UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAM
AMAT 307 Fall 2014

DATE: December 9 **Time: 2 hours**

Student ID Number:	Last Name:	First Name:

There are 25 questions.
The last page is a table of Laplace transforms.

EXAMINATION RULES

1. This is a closed book examination.
2. A Schulich calculator is allowed for this test, **no other aids are allowed**.
3. Your answers must be entered on the scantron sheet
4. Scantron sheets must be filled out during the exam time limit. No additional time will be granted to fill in scantron form.
5. The use of personal electronic or communication devices is prohibited.
6. A University of Calgary Student ID card is required to write the Final Examination and could be requested for midterm examinations. If adequate ID isn't present the student must complete an Identification Form.
7. Students late in arriving will not be permitted after one-half hour of the examination time has passed.
8. No student will be permitted to leave the examination room during the first 30 minutes, nor during the last 15 minutes of the examination. Students must stop writing and hand in their exam immediately when time expires.
9. All inquiries and requests must be addressed to the exam supervisor.
10. Students are strictly cautioned against:

a. communicating to other students;
b. leaving answer papers exposed to view;
c. attempting to read other students' examination papers

11. Once the examination has been handed in for marking, a student cannot request that the examination be cancelled. Retroactive withdrawals from the course will be denied.
12. Failure to comply with these regulations will result in rejection of the examination paper.

1. The solution to the initial value problem $y' + y = e^{-t} \cos t, \quad y(0) = 3$, is

A) $3e^{-t} + e^{-t} \sin t$

B) $3e^{-t} - e^{-t} \sin t$

C) $e^{-t} + 2e^{-t} \cos t$

D) $2e^{t} + e^{t} \cos t$

E) $3e^{t} + e^{t} \sin t$

2. A tank contains 100 litres of water. A solution with salt concentration 0.004 kg per litre starts flowing into the tank at the rate 6 litres per minute, and well stirred mixture flows out of tank at the same rate 6 litres per minute. The amount of salt in the tank in the limit (as time grows and tends to infinity)

A) equals 0.1 kg

B) equals 0.2 kg

C) equals 0.3 kg

D) equals 0.4 kg

E) depends on the initial amount of salt in the tank

3. Consider the exact equation $(2x - y)dx + (2y - x)dy = 0, \quad y(1) = 3$. The solution is

A) $2x^2 - xy + y^2 = 8$

B) $x^2 - 2xy + y^2 = 4$

C) $x^2 + xy + y^2 = 13$

D) $x^2 - xy + y^2 = 7$

E) $x^2 - xy + 2y^2 = 16$

4. Apply Euler's approximation method once to the initial value problem $y' = 2\cos(ty)$, $y(0) = 1$, to find the approximate value $y(t_1) = y(h) \approx 1.1$. The step size h is

A) $h = 0.05$
B) $h = 0.1$
C) $h = 0.5$
D) $h = 0.55$
E) $h = 1$

5. The solution to the initial value problem: $y^{(4)} - 5y'' + 4y = 0$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$, $y^{(3)}(0) = 0$ is given by:

A) $e^{-x} - e^{x} + \frac{1}{2}e^{-3x} + \frac{1}{2}e^{-3x}$
B) $e^{-x} - e^{x} + \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$
C) $\frac{2}{3}e^{-x} + \frac{2}{3}e^{x} - \frac{1}{6}e^{2x} - \frac{1}{6}e^{-2x}$
D) $\frac{1}{3}e^{-x} + \frac{1}{3}e^{x} + \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$
E) None of the above

6. The solution to the initial value problem: $x^{(3)} - x' = t$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$ is given by:

A) $\cos(t) - \sin(t) + t^2e^t$
B) $\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - \frac{1}{2}t^2$
C) $2e^{-t} - e^{t} + t^2$
D) $e^{-t} - e^{t} + 1 - \frac{1}{2}t^2$
E) None of the above

7. The smallest order m of a homogeneous real constant coefficient linear ordinary differential equation with a solution $x(t) = 5t^2 - \ln(10)e^{2t}$ is

A) $m = 2$

B) $m = 3$

C) $m = 4$

D) $m = 5$

E) There is no such homogeneous equation

8. Let y_1, y_2 and y_3 be three solutions of the differential equation $ty^{(3)} + t\sin(t)y'' + y' + \ln(t)y = 0$, $t > 0$. Assume that the Wronskian of y_1, y_2 and y_3 at $t = \dfrac{\pi}{2}$ is equal to 5; $W(y_1, y_2, y_3)(\frac{\pi}{2}) = 5$. Then $W(y_1, y_2, y_3)(\pi)$ is equal to:

A) $5e^{1-\pi}$

B) $-5e^{-1}$

C) $5e$

D) $-e^{-1}$

E) $5e^{-1}$

9. A particular solution of the differential equation: $y'' + y = \sec(t)$, $0 < t < \frac{\pi}{2}$, is given by:

A) $\cos(t)\ln(\cos(t)) + t\sin(t)$

B) $\cos(t)\ln(\cos(t)) + \sin(t)$

C) $-\cos(t)\ln(\cos(t)) - t\sin(t)$

D) $-\cos(t)\ln(\cos(t)) + t\sin(t)$

E) None of the above

10. According to the method of undetermined coefficients, a particular solution to the equation $y'' - 4y' + 5y = te^{2t}\sin(t) + e^{2t}$ is of the following form, where $A_0, A_1, B_0, B_1, C_0, C_1$ are real constants:

A) $(A_0t + A_1)e^{2t}\cos(t) + (B_0t + B_1)e^{2t}\sin(t) + C_0e^{2t}$

B) $(A_0t + A_1)e^{2t}\cos(t) + (B_0t + B_1)e^{2t}\sin(t) + (C_0t + C_1)e^{2t}$

C) $A_0te^{2t}\cos(t) + B_0te^{2t}\sin(t) + C_0e^{2t}$

D) $A_0t^2e^{2t}\cos(t) + B_0t^2e^{2t}\sin(t) + C_0e^{2t}$

E) $(A_0t^2 + A_1t)e^{2t}\cos(t) + (B_0t^2 + B_1t)e^{2t}\sin(t) + C_0e^{2t}$

11. The equation $y^{(4)} - 2y^{(2)} + y = te^{-t} + e^t\cos(t)$ has a particular solution to the equation is of the following form, where A_0, A_1, B_0, B_1, C_0 are real constants:

A) $(A_0t^2 + A_1t)e^{-t} + B_0t^2e^t\cos(t) + C_0t^2e^t\sin(t)$

B) $(A_0t^2 + A_1t)e^{-t} + (B_0t + B_1)e^t\cos(t) + (C_0t + C_1)e^t\sin(t)$

C) $(A_0t + A_1)e^{-t} + B_0e^t\cos(t) + C_0e^t\sin(t)$

D) $(A_0t^3 + A_1t^2)e^{-t} + B_0e^t\cos(t) + C_0e^t\sin(t)$

E) $(A_0t^3 + A_1t^2)e^{-t} + B_0t^2e^t\cos(t) + C_0t^2e^t\sin(t)$

12. Suppose $e^t\cos(2t) - 2e^{-t}$ is a solution of $y''' + ay'' + by' + cy = 0$, where a, b and c are real constants, then

A) $a = -3,\ b = 7,\ c = -5$

B) $a = -1,\ b = 3,\ c = 5$

C) $a = 1,\ b = 3,\ c = -5$

D) $a = 3,\ b = 7,\ c = 5$

E) None of the above

13. The **largest** interval on which a unique solution to $\mathbf{y}' = \begin{bmatrix} t-1 & (t+2) \\ (t^2+9)^{-1} & (t-2)^{-1} \end{bmatrix} \mathbf{y} + \begin{bmatrix} (t+1)^{-1} \ln |t| \\ 1 \end{bmatrix}$,

$\mathbf{y}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is **guaranteed** to exist is

A) $(-1, 2)$
B) $(-3, 3)$
C) $(0, 2\ )$
D) $(-2, 2)$
E) The empty set

14. A 2×2 matrix $\mathbf{A}$ has an eigenvalue $2i$ with eigenvector $\begin{bmatrix} i \\ -1 \end{bmatrix}$. A particular solution $\mathbf{y}_p$ to the system

$\mathbf{y}' = \mathbf{A}\mathbf{y} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is

A) $\mathbf{y}_p(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

B) $\mathbf{y}_p(t) = \begin{bmatrix} \cos^2(2t) - \sin^2(2t) \\ 0 \end{bmatrix}$

C) $\mathbf{y}_p(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

D) $\mathbf{y}_p(t) = \begin{bmatrix} -2\sin(2t)\cos(2t) \\ 1 \end{bmatrix}$

E) None of the above

15. Which of the following linear differential equations can be written as the linear system $\mathbf{x}' = \mathbf{A}x$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cos(t) & -t & \sin(t) \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$ for some function $y = y(t)$?

A) $y''' - \cos(t)y'' + ty' - \sin(t)y = 0$

B) $y''' + \cos(t)y'' - ty' + \sin(t)y = 0$

C) $y''' - \sin(t)y'' + ty' - \cos(t)y = 0$

D) $y''' - \sin(t)y'' - ty' + \cos(t)y = 0$

E) $y''' + \sin(t)y'' - ty' + \cos(t)y = 0$

16. Solve $\mathbf{x}' = \mathbf{A}x$ where $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$ where you already know $\mathbf{A}$ has eigenvalues $\{1, 1, 2\}$, and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are eigenvectors for 1 and 2 respectively. The unique solution $\mathbf{x}(t)$ is

A) $\begin{bmatrix} 2te^t + 3e^{2t} \\ 2e^t - 4te^t + 3e^{2t} \\ -e^t - 4te^t + 3e^{2t} \end{bmatrix}$

B) $\begin{bmatrix} e^t + 6te^t - e^{2t} \\ 6e^t - e^{2t} \\ 3e^t - e^{2t} \end{bmatrix}$

C) $\begin{bmatrix} -2e^t + 5te^t + e^{2t} \\ 5e^{2t} \\ 2e^{2t} \end{bmatrix}$

D) $\begin{bmatrix} e^t - e^{2t} \\ 6e^t - e^{2t} \\ 3e^t - e^{2t} \end{bmatrix}$

E) None of the above

17. The Laplace transform of $u_2(t)(t^2-4)$ is

A) $e^{-2s}(\frac{2}{s^3}-\frac{4}{s})$

B) $e^{-2s}(\frac{2}{s^3}-\frac{4}{s^2})$

C) $e^{-2s}(\frac{2}{s^3}+\frac{4}{s})$

D) $e^{-2s}(\frac{2}{s^3}+\frac{4}{s^2})$

E) None of the above

18. The inverse Laplace transform of $\frac{1}{s(s^2+1)}$ is

A) $1+\sin(t)$

B) $1-\sin(t)$

C) $1+\cos(t)$

D) $1-\cos(t)$

E) $\cos(t)-\sin(t)$

19. The inverse Laplace transform of $e^{-2s}\left(\frac{s+6}{s^3-3s^2}\right)$ is

A) $u_2(t)[-5-2t+e^{3(t+2)}]$

B) $u_2(t)[-1-2t+e^{3t}]$

C) $u_2(t)[3-2t+e^{3(t-2)}]$

D) $e^{-2t}[3-2t+e^{3(t-2)}]$

E) $e^{-2t}[-1-2t+e^{3(t-2)}]$

20. Using the functions $u_c(t)$ the following piece-wise continuous function

$$f(t) = \begin{cases} -1, & t < 2 \\ t + \sin t - 1, & 2 \le t < 5 \\ t^3 + t + \sin t, & 5 \le t \end{cases}$$

can be re-written as

A) $f(t) = -1 + u_2(t)[t + \sin t] + u_5(t)[t^3 + 1]$

B) $f(t) = -1 + u_2(t)[t + \sin t - 1] + u_5(t)[t^3 + 1]$

C) $f(t) = u_2(t)[t + \sin t - 1] + u_5(t)[t^3 + 1]$

D) $f(t) = -1 + u_2(t)[t + \sin t - 1] + u_5(t)[t^3 + t + \sin t]$

E) $f(t) = u_2(t)[t + \sin t - 1] + u_5(t)[t^3 + t + \sin t]$

21. If $F(s)$ is the Laplace transform of the function

$$f(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 3, & t \ge 2 \end{cases}$$

then

A) $F(s) = \dfrac{-1 + 2e^{-s} + 2e^{-2s}}{s - 1}$

B) $F(s) = \dfrac{-1 + e^{-s} + 3e^{-2s}}{s}$

C) $F(s) = \dfrac{-1 + 2e^{-s} + 2e^{-2s}}{s}$

D) $F(s) = \dfrac{-1 + e^{-s} + 2e^{-2s}}{s + 1}$

E) $F(s) = \dfrac{1 - 2e^{-s} - 2e^{-2s}}{s}$

22. The largest open interval on which the power series $\sum_{n=1}^{\infty} \frac{2^n x^n}{(n+2)!}$ converges is

A) $(-\frac{1}{2}, \frac{1}{2})$
B) $(-1, 1)$
C) $(-2, 2)$
D) $(-\infty, \infty)$
E) There is no interval, the series converges for $x = 0$ only

23. The largest open interval on which the power series $\sum_{n=0}^{\infty} \frac{(x-3)^{2n}}{4^n}$ converges is

A) $(-\infty, \infty)$
B) $(-4, 4)$
C) $(-2, 2)$
D) $(-1, 7)$
E) $(1, 5)$

24. For the series solution $\sum_{n=0}^{\infty} a_n t^n$ of the equation $y'' + (t-1)y' + 3y = 0$ the recurrence relation satisfied by the coefficients $a_0, a_1, a_2, a_3, \ldots$ is

A) $n(n-1)a_n + (t-1)a_n + 3a_n = 0$

B) $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + (n+3)a_n = 0$

C) $(n+2)(n+1)a_n - a_{n+1} + (n+3)a_n = 0$

D) $(n+2)(n+1)a_{n+2} - na_{n+1} + 3a_n = 0$

E) $(n+2)(n+1)a_n - (n+1)a_n + (n+3)a_n = 0$

25. If $y = \sum_{k=0}^{\infty} a_k t^k$ then the first terms $y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$ of the solution to the initial value problem $y'' + ty' + 3y = 1 + 5t$, $y(0) = 1$, $y'(0) = 2$ are given by

A) $y(t) = 1 + 2t + 0t^2 + 3t^3 + \ldots$

B) $y(t) = 1 + 2t - \dfrac{3}{2}t^2 - \dfrac{4}{3}t^3 + \ldots$

C) $y(t) = 1 + 2t + \dfrac{3}{2}t^2 + \dfrac{4}{3}t^3 + \ldots$

D) $y(t) = 1 + 2t + t^2 + \dfrac{1}{2}t^3 + \ldots$

E) $y(t) = 1 + 2t - t^2 - \frac{1}{2}t^3 + \ldots$

Extra workspace

Laplace Transforms

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1	$1/s$
t^n	$n!/s^{n+1}$
$t^n e^{at}$	$n!/(s-a)^{n+1}$
$\sin(at)$	$a/(s^2 + a^2)$
$\cos(at)$	$s/(s^2 + a^2)$
$\sinh(at)$	$a/(s^2 - a^2)$
$\cosh(at)$	$s/(s^2 - a^2)$
$e^{at} f(t)$	$F(s-a)$
$u_a(t)$	e^{-as}/s
$u_a(t) f(t-a)$	$e^{-as} F(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$
$f(at)$	$(1/a)F(s/a)$

Notes: Here $n = 0,1,2,\ldots$

The function $u_a(t) = h(t-a)$ where $h(t)$ is the Heaviside function