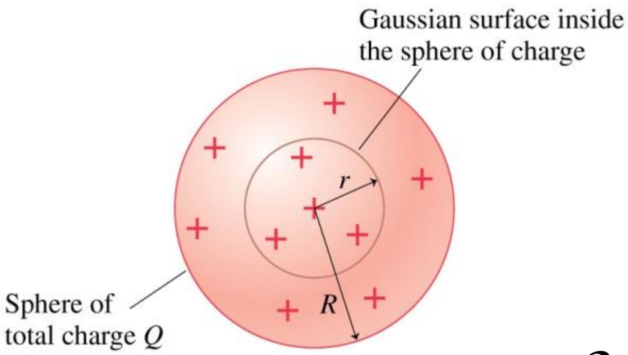
Last time

- More on properties of electric flux for a closed surface.
- More on Gauss's law
- Examples of calculation of flux for open surfaces
- Application of Gauss's law for point charges

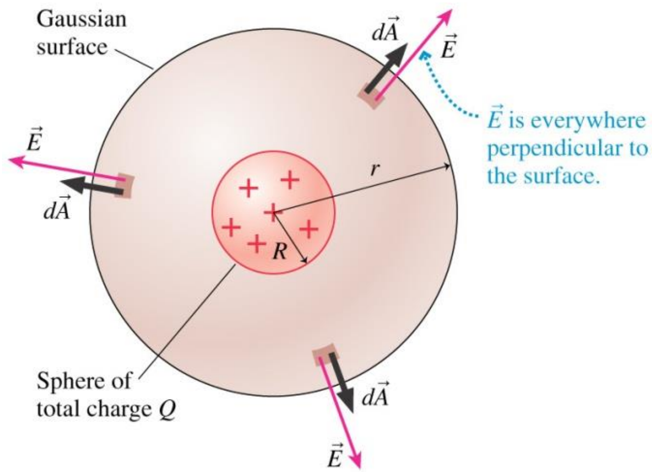
This time

- More on the electric field for a solid spherical volume of constant charge density
- Electric field for a spherical shell of constant charge density
- Electric field for a finite rod using Coulomb's law
- Electric field for an infinitely long rod using Gauss's law

Solid spherical volume (not a shell) of constant charge density



$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$



$$\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r} \qquad \text{For } r < R$$

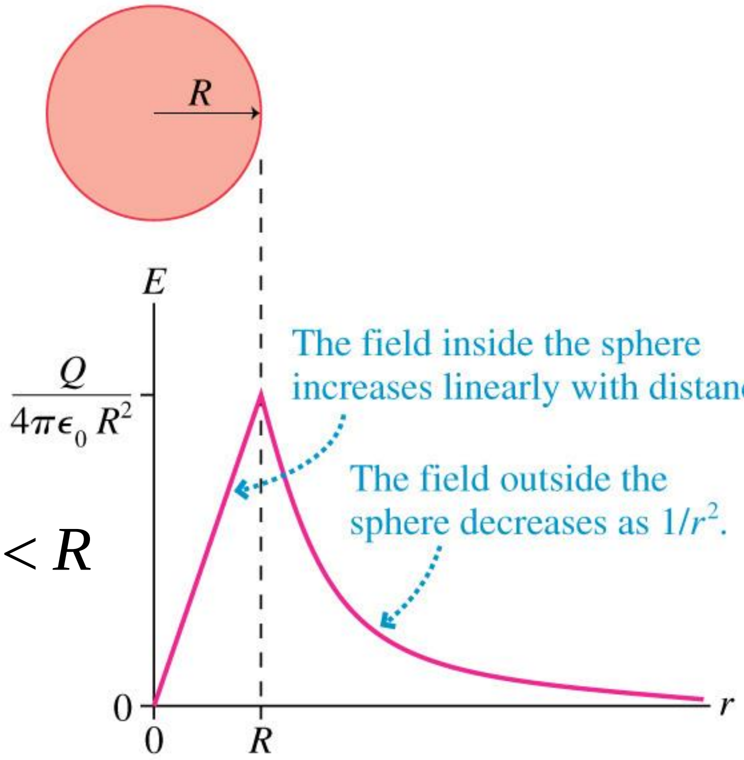
$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{For } r > R$$

Electric field vector must be continuous when r = R.

$$\vec{E}\Big|_{r=R} = \frac{\rho r}{3\varepsilon_0}\hat{r}\Bigg|_{r=R} = \frac{\frac{Q}{\frac{4}{3}\pi R^3}R}{3\varepsilon_0}\hat{r} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{r}$$

$$\vec{E}\Big|_{r=R} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}\Bigg|_{r=R} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{r}$$

Plot of electric field as a function of the distance from the center of a uniformly charged sphere (not a shell)



$$\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r} \qquad \text{For } r < R$$

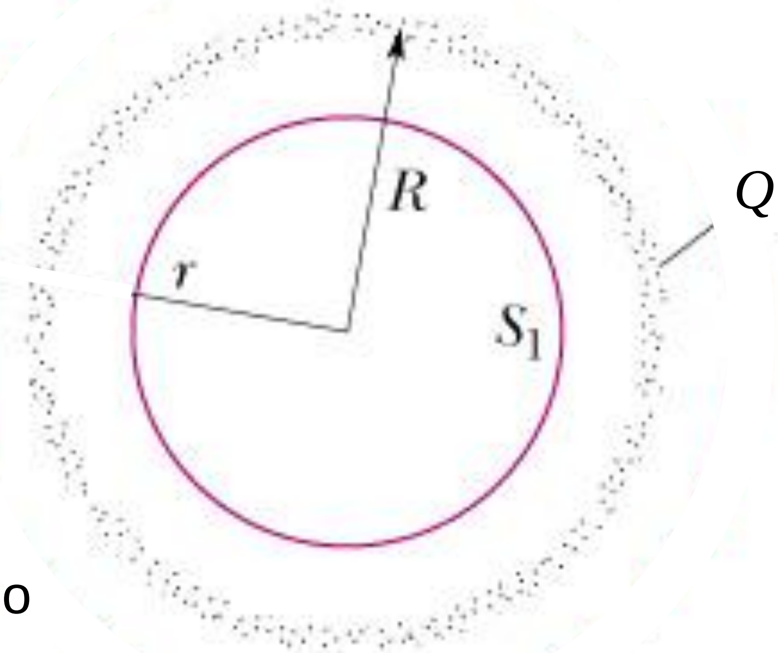
$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{For } r > R$$

Spherical shell (not solid) of constant charge density

For points inside the shell

$r < R$

$$\oint \vec{E} \bullet d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$



Gaussian is drawn as a spherical surface (S_1) with r<R and concentric with the spherical shell. It encloses no charge.

$$Q_{enclosed} = 0$$

$$\oint \vec{E} \bullet d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} = 0$$

$$\text{Since } d\vec{A} \neq 0 \Rightarrow \vec{E} = 0$$

For points outside the shell

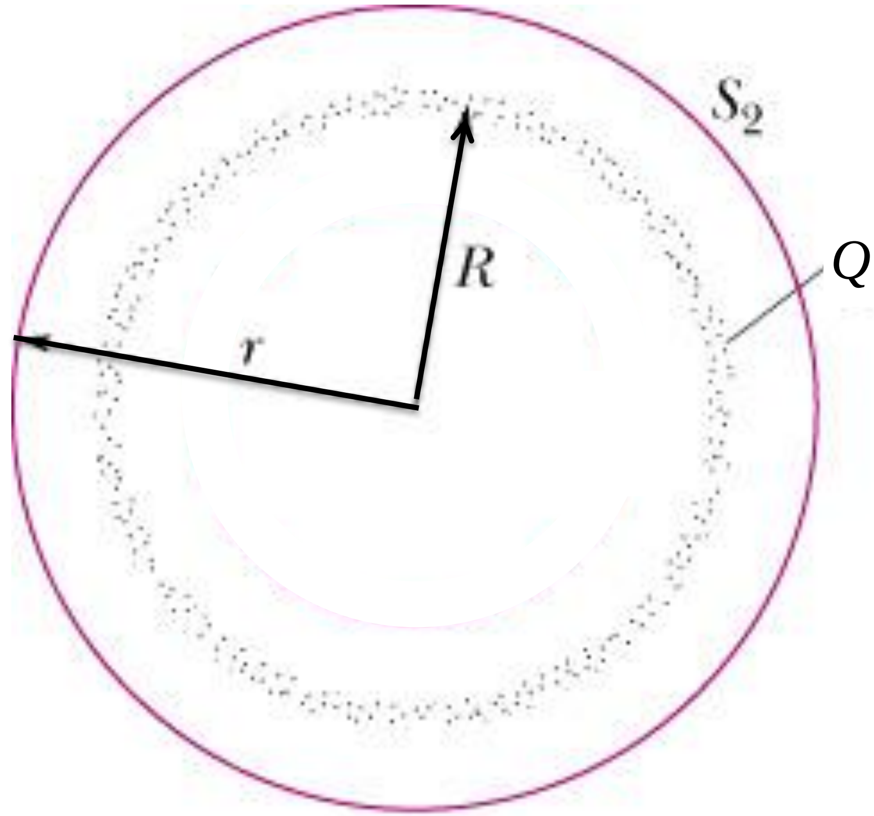
$r > R$

Electric field on the surface of the Gaussian is radial and has a constant magnitude.

$$\vec{E} = E\hat{r}$$

Infinitesimal area of the Gaussian is also radial.

$$d\vec{A} = dA\hat{r}$$



$$\vec{E} \bullet d\vec{A} = (E\hat{r}) \bullet (dA\hat{r}) = EdA$$

$$\oint \vec{E} \bullet d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

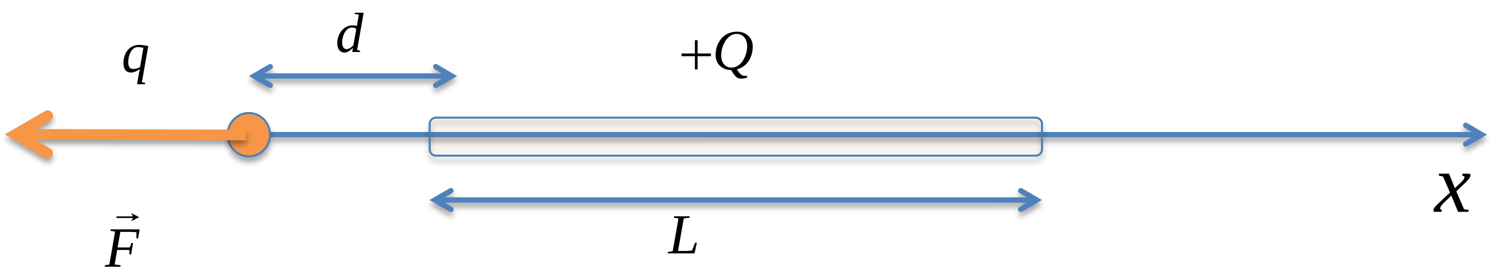
$$\oint \underbrace{E}_{\text{constant}} dA = \frac{Q}{\varepsilon_0}$$

$$\underbrace{E}_{\text{constant}} \underbrace{\oint dA}_{\text{surface area of the Gaussian}} = \frac{Q}{\varepsilon_0}$$

$$E 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

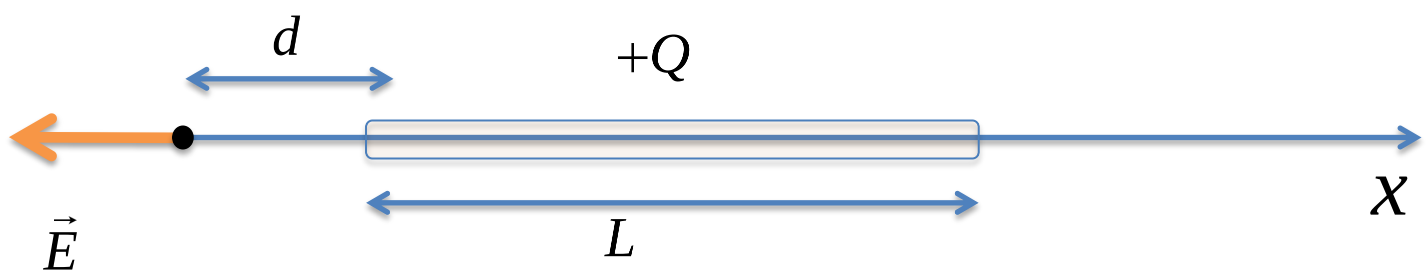
$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{For } r > R$$

Electric field for a charged rod at a point on the axis of the rod



$$\vec{F} = -k_e \frac{qQ}{d\left(d+L\right)}\hat{i}$$

From Lecture on Jan-25-2017



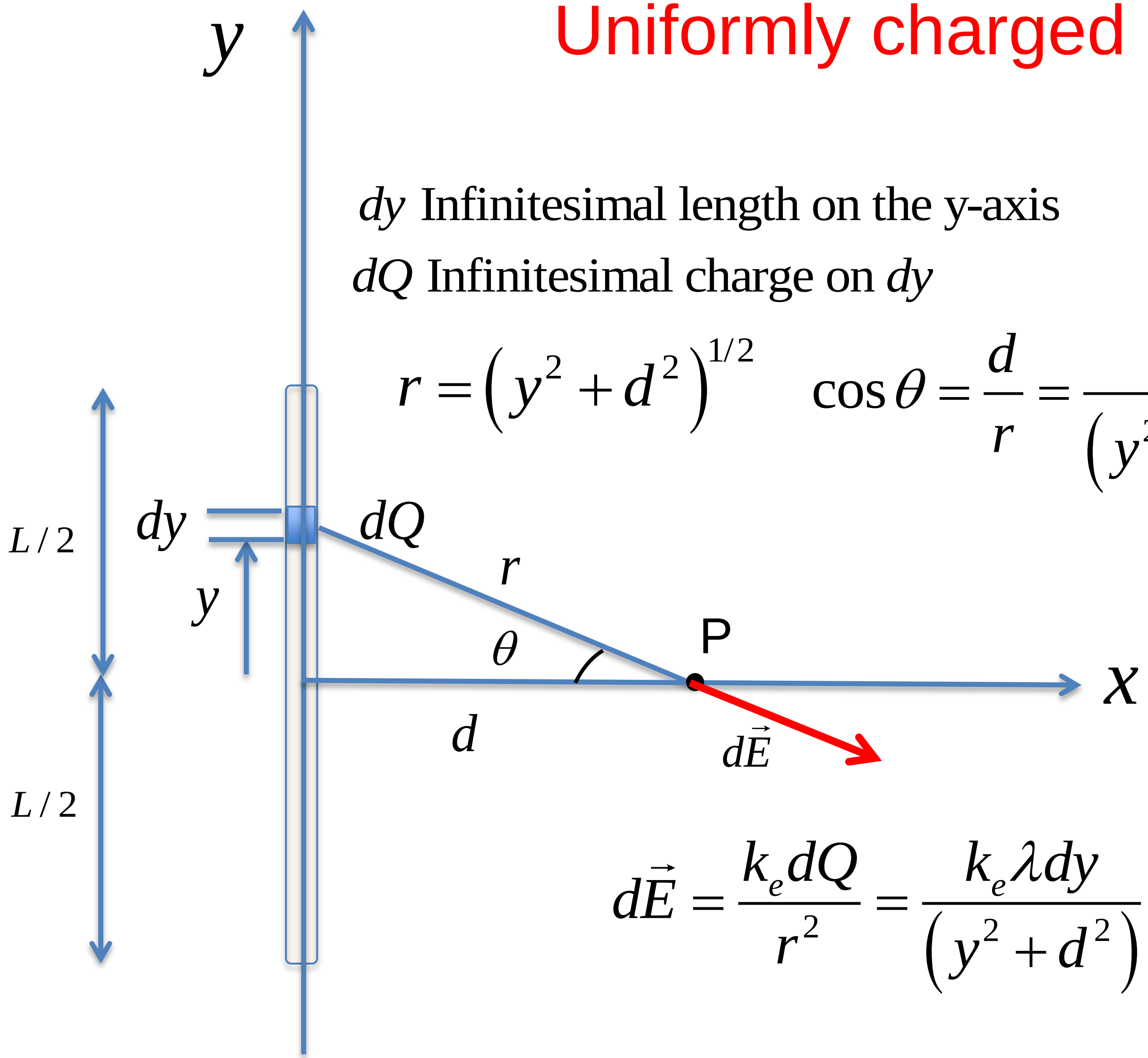
$$\vec{E} = -k_e \frac{Q}{d\left(d+L\right)}\hat{i}$$

Uniformly charged rod

dy Infinitesimal length on the y-axis

dQ Infinitesimal charge on dy

$$r = \left(y^2 + d^2\right)^{1/2} \qquad \cos\theta = \frac{d}{r} = \frac{d}{\left(y^2 + d^2\right)^{1/2}}$$

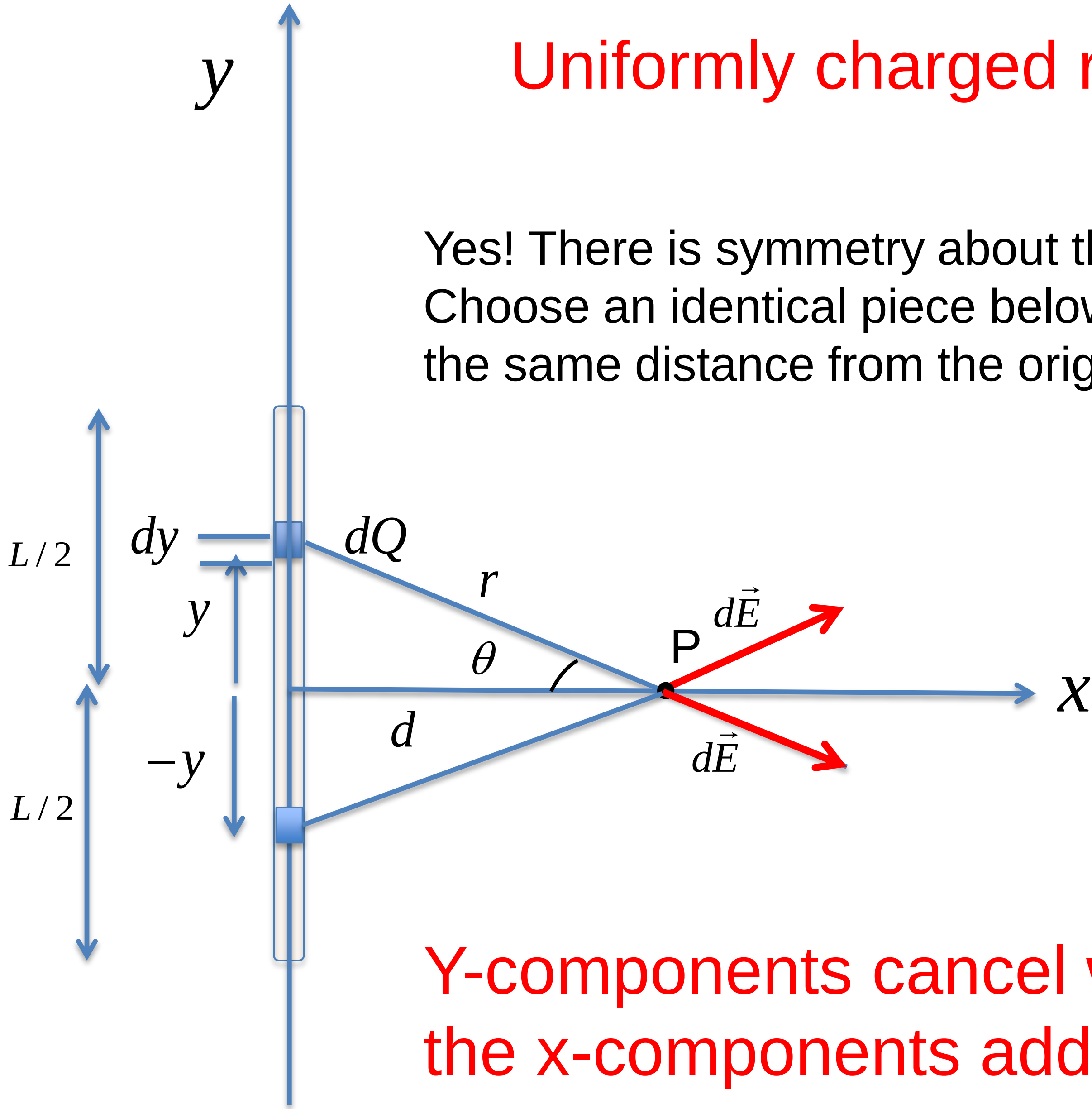


$$d\vec{E} = \frac{k_e dQ}{r^2} = \frac{k_e \lambda dy}{\left(y^2 + d^2\right)}$$

Is there any symmetry that we can take advantage of?

Uniformly charged rod

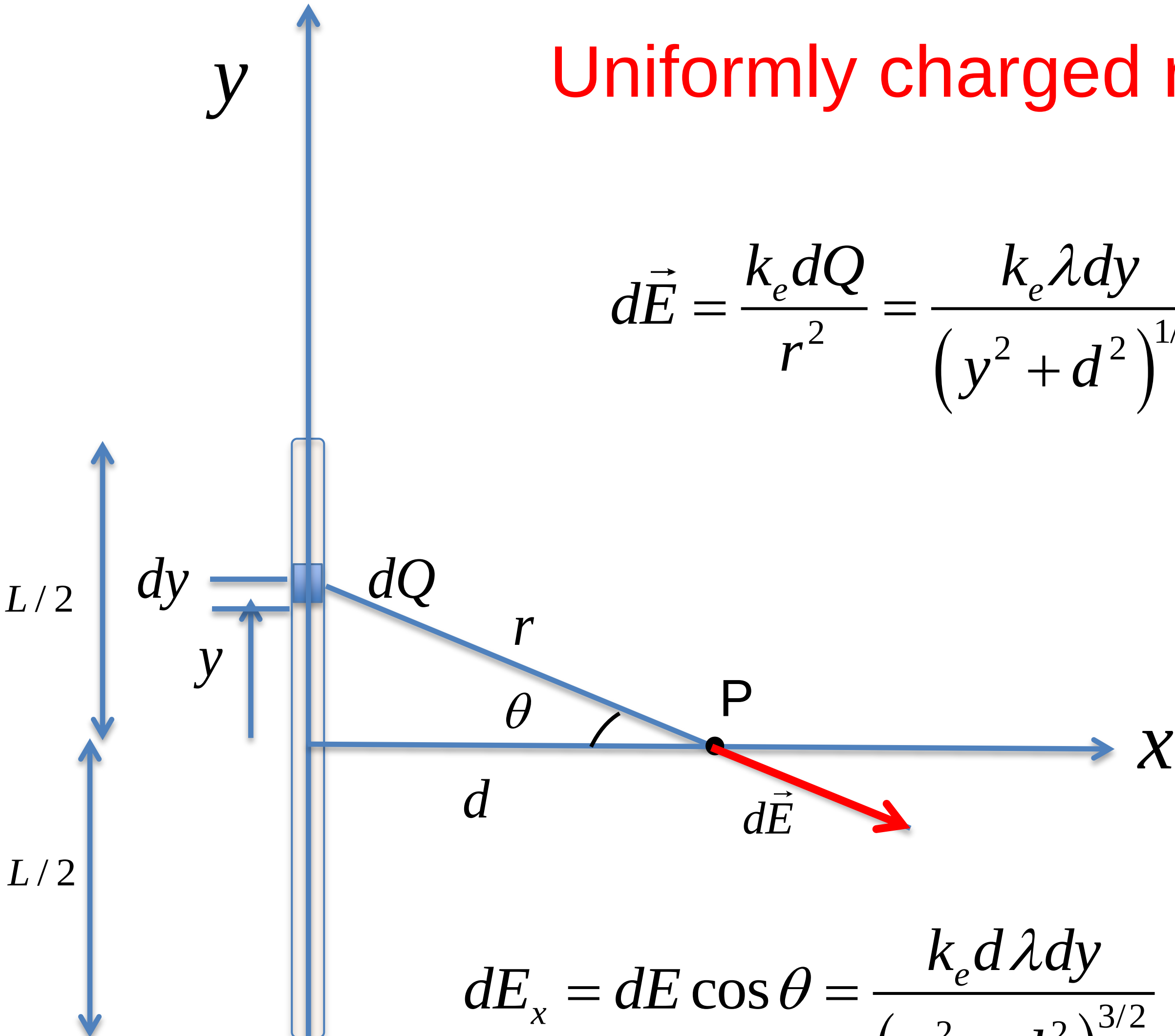
Yes! There is symmetry about the x-axis.
Choose an identical piece below the x-axis and the same distance from the origin.



Y-components cancel while the x-components add up.

$$E_y = 0, E_z = 0$$

Uniformly charged rod



$$d\vec{E} = \frac{k_e dQ}{r^2} = \frac{k_e \lambda dy}{\left(y^2 + d^2\right)^{1/2}}$$

$$dE_x = dE\cos\theta = \frac{k_e d\lambda dy}{\left(y^2 + d^2\right)^{3/2}}$$

$$E_x = \int_{-L/2}^{L/2} dE_x = k_e \lambda d \int_{-L/2}^{L/2} \frac{dy}{\left(y^2 + d^2\right)^{3/2}}$$

Just a math problem!

A math problem!

$$\int \frac{dy}{\left(y^2+d^2\right)^{3/2}} ?$$

$$y = d\tan\varphi$$

$$y^2+d^2 = d^2(1+\tan^2\varphi) = d^2/\cos^2\varphi$$

$$\cos^2\varphi = d^2/\left(y^2+d^2\right) \Rightarrow 1-\sin^2\varphi = d^2/\left(y^2+d^2\right) \Rightarrow \sin^2\varphi = y^2/\left(y^2+d^2\right)$$

$$\left(y^2+d^2\right)^{3/2} = d^3/\cos^3\varphi$$

$$dy = d(1+\tan^2\varphi)d\varphi = \frac{d}{\cos^2\varphi}d\varphi$$

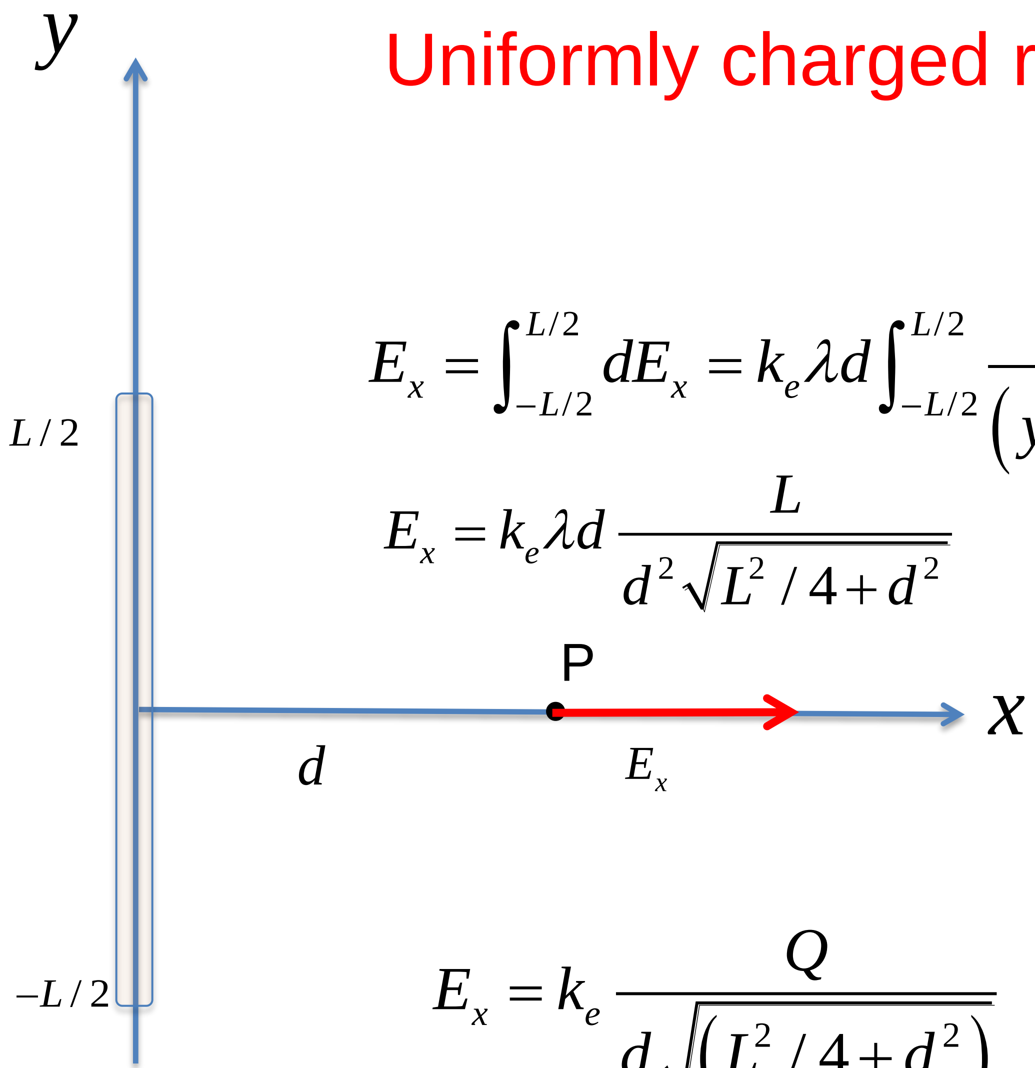
$$\int_{-L/2}^{L/2} \frac{dy}{\left(y^2+d^2\right)^{3/2}} = \frac{1}{d^2}\int \cos\varphi d\varphi = \frac{\sin\varphi}{d^2} = \frac{y}{d^2\sqrt{y^2+d^2}}\Bigg|_{-L/2}^{L/2}$$

$$\int_{-L/2}^{L/2} \frac{dy}{\left(y^2+d^2\right)^{3/2}} = \frac{L}{d^2\sqrt{L^2/4+d^2}}$$

Uniformly charged rod

$$E_x = \int_{-L/2}^{L/2} dE_x = k_e \lambda d \int_{-L/2}^{L/2} \frac{dy}{\left(y^2 + d^2\right)^{3/2}}$$

$$E_x = k_e \lambda d \frac{L}{d^2\sqrt{L^2/4 + d^2}}$$



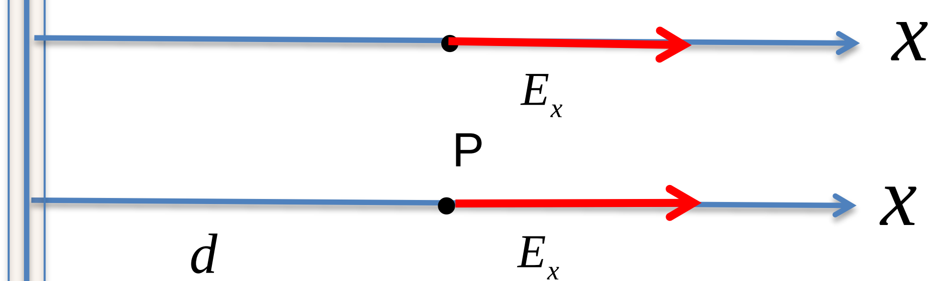
$$E_x = k_e \frac{Q}{d\sqrt{\left(L^2/4 + d^2\right)}}$$

An infinitely long rod

How would I use the results from a charged rod?

$$E_x = k_e \frac{Q}{d\sqrt{\left(L^2/4+d^2\right)}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{d\sqrt{\left(L^2/4+d^2\right)}}$$

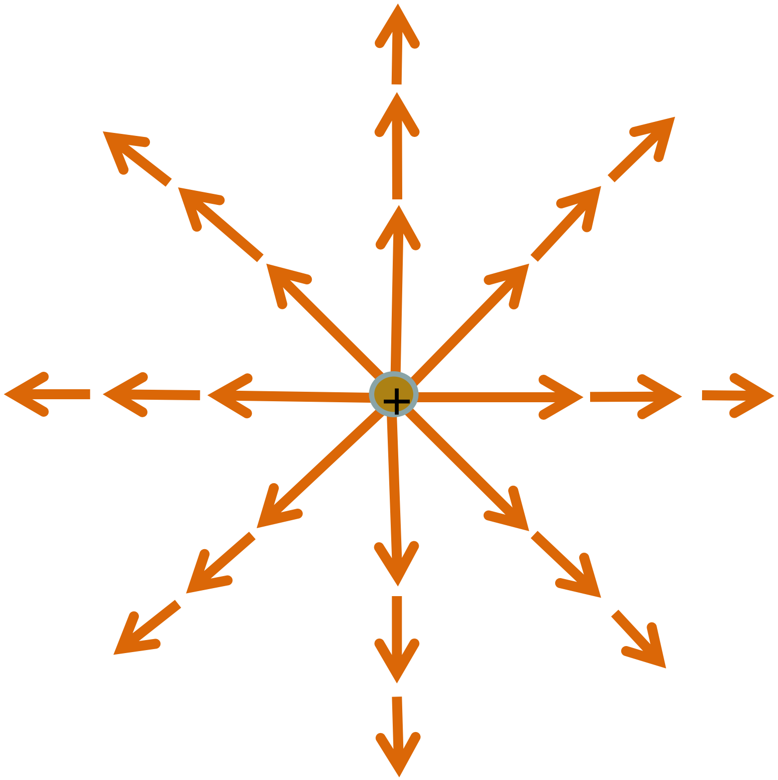
$$L \to \infty \qquad E_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{dL/2} = \frac{\lambda}{2\pi\varepsilon_0 d}$$



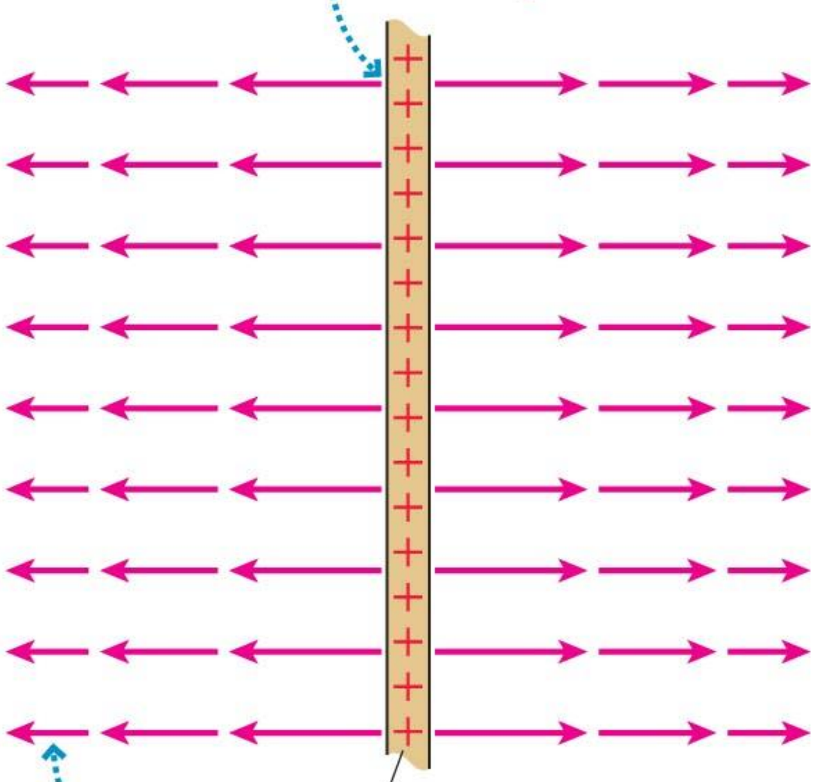
Now the question of where the x-axis located is irrelevant.
Electric field vector is perpendicular to the rod and decreases linearly with distance from the rod. This is also the case for points out of the page.

Did I say physics is wonderful?

The electric filed pattern of an infinite line of charge



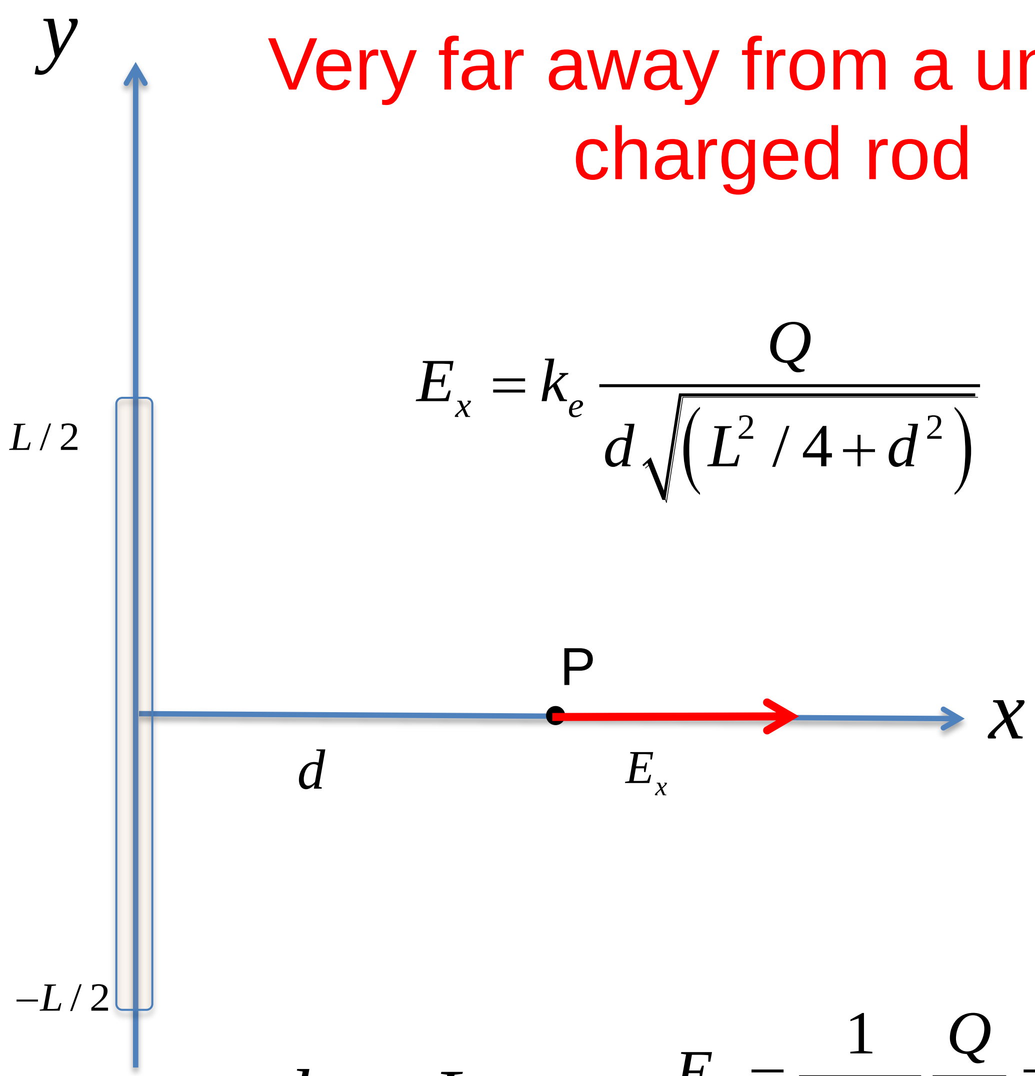
View from the top



$$E = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{d}$$

d is the vertical (radial) distance from the rod.

Very far away from a uniformly charged rod



$$E_x = k_e \frac{Q}{d\sqrt{\left(L^2/4 + d^2\right)}}$$

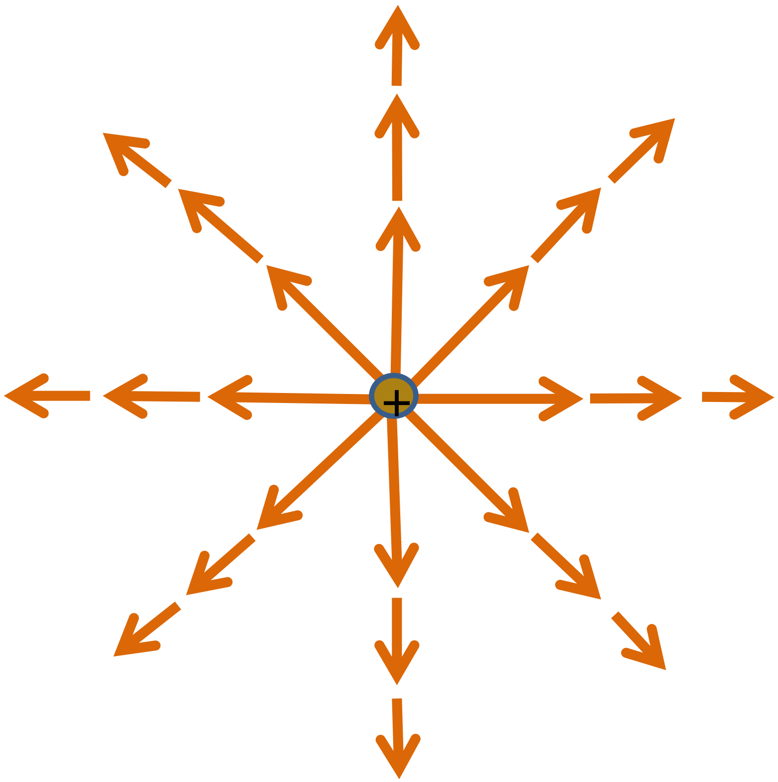
$$d >> L$$

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{Q}{dd} = \frac{Q}{4\pi\varepsilon_0 d^2}$$

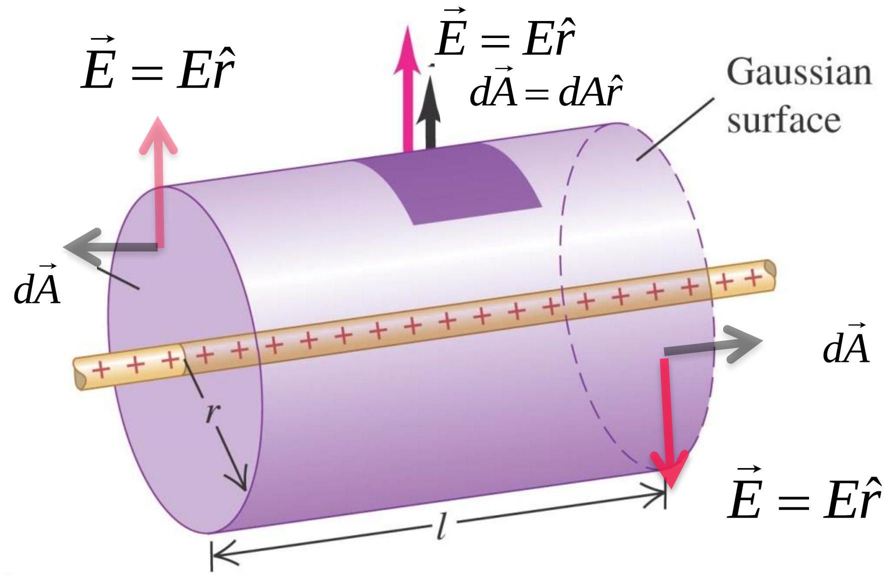
Appears as a point charge Q located at the center rod.

The field of an infinite line of charge using Gauss's law

A suitable Gaussian surface for an infinitely long wire is a cylinder whose axis coincides with line of charge



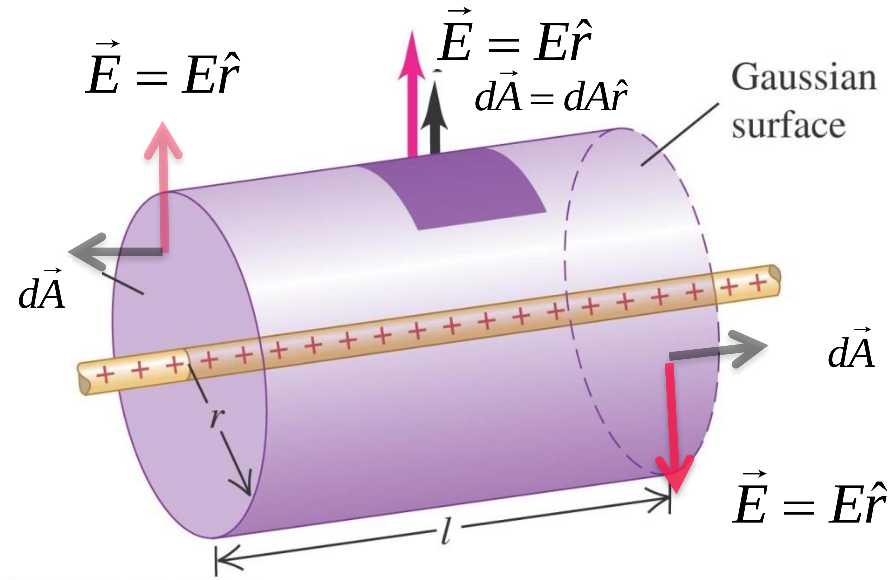
View from the top



Because the electric field is perpendicular to the line of charge, flux for the left and right caps of the Gaussian is zero.

$\vec{E} \bullet d\vec{A} = 0$ For the right and left caps

Electric field on the side of the Gaussian is radial and has a constant magnitude.



$$\vec{E} = E\hat{r}$$

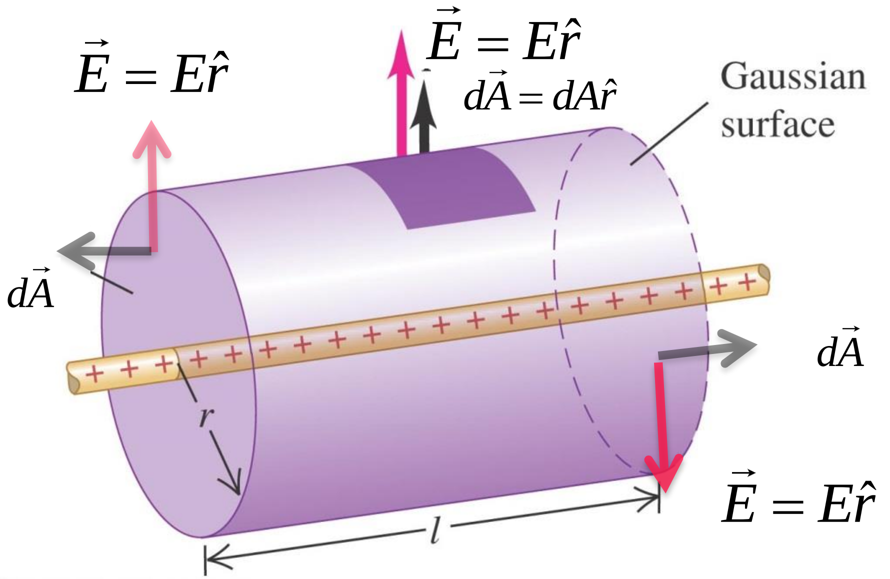
Infinitesimal area of the Gaussian for the side is also radial.

$$d\vec{A} = dA\hat{r}$$

$$\vec{E} \bullet d\vec{A} = \left(E\hat{r}\right) \bullet \left(dA\hat{r}\right) = \underset{\text{constant}}{E}\ dA \text{ For the side}$$

$$\oint \vec{E} \bullet d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} \qquad Q_{enclosed} = \lambda l$$

$$\int_{\text{right cap}} \underbrace{\vec{E} \bullet d\vec{A}}_{\substack{\parallel \\ 0}} + \int_{\text{left cap}} \underbrace{\vec{E} \bullet d\vec{A}}_{\substack{\parallel \\ 0}} + \int_{\text{side}} \underset{\text{constant}}{E}\ dA = \frac{\lambda l}{\varepsilon_0}$$



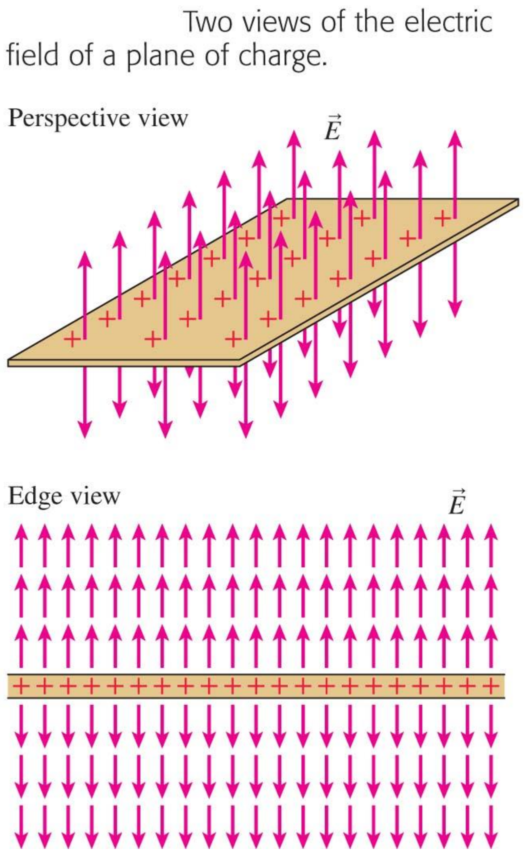
$$\underbrace{\int_{\text{right cap}} \vec{E}\bullet d\vec{A}}_{0} + \underbrace{\int_{\text{left cap}} \vec{E}\bullet d\vec{A}}_{0} + \int_{\text{side}} \underset{\text{constant}}{E}\, dA = \frac{\lambda l}{\varepsilon_0}$$

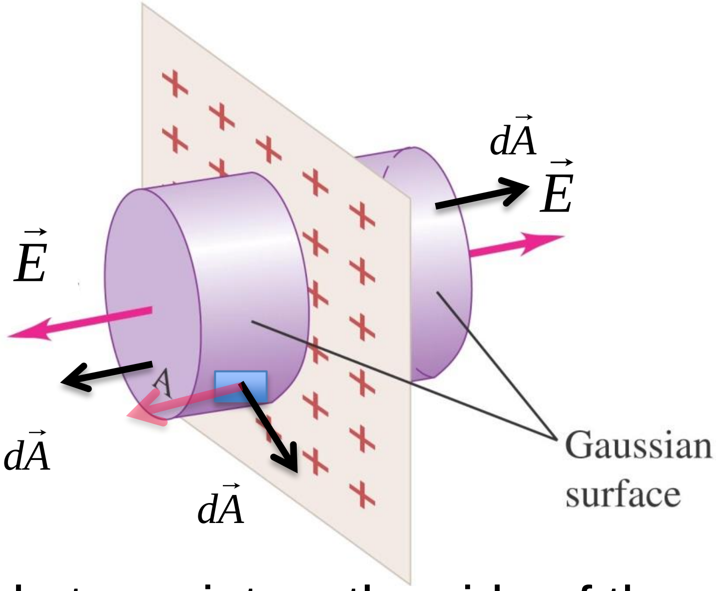
$$\underset{\text{constant}}{E} \quad \underbrace{\oint dA}_{\text{area of the side}} = \frac{\lambda l}{\varepsilon_0}$$

$$E2\pi r l = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} \text{ where } \hat{r} \text{ is } \perp \text{ to the line of charge}$$

The field of a large plane of charge

A Suitable Gaussian surface for an infinite sheet of charge is a cylinder or a rectangular cube.





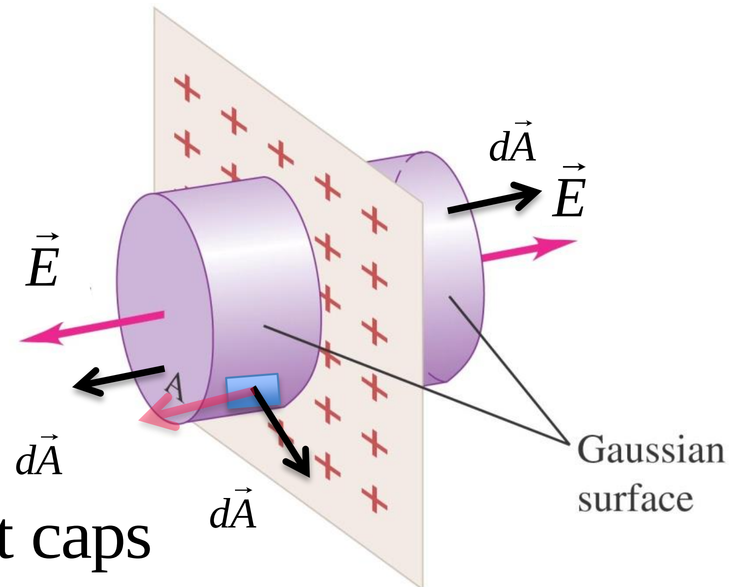
Electric field at a point on the side of the Gaussian is perpendicular to infinitesimal area vector, or parallel to the side resulting in no flux.

Electric field at a point on the caps is parallel to the infinitesimal area vector.

The field of a large plane of charge

A Suitable Gaussian surface for an infinite sheet of charge is a cylinder or a rectangular cube.

$$\oint \vec{E}\bullet d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} \qquad Q_{enclosed} = \sigma A$$



$$\vec{E}\bullet d\vec{A} = \underset{\text{constant}}{E}\; dA$$ For the right and left caps

$$\int_{\text{right cap}} \vec{E}\bullet d\vec{A} + \int_{\text{left cap}} \vec{E}\bullet d\vec{A} + \int_{\text{side}} \underset{\substack{\| \\ 0}}{\vec{E}\bullet d\vec{A}} = \frac{\sigma A}{\varepsilon_0} \qquad EA + EA = \frac{\sigma A}{\varepsilon_0}$$

$$\underset{\text{constant}}{E} \int_{\text{right cap}} dA + \underset{\text{constant}}{E} \int_{\text{left cap}} dA = \frac{\sigma A}{\varepsilon_0} \qquad E = \frac{\sigma}{2\varepsilon_0}$$