

5. A body cools from 60° C to 50° C in 15 minutes in air which is maintained at 30° C. How long will it take this body to cool from 100° C to 75° C in air that is maintained at 55° C? Assume Newton's Law of Cooling.
   Answer. 30 Minutes

6. Assuming Newton's Law of Cooling. If the temperature of the air is 300 K , and a substance cools from 380 K to 340 K in 15 minutes , find the temperature of the substance after an additional 45 minutes.
   Answer. 305 K

7. A cup of boiling water (100°C) is placed outside. One minute later the temperature of the water is 70°C. and after another minute its temperature is 49°C. What is the outside temperature (the medium) ?
   Answer. 0° C
   Solution. Let X be the constant temperature of the medium. Solving $T' = k(X - T)$, we obtain $T(t) = X + Ce^{-kt}$, taking into account the initial conditions, we get $T(t) = X + (100 - x)e^{-kt}$. Substituting the two conditions, we have $T(1) = X + (100 - X)e^{-k} = 70$, $T(2) = X + (100 - X)e^{-2k} = 49$. Thus $e^{-k} = (70 - X)/(100 - X)$, $e^{-2k} = (49 - X)/(100 - X)$. But $e^{-2k} = \left(e^{-k}\right)^2$, so

   $$\frac{(70 - X)^2}{(100 - X)^2} = \frac{49 - X}{100 - X} \quad \Rightarrow \quad (70 - X)^2 = (49 - X)(100 - X)$$

   $$\Rightarrow \quad 4900 - 140X + X^2 = 4900 - 149X + X^2 \quad \Rightarrow X = 0.$$

8. A tank contains 100 litres of water with 10 kg of salt. Pure tap water flows into the tank at a rate of 5 litres per minute, the mixture is well sturred and drains from the tank at the same rate. How much salt will be contained in the tank after an hour?

Answer. After an hour there will be $10e^{-3} \approx 0.5$kg of salt in the tank.
Solution. Let $Q(t)$ be the amount of salt at time t, $Q(0) = 10$. Since

$$Q'(t) = -5/100Q = -0.05Q,$$

so

$$Q(t) = Q(0)e^{-0.05t} = 10e^{-0.05t}$$

and

$$Q(60) = 10e^{-0.05\ 60} = 10e^{-3} \approx 0.498 \approx 0.5.$$

9. A tank contains 450 litres of brine made by dissolving 30 kg of salt in water. Saltwater containing $\frac{1}{9}$ kg of salt per litre runs in at the rate of 9 litre/min and the mixture kept uniform by stirring, runs out at the rate of 13.5 litre/min.
a) Find the amount of salt in tank at time t.
b) how much salt is in tank at the end of one hour?
Answer. a) $x(t) = \frac{1}{2}(100 - t) - \frac{1}{50000}(100 - t)^3$ b) 18.72 kg
Solution. Let $x(t)$ be the amount of salt at time $t \geq 0$, $x(0) = 30\ kg$. Then

$$\frac{dx}{dt} = c_{in}v_{in} - c_{out}v_{out},$$

where $c_{in} = \frac{1}{9}$, c_{out} are the concentrations of incoming and outcoming streams, $v_{in} = 9$ litre/min and $v_{out} = 13.5$ litre/min are the rates of the streams. The volume in the tank is decreasing by $13.5 - 9 = 4.5$ litre/min, so

$$V(t) = 450 - 4.5t, \quad c_{out} = \frac{x(t)}{V(t)} = \frac{x(t)}{450 - 4.5t}$$

and

$$\frac{dx}{dt} = \frac{1}{9} \cdot 9 - \frac{13.5x(t)}{450 - 4.5t} \quad \Rightarrow \quad \frac{dx}{dt} + \frac{3}{100 - t}x(t) = 1$$

(we have divided the numerator and the denominator by 4.5). This is a linear equation, with the integrating factor

$$\mu(t) = \exp\left\{\int \frac{3}{100 - t}\ dt\right\} = e^{-3\ln(100-t)} = (e^{\ln(100-t)})^{-3} = \frac{1}{(100 - t)^3}.$$

Multiplying by the integrating factor, we obtain

$$\frac{1}{(100 - t)^3}\frac{dx}{dt} + \frac{3}{(100 - t)^4} = \frac{dx}{dt}\left(\frac{1}{(100 - t)^3}x(t)\right) = \frac{1}{(100 - t)^3}.$$

Thus

$$\frac{1}{(100 - t)^3}x(t) = \int \frac{dt}{(100 - t)^3} = C + \frac{1}{2}\frac{1}{(100 - t)^2},$$

$$x(t) = C(100-t)^3 + \frac{1}{2}(100-t), \quad x(0) = 100^3 C + 50 = 30 \;\Rightarrow\; C = \frac{-20}{10^6} = -\frac{1}{50000},$$

and

$$x(t) = \frac{1}{2}(100-t) - \frac{1}{50000}(100-t)^3, \quad x(60) = \frac{40}{2} - \frac{1}{50000}40^3 = 20 - 1.28 = 18.72\ kg$$

10. A tank initially contains 40 litres of fluid in which there is dissolved 10 gram of salt. Starting at t 0, a brine containing 2 grams per litre of the dissolved salt flow into he tank at the rate of 2 litres/min. The mixture is kept uniform by continuous stirring nd the well-stirred mixture simultaneously flow out of the tank at the slower rate of 1 litre/min.

    Let $x = x(t)$ be the amount of salt in the tank at time $t \geq 0$.
    a) Set up the initial value problem satisfied by x. b) Find an expression for $x(t)$. c) How many grams of salt is in the tank at the end of 30 minutes ? d) what is the concentration of the brine after half an hour ? e) when is there 44 grams of salt in tank ?
    Answer. a) $\dot{x}(t) = 4 - x/(40+t)$, $x(0) = 10$
    b) $x(t) = 2(40+t) - 2800/(40+t)$ c) 100 gram d) $\frac{10}{7}$ gram / litre

11. A tank containing 0.5 m^3 of brine made by dissolving 40 kg of salt in water. Pure water runs into the tank at the rate of $3 \times 10^{-4}\ m^3/s$ and the mixture , kept uniform by stirring , runs out at the same rate.
    a) How much salt in tank at time $t > 0$? After 1 hour? b) How much salt in tank after a long time; that is as $t \to \infty$?
    Answer. a) $x(t) = 40e^{-0.0006t}$ b) zero
    Remark. If the rate of the incoming stream coincides with the rate of the outcoming stream, after a long time, as $t \to \infty$, the concentration in the tank approaches the concentration in the incoming stream.

12. A tank of volume $0.5m^3$ is filled with brine containing 30 kg of dissolved salt. Water runs into the tank at the rate of $15 \times 10^{-5}\ m^3/s$ and the mixture , kept uniform by stirring , runs out at the same rate.
    a) How much salt is in tank after one hour? b) What is the concentration of the brine after one hour?
    Answer. a) 10.91 kg b) 20.38 kg/m^3

13. A large tank initially contains 100 litres of brine made by dissolving 10 kg of salt. Pure water runs into the tank at the rate of 5 litres / min and the mixture , kept uniform by stirring , runs out at the rate of 2 litres/min.
    a) How much salt is in tank at the end of 15 min and what is the concentration at that time?
    b) If the capacity of the tank is 250 litres, what is the concentration at the instant the tank overflows?
    Answer. a) 7.8 kg b) 0.022 $kg/litre$

14. A 20 ohm resistor and a 5 henry inductor are connected in a series in an electric circuit in which there is initially a current flow of 20 amperes. Find expression for the current $I(t)$ at any time $t > 0$ if the emf is zero for $t > 0$.
Solution. For the resistor we apply Ohm's law: $E_r = RI = 20I$, for the inductor we use the fact that the voltage is proportional to the electric current change, with the inductunce $L = 5$ being the constant of proportionality $E_L = L\frac{dI}{dt}$. Summing up E_R and E_L, by Kirchhoff's law, we obtain $E = emf$, which is equal to zero by the condition of the problem:
$$E = E_R + E_L = RI + L\frac{dI}{dt} = 20I + 5\frac{dI}{dt} = 0.$$
We also have $I(0) = 20$. The initial value problem for the linear (also separable) equation is
$$5I'(t) + 20I = 0, \quad I(0) = 20.$$
Dividing by 5 and multiplying by e^{4t}, we get
$$I'(t)e^{4t} + 4I(t)e^{4t} = 0 \Rightarrow \frac{d}{dt}\left(Ie^{4t}\right) = 0$$
$$\Rightarrow I(t)e^{4t} = C \Rightarrow I(t) = C_1e^{-4t}, I(0) = C_1 = 20.$$
Finally, $I(t) = 20e^{-4t}$.

15. A capacitor of 0.005 farad is connected in a series with a 25 ohm resistor and a generator of an emf of 50 volts. If the switch is closed at $t = 0$ and the initial charge on the capacitor is zero, find an expression for the charge and the current at any time $t > 0$.
Solution. The voltage drop across the capacitor is $E_C = \frac{Q}{C} = \frac{Q}{0.005} = 200Q$, while the voltage drop across the resistor is $E_R = RI = 25I = 25\frac{dQ}{dt}$, as the electric current is the rate of change of the charge. Summing up E_C and E_R, by Kirchhoff's law, we obtain $E = emf$, which is 50:
$$E = E_C + E_R = 200Q + 25\frac{dQ}{dt} = 50 \Rightarrow Q'(t) + 8Q(t) = 2.$$
The initial condition is $Q(0) = 0$. Multiplying the linear equation by the integrating factor e^{8t}, we get
$$Q'(t)e^{4t} + 8e^{8t}Q(t) = 2e^{8t} \Rightarrow \frac{d}{dt}\left(e^{8t}Q(t)\right) = 2e^{8t}$$
$$\Rightarrow e^{8t}Q(t) = \frac{1}{4}e^{8t} + C_1 \Rightarrow Q(t) = \frac{1}{4} + C_1e^{-8t}$$
The initial condition $Q(0) = \frac{1}{4} + C_1 = 0$ gives $C_1 = -\frac{1}{4}$ and
$$Q(t) = \frac{1}{4}\left(1 - e^{-8t}\right).$$
Finally, differentiating, we obtain the current
$$I(t) = \frac{dQ}{dt} = 2e^{-8t}, \ t > 0.$$

16. An inductor of L henries, where $L = 0.05 + 0.001t$, $t \in [0, 100]$, is connected in a series with a 40 emf volts and a 10 ohm resistor. If the initial current in the circuit is zero, find an expression for the current $I(t)$, $t > 0$. What is the maximum current passing through the circuit?
Solution. By Kirchhoff's law, we have

$$E = E_L + E_R = L\frac{dI}{dt} + RI = (0.05 + 0.001t)\frac{dI}{dt} + 10I = 40,$$

and $I(0) = 0$. For the intial value problem

$$I'(t) + \frac{10}{0.05 + 0.001t}I = \frac{40}{0.05 + 0.001t}, \quad I(0) = 0,$$

the integrating factor is

$$\mu(t) = e^{\int \frac{10}{0.05+0.001t}\ dt} = e^{10^4 \ln(0.05+0.001t)} = (0.05 + 0.001t)^{10^4}.$$

Multiplying we have

$$(0.05 + 0.001t)^{10^4} I'(t) + 10(0.05 + 0.001t)^{10^4-1} I(t)$$

$$= \frac{d}{dt}\left((0.05 + 0.001t)^{10^4} I(t)\right) = 40(0.05 + 0.001t)^{9999}.$$

Integrating the right-hand side we obtain

$$(0.05+0.001t)^{10^4} I(t) = 40 \cdot 10^3 \cdot 10^{-4}(0.05+0.001t)^{10^4} + C_1 = 4(0.05+0.001t)^{10^4} + C_1$$

and

$$I(t) = 4 + C_1(0.05 + 0.001t)^{-10{,}000}, \quad I(0) = 4 + C_1 0.05^{-10{,}000} = 0,$$

thus $C_1 = -4\ 0.05^{10{,}000}$. The current $I(t)$ is

$$I(t) = 4 - 4\ 0.05^{10{,}000}(0.05 + 0.001t)^{-10{,}000} = 4 - 4\left(\frac{0.05 + 0.001t}{0.05}\right)^{10{,}000}$$

$$I(t) = 4 - 4(1 + 0.02t)^{-10{,}000}.$$

The current is monotonically increasing and, at $t = 100$, has the maximum value of

$$4\left(1 - 3^{-10{,}000}\right)$$

which is very close to 4 *amps*.

17. A resistor of R ohms, where $R = 3 + 0.5t$, $t \in [0, 100]$ is connected in a series with a 0.5 farad capacitor and a generator of an emf of $(3+0.5t)^{-3}$ volts. If the initial charge on the capacitor is zero, find the charge and current as a function of time. What is the

maximum charge?
Solution. By Kirchhoff's law, we have

$$E = E_R + E_C = RI + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{1}{C}Q = (3+0.5t)Q'(t) + \frac{1}{0.5}Q = (3+0.5t)^{-3}.$$

As the initial charge on the capacitor is zero, we have the initial value problem

$$Q'(t) + \frac{2}{3+0.5t}Q = (3+0.5t)^{-4}, \quad Q(0) = 0.$$

Since $\displaystyle\int \frac{2\ dt}{3+0.5t} = 4\ln(3+0.5t)$, the integrating factor is

$$\mu(t) = e^{4\ln(3+0.5t)} = \left(e^{\ln(3+0.5t)}\right)^4 = (3+0.5t)^4.$$

We multiply:

$$(3+0.5t)^4Q'(t)+2(3+0.5t)^3Q(t) = \frac{d}{dt}\left((3+0.5t)^4Q(t)\right) = 1 \Rightarrow (3+0.5t)^4Q(t) = t+C_1.$$

The initial condition gives $(3+0)0 = 0 + C_1$, or $C_1 = 0$ and $Q(t) = \dfrac{t}{(3+0.5t)^4}$,

$$I(t) = \frac{dQ}{dt} = \frac{(3+0.5t)^4 - 2t(3+0.5t)^3}{(3+0.5t)^8} = \frac{3-1.5t}{(3+0.5t)^5} = \frac{6-3t}{2(3+0.5t)^5}.$$

The maximal charge is attained when $I(t) = 0$, or $6 - 3t = 0$, or $t = 2$, which is $Q(t) = \dfrac{2}{(3+0.5\cdot 2)^4} = \dfrac{1}{128}$.

18. An inductor of 0.1 henry, a resistor of 10 ohms and an *emf* $E(t)$ volts are connected in a series, where

$$E(t) = \begin{cases} 10, & 0 \le t \le 5, \\ 0, & t > 5 \end{cases}$$

Find the current $I(t)$, assuming $I(0) = 0$.
Solution. By Kirchhoff's law, we have

$$E = E_L + E_R = L\frac{dI}{dt} + RI = 0.1I'(t) + 10I.$$

First we solve the initial value problem for $t \in [0,5]$:

$$0.1I'(t) + 10I = 10, \text{ or } I' + 100I = 100, \ I(0) = 0.$$

Multiplying by the integrating factor $\mu(t) = e^{100t}$, we have

$$e^{100t}I'(t) + 100e^{100t}I(t) = \frac{d}{dt}\left(e^{100t}I(t)\right) = 100e^{100t}$$

$$\Rightarrow \quad e^{100t} I(t) = e^{100t} + C_1 \quad \Rightarrow \quad I(t) = 1 + C_1 e^{-100t}$$

Substituting $I(0) = 1 + C_1 = 0$, we get $C_1 = -1$ and

$$I(t) = 1 - e^{-100t},\ t \in [0,5], \quad I(5) = 1 - e^{-500}.$$

Further, we solve the second initial value problem

$$I' + 100I = 0,\ I(5) = 1 - e^{-500}.$$

Similarly, we have $I(t) = C_2 e^{-100t}$, $I(5) = C_2 e^{-500} = 1 - e^{-500}$, which gives $C_2 = e^{500} - 1$. Finally,

$$I(t) = \begin{cases} 1 - e^{-100t}, & 0 \le t \le 5, \\ (e^{500} - 1)\, e^{-100t}, & t > 5 \end{cases}$$