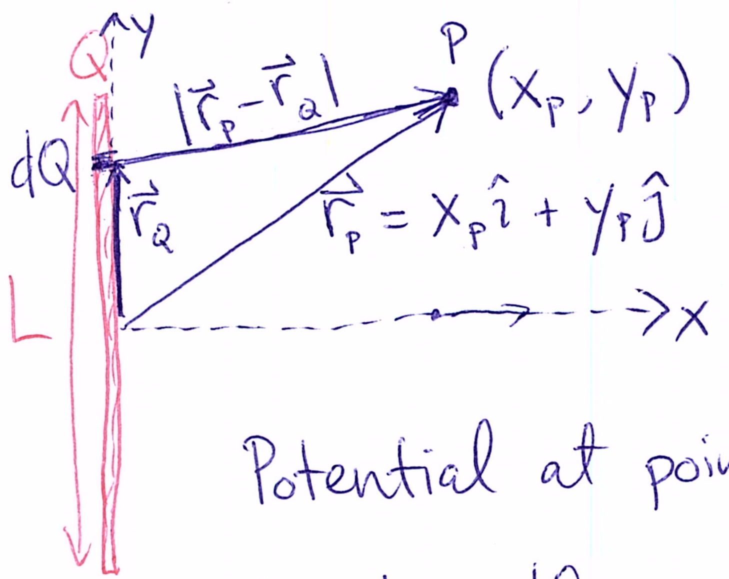
Electric Potential of a charged line



$$\lambda = \frac{Q}{L}$$

$$\vec{r}_Q = y\hat{\jmath}$$

$$\vec{r}_p - \vec{r}_Q = (x_p - 0)\hat{\imath} + (y_p - y)\hat{\jmath}$$

Potential at point P due to dQ is

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dQ}{|\vec{r}_p - \vec{r}_Q|} = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dy}{\sqrt{x_p^2 + (y_p - y)^2}}$$

$$V_p = \int_{-L/2}^{L/2} \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dy}{\sqrt{x_p^2 + (y_p - y)^2}}$$

let $y - y_p = Y$

$dy = dY$

$$V_p = \int_{\boxed{-L/2 - y_p}}^{\boxed{L/2 - y_p}} \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dY}{\sqrt{x_p^2 + Y^2}}$$

call this Y_f

call this Y_i

look up the integral:

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(2\left(\sqrt{a^2 + x^2} + x\right)\right)\Big|_{x_1}^{x_2}$$

here $x = Y$ and $a = x_p$

$$V_p = \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(2\left(\sqrt{x_p^2 + Y_f^2} + Y_f\right)\right) - \ln\left(2\left(\sqrt{x_p^2 + Y_i^2}\right) + Y_i\right)\right]$$

$$V_p = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\cancel{2}\left(\sqrt{x_p^2 + (L/2 - y_p)^2} + L/2 - y_p\right)}{\cancel{2}\left(\sqrt{x_p^2 + (-L/2 - y_p)^2} + (-L/2 - y_p)\right)}\right)$$

relabel $x_p = x$ and $y_p = y$

$$V(x,y) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{\sqrt{x^2 + (L/2 - y)^2} + L/2 - y}{\sqrt{x^2 + (L/2 + y)^2} - (L/2 + y)} \right)$$

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath}$$

look at $\vec{E}$ when $y=0$. Use symmetry to say y-component doesn't matter. We can set $y=0$ in the potential <u>before taking the derivative</u>.

$$V(x) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{\sqrt{x^2 + (L/2)^2} + L/2}{\sqrt{x^2 + (L/2)^2} - L/2} \right)$$

$$E_x = -\frac{dV}{dx} = -\frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+(L/2)^2} + L/2}\left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+(L/2)^2}} \right) - \frac{1}{\sqrt{x^2+(L/2)^2} - L/2}\left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+(L/2)^2}} \right) \right]$$

$$E_x = -\frac{\lambda}{4\pi\epsilon_0}\frac{x}{\sqrt{x^2+(L/2)^2}}\left[\frac{\cancel{\sqrt{x^2+(L/2)^2}} - L/2 - \left(\cancel{\sqrt{x^2+(L/2)^2}} + L/2\right)}{\left(\underbrace{\sqrt{x^2+(L/2)^2}}_{} + \underbrace{L/2}_{}\right)\left(\underbrace{\sqrt{x^2+(L/2)^2}}_{} - \underbrace{L/2}_{}\right)} \right]$$

$$(a+b)\cdot(a-b) = a^2 - b^2$$

$$E_x = +\frac{Q^{\leftarrow \lambda L}}{4\pi\epsilon_0}\frac{\cancel{x}}{\sqrt{x^2+(L/2)^2}}\left[\frac{1}{\underbrace{x^{\cancel{2}} \cancel{+ (L/2)^2}}_{} \underbrace{\cancel{- (L/2)^2}}_{}} \right] = \boxed{\frac{Q}{4\pi\epsilon_0 x\sqrt{x^2+(L/2)^2}}}$$

Along the y-axis: x-component doesn't matter.

If we just set $x=0$

$$V(y) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{\overbrace{|(L/2 - y)|}^{y-L/2} + L/2 - y}{|(L/2+y)| - (L/2+y)} \right)$$ problem!

restrict to above the rod $(y > L/2)$

mathematically correct: keep x in expression, take derivative then take limit as $x \to 0$. L'Hôpital's rule to the rescue!

$$\lim_{x\to 0} V(x,y) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\underbrace{\lim_{x\to 0} \frac{\sqrt{x^2+(y-L/2)^2}+(L/2-y)}{\sqrt{x^2+(y+L/2)^2}-(L/2+y)}} \right)$$

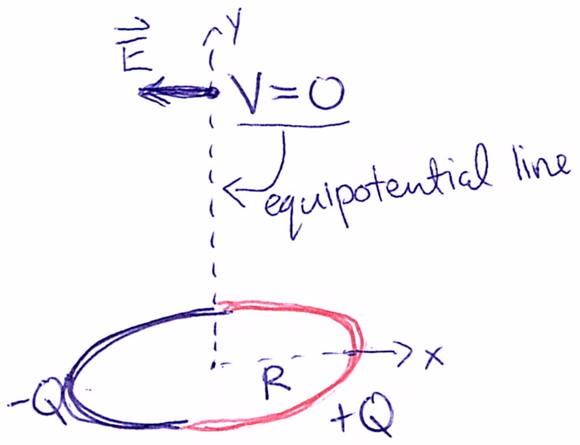
$$= \lim_{x\to 0} \frac{\frac{d}{dx}\left(\sqrt{x^2+(y-L/2)^2}+(L/2-y)\right)}{\frac{d}{dx}\left(\sqrt{x^2+(y+L/2)^2}-(L/2+y)\right)}$$

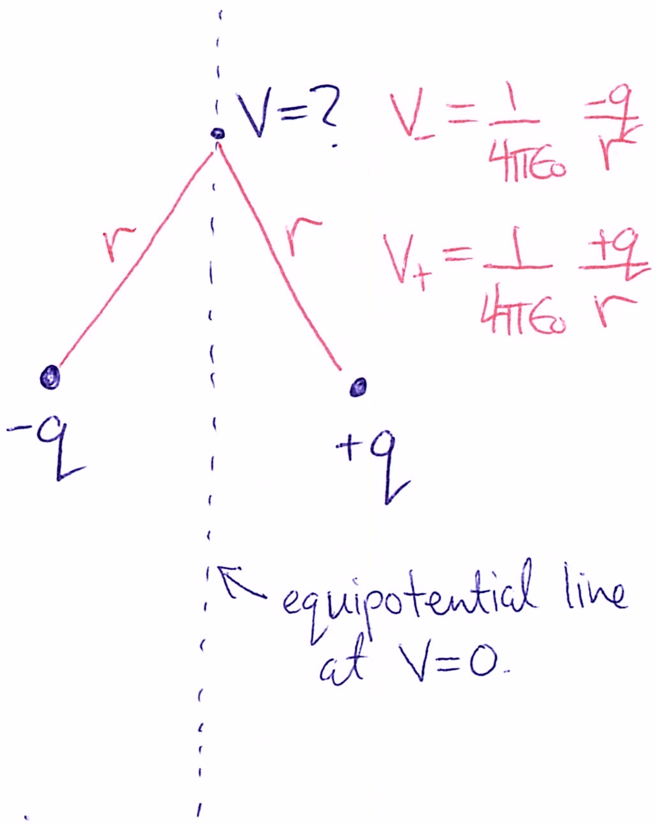
$$\lim_{x\to 0} V(x,y) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\lim_{x\to 0} \frac{2\cancel{x}}{2\sqrt{x^2+(y-L/2)^2}} \frac{\cancel{2}\sqrt{x^2+(y+L/2)^2}}{\cancel{2x}} \right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{y+L/2}{y-L/2} \right)$$

Along the y-axis (for $y > L/2$) $V(y) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{y+L/2}{y-L/2}\right)$

$$E_y = -\frac{dV}{dy} = \frac{-\lambda}{4\pi\epsilon_0}\left[\frac{1}{y+L/2} - \frac{1}{y-L/2}\right]$$

$$= \frac{-\lambda}{4\pi\epsilon_0}\left[\frac{(\cancel{y}-L/2)-(\cancel{y}+L/2)}{(y+L/2)(y-L/2)}\right] \Rightarrow \boxed{E_y = \frac{1}{4\pi\epsilon_0}\frac{Q}{(y^2-(L/2)^2)}}$$





$$V_- = \frac{1}{4\pi\epsilon_0}\frac{-q}{r}$$

$$V_+ = \frac{1}{4\pi\epsilon_0}\frac{+q}{r}$$

Electric field ≠ 0 even though V=0

In this case we can't set $x=0$ in the potential before taking the derivatives because $\vec{E}$ points in the x-direction so we would falsely conclude $\vec{E}=\vec{0}$ if we did that.