Math 375 **Fall 2016**

Linear Systems of First Order Differential Equations

Worksheet # 4 **November 21-25**

The problems on this worksheet refer to material from sections §§7.1, and, 7.4 of our text.
Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Matrix Form

01. Express each of the following system in matrix form $\vec{Y}' = \boldsymbol{Q}(t)\,\vec{Y} + \vec{F}(t)$

a. $\begin{cases} y_1' = y_1 + (2t+1)\,y_2 + \dfrac{1}{t^2+1} \\ y_2' = t\,y_1 + \tan(t)\,y_2 + \cosh(t) \end{cases}$ **b*.** $\begin{cases} y_1' = 2\,y_1 + t\,y_2 - 3\,y_3 + t \\ y_2' = -y_1 + \cos(t)\,y_2 + \sec(t) \\ y_3' = t\,y_1 + 4\,y_3 + \ln(t) \end{cases}$

02. Rewrite each of the differential equations as a first order linear system

a. $t\,y'' - 2\,y' + (1 - \mathrm{e}^t)\,y = \sin(t)$ **b*.** $y''' - t\,y'' - \mathrm{e}^t\,y' + y = \ln(t)$

Existence and Uniqueness

03*. Consider the initial value problem $\begin{cases} y_1' = t\,y_1 + 2\,y_2 + \ln(5-t) \\ y_2' = 3\,y_1 - \dfrac{t}{t-1}\,y_2 + \csc(t) \\ y_1(t_0) = 1 \text{ and } y_2(t_0) = -1 \end{cases}$

For each of the following cases, find the largest open interval where the solution to the initial value problem is garanteed to be defined.

a. $t_0 = -1$ **b.** $t_0 = 2$ **c.** $t_0 = 4$

04. Find the largest interval (a,b) such that a unique solution to the initial value problem $\begin{cases} (t+1)^2\,y_1' &= \cos(t)\,y_1 + y_2 + 2 \\ \sin(t)\,y_2' &= \cos(t)\,y_1 + y_2 + \sec t \\ y_1(1) = 3 & \& \quad y_2(1) = 2 \end{cases}$, is guaranteed to exist.

Simple real eigenvalues

05. The coefficient matrix of the system $\begin{cases} y_1' = -2\,y_1 + y_2 \\ y_2' = y_1 - 2\,y_2 \end{cases}$ has eigenvalues $\lambda_1 = -3$, $\lambda_2 = -1$, and corresponding eigenvectors $\vec{V}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Write down the general solution of the system, then find the solution $\vec{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ that satisfies $\vec{Y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

06. Solve the initial value problem $\begin{cases} \vec{Y}' = \boldsymbol{A}\,\vec{Y} \\ \vec{Y}(0) = \vec{Y}_0 \end{cases}$ in each of the following cases.

a. $\boldsymbol{A} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$, $\vec{Y}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ **b*.** $\boldsymbol{A} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, $\vec{Y}_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

07*. Consider the system $\begin{cases} y_1' = \quad y_1 + \ y_2 + \ y_3 \\ y_2' = \ \ 2\,y_1 + \ y_2 - \ y_3 \\ y_3' = -8\,y_1 - 5\,y_2 - 3\,y_3 \end{cases}$ and let $\boldsymbol{A}$ be its coefficient matrix. If you know that $\boldsymbol{A}$ has eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = 2$, and corresponding eigenvectors $\vec{V}_1 = \begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}$, $\vec{V}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, write down the general solution of the system, then find the solution $\vec{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$ that satisfies $\vec{Y}(0) = \begin{bmatrix} 1 \\ -2 \\ 8 \end{bmatrix}$

08. Solve the initial value problem $\begin{cases} \vec{Y}' = \boldsymbol{A}\,\vec{Y} \\ \vec{Y}(0) = \vec{Y}_0 \end{cases}$ where $\boldsymbol{A} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}$, and $\vec{Y}_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Simple Complex Eigenvalues

09. Given that the coefficient matrix of the system $\begin{cases} y_1' = \ y_1 - y_2 \\ y_2' = 5\,y_1 - 3\,y_2 \end{cases}$ has eigenvalue $\lambda_1 = -1 + i$, and corresponding eigenvector $\vec{V}_1 = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$, find the general solution of the system.

10. Solve the initial value problem $\begin{cases} \vec{Y}' = \boldsymbol{A}\,\vec{Y} \\ \vec{Y}(0) = \vec{Y}_0 \end{cases}$ in each of the following cases.

a. $\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, $\vec{Y}(0) = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ **b*.** $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$, $\vec{Y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$