First Order Homogeneous Differential Equations

The differential equation $F(x, y, y') = 0$ is called homogeneous if it can be rewritten in the form

$$y' = g\left(\frac{y}{x}\right)$$

For instance $y' = \sin\left(\frac{y}{x}\right)$, is homogeneous

$$y' = \frac{x^2 + 3xy - y^2}{x^2 + y^2},$$ is also homogeneous. Indeed, factoring out x^2 from both the numerator and denominator, we get

$$y' = \frac{x^2\left(1 + 3\frac{y}{x} - \frac{y^2}{x^2}\right)}{x^2\left(1 + \frac{y^2}{x^2}\right)} = \frac{1 + 3\frac{y}{x} - \left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^2}$$

To solve $\quad y' = g\left(\frac{y}{x}\right)$, simply make the substitution

$$u = \frac{y}{x} \quad \Rightarrow \quad y = xu \quad \Rightarrow \quad y' = u + xu'$$

which transforms the differential equation into

$$u + xu' = g(u) \quad \Leftrightarrow \quad x\frac{du}{dx} = g(u) - u$$

Assuming $g(u) - u \neq 0$, the equation becomes

$$\frac{1}{g(u) - u}\, du = \frac{1}{x}\, dx$$

This is a separable equation. Solve it and go back to y

Don't forget to look for the solutions that might come from the case $\quad g(u) - u = 0$

Example Solve $y' = \dfrac{x^2+2y^2}{xy}$

solution Factoring out x^2, we have

$$y' = \frac{x^2\left(1+2\frac{y^2}{x^2}\right)}{x^2 \cdot \frac{y}{x}} = \frac{1+2\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}$$

This shows that the equation is homogeneous.

To solve, we set $\frac{y}{x} = u \Rightarrow y = xu \Rightarrow y' = u + xu'$

Substituting, leads to

$$u + xu' = \frac{1+2u^2}{u} \Leftrightarrow x\frac{du}{dx} = \frac{1+2u^2}{u} - u = \frac{1+u^2}{u}$$

$$\Rightarrow \frac{u}{1+u^2}du = \frac{1}{x}dx \quad \Rightarrow \frac{2u}{u^2+1}du = \frac{2}{x}dx$$

Integrating both sides, we get

$$\ln(u^2+1) = 2\ln|x| + C \iff \ln(u^2+1) = \ln(x^2) + C$$

Taking the exponential of both sides leads to

$u^2+1 = x^2 \cdot e^C$. Renaming e^C as C and going back to y, we have

$$\left(\frac{y}{x}\right)^2 + 1 = C x^2 \Rightarrow y^2 + x^2 = C x^4$$