Math 375 **Fall 2016**

Higher Order Linear Differential Equations

Worksheet # 2 **Part 1** **October 10 - 14**

The problems on this worksheet refer to material from section §4.1 of your text. Solutions to all problems will be available on the course's D2L website Friday, October 14. Please report any typos, omissions and errors to `aiffam@ucalgary.ca`

Basics

01. Which of the following is a linear second order differential equation

a*. $y'' - 5\,t\,y = t\,y' - 25$ **b*.** $\big(\sin(t)\,y'\big)' + 2\,t\,y^2 = 0$

c*. $y'' + t\,y' - \dfrac{\sin(y)}{y} = \ln(t)$ **d*.** $\dfrac{y' + t\,y}{1 + y''} = \mathrm{e}^t$

02. Given that the general solution of $y'' + 4\,y = 0$ is $C_1\,\cos(2\,t) + C_2\,\sin(2\,t)$, solve the initial value problem $\begin{cases} y'' + 4\,y = 0 \\ y(\pi/2) = -1 \;\&\; y'(\pi/2) = 2\sqrt{3} \end{cases}$

Express your answer in the form $y = A\,\cos\big(2\,t + \phi\big)$

03*. Consider the differential equation $t^2\,y'' - 3\,t\,y' - 5\,y = 0$. Verify that $y_1 = t^5$ and $y_2 = \dfrac{1}{t}$ are solutions of the equation, then solve the initial value problem

$$\begin{cases} t^2\,y'' - 3\,t\,y' - 5\,y = 0 \\ y(1) = 4 \;\&\; y'(1) = 2 \end{cases}$$

04. The 2^{nd} order linear differential equation $y'' + p(t)\,y' + q(t)\,y = 0$, has solutions $y_1(t) = \dfrac{1}{t}$ and $y_2(t) = \mathrm{e}^t$. Find $p(t)$ and $q(t)$.

Existence and Uniqueness

05. Determine the largest open interval over which the unique solution of the given initial value problem is guaranteed to exist.

a*. $\begin{cases} t\,y'' + \dfrac{1}{t^2 - 9}y' + y = t \\ y(1) = 0 \;\&\; y'(1) = 2 \end{cases}$ **b*.** $\begin{cases} t\,y'' + \dfrac{1}{t^2 - 9}y' + y = t \\ y(4) = 0 \;\&\; y'(4) = 2 \end{cases}$

c. $\begin{cases} y'' + y' + y = \sec(t) \\ y(\pi/4) = 1 \;\&\; y'(\pi/4) = -1 \end{cases}$ **d.** $\begin{cases} y'' + t\,y' + \ln(1 - t)\,y = \ln(2 + t) \\ y(0) = 1 \;\&\; y'(0) = 2 \end{cases}$

Reduction of order

06*. Consider the 2^{nd} order linear differential equation $y'' - 4\,t\,y' + \left(4\,t^2 - 2\right) y = 0$

a. Verify that $y_1(t) = \mathrm{e}^{t^2}$ is a solution.

b. Use the reduction of order method to find the general solution.

07. Consider the 2^{nd} order linear differential equation $t^2\left(1 - \ln(t)\right) y'' + t\,y' - y = 0$

a. Verify that $y_1(t) = t$ is a solution

b. Use the reduction of order method to find the general solution.

Answers and Solutions

01. Recall that the second order differential equation $F\left(t, y, y', y''\right) = 0$, is called linear if **it is** or **can be rewritten** in the form $a(t)\,y'' + b(t)\,y' + c(t)\,y = f(t)$. Based on this definition, only (a) and (d) are linear.

02. We need to select the constants C_1 and C_2 so that $y(t) = C_1\cos(2\,t) + C_2\sin(2\,t)$, satisfies the initial conditions. We have $y'(t) = -2\,C_1\sin(2\,t) + 2\,C_2\cos(2\,t)$. It follows

$$\begin{cases} y(\pi/2) = -1 \\ y'(\pi/2) = 2\sqrt{3} \end{cases} \iff \begin{cases} C_1\cos(\pi) + C_2\sin(\pi) = -1 \\ -2\,C_1\sin(\pi) + 2\,C_2\cos(\pi) = 2\sqrt{3} \end{cases} \iff \begin{cases} C_1 = 1 \\ C_2 = -\sqrt{3} \end{cases}$$

Hence the solution $y(t) = \cos(2\,t) - \sqrt{3}\sin(2\,t) = \big(1\big)\cdot\cos(2\,t) + \big(-\sqrt{3}\big)\cdot\sin(2\,t)$

Multiplying and dividing by the amplitude $A = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$, leads to

$$y(t) = 2\left(\frac{1}{2}\cos(2\,t) - \frac{\sqrt{3}}{2}\sin(2\,t)\right) = 2\left(\cos(\pi/3)\cos(2\,t) - \sin(\pi/3)\sin(2\,t)\right) = 2\cos\left(2\,t + \pi/3\right)$$

where in the last step, we made use of the identity $\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \cos(\alpha+\beta)$

03. We have

$t^2\,y_1'' - 3\,t\,y_1' - 5\,y_1 = t^2\left(20\,t^3\right) - 3\,t\left(5\,t^4\right) - 5\left(t^5\right) = 20\,t^5 - 15\,t^5 - 5\,t^5 = 0$

$t^2\,y_2'' - 3\,t\,y_2' - 5\,y_2 = t^2\left(2\,t^{-3}\right) - 3\,t\left(-t^{-2}\right) - 5\left(t^{-1}\right) = 2\,t^{-1} + 3\,t^{-1} - 5\,t^{-1} = 0$

Hence both $y_1 = t^5$ and $y_2 = t^{-1}$, are solutions of the differential equation $t^2\,y'' - 3\,t\,y' - 5\,y = 0$.

The two solutions are linearly independent in $\left(0\,,+\infty\right)$, since their ratio $\dfrac{y_1(t)}{y_2(t)} = \dfrac{t^5}{1/t} = t^6$ is not a constant. It follows that the general solution is $y(t) = C_1\,y_1(t) + C_2\,y_2(t) = C_1\,t^5 + C_2\,t^{-1}$

To solve the initial value problem, we need to determine the constants C_1 and C_2 so that the initial conditions are satisfied. From $y(t) = C_1\,t^5 + C_2\,t^{-1}$, it follows $y'(t) = 5\,C_1\,t^4 - C_2\,t^{-2}$.

Thus $\begin{cases} y(1) = 4 \\ y'(1) = 2 \end{cases} \iff \begin{cases} C_1 + C_2 = 4 \\ 5\,C_1 - C_2 = 2 \end{cases}$ Making use of Cramer's Rule, we have

$$C_1 = \frac{\begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix}} = \frac{-4-2}{-1-5} = 1 \qquad \text{and} \qquad C_2 = \frac{\begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix}} = \frac{2-20}{-1-5} = 3$$

Hence the solution of the initial value problem is $y(t) = C_1\,t^5 + C_2\,t^{-1} = t^5 + 3\,t^{-1} = t^5 + \dfrac{3}{t}$

04. First y_1 and y_2, being solutions of the differential equation, we have

$$\begin{cases} y_1'' + y_1'\, p(t) + y_1\, q(t) = 0 \\ y_2'' + y_2'\, p(t) + y_2\, q(t) = 0 \end{cases} \iff \begin{cases} y_1'\, p(t) + y_1\, q(t) = -y_1'' \\ y_2'\, p(t) + y_2\, q(t) = -y_2'' \end{cases}$$

Substituting $\dfrac{1}{t}$ for y_1, and e^t for y_2, the system becomes

$$\begin{cases} -\dfrac{1}{t^2}\, p(t) + \dfrac{1}{t}\, q(t) = -\dfrac{2}{t^3} \\ \mathrm{e}^t\, p(t) + \mathrm{e}^t\, q(t) = -\mathrm{e}^t \end{cases} \iff \begin{cases} p(t) - t\, q(t) = \dfrac{2}{t} \\ p(t) + q(t) = -1 \end{cases}$$

By Cramer's rule, we get

$$p(t) = \frac{\begin{vmatrix} 2/t & -t \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -t \\ 1 & 1 \end{vmatrix}} = \frac{2/t - t}{1+t} = \frac{2-t^2}{t\,(t+1)} \qquad q(t) = \frac{\begin{vmatrix} 1 & 2/t \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -t \\ 1 & 1 \end{vmatrix}} = \frac{-1-2/t}{1+t} = \frac{-t-2}{t\,(t+1)}$$

The equation $y'' + p(t)\, y' + q(t)\, y = 0$, then becomes

$$y'' + \frac{2-t^2}{t\,(t+1)}\, y' + \frac{-t-2}{t\,(t+1)}\, y = 0 \qquad \text{or else} \qquad t\,(t+1)\, y'' - \big(t^2 - 2\big)\, y' - \big(t+2\big)\, y = 0$$

Recall: that if $p(t)$, $q(t)$, and $g(t)$ are continuous at and around $t = t_0$, then the initial value problem $\begin{cases} y'' + p(t)\, y' + q(t)\, y = g(t) \\ y\big(t_0\big) = y_0 \quad \& \quad y'\big(t_0\big) = y_1 \end{cases}$ has a unique solution that is defined (at least) in the largest interval $\big(a\,,\, b\big)$ that contains t_0, and where $p(t)$, $q(t)$, $g(t)$ are all continuous.

05ab. Rewrite the equation in normal form as $y'' + \dfrac{1}{t\,(t-3)\,(t+3)}\, y' + \dfrac{1}{t}\, y = 1$

$p(t) = \dfrac{1}{t\,(t-3)\,(t+3)}$ is continuous in $\big(-\infty\,,\,-3\big)$, $\big(-3\,,\,0\big)$, $\big(0\,,\,3\big)$, and $\big(3\,,\,+\infty\big)$

$q(t) = \dfrac{1}{t}$ is continuous in $\big(-\infty\,,\,0\big)$, and $\big(0\,,\,+\infty\big)$

$g(t) = 1$ is continuous in $\big(-\infty\,,\,+\infty\big)$

a. The largest open interval that contains the initial time $t = 1$ where $p(t)$, $q(t)$, and $g(t)$ are all continuous is $\big(0\,,\,3\big)$. By the existence and uniqueness theorem, the solution of the I.V.P. is guaranteed to exist in $\big(0\,,\,3\big)$

b. The largest open interval that contains the initial time $t = 4$ where $p(t)$, $q(t)$, and $g(t)$ are all continuous is $\big(3\,,\,+\infty\big)$. By the existence and uniqueness theorem, the solution of the I.V.P. is guaranteed to exist in $\big(3\,,\,+\infty\big)$

05c. The equation is already in normal form $y'' + y' + y = \sec(t)$

$p(t) = 1$, $q(t) = 1$ are continuous in $\big(-\infty\,,\,,+\infty\big)$

$g(t) = \sec(t) = \dfrac{1}{\cos(t)}$ is continuous in $\cdots\,,\ \big(-3\,\pi/2\,,\,-\pi/2\big)\,,\ \big(-\pi/2\,,\,\pi/2\big)\,,\ \big(\pi/2\,,\,3\,\pi/2\big)\,,\cdots$

The largest open interval that contains the initial time $t = \pi/4$ where $p(t)$, $q(t)$, and $g(t)$ are all continuous is $\big(-\pi/2\,,\,\pi/2\big)$. By the existence and uniqueness theorem, the solution of the I.V.P. is guaranteed to exist in $\big(-\pi/2\,,\,\pi/2\big)$

05d. The equation is already in normal form $y'' + t\,y' + \ln(1-t)\,y = \ln(2+t)$
$p(t) = t$, is continuous in $\big(-\infty\,,\,+\infty\big)$
$q(t) = \ln(1-t)$ is continuous in $\big(-\infty\,,\,,1\big)$
$g(t) = \ln(2+t)$ is continuous in $\big(-2\,,\,+\infty\big)$
The largest open interval that contains the initial time $t=0$ where $p(t)$, $q(t)$, and $g(t)$ are all continuous is $\big(-2\,,\,1\big)$. By the existence and uniqueness theorem, the solution of the I.V.P. is guaranteed to exist in $\big(-2\,,\,1\big)$

Reduction of Order Method

Assuming that $z = z(t)$ is a given solution of the **homogeneous** differential equation $a(t)\,y'' + b(t)\,y' + c(t)\,y = 0$, the substitution $y = z(t)\,u$, converts the differential equation $a(t)\,y'' + b(t)\,y' + c(t)\,y = f(t)$, whether $f(t) \equiv 0$ or not, into

$$a(t)\,z(t)\cdot u'' + \big(2\,a(t)\,z'(t) + b(t)\,z(t)\big)\cdot u' = f(t)$$

or equivalently

$$\begin{cases} a(t)\,z(t)\cdot v' + \big(2\,a(t)\,z'(t) + b(t)\,z(t)\big)\cdot v = f(t) \\ u' = v \end{cases}$$

This way of reducing a second order linear differential equation to two first order linear differential equations is known as the reduction of order method.

06a. We have $y_1 = \mathrm{e}^{t^2} \implies y_1' = 2\,t\,\mathrm{e}^{t^2} \implies y_1'' = \big(4\,t^2 + 2\big)\,\mathrm{e}^{t^2}$. It follows
$y_1'' - 4\,t\,y_1' + \big(4\,t^2 - 2\big)\,y_1 = \big(4\,t^2 + 2\big)\,\mathrm{e}^{t^2} - 4\,t\,2\,t\,\mathrm{e}^{t^2} + \big(4\,t^2 - 2\big)\,\mathrm{e}^{t^2} = \big(4\,t^2 + 2 - 8\,t^2 + 4\,t^2 - 2\big)\,\mathrm{e}^{t^2} = 0$

06b. The reduction of order method, as mentioned above, looks for the general solution in the form $y(t) = y_1(t)\,v(t) = \mathrm{e}^{t^2}\,v$.
We have

$$y' = 2\,t\,\mathrm{e}^{t^2}\,v + \mathrm{e}^{t^2}\,v' \implies y'' = \big(4\,t^2 + 2\big)\,\mathrm{e}^{t^2}\,v + 4\,t\,\mathrm{e}^{t^2}\,v' + \mathrm{e}^{t^2}\,v''$$

Substituting into the differential equation, we have

$$\mathrm{e}^{t^2}\,v'' + 4\,t\,\mathrm{e}^{t^2}\,v' + \big(4\,t^2 + 2\big)\,\mathrm{e}^{t^2}\,v - 4\,t\,\mathrm{e}^{t^2}\,v' - 8\,t^2\,\mathrm{e}^{t^2}\,v + \big(4\,t^2 - 2\big)\,\mathrm{e}^{t^2}\,v = 0$$

or else

$$\mathrm{e}^{t^2}\,v'' = 0 \iff v'' = 0 \iff v = C_1\,t + C_2$$

Hence the general solution

$$y(t) = y_1(t)\,v(t) = \mathrm{e}^{t^2}\,\big(C_1\,t + C_2\big) = C_1\,t\,\mathrm{e}^{t^2} + C_2\,\mathrm{e}^{t^2}$$

07a. We have $y_1 = t \implies y_1' = 1 \implies y_1'' = 0$ It follows
$t^2\left(1 - \ln(t)\right) y_1'' + t\, y_1' - y_1 = t^2\left(1 - \ln(t)\right)\left(0\right) + t\left(1\right) - t = 0 + t - t = 0$

07b. Using the reduction of order method, we look for the general solution in the form

$$y(t) = y_1(t)\, v(t) = t\, v$$

We have

$$y' = v + t\, v' \implies y'' = 2\, v' + t\, v''$$

Substituting into the differential equation, leads to

$$t^2\left(1 - \ln(t)\right)\left(2\, v' + t\, v''\right) + t\left(v + t\, v'\right) - t\, v = 0 \iff t^3\left(1 - \ln(t)\right) v'' + t^2\left(3 - 2\,\ln(t)\right) v'$$

or else

$$v'' + \frac{3 - 2\,\ln(t)}{1 - \ln(t)}\, \frac{1}{t}\, v'$$

Setting $u = v'$, the equation becomes

$$u' + \frac{3 - 2\,\ln(t)}{1 - \ln(t)}\, \frac{1}{t}\, u$$

This is a first order linear differential. An integrating factor is $\mu = \dfrac{t^2}{1 - \ln(t)}$.

Hint: use the substitution $s = 1 - \ln(t)$ to compute $\displaystyle\int \frac{3 - 2\,\ln(t)}{1 - \ln(t)}\, \frac{1}{t}\, \mathrm{d}t$.

Multiplying by the integrating factor and solving for u, we obtain $u = C_1\, \dfrac{1 - \ln(t)}{t^2}$. It follows
$v' = C_1\, \dfrac{1 - \ln(t)}{t^2} \implies v = C_1\, \dfrac{\ln(t)}{t} + C_2$ Hence the general solution is

$$y = t\, v = t\left(C_1\, \frac{\ln(t)}{t} + C_2\right) = C_1\, \ln(t) + C_2\, t$$