Last time

- Capacitance as a geometric quantity
- Capacitance for a parallel plate capacitor
- Capacitance for a solid spherical conductor
- Energy stored in a parallel plate capacitor

This time

- Spherical capacitor
- Energy density stored in a spherical capacitor
- Cylindrical capacitor and co-axial cable

Capacitance

Capacitance of a conductor is defined as

$$C = \frac{Q}{\Delta V}$$

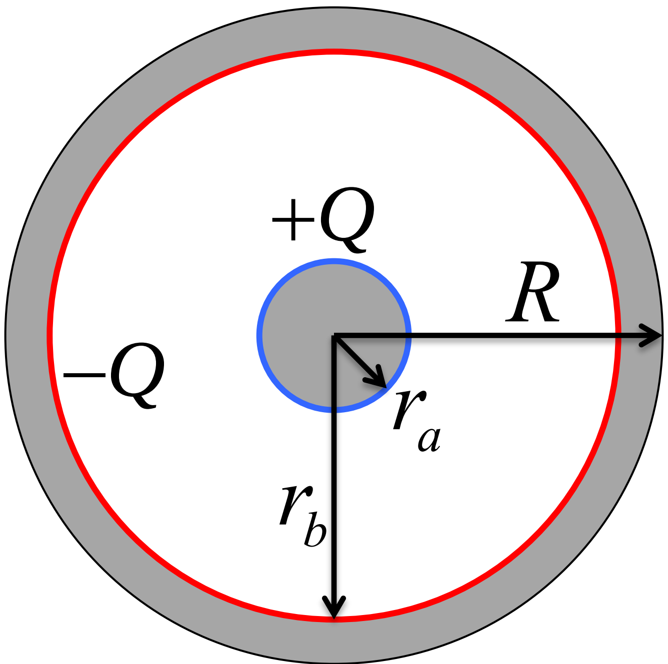
For a parallel plate capacitor

$$\Delta V_C = \frac{\sigma}{\varepsilon_0} d = \left(\frac{d}{A\varepsilon_0} \right) Q$$

$$C = \frac{Q}{\Delta V} = \frac{\varepsilon_0 A}{d}$$

C is a geometric factor. Roughly speaking capacitance is the ability of a geometrical shape to store charge.

Spherical Capacitor



1) What is the E-field everywhere?
2) What is V everywhere?
3) What is ΔV between the plates?

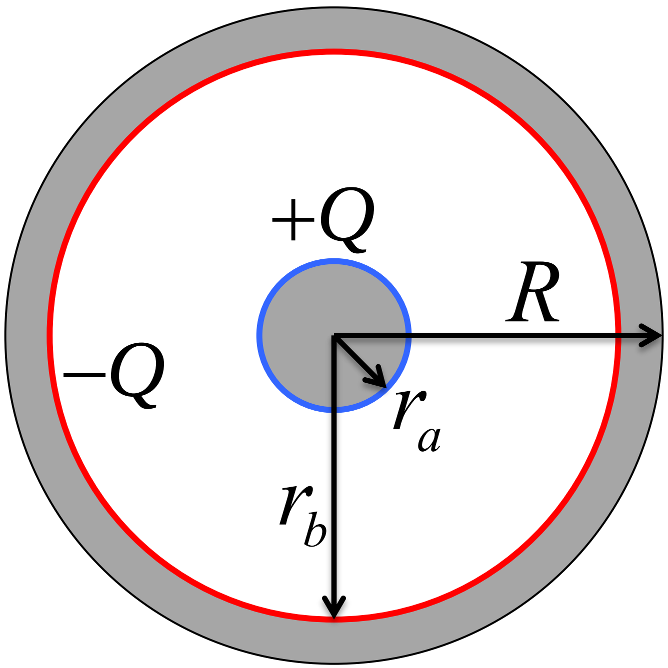
$$\boxed{V = 0 \quad \text{at infinity}}$$

$$r > r_b, \quad \Delta V = -\int_{\infty}^{R} \vec{E} \cdot d\vec{r} = 0 \qquad \boxed{V_{r>r_b} = 0}$$

$$r_b > r > r_a, \quad V_b - V_a = -\int_{r_a}^{r_b} \vec{E} \cdot d\vec{r} = -\int_{r_a}^{r_b} \frac{Q}{4\pi\varepsilon_0 r^2} dr \boxed{= \frac{+Q}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right) < 0}$$

$$r \le r_a, \quad \Delta V = -\int_{\infty}^{R} \vec{E} \cdot d\vec{r} = 0 \qquad \boxed{V_{r \le r_a} = \frac{+Q}{4\pi\varepsilon_0 r_a}}$$

Spherical Capacitor



1) What is the E-field everywhere?
2) What is V everywhere?
3) What is ΔV between the plates?
4) How can we relate ΔV to the charge on the plates?

$$\Delta V_C = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{r_b - r_a}{r_b r_a}\right)$$

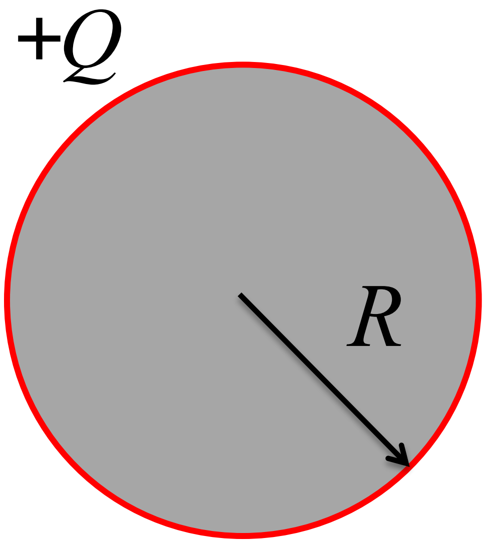
Rewrite this relation as

$$Q = \left(\frac{4\pi\varepsilon_0 r_b r_a}{r_b - r_a}\right) V_C$$

$$\boxed{C = \frac{Q}{\Delta V_C}}$$

$$C = \left(\frac{4\pi\varepsilon_0 r_b r_a}{r_b - r_a}\right)$$

Isolated Sphere as a Capacitor



Capacitors need two plates in general for the field lines to end. In the case of a sphere, we can consider the other plate to be at infinity and define the capacitance of an isolated sphere with charge Q. This will not work for an infinite cylinder as we will see later.

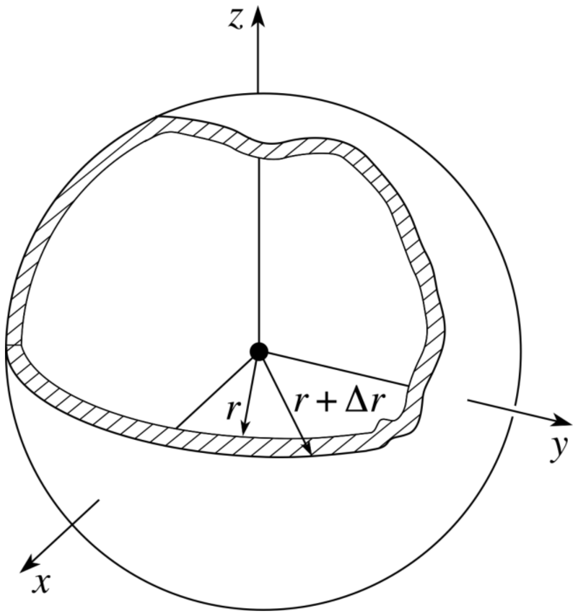
Start with expression for spherical capacitor with $r_a = R$, $r_b = \infty$:

$$Q = \left(\frac{4\pi\varepsilon_0 r_b r_a}{r_b - r_a} \right) \Delta V_C \rightarrow \left(4\pi\varepsilon_0 R\right) \Delta V_C \qquad \boxed{C = 4\pi\varepsilon_0 R}$$

Calculate the volume of the spherical shell shown below.

$$r_a \leq r \leq r_b$$

$$d\mathrm{v} = 4\pi r^2 dr$$



Energy stored in spherical conductor

Calculate the energy content of a thin spherical shell having an inner radius r and an outer radius r+dr in the region between the two conductors.

$$du = \frac{1}{2}\varepsilon_0 E^2 d\mathrm{v}$$

$$d\mathrm{v} = 4\pi r^2 dr$$

$$du = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$U = \int_{r_a}^{r_b} du = \frac{Q^2}{8\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2}$$

$$U = \frac{Q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r_b} + \frac{1}{r_a}\right)$$

$$U = \frac{Q^2}{8\pi\varepsilon_0}\frac{(r_b - r_a)}{r_a r_b}$$

An alternate method

$$C = \frac{Q}{V_a - V_b} = \frac{4\pi\varepsilon_0 r_a r_b}{\left(r_b - r_a\right)}$$

$$U = \frac{Q^2}{2C} = \frac{Q^2}{2\dfrac{4\pi\varepsilon_0 r_a r_b}{\left(r_b - r_a\right)}}$$

$$U = \frac{Q^2\left(r_b - r_a\right)}{8\pi\varepsilon_0 r_a r_b}$$

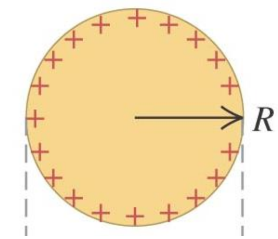
Supercapacitors typically store 10 to 100 times more energy per unit volume or mass than electrolytic capacitors, can accept and deliver charge much faster than batteries, and tolerate many more charge and discharge cycles than rechargeable batteries. However, they cannot tolerate high voltages and are used in the low voltage applications.

1 F capacitor supercapacitor

Capacitance

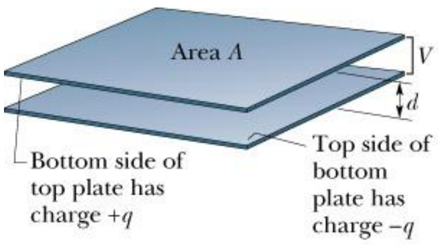
Capacitance of an isolated spherical conductor

$$C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$



$$C = \frac{\varepsilon_0 A}{d}$$

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

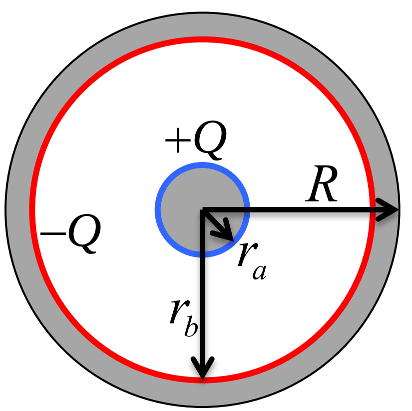


$$\boxed{u = \frac{1}{2}\varepsilon_0 E^2}$$

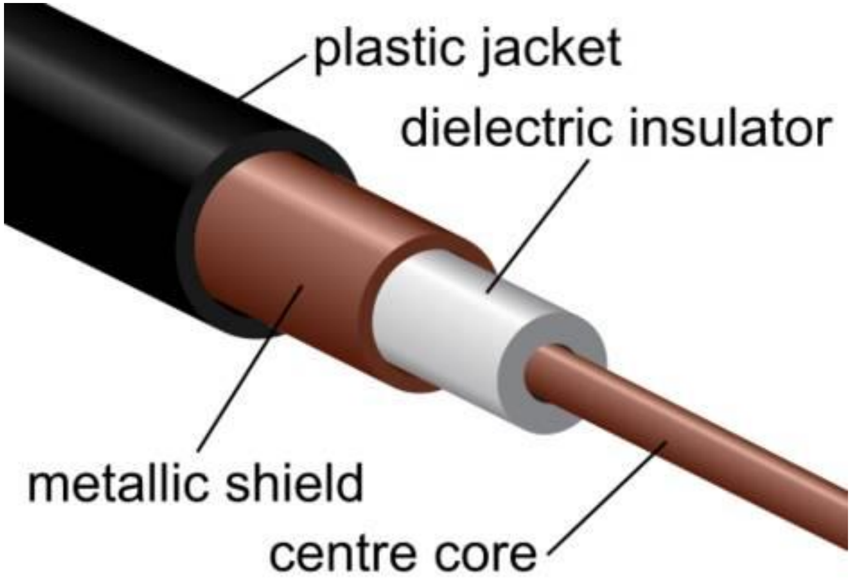
General result.

$$C = \left(\frac{4\pi\varepsilon_0 r_b r_a}{r_b - r_a}\right)$$

$$U = \frac{Q^2 (r_b - r_a)}{8\pi\varepsilon_0 r_a r_b}$$

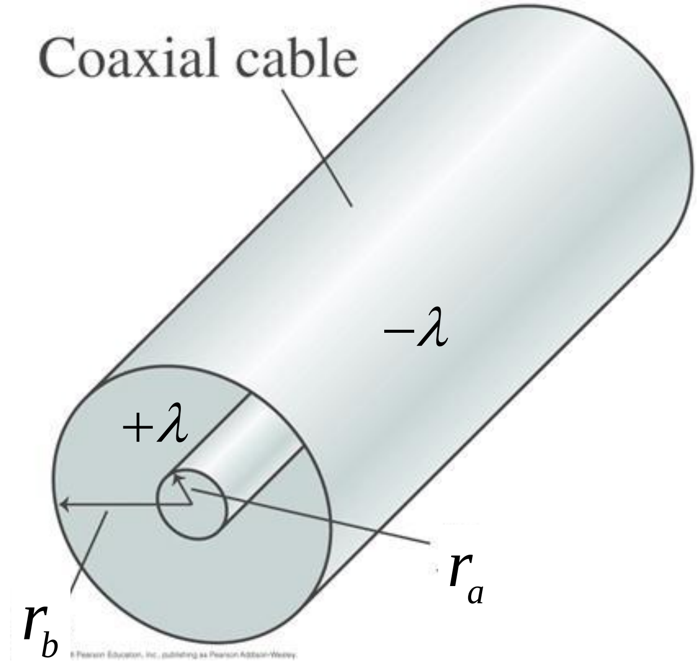


Coaxial cable



λ represents charge per unit length

The inner conductor is then an infinitely long conductor with a uniform charge density



$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

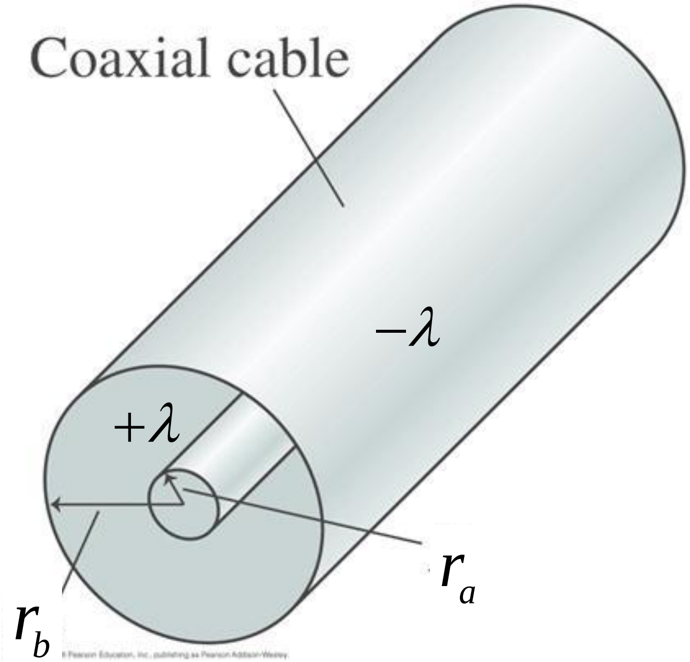
$$V_b - V_a = -\int_a^b \vec{E} \bullet d\vec{l} \qquad d\vec{l} = dr\hat{r}$$

$$V_b - V_a = -\int_{r_a}^{r_b} \left(\frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} \right) \bullet (dr\hat{r}) = -\frac{\lambda}{2\pi\varepsilon_0} \int_{r_a}^{r_b} \frac{dr}{r}$$

$$V_b - V_a = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_b}{r_a}$$

$$V_b - V_a = -\frac{Q}{2\pi\varepsilon_0 L}\ln\frac{r_b}{r_a}$$

$$V_a - V_b = \frac{Q}{2\pi\varepsilon_0 L}\ln\frac{r_b}{r_a}$$



$$C = \frac{Q}{V_a - V_b} = \frac{2\pi\varepsilon_0 L}{\ln\left(r_b / r_a\right)}$$

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(r_b / r_a\right)}$$

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln\left(r_b / r_a\right)}$$

The inner cylinder is at a higher potential than the outer one.

C is proportional of the length of the coax. It also depends on r_a and r_b.

A typical has a capacitance per unit length of 69 pF/m.

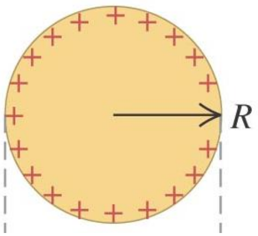
General result

$$u = \frac{1}{2}\varepsilon_0 E^2$$

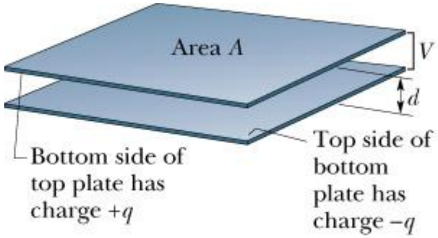
$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

Capacitance

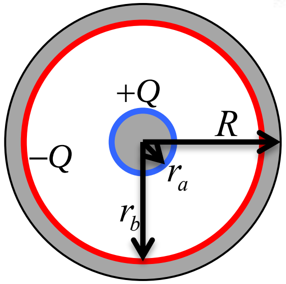
$$C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$



$$C = \frac{\varepsilon_0 A}{d}$$



$$C = \frac{4\pi\varepsilon_0 ab}{(b-a)}$$



$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln(r_b / r_a)}$$