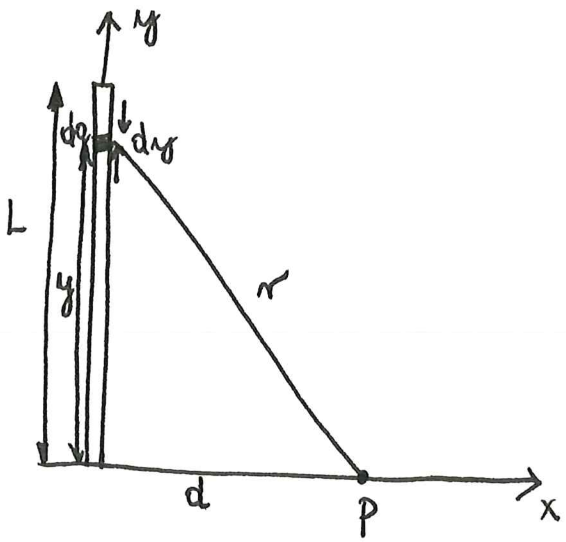
POTENTIAL DUE TO FINITE LINE OF CHARGE



λ - constant

$$dq = \lambda\, dy$$

$$dV = \frac{1}{4\pi\varepsilon_0} \frac{dq}{\sqrt{y^2+d^2}}$$

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$$V = \frac{\lambda}{4\pi\varepsilon_0} \int_0^L \frac{dy}{\sqrt{y^2+d^2}}$$

From calculus: $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right)$

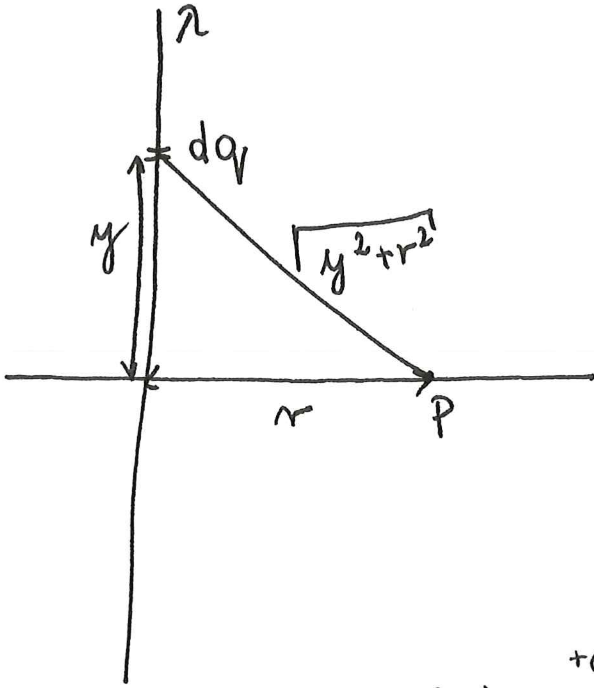
$$V = \frac{\lambda}{4\pi\varepsilon_0} \left[\ln\left(y + \sqrt{y^2+d^2}\right)\right]_0^L$$

$$V = \frac{\lambda}{4\pi\varepsilon_0} \left[\ln\left(L + \sqrt{L^2+d^2}\right) - \ln\left(0 + \sqrt{0^2+d^2}\right)\right]$$

$$\boxed{V = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{L + \sqrt{L^2+d^2}}{d}\right)}$$

POTENTIAL OF A LINE OF CHARGE

$\lambda =$ constant



1. Potential due to dq.

Note: this assumes $V = 0$ at infinity.

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{y^2+r^2}}$$

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy}{\sqrt{y^2+r^2}}$$

$$V(r) = \int_{-\infty}^{+\infty}\frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy}{\sqrt{y^2+r^2}}$$

(The integrand is symmetric, so integral from − infinity to + infinity is twice the integral from 0 to + infinity)

$$V(r) = \frac{2\lambda}{4\pi\varepsilon_0}\int_0^{+\infty}\frac{dy}{\sqrt{y^2+r^2}} = \frac{2\lambda}{4\pi\varepsilon_0}\int_0^{\infty}\frac{dy}{r\sqrt{\left(\frac{y}{r}\right)^2+1}} = \frac{2\lambda}{4\pi\varepsilon_0}\int_0^{\infty}\frac{dy}{r\sqrt{1+\left(\frac{y}{r}\right)^2}}$$

r factored out

From calculus: $\displaystyle\int\frac{dX}{\sqrt{1+X^2}} = \ln\left(\sqrt{1+X^2}+X\right)$

Changing variables: $X = \frac{y}{r}$ and $dX = \frac{dy}{r}$ – same limits

Substitute X in place for y/r

$$V(r) = \frac{2\lambda}{4\pi\varepsilon_0}\int_0^{\infty}\frac{dX}{\sqrt{1+X^2}} \qquad V(r) = \frac{2\lambda}{4\pi\varepsilon_0}\ln\left(\sqrt{1+X}+X\right)\Big|_0^{\infty}$$

Substitute y/r in place for X

$$V(r) = \frac{2\lambda}{4\pi\varepsilon_0}\ln\left(\sqrt{1+\frac{y}{r}}+\frac{y}{r}\right)\Big|_0^{\infty}$$

Look at the $y \to \infty$ limit

$$V(r) = \frac{2\lambda}{4\pi\varepsilon_0} \ln\left(\lim_{y\to\infty} \sqrt{1+\left(\frac{y}{r}\right)^2} + y/r\right)$$

$$V(r) = \frac{2\lambda}{4\pi\varepsilon_0} \ln\left(\lim_{y\to\infty} \frac{y}{r}\sqrt{\left(\frac{r}{y}\right)^2+1} + y/r\right)$$

This gives an infinite answer regardless of the value of r.

$\boxed{\text{Fix: change the zero of } V \text{ by adding a constant } (V_0)}$

$$V(r) = \frac{2\lambda}{4\pi\varepsilon_0} \ln\left(\frac{y}{r}\sqrt{\left(\frac{r}{y}\right)^2+1} + y/r\right) + \underline{V_0}$$

Factor out the divergent piece (y/r) and for the well behaved piece in square brackets, set $y =$ infinity.

$$V(r) = \frac{\cancel{2}\lambda}{\cancel{4}_{2}\pi\varepsilon_0} \ln\left(\frac{y}{r}\overbrace{\underbrace{\left[\sqrt{1+\left(\frac{r}{y}\right)^2_{\to 0}} + 1\right]}_{\text{set } y\to\infty}}^{=2}\right) + V_0$$

$$V(r) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{2y}{r}\right) + V_0$$

Choose the constant V_0 such that the infinity (y) cancels out.

$$V(r) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{2y}{r}\right) - \boxed{\frac{\lambda}{2\pi\varepsilon_0} \ln(2y)}\ V_0$$

$$V(r) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{\cancel{2y}}{r} \cdot \frac{1}{\cancel{2y}}\right)$$

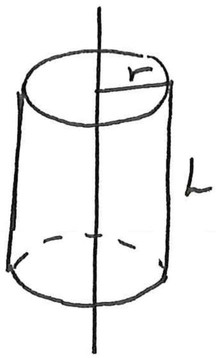
After this convoluted process, we arrive at a useful expression:

$$V(r) = \frac{-\lambda}{2\pi\varepsilon_0} \ln(r)$$

$$E_r = -\frac{\partial V}{\partial r} \implies E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Alternative, much easier way!

1. Use Gauss' law to find E



$$\oint \vec{E} \circ d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$E \cdot A = \frac{q_{enc}}{\varepsilon_0}$$

$$A = 2\pi r \cdot L \quad \& \quad q_{enc} = \lambda \cdot L$$

$$E \cdot 2\pi r \cdot \cancel{L} = \frac{\lambda \cdot \cancel{L}}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Use the link between E and V to find ΔV

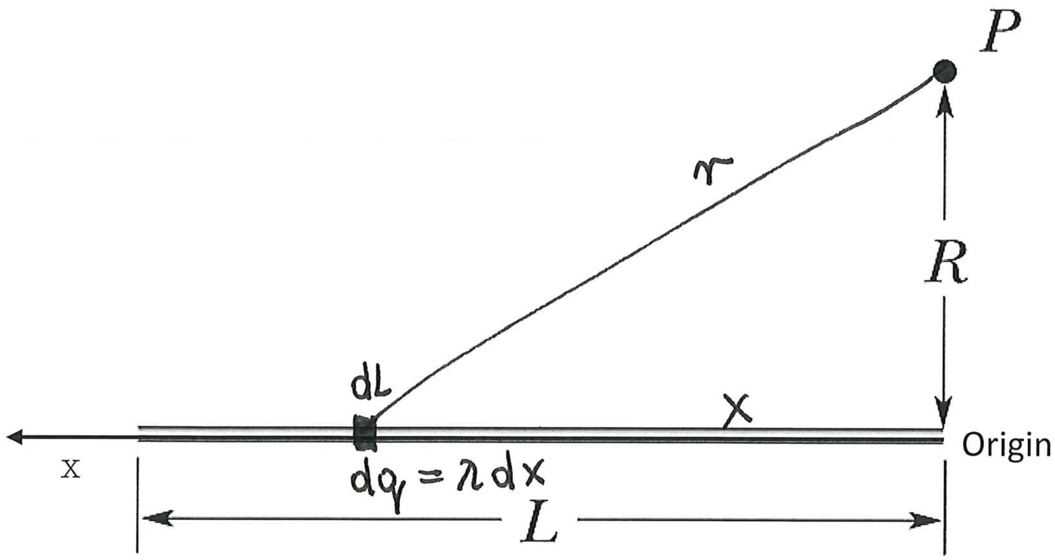
$$\Delta V_{AB} = -\int_A^B \vec{E} \circ d\vec{r}$$ radially away from the line

$$\Delta V_{AB} = -\int_A^B \frac{\lambda}{2\pi\varepsilon_0 r} dr = \frac{-\lambda}{2\pi\varepsilon_0} \ln r \Big|_{r_A}^{r_B}$$

$$\Delta V_{AB} = \frac{-\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_B}{r_A}\right)$$

Since this gives the potential difference between points A & B, we did not need to worry about where $V = 0$.

[4 marks] In the figure below, point P is at perpendicular distance R from the end of a finite line of charge with a constant charge distribution, λ.



30. Write down an expression for the contribution to the electric potential at point P from a small segment of the line of charge. Label all relevant variables in the diagram provided. [2 marks]

$$dV = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, dx}{\sqrt{x^2 + R^2}}$$

31. Calculate the electric potential at point P due to the entire line of charge (evaluate the integral) [2 marks]

$$V_P = \frac{\lambda}{4\pi\varepsilon_0} \int_0^L \frac{dx}{\sqrt{x^2+R^2}} = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(x + \sqrt{x^2+R^2}\right)\Big|_0^L$$

$$V_P = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{L + \sqrt{L^2+R^2}}{R}\right)$$

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