Some Applications of First Order Differential Equations

Decay and Growth Problems

These problems usually involve a quantity $Q = Q(t)$ that depends on time t whose rate of change $Q'(t)$ is proportional to $Q(t)$ itself. In mathematical terms this is written as

$$Q'(t) = k\,Q(t)$$

where k is some constant.

Radioactive Materials

When the quantity is the mass $m = m(t)$ of a radioactive material at time t, it is known that

- the rate of change $m'(t)$ is proportional to $m(t)$
- the mass decays with time i.e., $m'(t) < 0$

Hence $m = m(t)$ satisfies $\begin{cases} m' = -km \\ m(t_0) = m_0 \end{cases}$

here m_0 is the mass of the sample quantity at time $t = t_0$, and $k > 0$ is a constant called the decay constant.

This is a 1st order linear diff. eq. Solving we get

$$\boxed{m(t) = m_0 e^{-k(t-t_0)}}$$

The time τ it takes the quantity to lose half its mass is called the <u>half-life</u> of the material.

By definition τ satisfies

$$m(t_0+\tau) = \frac{1}{2} m_0 \iff m_0 e^{-k(t_0+\tau-t_0)} = \frac{m_0}{2} \iff e^{-k\tau} = \frac{1}{2}$$

$$\iff e^{k\tau} = 2 \iff k\tau = \ln(2)$$

Hence the half life and the decay constant of a radioactive material are related by $\boxed{k\tau = \ln(2)}$

Using this relation we can rewrite $m(t) = m_0 e^{-k(t-t_0)}$ as

$$m(t) = m_0 e^{-k\tau \cdot \frac{t-t_0}{\tau}} = m_0 \left(e^{-k\tau}\right)^{\frac{t-t_0}{\tau}} = m_0 \left(\frac{1}{2}\right)^{\frac{t-t_0}{\tau}}$$

If we take $t_0 = 0$, then

$$\boxed{m(t) = m_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}}}$$

Example Radium-226 is a radioactive substance with half-life $\tau = 1720$ years. Find the time required for a sample mass of Radium-226 to decrease to 25% of its original mass.

If $m(t)$ is the mass of the sample at time t, then $m(t) = m_0 e^{-k(t-t_0)}$. If we take $t_0 = 0$, then

$$m(t) = m_0 e^{-kt}, \quad \text{with } k\tau = \ln(2) \iff k = \frac{\ln(2)}{\tau} = \frac{\ln(2)}{1720}$$

Hence $m(t) = m_0 e^{-\frac{\ln(2)}{1720}t}$. If s is the time it takes for the mass to decrease to 25% of its original mass, then

$$m(s) = \frac{1}{4} m_0 \iff m_0 e^{-\frac{\ln(2)}{1720}s} = \frac{1}{4} m_0 \iff e^{-\frac{\ln(2)}{1720}s} = \frac{1}{4} \iff$$

$$-\frac{\ln(2)}{1720}s = \ln\left(\frac{1}{4}\right) = -\ln(4) = -2\ln(2) \Rightarrow s = 2(1720) = 3440 \text{ years.}$$

Population Growth

When the quantity is the number $p = p(t)$ of individuals in a colony of birds, fishes, bacterium, ... the constant of proportionality k usually represents the average rate contribution of a single individual to the rate of change of the population. $p(t)$ satisfies

$$\begin{cases} p' = kp \\ p(t_0) = p_0 \end{cases}$$

solving, we get $\boxed{p(t) = p_0 e^{k(t-t_0)}}$

Definition

The time T it takes the colony to double its size is called the <u>doubling time</u> of the colony.

T satisfies

$$p(t_0+T) = 2p_0 \iff p_0 e^{k(t_0+T-t_0)} = 2p_0 \iff e^{kT} = 2$$

$$\iff \boxed{kT = \ln(2)}$$

Example

The doubling time of a colony of bacterium is 5 days. How long will it take the colony to triple its size?

We have $p(t) = p_0 e^{k(t-t_0)}$ and the doubling time is $T=5$

It follows: $kT = \ln(2) \Rightarrow k = \frac{\ln(2)}{T} = \frac{\ln(2)}{5}$. Hence $p(t) = p_0 e^{\frac{\ln(2)}{5}(t-t_0)}$. If s is the time it takes the colony to triple its size, then $p(t_0+s) = 3p_0 \iff p_0 e^{\frac{\ln(2)}{5}(t_0+s-t_0)} = 3p_0 \iff$

$$e^{\frac{\ln(2)}{5}s} = 3 \iff \frac{\ln(2)}{5}s = \ln(3) \iff s = \frac{5\ln(3)}{\ln(2)} \approx 7.9 \text{ days}$$