Department of Mathematics and Statistics
MATH 375
Handout # 4 - ANSWERS, HINTS, SOLUTIONS
Second Order Linear Equations

1. Solve the initial value problem

$$\frac{d^2y}{dx^2} = e^x - \frac{1}{(1+x)^2}, \quad y(0) = 1, \quad y'(0) = -2.$$

   Hint: integrate twice
   Answer. $y = e^x + \ln(1+x) - 4x$

2. Prove that the following functions are linearly independent:

   a) e^{2x} and e^{-2x}
   b) e^{2x} and e^{5x}
   c) $\cos(3x)$ and $\sin(3x)$
   Solution. a) The Wronskian

$$\left| \begin{array}{cc} e^{2x} & e^{-2x} \\ (e^{2x})' & (e^{-2x})' \end{array} \right| = \left| \begin{array}{cc} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{array} \right| = e^{2x}(-2e^{-2x}) - e^{-2x}2e^{2x} = -4 \neq 0,$$

   so the functions are linearly independent. Similarly, in b) the Wronskian is $3e^{7x} \neq 0$ and in c) is $3 \neq 0$.

3. Two functions $y_1 = x$ and $y_2 = x^3$ are solutions of the differential equation

$$y'' + p(x)y' + q(x) = 0.$$

   Can $p(x)$ and $q(x)$ be continuous on $(-\infty, \infty)$? If not, what is the discontinuity point?
   Solution. The Wronskian is $2x^3$ which is distinct from zero everywhere but at $x = 0$. If the coefficients $p(x)$ and $q(x)$ were continuous everywhere then the Wronskian would be either zero everywhere or nonzero everywhere (the latter is satisfied if the interval does not include zero). Thus at least one of coefficients is discontinuous at $x = 0$.

4. Find the general solution of the following differential equations

   a) $y'' - 5y' + 6y = 0$
   Solution. $r^2 - 5r + 6 = 0 \Rightarrow r_1 = 3,\ r_2 = 2 \Rightarrow y = C_1e^{3x} + C_2e^{2x}$.
   b) $y'' - 25y = 0$
   Solution. $r^2 - 25 = 0 \Rightarrow r_1 = 5,\ r_2 = -5 \Rightarrow y = C_1e^{5x} + C_2e^{-5x}$.
   c) $y'' + y' = 0$
   Solution. $r^2 + r = r(r+1) = 0 \Rightarrow r_1 = 0,\ r_2 = -1 \Rightarrow y = C_1 + C_2e^{-x}$.
   d) $y'' + 2y' + y = 0$

Solution. $r^2+2r+1=(r+1)^2=0 \Rightarrow r_1=r_2=-1 \Rightarrow y=C_1e^{-x}+C_2xe^{-x}$.
e) $y''-6y'+9y=0$
Solution. $r^2-6r+9=0 \Rightarrow r_1=r_2=3 \Rightarrow y=C_1e^{3x}+C_2xe^{3x}$.
f) $y''+5y=0$
Solution. $r^2+5=0 \Rightarrow r=\pm\sqrt{5}i \Rightarrow y=C_1\cos(\sqrt{5}x)+C_2\sin(\sqrt{5}x)$.
g) $y''+2y'+5y=0$
Solution. $r^2+2r+5=0 \Rightarrow r=\frac{1}{2}(-2\pm\sqrt{-16})=-1\pm2\sqrt{-1}=-1\pm2i \Rightarrow y=e^{-x}(C_1\cos 2x+C_2\sin 2x)$.
h) $y''-y'+y=0$
Solution. $r^2-r+1=0 \Rightarrow r=\frac{1}{2}(1\pm\sqrt{1-4})=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i \Rightarrow y=e^{0.5x}[C_1\cos(\frac{\sqrt{3}}{2}x)+C_2\sin(\frac{\sqrt{3}}{2}x)]$.

5. Find the solution of $y''+y=0$ satisfying $y(\frac{\pi}{2})=1$, $y'(\frac{\pi}{2})=0$.
Solution. $r^2+1=0 \Rightarrow r=\pm i$, thus the general solution is $y=C_1\cos x+C_2\sin x$. If $x=\frac{\pi}{2}$, $y=1$ then $1=C_2$; $y'=-C_1\sin x+C_2\cos x$, so $x=\frac{\pi}{2}$, $y'=0$ implies $0=-C_1+0 \Rightarrow C_1=0$,$C_2=1$. Thus $y=\sin x$.

6. Find an equation $y''+by'+cy=0$, such that it has following solutions:
a) $y_1=e^x$, $y_2=e^{-x}$
Solution. The roots of the characteristic equation are $r_1=1, r_2=-1$, so the characteristic equation is $(r-r_1)(r-r_2)=0$, hence $(r-1)(r+1)=r^2-1=0$. Consequently, the differential equation is $y''-y=0$.
b) $y_1=e^{2x}$, $y_2=xe^{2x}$
Solution. The roots of the characteristic equation are $r_1=r_2=2$, so the characteristic equation is $(r-2)^2=r^2-4r+4=0$, thus the differential equation is $y''-4y'+4y=0$.

7. For which c all solutions of the equation $y''+cy=0$ are periodic?
Solution. If $c<0$, then the solutions of the characteristic equation are $r_{1,2}=\pm\sqrt{c}$ and the general solution is $y=C_1e^{\sqrt{c}x}+C_2e^{-\sqrt{c}x}$, which is not periodic. If $c=0$, then the solutions of the characteristic equation are $r_{1,2}=0$ and the general solution is $y=C_1+C_2x$ which is also not periodic for any C_1, C_2. If $c>0$, then the solutions of the characteristic equation are $r_{1,2}=\pm\sqrt{c}i$ and the general solution is $y=C_1\cos(\sqrt{c}x)+C_2\sin(\sqrt{c}x)$ which is periodic for any C_1, C_2. So all solutions of the equation $y''+cy=0$ are periodic for $c>0$.

8. Find the general solution of the following differential equations
a) $y''-2y'-3y=e^{4x}$
Solution. Since $r^2-2r-3=0 \Rightarrow r=3,-1$, then the general solution of the homogeneous equation is $y_{hom}=C_1e^{3x}+C_2e^{-x}$. The particular solution of the non-homogeneous equation has the form $y_{part}=Ae^{4x}$. Let us find A by substitution. $y'=4Ae^{4x}, y''=16Ae^{4x}$, thus

$$16Ae^{4x}-8Ae^{4x}-3Ae^{4x}=5Ae^{4x}=e^{4x} \Rightarrow A=\frac{1}{5}$$

$$\Rightarrow y = y_{hom} + y_{part} = C_1 e^{3x} + C_2 e^{-x} + \frac{1}{5} e^{4x}.$$

b) $y'' - y = -x^2$

Solution. Since $r^2 - 1 = 0 \Rightarrow r = \pm 1$, then the general solution of the homogeneous equation is $y_{hom} = C_1 e^x + C_2 e^{-x}$. The particular solution of the nonhomogeneous equation has the form $y_{part} = Ax^2 + Bx + C$. Let us find A, B, C by substitution. $y' = 2Ax + B$, $y'' = 2A$, thus

$$2A - Ax^2 - Bx - C = -x^2 \Rightarrow -A = -1, -B = 0, 2A - C = 0 \Rightarrow A = 1, C = 2$$

$$\Rightarrow y = y_{hom} + y_{part} = C_1 e^x + C_2 e^{-x} + x^2 + 2.$$

c) $y'' + 3y' - 4y = xe^{-x}$

Solution. Since $r^2 + 3r - 4 = 0 \Rightarrow r = -4, 1$, then the general solution of the homogeneous equation is $y_{hom} = C_1 e^{-4x} + C_2 e^x$. The particular solution of the nonhomogeneous equation has the form $y_{part} = (Ax+B)e^{-x}$. Let us find A, B by substitution. $y' = (A - Ax - B)e^{-x}$, $y'' = (Ax + B - 2A)e^{-x}$, thus

$$e^{-x}(Ax+B-2A+3A-3Ax-3B-4Ax-4B) = e^{-x}(-6Ax+A-6B) = xe^{x} \Rightarrow -6A = 1,$$

$$A - 6B = 0 \Rightarrow A = -\frac{1}{6}, B = \frac{1}{6}A = -\frac{1}{36},$$

$$y = y_{hom} + y_{part} = C_1 e^{-4x} + C_2 e^x - \left(\frac{1}{6}x + \frac{1}{36}\right) e^{-x}.$$

d) $y'' - 4y' + 3y = 20\cos 2x + 35\sin 2x$

Solution. Since $r^2 - 4r + 3 = 0 \Rightarrow r = 1, 3$, then the general solution of the homogeneous equation is $y_{hom} = C_1 e^x + C_2 e^{3x}$. The particular solution of the nonhomogeneous equation has the form $y_{part} = A\cos 2x + B\sin 2x$. Let us find A, B by substitution. $y' = -2A\sin 2x + 2B\cos 2x$, $y'' = -4A\cos 2x - 4B\sin 2x$, thus
$(-4A\cos 2x - 4B\sin 2x + 8A\sin 2x - 8B\cos 2x + 3A\cos 2x + 3B\sin 2x)$
$= 20\cos 2x + 35\sin 2x \Rightarrow -A - 8B = 20, -B + 8A = 35 \Rightarrow -8A - 64B = 160$, the sum is $-65B = 195$, thus $B = -3$ and $A = -8B - 20 = 24 - 20 = 4$. Consequently,

$$y = y_{hom} + y_{part} = C_1 e^x + C_2 e^{3x} + 4\cos 2x - 3\sin 2x.$$

e) $y'' - 4y' + 4y = 2\sin 2x + x$

Solution. Since $r^2 - 4r + 4 = 0 \Rightarrow r = 2$, then the general solution of the homogeneous equation is $y_{hom} = C_1 e^{2x} + C_2 x e^{2x}$. The particular solution is a sum of two, one for the right hand side $2\sin x$, which has the form $y_1 = A\cos 2x + B\sin 2x$, $y' = -2A\sin 2x + 2B\cos 2x$, $y'' = -4A\cos 2x - 4B\sin 2x$, the substitution gives

$$-4A\cos 2x - 4B\sin 2x + 8A\sin 2x - 8B\cos 2x + 4A\cos 2x + 4B\sin 2x = 2\sin 2x$$

$\Rightarrow -8B = 0, 8A = 2 \Rightarrow A = \frac{1}{4}, B = 0 \Rightarrow y_1 = \frac{1}{4}\cos 2x$, and another for the right hand side x, which has the form $y_2 = Cx + D$, $y' = C$, $y'' = 0$, the substitution gives $-4C + 4Cx + 4D = x \Rightarrow 4C = 1, 4D - 4C = 0 \Rightarrow C = D = \frac{1}{4}$, thus $y_2 = \frac{1}{4}x + \frac{1}{4}$ and

$$y = y_{hom} + y_{part} = y_{hom} + y_1 + y_2 = C_1e^{2x} + C_2xe^{2x} + \frac{1}{4}\cos 2x + \frac{1}{4}x + \frac{1}{4}.$$

f) $y'' - y = 2e^x$

Solution. Since $r^2 - 1 = 0 \Rightarrow r = \pm 1$, the general solution of the homogeneous equation is $y_{hom} = C_1e^x + C_2e^{-x}$. Since $r = 1$ is a solution of the characteristic equation with multiplicity one, then we look for the particular solution in the form $y_{part} = Axe^x$, then $y' = (A + Ax)e^x$, $y'' = (2A + Ax)e^x$. The substitution gives

$$(2A + Ax - Ax)e^x = 2Ae^x = 2e^x \Rightarrow A = 1 \Rightarrow y = y_{hom} + y_{part} = C_1e^x + C_2e^{-x} + xe^x.$$

9. Find the solution of $y'' + y = -\sin 2x$ satisfying $y(\pi) = 1$, $y'(\pi) = 1$.

Solution. $r^2 + 1 = 0$ implies $r = \pm i$; the particular solution of the equation $y'' + y = -\sin 2x$ has the form $y = A\cos 2x + B\sin 2x$, so $y'' = -2A\cos 2x - 2B\sin 2x$, thus

$$-4A\cos 2x - 4B\sin 2x + A\cos 2x + B\sin 2x = -\sin 2x \Rightarrow -3A = 0, -3B = -1$$

$$\Rightarrow y = \frac{1}{3}\sin 2x,$$

the general solution is $y = C_1\cos x + C_2\sin x + \frac{1}{3}\sin 2x$, $y' = -C_1\sin x + C_2\cos x + \frac{2}{3}\cos 2x$, after substituting $y(\pi) = 1$, $y'(\pi) = 1$ we have $1 = C_1(-1), 1 = -C_2 + \frac{2}{3} \Rightarrow C_1 = -1$, $C_2 = -\frac{1}{3}$. So $y = -\cos x - \frac{1}{3}\sin x + \frac{1}{3}\sin 2x$.

10. Find the general solution of the following equations

a)$y'' + 2y' + y = \frac{1}{x}e^{-x}$

Answer. $y = C_1e^{-x} + C_2xe^{-x} + x\ln|x|e^{-x}$.

Solution. Since the solutions of the characteristic equation r^2+2r+1 are $r_{1,2} = r = -1$ and the general solution of the homogeneous equation is $y = C_1y_1 + C_2y_2 = C_1e^{-x} + C_2xe^{-x}$, then we will look for the general solution in the form $y = C_1(x)y_1 + C_2(x)y_2$. The method of variation of parameters leads to the system

$$C_1'y_1 + C_2'y_2 = 0, C_1'y_1' + C_2'y_2' = \frac{1}{x}e^{-x}.$$

In our case $y_1' = -e^{-x}$, $y_2' = (1 - x)e^{-x}$, so the system is

$$C_1'e^{-x} + C_2'xe^{-x} = 0, -C_1'e^{-x} + C_2'(1 - x)e^{-x} = \frac{1}{x}e^{-x},$$

thus after the addition of these equations we have $C_2'e^{-x} = \frac{1}{x}e^{-x} \Rightarrow C_2'(x) = \frac{1}{x} \Rightarrow C_2(x) = \ln|x| + C_2$. Then, from the first equation, $C_1'e^{-x} = -C_2'xe^{-x}$, thus $C_1' =$

$-C_2'x = -1 \Rightarrow C_1(x) = -x + C_1$. Consequently, $y = C_1(x)y_1 + C_2(x)y_2 = (C_1 - x)e^{-x} + (C_2 + \ln|x|)xe^{-x}$

$$= C_1e^{-x} + C_2xe^{-x} - xe^{-x} + x\ln|x|e^{-x} = C_1e^{-x} + \bar{C}_2xe^{-x} + x\ln|x|e^{-x}$$

b)$y'' - 4y' + 5y = \frac{1}{\cos x}e^{2x}$
Answer. $y = C_1e^{2x}\cos x + C_2e^{2x}\sin x + \ln|\cos x|e^{2x}\cos x + xe^{2x}\sin x$.
Solution. Since the solutions of the characteristic equation $r^2 - 4r + 5$ are $r_{1,2} = \frac{1}{2}(4 \pm \sqrt{16-20}) = 2 \pm i$ and the general solution of the homogeneous equation is $y = C_1y_1 + C_2y_2 = C_1e^{2x}\cos x + C_2e^{2x}\sin x$, then we will look for the general solution in the form $y = C_1(x)y_1 + C_2(x)y_2$, where the system

$$C_1'y_1 + C_2'y_2 = 0, C_1'y_1' + C_2'y_2' = \frac{1}{\cos x}e^{2x}$$

is satisfied. In our case $y_1 = e^{2x}\cos x, y_2 = e^{2x}\sin x$, $y_1' = 2e^{2x}\cos x - e^{2x}\sin x$, $y_2' = 2e^{2x}\sin x + e^{2x}\cos x$. Thus the system has the form

$$(C_1'\cos x + C_2'\sin x)e^{2x} = 0, \quad [C_1'(2\cos x - \sin x) + C_2'(2\sin x + \cos x)]\,e^{2x} = \frac{1}{\cos x}e^{2x},$$

since $e^{2x} > 0$, this is equivalent to

$$C_1'\cos x + C_2'\sin x = 0, \quad C_1'(2\cos x - \sin x) + C_2'(2\sin x + \cos x) = \frac{1}{\cos x}.$$

Let us multiply the first equation by two and subtract the second equation:

$$C_1'\sin x - C_2'\cos x = -\frac{1}{\cos x}, \quad C_1'\cos x + C_2'\sin x = 0.$$

We multiply the first equation by $\cos x$, the second equation by $\sin x$ and add:
$-C_2'\cos^2 x - C_2'\sin^2 x = -1 \Rightarrow C_2' = 1 \Rightarrow C_2(x) = C_2 + x$. By the second equation

$$C_1' = -\frac{1}{\cos x}C_2'\sin x = -\frac{\sin x}{\cos x} \Rightarrow C_1(x) = C_1 + \ln|\cos x|.$$

Consequently,$y = C_1(x)y_1 + C_2(x)y_2 = (C_1 + \ln|\cos x|)e^{2x}\cos x + (C_2 + x)e^{2x}\sin x$

$$= C_1e^{2x}\cos x + C_2e^{2x}\sin x + \ln|\cos x|e^{2x}\cos x + xe^{2x}\sin x.$$

11. Find b and c such that, for some real A, the function $y = Ax^2e^{4x}$ is a solution of the differential equation

$$y'' + by' + cy = 5e^{4x}.$$

Solution. The right hand side is e^{4x} and the particular solution is Ax^2e^{4x} which can be only in the case when both roots of the characteristic equation are $r_1 = r_2 = r = 4$, or the characteristic equation is $(r-4)^2 = r^2 - 8r + 16 = 0$, thus $b = -8$, $c = 16$.

12. Find the Wronskian for the two solutions y_1 and y_2 of the equation

$$y'' + \frac{1}{1+t} y'(t) + q(t)y(t) = 0,$$

where $q(t)$ is continuous for $t \geq 0$, corresponding to the initial values
a) $y_1(0) = 2,\ y_1'(0) = -3, \quad y_2(0) = -1,\ y_2'(0) = 2$
b) $y_1(0) = 2,\ y_1'(0) = -3, \quad y_2(0) = -4,\ y_2'(0) = 6$
Solution. a) The Wronskian at $t = 0$ is

$$W[y_1, y_2](0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} = 2 \cdot 2 - (-1)(-3) = 1.$$

By Abel's theorem,

$$W[y_1, y_2](t) = W[y_1, y_2](0) e^{\int_0^t -p(s)\ ds}$$

$$= W[y_1, y_2](0) e^{\int_0^t -ds/(1+s)} = 1 \cdot e^{-\ln(1+t)+0} = \frac{1}{1+t}.$$

b) In b), $W[y_1, y_2](0) = 0$, the solutions are linearly dependent and $W[y_1, y_2](t) = 0$ for any t.

13. Find p such that the Wronskian for the two solutions y_1 and y_2 of the equation

$$y'' + p(t)y'(t) + qy(t) = 0,$$

where q is a constant, corresponding to the initial values $y_1(0) = 1,\ y_1'(0) = 0,\ \ y_2(0) = 0,\ y_2'(0) = 1$ is

$$W[y_1, y_2](t) = e^{7t^2}.$$

Solution. $W[y_1, y_2](0) = 1$, so by Abel's theorem,

$$-\int_0^t p(s)\ ds = 7t^2,$$

by the Fundamental Theorem of Calculus (we differentiate both parts), $p(t) = -14t$.

14. Find all solutions of the following boundary value problems for $y(t)$:
a) $y'' + 16y = 0,\ y(0) = 0,\ y\left(\frac{\pi}{8}\right) = 0$
b) $y'' + 16y = 0,\ y(0) = 0,\ y\left(\frac{\pi}{4}\right) = 0$
c) $y'' + y = 0,\ y(0) = 0,\ y(\pi) = 2$
d) $y'' + y = 0,\ y'(0) = 0,\ y'(\pi) = 0$
Solution. a) The solutions of the characteristic equation $r^2 + 16 = 0$ are $\pm 4i$, the general solution is $y(t) = C_1 \cos(4t) + C_2 \sin(4t)$. Thus $y(0) = C_1 = 0$ implies $y(t) = C_2 \sin(4t)$. Substituting $y(\pi/8) = C_2 \sin(\pi/2) = C_2 = 0$, we get $C_1 = C_2 = 0$, the boundary value problem has only the trivial solution.

b) The general solution is $y(t) = C_1 \cos(4t) + C_2 \sin(4t)$. Thus $y(0) = C_1 = 0$ implies $y(t) = C_2 \sin(4t)$. Substituting $y(\pi/4) = C_2 \sin(\pi) = 0$, we get that C_2 can be any. So $y(t) = C_2 \sin(4t)$ for any $C_2 \in \mathbb{R}$.
c) The general solution is $y(t) = C_1 \cos(t) + C_2 \sin(t)$, $y' = -C_1 \sin(t) + C_2 \cos(t)$. Thus $y'(0) = C_2 = 0$ implies $y(t) = C_1 \cos(t)$. Substituting $y'(\pi) = -C_1 \sin(\pi) = 0 \neq 2$. The boundary value problem has no solutions.
d) The general solution is $y(t) = C_1 \cos(t) + C_2 \sin(t)$, $y' = -C_1 \sin(t) + C_2 \cos(t)$. Thus $y'(0) = C_2 = 0$ implies $y(t) = C_1 \cos(t)$. Substituting $y'(\pi) = -C_1 \sin(\pi) = 0$, we conclude that the boundary value problem has solutions $y(t) = C_1 \cos(t)$ for any $C_1 \in \mathbb{R}$.

15. a) Check that $y_1 = \cosh(kt)$ and $\sinh(kt)$ are solutions of the equation $y'' - k^2 y = 0$.
b) Solve the boundary value problem $y'' - 9y = 0$, $y(0) = 0$, $y\left(\frac{\pi}{3}\right) = 1$
c) Solve the boundary value problem $y'' - 25y = 0$, $y'(0) = 0$, $y'(3) = 10$
Solution. a) Recall that the characteristic equation $r^2 - k^2 = 0$ has the roots $\pm k$, e^{kx} and e^{-kt} are solutions and $\cosh(kt) = \dfrac{e^{kx} + e^{-kt}}{2}$, $\sinh(kt) = \dfrac{e^{kx} - e^{-kt}}{2}$. The general solution can be written in the form $y(t) = C_1 \cosh(kt) + C_2 \sinh(kt)$, where $\sinh(0) = 0$, $\cosh(0) = 1$, $\cosh'(kt) = k \sinh(kt)$, $\sinh'(kt) = k \cosh(kt)$.
b) The general solution can be written in the form $y(t) = C_1 \cosh(3t) + C_2 \sinh(3t)$, so $y(0) = C_1 \cosh(0) + C_2 \sinh(0) = C_1 = 0$, $y\left(\frac{\pi}{3}\right) = C_2 \sinh\left(\frac{\pi}{3}\right) = 1$, and $C_2 = 1/\sinh(\pi/3)$. The solution is $y = \dfrac{\sinh(3t)}{\sinh(\pi/3)}$.
c) The general solution can be written in the form $y(t) = C_1 \cosh(5t) + C_2 \sinh(5t)$, $y'(t) = 5C_1 \sinh(5t) + 5C_2 \cosh(5t)$, $y'(0) = 5C_1 \sinh(0) + 5C_2 \cosh(0) = 5C_2 = 0$, thus $C_2 = 0$. Substituting $y'(t) = 5C_1 \sinh(15) = 10$, we get $C_1 = 2/\sinh(15)$ and the solution is $y = \dfrac{2\cosh(5t)}{\sinh(15)}$.