Math 375 **Fall 2016**

Laplace Transform

Worksheet # 3 **Part 2** **November 14-18**

The problems on this worksheet refer to material from sections §6.1, §6.2, 6.3, and §6.4 of your text. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Inverse Transform of Rational Functions

01. Compute the inverse Laplace transform of each of the following

a. $\dfrac{2}{s^2 - 6\,s}$ **b.** $\dfrac{3\,s - 14}{s^2 - 4\,s + 8}$ **c*.** $\dfrac{8\,s + 20}{s^2 - 12\,s + 32}$

d*. $\dfrac{3\,s + 2}{(s+1)\,(s+2)\,\left(s^2+1\right)}$ **e.** $\dfrac{3\,s^2 + 2\,s + 1}{\left(s^2+1\right)\left(s^2 + 2\,s + 2\right)}$ **f.** $\dfrac{4}{s\left(s^4 - 1\right)}$

g. $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)}$ **h.** $\dfrac{3\,s^2 + 24\,s + 15}{s\left(s^2 + 8\,s + 15\right)}$

The Second Shift Formula for the Inverse Transform

02. Determine the inverse Laplace transform of

a. $\dfrac{\mathrm{e}^{-5\,s}}{(s-2)^4}$ **b.** $\dfrac{8\,\mathrm{e}^{-3\,s}}{s^2+4}$ **c*.** $\dfrac{(s+2)\,\mathrm{e}^{-\pi\,s}}{s^2 - 4\,s + 13}$

Inverse Transform of a Derivative

03. Compute the inverse Laplace transform of

a*. $\dfrac{s}{(s^2+1)^2}$ **b.** $\dfrac{s^2 - 1}{\left(s^2+1\right)^2}$ **c.** $\ln\left(\dfrac{s^2+1}{s^2+4}\right)$

d. $\tan^{-1}\left(\dfrac{3}{s+2}\right)$

Solving Initial Value Problems

04. Use Laplace transform to solve the initial value problems.

a. $\begin{cases} y'' + 5\,y' + 6\,y = 2\,\mathrm{e}^{-t} \\ y(0) = 1, \quad y'(0) = 3 \end{cases}$ **b.** $\begin{cases} y'' + 4\,y = 8\,\sin(2\,t) + 9\,\cos(t) \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

c*. $\begin{cases} y'' - 3\,y' + 2\,y = g(t) \\ y(0) = -3, \quad y'(0) = 1 \end{cases}$ with $g(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 2 \\ -1 & \text{if } t \ge 2 \end{cases}$

05. Use Laplace transform method to solve the initial value problems.

a. $\begin{cases} y'' + 2\,y' + y = g(t) \\ y(0) = 3, \;\; y'(0) = -1 \end{cases}$ with $g(t) = \begin{cases} \mathrm{e}^t & \text{if } \; 0 \le t < 1 \\ \mathrm{e}^t - 1 & \text{if } \; t \ge 1 \end{cases}$

b. $\begin{cases} y'' + 9\,y = g(t) \\ y(0) = 0, \;\; y'(0) = 0 \end{cases}$ with $g(t) = \begin{cases} \cos(t) & \text{if } \; 0 \le t < 3\pi/2 \\ \sin(t) & \text{if } \; t \ge 3\pi/2 \end{cases}$

Answers and Solutions

Computing an inverse Laplace transform of a rational function may require decomposing it into partial fractions. Furthermore, computing an inverse Laplace transform of the irreducible partial fraction $\boxed{\dfrac{A\,s + B}{s^2 + b\,s + c}}$ may require completing the square and the use of the first shift formula $\boxed{\mathcal{L}^{-1}\left\{F(s+a)\right\}(t) = \mathrm{e}^{-a\,t}\,\mathcal{L}^{-1}\left\{F(s)\right\}(t)}$

01a. We first completely factor out the denominator as $s^2 - 6 = s\,(s-6)$. Next we use partial fraction decomposition to write $\dfrac{2}{s^2 - 6\,s} = \dfrac{A}{s} + \dfrac{B}{s-6}$. Clearing out the denominators, we obtain $2 = A\,(s-6) + B\,s$. Setting in turn $s = 0,\,1,$ leads to $A = -1/3,$ and $B = 1/3$. Hence

$$\frac{2}{s^2 - 6\,s} = \frac{-1/3}{s} + \frac{1/3}{s-6}$$

It follows

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2 - 6\,s}\right\}(t) = -\frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}(t) = -\frac{1}{3} + \frac{1}{3}\mathrm{e}^{6\,t}$$

01b. The quadratic in the the denominator is irreducible, so $\dfrac{3\,s - 14}{s^2 - 4\,s + 8}$ is already a partial fraction. Completing the square, we have $\dfrac{3\,s - 14}{s^2 - 4\,s + 8} = \dfrac{3\,s - 14}{(s-2)^2 + 4} = \dfrac{3\,(s-2) - 8}{(s-2)^2 + 4}$, By the first shift formula for the inverse transform $\mathcal{L}^{-1}\left\{F(s-a)\right\}(t) = \mathrm{e}^{a\,t}\,\mathcal{L}^{-1}\left\{F(s)\right\}(t)$, we have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3\,s - 14}{s^2 - 4\,s + 8}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{3\,(s-2) - 8}{(s-2)^2 + 4}\right\}(t) = \mathrm{e}^{2\,t}\,\mathcal{L}^{-1}\left\{\frac{3\,s - 8}{s^2 + 4}\right\}(t) \\
&= \mathrm{e}^{2\,t}\,\mathcal{L}^{-1}\left\{3\,\frac{s}{s^2+4} - 8\,\frac{1}{s^2+4}\right\}(t) \\
&= \mathrm{e}^{2\,t}\left(3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 8\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t)\right) \\
&= \mathrm{e}^{2\,t}\left(3\,\cos(2\,t) - 4\,\sin(2\,t)\right)
\end{aligned}$$

01c. Factoring out the denominator, we have $x^2 - 12\,s + 32 = (s-4)\,(s-8)$. Using partial fraction decomposition we have $\dfrac{8\,s + 20}{s^2 - 12\,s + 32} = \dfrac{A}{s-4} + \dfrac{B}{s-8}$. Clearing out the denominators, we get $8\,s + 20 = A\,(s-8) + B\,(s-4)$. Setting in turn $s = 4,\,8,$ leads to $A = -13,\; B = 21$. Hence $\dfrac{8\,s + 20}{s^2 - 12\,s + 32} = \dfrac{-13}{s-4} + \dfrac{21}{s-8}$. It follows

$$\mathcal{L}^{-1}\left\{\frac{8\,s+20}{s^2-12\,s+32}\right\}(t)=\mathcal{L}^{-1}\left\{-13\,\frac{1}{s-4}(t)+21\,\frac{1}{s-8}\right\}(t)$$
$$=-13\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}(t)+21\,\mathcal{L}^{-1}\left\{\frac{1}{s-8}\right\}(t)=-13\,\mathrm{e}^{4\,t}+21\,\mathrm{e}^{8\,t}$$

01d. Decomposing into partial fractions, we have $\dfrac{3\,s+2}{(s+1)\,(s+2)\,(s^2+1)}=\dfrac{A}{s+1}+\dfrac{B}{s+2}+\dfrac{C\,s+D}{s^2+1}$. Clearing out the denominators, we get

$$3\,s+2=A\,(s+2)\,(s^2+1)+B\,(s+1)\,(s^2+1)+(C\,s+D)\,(s+1)\,(s+2)\qquad\cdots\qquad(◀)$$

Setting in turn $s=-1,\ -2,$ leads to $A=-1/2,\ B=4/5$. Substituting back into (◀), we get

$$3\,s+2=-\frac{1}{2}\,(s+2)\,(s^2+1)+\frac{4}{5}\,(s+1)\,(s^2+1)+(C\,s+D)\,(s+1)\,(s+2)\qquad\cdots\qquad(▶)$$

Since there is no s^3 term on the left of (▶), the coefficient of s^3 from the right side should be zero: $-\frac{1}{2}+\frac{4}{5}+C=0 \iff C=-\frac{3}{10}$. To find the constant D, we simply set $s=0$ to get $2=-1+\frac{4}{5}+2\,D \iff D=\frac{11}{10}$. Hence

$$\frac{3\,s+2}{(s+1)\,(s+2)\,(s^2+1)}=\frac{-1/2}{s+1}+\frac{4/5}{s+2}+\frac{-(3/10)\,s+(11/10)}{s^2+1}$$
$$=-\frac{1}{2}\,\frac{1}{s+1}+\frac{4}{5}\,\frac{1}{s+2}-\frac{3}{10}\,\frac{s}{s^2+1}+\frac{11}{10}\,\frac{1}{s^2+1}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{3\,s+2}{(s+1)\,(s+2)\,(s^2+1)}\right\}(t)=-\frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)+\frac{4}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t)$$
$$-\frac{3}{10}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t)+\frac{11}{10}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t)$$
$$=-\frac{1}{2}\,\mathrm{e}^{-t}+\frac{4}{5}\,\mathrm{e}^{-2\,t}-\frac{3}{10}\,\cos(t)+\frac{11}{10}\,\sin(t)$$

01e. Using partial fraction decomposition, we write $\dfrac{3\,s^2+2\,s+1}{\left(s^2+1\right)\left(s^2+2\,s+2\right)}=\dfrac{A\,s+B}{s^2+1}+\dfrac{C\,s+D}{s^2+2\,s+2}$. Clearing out the denominators, leads to

$$3\,s^2+2\,s+1=(A\,s+B)\,(s^2+2\,s+2)+(C\,s+D)\,(s^2+1)\qquad\cdots\qquad(*)$$

Expanding and collecting like power terms, we obtain

$$3\,s^2+2\,s+1=(A+C)\,s^3+(2\,A+B+D)\,s^2+(2\,A+2\,B+C)\,s+(2\,B+D)=0$$

That is equivalent to the system

$$\left\{\begin{array}{l}A+C=0\\2\,A+B+D=3\\2\,A+2\,B+C=2\\2\,B+D=1\end{array}\right.\iff\left\{\begin{array}{l}A=6/5\\B=2/5\\C=-6/5\\D=1/5\end{array}\right.$$

Hence

$$\frac{3\,s^2+2\,s+1}{\left(s^2+1\right)\left(s^2+2\,s+2\right)}=\frac{(6/5)\,s+(2/5)}{s^2+1}+\frac{(-6/5)\,s+(1/5)}{s^2+2\,s+2}$$

which we can rewrite as

$$
\begin{aligned}
F(s) = \frac{3\,s^2+2\,s+1}{\left(s^2+1\right)\left(s^2+2\,s+2\right)} &= \frac{6}{5}\;\frac{s}{s^2+1} + \frac{2}{5}\;\frac{1}{s^2+1} + \frac{(-6/5)\,s+1/5}{(s+1)^2+1} \\
&= \frac{6}{5}\;\frac{s}{s^2+1} + \frac{2}{5}\;\frac{1}{s^2+1} + \frac{(-6/5)\,(s+1)+7/5}{(s+1)^2+1}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{F(s)\right\}(t) &= \frac{6}{5}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{(-6/5)\,(s+1)+7/5}{(s+1)^2+1}\right\}(t) \\
&= \frac{6}{5}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) + \mathrm{e}^{-t}\;\mathcal{L}^{-1}\left\{\frac{(-6/5)\,s+7/5}{s^2+1}\right\}(t) \\
&= \frac{6}{5}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) \\
&\qquad + \mathrm{e}^{-t}\;\left(-\frac{6}{5}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + \frac{7}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t)\right) \\
&= \frac{6}{5}\;\cos(t) + \frac{2}{5}\;\sin(t) + \mathrm{e}^{-t}\;\left(-\frac{6}{5}\;\cos(t) + \frac{7}{5}\;\sin(t)\right) \\
&= \frac{6}{5}\;\cos(t) + \frac{2}{5}\;\sin(t) - \frac{6}{5}\;\mathrm{e}^{-t}\;\cos(t) + \frac{7}{5}\;\mathrm{e}^{-t}\;\sin(t)
\end{aligned}
$$

01f. We decompose $\dfrac{4}{s\left(s^4-1\right)}$ into partial fraction to get

$$
\frac{4}{s\left(s^4-1\right)} = \frac{4}{s\,(s-1)(s+1)\left(s^2+1\right)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D\,s+E}{s^2+1}
$$

Clearing out the denominators, leads to

$$
4 = A\left(s^4-1\right) + B\,s\,(s+1)\left(s^2+1\right) + C\,s\,(s-1)\left(s^2+1\right) + s\left(s^2-1\right)(D\,s+E) \quad \cdots \quad (\blacktriangle)
$$

Setting in turn $s = 0,\ 1,\ -1$, leads to $A = -4,\ B = 1,\ C = 1$. Substituting back into $(\blacktriangle)$, and expanding, leads to

$$
4 = -2\,s^4+2\,s^2+4+s\left(s^2-1\right)(D\,s+E) = (D-2)\,s^4+E\,s^3+(2-D)\,s-E\,s+4 \implies \begin{cases} D-2=0 \\ E=0 \end{cases}
$$

A faster way of computing D and E is to identify the coefficients of the s^4 and s^3 terms on both sides of the equation $4 = -2\,s^4+2\,s^2+4+s\left(s^2-1\right)(D\,s+E)$, to get $0 = -2+D$ & $0 = E$, respectively.

Hence $\dfrac{4}{s\left(s^4-1\right)} = -\dfrac{4}{s} + \dfrac{1}{s-1} + \dfrac{1}{s+1} + \dfrac{2\,s}{s^2+1}$, from which it follows

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{4}{s\left(s^4-1\right)}\right\}(t) &= -4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) \\
&\qquad + 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) \\
&= -4 + \mathrm{e}^{t} + \mathrm{e}^{-t} + 2\,\cos(t)
\end{aligned}
$$

01g. The partial fraction decomposition of our rational function is

$$\frac{s}{\left(s^2+1\right)\left(s^2+4\right)} = \frac{A\,s+B}{s^2+1} + \frac{C\,s+D}{s^2+4}$$

Clearing out the denominators, we get

$$\begin{aligned} s &= (A\,s+B)\left(s^2+4\right) + (C\,s+D)\left(s^2+1\right) \\ &= (A+C)\,s^3 + (B+D)\,s^2 + (4\,A+C)\,s + (4\,B+D) \end{aligned}$$

Hence the system $\left\{\begin{array}{l} A+C=0 \\ 4\,A+C=1 \\ B+D=0 \\ 4\,B+D=0 \end{array}\right. \iff \left\{\begin{array}{l} A=1/3 \\ B=0 \\ C=-1/3 \\ D=0 \end{array}\right.$ Thus $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)} = \dfrac{1}{3}\,\dfrac{s}{s^2+1} - \dfrac{1}{3}\,\dfrac{s}{s^2+4}$, from which it follows that

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+1\right)\left(s^2+4\right)}\right\}(t) &= \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) - \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t) \\ &= \frac{1}{3}\,\cos(t) - \frac{1}{3}\,\cos(2\,t) \end{aligned}$$

Notice that the decomposition could have been made easier by noticing that beacause the left side of $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)} = \dfrac{A\,s+B}{s^2+1} + \dfrac{C\,s+D}{s^2+4}$ is odd, the right side should be odd as well. Hence $B=0$ and $E=0$.

01h. First completely factor out the denominator into $s\left(s^2+8\,s+15\right) = s\,(s+3)\,(s+5)$. A partial fraction decomposition is then $\dfrac{3\,s^2+24\,s+15}{s\left(s^2+8\,s+15\right)} = \dfrac{A}{s} + \dfrac{B}{s+3} + \dfrac{C}{s+5}$. Clearing out the denominators, we get $3\,s^2+24\,s+15 = A\,(s+3)\,(s+5) + B\,s\,(s+5) + C\,s\,(s+3)$
Setting in turn $s=0,-3,-5,$ leads to $A=1,\ B=5,\ C=-3$. Hence

$$\frac{3\,s^2+24\,s+15}{s\left(s^2+8\,s+15\right)} = \frac{1}{s} + \frac{5}{s+3} - \frac{3}{s+5}$$

It follows

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{3\,s^2+24\,s+15}{s\left(s^2+8\,s+15\right)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t) - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}(t) \\ &= 1 + 5\,\mathrm{e}^{-3\,t} - 3\,\mathrm{e}^{-5\,t} \end{aligned}$$

Recall the second shift formula for the inverse Laplace transform

$$\boxed{\mathcal{L}^{-1}\left\{F(s)\,\mathrm{e}^{-a\,s}\right\}(t) = \mathcal{L}^{-1}\left\{F(s)\right\}(t-a)\;u_a(t)}$$

where $\mathcal{L}^{-1}\{F(s)\}(t-a)$ means compute $\mathcal{L}^{-1}\{F(s)\}(t)$, then change t into $t-a$

02a. Making use of the second shift formula for the inverse transform, we have

$$\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-5\,s}}{(s-2)^4}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\}(t-5)\;u_5(t)$$

with

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\}(t)=\frac{1}{3!}\,\mathrm{e}^{2\,t}\,t^3 \implies \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\}(t-5)=\frac{1}{6}\,\mathrm{e}^{2\,(t-5)}\,(t-5)^3$$

It follows

$$\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-5\,s}}{(s-2)^4}\right\}(t)=\frac{1}{6}\,\mathrm{e}^{2\,(t-5)}\,(t-5)^3\,u_5(t)=\frac{1}{6}\,(t-5)^3\,\mathrm{e}^{2\,(t-5)}\,u_5(t)$$

02b. Making use of the shift formula for the inverse transform, we have

$$\mathcal{L}^{-1}\left\{\frac{8\,\mathrm{e}^{-3\,s}}{s^2+4}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{8}{s^2+4}\right\}(t-3)\;u_3(t)$$

with

$$\mathcal{L}^{-1}\left\{\frac{8}{s^2+4}\right\}(t)=8\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t)=4\,\sin(2\,t) \implies \mathcal{L}^{-1}\left\{\frac{8}{s^2+4}\right\}(t-3)=4\,\sin\big(2\,(t-3)\big)$$

It follows

$$\mathcal{L}^{-1}\left\{\frac{8\,\mathrm{e}^{-3\,s}}{s^2+4}\right\}(t)=4\,\sin\big(2\,(t-3)\big)\,u_3(t)=4\,\sin(2\,t-6)\,u_3(t)$$

02c. We use the second shift formula for the inverse transform, to get

$$\mathcal{L}^{-1}\left\{\frac{(s+2)\,\mathrm{e}^{-\pi\,s}}{s^2-4\,s+13}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-4\,s+13}\right\}(t-\pi)\;\;u_\pi(t)$$

Making use of the first shift formula for the inverse transform, we can write

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-4\,s+13}\right\}(t)&=\mathcal{L}^{-1}\left\{\frac{(s-2)+4}{(s-2)^2+9}\right\}(t)=\mathrm{e}^{2\,t}\,\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+9}\right\}(t)\\&=\mathrm{e}^{2\,t}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}(t)+4\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}(t)\right)\\&=\mathrm{e}^{2\,t}\left(\cos(3\,t)+\frac{4}{3}\,\sin(3\,t)\right)\end{aligned}$$

It follows

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{(s+2)\,\mathrm{e}^{-3\,s}}{s^2-4\,s+13}\right\}(t)&=\mathrm{e}^{2\,(t-\pi)}\left(\cos\big(3\,(t-\pi)\big)+\frac{4}{3}\,\sin\big(3\,(t-\pi)\big)\right)\,u_\pi(t)\\&=\mathrm{e}^{2\,t-2\,\pi}\left(\cos(3\,t-3\,\pi)+\frac{4}{3}\,\sin(3\,t-3\,\pi)\right)\,u_\pi(t)\\&=\mathrm{e}^{2\,t-2\,\pi}\left(-\cos(3\,t)-\frac{4}{3}\,\sin(3\,t)\right)\,u_\pi(t)\\&=-\left(\cos(3\,t)+\frac{4}{3}\,\sin(3\,t)\right)\,\mathrm{e}^{2\,t-2\,\pi)}\,u_\pi(t)\end{aligned}$$

Recall the inverse Laplace transform of a derivative formula:

$$\mathcal{L}^{-1}\left\{F'(s)\right\}(t) = -t\,\mathcal{L}^{-1}\left\{F(s)\right\}(t) \quad \text{or} \quad \mathcal{L}^{-1}\left\{F(s)\right\}(t) = -\frac{1}{t}\,\mathcal{L}^{-1}\left\{F'(s)\right\}(t)$$

Use the first formula if the antiderivative of your function is simpler, and use the second formula if the derivative of your function is simpler.

03a. From $\displaystyle\int \frac{s}{\left(s^2+1\right)^2}\,\mathrm{d}s = -\frac{1}{2}\,\frac{1}{s^2+1} + C$, it follows $\displaystyle\frac{s}{\left(s^2+1\right)^2} = \left(-\frac{1}{2}\,\frac{1}{s^2+1}\right)'$

Using the inverse Laplace transform of a derivative formula we can write

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+1\right)^2}\right\}(t) &= \mathcal{L}^{-1}\left\{\left(-\frac{1}{2}\,\frac{1}{s^2+1}\right)'\right\}(t) = -t\,\mathcal{L}^{-1}\left\{-\frac{1}{2}\,\frac{1}{s^2+1}\right\}(t)\\ &= \frac{1}{2}\,t\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) = \frac{1}{2}\,t\,\sin(t)\end{aligned}$$

03b. From $\displaystyle\int \frac{s^2-1}{\left(s^2+1\right)^2}\,\mathrm{d}s = -\frac{s}{s^2+1} + C$, it follows $\displaystyle\frac{s^2-1}{\left(s^2+1\right)^2} = -\left(\frac{s}{s^2+1}\right)'$.

Using the inverse Laplace transform of a derivative formula, we have

$$\mathcal{L}^{-1}\left\{\frac{s^2-1}{\left(s^2+1\right)^2}\right\}(t) = -\mathcal{L}^{-1}\left\{\left(\frac{s}{s^2+1}\right)'\right\}(t) = t\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) = t\,\cos(t)$$

03c. We have

$$\ln\frac{s^2+1}{s^2+4} = \ln\left(s^2+1\right) - \ln\left(s^2+4\right) \implies \left(\ln\frac{s^2+1}{s^2+4}\right)' = \frac{2\,s}{s^2+1} - \frac{2\,s}{s^2+4}$$

Making use of the second form of the inverse Laplace transform of a derivative formula, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\ln\frac{s^2+1}{s^2+4}\right\}(t) &= -\frac{1}{t}\,\mathcal{L}^{-1}\left\{\left(\ln\frac{s^2+1}{s^2+4}\right)'\right\}(t) = -\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{2\,s}{s^2+1} - \frac{2\,s}{s^2+4}\right\}(t)\\ &= -\frac{1}{t}\left(2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) - 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t)\right)\\ &= -\frac{1}{t}\Big(2\,\cos(t) - 2\,\cos(2\,t)\Big) = \frac{2}{t}\left(\cos(2\,t) - \cos(t)\right)\end{aligned}$$

03d. Making use of the second form of the inverse Laplace transform of a derivative formula, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\tan^{-1}\frac{3}{s+2}\right\}t &= -\frac{1}{t}\,\mathcal{L}^{-1}\left\{\left(\tan^{-1}\frac{3}{s+2}\right)'\right\}(t)\\ &= -\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{-3}{(s+2)^2+9}\right\}(t) \quad \text{Next, we use the first shift formula to get}\\ &= -\frac{1}{t}\,\mathrm{e}^{-2\,t}\,\mathcal{L}^{-1}\left\{\frac{-3}{s^2+9}\right\}(t) = \frac{1}{t}\,\mathrm{e}^{-2\,t}\,\sin(3\,t)\end{aligned}$$

04a. Take the Laplace Transform of both sides of the equation to get

$$\left(s^2+5\,s+6\right)\,\mathcal{L}\left\{y\right\}(s)-s\,y(0)-y'(0)-5\,y(0)=\frac{2}{s+1}\iff\mathcal{L}\left\{y\right\}(s)=\frac{s^2+9\,s+10}{(s+1)\left(s^2+5\,s+6\right)}$$

Thus

$$y(t)=\mathcal{L}^{-1}\left\{\frac{s^2+9\,s+10}{(s+1)\left(s^2+5\,s+6\right)}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{4}{s+2}-\frac{4}{s+3}\right\}(t)==\mathrm{e}^{-t}+4\,\mathrm{e}^{-2\,t}-4\,\mathrm{e}^{-3\,t}$$

04b. Take the Laplace Transform of both sides of the equation to get

$$\left(s^2+4\right)\,\mathcal{L}\left\{y\right\}(s)-s\,y(0)-y'(0)=\frac{16}{s^2+4}+\frac{9\,s}{s^2+1}\iff\mathcal{L}\left\{y\right\}(s)=\frac{3\,s}{s^2+1}-\frac{2\,s}{s^2+4}+\frac{16}{\left(s^2+4\right)^2}$$

Thus

$$y(t)=\mathcal{L}^{-1}\left\{\frac{3\,s}{s^2+1}-\frac{2\,s}{s^2+4}+\frac{16}{\left(s^2+4\right)^2}\right\}(t)=3\,\cos(t)-2\,\cos(2\,t)+\mathcal{L}^{-1}\left\{\frac{16}{\left(s^2+4\right)^2}\right\}(t)$$

To compute the remaining inverse Laplace transform, we write $\dfrac{16}{\left(s^2+4\right)^2}=\dfrac{2}{s^2+4}+2\left(\dfrac{s}{s^2+4}\right)'$, and make use of the formula $\mathcal{L}^{-1}\left\{F'(s)\right\}(t)=-t\,\mathcal{L}^{-1}\left\{F(s)\right\}(t)$, to obtain

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{16}{\left(s^2+4\right)^2}\right\}(t)&=\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}(t)+2\,\mathcal{L}^{-1}\left\{\left(\frac{s}{s^2+4}\right)'\right\}(t)=\sin(2\,t)-2\,t\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t)\\&=\sin(2\,t)-2\,t\,\cos(2\,t)\end{aligned}$$

Hence

$$y(t)=3\,\cos(t)-2\,\cos(2\,t)+\mathcal{L}^{-1}\left\{\frac{16}{\left(s^2+4\right)^2}\right\}(t)=3\,\cos(t)-2\,\cos(2\,t)+\sin(2\,t)-2\,t\,\cos(2\,t)$$

or else

$$y(t)=\sin(2\,t)-2\,(t+1)\,\cos(2\,t)+3\,\cos(t)$$

04c. Take the Laplace Transform of both sides of the equation to get

$$\begin{gathered}\left(s^2-3\,s+2\right)\,\mathcal{L}\left\{y\right\}(s)-s\,y(0)-y'(0)+3\,y(0)=\mathcal{L}\left\{g(t)\right\}(s)\\\left(s^2-3\,s+2\right)\,\mathcal{L}\left\{y\right\}(s)+3\,s-10=\mathcal{L}\left\{g(t)\right\}(s)\\\mathcal{L}\left\{y\right\}(s)=\frac{-3\,s+10}{s^2-3\,s+2}+\frac{1}{s^2-3\,s+2}\,\mathcal{L}\left\{g(t)\right\}(s)\end{gathered}$$

To compute $\mathcal{L}\left\{g(t)\right\}(s)$, we first express $g(t)$ in terms of unit step functions as

$$g(t)=0-0\,u_1(t)+1\,u_1(t)-1\,u_2(t)-1\,u_2(t)=u_1(t)-2\,u_2(t)$$

It follows $\mathcal{L}\left\{g(t)\right\}(s)=\mathcal{L}\left\{u_1(t)\right\}(s)-2\,\mathcal{L}\left\{u_2(t)\right\}(s)=\dfrac{\mathrm{e}^{-s}}{s}-2\,\dfrac{\mathrm{e}^{-2\,s}}{s}$. Substituting into the expression of $\mathcal{L}\left\{y\right\}(s)$, we get

$$\mathcal{L}\left\{y\right\}(s)=\frac{-3\,s+10}{s^2-3\,s+2}+\frac{\mathrm{e}^{-s}}{s\left(s^2-3\,s+2\right)}-\frac{2\,\mathrm{e}^{-2\,s}}{s\left(s^2-3\,s+2\right)}$$

Taking the inverse Laplace transform, we get

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-3\,s+10}{s^2-3\,s+2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-s}}{s\,(s^2-3\,s+2)}\right\}(t) - 2\,\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-2\,s}}{s\,(s^2-3\,s+2)}\right\}(t)$$

But,

$$\mathcal{L}^{-1}\left\{\frac{-3\,s+10}{s^2-3\,s+2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{-7}{s-1} + \frac{4}{s-2}\right\}(t) = -7\,\mathrm{e}^t + 4\,\mathrm{e}^{2\,t}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-s}}{s\,(s^2-3\,s+2)}\right\}(t) &= u_1(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2-3\,s+2)}\right\}(t-1) \\
&= u_1(t)\,\mathcal{L}^{-1}\left\{\frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2}\right\}(t-1) \\
&= u_1(t)\,\left(\frac{1}{2} - \mathrm{e}^{t-1} + \frac{1}{2}\,\mathrm{e}^{2\,(t-1)}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-2\,s}}{s\,(s^2-3\,s+2)}\right\}(t) &= u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s\,(s^2-3\,s+2)}\right\}(t-2) \\
&= u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2}\right\}(t-2) \\
&= u_2(t)\,\left(\frac{1}{2} - \mathrm{e}^{t-2} + \frac{1}{2}\,\mathrm{e}^{2\,(t-2)}\right)
\end{aligned}$$

Thus $y(t) = -7\,\mathrm{e}^t + 4\,\mathrm{e}^{2\,t} + u_1(t)\,\left(\frac{1}{2} - \mathrm{e}^{t-1} + \frac{1}{2}\mathrm{e}^{2\,(t-1)}\right) - 2\,u_2(t)\,\left(\frac{1}{2} - \mathrm{e}^{t-2} + \frac{1}{2}\mathrm{e}^{2\,(t-2)}\right)$

or else $\quad y(t) = -7\,\mathrm{e}^t + 4\,\mathrm{e}^{2\,t} + \frac{1}{2}\,\left(1 - \mathrm{e}^{t-1}\right)^2\,\left(u_1(t) - 2\,u_2(t)\right)$

05a. Let's start by writing the function $g(t)$ in terms of unit step functions. We have

$$g(t) = \mathrm{e}^t + \left(\left(\mathrm{e}^t - 1\right) - \mathrm{e}^t\right)\,u_1(t) = \mathrm{e}^t - u_1(t)$$

Taking the Laplace Transform of both sides of the differential equation, we get

$$\left(s^2\,\mathcal{L}\left\{y(t)\right\}(s) - s\,y(0) - y'(0)\right) + 2\,\left(s\,\mathcal{L}\left\{y(t)\right\}(s) - y(0)\right) + \mathcal{L}\left\{y(t)\right\}(s) = \mathcal{L}\left\{\mathrm{e}^t\right\}(s) - \mathcal{L}\left\{u_1(t)\right\}$$

$$\left(s^2\,\mathcal{L}\left\{y(t)\right\}(s) - 3\,s + 1\right) + 2\,\left(s\,\mathcal{L}\left\{y(t)\right\}(s) - 3\right) + \mathcal{L}\left\{y(t)\right\}(s) = \frac{1}{s-1} - \frac{1}{s}\,\mathrm{e}^{-s}$$

$$\left(s^2 + 2\,s + 1\right)\,\mathcal{L}\left\{y\right\}(s) = 3\,s + 5 + \frac{1}{s-1} - \frac{1}{s}\,\mathrm{e}^{-s} = \frac{3\,s^2 + 2\,s - 4}{s-1} - \frac{1}{s}\,\mathrm{e}^{-s}$$

$$\mathcal{L}\left\{y\right\}(s) = \frac{3\,s^2 + 2\,s - 4}{(s-1)\,(s+1)^2} - \frac{1}{s\,(s+1)^2}\,\mathrm{e}^{-s}$$

Taking the inverse Laplace transform of both sides, we get

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{3\,s^2 + 2\,s - 4}{(s-1)\,(s+1)^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s\,(s+1)^2}\,\mathrm{e}^{-s}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{1/4}{s-1} + \frac{11/4}{s+1} + \frac{3/2}{(s+1)^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s\,(s+1)^2}\right\}(t-1)\,u_1(t) \\
&= \frac{1}{4}\,\mathrm{e}^t + \frac{11}{4}\,\mathrm{e}^{-t} + \frac{3}{2}\,t\,\mathrm{e}^{-t} - \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right\}(t-1)\,u_1(t) \\
&= \frac{1}{4}\,\mathrm{e}^t + \frac{11}{4}\,\mathrm{e}^{-t} + \frac{3}{2}\,t\,\mathrm{e}^{-t} - \left(1 - \mathrm{e}^{-(t-1)} - (t-1)\,\mathrm{e}^{-(t-1)}\right)\,u_1(t)
\end{aligned}$$

Hence $\quad y(t) = \frac{1}{4}\,\mathrm{e}^t + \frac{11}{4}\,\mathrm{e}^{-t} + \frac{3}{2}\,t\,\mathrm{e}^{-t} - \left(1 - t\,\mathrm{e}^{-(t-1)}\right)\,u_1(t)$

05b. We start by writing the function $g(t)$ in terms of unit step functions. We have

$$g(t) = \cos(t) + \big(\sin(t) - \cos(t)\big)\, u_{3\pi/2}(t)$$

Taking the Laplace Transform of both sides of the differential equation, we get

$$\Big(s^2\,\mathcal{L}\{y(t)\}(s) - s\,y(0) - y'(0)\Big) + 9\,\mathcal{L}\{y(t)\}(s) = \mathcal{L}\Big\{\cos(t) + \big(\sin(t) - \cos(t)\big)\, u_{3\pi/2}(t)\Big\}(s)$$

$$\big(s^2+9\big)\,\mathcal{L}\{y(t)\}(s) = \frac{s}{s^2+1} + \mathcal{L}\{\sin(t+3\,\pi/2) - \cos(t+3\,\pi/2)\}(s)\ \mathrm{e}^{-(3\,\pi/2)\ s}$$

$$\begin{aligned}\big(s^2+9\big)\ \mathcal{L}\{y\}(s) &= \frac{s}{s^2+1} + \mathcal{L}\{-\cos(t) - \sin(t)\}(s)\ \mathrm{e}^{-(3\,\pi/2)} \\ &= \frac{s}{s^2+1} - \left(\frac{s}{s^2+1} + \frac{1}{s^2+1}\right)\ \mathrm{e}^{-(3\,\pi/2)\ s} = \frac{s}{s^2+1} - \frac{s+1}{s^2+1}\ \mathrm{e}^{-(3\,\pi/2)\ s}\end{aligned}$$

Hence

$$\mathcal{L}\{y\}(s) = \frac{s}{(s^2+1)\,(s^2+9)} - \frac{s+1}{(s^2+1)\,(s^2+9)}\ \mathrm{e}^{-(3\,\pi/2)\ s}$$

Taking the inverse Laplace transform of both sides, we get

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)\,(s^2+9)}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)\,(s^2+9)}\ \mathrm{e}^{-(3\,\pi/2)\ s}\right\}(t)$$

Now

$$\mathcal{L}^{-1}\left\{\frac{s}{\big(s^2+1\big)\,\big(s^2+9\big)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{8}\ \left(\frac{s}{s^2+1} - \frac{s}{s^2+9}\right)\right\}(t) = \frac{1}{8}\,\Big(\cos(t) - \cos(3\,t)\Big)$$

and

$$\begin{aligned} E &= \mathcal{L}^{-1}\left\{\frac{s+1}{\big(s^2+1\big)\,\big(s^2+9\big)}\ \mathrm{e}^{-(3\,\pi/2)\ s}\right\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{s+1}{\big(s^2+1\big)\,\big(s^2+9\big)}\right\}\big(t - \frac{3\,\pi}{2}\big)\ \ u_{3\,\pi/2}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{8}\ \left(\frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{s}{s^2+9} - \frac{1}{s^2+9}\right)\right\}\big(t - \frac{3\,\pi}{2}\big)\ \ u_{3\,\pi/2}(t) \\ &= \frac{1}{8}\ \left(\cos(t - \frac{3\,\pi}{2}) + \sin(t - \frac{3\,\pi}{2}) - \cos\Big(3\,(t - \frac{3\,\pi}{2})\Big) - \frac{1}{3}\ \sin\Big(3\,(t - \frac{3\,\pi}{2})\Big)\right)\ \ u_{3\,\pi/2}(t) \\ &= \frac{1}{8}\ \left(-\sin(t) + \cos(t) - \sin(3\,t) + \frac{1}{3}\ \cos(3\,t)\right)\ \ u_{3\,\pi/2}(t)\end{aligned}$$

Hence

$$y(t) = \frac{1}{8}\,\Big(\cos(t) - \cos(3\,t)\Big) - \frac{1}{8}\ \left(-\sin(t) + \cos(t) - \sin(3\,t) + \frac{1}{3}\ \cos(3\,t)\right)\ \ u_{3\,\pi/2}(t)$$