Prop: if $x_1 \dots x_n$ are the only free vars in A
and $s \approx_{x_i} s'$ for $i=1 \dots n$ then $\mathcal{M}, s \vDash A$ iff $\mathcal{M}, s' \vDash A$

Proof: By induction on A

Induction Basis: A is atomic

i) $A = \bot$ $\quad \mathcal{M}, s \nvDash \bot$ iff $\mathcal{M}, s' \nvDash \bot$ ✓

ii) $A = P^n(t_1, \dots, t_n)$

WTS: $\mathcal{M}, s \vDash P^n(t_1 \dots t_n)$ iff $\mathcal{M}, s' \vDash P^n(t_1 \dots t_n)$

$\mathcal{M}, s \vDash P^n(t_1 \dots t_n)$ iff (By Defn of $\vDash$)

$\langle Val^{\mathcal{M}}_s(t_1), \dots Val^{\mathcal{M}}_s(t_n) \rangle \in (P^n)^{\mathcal{M}}$

$\left(A^2_0 (\underbrace{f^1_0(c_0)}_{t_1}, \underbrace{f^2_0(c_0, v_0)}_{t_2}) \right)$ $\quad \mathcal{N}$

$s(v_0) = 5$

? $\mathcal{N}, s \vDash 0' < (0 + v_0)$

yes

$\langle Val^{\mathcal{N}}_s(0'), Val^{\mathcal{N}}_s(0+v_0) \rangle$

$\langle 1 \quad , \quad 5 \rangle \in (A^2_0)^{\mathcal{N}}$

↓

$<$ relation

∴ true

Lemma: if $s(x_i) = s'(x_i)$ for all x in t,

$\underline{Val^{\mathcal{M}}_s(t) = Val^{\mathcal{M}}_{s'}(t)}$

$\langle Val^{\mathcal{M}}_{s'}(t_1), \dots, Val^{\mathcal{M}}_{s'}(t_n) \rangle \in (P^n)^{\mathcal{M}}$

iff (By defn of $\vDash$

$\mathcal{M}, s' \vDash P^n(t_1, \dots, t_n)$

inductive steps

i) $A = \neg B$ $\quad$ iii) $A = (B \vee C)$ $\quad$ v) $A = \exists x B$

ii) $A = (B \wedge C)$ $\quad$ iv) $A = (B \to C)$ $\quad$ vi) $A = \forall x B$

(iii) $\mathcal{M}, s \vDash (B \vee C)$ iff (By def of $\vDash$)

$\underbrace{\mathcal{M}, s \vDash B}$ or $\underbrace{\mathcal{M}, s \vDash C}$ (or Both)

By IH iff $\quad$ iff

$\mathcal{M}, s' \vDash B$ $\underline{\underline{\text{or}}}$ $\mathcal{M}, s' \vDash C$

B/c every free variable of B is free in A

$\mathcal{M}, s' \vDash (B \vee C)$

v) $m, s \models \exists x B$ iff (By defn $\models$)

for at least one x-variant of s, $\bar{s} \hat{x} s$

$m, \bar{s} \models B$

free vars of B = free vars of A + X

$\bar{s} \hat{x} s$ so $\bar{s}(y) = s(y)$ for $y \neq x$

for some $\bar{s}' \hat{x} s'$, $m, \bar{s}' \models B$

Suppose there is a $\bar{s} \hat{x} s$ ST $m, \bar{s} \models B$

WTS: there is an $\bar{s}' \hat{x} s'$ ST $m, \bar{s}' \models B$

consider $\bar{s}$. let $\bar{s}'$ Be:

$$\bar{s}'(y) = \begin{cases} s'(y) & \text{if } y \neq x \\ \bar{s}(y) & \text{if } y = x \end{cases}$$

1) $\bar{s}$ and $\bar{s}'$ assign the same objects to the free variables of B

if z is free in A then $z \neq x$, so $\bar{s}(z) = s(z)$

since $\bar{s} \hat{x} s$

then also $s'(z) = s(z)$ By hypoth of the prop

By Defn of $\bar{s}'$, $\bar{s}'(z) = s'(z)$

together $\bar{s}(z) = \bar{s}'(z)$

if z is free in B But not in A, i.e) $z = x$, i.e) WTS $\bar{s}(x) = \bar{s}'(x)$

this is true by how $\bar{s}'$ is defined

By IH, $m, \bar{s}' \models B$

verify that $\bar{s}' \hat{x} s'$: By Def of $\bar{s}'$

so there is an $\bar{s}' \hat{x} s'$ ST $m, \bar{s}' \models B$

so $m, s' \models \exists x B$

homework is to do for case 6

truth + consequence

Defn: A is A <u>sentence</u> if it has no free variables

$\mathfrak{M} \models A$ iff $\mathfrak{M}, s \models A$ for any variable assignments

Defn Γ set of sentences

A is a sentence

Γ <u>entails</u> A, $\Gamma \models A$ iff for all

$\mathcal{L}$-structures $\mathfrak{M}$, if $\mathfrak{M} \models B$ for all $B \in \Gamma$,

then $\mathfrak{M} \models A$

A is true in every structure that makes all sentences in Γ true

Defn A is <u>valid</u> iff $\mathfrak{M} \models A$ for all $\mathfrak{M}$

<u>Semantic Deduction Theorem</u>

$\Gamma \cup \{A\} \models B$ iff $\Gamma \models (A \to B)$

(if $\Gamma = \emptyset$, this is: $A \models B$ iff $\models (A \to B)$)