Last time:

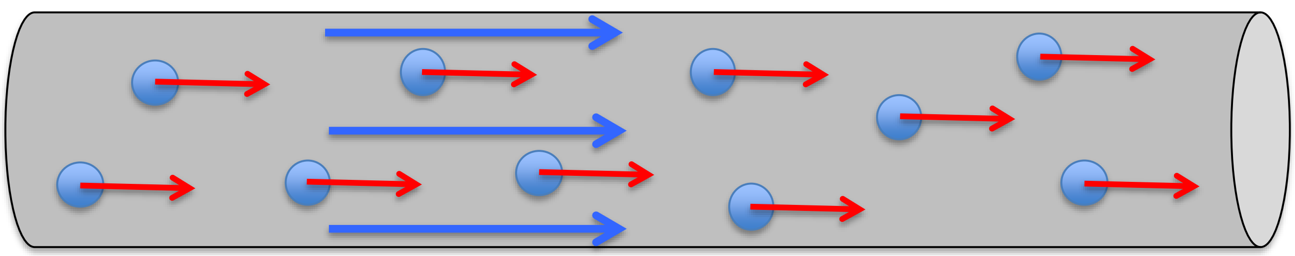
- Subtlety with capacitors: series or parallel?
- Linear dielectric materials: an atomic perspective
- Effect of dielectrics on capacitance
- Applications: fuel gauges, touch screens, wall climbers

Today:

- Electric current: a microscopic picture
- Current density (a vector) vs current (a scalar)
- Electric fields in conductors and electron drift speed

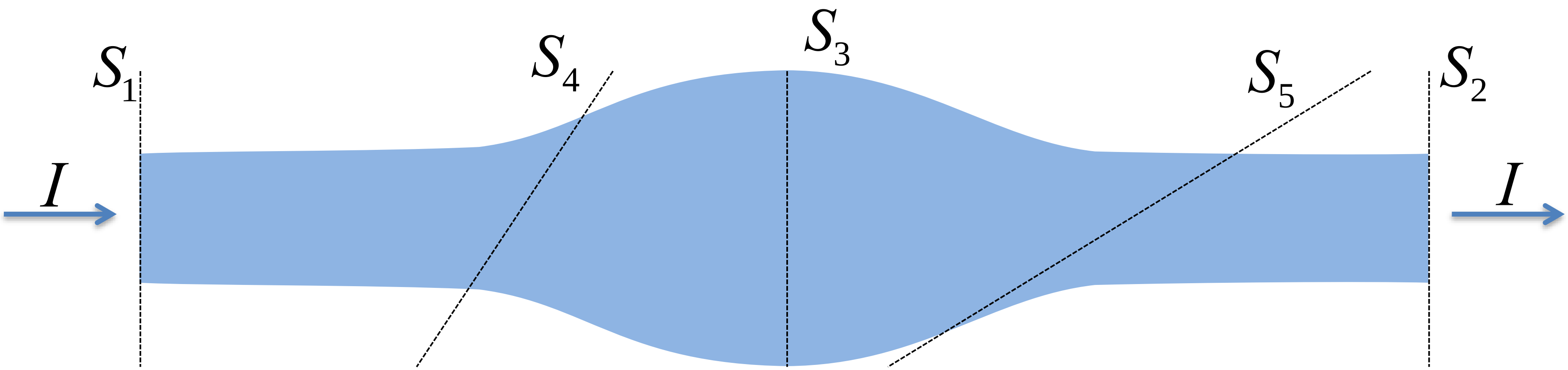
Where we're going

We will be using what we have been building up to talk about **moving charges** in electric circuits. This is no longer electrostatic equilibrium, so **conductors are allowed to have non-zero electric field inside** (this is what causes the charges to move).



First, we will take a closer look at what happens inside conductors and use this to define what an electric current is

Definition of current



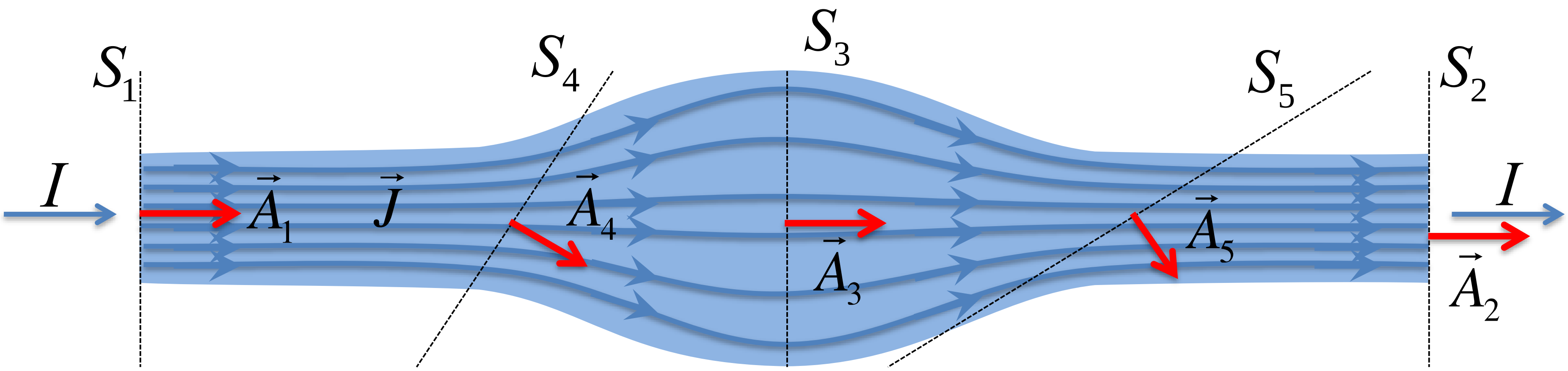
Total amount of charge flowing past this surface in a time Δt

$$\boxed{I = \frac{dq}{dt}}$$

Total amount of charge flowing past this surface in the same time Δt

Total amount of charge flowing through **ANY** surface in a time Δt must be constant, otherwise charges would begin to accumulate. **Current in a wire is constant**.

Current and Current Density



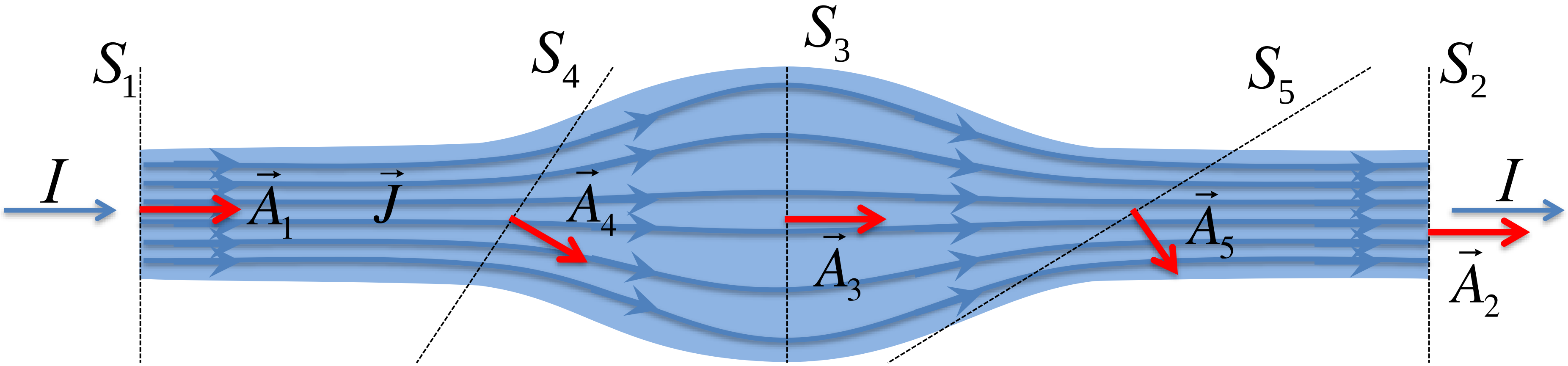
Total amount of charge flowing through **ANY** surface in a time Δt must be constant. This should be reminiscent of **FLUX**.

$$I = \oint_S \vec{J} \cdot d\vec{A}$$

The current in a wire is the flux of charge carriers (i.e. electrons) through a surface.

Since the current is constant, the flux through any cross-sectional surface must be the same.

Current and Current Density

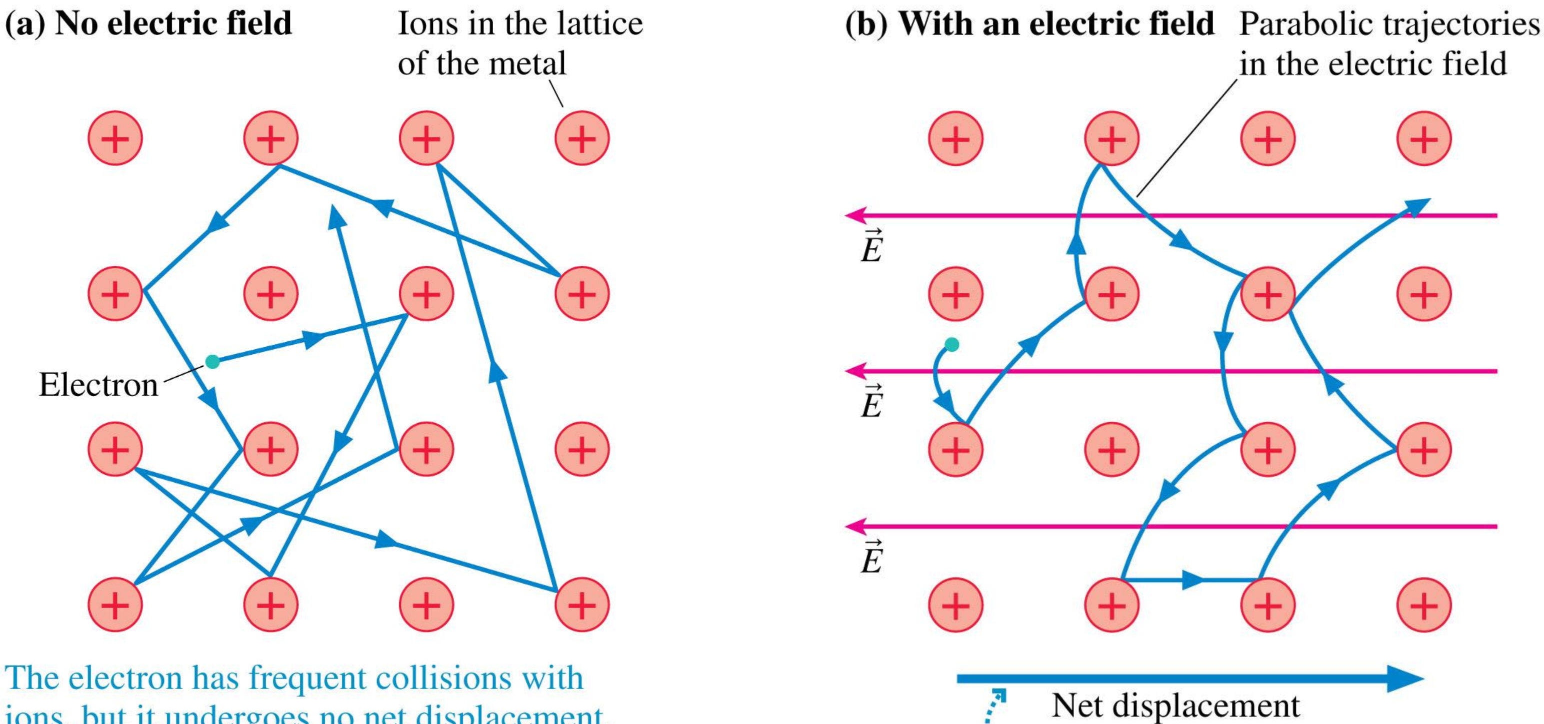


We call $\vec{J}$ the **current density**. It encodes information about:

- The density of conduction electrons in the conductor
- The net velocity of these conduction electrons

The current I is then interpreted as the number of charges passing through a surface in a specified direction. Note: current density is a vector, current is a scalar.

Inside a conductor

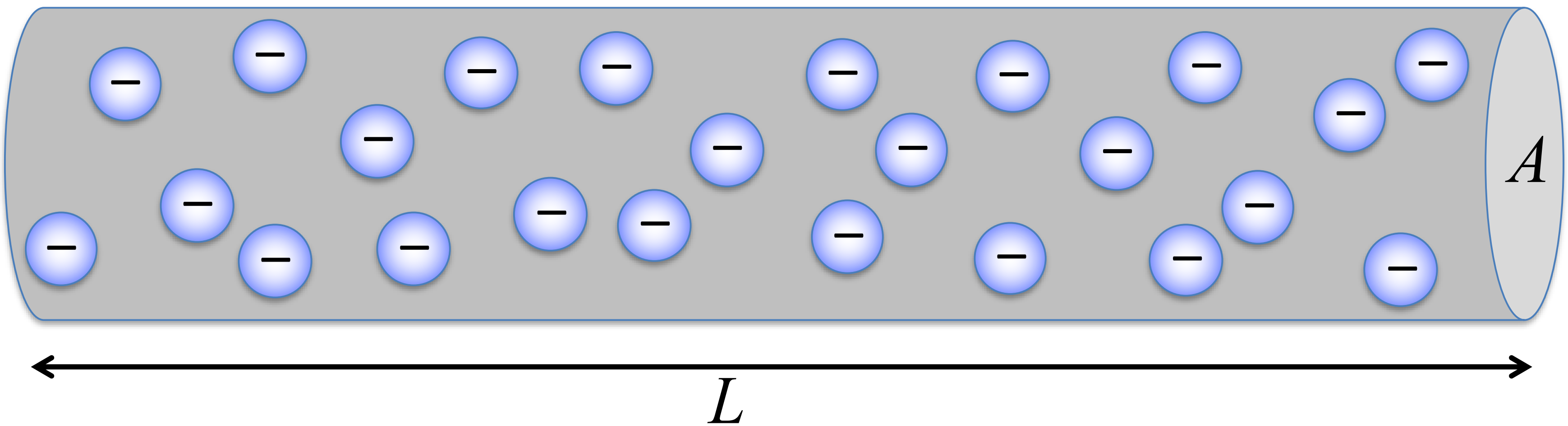


The electron has frequent collisions with ions, but it undergoes no net displacement.

A net displacement in the direction opposite to $\vec{E}$ is superimposed on the random thermal motion.

Net result: electrons move at an average net “drift speed” v_d

Current Density



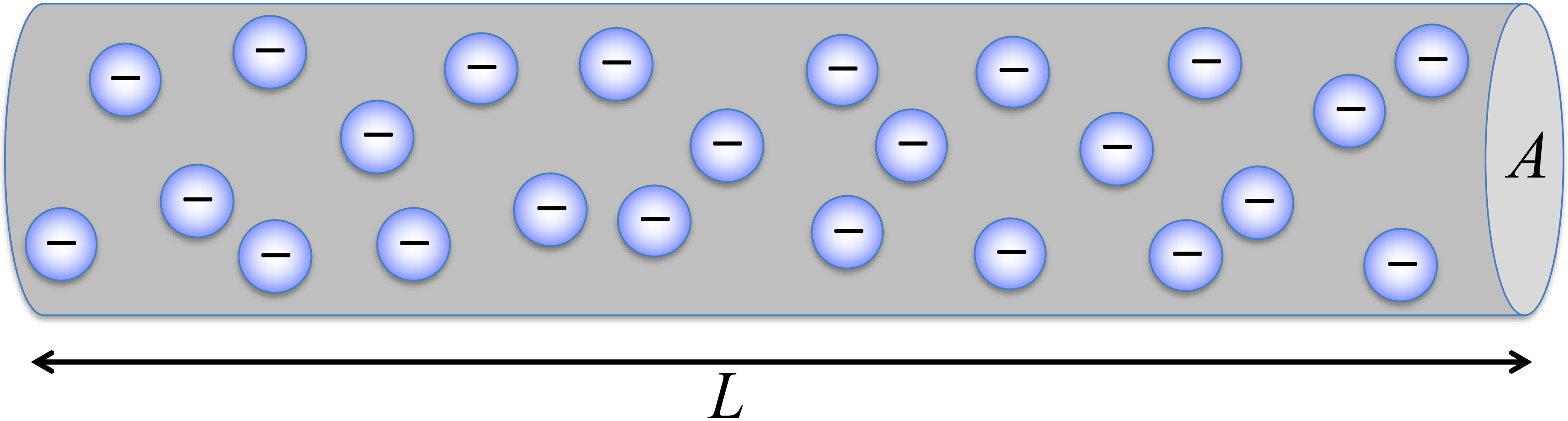
If the volume density of conduction electrons is n_e, then the amount of charge contained in a length L of the wire is

$$q = n(AL)e$$

The time it takes each charge to travel a distance L is $t = L/v_d$, so the current is

$$i = \frac{q}{t} = \frac{n(AL)e}{L/v_d} = nAev_d$$

Current Density

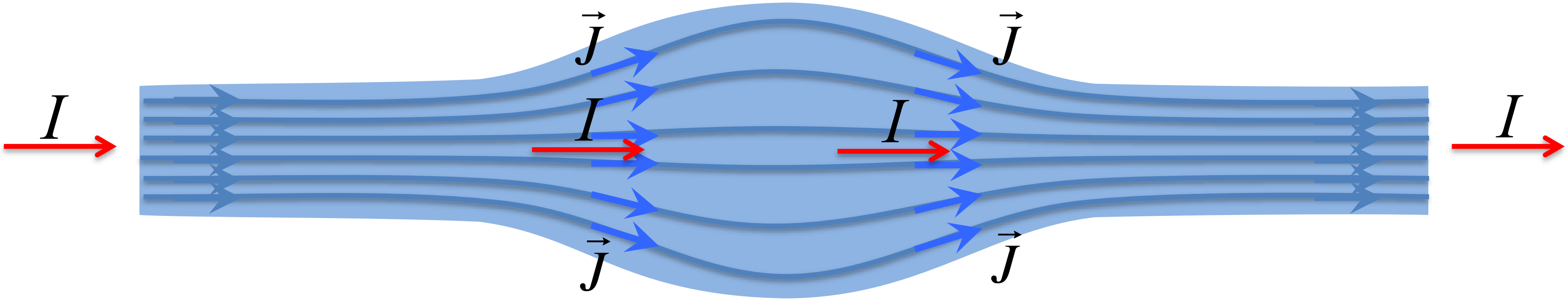


$$i = nAev_d = JA$$

The current density is then seen to be given by the **charge density** ne and the **drift velocity** (average velocity of the electrons)

$$\vec{J} = ne\vec{v}_d$$

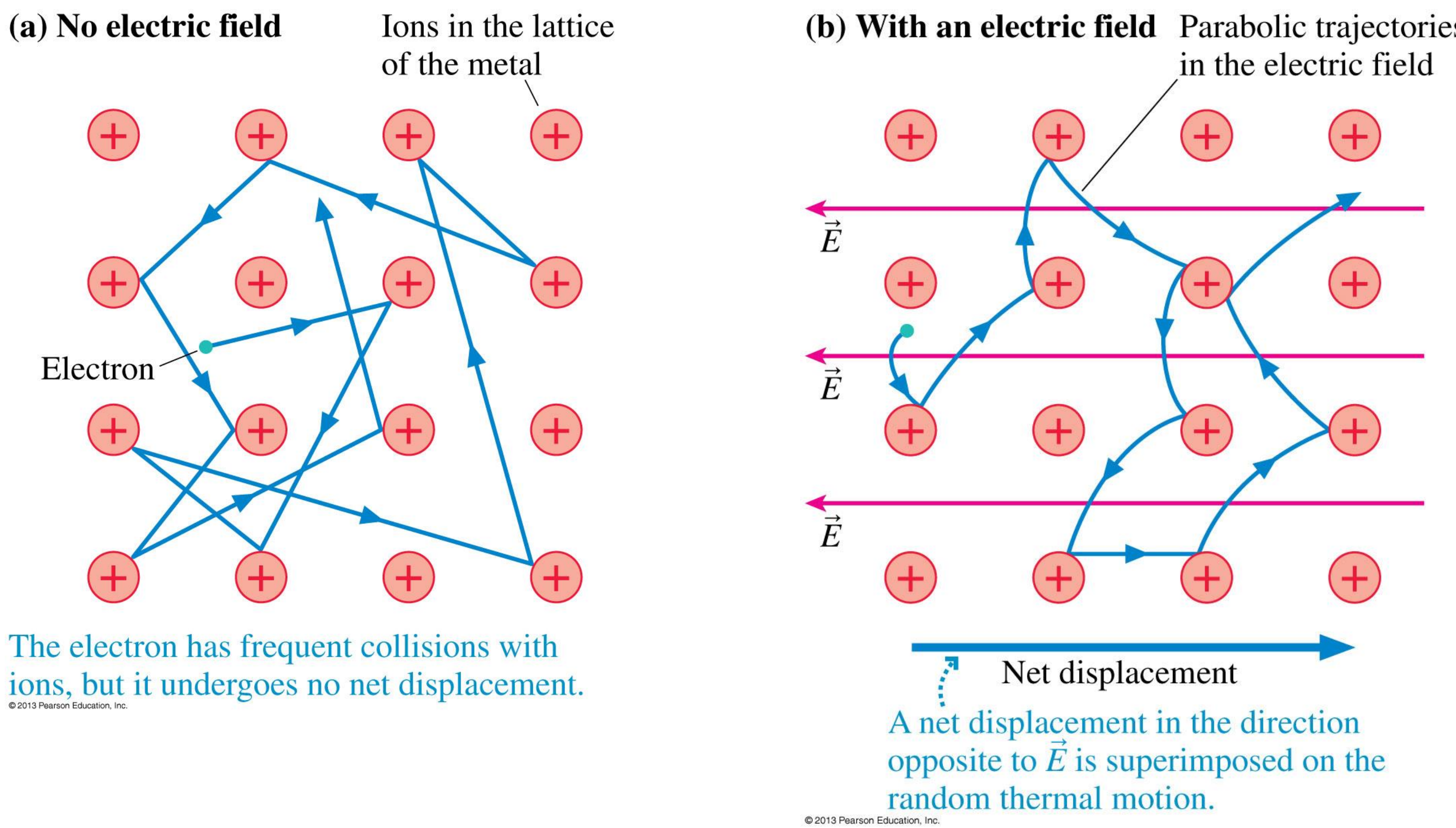
Current and Current Density



Because it is a vector, $\vec{J}$ is always in the direction of the "streamlines" of the electrons at any given location in the wire.

The current I, on the other hand, is a scalar and so it just has a magnitude. The direction we typically associate with it is the average displacement of all the charges in the wire, and so always points along the general direction of the wire.

Inside a conductor

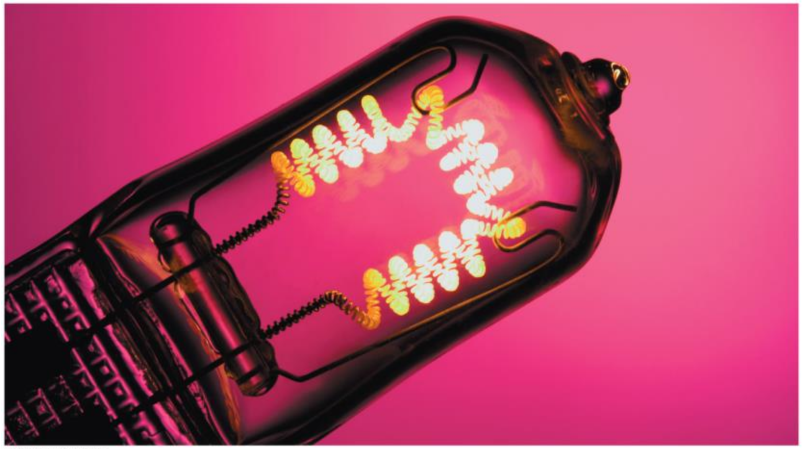


Resistance

Resistance is a property of conductors that are not ideal:

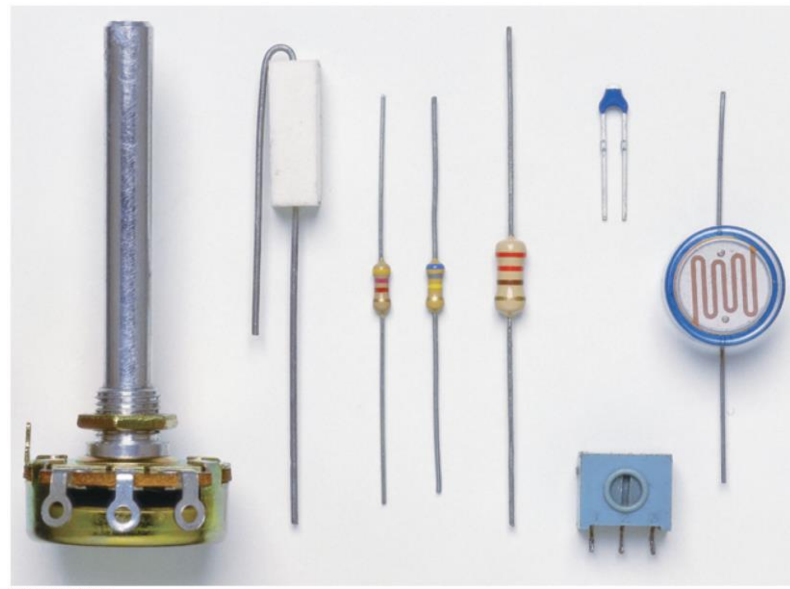
- Electrons have frequent collisions with atomic nuclei. At constant temperature, this process is in thermal equilibrium.
- When a voltage difference is created across the conductor, this accelerates the electrons, making their collisions more energetic.
- This gets dissipated as heat inside the metal

Tungsten filament:



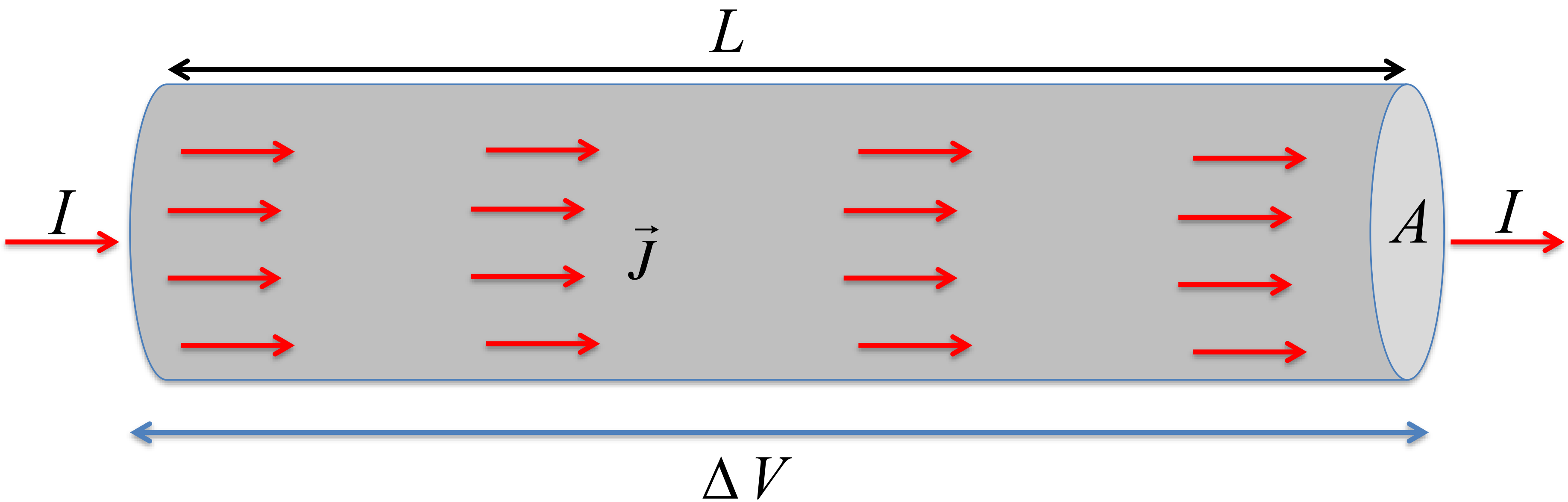
Resistors

A resistor is any circuit element that dissipates energy. Light bulbs are the classic example, but there are others:



How much energy is dissipated by a given resistor is encoded in a property called its resistance *R*. The resistance is dependent on the particular material used as well as the geometry.

How can we quantify this



Ohm's Law states that the current density inside the conductor is related to the electric field causing the charges to move via

$$\vec{E} = \rho \vec{J}$$

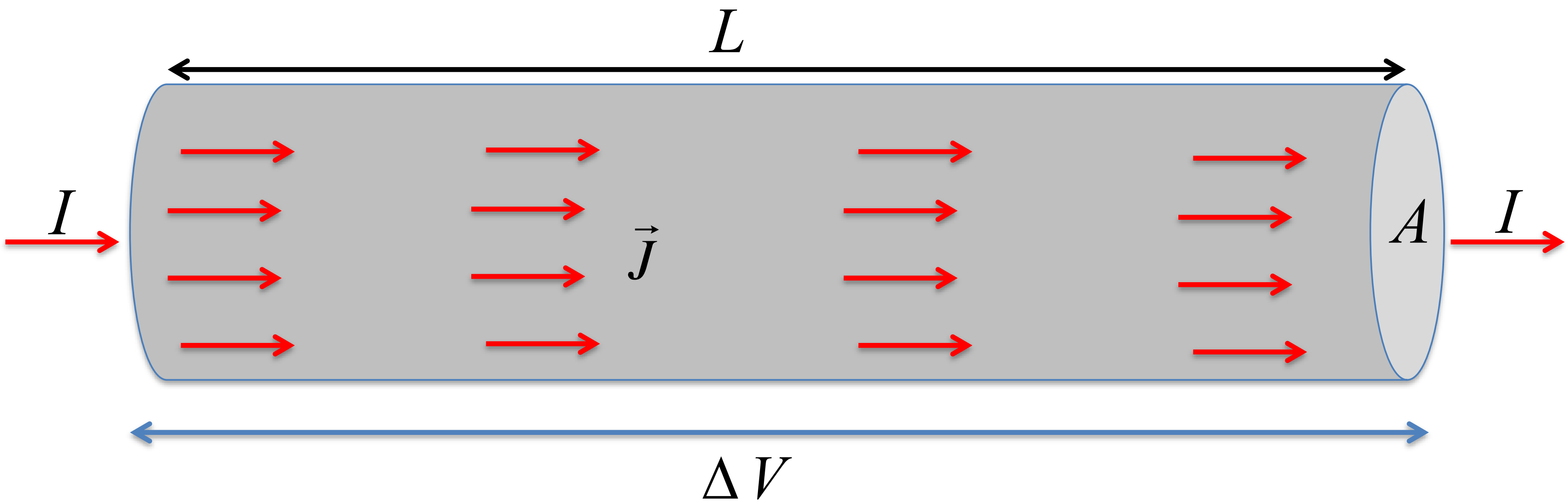
The resistivity is a physical property of the material that makes up the resistor

$$r = \frac{E}{J} = \frac{\Delta V / L}{I / A}$$

Using the resistivity, we can define a geometric quantity of the resistor:

$$r \frac{L}{A} = \frac{\Delta V}{I}$$

How can we quantify this



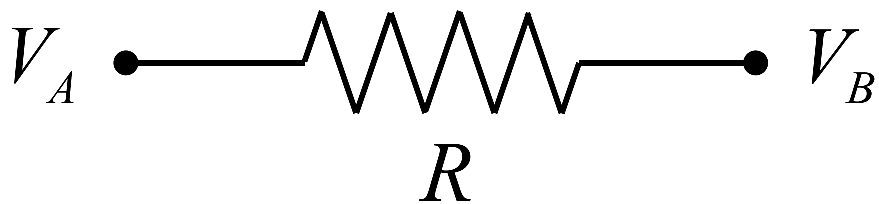
We introduce the resistance, which is dependent on ρ of the material and on the geometry of the resistor

$$\rho \frac{L}{A} \equiv R \qquad \rho \frac{L}{A} = \frac{\Delta V}{I} \qquad R = \frac{\Delta V}{I}$$

This gives us the familiar form of Ohm's Law: $\boxed{\Delta V = IR}$

Ohm’s Law

When a voltage difference ΔV is applied across a resistor R, the voltage difference causes electrons to flow through the resistor



This flow of electrons is the electric current I. These quantities are related by Ohm’s Law:

$$\Delta V = IR$$

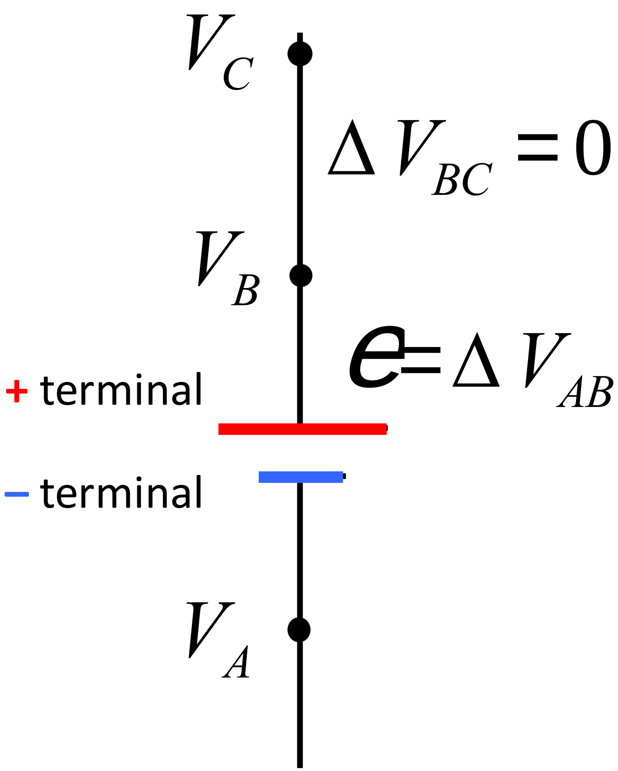
Current convention: the flow of positive charge (opposite the flow of negative charge)

$$(+e)(+\vec{v}) = (-e)(-\vec{v})$$

Both define a current I pointing to the right, so it is more convenient to think in terms of + charges moving

Ideal wires & batteries

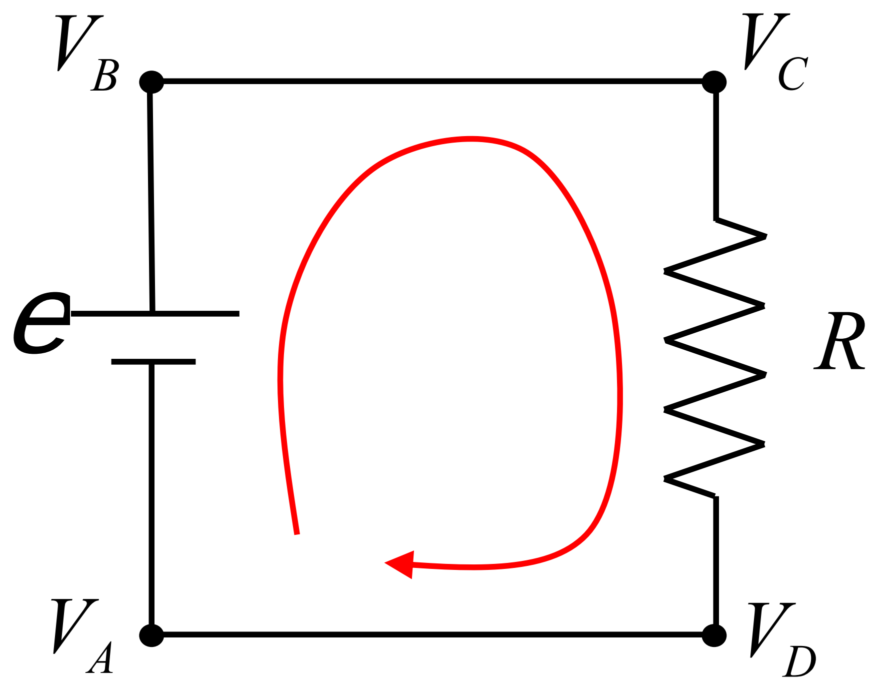
Real wires always have some **resistance** to them, but it is usually **small enough** that we can **ignore** it.
In this class we will <u>usually</u> treat wires as **ideal**, meaning $\Delta V = 0$ across any wire segment even if there is a current flowing.



A battery is any **source** that supplies a **voltage difference** in an electric circuit. The voltage is either specified by *V* or by the symbol e which stands for **electromotive force** (EMF)
Real batteries also have a resistance to them and we will see later how to account for this.

A Basic Circuit

The simplest circuit has an ideal battery, ideal wires, and a single resistor.



Kirchhoff's Loop Rule:

The sum of the voltage differences around a closed loop in a circuit must be zero.

(conservation of energy)

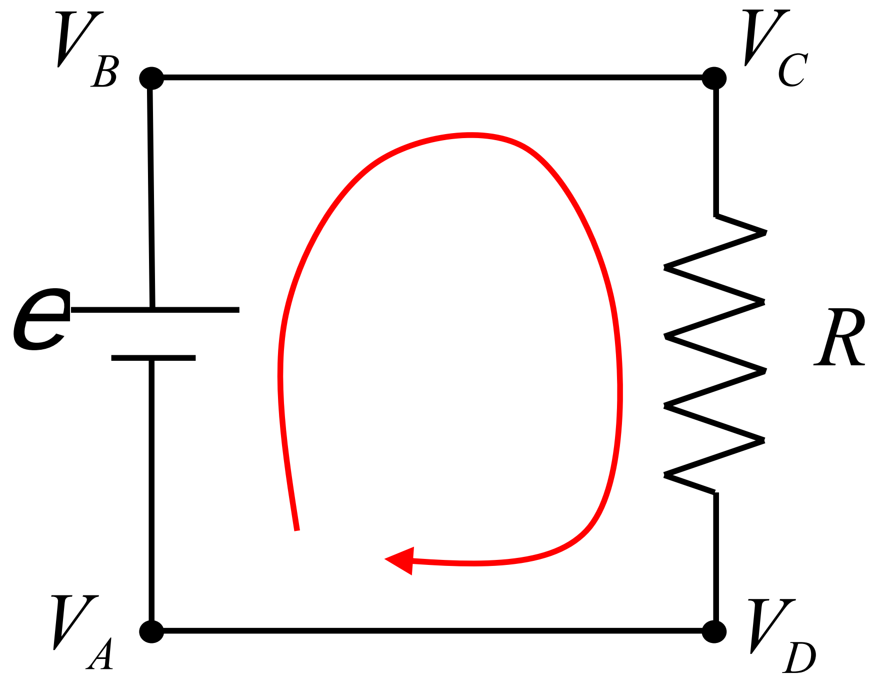
$$\boxed{\Delta V_{AB} + \Delta V_{BC} + \Delta V_{CD} + \Delta V_{DA} = 0}$$

$$(V_B - V_A) + (V_C - V_B) + (V_D - V_C) + (V_A - V_D) = 0$$

A Basic Circuit

The voltage across a resistor is **negative** if you are going around the loop in the **direction of the flow of current**.

Current flows **from the negative** terminal **to the positive** terminal



ideal wires

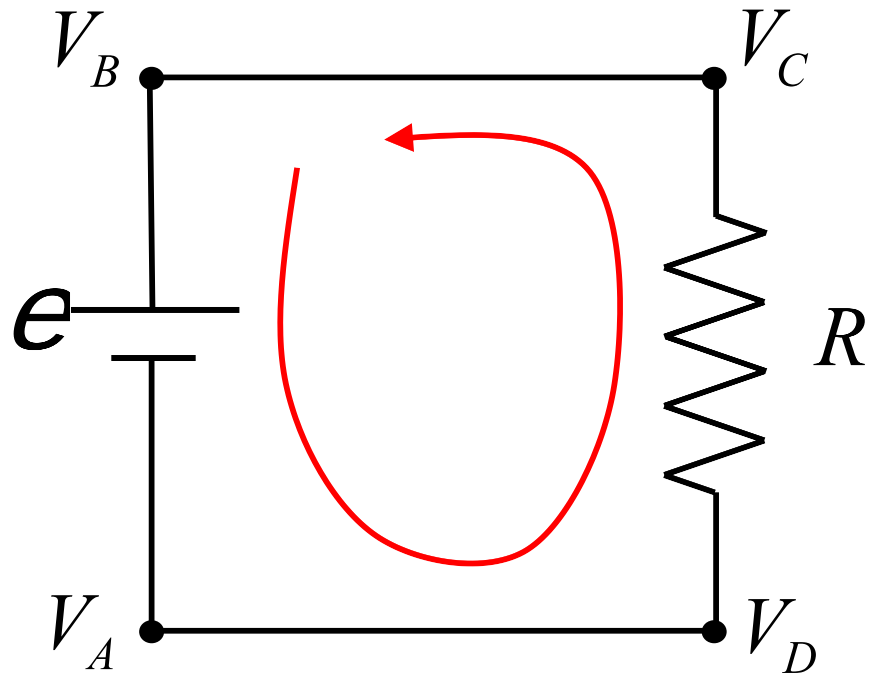
$$\Delta V_{AB} + \cancel{\Delta V_{BC}} + \Delta V_{CD} + \cancel{\Delta V_{DA}} = 0$$

$$e - IR = 0$$

Ohm's Law

A Basic Circuit

The voltage across a resistor is **positive** if you are going around the loop in the **opposite direction of the flow of current**. Voltage across a battery is **negative** going **from positive to negative**



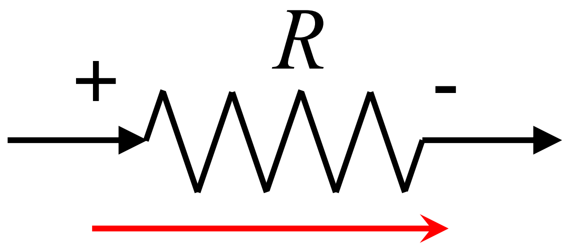
ideal wires

$$\Delta V_{BA} + \cancel{\Delta V_{AD}} + \Delta V_{DC} + \cancel{\Delta V_{CB}} = 0$$

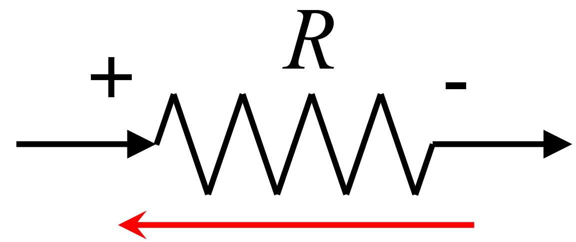
$$-e + IR = 0$$

Same as before

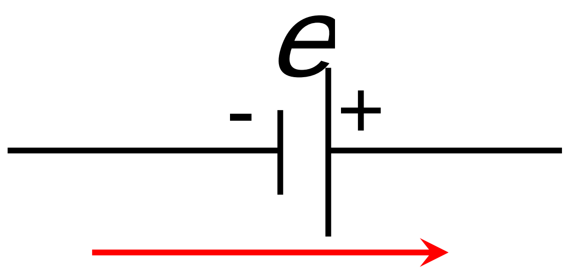
Kirchhoff’s Loop Rule



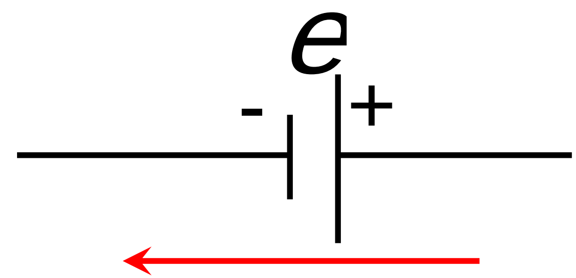
Higher to lower V: $\Delta V = - IR$



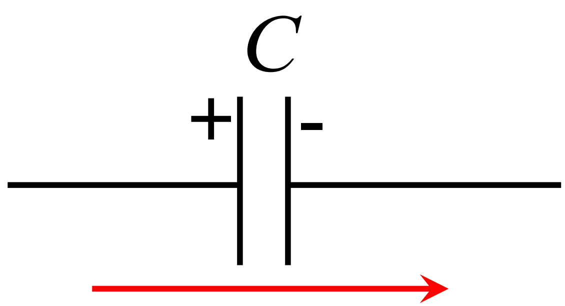
Lower to higher V: $\Delta V = + IR$



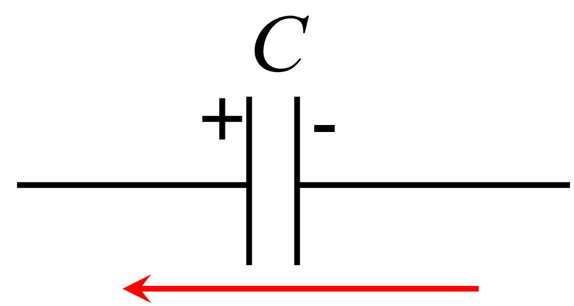
Lower to higher V: $\Delta V = + e$



Higher to lower V: $\Delta V = - e$



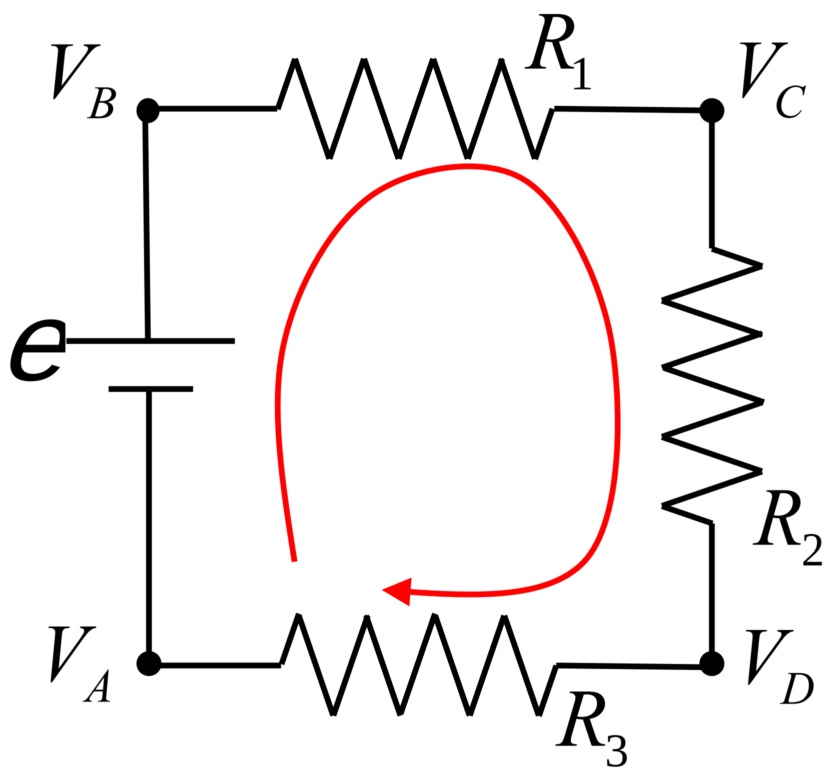
Higher to lower V: $\Delta V = - \dfrac{Q}{C}$



Lower to higher V: $\Delta V = + \dfrac{Q}{C}$

Resistors in Series

A slightly more complicated circuit has multiple resistors in series



Kirchhoff's Loop Rule:

$$\Delta V_{AB} + \Delta V_{BC} + \Delta V_{CD} + \Delta V_{DA} = 0$$

Current through each R is same

$$e - IR_1 - IR_2 - IR_3 = 0$$

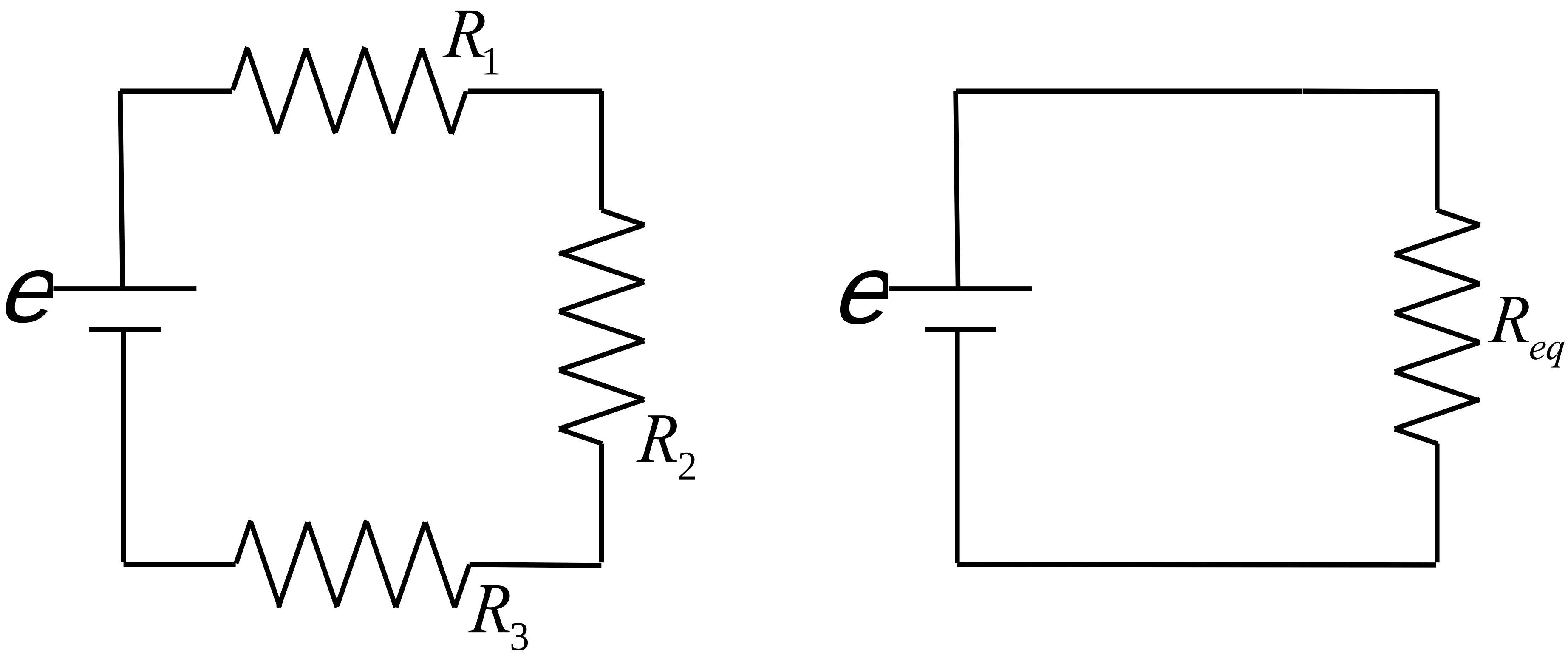
Rewrite this as

$$e - I\left(R_1 + R_2 + R_3\right) = 0$$

Define an equivalent resistance

$$e - IR_{eq} = 0$$

Resistors in Series

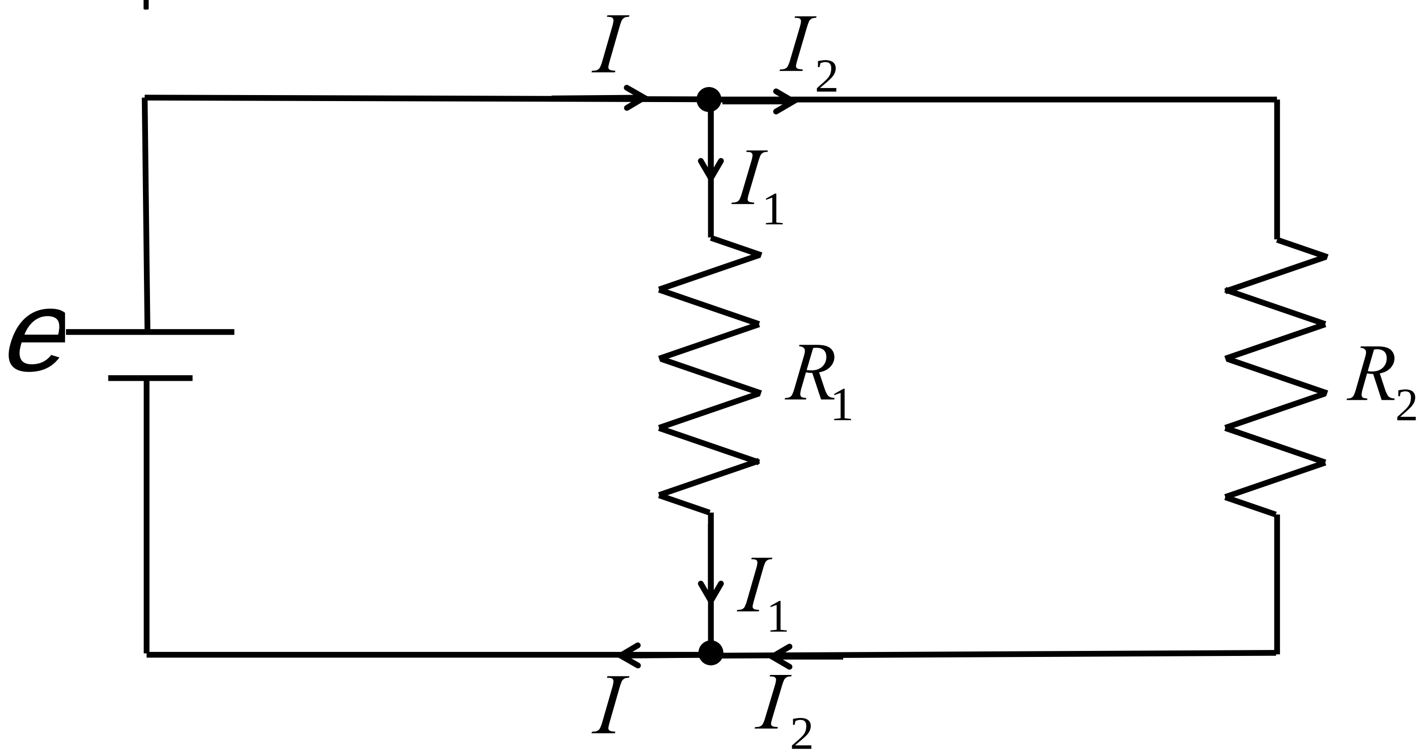


Resistors in series act like a single equivalent resistor:

$$R_{eq} = R_1 + R_2 + R_3$$

Resistors in Parallel and Kirchhoff's junction rule

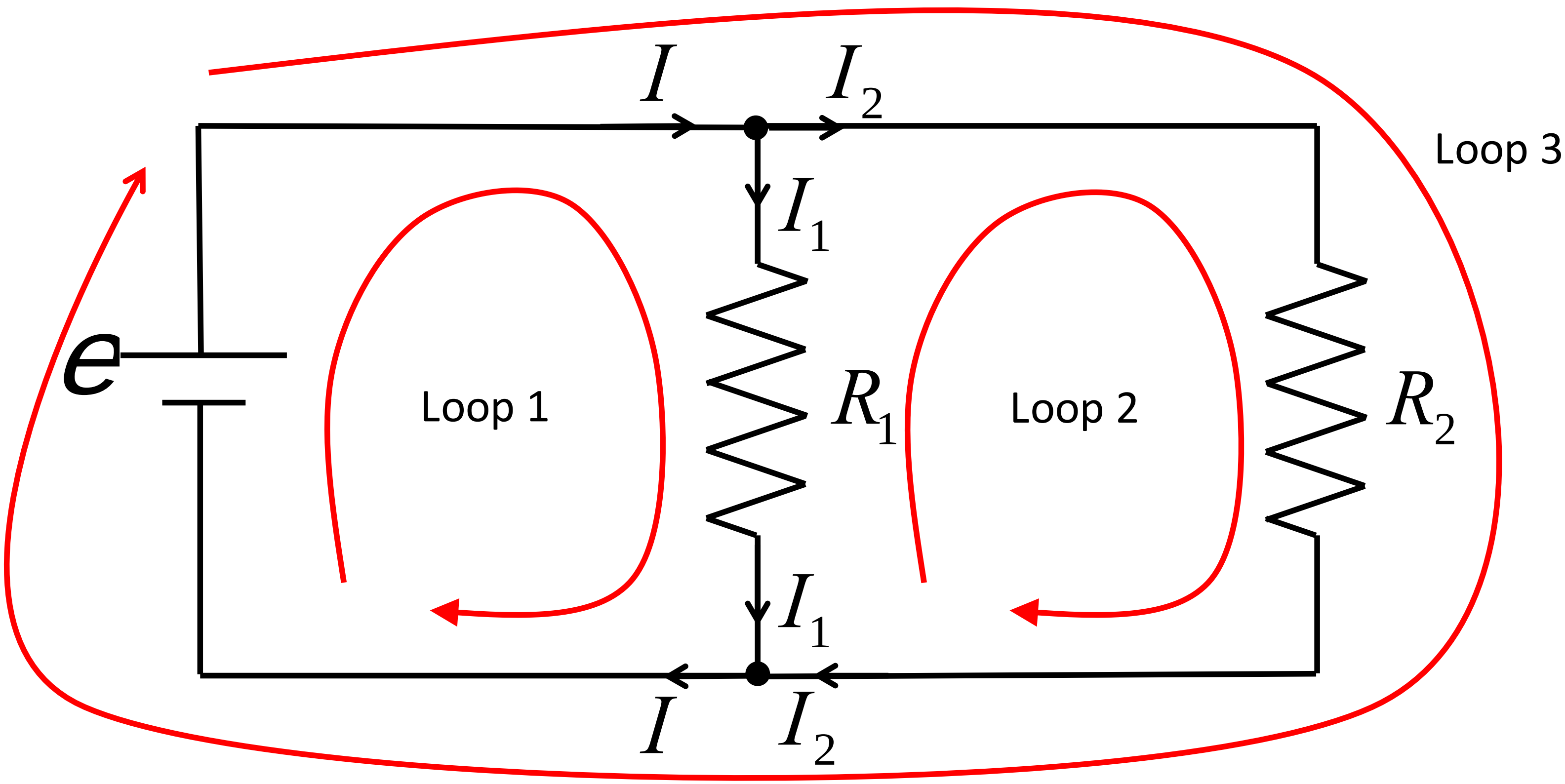
A slightly more complicated circuit has multiple branches with resistors in parallel



Current is the flow of charges. Charge has to be conserved.

Current into junction = current out of junction $\boxed{I = I_1 + I_2}$

Resistors in Parallel



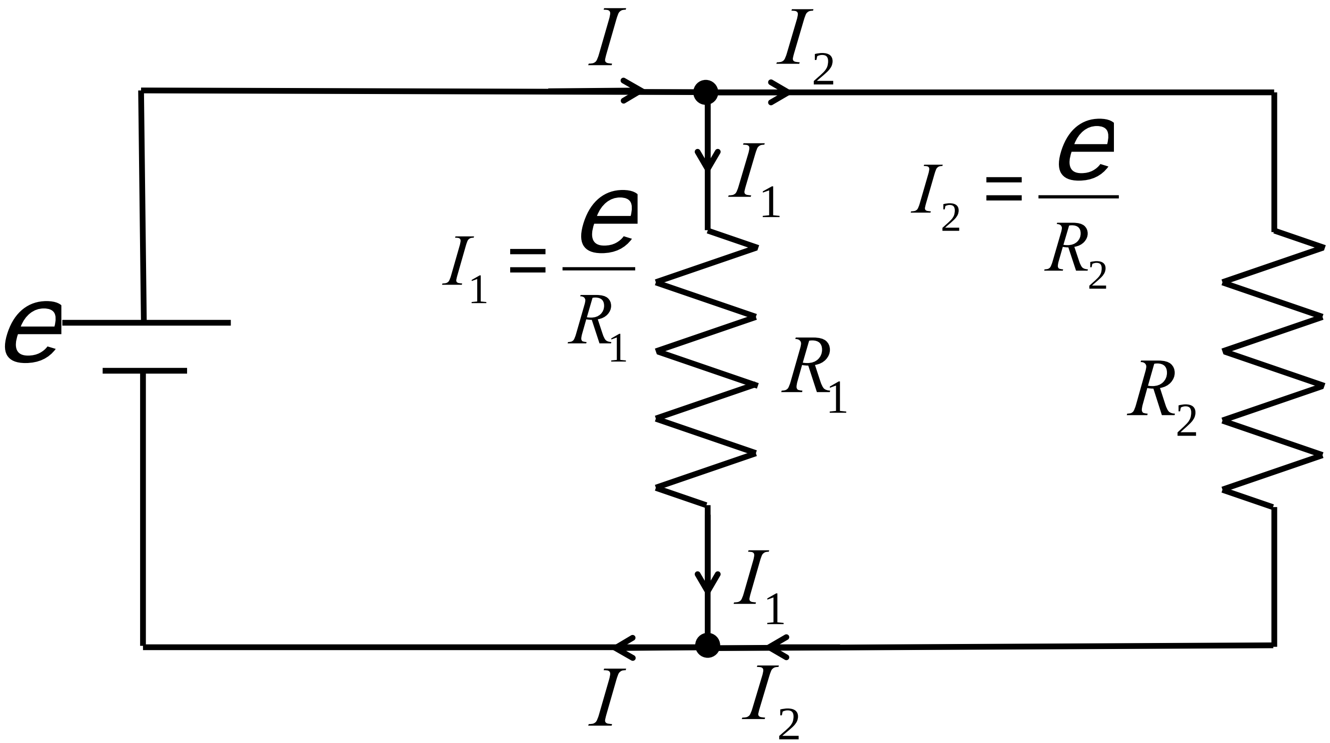
Loop 1: $\mathcal{E} - I_1R_1 = 0$

Loop 2: $I_1R_1 - I_2R_2 = 0$

Loop 3: $\mathcal{E} - I_2R_2 = 0$

$$I_1 = \frac{\mathcal{E}}{R_1} \qquad I_2 = \frac{\mathcal{E}}{R_2}$$

Resistors in Parallel



$$I = I_1 + I_2$$

$$I = \frac{\varepsilon}{R_1} + \frac{\varepsilon}{R_2}$$

$$= \varepsilon\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= \frac{\varepsilon}{R_{eq}}$$

Resistors in parallel: $$R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

Summary of Resistors

Ohm's Law

$$\Delta V_R = IR$$

Resistors in Series: have the same current running through them

$$R_{eq} = R_1 + R_2 + \ldots + R_N$$

Resistors in Parallel: have the same voltage across them

$$R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_N} \right)^{-1}$$

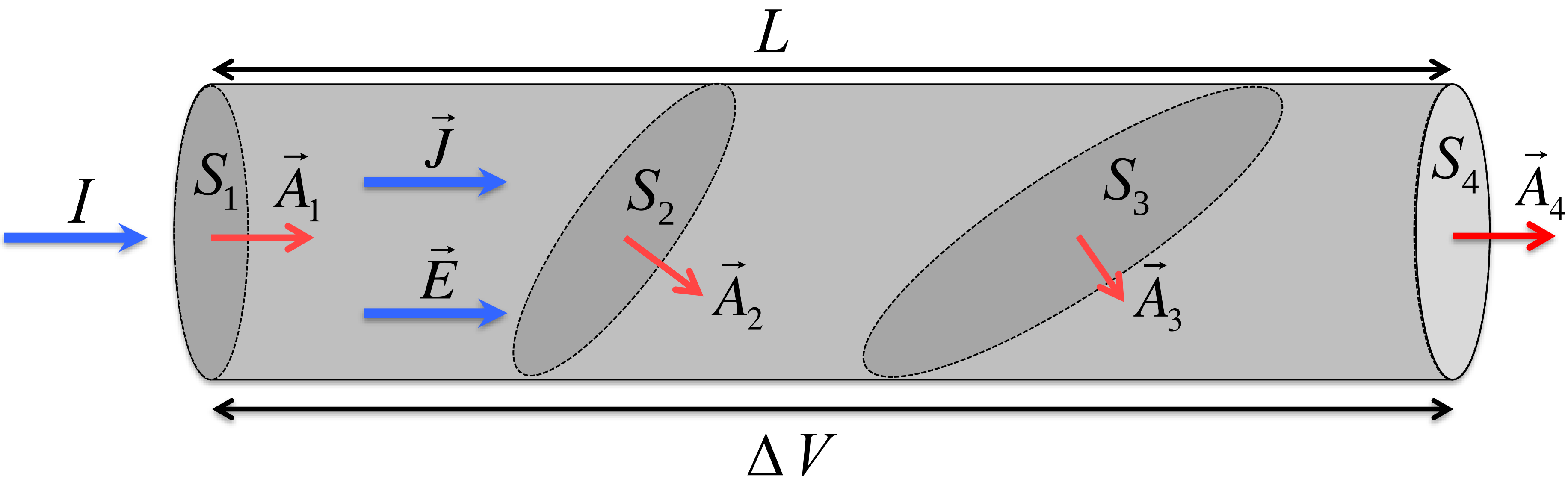
Last time:

- Resistance as geometric quantity
- Resistors in series
- Resistors in parallel

Today:

- Recap of resistivity + ohmic vs non-ohmic materials
- Microscopic view of resistivity
- Temperature dependence of resistivity/resistance
- Ideal vs non-ideal batteries

Quick Recap of Resistivity



Current as flux of J

$$I = \int_S \vec{J} \cdot d\vec{A}$$

Ohm's Law

$$\vec{E} = \rho \vec{J}$$

Current as flux of E

$$I = \frac{1}{\rho} \int_S \vec{E} \cdot d\vec{A} = \frac{EA}{\rho}$$

Rewrite E = ΔV/L

$$I = \frac{\Delta V}{L} \frac{A}{r}$$

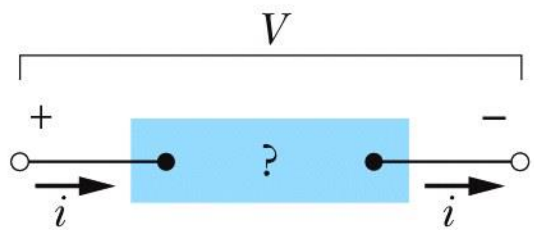
Familiar Ohm's Law

$$r \frac{L}{A} = \frac{\Delta V}{I}$$

Resistance

$$R = r \frac{L}{A}$$

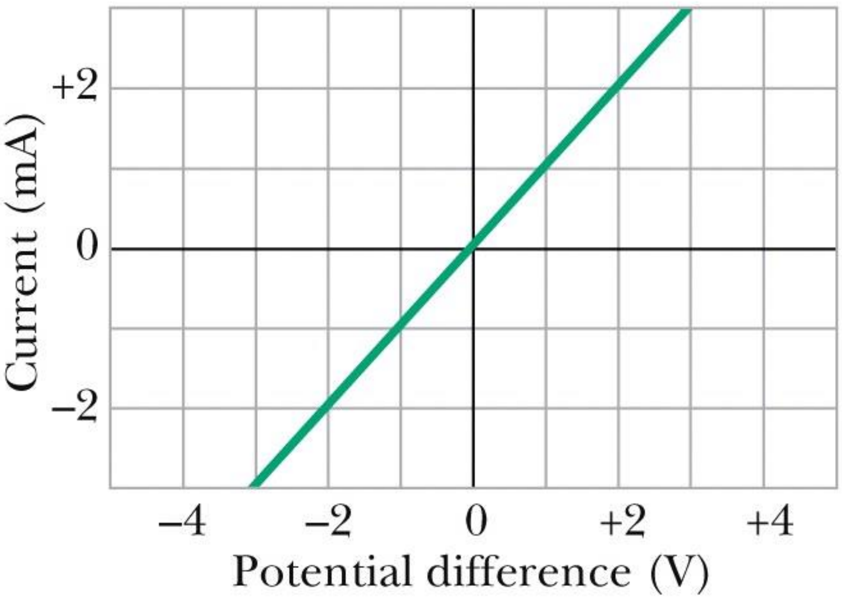
Ohmic vs non-Ohmic devices



(a)

Materials with isotropic electrical properties

Materials with anisotropic electrical properties (pn junction diode)



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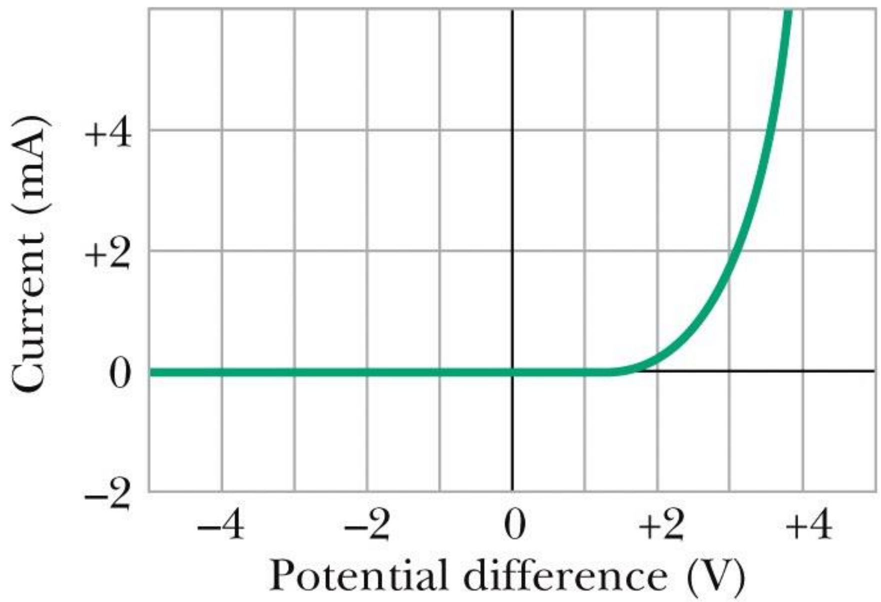

(c)

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TopHat Question

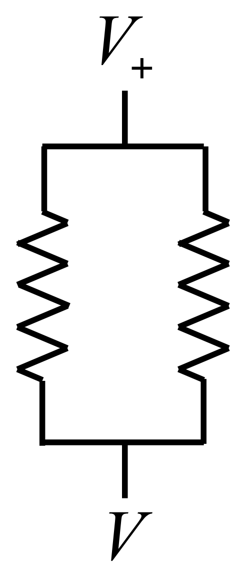
A cylindrical resistor is composed of 1/3 copper and 2/3 Tungsten as shown ($\rho_{Cu} = 1.69 \times 10^{-8}\ \Omega m$, and $\rho_W = 5.25 \times 10^{-8}\ \Omega m$). The radius of the cylinder is R = 0.105 mm, and it is L = 5.26 cm long.

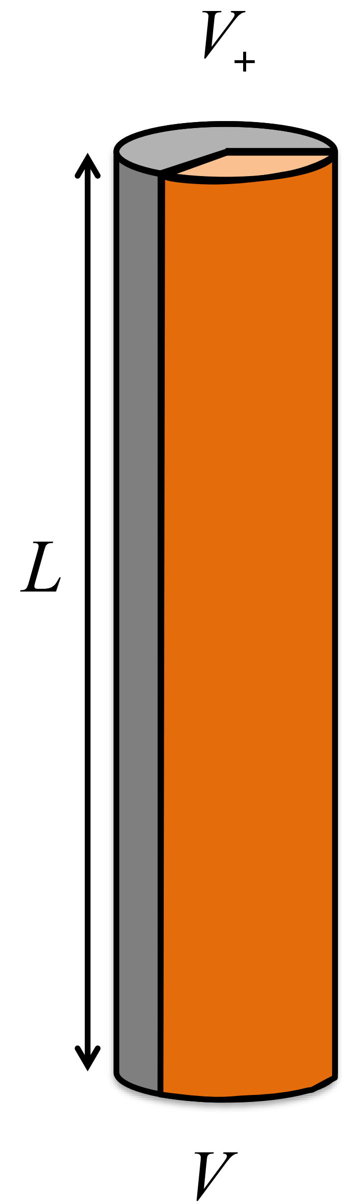
Each piece has a resistance. Are these resistors in series or in parallel?

A. Series

B. Parallel

C. Neither





TopHat Question

A cylindrical resistor is composed of 1/3 copper and 2/3 Tungsten as shown (ρ_{Cu} = 1.69x10^{-8} Ωm, and ρ_W = 5.25x10^{-8} Ωm). The radius of the cylinder is R = 0.105 mm, and it is L = 5.26 cm long.

What is the total resistance of this device?

A. 0.0468 Ω

B. 0.197 Ω

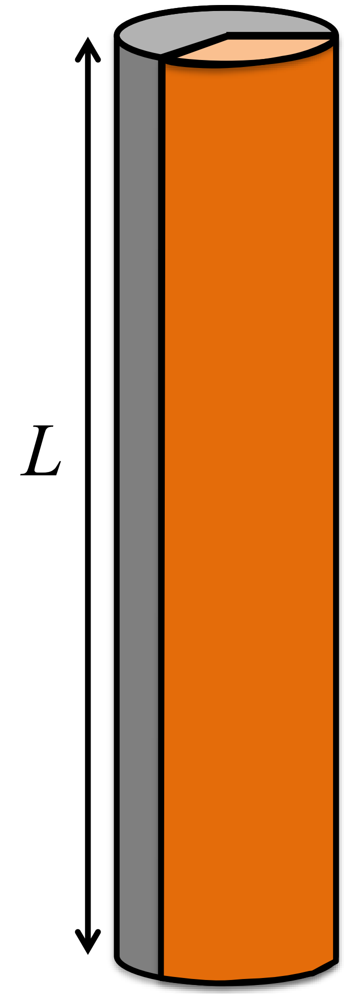
C. 0.105 Ω

D. 0.0729 Ω

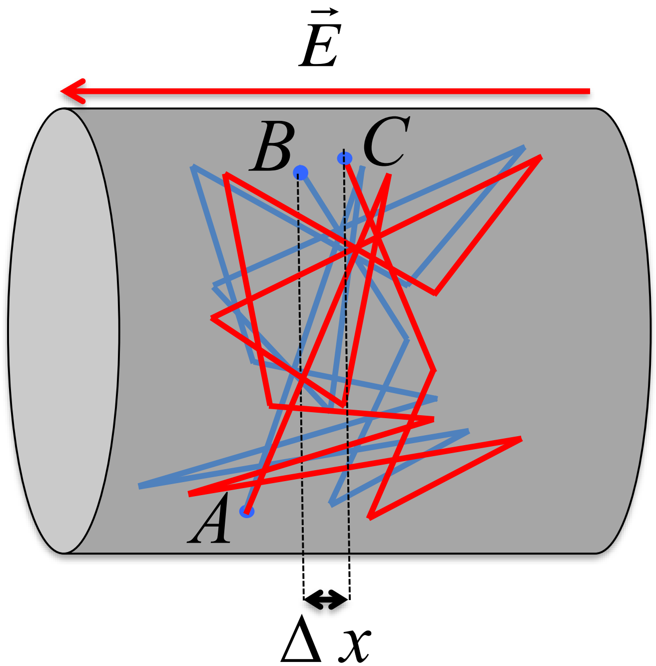
$$R = \left(\frac{1}{R_{Cu}} + \frac{1}{R_W} \right)^{-1}$$

$$R_{Cu} = \rho_{Cu} \frac{L}{\frac{1}{3} \pi R^2}$$

$$R_W = \rho_W \frac{L}{\frac{2}{3} \pi R^2}$$



Microscopic view of resistivity



Electrons bounce around inside the metal at speeds very high speeds on the order of 0.5% light speed.

When an electric field is applied in the conductor, there is a net force on the electrons, leading to an average "drift speed" of $v_d = 0.5\ \mu\text{m/s}$

The acceleration felt by the electrons from the E-field is

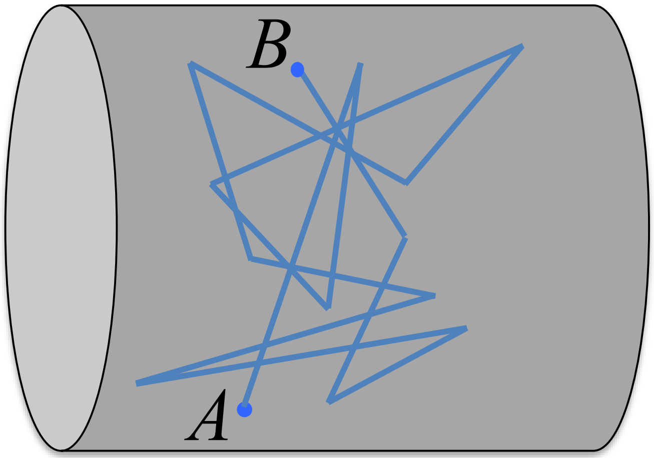
$$a_x = \frac{eE}{m}$$

So the average drift speed of the electrons will be given by

$$v_d = at = \frac{eEt}{m}$$ but we found before: $$v_d = \frac{J}{ne}$$

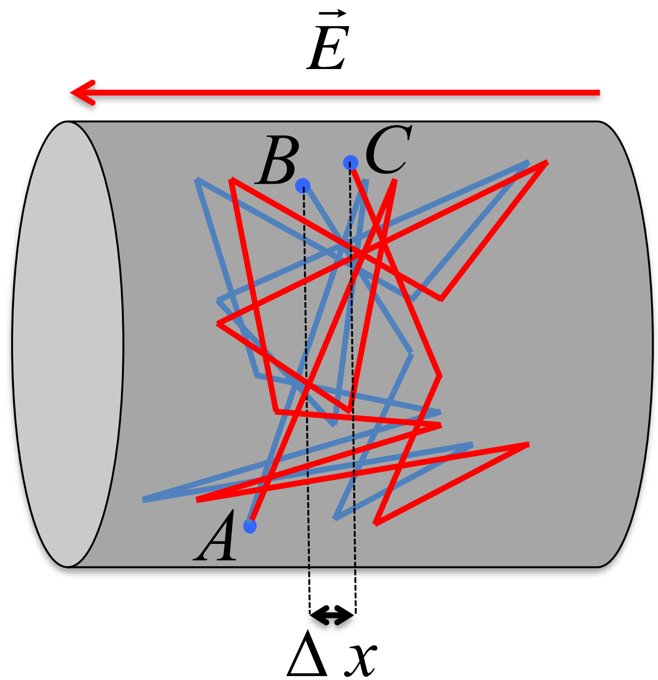
Microscopic view of resistivity

No E-field applied



$$v_d = \frac{eE\tau}{m}$$

$$v_d = \frac{J}{ne}$$



The average time between collisions is τ and is called the *mean free time*. Equating the two expressions for the drift speed, we get:

$$\frac{eE\tau}{m} = \frac{J}{ne} \quad \text{Rearrange this to find} \quad E = \left(\frac{m}{ne^2\tau}\right)J$$

This gives a microscopic picture of resistivity:

$$\boxed{\rho = \frac{m}{ne^2\tau}}$$

Consequence of this microscopic view

When the temperature of a metal increases, its volume increases (thermal expansion) according to

$$\frac{\Delta V}{V_0} = \alpha_V \frac{\Delta T}{T_0} \qquad \alpha_V = \text{vol. coefficient of thermal expansion}$$

The resistivity depends on the conduction electron number *density* and hence implicitly depends on the volume of the metal

$$\rho = \frac{m}{ne^2\tau} = \frac{mV}{Ne^2\tau} \qquad m, N, e, \text{ and } \tau \text{ are unaffected by } T$$

The resistivity is a temperature dependent property

$$\boxed{\alpha = \frac{\alpha_V}{T_0}}$$

$$\Delta\rho = \frac{m\Delta V}{Ne^2\tau} = \frac{mV_0}{Ne^2\tau}\left(\frac{\Delta V}{V_0}\right) \qquad \rho - \rho_0 = \rho_0\alpha\left(T - T_0\right)$$

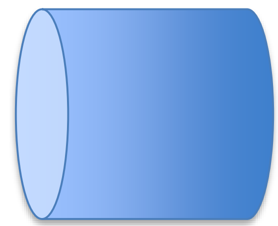
This is why the resistance of a device depends on temperature

Temperature Dependent Resistance

$$\frac{\Delta A}{A_0} = a_A \frac{\Delta T}{T_0} = \tfrac{2}{3} a_V \frac{\Delta T}{T_0}$$

$$\frac{\Delta L}{L_0} = a_L \frac{\Delta T}{T_0} = \tfrac{1}{3} a_V \frac{\Delta T}{T_0}$$

$$\frac{\Delta R}{R_0} = \frac{\Delta r}{r_0} + \frac{\Delta L}{L_0} - \frac{\Delta A}{A_0}$$



$T_0 \quad L_0 \quad A_0$

$T \quad L \quad A$

$$\frac{\Delta R}{R_0} = a_V \left(1 + \tfrac{1}{3} - \tfrac{2}{3}\right) \frac{\Delta T}{T_0}$$

$$\frac{\Delta R}{R_0} = \tfrac{2}{3} a_V \frac{\Delta T}{T_0}$$

$$\boxed{R - R_0 = R_0 \left(\tfrac{2}{3} a\right)\left(T - T_0\right)}$$

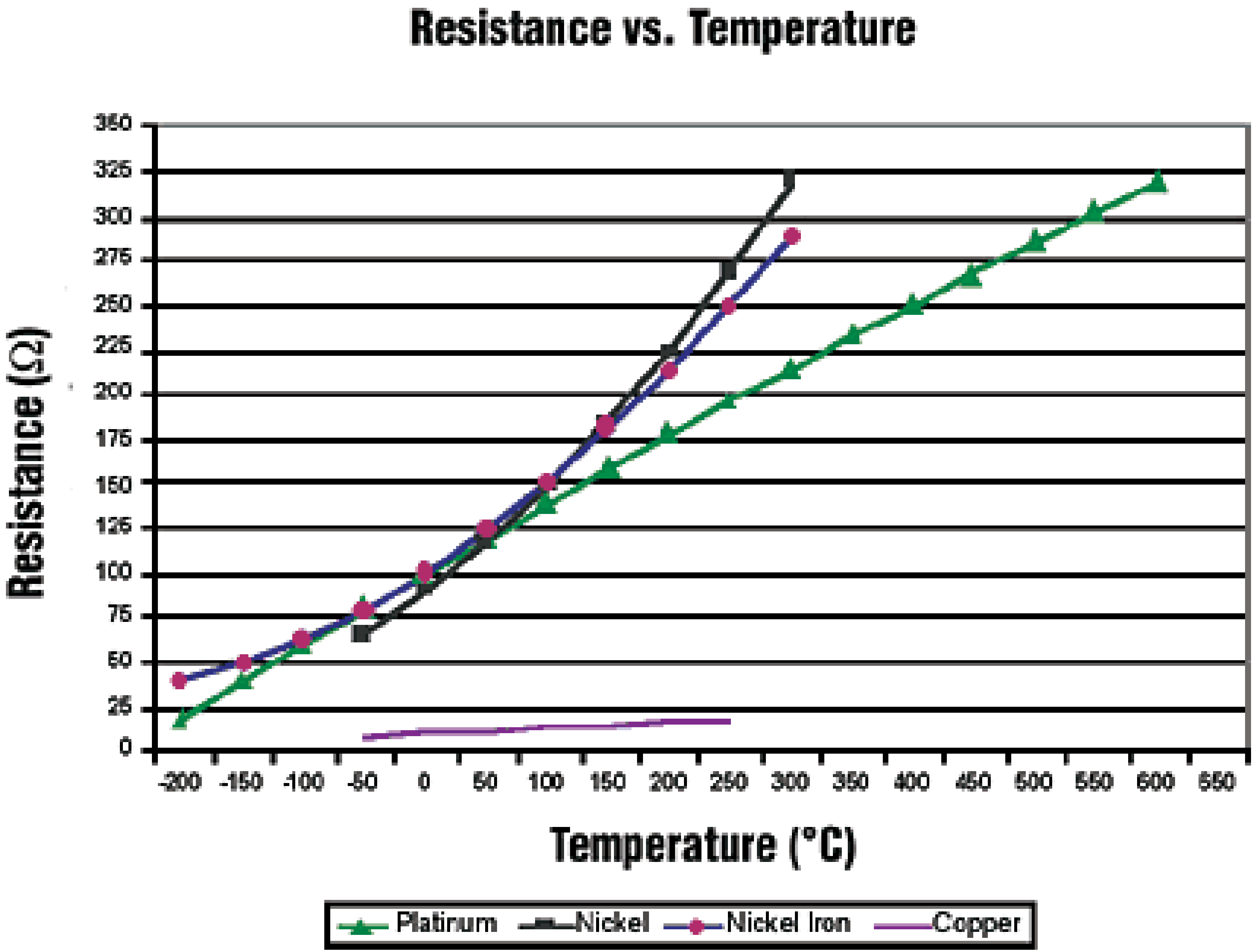
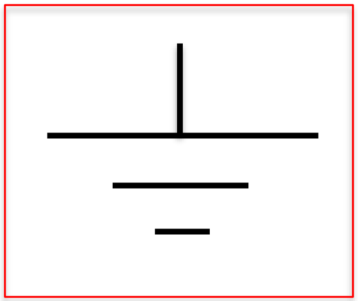
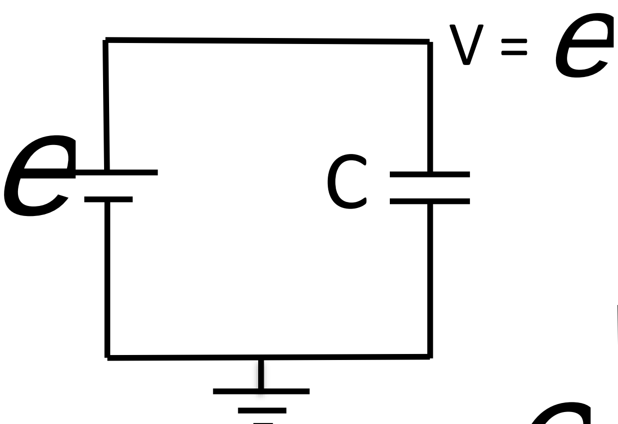


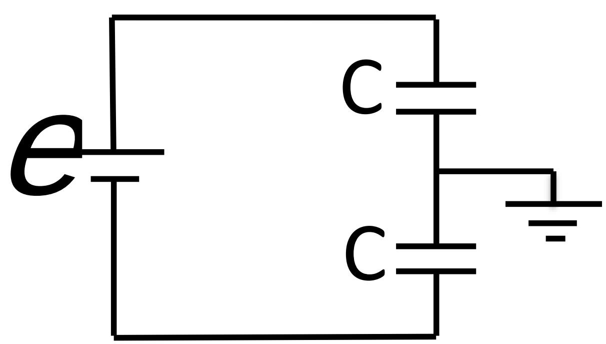
Image from http://archives.sensorsmag.com/articles/0503/52/main.shtml

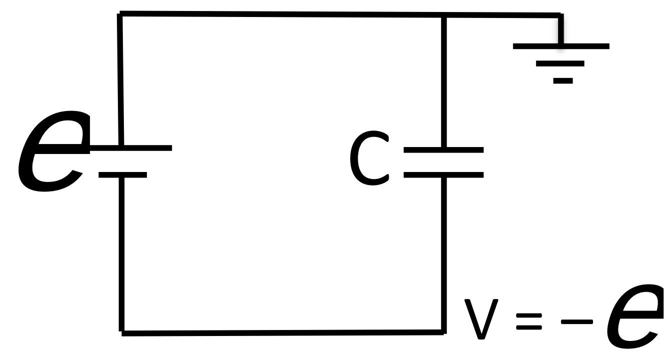
Grounding



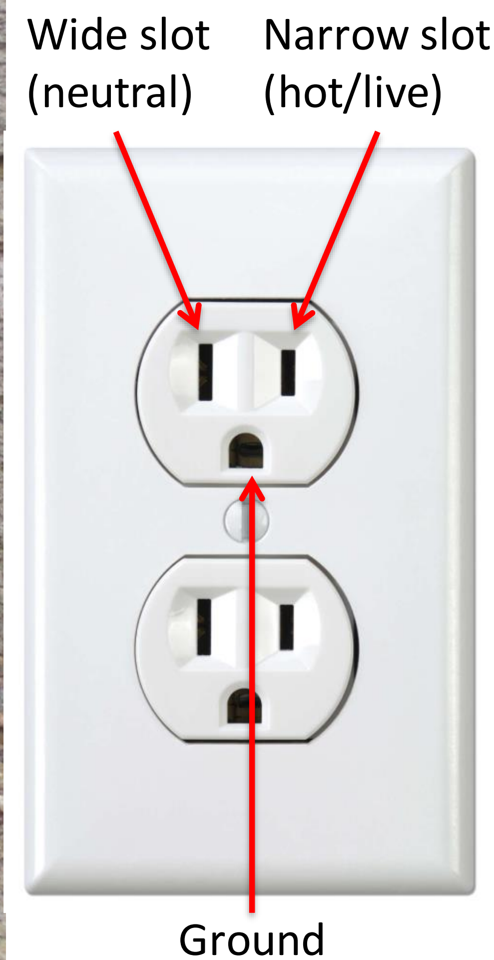
This symbol is called “ground” and it represents the place in the circuit where V = 0. Very important in AC circuits but also important in DC circuits.



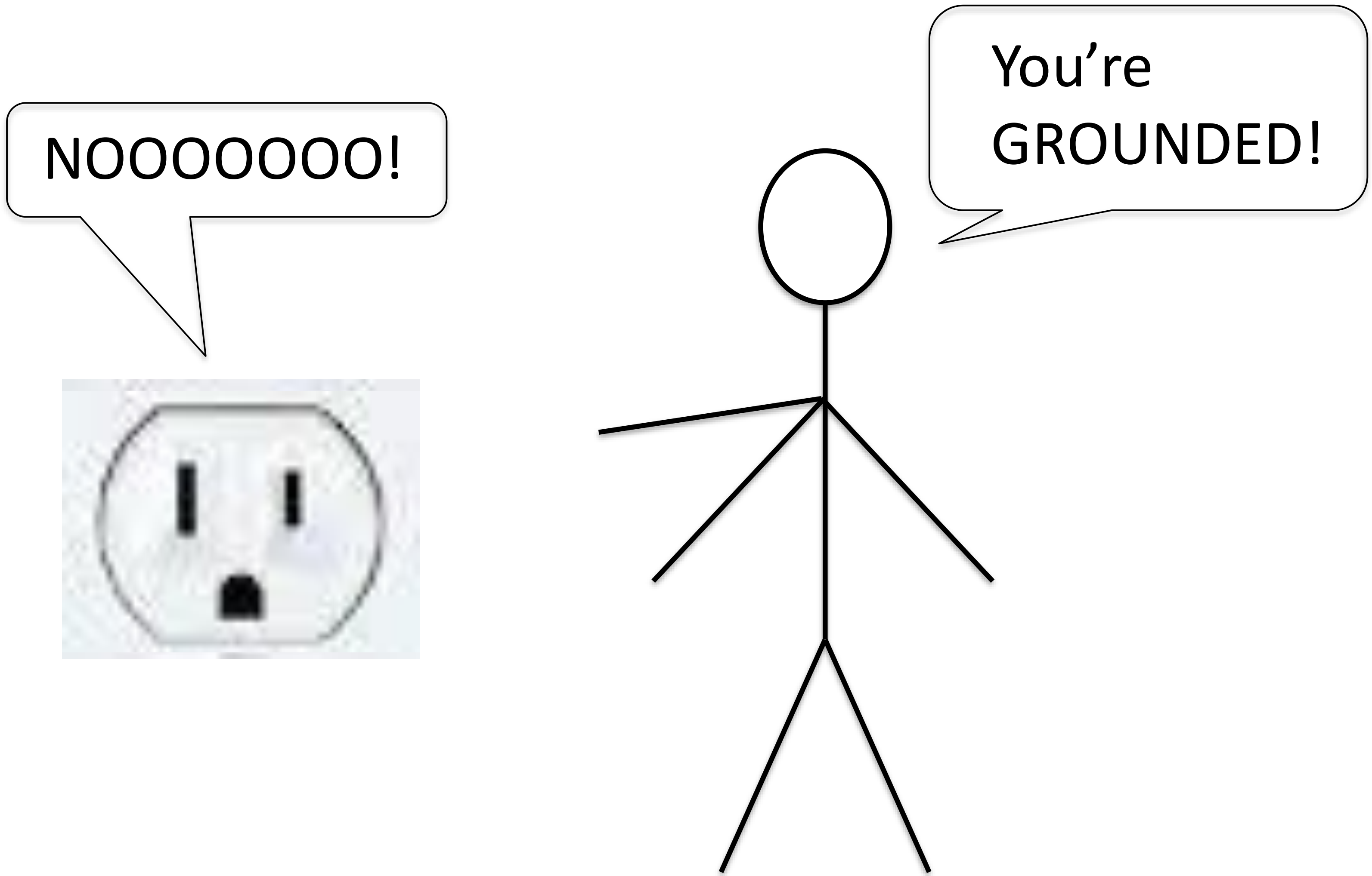




Grounding



How do you Ground an Outlet?



Power in circuits

Recall that **POWER** is the **rate at which work is done**.

$$P = \frac{W}{\Delta t}$$

A battery with voltage ΔV raises the **potential energy** of a single charge q by an amount $q\Delta V$. This is the **work done** by the battery. For N charges

$$P = \frac{Nq\Delta V}{\Delta t} = \left(\frac{Nq}{\Delta t}\right)\Delta V$$

$Nq/\Delta t$ is the number of **charges passing through the battery** in time Δt, i.e. it is the **current**

$$\boxed{P = I\Delta V}$$

Last time:

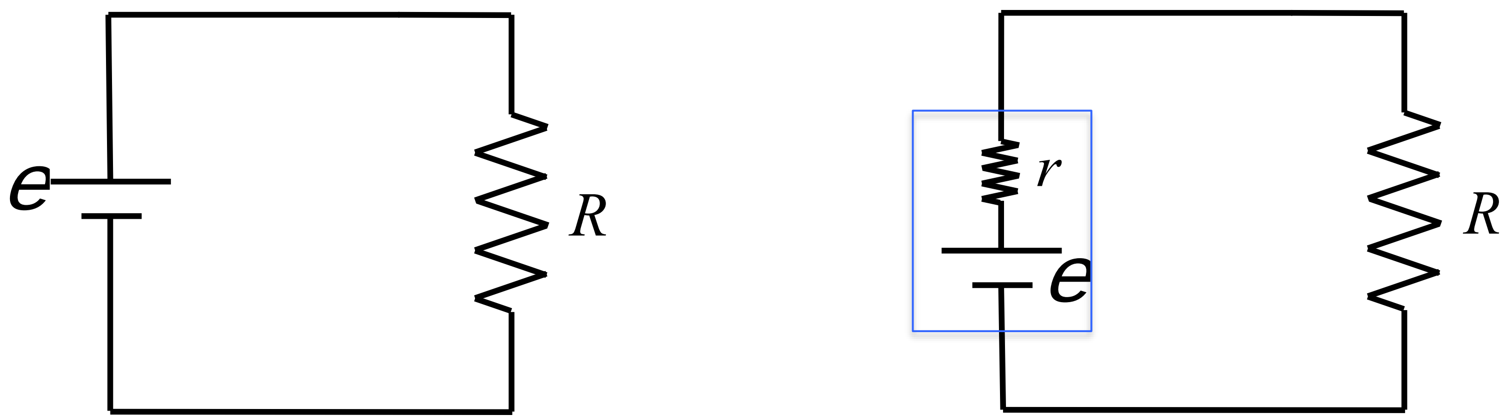
- Recap of resistivity + ohmic vs non-ohmic materials
- Microscopic view of resistivity
- Temperature dependence of resistivity/resistance

Today:

- Ideal vs non-ideal batteries
- RC circuits (charging/discharging capacitors)
- RC time constant and its meaning
- Early and late time behaviour of RC circuits

Non-ideal Batteries: internal resistance

Every voltage source has **some** internal resistance to it. Usually this can be ignored but not always



The internal resistance simply acts as a resistor in series with the rest of the circuit.

$$e - Ir - IR = 0$$

$$I = \frac{e}{(r+R)}$$

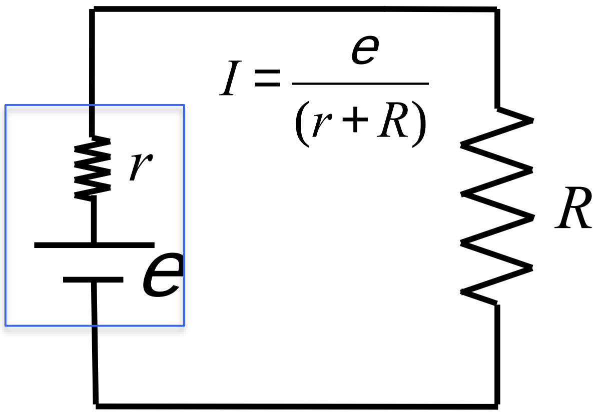
$$P_e = Ie = \frac{e^2}{R+r}$$

Power output required by the emf source

$$P_R = I^2 R = \frac{e^2 R}{(R+r)^2}$$

Power dissipated by the resistive load

Non-ideal Batteries: internal resistance



Conservation of energy requires power in must be equal to power out, but power by emf is not power dissipated by R.

$$P_e = Ie = \frac{e^2}{R+r} \qquad P_R = I^2 R = \frac{e^2 R}{(R+r)^2}$$

Resolution: power dissipated by emf

$$P_r = I^2 r = \frac{e^2 r}{(R+r)^2}$$

The emf must do more work because it fights against its own internal resistance

Now we can verify that power in = power out

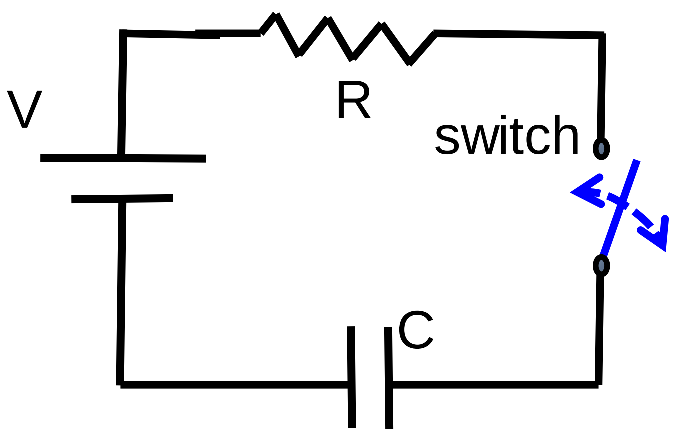
$$P_e = P_r + P_R = \frac{e^2 r}{(R+r)^2} + \frac{e^2 R}{(R+r)^2} = \frac{e^2 (R+r)}{(R+r)^2} = \frac{e^2}{R+r}$$

RC Circuits (resistors + capacitors)

So far we have considered only steady and continuous currents. However, many important circuit applications use a combination of capacitors and resistors to produce time dependent currents.

Examples: Intermittent car wipers, filtering (or 'cleaning') wireless signals in a cordless phone, remote control, etc.

Simple RC Series Circuit



When the switch is <u>open</u> -- no current can flow in the circuit.
The charge and the voltage on the capacitor both remain zero.

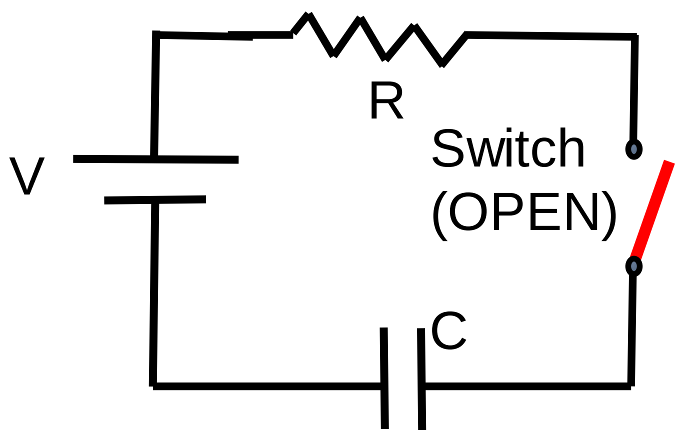
After the switch has been <u>closed</u> for a long time: Charge on capacitor is $Q = CV$
Voltage across capacitor is *V* and no current flows in the circuit.

What happens immediately after switch is closed or opened?

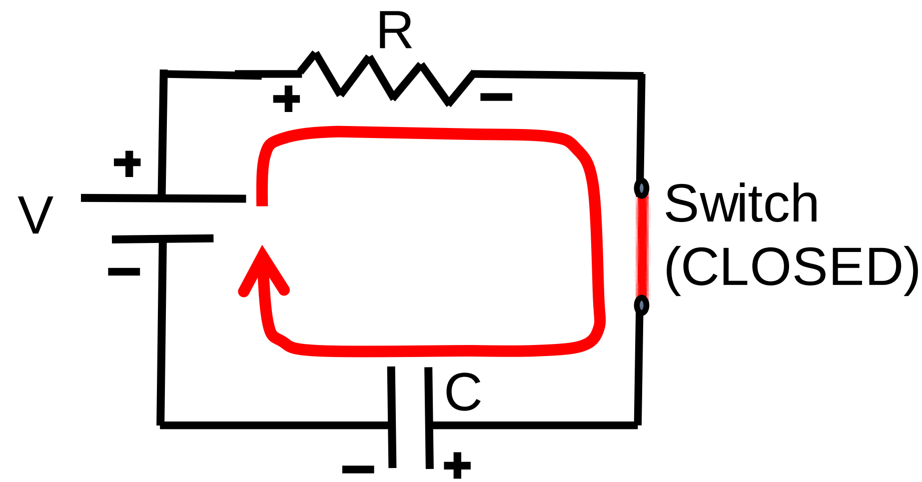
We get time dependent currents!

CASE 1: Charging the capacitor

Switch is open for a long time before t=0 $i(0-) = 0$



Switch is suddenly closed at t=0



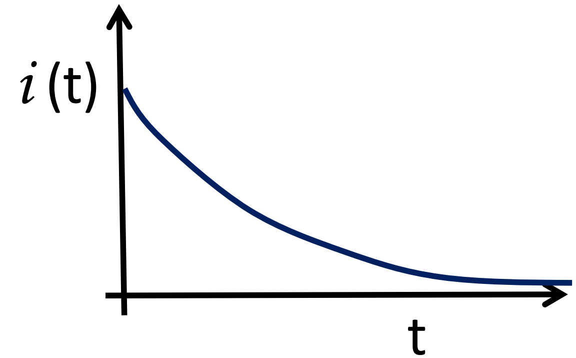
Once the switch is closed, Kirchhoff's voltage rule gives: $V - iR - \frac{q}{C} = 0$

Taking the derivative: → $0 - R\frac{di}{dt} - \frac{1}{C}\frac{dq}{dt} = 0$ but $i = \frac{dq}{dt}$

Note: Immediately after the switch is closed, the capacitor is the same as a wire segment. There is no charge on the capacitor so $V_C = 0$ and $i(0) = V/R$

CASE 1: Charging the capacitor

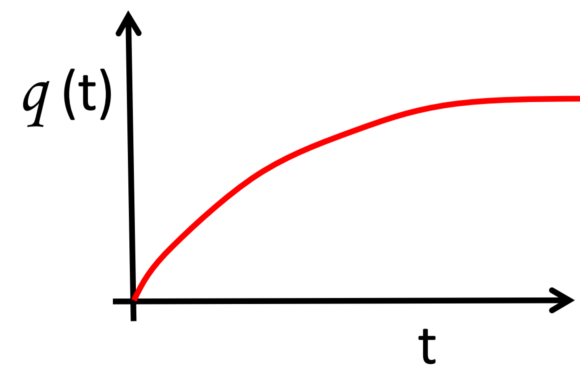
Hence $$-R\frac{di}{dt}-\frac{1}{C}i=0 \Rightarrow \frac{di}{i}=-\frac{1}{RC}\,dt$$



Integration → $$i(t)=i(0)\mathrm{e}^{-\left(\frac{t}{RC}\right)}$$

Substitution → $$i(t)=\frac{V}{R}\mathrm{e}^{-\left(\frac{t}{RC}\right)}=\frac{dq}{dt}$$

Therefore we obtain $$q=-CV\,e^{-\left(\frac{t}{RC}\right)}+\text{constant}$$



Since q(0) = 0 we obtain the constant = CV = Q_f

$$\Rightarrow q(t)=Q_f\left(1-e^{-\left(\frac{t}{RC}\right)}\right)$$

Top Hat Question

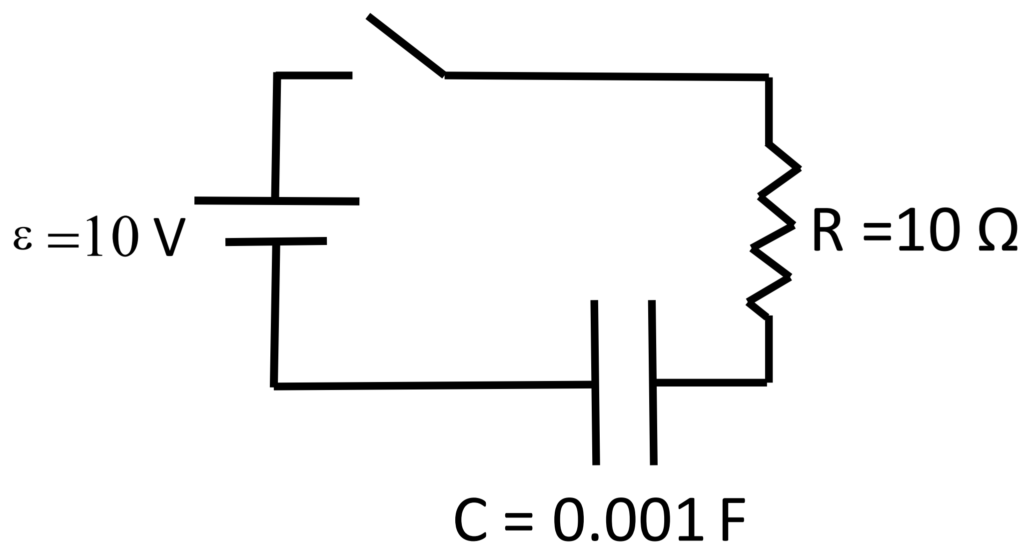
An RC circuit is shown below. Initially the switch is open and the capacitor is uncharged. At time t=0, the switch is closed. What is the voltage across the capacitor ***immediately*** after the switch is closed (time = 0+)?

A. 0.0 V

B. 10 V

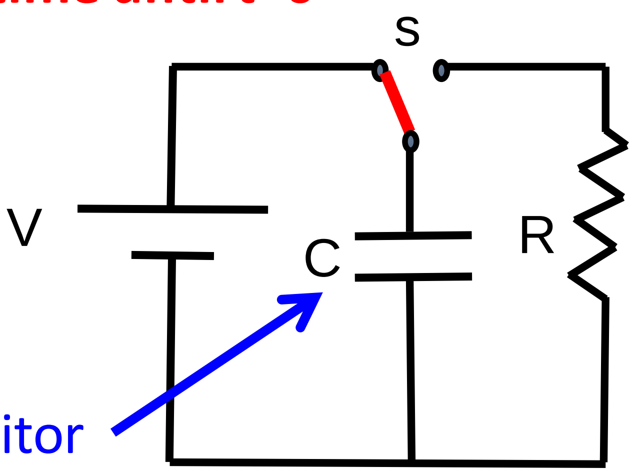
C. 5.0 V

D. 1.0 V



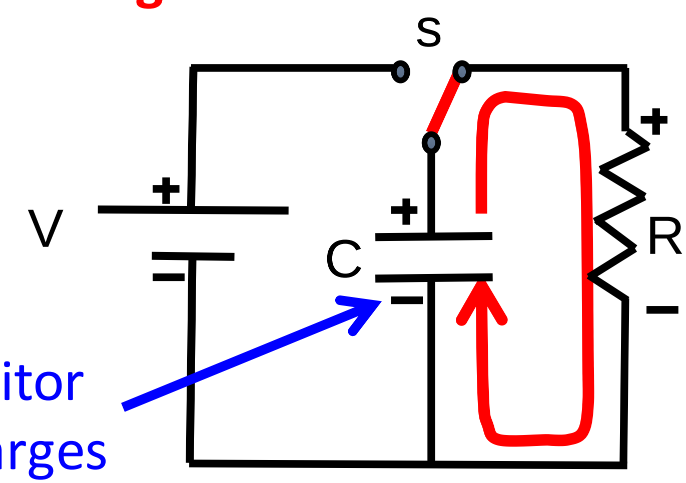
CASE 2: Discharging the capacitor

Switch is connected to the left for a long time until t=0-



Capacitor charges up to voltage V

Switch is suddenly flipped to the right at t=0+



Capacitor discharges

Once the switch is flipped to right, Kirchhoff's loop rule gives: $\dfrac{q(t)}{C} - iR = 0$

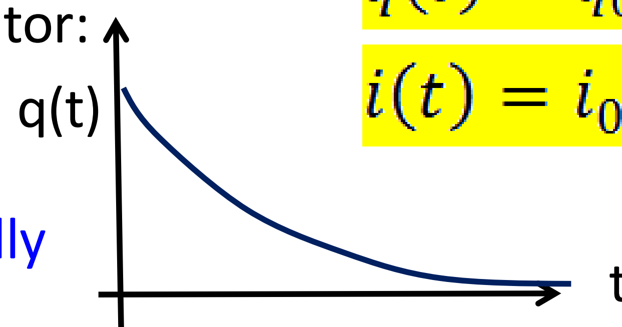
i.e. $\dfrac{q}{RC} = i = -\dfrac{dq}{dt}$ NOTE here dq < 0

Solving for the charge q(t) on the capacitor:

with $q_0 = CV$

$q(t) = q_0 e^{-t/RC}$

$i(t) = i_0 e^{-t/RC}$



Note – both q and i decrease exponentially

All the capacitors in the circuit below are initially uncharged.

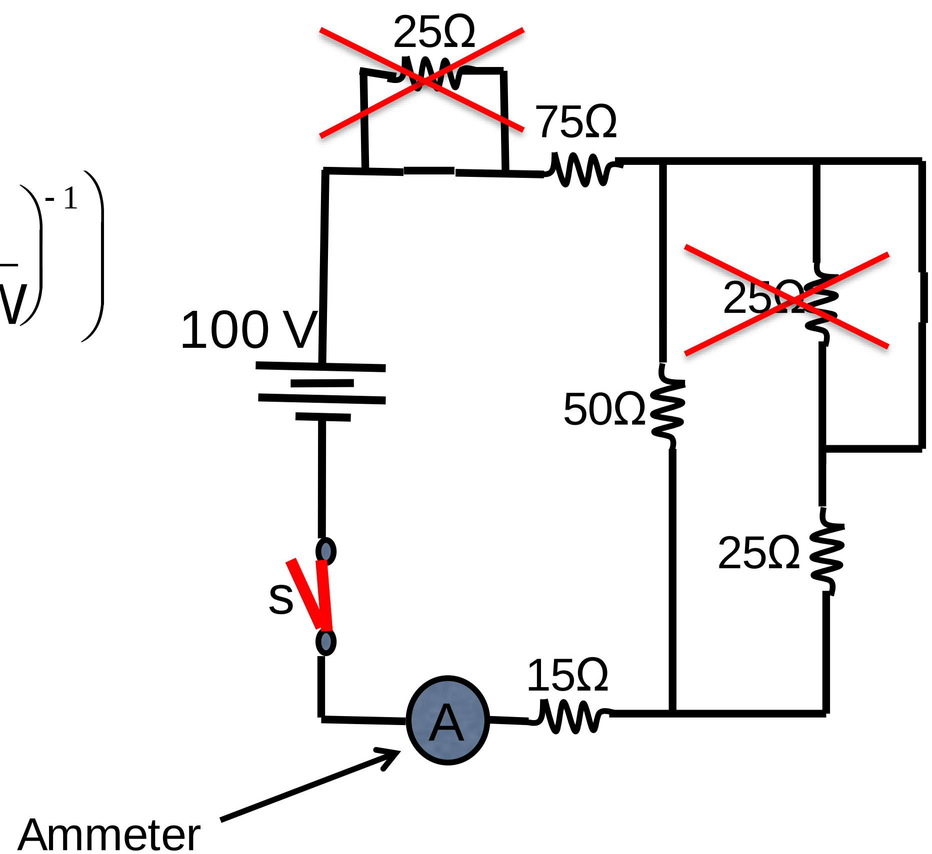
(a) Find the reading of the ammeter just after the switch S is closed.

(b) Find the reading of the ammeter after the switch S has been closed for a long time.

$$R_{eq} = \left(75\text{W} + 15\text{W} + \left(\frac{1}{50\text{W}} + \frac{1}{25\text{W}} \right)^{-1} \right)$$

$$= 107\text{W}$$

$$\boxed{I = \frac{100V}{107\text{W}} = 0.93A}$$



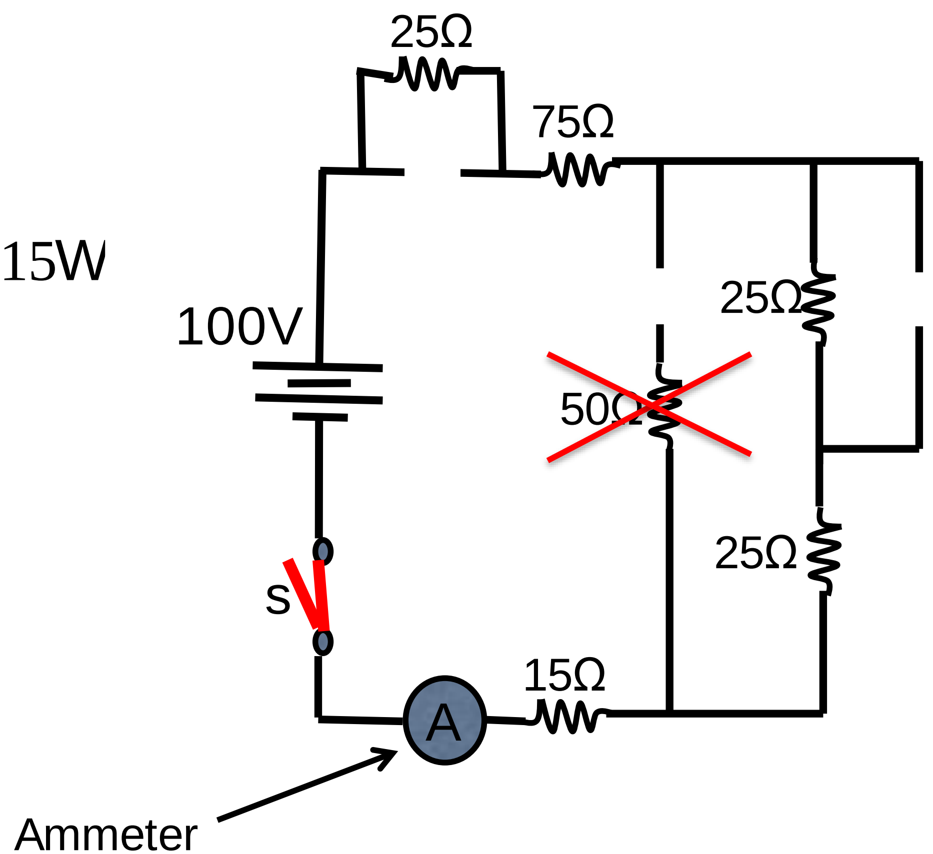
All the capacitors in the circuit below are initially uncharged.

(a) Find the reading of the ammeter just after the switch S is closed.

(b) Find the reading of the ammeter after the switch S has been closed for a long time.

$$R_{eq} = 25W + 75W + 25W + 25W + 15W$$
$$= 165W$$

$$\boxed{I = \frac{100V}{165W} = 0.61A}$$



Last time:

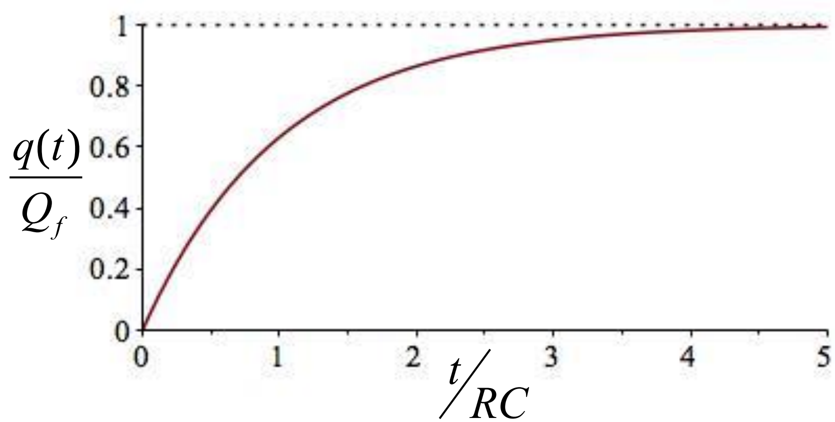
- Ideal vs non-ideal batteries
- RC circuits (charging/discharging capacitors)
- Early and late time behaviour of RC circuits

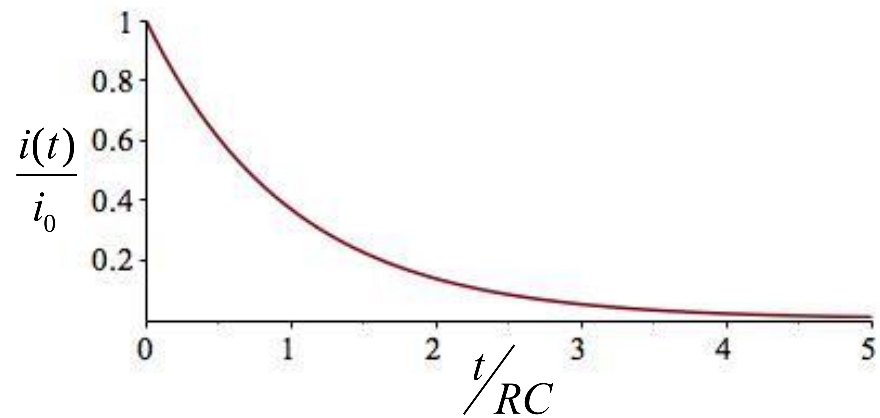
Today:

- RC time constant and its meaning
- Charging/discharging capacitors (doc cam calculation)

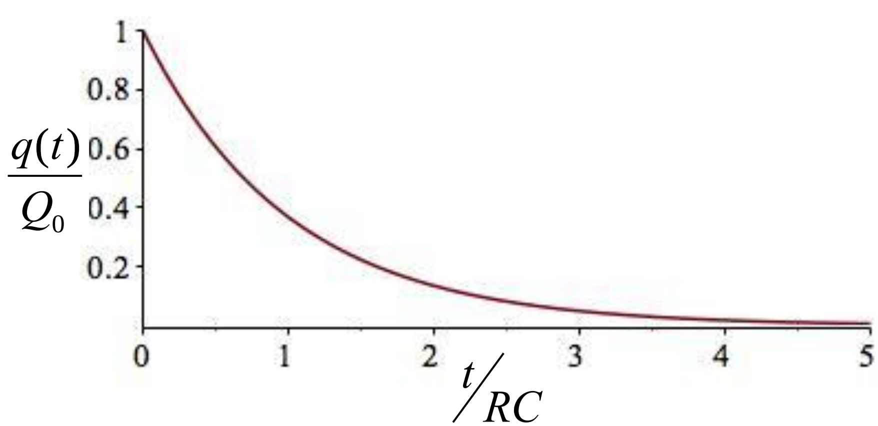
Charging/Discharging Capacitors

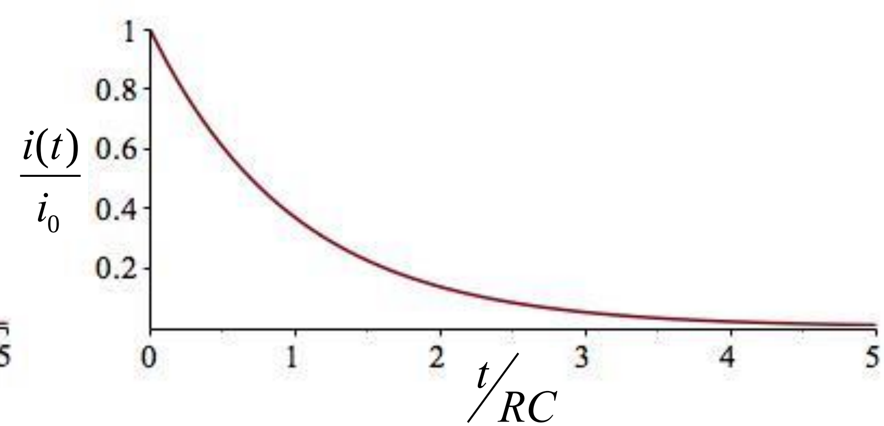
Charging: $q(t) = Q_f\left(1 - e^{-\frac{t}{RC}}\right)$ $i(t) = i_0 e^{-\frac{t}{RC}}$





Discharging: $q(t) = Q_0 e^{-\frac{t}{RC}}$ $i(t) = i_0 e^{-\frac{t}{RC}}$





The RC time constant

The constant RC pops up in the exponential factor for both charging and discharging capacitors. What does it represent?

The units of RC is seconds:

$$[RC] = \frac{V}{A}\frac{C}{V} = \frac{C}{C/s} = s$$

We call RC the "RC time constant" and it tells us how quickly a capacitor can charge or discharge.

$$RC^{\circ}\ t$$

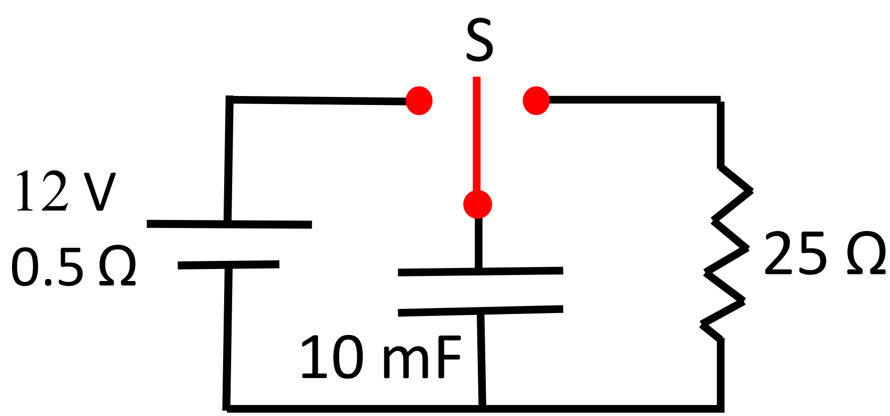
After a time τ, the charge on a discharging capacitor is reduced by a factor of 1/e. After a time $N\tau$, it is reduced by a factor of $1/e^N$

$$q(t) = Q_0 e^{-\frac{t}{t}}$$

Document Camera Calculation

An RC circuit is shown below. Initially the switch is open and the capacitor is uncharged. At time t = 0 s, the switch is thrown to the left, connecting the capacitor to the battery. At time t = 15 ms the switch is thrown to the right, connecting the capacitor to the resistor.

1) How much charge builds up on the capacitor while it is connected to the battery?
2) What is the voltage across the resistor as a function of time as the capacitor discharges?
3) What is the ratio of the charging time to discharging time?



Top Hat Question

An RC circuit is shown below. Initially the switch is open and the capacitor is fully charged. At time t = 0, the switch is closed.

How much charge is left on the capacitor plates after t = 10 ms?

A. 0.67 μC

B. 14 μC

C. 37 μC

D. 4.5 nC