Last time

- AC generator
- DC generator
- Mutual inductance

This time

- Self inductance
- Energy density for a magnetic field
- RL circuits and their applications

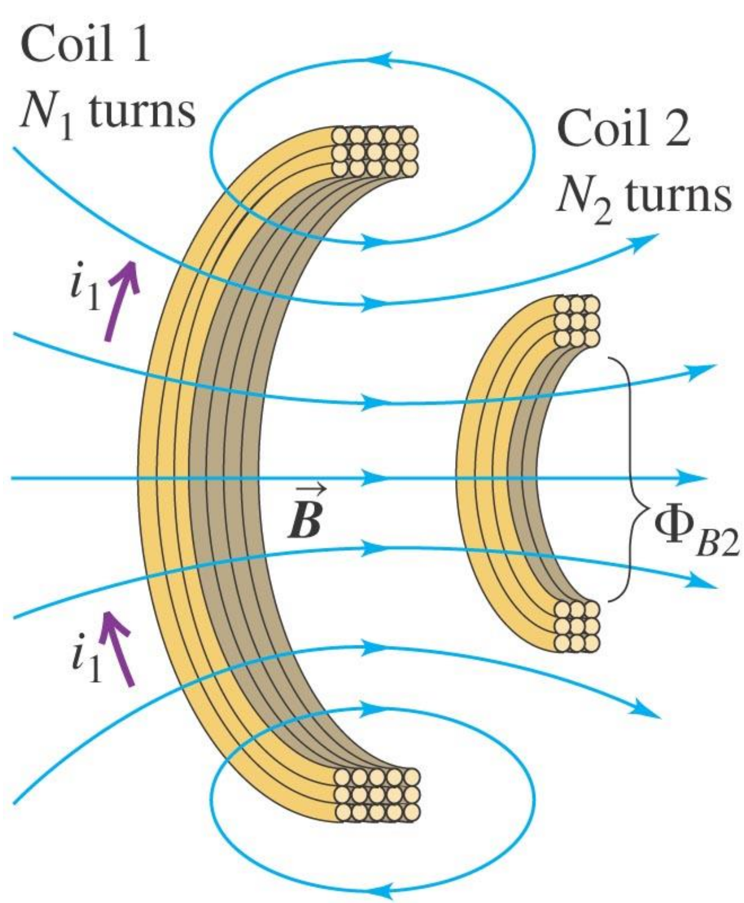
Introduction

– A charged coil can create a field that will induce a current in a neighboring coil.
– Inductance can allow a sensor to trigger the traffic light to change when the car arrives at an intersection.

Mutual inductance

– A coil in a device which generates a field that creates a current in a neighboring coil. You could use this principle to create a charger that restores a battery without an actual connection, just proximity.

Mutual inductance: If the current in coil 1 is changing, the changing flux through coil 2 induces an emf in coil 2.



Mutual inductance

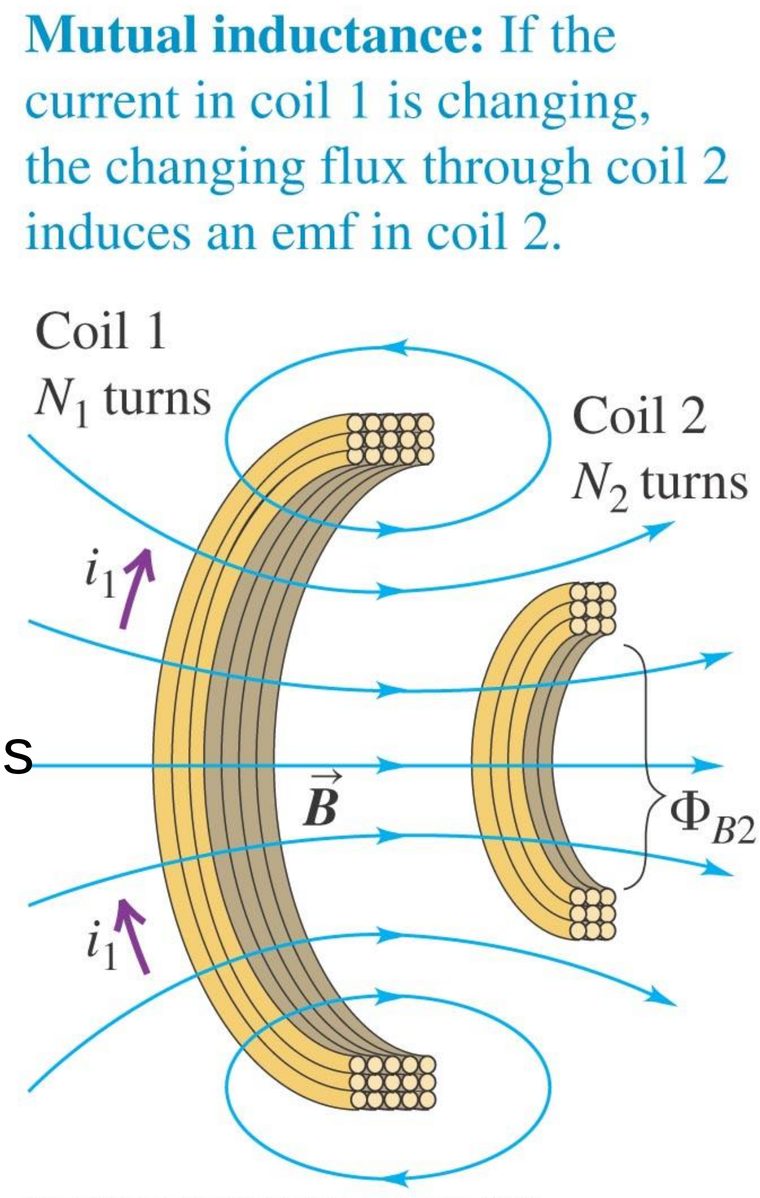
Φ_{B_2} is proportional to the magnetic field generated by the first coil. This in turn is proportional to current i_1 .

$$\varepsilon_2 = -N_2 \frac{d\Phi_{B_2}}{dt}$$

$$N_2 \Phi_{B_2} = M_{21} i_1$$ For linear cases

$$N_2 \frac{d\Phi_{B_2}}{dt} = M_{21} \frac{di_1}{dt}$$

$$\varepsilon_2 = -M_{21} \frac{di_1}{dt} \qquad M_{21} = \frac{N_2 \Phi_{B_2}}{i_1}$$



Mutual inductance

Considering the situation for the opposite case in which the current in the second coil causes a changing flux in the first coil

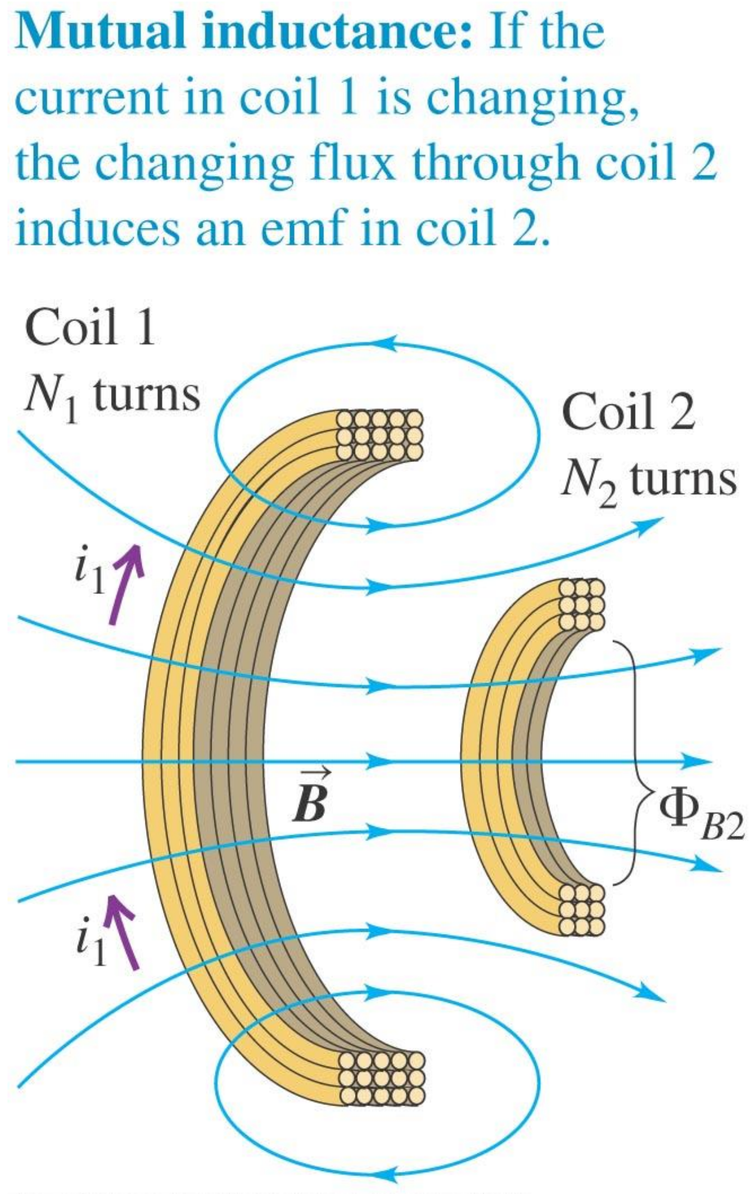
$$\varepsilon_1 = -N_1 \frac{d\Phi_{B_1}}{dt}$$

$$N_1 \Phi_{B_1} = M_{12} i_2 \qquad N_1 \frac{d\Phi_{B_1}}{dt} = M_{12} \frac{di_2}{dt}$$

$$\varepsilon_1 = -M_{12} \frac{di_2}{dt} \qquad M_{12} = \frac{N_1 \Phi_{B_1}}{i_2}$$

In can be shown that

$$M_{12} = M_{21} = M$$



$$\varepsilon_1 = -M \frac{di_2}{dt} \qquad \varepsilon_2 = -M \frac{di_1}{dt}$$

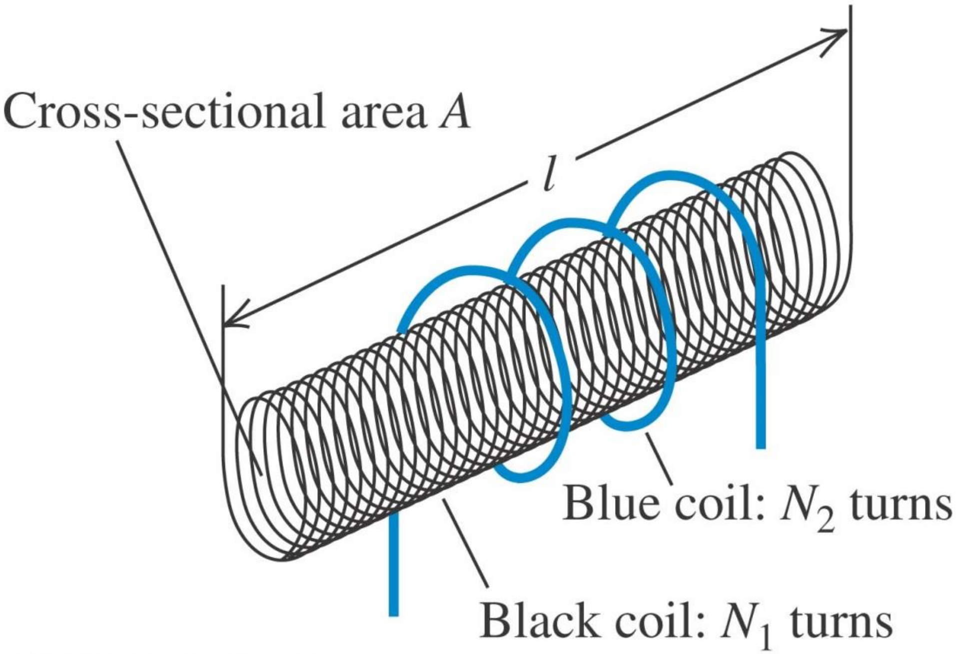
$$\frac{N_1 \Phi_{B_1}}{i_2} = \frac{N_2 \Phi_{B_2}}{i_1}$$

The negative signs are a reflection of Lenz's law. It indicates that a change in current in one coil causes a change of flux through the second coil, inducing an emf in the second coil whose induced current opposes the flux change.

1 H = 1 Wb/1 A = 1 V.s/A = 1 Ω.s = 1 J/A^2

Mutual inductance—examples

Although it appears that two solenoids have different cross sectional area in the figure. In practice, the coils are wound on the top of each other, thus the cross sectional areas for the coils are the same.



$$B_1 = \mu_0 n_1 i_1 = \frac{\mu_0 N_1 i_1}{l}$$

$$M = \frac{N_2 \Phi_{B_2}}{i_1} = \frac{N_2 A B_1}{i_1} = \frac{N_2 A}{i_1} \frac{\mu_0 N_1 i_1}{l} = \frac{\mu_0 A N_1 N_2}{l}$$

Mutual inductance of any two coils is always proportional to the product of the number of turns in the coils and the geometry of the two coils, and not on the current. This is similar to capacitance and resistance.

Example

$$M = \frac{\mu_0 A N_1 N_2}{l}$$

$$M = \frac{\left(4\pi \times 10^{-7}\ \text{Wb/A.m}\right)\left(0.01\ \text{m}^2\right)\left(1000\right)(10)}{0.5\ \text{m}}$$

$$M = 25\ \mu\text{H}$$

Mutual inductance can be a nuisance in circuits, since variation in current in one circuit induces unwanted emfs in other nearby circuits.

To minimize the effect multiple-circuit systems must be designed so that M is as small as possible. For example two coils would be placed far apart or with their planes perpendicular.

Example

$$i_2 = \left(2\times10^6\ \text{A/s}\right)t$$

What is the average flux through each turn of the solenoid caused by the current in the outer coil at time $t = 3.0\ \mu\text{s}$ and what is the induced emf?

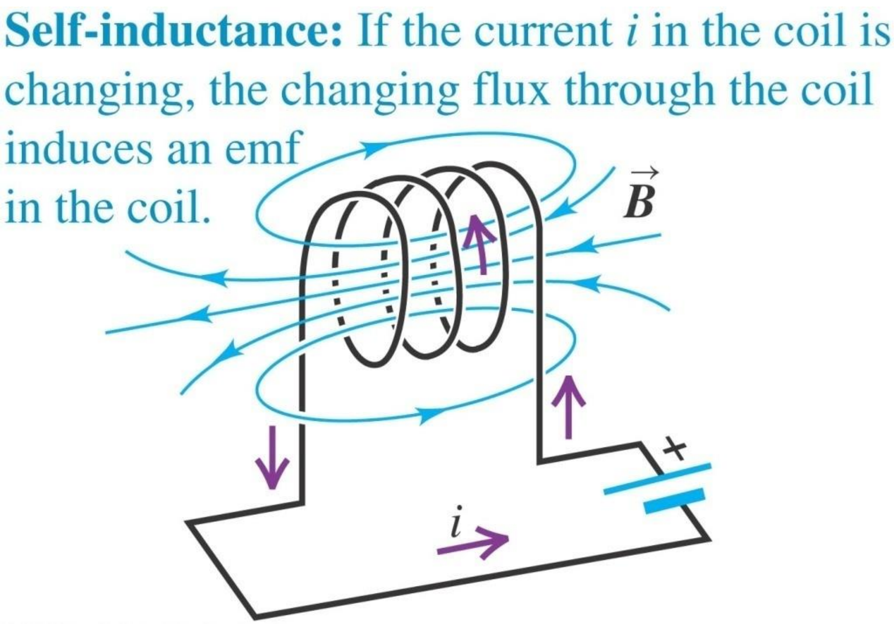
$$\Phi_{B_1} = \frac{Mi_2}{N_1} = \frac{\left(25\ \mu\text{H}\right)(6.0\ \text{A})}{1000} = 1.5\times10^{-7}\ \text{Wb}$$

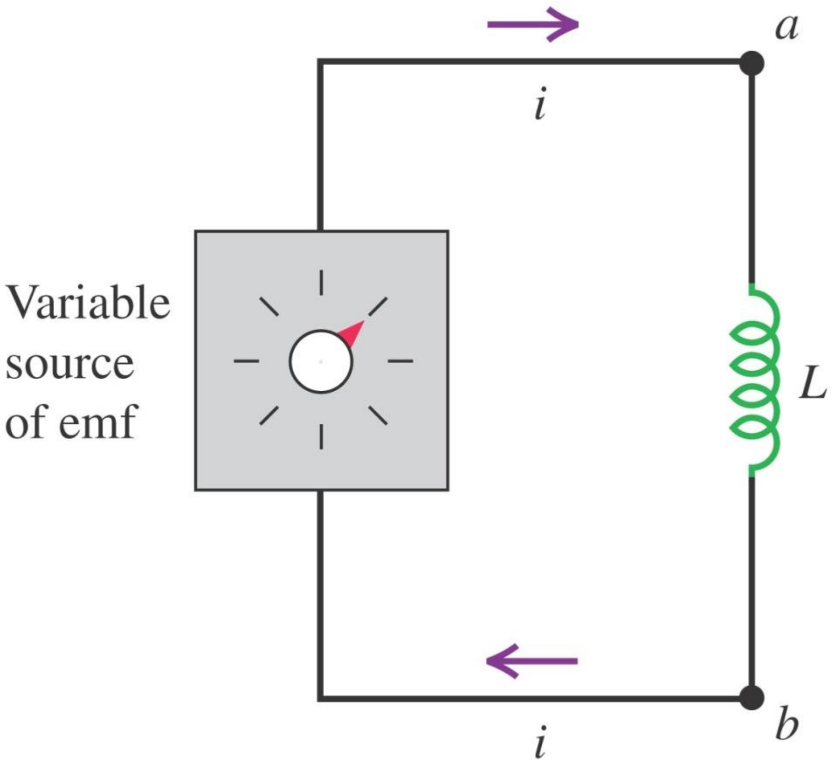
$$\varepsilon_1 = -M\frac{di_2}{dt} = -\left(25\ \mu\text{H}\right)\frac{d}{dt}\left(2\times10^6\ \text{A/s}\right)t$$

$$\varepsilon_1 = -50\ \text{V}$$

For large changes in current the induced emf is substantial.

Self-inductance





Inductor or a choke!

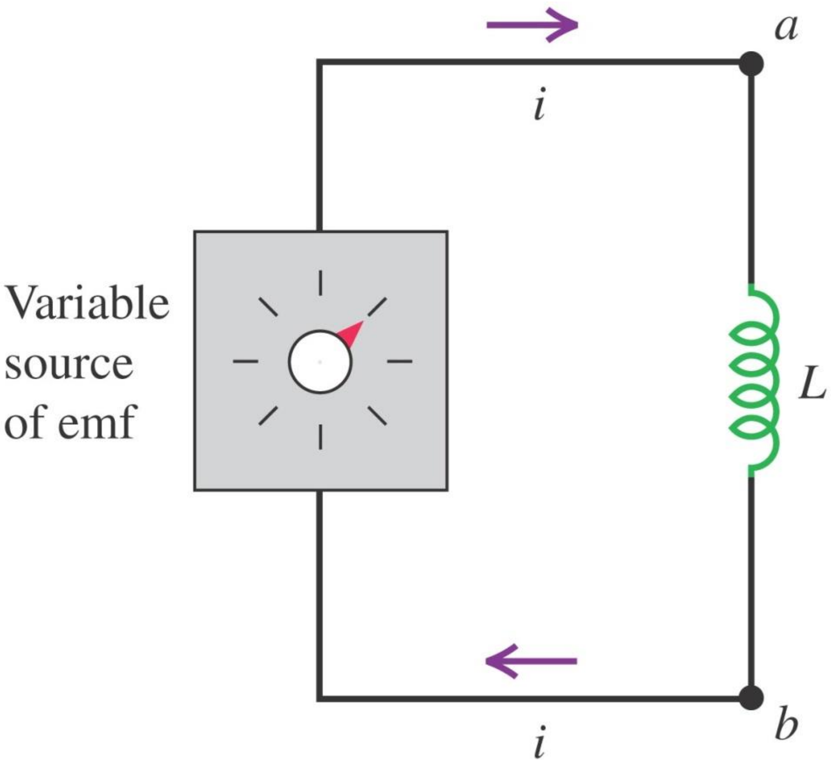
The purpose of a inductor is to oppose any variations in the currents.

In a direct current circuit, inductors are used to maintain a steady current despite any fluctuations in the applied emf.

In an alternating-current circuit, an inductor tends to suppress variations of current that are more rapid than desired.

How do we treat an inductor in a circuit and particularly in a loop?

$$L = \frac{N\Phi_B}{i} \qquad \text{(Self inductance)}$$



$$N\frac{d\Phi_B}{dt} = L\frac{di}{dt}$$

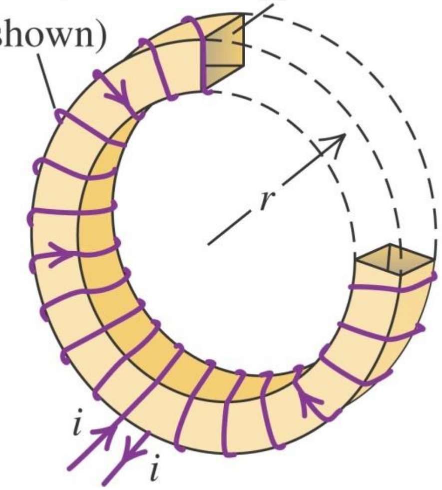
$$\varepsilon = -L\frac{di}{dt}$$

$$V_{ab} = L\frac{di}{dt}$$

A fluorescent lamp is a highly non-ohmic system. The current can grow rapidly so much that it could damage the circuit. An inductor is used to protect the circuit.

Inductors are also used in street lights and neon lamps.

Self inductance of a toroid



$$B = \mu_0 Ni / 2\pi r$$

$$\Phi_M = BA = \left(\mu_0 Ni / 2\pi r\right) A$$

$$L = \frac{N\Phi_M}{i} = \frac{N\left(\mu_0 Ni / 2\pi r\right) A}{i} = \frac{\mu_0 N^2 A}{2\pi r}$$

Self inductance is proportional to N^2.

$$L = \frac{\left(4\pi \times 10^{-7}\ \text{Wb/A.m}\right)\left(200^2\right)\left(5.0\times 10^{-4}\ \text{m}^2\right)}{2\pi(0.01\ \text{m})} = 40\ \mu\text{H}$$

$$i = \left(2\times 10^6\ \text{A/s}\right)t \qquad |\varepsilon| = L\left|\frac{di}{dt}\right| = \left(40\ \mu\text{H}\right)\frac{d}{dt}\left(2\times 10^6\ \text{A/s}\right)t = 80\ \text{V}$$

Energy for a parallel plate capacitor

Calculate the work done to accumulate an additional charge dq on the plates.

Assume that the plates have an initial charge of q. Then

$$V = \frac{q}{C}$$

The work done to transfer the additional charge dq is given by

$$dW = Vdq$$

The work required to charge a capacitor from zero to Q is

$$W = \int_0^Q Vdq = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C}$$

The energy stored in the capacitor is the

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \qquad V = \frac{Q}{C}$$

Energy density is given by

$$u = \frac{U}{\text{volume}} = \frac{CV^2}{2Ad} = \frac{\varepsilon_0 \dfrac{A}{d}(Ed)^2}{2Ad} = \frac{1}{2}\varepsilon_0 E^2$$

$$u = \frac{1}{2}\varepsilon_0 E^2$$

This is a general result!

Energy stored in an inductor

$$P = V_{ab} i = L \frac{di}{dt} i$$

$$P = \frac{dU}{dt}$$

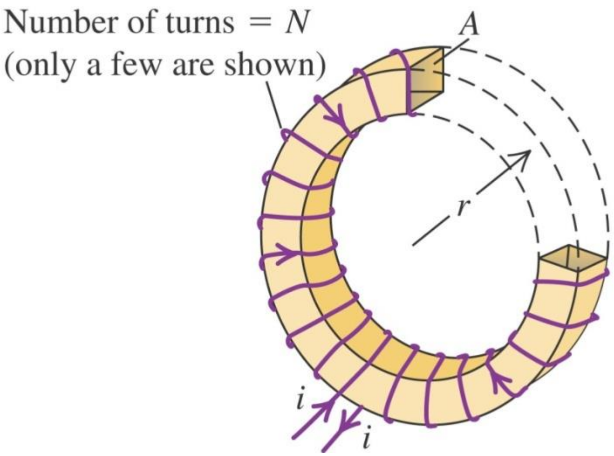
$$\frac{dU}{dt} = Li \frac{di}{dt} \Rightarrow dU = Lidi$$

When current is increased from 0 to I, $\frac{1}{2}LI^2$ is stored in the inductor.

$$U = L \int_0^I idi = \frac{1}{2} LI^2$$

When current is decreased from I to 0, the inductor acts as a source of energy.

$$U = \frac{1}{2} L I^2 \qquad L = \frac{\mu_0 N^2 A}{2\pi r}$$



$$U = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{2\pi r} I^2$$

$$u = \frac{U}{2\pi r A} = \frac{1}{2} \frac{\mu_0 N^2}{(2\pi r)^2} I^2 \qquad B = \mu_0 N I / 2\pi r$$

$$\boxed{u_B = \frac{1}{2} \frac{B^2}{\mu_0}}$$

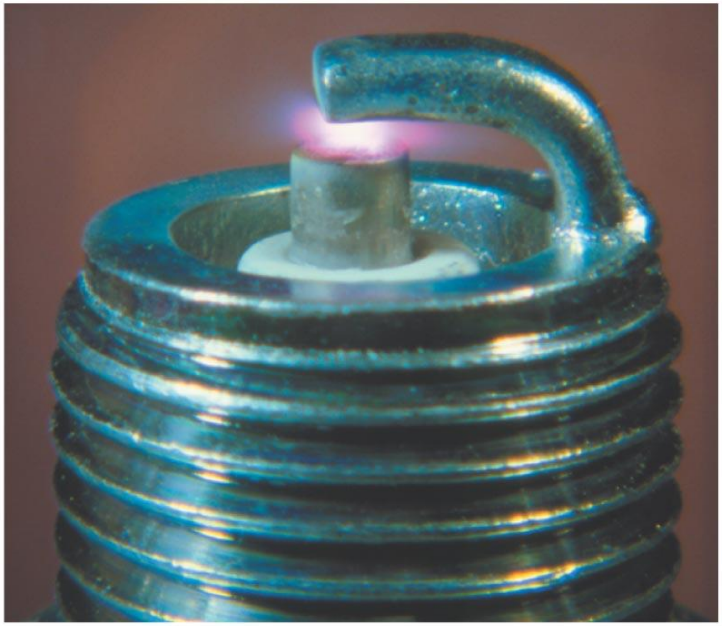
These are general results.

$$\boxed{u_E = \frac{1}{2} \varepsilon_0 E^2} \qquad \boxed{u_{EM} = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \frac{B^2}{\mu_0}}$$

Primary 250 turns

Secondary 25000 turns

The current is quickly dropped to zero to generate an emf of tens of thousands of volts, generating a powerful pulse of current that travels through the secondary coil to the spark plug, igniting the fuel-air mixture in the engine’s cylinders.



$$U = \frac{1}{2} L I^2$$

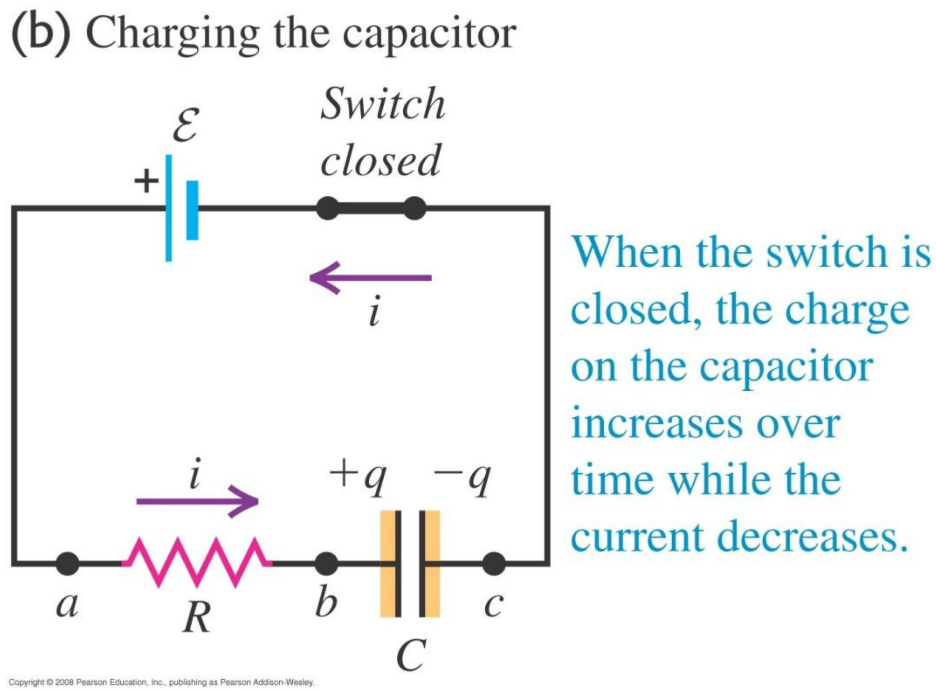
Store energy during low demand hours in an inductor.
Use the energy stored during the high demand hours.

$$L = \frac{2U}{I^2} = \frac{2 \times (1\ \mathrm{kW.h})}{200\ \mathrm{A}} = \frac{2 \times (3600 \times 10^3\ \mathrm{J})}{200\ \mathrm{A}} = 180\ \mathrm{H}$$

Large inductance means large heat losses due to

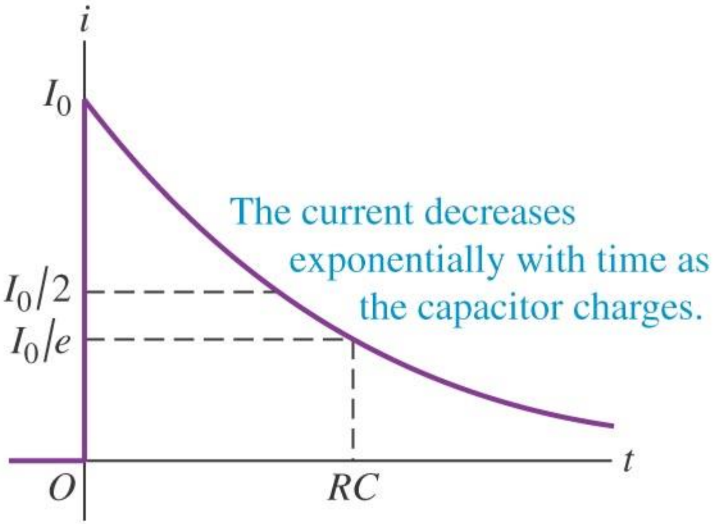
$$U_{\text{heat loss}} = R I^2$$

Superconductors have very low R but because they must be cooled to 100 K or lower to become superconducting, the scheme is impractical at the present time.



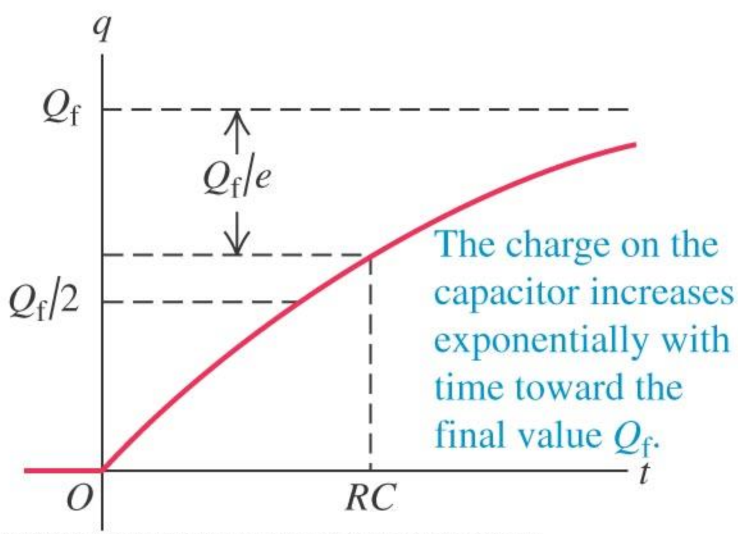
$$\frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{q}{RC} = -\frac{1}{RC}(q - \varepsilon C)$$

(a) Graph of current versus time for a charging capacitor



$$i = I_0 e^{-t/RC}$$

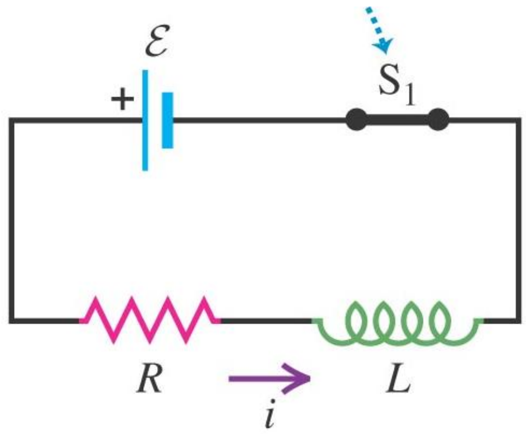
(b) Graph of capacitor charge versus time for a charging capacitor



$$q = \varepsilon C\left(1 - e^{-t/RC}\right) = Q_f\left(1 - e^{-t/RC}\right)$$

RL CIRCUITS

An inductor in a circuit help to prevent rapid changes in current to occur due to self-inductance and Lenz's law.



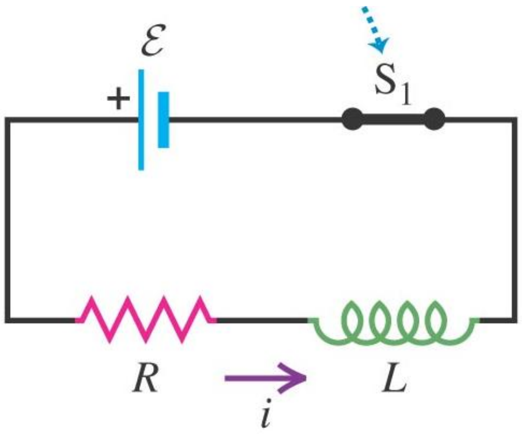
Useful when steady currents are desirable but the external source has a fluctuating emf.

Note that an inductor has an associated resistance.

The current must begin to grow at a rate that depends on self inductance.

RL CIRCUITS

$$t = \text{intermediate}$$



$$\varepsilon - iR - L\frac{di}{dt} = 0$$

$$\frac{di}{i - (\varepsilon / R)} = -\frac{R}{L}dt$$

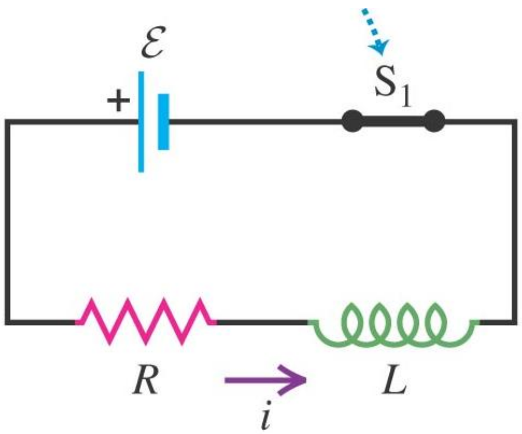
$$\int_0^i \frac{di}{i - (\varepsilon / R)} = -\int_0^t \frac{R}{L}dt \qquad \ln\left(\frac{i - (\varepsilon / R)}{-\varepsilon / R}\right) = -\frac{R}{L}t$$

$$i = \left(\frac{\varepsilon}{R}\right)\left(1 - e^{-\frac{R}{L}t}\right)$$

RL CIRCUITS

$t = 0$ at the time swicth is closed.

$$i = \left(\frac{\varepsilon}{R}\right)\left(1 - e^{-\frac{R}{L}t}\right)$$



$$i_0 = 0$$

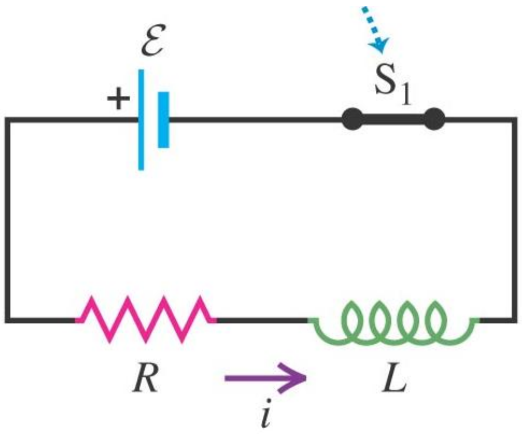
$$\left(\frac{di}{dt}\right)_{t=0} = \frac{\varepsilon}{R}\left(\frac{R}{L}e^{-\frac{R}{L}t}\right)_{t=0} = \frac{\varepsilon}{L} \qquad V_L = L\left(\frac{di}{dt}\right)_{t=0} = \varepsilon$$

The greater the inductance the more slowly the current increases.

RL CIRCUITS

$$t = \infty$$

$$i = \left(\frac{\mathcal{E}}{R}\right)\left(1 - e^{-\frac{R}{L}t}\right)$$



$$i_{t=\infty} = \frac{\mathcal{E}}{R}$$

$$V_L = L\frac{di}{dt} = 0$$

When current reaches its maximum value, all of the voltage will appear across the resistor and none across the inductor.

RL CIRCUITS

$$i = \left(\frac{\mathcal{E}}{R}\right)\left(1 - e^{-\frac{R}{L}t}\right)$$

$$t = 0 \Rightarrow i = 0$$

$$t = \infty \Rightarrow i = \frac{\mathcal{E}}{R}$$

$$\tau = \frac{L}{R}$$