Math 375		Fall 2016
	Higher Order Linear Differential Equations	
Worksheet # 2	Part 2	October 17 - 21

This week's tutorial will be used to write the quiz. As a result, you are expected to go over the problems on your own. The solutions to all problems are included and start on the next page. If you need help with any of the problems, do not hesitate to talk to your instructor and/or your TA. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

The Wronskian

01. Compute the wronskian $W(t)$ of each of the following set of functions.

a. $\left\{ e^{rt},\ t\,e^{rt} \right\}$ **b*.** $\left\{ e^{rt}\cos(s\,t),\ e^{rt}\sin(s\,t) \right\}$

Assume that $r,\ s \neq 0$, are real constants.

02. Write Abel's formula for each of the following equations.

a. $3\,y'' - 2\,y' + t\,y = 0$ **b*.** $t\,y'' + (5\,t - 2)\,y' + t\,e^t\,y = 0$

03*. Consider the linear homogeneous differential equation $y'' + \left(t^2 + 1\right)y' - 2\,y = 0$ Suppose that $y_1(t)$ and $y_2(t)$, are solutions of the differential equation, and let $W(t)$ be their wronskian. Given that $W(1) = 2$, use Abel's formula to evaluate $W(2)$.

04. Let $y_1(t)$ and $y_2(t)$ be two solutions of $y'' + p(t)\,y' + q(t)\,y = 0$, in (a,b), and let $W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$ be their wronskian

a. Show that $W'(t) + p(t)\,W(t) = 0$

b. Show that $W(t) = K\,e^{-\int p(t)\,dt}$, where K is a real constant.

Second Order Homogeneous With Constant Coefficients

05. Solve the following second order linear differential equations.

a*. $2\,y'' - y' - 3\,y = 0$ **b*.** $9\,y'' - 12\,y' + 4\,y = 0$ **c*.** $y'' - 2\,y' + 5\,y = 0$

d. $3\,y'' - 2\,y' - 8\,y = 0$ **e.** $4\,y'' + 4\,y' + y = 0$ **f.** $4\,y'' - 12\,y' + 25\,y = 0$

06. A second order linear homogeneous differential equation with constant coefficients has the given function as a solution, write down the normal form of the equation.

a. $2\,e^{-5t} + 3\,e^{-2t}$ **b.** $5\,e^{-2t} - 2$ **c*.** $2\,e^{3t}\cos(2\,t)$

Answers and Solutions

01a. We have

$$W(t) = \begin{vmatrix} e^{rt} & t\,e^{rt} \\ r\,e^{rt} & (1+r\,t)\,e^{rt} \end{vmatrix} = e^{rt}\,e^{rt} \begin{vmatrix} 1 & t \\ r & 1+r\,t \end{vmatrix} = e^{2\,r\,t}\Big(1 + r\,t - r\,t\Big) = e^{2\,r\,t}$$

01b. We have

$$W(t) = \begin{vmatrix} e^{2t}\cos(3t) & e^{2t}\sin(3t) \\ e^{2t}\big(2\cos(3t) - 3\sin(3t)\big) & e^{2t}\big(3\cos(3t) + 2\sin(3t)\big) \end{vmatrix}$$

$$= e^{2t}\,e^{2t} \begin{vmatrix} \cos(3t) & \sin(3t) \\ 2\cos(3t) - 3\sin(3t) & 3\cos(3t) + 2\sin(3t) \end{vmatrix}$$

$$\overset{R_2 - 2\,R_1}{=\!=\!=} e^{4t} \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = e^{4t}\big(3\cos^2(3t) + 3\sin^2(3t)\big) = 3\,e^{4t}$$

Abel's Formula

If $y_1(t)$ and $y_2(t)$ are any two solutions of $y'' + p(t)\,y' + q(t)\,y = 0$, and if $W(t)$ is the wronskian of the two solutions, then $W(t) = K\,e^{-\int p(t)\,dt}$, for some constant K.

02a. We first rewrite the differential equation in normal form as $y'' - \dfrac{2}{3}\,y' + \dfrac{t}{3}\,y = 0$. Then

$$W(t) = K\,e^{-\int -2/3\,dt} = K\,e^{2\,t/3}$$

02b. We need to rewrite the equation in normal form as $y'' + \left(5 - \dfrac{2}{t}\right)\,y' + e^t\,y = 0$. It follows

$$W(t) = K\,e^{-\int \big(5 - 2/t\big)\,dt} = K\,e^{-5\,t + 2\,\ln|t|} = K\,e^{-5\,t}\,e^{\ln(t^2)} = K\,t^2\,e^{-5\,t}$$

03. By Abel's formula

$$W\big(t\big) = K\,e^{-\int p(t)\,dt} = K\,e^{-\int (t^2+1)\,dt} = K\,e^{-t-t^3/3}, \qquad \text{for some constant} \qquad K$$

To compute the constant, we set $t = 1$, and equate the result to 2, to get

$$W(1) = 2 \iff K\,e^{-1-1/3} = 2 \implies K\,e^{-4/3} = 2 \implies K = 2\,e^{4/3}$$

It follows $W(t) = 2\,e^{4/3}\,e^{-t-t^3/3}$, and

$$W\big(2\big) = 2\,e^{4/3}\,e^{-2-8/3} = 2\,e^{4/3}\,e^{-14/3} = 2\,e^{-10/3}$$

04a. Differentiating $W(t) = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t)$, we get

$$\begin{aligned} W'(t) &= y_1'(t)\, y_2'(t) + y_1(t)\, y_2''(t) - y_2'(t)\, y_1'(t) - y_2(t)\, y_1''(t) \\ &= y_1(t)\, y_2''(t) - y_2(t)\, y_1''(t) \qquad \text{remember both } y_1(t) \text{ and } y_2(t) \text{ are solutions of} \\ &\qquad\qquad\qquad\qquad\qquad\qquad y'' + p(t)\, y' + q(t)\, y = 0 \iff y'' = -p(t)\, y' - q(t)\, y \\ &= y_1(t) \Big(-p(t)\, y_2'(t) - q(t)\, y_2(t) \Big) - y_2(t) \Big(-p(t)\, y_1'(t) - q(t)\, y_1(t) \Big) \\ &= -p(t)\, y_1(t)\, y_2'(t) + p(t)\, y_2(t)\, y_1'(t) = -p(t) \Big(y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t) \Big) \\ &= -p(t)\, W(t) \end{aligned}$$

Hence

$$W'(t) + p(t)\, W(t) = 0$$

04b. $W'(t) + p(t)\, W(t) = 0$ is a first order linear differential equation. An integrating factor is $\mu(t) = \mathrm{e}^{\int p(t)\,\mathrm{d}t}$. Multiplying both sides of the equation by $\mu(t)$, we have

$$\Big(\mathrm{e}^{\int p(t)\,\mathrm{d}t}\, W(t) \Big)' = 0 \implies \mathrm{e}^{\int p(t)\,\mathrm{d}t}\, W(t) = K \implies W(t) = K\, \mathrm{e}^{-\int p(t)\,\mathrm{d}t}$$

05a. The characteristic equation is $2\lambda^2 - \lambda - 3 = 0$. The roots are $\lambda_1 = -1$ and $\lambda_2 = \frac{3}{2}$. Hence a fundamental set of solutions is $\Big\{ \mathrm{e}^{-t} \,,\ \mathrm{e}^{3t/2} \Big\}$, and the general solution is

$$C_1\, \mathrm{e}^{-t} + C_2\, \mathrm{e}^{3t/2}$$

05b. The characteristic equation is $9\lambda^2 - 12\lambda + 4 = 0$. The roots are $\lambda_1 = \lambda_2 = \frac{2}{3}$. Hence a fundamental set of solutions is $\Big\{ \mathrm{e}^{2t/3} \,,\ t\,\mathrm{e}^{2t/3} \Big\}$, and the general solution is

$$C_1\, \mathrm{e}^{2t/3} + C_2\, t\,\mathrm{e}^{2t/3}$$

05c. The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$. The roots are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Hence a fundamental set of solutions is $\Big\{ \mathrm{e}^{t} \cos\big(2t\big) \,,\ \mathrm{e}^{t} \sin\big(2t\big) \Big\}$, and the general solution is

$$C_1\, \mathrm{e}^{t} \cos\big(2t\big) + C_2\, \mathrm{e}^{t} \sin\big(2t\big)$$

05d. The characteristic equation is $3\lambda^2 - 2\lambda - 8 = 0$. The roots are $\lambda_1 = -\frac{4}{3}$ and $\lambda_2 = 2$. Hence a fundamental set of solutions is $\Big\{ \mathrm{e}^{-4t/3} \,,\ \mathrm{e}^{2t} \Big\}$, and the general solution is

$$C_1\, \mathrm{e}^{-4t/3} + C_2\, \mathrm{e}^{2t}$$

05e. The characteristic equation is $4\lambda^2 + 4\lambda + 1 = 0$. The roots are $\lambda_1 = \lambda_2 = -\frac{1}{2}$. Hence a fundamental set of solutions is $\Big\{ \mathrm{e}^{-t/2} \,,\ t\,\mathrm{e}^{-t/2} \Big\}$, and the general solution is

$$C_1\, \mathrm{e}^{-t/2} + C_2\, t\,\mathrm{e}^{-t/2}$$

05f. The characteristic equation is $4\lambda^2 - 12\lambda + 25 = 0$. The roots are $\lambda_1 = \dfrac{3}{2} + 2i$ and $\lambda_2 = \dfrac{3}{2} - 2i$. Hence a fundamental set of solutions is $\left\{ e^{3t/2} \cos(2t) \, , \, e^{3t/2} \sin(2t) \right\}$, and the general solution is

$$C_1 e^{3t/2} \cos(2t) + C_2 e^{3t/2} \sin(2t)$$

06a. The fact that $2e^{-5t} + 3e^{-2t}$ is a solution of the second order, linear, homogeneous, constant coefficients differential equation $y'' + by' + cy = 0$, implies that $\lambda_1 = -5$ and $\lambda_2 = -2$, are roots of the characteristic equation. Hence the characteristic equation is given by

$$(\lambda + 5)(\lambda + 2) = 0 \iff \lambda^2 + 7\lambda + 10 = 0$$

It follows that the differential equation, in normal form is $y'' + 7y' + 10y = 0$.

06b. The fact that $5e^{-2t} - 2$ is a solution of the second order, linear, homogeneous, constant coefficients differential equation $y'' + by' + cy = 0$, implies that $\lambda_1 = -2$ and $\lambda_2 = 0$, are roots of the characteristic equation. Hence the characteristic equation is given by

$$(\lambda + 2)(\lambda - 0) = 0 \iff \lambda^2 + 2\lambda = 0$$

It follows that the differential equation, in normal form is $y'' + 2y' = 0$.

06c. The fact that $e^{3t}\cos(2t)$, is a solution shows that the roots of the characteristic equation are $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$. Consequently the characteristic equation is

$$\big(\lambda - (3 + 2i)\big)\big(\lambda - (3 - 2i)\big) = 0 \iff \lambda^2 - 6\lambda + 13 = 0$$

Hence the differential equation (in normal form) is

$$y'' - 6y' + 13y = 0$$