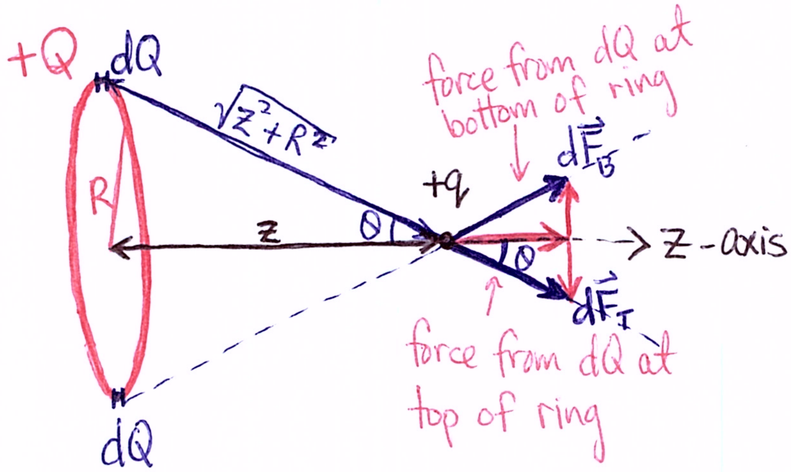
Electric force due to ring of charge



To find the electric force on q from the ring, break the ring into small charges, each with charge dQ.

Step 1: draw the line connecting dQ with q

Step 2: determine the direction of the force:

→ like charges, so $\vec{F}$ must point <u>away</u> from dQ

Step 3: <u>USE SYMMETRY</u>. For each dQ, there is a diametrically opposite dQ whose force on q cancels one component.

Step 4: Calculate the only component you need to; here it is the z-component. Do this for a single charge element dQ.

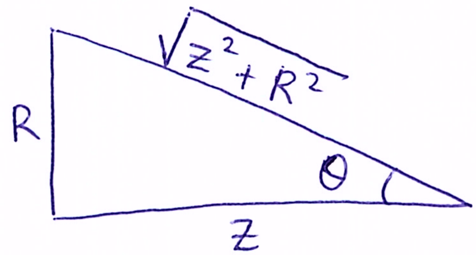
$$|dF_z| = |dF| \cos\theta$$

↑ from Coulomb's Law

← from geometrical information.

$$|dF| = \frac{1}{4\pi\epsilon_0}\frac{q\,dQ}{\left(\sqrt{z^2+R^2}\right)^2} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q\,dQ}{z^2+R^2}}$$

We can get $\cos\theta$ from the triangle



$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\Rightarrow \boxed{\cos\theta = \frac{z}{\sqrt{R^2+z^2}}}$$

Combine these two together to get

$$|dF_z| = \frac{1}{4\pi\epsilon_0}\frac{q\,dQ}{(z^2+R^2)}\frac{z}{\sqrt{z^2+R^2}} = \frac{1}{4\pi\epsilon_0}\frac{qz\,dQ}{(z^2+R^2)^{3/2}}$$

Two ways to proceed:

1. Recognize that z and R are the same for each element dQ, so they can be treated as constant. Can integrate dQ directly:

$$\therefore\ |F_z| = \int_0^Q \frac{1}{4\pi\epsilon_0}\frac{qz\,dQ}{(z^2+R^2)^{3/2}} = \frac{1}{4\pi\epsilon_0}\frac{qz}{(z^2+R^2)^{3/2}}\underbrace{(Q-0)}_{\int_0^Q dQ}$$

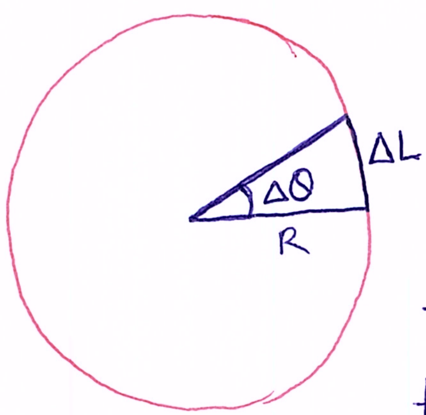
$$\boxed{|F_z| = \frac{1}{4\pi\epsilon_0}\frac{Qqz}{(z^2+R^2)^{3/2}}}$$

2. This way is more systematic, so I would recommend using it until you get comfortable with the proceedure. Use the linear charge density to integrate over a spatial variable.

definition: $\lambda = \frac{Q}{L}$ ← total charge, ← total length

$L_{ring} = 2\pi R$ so $\lambda = \frac{Q}{2\pi R}$ (*)

for some small segment ΔL of the ring, there is a charge $\Delta Q = \lambda \, \Delta L$



definition: $\Delta L = R \Delta \theta$

arclength of circle

change in angle

don't confuse this θ with the previous θ from the triangles. This one has to do with angular position around the ring.

Taking all of these quantities to be smaller and smaller, we get infinitesimal quantities.

$$dQ = \lambda \, dL = \lambda (R \, d\theta) = \lambda R \, d\theta$$

$$|dF_z| = \frac{1}{4\pi\epsilon_0} \frac{qz}{(z^2+R^2)^{3/2}} \lambda R \, d\theta$$

everything here is constant for every position θ around the ring.

$$|F_z| = \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{qz}{(z^2+R^2)^{3/2}} \lambda R \, d\theta = \frac{1}{4\pi\epsilon_0} \frac{qz\lambda R}{(z^2+R^2)^{3/2}} \int_0^{2\pi} d\theta$$

$$|F_z| = \frac{1}{4\pi\epsilon_0} \frac{qz\lambda R}{(z^2+R^2)^{3/2}} (2\pi)$$

but $2\pi R\lambda = Q$ from Eq (*)

$$\therefore \boxed{|F_z| = \frac{1}{4\pi\epsilon_0} \frac{Qqz}{(z^2+R^2)^{3/2}}}$$

same as before.

Limits: What does the force look like when very far away from the ring ($z \gg R$)?
The ring would look like a point, so the force should reduce to a point charge of charge Q.
Check: (note $z^2+R^2 \approx z^2$)

$$|F_z| \approx \frac{1}{4\pi\epsilon_0} \frac{Qqz}{(z^2)^{3/2}} = \frac{1}{4\pi\epsilon_0} \frac{Qqz}{z^3} = \boxed{\frac{1}{4\pi\epsilon_0} \frac{Qq}{z^2}}$$

This is indeed the electric force due to a point charge Q.

What about when $z \ll R$? That is, near the center of the ring? (note $z^2+R^2 \approx R^2$)

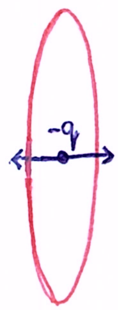
$$|F_z| \approx \frac{1}{4\pi\epsilon_0} \frac{Qqz}{(R^2)^{3/2}} = \boxed{\frac{1}{4\pi\epsilon_0} \frac{Qqz}{R^3}}$$

the force is linear in z.

Consider the force on a charge $-q$ for $z \ll R$:

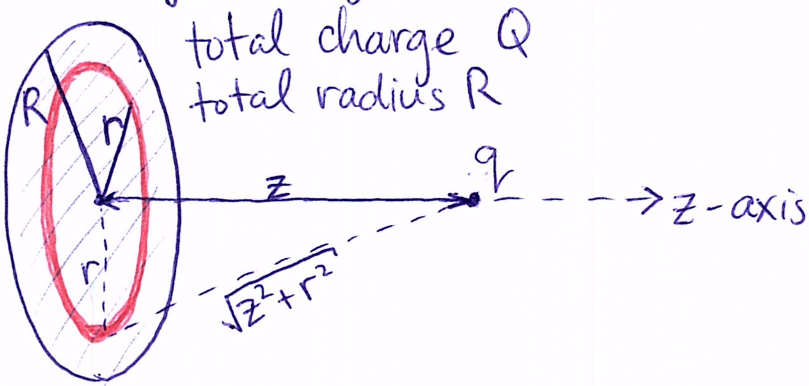
$$F_z = -\underbrace{\left(\frac{1}{4\pi\epsilon_0}\frac{Qq}{R^3}\right)}_{\text{call this } k_s} z \implies F_z = -k_s z$$

This is Hooke's Law! A negative charge acts like it is attached to a spring.



The negative charge would oscillate back and forth around the center of the ring. This is called simple harmonic motion.

Now let's look at the electric force due do a uniformly charged disk.



Introduce surface charge density

$$\boxed{\eta = \frac{Q}{\pi R^2}} = \frac{(\text{total } Q)}{(\text{total } A)}$$

Treat the disk as a series of concentric rings of charge. Each ring has thickness Δr and charge ΔQ

Area of ring:

$$A_2 - A_1 = \pi r_2^2 - \pi r_1^2$$

$$\Delta A = \pi(r_2^2 - r_1^2)$$

We also have $\Delta r = r_2 - r_1$, so we can rewrite

$$\Delta A = \pi (r_2 - r_1)(r_2 + r_1) \quad \text{(factoring } (r_2^2 - r_1^2)\text{)}$$

$$= \pi \, \Delta r \, (r_2 + \underbrace{(r_2 - \Delta r)}_{r_1})$$

Assume Δr is very small:

$$\Delta A = 2\pi r_2 \, \Delta r + \underbrace{\text{term proportional to } \Delta r^2}_{\text{can ignore if } \Delta r^2 \ll \Delta r}$$

Now how much charge is there in ΔA?

$$\Delta Q = (2\pi r_2 \, \Delta r)\, \eta \qquad \text{now call } r_2 \text{ simply } r.$$

Force on charge due to ring of radius r:

from previous calculation

$$\Delta F_z = \frac{1}{4\pi\epsilon_0} \frac{q z \, \Delta Q}{(z^2 + r^2)^{3/2}} = \frac{1}{4\pi\epsilon_0} \frac{q z \, 2\pi r \eta \, \Delta r}{(z^2 + r^2)^{3/2}}$$

Take Δr smaller and smaller and sum up all rings:

$$F_z = \int_0^R \frac{1}{4\pi\epsilon_0} \frac{q z \, 2\pi r \eta \, dr}{(z^2 + r^2)^{3/2}}$$

R ← integrate to outer edge of disk.

0 ← integrate from center of disk

bring out all constants

$$F_z = \frac{1}{2\cancel{4\pi}\epsilon_0} q z \cancel{2\pi} \eta \int_0^R \frac{r\,dr}{(z^2 + r^2)^{3/2}}$$

replace η with $\frac{Q}{\pi R^2}$. Look up integral in a table.

This is the place you should be able to get to.

$$F_z = \frac{q z}{2\epsilon_0} \frac{Q}{\pi R^2} \left[-\frac{1}{\sqrt{z^2 + r^2}} \right] \Bigg|_0^R$$

$$F_z = \frac{qQz}{2\pi\epsilon_0 R^2}\left[\left(-\frac{1}{\sqrt{z^2+R^2}}\right)-\left(-\frac{1}{\sqrt{z^2+0^2}}\right)\right]$$

$$F_z = \frac{qQz}{2\pi\epsilon_0 R^2}\left[-\frac{1}{\sqrt{z^2+R^2}}+\frac{1}{z}\right]$$

now multiply the factor of z into brackets.

$$\boxed{F_z = \frac{qQ}{2\pi\epsilon_0 R^2}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right]}$$

Take the limit of small $z \ll R$: $z^2+R^2 \approx R^2$

$$F_z \approx \frac{qQ}{2\pi\epsilon_0 R^2}\left[1-\frac{z}{R}\right] \approx \boxed{\frac{qQ}{2\pi\epsilon_0 R^2}} \qquad \left(\frac{z}{R} \ll 1\right)$$

The force is CONSTANT if you are close enough to the disk.

Sufficiently far away from the disk, it looks like a point charge. Take the limit $z \gg R$.

Try what we've been doing so far:

$$F_z \approx \frac{qQ}{2\pi\epsilon_0 R^2}\left[1-\frac{z}{z}\right] = 0$$

doesn't work!

Resolution: do a binomial (Taylor series) expansion of F_z. We need a small parameter, so choose $\left(\frac{R}{z}\right)^2 = x$

$$F_z = \frac{qQ}{2\pi\epsilon_0 R^2}\left[1 - \frac{\cancel{z}}{\cancel{z}\sqrt{1+\left(\frac{R}{z}\right)^2}}\right]$$

(used the identity $\sqrt{z^2+R^2} = z\sqrt{1+\left(\frac{R}{z}\right)^2}$)

$$F_z = \frac{qQ}{2\pi\epsilon_0 R^2}\left[1 - (1+x)^{-1/2}\right]$$

(replaced $(R/z)^2$ with x, a dummy variable)

Taylor Series: $f(x) = (1+x)^{-1/2}$

then $f(x) \approx f(0) + f'(0)\,x + \frac{1}{2}f''(0)x^2 + \ldots$

we will only need this term

$$f(x) \approx (1+0)^{-1/2} + \left(\frac{-1}{2(1+0)^{3/2}}\right)x = 1 - \frac{1}{2}x$$

$$F_z \approx \frac{qQ}{2\pi\epsilon_0 R^2}\left[1 - \left(1 - \frac{1}{2}x\right)\right]$$

$$F_z = \frac{1}{2\pi\epsilon_0}\frac{qQ}{R^2}\left[\cancel{1} - \cancel{1} + \frac{1}{2}\frac{R^2}{z^2}\right]$$

(replaced x with $\frac{R^2}{z^2}$)

$$\boxed{F_z = \frac{1}{4\pi\epsilon_0}\frac{qQ}{z^2}}$$

This is exactly the force due to a point charge.