Department of Mathematics and Statistics
MATH 375
Handout # 2 - ANSWERS, HINTS, SOLUTIONS
First Order Ordinary Differential Equations

1. Find the general solution of the following differential equations
   a) $xy' - y = y^3$ b) $xyy' = 1 - x^2$
   c) $y' \tan x = y$ d) $y' = 10^{x+y}$
   Answer. $x = \dfrac{Cy}{\sqrt{y^2+1}}$, $y = 0$ b) $y^2 = \ln\left(C^2x^2\right) - x^2$, or $y = \pm\sqrt{\ln\left(C^2x^2\right) - x^2}$
   c) $y = C\sin x$ d) $y = -\log_{10}\left(C - 10^x\right)$.
   Solution. a) The equation is separable $xy' = y + y^3 = y(y^2+1) \Rightarrow x\frac{dy}{dx} = y(y^2+1) \Rightarrow$

$$\frac{dy}{y(y^2+1)} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y(y^2+1)} = \int \frac{dx}{x} = \ln|x| + C_1 = \ln|x| + \ln|C_2| = \ln|C_2x|.$$

   We present $\frac{1}{y(y^2+1)}$ as a sum of partial fractions:

$$\frac{1}{y(y^2+1)} = \frac{Ay+B}{y^2+1} + \frac{C}{y} \Rightarrow Ay^2 + By + Cy^2 + C = 1 \Rightarrow C = 1, B = 0, A + C = 0 \Rightarrow A = -1,$$

$$\int \frac{dy}{y(y^2+1)} = -\int \frac{ydy}{y^2+1} + \int \frac{dy}{y} = -\frac{1}{2}\ln\left(y^2+1\right) + \ln|y| = \ln\left|\frac{y}{\sqrt{y^2+1}}\right| = \ln|C_2x|$$

$$\Rightarrow \frac{y}{\sqrt{y^2+1}} = \pm C_2x \Rightarrow x = \pm\frac{y}{C_2\sqrt{y^2+1}} = \frac{Cy}{\sqrt{y^2+1}}$$

   if C has an arbitrary sign. In the process of the solution we divided by x, y, y^2+1. The latter expression never vanishes, $x = 0$ is not a solution, $y = 0$ is a solution.
   Remark. In future, you can also try to solve it as a Bernoulli equation.
   b) $yy' = \frac{1-x^2}{x} = \frac{1}{x} - x \Rightarrow$

$$y\ dy = \left(\frac{1}{x} - x\right)\ dx \Rightarrow \int y\ dy = \int \left(\frac{1}{x} - x\right)\ dx$$

$$\Rightarrow \frac{1}{2}y^2 = \ln|x| - \frac{1}{2}x^2 + C_1 = \ln|x| + \ln C_2 - \frac{1}{2}x^2 = \ln|C_2x| - \frac{1}{2}x^2$$
$$\Rightarrow y^2 + x^2 = 2\ln|C_2x| = \ln\left(C_2^2x^2\right) = \ln\left(C^2x^2\right).$$

   We divided only by x, $x = 0$ is not a solution.
   c) $\frac{1}{y}y' = \frac{1}{\tan x} = \cot x \Rightarrow \frac{dy}{y} = \cot x\ dx \Rightarrow \int \frac{dy}{y} = \int \cot x\ dx$

$$\Rightarrow \ln|y| = \ln|\sin x| + C_1 = \ln|\sin x| + \ln C_2 = \ln|C_2\sin x| \Rightarrow y = \pm C_2\sin x$$

   or $y = C\sin x$, C is positive or negative. We divided by y, it is easily checked that $y = 0$ is a solution, so C can be a zero as well.
   d) $y' = 10^x10^y \Rightarrow \frac{dy}{10^y} = 10^x\ dx$

$$\Rightarrow \int 10^{-y}\ dy = \int 10^x\ dx \Rightarrow -\frac{1}{\ln 10}10^{-y} = \frac{1}{\ln 10}10^x + C_1 \Rightarrow 10^{-y} = -10^x - \ln 10C_1 = C - 10^x$$

$$\Rightarrow -y = \log_{10}\left(C - 10^x\right) \Rightarrow y = -\log_{10}\left(C - 10^x\right) = \log_{10}\left(\frac{1}{C - 10^x}\right).$$

2. Find the solution of the following initial value problems
a) $(x^2-1)y' + 2xy^2 = 0, \quad y(0) = 1$
b) $y' \sin x = y \ln y, \quad y(\frac{\pi}{2}) = 1$
Answer. a) $y(\ln|x^2-1|+1) = 1$ b) $y = 1$. **Solution.** $y' = -\frac{2xy^2}{x^2-1} \Rightarrow -\frac{dy}{y^2} = \frac{2x\ dx}{x^2-1}$

$$\Rightarrow -\int \frac{dy}{y^2} = \int \frac{2x\ dx}{x^2-1} \Rightarrow \frac{1}{y} = \ln|x^2-1| + C$$

$x = 0, y = 1 \Rightarrow 1 = \ln 1 + C = C \Rightarrow C = 1 \Rightarrow \frac{1}{y} = \ln|x^2-1| + 1 \Rightarrow y(\ln|x^2-1|+1) = 1$, or $y = \frac{1}{\ln|x^2-1|+1}$.
b) $y' \sin x = y \ln y, \quad y(\frac{\pi}{2}) = 1$

Solution. $$\frac{y'}{y\ln y} = \frac{1}{\sin x} \Rightarrow \frac{dy}{y \ln y} = \frac{dx}{\sin x} \Rightarrow \int \frac{dy}{y\ln y} = \int \frac{dx}{\sin x}$$

Here the integral in the left hand side is computed by the substitution $u = \ln y$, while the integral in the right hand side is computed using the table (formula 20). Thus

$$\Rightarrow \ln|\ln y| = \ln\left[\frac{1}{\sin x} - \cot x\right] + C_1 = \ln\left[C_2\left(\frac{1}{\sin x} - \cot x\right)\right]$$

$$\Rightarrow |\ln y| = C_2\left(\frac{1}{\sin x} - \cot x\right) \Rightarrow \ln y = C\left(\frac{1}{\sin x} - \cot x\right)$$

Subsititute $x = \frac{\pi}{2}$,$y = 1$:

$$\ln 1 = C(1-0) = 0 \Rightarrow C = 0 \Rightarrow \ln y = 0 \Rightarrow y = 1.$$

Remark. It is to be noted that $y = 1$ is a special (singular) solution; when solving the equation we assume C is positive or negative; however in the process of the solution we divide by $\ln y$; it is easily checked that $y = 1$ is a solution, so C can be equal to zero as well.

3. Find the general solution of the following differential equations
a) $y' + y = xy^3$
b) $x^2 y' = y(x+y)$
c) $y' + x\sqrt[3]{y} = 3y$
Hints and answers. All these equations are Bernoulli equations.
a) Divide by y^3 and make the substitution $z = 1/y^2$. The final answer is $y^2(Ce^{2x}+x+0.5) = 1$.
b) Divide by y^2 and make the substitution $z = 1/y$. The final answer is $y\ln(Cx) = -x$.
c) Divide by $\sqrt[3]{y}$ and make the substitution $z = \sqrt[3]{y^2}$. The final answer is $y^{2/3} = Ce^{2x} + \frac{x}{3} + \frac{1}{6}$.

4. Find the general solution of the following differential equations

$$a)\ y' = \frac{2xy}{x^2+y^2}$$

Solution. The equation is homogeneous nonlinear

$$\frac{2kxky}{(kx)^2+(ky)^2} = \frac{k^2 2xy}{k^2x^2+k^2y^2} = \frac{k^2 2xy}{k^2(x^2+y^2)} = \frac{2xy}{x^2+y^2},$$

so it is solved by the substitution $t = \frac{y}{x}$, $y = tx$, $y' = t'x + t$

$$t'x + t = \frac{2xtx}{x^2 + t^2x^2} = \frac{2t}{1+t^2} \Rightarrow t'x = \frac{2t}{1+t^2} - t = \frac{2t - t - t^3}{1+t^2} = \frac{t - t^3}{1+t^2}$$

$$\frac{1+t^2}{t-t^3}\ dt = \frac{dx}{x} \Rightarrow \int \frac{1+t^2}{t-t^3}\ dt = \int \frac{dx}{x} = \ln|x|$$

Let us present $\frac{1+t^2}{t-t^3}$ as a sum of partial fractions

$$1 + t^2 t(1-t)(1+t) = \frac{A}{t} + \frac{B}{1-t} + \frac{C}{1+t} \Rightarrow A(1-t)(1+t) + Bt(1+t) + Ct(1-t) = 1 + t^2$$

Substituting $t = 0$ we get $A = 1$, $t = 1 \Rightarrow 2B = 2 \Rightarrow B = 1$, $t = -1 \Rightarrow -2C = 2 \Rightarrow C = -1$, thus

$$\int \frac{1+t^2}{t-t^3}\ dt = \int \frac{dt}{t} + \int \frac{dt}{1-t} - \int \frac{dt}{1+t} = \ln|t| - \ln|1-t| - \ln|1+t| + \ln C_2 = \ln\left|C_2 \frac{t}{1-t^2}\right|$$

$$\Rightarrow \frac{C_2 t}{1-t^2} = x \Rightarrow \frac{C_2 \frac{y}{x}}{1 - (\frac{y}{x})^2} = \frac{C_2 yx}{x^2 - y^2} = x \Rightarrow x^2 - y^2 = Cy$$

Answer. $x^2 - y^2 = Cy$.
Remark. In the process of solution we divided by x,t,$t-1$,$1+t$; $x = 0$ is not a solution, $t = 0$ leads to $y = 0$, which is a singular solution (is not included in the general solution), $t - 1$ and $1 + t$ mean $y = x$ and $y = -x$, respectively, which can be included in the general solution (C=0).

$$b)\ x - y\cos\left(\frac{y}{x}\right) + x\cos\left(\frac{y}{x}\right) y' = 0$$

Solution. If we divide the left hand side by x, then the left hand side obviously depends on $\frac{y}{x}$ only, so the equation is homogeneous nonlinear, so it is solved by the substitution $t = \frac{y}{x}$, $y = tx$, $y' = t'x + t$:

$$1 - \frac{y}{x}\cos\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right) y' = 0 \Rightarrow 1 - t\cos t + \cos t(t'x + t) = 1 - t\cos t + t'x\cos t + t\cos t$$

$$= 1 + t'x\cos t = 0 \Rightarrow t'\cos t = -\frac{1}{x} \Rightarrow \cos t\ dt = -\frac{dx}{x} \Rightarrow \int \cos t\ dt = -\int \frac{dx}{x}$$

$$\Rightarrow \sin t = C_1 - \ln|x| = -\ln C_2 - \ln|x| = \ln(cx) \Rightarrow \sin\left(\frac{y}{x}\right) = -\ln(Cx)$$

Answer. $\sin\left(\frac{y}{x}\right) = -\ln(Cx)$.
Remark. We divided by x; $x = 0$ is not a solution.

5. Find the solution of the initial value problem $2xyy' + x^2 - y^2 = 0$, $y(1) = 0$.
Solution. Since $2xyy' = y^2 - x^2$, $y' = \frac{y^2}{2xy} - \frac{x^2}{2xy} = \frac{1}{2}\frac{y}{x} - \frac{1}{2}\frac{x}{y}$, then the equation is homogeneous nonlinear. Afterthe substitution $t = \frac{y}{x}$, $y = tx$, $y' = t'x + t$ we have

$$t'x + t = \frac{1}{2}t - \frac{1}{2t} \Rightarrow t'x = -\frac{1}{2}t - \frac{1}{2t} = -\frac{t^2+1}{2t} \Rightarrow \frac{2tdt}{t^2+1} = -\frac{dx}{x} \Rightarrow \int \frac{2tdt}{t^2+1} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln(t^2+1) = -\ln|x| + C_1 = -\ln|x| + \ln C_2 = \ln\frac{C}{x} \Rightarrow \frac{C}{x} = t^2+1 = \frac{y^2}{x^2}+1$$

$$x=1, y=0 \Rightarrow \frac{C}{1} = 0+1 \Rightarrow C=1 \Rightarrow \frac{y^2}{x^2}+1 = \frac{1}{x}, \text{ or } y^2 = x^2\left(\frac{1}{x}-1\right) = x - x^2$$

Answer. Anyone of two solutions $y = \pm\sqrt{x-x^2}$.

6. If possible, find the values of α and β such that the equation

$$xy^\beta y' = 3x^\alpha + x^3y^3$$

is
a) linear;
b) separable;
c) homogeneous;
d) Bernoulli?
Answers. Since y^3 is involved in the equation, there are no α and β such that the equation is linear. The equation is separable for $\alpha = 3$ and any β, homogeneous for $\alpha = 6$ and $\beta = 5$, Bernoulli for $\beta = -1$ or $\beta = 2$ and any α.

7. Find α and β such that the following equations are exact:
a) $x^\alpha y^2 + x^3y^\beta y' = 0$
b) $6x^\beta e^y\ dx + x^{\beta+\alpha}e^y\ dy = 0$
c) $e^{\alpha x+\beta y}(3+y') = 0$
Answers and solutions. a) The equation is exact if

$$\frac{\partial}{\partial y}(x^\alpha y^3) = \frac{\partial}{\partial x}(x^3y^\beta),$$

or $3x^\alpha y^2 = 3x^2y^\beta$, so the equation is exact for $\alpha = \beta = 2$ only.
b) $\alpha = 1,\ \beta = 5$ c) any $\alpha = 3\beta \in \mathbb{R}$.

8. Find the general solution of the following differential equations
a) $(2-9xy^2)x\ dx + (4y^2-6x^3)y\ dy = 0$ b) $1 + y^2\sin 2x - 2yy'\cos^2 x = 0$
c) $x\ dx + y\ dy = \dfrac{xdy - ydx}{x^2+y^2}$
Answers. All these equations are exact.
a) $x^2 - 3x^3y^2 + y^4 = C$
b) $x - y^2\cos^2 x = C$
c) $x^2 + y^2 - 2\arctan\frac{y}{x} = C$
Solution. Let us present a complete solution for a). The equation is exact since

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial y}(2x - 9x^2y^2) = -18x^2y = \frac{\partial}{\partial x}(4y^3 - 6x^3y) = \frac{\partial}{\partial x}N(x,y).$$

Integrating $M(x,y)$ in x, we obtain

$$F(x,y) = \int (2x - 9x^2y^2)\ dx = x^2 - 3x^3y^2 + g(y).$$

Differentiating the result in y and comparing to $N(x,y) = 4y^3 - 6x^3y$ we have

$$\frac{\partial}{\partial y}(x^2 - 3x^3y^2 + g(y)) = -6x^3y + g'(y) = 4y^3 - 6x^3y \Rightarrow g'(y) = 4y^3,$$

so $g(y) = y^4$, $F(x,y) = x^2 - 3x^3y^2 + y^4$ and the general solution is

$$F(x,y) = x^2 - 3x^3y^2 + y^4 = C.$$

9. Find the general solution of the following differential equations.

a) $y' = 2xy + x$
b) $2xy' = y$
$x^2 + 2xy^3 + (y^2 + 3x^2y^2)y' = 0$
Answer. a) $y = Ce^{x^2} - \frac{1}{2}$ b) $y^2 = Cx$, $C \in \mathbb{R}$ c) $x^3 + 3x^2y^3 + y^3 = C$
Solution. a) The equation is both linear and separable; for example, we can solve it as a linear equation, the integrating factor is $e^{\int -2x\ dx} = e^{-x^2}$, after multiplying we have

$$e^{-x^2}y' - 2xe^{-x^2}y = \frac{d}{dx}\left(e^{-x^2}y\right) = xe^{-x^2}.$$

We can compute the integral using the substitution $u = -x^2$, $du = -2x\ dx$, of referring to the computation above:

$$e^{-x^2}y = \int xe^{-x^2}\ dx = -\frac{1}{2}e^{-x^2} + C \ \Rightarrow\ y = e^{x^2}\left(C - \frac{1}{2}e^{-x^2}\right) = Ce^{x^2} - \frac{1}{2}.$$

b) The equation is separable:

$$2x\frac{dy}{dx} = y \Rightarrow \frac{2dy}{y} = \frac{dx}{x} \Rightarrow 2\ln|y| = \ln y^2 = \ln|x| + C_1$$

$$= \ln|x| + \ln C_2 = \ln(C_2|x|) = \ln(Cx),\ \ C \neq 0$$

$\Rightarrow y^2 = Cx$; in the process of the solution we divided by x and y; $x = 0$ is not a solution, while $y = 0$ is a solution, which can be included into the general solution if we assume $C = 0$.
c) This is an exact equation, with

$$M(x,y) = x^2 + 2xy^3, N(x,y) = y^2 + 3x^2y^2, \quad \text{since} \quad \frac{\partial M}{\partial y} = 6xy^2, \frac{\partial N}{\partial x} = 6xy^2.$$

$$\text{Thus} \quad \frac{\partial\phi(x,y)}{\partial x} = M(x,y) = x^2 + 2xy^3, \phi(x,y) = \int (x^2 + 2xy^3)dx = \frac{1}{3}x^3 + x^2y^3 + C(y)$$

$$\frac{\partial\phi(x,y)}{\partial y} = 3x^2y^2 + C'(y) = N(x,y) = y^2 + 3x^2y^2 \Rightarrow C(y) = \int y^2\ dy = \frac{1}{3}y^3.$$

Finally, $\phi(x,y) = \frac{1}{3}x^3 + x^2y^3 + C(y) = \frac{1}{3}x^3 + x^2y^3 + \frac{1}{3}y^3$ and the general solution is

$$\frac{1}{3}x^3 + x^2y^3 + \frac{1}{3}y^3 = C \quad \text{or} \quad x^3 + 3x^2y^3 + y^3 = C.$$

10. Use the appropriate existence and uniqueness theorem to find the largest interval (a, b) on which the solution to each the following equations is guaranteed to exist:

a) $y' + \dfrac{t}{t^2-1}y = \sqrt{5-t}, \;\; y(4) = -3$

b) $(t-6)y' + ty = \ln\left(t - \dfrac{4}{t}\right), \;\; y(3) = 7$

Solution. By the existence and uniqueness theorem, the linear equation $y' + p(t)y = q(t)$, $y(t_0) = y_0$ has a unique solution on the interval (a, b) containing t_0 if $p(t)$ and $q(t)$ are continuous on (a, b).

a) $\dfrac{t}{t^2-1}$ is continuous everywhere but at $t = \pm 1$, $\sqrt{5-t}$ is continuous on $(-\infty, 5)$. The largest interval containing 4 on which both are continuous is $(1, 5)$.

b) To bring to the standard form, we have to divide by $t-6$. The function $\dfrac{t}{t-6}$ is continuous everywhere but at $t = 6$, while $\ln\left(t - \frac{4}{t}\right)/(t-6)$ is defined and continuous for $t - \frac{4}{t} > 0$, $t \neq 6$, or $(t+2)(t-2)/t > 0$, $t \in (-2, 0) \cup (2, 6) \cup (6, \infty)$. The largest interval containing 3 is $(2, 6)$.

11. Find the general solution of the following differential equations

a) $(1 + 2y)y' + 2(y + y^2) = 0$;

b) $\left(x^2 + \dfrac{x}{\cos^2 y}\right)y' + 3xy + 2tany = 0$.

Solution. a) Since $M(x, y) = 2(y + y^2)$, $N(x, y) = 1 + 2y$ and

$$\frac{M_y - N_x}{N} = \frac{2 + 4y - 0}{1 + 2y} = 2$$

does not depend on x, then the integrating factor can be found as

$$\ln \mu(x) = \int 2 \; dx = 2x \Rightarrow \mu(x) = e^{2x},$$

the equation

$$e^{2x}(1 + 2y)y' + 2e^{2x}(y + y^2) = 0$$

is exact: $\dfrac{\partial}{\partial x}(e^{2x}(1 + 2y)) = 2e^{2x}(1 + 2y) = \dfrac{\partial}{\partial y}(2e^{2x}(y + y^2))$. Its solution is

$$e^{2x}(y + y^2) = C.$$

b) Since $M(x, y) = 3xy + 2tany$, $N(x, y) = x^2 + \dfrac{x}{\cos^2 y}$ and

$$\frac{M_y - N_x}{N} = \frac{3x + 2/\cos^2 y - 2x - 1/\cos^2 y}{x^2 + x/\cos^2 y} = \frac{x\cos^2 y + 1}{x(x\cos^2 y + 1)} = \frac{1}{x},$$

so

$$\ln \mu(x) = \int \frac{1}{x} dx = \ln|x| \Rightarrow \mu(x) = x.$$

The equation

$$\left(x^3 + \frac{x^2}{\cos^2 y}\right)y' + 3x^2 y + 2xtany = 0$$

is exact: $\dfrac{\partial}{\partial x}\left(x^3+\dfrac{x^2}{\cos^2 y}\right)=3x^2+\dfrac{2x}{\cos^2 y}=\dfrac{\partial}{\partial y}(3x^2y+2xtany)$. Its solution is

$$x^3y+x^2tany=C.$$