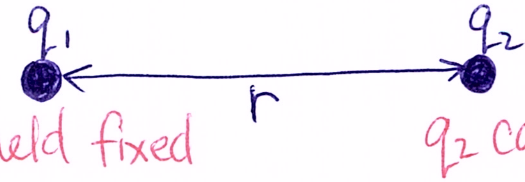
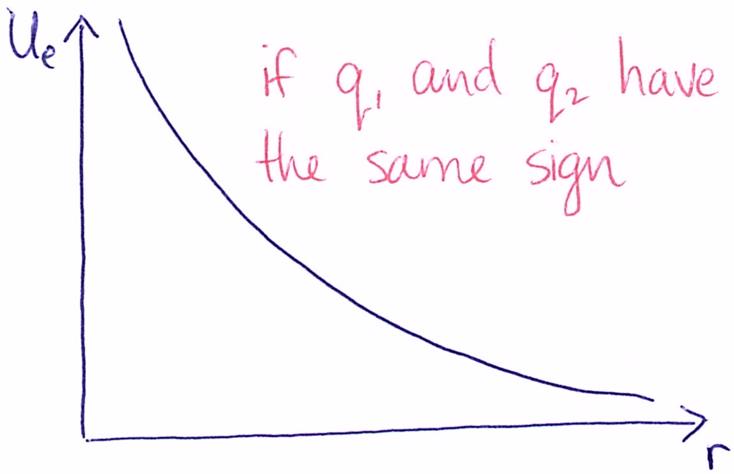
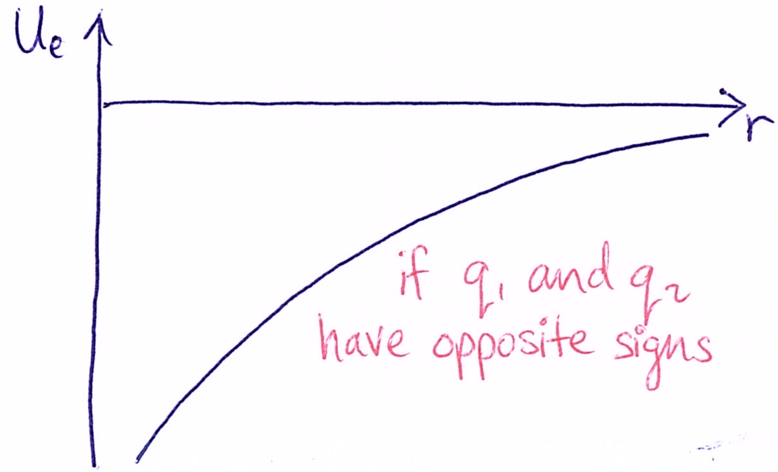
Electric Potential Energy

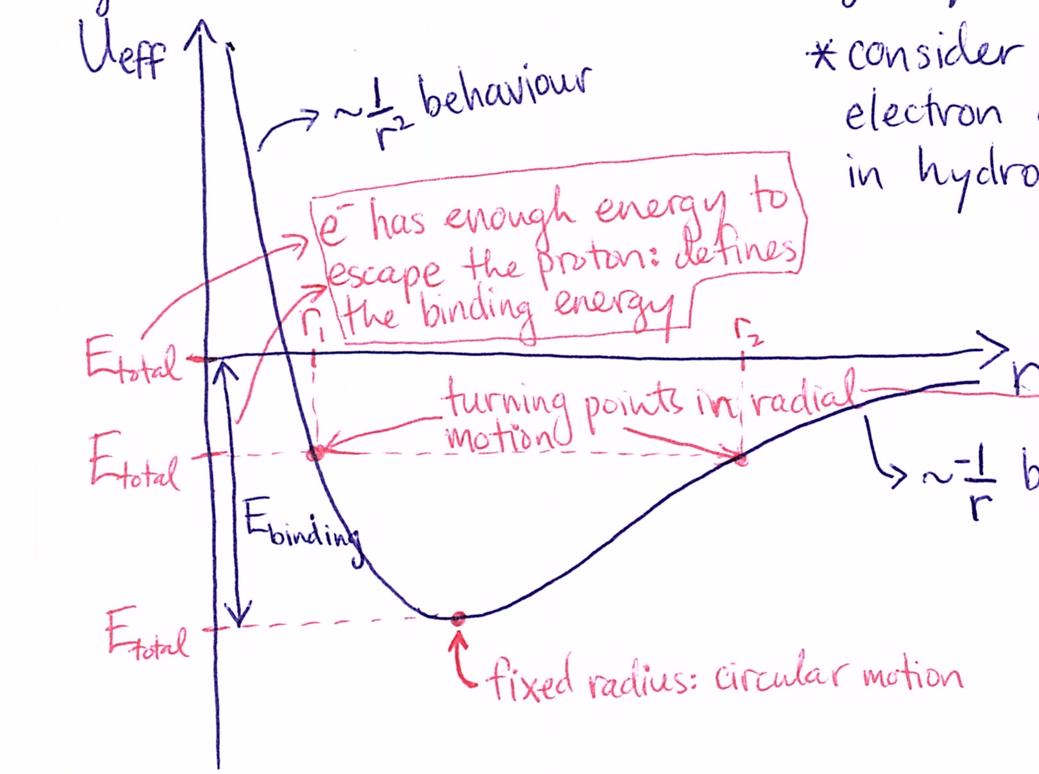


$$U_e = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$$

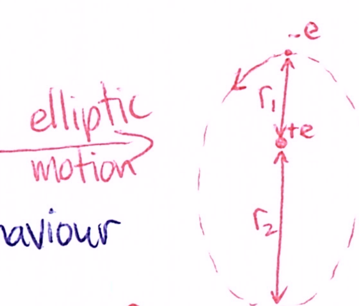


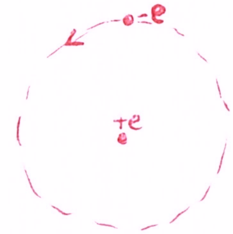


If q_2 has motion that is not radial, then it experiences an effective potential energy due to its angular momentum about charge q_1. The result:



*consider this as an electron orbiting a proton in hydrogen atom.





There is a general relationship between the force on a charge and its potential energy:

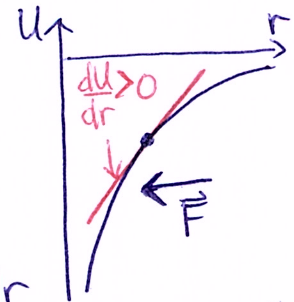
$$\vec{F}_e = -\frac{dU_e}{dr}\hat{r} \Rightarrow \vec{F}_e = -\frac{d}{dr}\left(\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}\right)\hat{r}$$

$$\vec{F}_e = -\frac{1}{4\pi\epsilon_0}\left(-\frac{q_1 q_2}{r^2}\right)\hat{r}$$

$$\vec{F}_e = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$ ✓ works for the case we know

Graphically:

q_1, q_2 same sign, $\vec{F}$ is repulsive (toward larger r)



q_1, q_2 opposite sign, $\vec{F}$ is attractive (toward smaller r)

In general, for a conservative force $\vec{F}$ there is a potential energy associated with is such that

$$\vec{F} = -\underbrace{\nabla}_{\text{"gradient"}} U \underbrace{\equiv}_{\text{"defined as" or "equivalent to"}} -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k} \qquad (*)$$

Let's write the force on a charge q sitting in an electric field $\vec{E}$ as $\vec{F} = q\vec{E}$ and write the potential energy in terms of the electric potential as $U = qV$

(*) becomes $\quad q\vec{E} = -\frac{\partial}{\partial x}(qV)\hat{i} - \frac{\partial}{\partial y}(qV)\hat{j} - \frac{\partial}{\partial z}(qV)\hat{k}$

but q is constant, so write this as

$$q\vec{E} = -q\frac{\partial V}{\partial x}\hat{i} - q\frac{\partial V}{\partial y}\hat{j} - q\frac{\partial V}{\partial z}\hat{k}$$

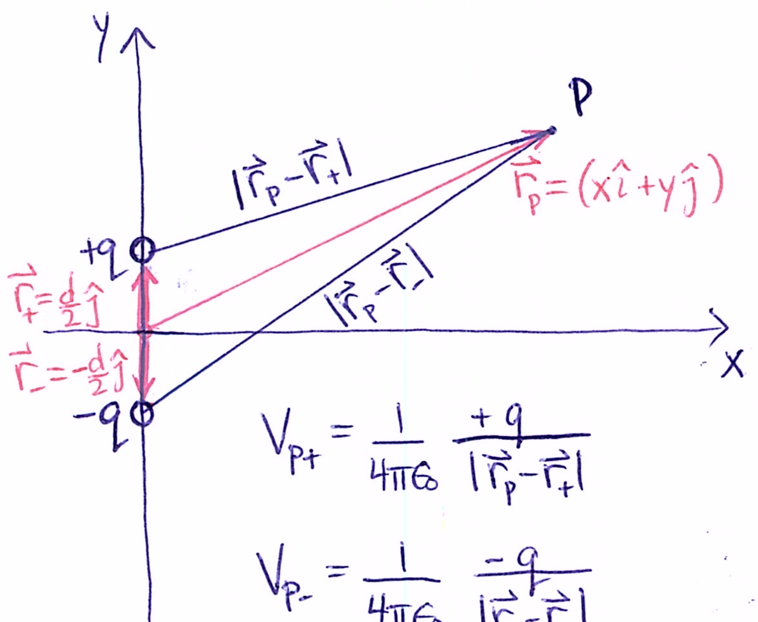
$$\cancel{q}\vec{E} = \cancel{q}\left(-\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}\right)$$

Therefore $\vec{E}$ is related to V by

$$\boxed{\vec{E} = -\nabla V = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}}$$

The electric potential encodes exactly the same information as the electric field but it is much easier to calculate because it is a SCALAR!

Example: Electric potential of a dipole at arbitrary (x,y)



$\vec{r}_p - \vec{r}_+$ = displacement vector from $+q$ to P.

$\vec{r}_p - \vec{r}_-$ = displacement vector from $-q$ to P.

Potential at point P:

$$V_{P+} = \frac{1}{4\pi\epsilon_0}\frac{+q}{|\vec{r}_p - \vec{r}_+|}$$

$$V_{P-} = \frac{1}{4\pi\epsilon_0}\frac{-q}{|\vec{r}_p - \vec{r}_-|}$$

$$V_p = V_{P+} + V_{P-}$$

$$|\vec{r}_p - \vec{r}_+| = \sqrt{(x-0)^2 + (y-d/2)^2}$$

$$|\vec{r}_p - \vec{r}_-| = \sqrt{(x-0)^2 + (y+d/2)^2}$$

$$\boxed{V_p = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+(y-d/2)^2}} - \frac{1}{\sqrt{x^2+(y+d/2)^2}}\right]}$$

This was easily obtained. Now to get the x-component of the electric field <u>anywhere in the (x,y) plane</u> we take $E_x = -\frac{\partial V}{\partial x}$ and similarly $E_y = -\frac{\partial V}{\partial y}$. This is <u>MUCH</u> easier than attempting to calculate $\vec{E}$ directly because you would need to decompose two vectors into their x- and y-components, then appropriately add or subtract them; try it, it's a mess!

Ideal dipole: take limit $d \to 0$ while keeping $p \equiv qd$ finite.

$$V_p = \lim_{d\to 0}\left[\frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+(y-d/2)^2}} - \frac{1}{\sqrt{x^2+(y+d/2)^2}}\right)\right]$$

for convenience, switch to polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x^2+(y-d/2)^2 = x^2+y^2-yd+d^2/4 = r^2 - rd\sin\theta + \cancel{d^2/4}$$
$$x^2+(y+d/2)^2 = x^2+y^2+yd+d^2/4 = r^2 + rd\sin\theta + \cancel{d^2/4}$$

for small d, order d^2 is even smaller

$$V_p = \lim_{d\to 0}\left[\frac{q}{4\pi\epsilon_0}\left(\frac{1}{r\sqrt{1-\frac{d\sin\theta}{r}}} - \frac{1}{r\sqrt{1+\frac{d\sin\theta}{r}}}\right)\right]$$

* I have factored out r in each square root because I want to do a binomial expansion of each term.

$$V_p = \lim_{d\to 0}\left[\frac{q}{4\pi\epsilon_0 r}\left(\left(1-\frac{d\sin\theta}{r}\right)^{-1/2} - \left(1+\frac{d\sin\theta}{r}\right)^{-1/2}\right)\right]$$

Both terms are now in the form $(1 \pm X)^{-1/2}$ for small X: can expand!

$$V_p = \lim_{d \to 0} \left[\frac{q}{4\pi\epsilon_0 r} \left(\left(\cancel{1} - \left(-\frac{1}{2}\right) \frac{d\sin\theta}{r} \right) - \left(\cancel{1} + \left(-\frac{1}{2}\right) \frac{d\sin\theta}{r} \right) \right) \right]$$

order d^2 terms will become zero in the limit $d \to 0$

$$V_p = \lim_{d \to 0} \left[\frac{q}{4\pi\epsilon_0 r} \left(\frac{1}{2} \frac{d\sin\theta}{r} + \frac{1}{2} \frac{d\sin\theta}{r} \right) \right]$$

now set $qd = p$

$$V_p = \lim_{d \to 0} \left[\frac{p\sin\theta}{4\pi\epsilon_0 r^2} \right]$$

all the terms that become zero in the limit have been suppressed in this calculation.

$$\Rightarrow V_p = \frac{p\sin\theta}{4\pi\epsilon_0 r^2}$$

now switch back to x, y: $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

$r^2 = x^2 + y^2$

$$\boxed{V(x,y) = \frac{py}{4\pi\epsilon_0 (x^2+y^2)^{3/2}}}$$

potential due to an ideal dipole. Derivatives give x- and y- components of $\vec{E}$.