PHY 2049: Introductory Electromagnetism

Notes From the Text

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1 Introduction

The subject of this course is the physics of electricity and magnetism. We have many different concepts and ideas to study before we will have a firm understanding of electromagnetism, but it is interesting to note from the outset that the entire theory can be written down in five fairly simple equations.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} + \mu_0 \mathbf{J}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

2 Chapter 21: Electric Charge

2.1 Electric Charge

Electric charge is an intrinsic quantity of a particle. We can not 'see' this charge, but we can experience its effects through the force that it exerts when placed near another charged object. Unlike gravity (which is always attractive), charged objects can exert both attractive and repulsive forces on each other. We distinguish the two types of charge by calling one negative and the other positive. Charges with the same sign repel each other while charges with opposite signs attract each other. Note that the direction of the force always lies along the line connecting the two charges. Recall that an atom is made up of a positively charged nucleus and negatively charged electrons, so that most atoms are electrically neutral.

We can classify materials by their ability to transmit charge, i.e. how easily charge flows through them. Materials which allow charge to flow essentially unimpeded are called conductors, while those which do not allow charge to flow are called insulators. Most metals are good conductors which is the reason that electrical wires and circuits are made up of metal components. Plastics are generally insulators however which is why most wires are wrapped in a plastic coating.

2.2 Coulomb's Law

We can describe the force between two charged particles through Coulomb's Law:

$$\vec{F} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q_1 q_2}{r^2}\hat{r} \tag{1}$$

where ϵ_0 is a universal constant, q_1 and q_2 are the charge on each particle, and r is the distance between the two. Notice the similarities to Newton's Law of Gravitation,

$$\vec{F}_g = (G)\,\frac{m_1 m_2}{r^2}\hat{r} \tag{2}$$

The constant which appears in Coulomb's Law is called the permittivity of free space and is written in such a strange way because ϵ_0 shows up in other places alone.

Given the similarity between these two force laws, we should expect other similarities as well. One of these similarities occurs with spherically symmetric objects, which have the following properties: 1. A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all of the charge were located at its center. 2. A charged particle inside of a shell of uniform charge does not experience any net force from the shell. 3. Any excess charge placed on a spherical conductor will rearrange itself until it is uniformly distributed on the outside of the conductor. The first two properties are exactly analogous to a spherical shell of uniform mass, while the last one has no gravitational analogy because there is no gravitational analogue to a conductor.

2.3 Quantization and Conservation of Charge

In Benjamin Franklin's day, electric charge was thought to be a continuous fluid, but it has since been observed that charge is quantized. That is to say, charge comes in (albeit small) indivisible packages. The elementary charge given in Coulombs is:

$$e = 1.602 \times 10^{-19}\, C$$

This fundamental unit of charge is the charge which is carried by the electron and the proton. The electron has negative one elementary charge while the proton has positive one elementary charge.

Similar to energy and linear and angular momentum, charge is conserved. This means (at least for this class) that charge can neither be created or destroyed, and that if a charged particle disappears from one area, it must show up in another.

2.4 Problems

Problem 21.46

In the figure, three identical conducting spheres form an equilateral triangle of side length $d = 20.0\, cm$. The sphere radii are much smaller than d, and the sphere charges are $q_A = -2.00\, nC$, $q_B = -4.00\, nC$, and $q_C = +8.00\, nC$.

a)What is the magnitude of the electrostatic force between sphere A and C?

The following steps are then taken: A and B are connected by a thin wire and then disconnected; B is grounded by the wire, and the wire is then removed; B and C are connected by the wire and then disconnected. What now are the magnitudes of the electrostatic force

b) between spheres A and C and

c) between spheres B and C?

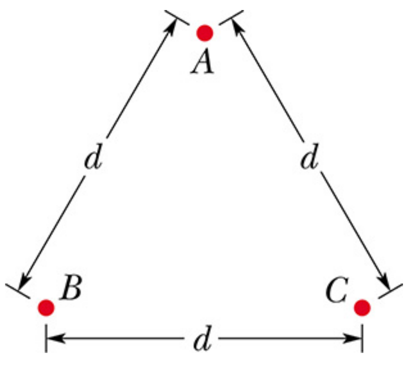

Figure 1: Problem 21.40

Solution

Part a) As noted earlier in these notes, charge will spread out uniformly over a conducting sphere (if we assume that it is unaffected by the presence of the other spheres). Hence, we can treat the spheres as point charges placed at the center of the spheres. For part a we are simply asked to calculate the force between spheres A and C, which is a simple application of Coulomb's law.

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_A\, q_C}{d^2} = 3.60\, \mu N$$

Parts b) & c) When A and B are connected by the wire, the excess charge on the two spheres will equalize, leaving both with a charge of $-3.00\,nC$. The ground is charge neutral, so all of the excess charge on sphere B will dissipate into the ground leaving it with zero charge. Finally, when B and C are connected by the wire they too will equalize leaving both with a charge of $+4.00\,nC$. So the final state after the three successive steps is: $q_A = -3.00\,nC$, $q_B = +4.00\,nC$, and $q_C = +4.00\,nC$. Parts b and c are now just a simple application of Coulomb's law.

$$F_{AC} = \frac{1}{4\pi\epsilon_0}\frac{q_A\,q_C}{d^2} = 2.70\,\mu N$$

$$F_{BC} = \frac{1}{4\pi\epsilon_0}\frac{q_B\,q_C}{d^2} = 3.60\,\mu N$$

3 Chapter 22: Electric Fields

3.1 The Electric Field

Recall from our discussion of gravitation, that we could divide both sides of Newton's law by the mass of the object upon which the force was acting to obtain the gravitational acceleration.

$$\vec{g} = -G\frac{m}{r^2}\hat{r}$$

This form of Newton's law assigns a direction and magnitude of acceleration to every point in space. Although we did not talk about it at the time, we can think of this as a gravitational field. Similarly, we can think of the temperature in a room as a (directionless) field. The temperature field might have lower values near the doors and windows and higher values near the heater vents.

In this chapter we will extend this concept of a field to electric charge. Similar to gravitation, we will define the electric field as the force per unit charge.

$$\vec{E} = \frac{\vec{F}}{q_0} \tag{3}$$

In the case of a point charge, then the electric field is simply:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{4}$$

We can use the concept of field lines (originally thought of by Faraday) to visualize the electric field. The density of the lines in a certain region gives the relative strength of the field, and the direction of the field lines gives the direction of the field at that location. Electric field lines originate at positive charges and terminate on negative charges.

3.2 The Electric Field of a Dipole

A perfect dipole is two point charges of equal and opposite charge located very close together. Understanding dipoles is important because many atoms act like dipoles even though they are not perfect dipoles (such as H_2O). To find the field of a dipole, we begin by writing down the field of two point charges located a distance d apart. For simplicity, we restrict ourselves to finding the field along the dipole axis. After some algebraic manipulation we find the field to be:

$$E = \frac{1}{2\pi\epsilon_0}\frac{p}{z^3} \tag{5}$$

where $p = q\,d$ is called the dipole moment of the electric dipole and z is the distance from the center of the dipole along the axis of the dipole.

Note that in deriving this formula we must make the approximation that $d \ll z$. As noted before, most dipoles which we talk about will be atoms and hence, this is not a bad aproximation.

3.3 The Electric Field for Continuous Charge Distributions

For a continuous charge distribution, we can write down the differential bit of electric field due to a differential bit of charge as:

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2} \tag{6}$$

Using this knowledge, we can find the total electric field by summing up (integrating) all of the contributions from the differential elements of charge. The challenge in doing so usually comes down to expressing the charge in a way such that it can be integrated over.

For line charges we can use the linear charge density to express dq in terms of an integration variable. For straight line charges:

$$dq = \lambda\, dx$$

and for circular line charges:

$$dq = \lambda\, R\, d\theta$$

Using the second form, we can derive the electric field due to a ring of charge as measured on the axis of the ring:

$$E = \frac{1}{4\pi\epsilon_0}\frac{qz}{(z^2 + R^2)^{3/2}} \tag{7}$$

Similarly, for a charged surface such as a disk or a plane, we can use the surface charge density σ to express dq in terms of an integration variable. In the case of a charged disk:

$$dq = \sigma\, dA = \sigma\,(2\pi r)dr$$

and a charged plane:

$$dq = \sigma\, dA = \sigma\, dx\, dy$$

where it is understood that the second case implies a double integration.

3.4 Point Charges and Dipoles in External Fields

Now that we have talked about the electric fields of various charge distributions, the natural question to ask next is: how do these objects act in external electric fields? For a point charge, the answer is simple. By turning around the definition of the electric field (equation (3)) we can easily see that the force on a point charge in an electric field is:

$$\vec{F} = q\,\vec{E} \tag{8}$$

So, a point charge in an electric field will experience a force which is in the same or opposite (depending on the sign of the charge) direction as the field and proportional to both the size of the charge and the strength of the field.

A dipole in an external electric field is not so simple. The force on a dipole can be calculated from the above equation by treating the two point charges of the dipole separately. Note that if the electric field is constant, then the net force will be zero because the two point charges will have opposite sign. By drawing a free body diagram we can see that the net torque on a dipole will not necessarily be zero in a constant field.

$$\tau = \vec{p} \times \vec{E} \tag{9}$$

Notice that this torque tends to align the dipole with the field.

The potential energy of a dipole in an external field is:

$$U = -\vec{p} \cdot \vec{E} \tag{10}$$

It is important to note that the zero of the potential energy is at 90^o, but that the minimum occurs when the dipole is aligned with the field.

3.5 Problems

Problem 22.25

In figure 2, two curved plastic rods, one of charge $+q$ and the other of charge $-q$, form a circle of radius $R = 8.50\,cm$ in an xy plane. The x axis passes through both of the connecting points, and the charge is distributed uniformly on both rods. If $q = 15.0\,pC$, what are the **a)** magnitude and **b)** dirction (relative to the positive direction of the x axis) of the electric field $\vec{E}$ produced at P, the center of the circle?

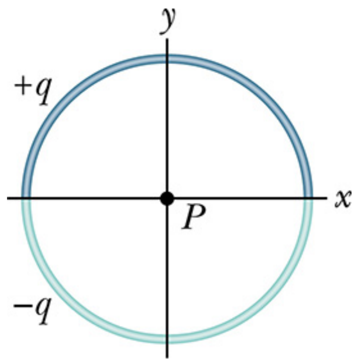

Figure 2: Problem 22.25

Solution

Part a) As with all of these problems, we begin with the differential form of the electric field.

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}$$

We can relate the differential element of charge dq to the differential element of length along the hoop via the linear charge density. The charge density will be the same for the top and the bottom portion of the ring with opposite signs $\lambda = \pm\frac{q}{\pi R}$. We can use this to describe the differential element of charge in terms of the differential element of length $d\theta$.

$$dq = \lambda\, R\, d\theta = \pm\frac{q}{\pi}\, d\theta$$

We must be careful here. Our goal is to sum up all of the contributions from all of the differential elements of charge to the field at the center of the ring. The electric field is a vector quantity however, and so we can not simply sum it up as if it was a scaler. The symmetry of the problem allows us to greatly simplify the problem though. If we consider an element of charge dq located at a position θ (where θ is chosen to be 0 at the positive x axis) then it will have a corresponding piece of charge at $\pi - \theta$ which will cancel out the x portion of the field. Hence, only the y component of the field will contribute for each differential piece of charge. The y component of the field is a scaler quantity and hence can be integrated in the normal way.

$$dE_y = \frac{1}{4\pi\epsilon_0}\frac{dq}{R^2}\sin\theta = \pm\frac{q}{4\pi^2\epsilon_0 R^2}\sin\theta\, d\theta$$

Integrating gives the total electric field

$$\begin{aligned}
E &= \frac{q}{4\pi^2\epsilon_0 R^2}\left[\int_0^{\pi}\sin\theta\, d\theta \;-\; \int_{\pi}^{2\pi}\sin\theta\, d\theta\right] \\
&= \frac{q}{4\pi^2\epsilon_0 R^2}\left[-\cos\theta|_0^{\pi} \;-\; -\cos\theta|_{\pi}^{2\pi}\right] \\
&= \frac{q}{\pi^2\epsilon_0 R^2} = 2.02\,\frac{N}{C}
\end{aligned}$$

Part b) We have already argued that the field must point in the y direction, we must now decide whether it points in the $+\hat{y}$ or the $-\hat{y}$ direction. Recalling that field lines originate on positive charges and terminate on negative charges, it should be apparent that the field points in the $-\hat{y}$ direction.

Problem 22.51

Two large parallel copper plates are $5.0\,cm$ apart and have a uniform electric field between them as depicted in figure 3. An electron is released from the negative plate at the same time that a proton is released from the positive plate. Neglect the force of the particles on each other and find their distance from the positive plate when they pass each other. (Does it surprise you that you need not know the electric field to solve this problem?)

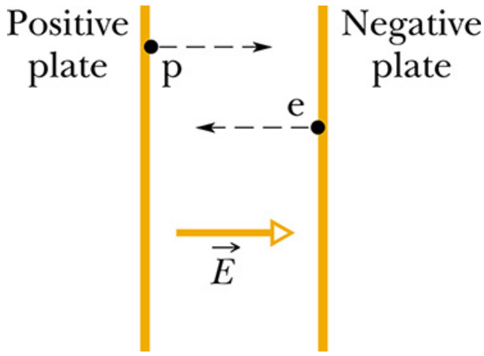

Figure 3: Problem 22.51

Solution

This is really more of a mechanics problem than an electrodynamics one. All we really need to recall from this chapter is that the force on a particle in the presence of an electric field is given by:

$$F = q\,E$$

Clearly, both particles are going to experience the same force, but it will be in opposite directions. The acceleration that each will experience is:

$$a_e = -\frac{q}{m_e}E \qquad \& \qquad a_p = \frac{q}{m_p}E$$

Since the accelerations are constant and both particles are released from rest, we can express the positions of both as functions of time.

$$x_e = d - \frac{q}{m_e}E\,t^2 \qquad \& \qquad x_p = \frac{q}{m_p}E\,t^2$$

The question that we are asking is thus, when does $x_e = x_p = x_0$. Setting both equal to x_0 and solving the system of equations gives:

$$x_0 = d - \frac{q}{m_e}E\left(\frac{m_p\,x_0}{q\,E}\right) \quad \Rightarrow \quad x_0 = \left(\frac{m_e}{m_e + m_p}\right)d = 27.3\,\mu m$$

This result is independent of the strength of the electric field because the electron and the proton have equal and opposite charge.

4 Chapter 23: Gauss's Law

4.1 Flux

Gauss's law is a very powerful tool for solving highly symmetric problems. Before we can discuss Gauss's law though, we need to discuss the concept of flux.

It is easy to begin thinking about flux from the perspective of fluids. If we consider a pipe with liquid flowing through it, then the flux of liquid through the pipe is the amount of water which flows through in a given amount of time. If we choose a larger pipe with the same rate of flow, then the flux would be higher. So our first conclusion is that the amount of flux is proportional to the area chosen as well as the flow rate. If we then asked the question: what is the flux through a patch of area which is inside of the pipe but perpendicular to the direction of flow? In this case, there would be zero fluid flowing through the patch of area and hence, the flux would be zero. So our second conclusion is that the amount of flux is related to the angle between the patch of area and the direction of flow. This second conclusion hints to us that flux will have a dot product relationship between the flow direction and the normal to the surface area patch in question.

With electromagnetic flux, there is nothing flowing, and the fields are not contained within a walled vessel such as a pipe. We can, however, still evaluate the electric flux through an imaginary surface (called a Gaussian surface). The electric flux through our imaginary surface is given by:

$$\Phi = \oint \vec{E} \cdot d\vec{A} \tag{11}$$

Lets consider what this equation is telling us. The integrand $\vec{E} \cdot d\vec{A}$ says that we take a differential patch of area on our Gaussian surface and evaluate its dot product with the electric field. The question at hand is: how do we assign a direction to a patch of surface area? There are two natural directions for a closed surface, inward and outward. By convention the direction of $d\vec{A}$ is chosen to be normal to the patch of surface area pointing outwards. In this case outward is unambiguous because we are dealing with closed surfaces (that is the meaning of the little circle on the integral sign $\oint$). Finally, after we have evaluated all of these dot products, then we sum all of them up via integration.

This may seem at first like a daunting process, and indeed it is in many, if not most, situations. We will soon see though that in highly symmetric situations, we can essentially crack open the integral sign and easily calcualte the flux.

4.2 Gauss's Law

Gauss's law tells us that the net electric flux through *any* real or imaginary surface is directly related to the amount of charge enclosed by that surface.

$$\Phi = \frac{q}{\epsilon_0} \quad \Rightarrow \quad \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \tag{12}$$

Note that this equation is only exactly true in a vacuum, and holds approximately in air. We will later modify it to include other materials.

As a simple application of Gauss's law, lets calculate the electric field of a point charge of charge $+q$ and check to see that it agrees with Coulomb's law. We will choose as our Gaussian surface a sphere of radius r which has the point charge at its center. Note that this is generally how we exploit Gauss's law; we choose a Gaussian surface whose symmetry mimics that of the object in question. Because of the symmetry of the situation, we can conclude that the electric field must point radially outwards for if it pointed in any other direction, then the symmetry of the situation would be broken. Since the electric field points radially outward at every point on the surface $\vec{E} \cdot d\vec{A}$ becomes $E\,dA$. Furthermore, the symmetry of the problem demands that $|\vec{E}|$ be constant over the surface, so it can come outside of the integral.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \quad \Rightarrow \quad \oint E\,dA = \frac{q}{\epsilon_0} \quad \Rightarrow \quad E \oint dA = \frac{q}{\epsilon_0}$$

The integral of dA over the entire surface is simple the total surface area. Hence, the electric field of the point charge is given by:

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

which agrees perfectly with our calculation of the electric field from Coulomb's law.

4.3 Aside: Conductors

Gauss's law can be used to prove many important properties about conductors. I will forgo the proofs here and simply state their key properties. The proofs can be found on pg. 613 of the textbook.

- $\vec{E} = 0$ within a conductor.
- A cavity within a conductor is shielded from external fields.
- The exterior is not however shielded from interior fields.
- The electric field at the surface of a conductor must point normal to the surface.
- Any excess charge placed on a conductor must reside on the outside surface.

The first point does not even require Gauss's law for proof, but is simply a restatement of the defining property of conductors. By definition, a conductor allows charge to flow freely. If there were an electric field within the conductor, then there would be a force on the charges and they would move until there was no electric field.

The second statement tells us that if we place a charge outside of a conducting shell, then the cavity within the conductor will not know that the charge is there. This is the idea behind Faraday cages. Strangely enough however, if we place the charge inside of the conductor, then the outside world will still know that it is there. This is kind of like a one way mirror, charges outside can see charges inside, but charges outside are blind to all of the charges in the outside world.

The fourth item is very similar to the first in that it does not rely on Gauss's law. If there is a tangential electric field at the surface of the conductor, then the charges will feel a force and will rearrange themselves until the field is gone.

The final item is somewhat natural. The excess charges on the conductor will try to get as far away from each other as possible and hence will move to the outside.

One final point about conductors (which is also derived from Gauss's law) is that the normal component of the electric field at the surface of the conductor is proportional to the surface charge density. We can use this to calculate the electric field if we know the charge density or the charge density if we know the field.

$$E = \frac{\sigma}{\epsilon_0} \tag{13}$$

4.4 Applications of Gauss's Law

As stated earlier, Gauss's law is not very useful unless we can exploit the symmetry of the situation. In order to exploit the symmetry of the different situations which we will encounter, our Gaussian surface will generally need to mimic the symmetry of the charged object. The two shapes which will almost always be useful are the sphere and the cylindrical can. Lets calculate the electric field of a few objects to get a feel for these ideas.

4.4.1 A Line of Charge: Cylindrical Symmetry

As noted before, we want to choose a Gaussian surface which mimics the symmetry of the line of charge. In this case we choose a cylinder as shown in figure 4. Recall Gauss's law:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

Because of the symmetry of the situation, we can see that the electric field from the line of charge must point radially outward. Since the field points radially outward, $\vec{E} \cdot d\vec{A}$ will be zero for the endcaps of the cylinder and will be equal to EdA on the rest of the surface. Symmetry also implies that the electric field must be constant over the entire surface (excluding the endcaps). We can therefore pull E outside of the integral with the understanding that dA is only to be integrated over the wall and no the endcaps.

$$E = \frac{q_{enc}}{\epsilon_0 \oint dA} = \frac{q_{enc}}{\epsilon_0\, 2\pi r h} = \frac{\lambda\, h}{\epsilon_0\, 2\pi r h} = \frac{\lambda}{\epsilon_0\, 2\pi r}$$

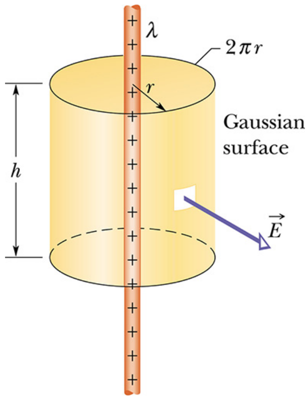

Figure 4: Line of charge with cylindrical Gaussian surface.

4.4.2 A Charged Plane: Planer Symmetry

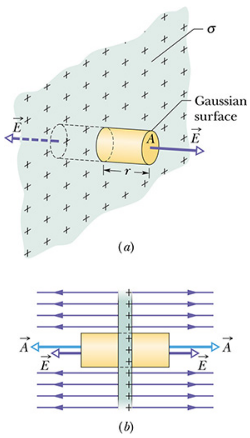

Figure 5: Charged plane with cylindrical Gaussian surface.

In this case, we again want to choose a Gaussian surface which mimics the symmetry of the problem. We will choose a cylinder as we did in the last problem, but we could just as easily have chosen a box or a cube. By symmetry, the electric field must point away from the plane (normal) and must be constant over the entire surface. Furthermore, symmetry implies that the electric field must be the same on opposite sides of the surface at equal distances from the plane. Since the field points normally away from the plane, the only flux through our Gaussian surface will be through the endcaps. The charge enclosed by the surface will be $q_{enc} = \sigma\, A$ where A is the are of the endcaps of the cylinder. Plugging all of this into Gauss's law gives:

$$E = \frac{q_{enc}}{\epsilon_0 \ \oint dA} = \frac{\sigma\, A}{\epsilon_0\ 2A} = \frac{\sigma}{2\epsilon_0}$$

Notice that this differs by a factor of 2 from the equation for the electric field at the surface of a conductor. The factor of 2 comes in because inside of a conductor, the electric field is zero and hence the total area which contributes to the flux is only A instead of $2A$. Otherwise, the derivation is the same.

4.4.3 A Uniformly Charged Sphere: Spherical Symmetry

In this case we choose a spherical Gaussian surface centered at the center of the sphere to mimic the symmetry. Lets first consider the field outside of the sphere. Because of symmetry, the field must point radially outward and must be constant at equal distances from the sphere. The total charge enclosed by the surface is simply the total charge on the sphere q. Hence, by Gauss's law, the field outside must be:

$$E = \frac{q}{\epsilon_0 \, 4\pi r^2}$$

which is exactly the same field as a point charge centered at the center of the sphere.

Inside of the sphere the same symmetry arguments hold, but the total charge enclosed by the sphere will be proportional to the volume of the sphere.

$$q_{enc} = \rho \, V = \frac{q}{\frac{4}{3}\pi R^3} \, \frac{4}{3}\pi r^3 = q \, \frac{r^3}{R^3}$$

Plugging this into Gauss's law gives the electric field inside of a uniformly charged sphere.

$$E = \left(\frac{q}{\epsilon_0 \, 4\pi R^3} \right) r$$

4.5 Problems

Problem 23.8

Figure 6 shows two nonconducting spherical shells fixed in place. Shell 1 has uniform surface charge density $+6.0 \, \frac{\mu C}{m^2}$ on its outer surface and radius $3.0 \, cm$; shell 2 has uniform surface charge density $+4.0 \, \frac{\mu C}{m^2}$ on its outer surface and radius $2.0 \, cm$; the shell centers are separated by $L = 10 \, cm$. In unit-vector notation, what is the net electric field at $x = 2.0 \, cm$?

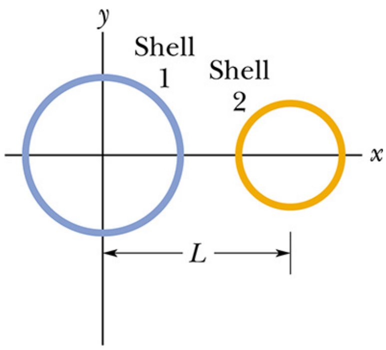

Figure 6: Two charged shells.

Solution

Notice that the point in question is inside of shell 1. Lets first consider shell 1 by itself for a minute. Without shell 2 in the picture, this is a spherically symmetric situation and so the electric field from shell 1 must be uniform and be directed radially. If we no draw a spherical Gaussian surface inside of the shell, the uniformity of the field implies that the electric field must contribute the same amount of flux at all points on the Gaussian surface. The Gaussian surface encloses no charge though since it is inside of the shell. Hence, the electric inside of shell 1 *due to shell 1* must be zero. We can therefore neglect the presence of shell 1 all together.

As was discussed earlier in the notes. The spherical symmetry of shell 2 implies that the field outside of the shell looks like a point charge centered at its center. The total charge of the imaginary point charge is $4\pi R_2^2 \sigma_2$. Hence, the electric field outside of shell 2 due to shell 2 is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{4\pi R_2^2 \sigma_2}{r^2} \hat{r} = \frac{\sigma_2}{\epsilon_0} \frac{R_2^2}{r^2} \hat{r}$$

Plugging in $r = L - 2 \, cm$ gives the electric field at $x = 2 \, cm$.

$$\vec{E} = -\frac{\sigma_2}{\epsilon_0} \frac{R_2^2}{(L - 2 \, cm)^2} \hat{x} = 2.82 \times 10^4 \, \frac{V}{m}$$

Problem 23.27

Figure 7 is a section of a conducting rod of radius $R_1 = 1.3\,mm$ and length $L = 11\,m$ inside of a thin-walled coaxial conducting cylindrical shell of radius $R_2 = 10\,R_1$ and the same length L. The net charge on the rod is $Q_1 = +3.4 \times 10^{-12}\,C$; that on the shell is $Q_2 = -2\,Q_1$. What are the **a)** magnitude E and **b)** direction of the electric field at radial distance $r = 2\,R_2$? What are **c)** E and **d)** the direction at $r = 5\,R_1$? What is the charge on the **e)** interior and **f)** exterior surface of the shell?

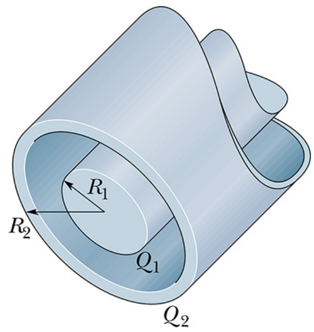

Figure 7: Two coaxial cylindrical conductors.

Parts a) and b) Because of the cylindrical symmetry of the problem, we can see immediately that the most useful Gaussian surface will be the cylinder. Notice that the problem states that this is a section from a much longer rod and hence we can treat this as an infinite rod. This allows us to exploit the cylindrical symmetry of the problem and say that the field from both pieces must be directed radially and must be constant at equal radial distances from the center of the conductor.

For the first two parts of the problem, we use a Gaussian cylinder of radius $2\,R_2$. The total charge enclosed by this cylinder is $Q_1 + Q_2 = -Q_1$. We can now use Gauss's law to calculate the field.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \quad \Rightarrow \quad \vec{E} = \frac{-Q_1}{\epsilon_0\, 40\pi R_1 L}\hat{r} = -.214\,\tfrac{V}{m}\,\hat{r}$$

Parts c) and d) These two parts are very similar to the last two except in this case we place our Gaussian surface at $r = 5\,R_1$.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \quad \Rightarrow \quad \vec{E} = \frac{Q_1}{\epsilon_0\, 10\pi R_1 L}\hat{r} = +.855\,\tfrac{V}{m}\,\hat{r}$$

Parts e) and f) These two parts may seem tricky at first, but they are actually very easy. If we place our Gaussian cylinder inside of conducting shell (in between the inner and the outer surfaces), then the boundary of the surface is within the conducting medium. Since the electric field within a conductor must always be zero, Gauss's law together with our symmetry arguments tells us that the total charge enclosed must be zero. Hence, the inner wall of the conductor must contain $Q_{in} = -Q_1$ which implies that the outer wall must have $Q_{out} = Q_2 - -Q_1 = -Q_1$.

Problem 23.50

In figure 8, a nonconducting spherical shell of inner radius $a = 2\,cm$ and outer radius $b = 2.4\,cm$ has (within its thickness) a positive volume charge density $\rho = A/r$, where A is a constant and r is the distance form the center of the shell. In addition, a small ball of charge $q = 45\,fC$ is located at the center. What value should A have if the electric field in the shell ($a \leq r \leq b$) is to be uniform?

From the symmetry of the problem we can clearly see that a spherical Gaussian surface is appropriate here. We need to calculate the electric field within the medium of the shell so we will place our Gaussian sphere there. As usual, we exploit the symmetry of the problem by observing that the field must point radially and that it must be a constant at equal radii. The charge enclosed

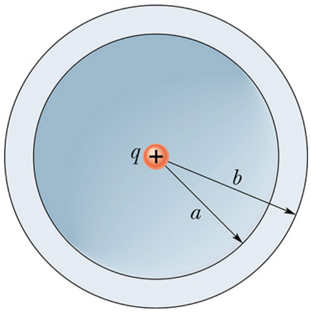

Figure 8: Charged shell with a positive point charge at its center.

by our Gaussian surface of radius r includes the central point charge as well as the charge from the enclosed portion of the shell. Lets first calculate the charge from the shell enclosed by the Gaussian sphere. This is not a simple as multiplying the charge density times the volume because the charge density is a funtion of r. We must therefore sum up all of the contributions at each value of r (i.e. integrate).

$$Q_{enc,shell} = \int \rho \, dV = \int_a^r \frac{A}{r'} 4\pi r'^2 \, dr' = 4\pi A \int_a^r r' \, dr' = 2\pi A \left(r^2 - a^2\right)$$

Plugging this into Gauss's law and exploiting the symmetry gives the electric field with in the shell.

$$\begin{aligned} E &= \frac{Q_{enc}}{\epsilon_0 A} \\ &= \frac{1}{\epsilon_0 \, 4\pi r^2} \left[q + 2\pi A \left(r^2 - a^2\right)\right] \\ &= \frac{A}{2\epsilon_0} + \frac{q - 2\pi A a^2}{4\pi\epsilon_0 r^2} \end{aligned}$$

The question at hand is what does the value of A need to be such that E is independent of r? Clearly, the first term is a constant, but the second term has some r dependence. In order for E to be a constant, the numerator in the second term must therefore vanish.

$$q - 2\pi A a^2 = 0 \quad \Rightarrow \quad A = \frac{q}{2\pi a^2} = 1.79 \times 10^{-11} \, \tfrac{C}{m^2}$$

5 Chapter 24: Electric Potential

5.1 Electric Potential and Electric Potential Energy

In this chapter we will be talking about the electric potential. A good way to introduce the electric potential is through the electric potential energy. We must be careful here though, because the two are not the same thing, though they are simply related.

Lets recall for a minute how we approached the electric field. We began by writing down the force law for charged objects (Coulomb's law). We then decided that it would be nice to be able to write down what the force would be in all regions of space if we placed a hypothetical 'test charge' there. We found this by simply taking out the 'test charge' from Coulomb's law and calling the new entity the electric field. Note that although they are simply related, the two things are very different. The force law tells us the force between two objects, while the field tells us what the force would be anywhere in space if we were to place a charge there.

The electric potential and the electric potential energy are very similarly related. The potential energy is a quantity that exists between two charges (or charge objects), while the potential is something which exists independent of the second charge. Because potential is related to potential energy, we should expect that only changes of potential are important and that there is no absolute potential. Although this is true, we will often choose (and it will be built into some formulas) the potential to be zero at infinity.

$$\Delta V = \frac{\Delta U}{q} = \frac{-W}{q} \qquad\qquad V = \frac{-W_\infty}{q}$$

In the equations above we have used the work energy theorem to relate changes in potential energy to the work done on a particle. In this language, the second equation says that the 'absolute' potential is the work required to bring a charged particle in from very far away divided by the charge of the particle.

It is important to note that the electric potential is a scaler quantity. This is nice because as we will see later on, the potential is very simply related to the electric field. This is nice because scaler quantities are much easier to work with than vector quantities and hence, we will have an easy way of calculating the electric field.

5.2 Calculating the Potential from the Field

Before we talk about how to calculate the potential directly, lets look at the connections between the potential and the electric field. The relation between the two is rather simple, but it involves the concept of a line integral. A line integral is very similar to a regular one dimensional integral, but the path of integration may not lie along a particular axis and it may not even be a straight line.

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s} \tag{14}$$

In this expression, $d\vec{s}$ is a little differential piece of the line along which we want to evaluate the integral. This may seem very foreign at first, but it is actually quite natural. If we take our path of integration to be the x-axis, then $d\vec{s} = dx\hat{x}$ and the integral reduces to integrating the x component of the electric field along the x axis.

One of the very nice features about calculating the potential in this way is that the path of integration does not matter. This is because the electrostatic force is a conservative force similar to gravity. We can therefore choose any path that we like when calculating the potential in this way.

5.2.1 Potential Due to a Point Charge

Lets now use this formula to calculate the potential of a point charge. We will choose (as previously mentioned) the potential to be zero at infinity. For our path we will choose the simplest one possible: we will begin at a point a distance r away from our point charge and go out radially from the charge to infinity.

$$\begin{aligned} V_f - V_i &= -\int_r^\infty \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\right)\hat{r}\cdot d\vec{r}' \\ &= -\frac{q}{4\pi\epsilon_0}\int_r^\infty \frac{1}{r'^2}dr' \end{aligned}$$

Since we are choosing the potential at infinity to be zero, $V_f = 0$ and we can drop the i subscript on V_i.

$$\begin{aligned} V &= \frac{q}{4\pi\epsilon_0}\int_r^\infty \frac{1}{r'^2}dr' \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{-1}{r'}\right]_r^\infty \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \end{aligned}$$

5.2.2 Superposition

Just like the electric field, the electric potential obeys the law of superposition. This means that if we have multiple point charges, we can simply add up the potential due to each of them to find the overall potential. Notice that since the potential is a scaler and not a vector, we do not have to bother breaking it up into components to sum it up as we do with the electric field. Also, similar to the electric field, we can break a continuous object down into differential pieces of charge and sum up the contributions to the potential via integration.

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} \tag{15}$$

This is extraordinarily useful because we do not have to concern ourselves whit the vectorness of the electric field, but we can still calculate the field for complex objects.

5.3 Calculating the Field from the Potential

The one missing piece (and one of the most important ones) is how we can connect the potential back to the electric field. Lets address that now. Recall that the fundamental theorem of calculus relates the concepts of integration and differentiation. Hence, from equation 14 we should expect the electric field to be related to the potential through a derivative. The one thing which complicates matters is that equation 14 involves a line integral. It turns out that each component of the field is given by the derivative along that direction. Let s be the direction in which we want to know the field, then the s component of the field is given by:

$$E_s = -\frac{\partial V}{\partial s} \tag{16}$$

Hence, if we want to know the full electric field in a Cartesian space then we need to take three seperate derivatives.

$$E_x = -\frac{\partial V}{\partial x} \quad E_y = -\frac{\partial V}{\partial y} \quad E_z = -\frac{\partial V}{\partial z}$$

For those of you familiar with three dimensional calculus, this can be written more compactly as: $\vec{E} = -\nabla V$

We now have a simple and powerful way of calculating the electric field of continuous objects without worrying about the vector properties of the electric field. We simply calculate the potential using equation 15 and then calculate the field from the potential.

5.3.1 One More Property of Conductors

We can learn a new fact about conductors from looking at equation 16. We showed previously that the electric field within a conductor must be zero. Since the filed must be zero, equation 16 implies that the potential must not change within the conductor. Hence, a conductor is an equipotential surface.

This is one of the other reasons that the potential concept is so useful. On a day to day basis, our experiences with electric fields is through electronics which are made up of conductors. Even though the electric fields within our electronics may be very complicated and at times even uncalculable, we know immediately that the potential of any conductor must be constant.

5.4 Problems

Problem 24.29

A plastic disk of radius $R = 64\,cm$ is charged on one side with a uniform surface charge density $\sigma = 7.73\frac{fC}{m^2}$, and then three quadrants of the disk are removed. The remaining quadrant is shown in figure 9. With $V = 0$ at infinity, what is the potential due to the remaining quadrant at point P, which is on the central axis of the original disk at distance $D = 25.9\,cm$ from the original center?

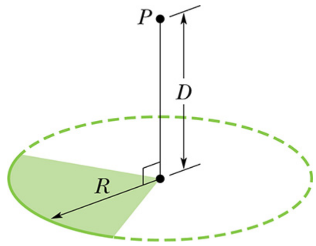

Figure 9: One quadrant of a charged disk.

This is clearly going to be an integration problem, and it will work out very similarly to the electric field problems from last chapter. We begin by writing down the differential bit of potential from a differential piece of charge on the disk.

$$dV = k\frac{dq}{r'}$$

Note that this definition of the differential piece of potential already has the assumption that $V = 0$ at infinity built in.

As usual, the challenge is to express the differential piece of charge dq in terms of the charge density and a variable of integration. From the definition of a surface charge density, we can see that $dq = \sigma\, dA$ where dA is a differential piece of area on the disk. There are a number of good ways to express the differential piece of area on a circular object. The easiest in this case is to take a sliver of area at the same radius (like a very small piece of a donut). Expressing the area in this manner leads to:

$$dq = \sigma\, dA = \sigma\,(\tfrac{\pi}{2}r)dr$$

We must be very careful at this point. The r which we have used to express dq is the distance *from the center of the disk* to the piece of area in question. The r' in the formula for the potential however is the distance *from point P* to the piece of area in question. Hence, before we can integrate we must express r' in terms of r. This is easy to do with the Pythagorean theorem: $r' = \sqrt{r^2 + D^2}$. We are now ready to put everything together and integrate.

$$V = \int_0^R k\frac{\sigma\frac{\pi}{2}r}{\sqrt{r^2+D^2}}dr = \frac{\sigma k\pi}{2}\int_0^R \frac{r\,dr}{\sqrt{r^2+D^2}} = \frac{\sigma}{8\epsilon_0}\int_0^R \frac{r\,dr}{\sqrt{r^2+D^2}}$$

This integral is most easily solved with a variable substitution. Let

$$u = r^2 + D^2 \quad\Rightarrow\quad du = 2r\,dr \quad\Rightarrow\quad u_- = D^2 \quad\&\quad u_+ = R^2 + D^2$$

which transforms the integral into:

$$\begin{aligned} V &= \frac{\sigma}{16\epsilon_0}\int_{D^2}^{R^2+D^2}\frac{du}{\sqrt{u}} \\ &= \frac{\sigma}{8\epsilon_0}\left[\sqrt{u}\right]_{D^2}^{R^2+D^2} \\ &= \frac{\sigma}{8\epsilon_0}\left[\sqrt{R^2+D^2} - D\right] \\ &= 4.71\times 10^{-5}\,V \end{aligned}$$

Problem 24.46

In figure 10, how much work must we do to bring a particle, of charge $Q = +16e$ and initially at rest, along the dashed line from infinity to the indicated point near two fixed particles of charges $q_1 = +4e$ and $q_2 = -\frac{q_1}{2}$? Distance $d = 1.4\,cm$, $\theta_1 = 43°$, and $\theta_2 = 60°$.

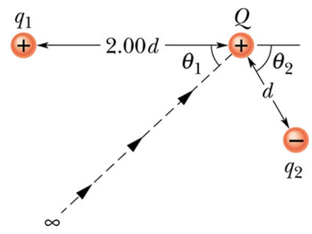

Figure 10: Three charged particles.

Recall that the potential is related to the potential energy in the same way that the electric field is related to the electric force. That is, we simply need to multiply the potential by the charge to get the potential energy. If we choose infinity as our reference point for the potential ($V_\infty = 0$), then the potential of each of the point charges is:

$$V = k\frac{q}{r}$$

Furthermore, since the electric force is a conservative force, the change in energy from infinity to the final location does not depend on the path taken. Since we choose $V_\infty = 0$, $U_\infty = q\,V_\infty = 0$ also. Hence, the final potential energy of the point charge gives the work done to get it there. The potential of of charges q_1 and q_2 at the location of Q is:

$$V_{fin} = k\left[\frac{q_1}{2d} + \frac{q_2}{d}\right] = 0$$

The final potential energy of charge Q is therefore:

$$U_{fin} = Q\,V_{fin} = 0$$

The total work done in moving the charge in from infinity to its current position is therefore 0.

Problem 24.66

Two isolated, concentric, conducting spherical shells have radii $R_1 = 0.5\,m$ and $R_2 = 1\,m$, uniform charges $q_1 = 2\,\mu C$ and $q_2 = 1\,\mu C$, and negligible thicknesses. What is the magnitude of the electric field E at radial distance a) $r = 4\,m$, b) $r = .7\,m$, and c)$r = .2\,m$? With $V = 0$ at infinity, what is V at d) $r = 4\,m$, e) $r = 1\,m$, f) $r = .7\,m$, g) $r = .5\,m$, h) $r = .2\,m$, and i) $r = 0$? j) Sketch $E(r)$ and $V(r)$.

We begin as usual when calculating electric fields with Gauss's law. By making the usual symmetry arguments, we can break open the integral and turn it into an algebraic statement.

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \quad\Rightarrow\quad E = \frac{Q_{enc}}{\epsilon_0 A} \quad\Rightarrow\quad E = \frac{Q_{enc}}{4\pi\epsilon_0 r^2}$$

where r is the radius of the appropriately chosen Gaussian sphere. Since the conducting shells are infinitely thin, the only important quantity in determining the charge enclosed by the surface is whether the radius in question is outside of or inside of each of the shells. Hence, the answers to the first three parts are: a) $1.69\times 10^3\,\frac{V}{m}$ b) $3.67\times 10^4\,\frac{V}{m}$, and c) 0.

There is not Gauss's law for the electric potential, so we will have to calculate it directly from the electric fields derived above. Recall that the potential is given in terms of the electric field (with $V = 0$ at infinity) by:

$$V = -\int_i^f \vec{E}\cdot d\vec{s}$$

Since the electric field points radially in all three regions, we will choose to come in along a radial path. Hence, this integral simplifies to:

$$V = -\int_\infty^r E(r')\,dr'$$

We must be careful here because our limits of integration will change as r changes.

d)

$$V = -\int_{\infty}^{r} \frac{q_1 + q_2}{4\pi\epsilon_0 r'^2} dr' = \frac{q_1 + q_2}{4\pi\epsilon_0\, r} = 6.73 \times 10^3\, V$$

e)

$$V = -\int_{\infty}^{R_2} \frac{q_1 + q_2}{4\pi\epsilon_0 r'^2} dr' = \frac{q_1 + q_2}{4\pi\epsilon_0\, R_2} = 2.74 \times 10^4\, V$$

f)

$$\begin{aligned} V &= -\int_{\infty}^{R_2} \frac{q_1 + q_2}{4\pi\epsilon_0 r'^2} dr' - \int_{R_2}^{r} \frac{q_1}{4\pi\epsilon_0\, r'} dr' \\ &= \frac{q_1 + q_2}{4\pi\epsilon_0\, R_2} + \frac{q_1}{4\pi\epsilon_0 r} - \frac{q_1}{4\pi\epsilon_0 R_2} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r} + \frac{q_2}{R_2}\right] = 3.47 \times 10^4\, V \end{aligned}$$

The integration works out very similarly for the other two parts giving:

g)

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r} + \frac{q_2}{R_2}\right] = 4.50 \times 10^4\, V$$

h)

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{R_1} + \frac{q_2}{R_2}\right] = 4.50 \times 10^4\, V$$

i)

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{R_1} + \frac{q_2}{R_2}\right] = 4.50 \times 10^4\, V$$

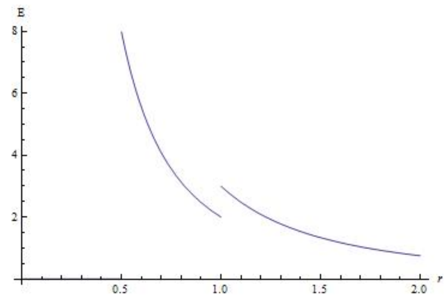

Figure 11: The Electric Field as a Function of r (The Vertical Axes is Arbitrarily Scaled).

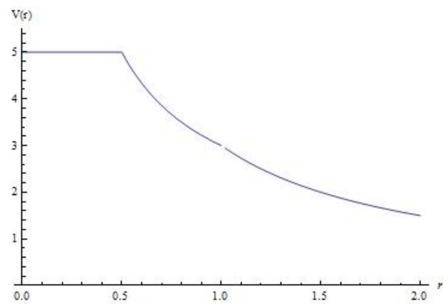

Figure 12: The Potential as a Function of r (The Vertical Axes is Arbitrarily Scaled).

6 Chapter 25: Capacitance

This chapter and the next one are basically a build up to chapter 27 where we will talk about the basic physics of electronics/circuits. The two build up chapters will focus on the individual components of circuits, such as capacitors and resistors, and chapter 27 will allow us put everything together and analyze how circuits work as a whole.

6.1 Capacitance

Conductors are the main constituent of electronic circuits and so we will seek to find new ways of talking about them in this chapter. When we put two conductors of equal and opposite charge near each other, we know from the last two chapter that there will be an electric field and hence a potential difference between them. It turns out that the relationship between the amount of (equal and opposite) charge placed on the two conductors is linear, by which we mean that the amount of charge is related to the potential by a constant. This constant of proportionality is what we call capacitance.

$$q = C\,V \tag{17}$$

In this equation q is the magnitude of the charge, so one conductor will have charge $+q$ and the other will have charge $-q$. It is important to note that capacitance C is a purely geometrical quantity which is entirely determined by the shape and size of the two conductors. Hence, when we derive relations for various geometries (such as the parallel-plate capacitor) we will not have to worry about re-deriving the capacitance that geometry ever again.

In order to charge a capacitor, we generally hook up opposite ends of a battery to each conductor via a circuit. The battery then removes electrons from the positive plate and places them on the negative plate until it has built up enough of a charge difference that the potential difference between the plates is equal to the potential difference of the battery. As we will see later on in this chapter, it takes time for the battery to build up this much charge and hence, charging (and discharging) a capacitor is a time dependent process. People who design circuits can exploit this time dependence to give their circuits a sense of time.

6.2 Calculating Capacitance

The general strategy for calculating capacitance goes like this:

1. Assume that there is a q on the capacitor plates. Note that this charge is hypothetical and will end up canceling out later in the calculation so it does not need to have a particular value.
2. Calculate the electric field $\vec{E}$ between the two capacitor plates in terms of this charge. This step generally involves using Gauss's law.
3. Knowing $\vec{E}$, calculate the potential difference between the two plates.
4. Calculate C by taking the ratio of the charge to the potential (equation 17). The charge should drop out altogether in this step.

Lets practice this technique with a few examples.

6.2.1 Parallel-Plate Capacitor

We will assume for simplicity that the plates of our capacitor are large enough or close enough together that we can neglect any fringe effects and treat the electric field in between the plates as being uniform. If we draw our little Gaussian pillbox so that it only encloses the charge on the positive plate of the capacitor, then we can use Gauss's law to calculate the field. The symmetry arguments in this case will be: 1) Since the field is uniform and the electric field inside of the

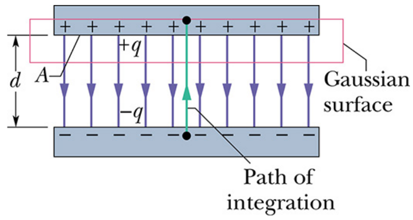

Figure 13: A Parallel-Plate Capacitor

conductor is zero, the only flux will be through the part of our Gaussian surface in between the two plates, and 2) Since the field is uniform, $\oint \vec{E}\cdot d\vec{A} = E\oint d\vec{A}$. Hence,

$$\oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0} \quad \Rightarrow \quad E = \frac{q}{\epsilon_0\, A}$$

where A is the area of one of the plates. Now that we know the field between the plates, we can calculate the potential between them.

$$V = \int_0^d \vec{E}\cdot d\vec{s} = \int_0^d \frac{q}{\epsilon_0\, A} ds = \frac{q\, d}{\epsilon_0\, A}$$

Now that we know the potential difference between the two plates, calculating the capacitance is simple.

$$C = \frac{q}{V} = \frac{q}{\frac{q\, d}{\epsilon_0\, A}} = \frac{\epsilon_0\, A}{d}$$

Notice that, as stated before, the capacitance is simply a geometrical quantity. In the case of a parallel-plat capacitor, it is proportional to the area of each plate and inversely proportional to the distance between the two plates.

6.2.2 Cylindrical Capacitor

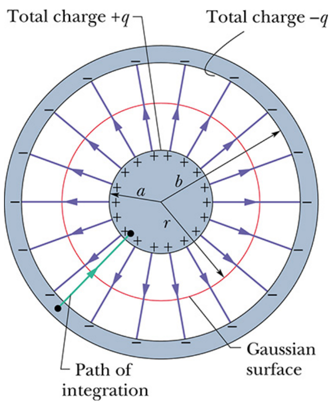

Figure 14: A Cylindrical Capacitor

We begin by assuming that there is a hypothetical charge q placed on the inner and outer conductor. Because of symmetry, the charge on the inner conductor will distribute itself uniformly on the outside edge of the cylinder. If we draw our Gaussian surface in the usual way and make the usual symmetry arguments, then the electric field between the plates is

$$\vec{E} = \frac{q}{\epsilon_0\, 2\pi r L}\hat{r}$$

Hence, the potential difference between the plates is

$$V = \int_a^b \frac{q}{\epsilon_0\, 2\pi r L} dr = \frac{q}{\epsilon_0\, 2\pi L} \ln\left(\frac{b}{a}\right)$$

The capacitance of the cylindrical capacitor of length L is therefore

$$C = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

Note that the most commonly encountered cylindrical capacitor is a shielded cable (like coaxial cable used for television transmissions). In these cases, it is more useful to talk about the capacitance per unit length so that one can measure the length of the cable to find the total capacitance. The capacitance per unit length is easily seen to be:

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

One more point of interest is that again, as before, the capacitance is strictly a geometrical quantity. Aside from constants, it depends solely on the length of the conductor and the inner and outer radii.

6.3 Capacitors in Parallel and in Series

As we begin to work with electrical circuits, it will be nice to have some simplification rules for circuits with many components. Here we will develop some rules for working with circuits with many capacitors. We first need to define some terminology. For simplicity, we will consider two capacitors C_1 and C_2, but the ideas will extend naturally to larger numbers of capacitors. The two capacitors are said to be in *parallel* if the positive plates of C_1 and C_2 are both wired to the positive side of the battery, and the negative plates of both are wired to the negative side of the battery (see figure 15). The two capacitors are said to be in *series* if they are wired such that the positive plate of C_1 is connected to the positive side of the battery, the negative plate of C_2 is connected to the negative side of the battery, and the other two plates are wired directly together (see figure 16).

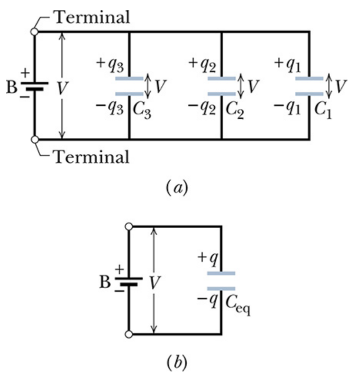

Figure 15: Three Capacitors in Parallel with their Simplified Equivalent Capacitor

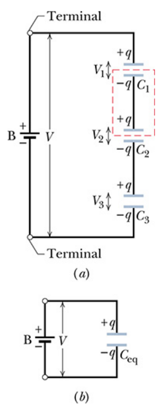

Figure 16: Three Capacitors in Series with their Simplified Equivalent Capacitor

Now lets look more closely at capacitors in parallel. Since all of the capacitors have one side hooked up to the positive terminal of the battery and one side hooked up to the negative side of the battery, *the potential difference across each capacitor must be the same.* Hence, if we consider the charge on each of the three capacitors in figure 15

$$q_1 = C_1 V \quad q_2 = C_2 V \quad q_3 = C_3 V$$

Since the total charge q on all three capacitors is given by $q = q_1 + q_2 + q_3$ we can write

$$q = C_1 V + C_2 V + C_3 V = (C_1 + C_2 + C_3)V \quad \Rightarrow \quad C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3$$

We can see how this would extend naturally to many capacitors in parallel. Hence, the general rule for n capacitors in parallel is that we can treat them as an equivalent single capacitor of capacitance

$$C_{eq} = \sum_{i=1}^{n} C_i \tag{18}$$

What about capacitors in series? When the three capacitors are wired in series, they have equal charge on all three capacitors. Also, the sum of the potential differences across all three capacitors must equal the potential difference of the battery. We can see why the capacitors must have equal charge by considering the circuit before and after attaching the battery. Before the battery is attached, the wires attaching each capacitor is electrically neutral (i.e. no charge built up on either plate). When we hook up the battery, charge will be drawn to one of the capacitor plates which will leave the other capacitor plate with a deficit of charge equal and opposite to that on the first capacitor. We can now figure out the equivalent capacitance of the system by writing down the potential across all three capacitors individually.

$$\begin{aligned}
V &= V_1 + V_2 + V_3 \\
&\frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} \\
&= q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) \\
\Rightarrow \quad C_{eq} &= \frac{q}{V} \\
&= \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} \\
\Rightarrow \quad \frac{1}{C_{eq}} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}
\end{aligned}$$

Again, we can see how this naturally extends to n capacitors.

$$\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i} \tag{19}$$

6.4 Energy Stored in an Electric Field

When we build up an electric field between the plates of a capacitor, we do work (or rather the battery does work). As we know, doing work on a system means that we are transferring energy to the system, so the work done to build up an electric field must store energy in the field. To calculate the work that we do when establishing a field between the capacitor plates we will appeal to the relationship between potential and work. Recall from last chapter that work is given by $\Delta W = q\Delta V$, a differential bit of work is therefore given by

$$dW = V dq'$$

where we have placed a prime on the charge dq' because we will soon integrate over it. We can use this to calculate the total work done when bringing a charge q to a capacitor

$$W = \int V\, dq' = \frac{1}{C}\int_{)}^{q} q'\, dq' = \frac{q^2}{2C}$$

Hence, the potential energy stored in the electric field between the plates must be

$$U = \frac{q^2}{2C} = \tfrac{1}{2}CV^2 \tag{20}$$

6.4.1 Energy Density

During the semester, we have encountered many different densities: mass density, volume charge density, and surface charge density to name a few. We will also end up encountering many more because it is always nice to talk about a quantity which extends over a finite region of space as a quantity which is defined at each point in space. Since the electric field between capacitor plates is extended over a finite region of space, it is natural to think about the energy stored in it in the same way. We will use a lowercase u to denote energy density. For a parallel plate capacitor

$$u = \frac{U}{Ad} = \frac{CV^2}{2Ad} = \tfrac{1}{2}\epsilon_0 \left(\frac{V}{d}\right)^2 = \boxed{\tfrac{1}{2}\epsilon_0 E^2} \tag{21}$$

Although we have derived this formula for a parallel plate capacitor, it is actually always valid. Hence, anytime we establish an electric field we can think of it as storing energy and that energy is given by equation 21

6.5 Dielectric Materials

Up until now in our discussion of electrostatics we have implicitly considered everything to be happening in a vacuum (or in air which is very close to a vacuum). Since we live in a world full of different materials, we need to understand what happens when we have electric fields inside of these materials.

Before we consider the specific mathematical changes that we must make to our formulas, lets consider what happens when we place an electric field in a material. Materials are made up of atoms and molecules, which in general have dipole moments. When there is no electric field present these dipole moments point in random directions and the net effective field is zero. As we saw back in chapter 23 a dipole in an electric field experiences a torque (equation 9). We can view this alignment of dipole moments as a net shift in the charge distribution of the material, with some excess positive charge moving towards the negative side of the electric field and some excess negative charge moving towards the positive side of the electric field. Dielectrics therefore have the effect of weakening the internal field.

The beauty of dielectrics is that *for weak fields* all we need to do is rewrite all of our equations by replacing ϵ_0 with $\kappa\epsilon_0$. The electric field of a point charge and Gauss's law therefore become

$$\vec{E} = \frac{1}{4\pi\kappa\epsilon_0}\frac{q}{r^2}\hat{r} \quad \& \quad \kappa\epsilon_0 \oint \vec{E}\cdot d\vec{A} = q_{enc}$$

6.6 Problems

Problem 25.13

In figure 17, a potential difference of $V = 100\,V$ is applied across a capacitor arrangement with capacitances $C_1 = 10\,\mu F$, $C_2 = 5\,\mu F$, and $C_3 = 4\,\mu F$. If capacitor 3 undergoes electrical breakdown so that it becomes equivalent to conducting wire, what is the increase in (a) the charge on capacitor 1 and (b) the potential difference across capacitor 1?

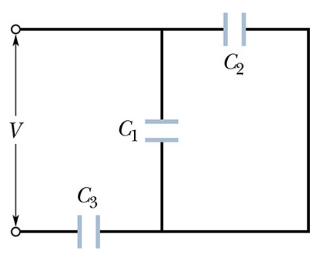

Figure 17: A Capacitor Setup with an Established Potential Difference

The best way to begin these problems is generally to consider the entire circuit as one equivalent capacitor at first. Lets find this equivalent capacitance. C_1 and C_2 are in parallel so they can be considered to be an equivalent capacitor of $C_{eq1} = C_1 + C_2$. Now that we have turned these into an equivalent system, we can consider this equivalent capacitor to be in series with C_3. The equivalent capacitance of the entire circuit is therefore

$$\frac{1}{C_{eq2}} = \frac{1}{C_3} + \frac{1}{C_{eq1}} \quad \Rightarrow \quad C_{eq2} = \frac{C_3(C_1 + C_2)}{C_1 + C_2 + C_3}$$

Our goal now is to find the initial charge and potential on C_1, but it will be easier to take it is steps. We will thus begin by finding the charge and potential across the equvalent capacitor C_{eq1}. The total charge across the entire circuit is

$$q_{tot} = C_{eq2} V$$

Since C_3 and C_{eq1} are in series, they both must have equal charge across them so $q_{eq1} = q_{tot} = C_{eq2}V$. We can use this fact to find the potential across C_{eq1} via

$$V_{eq1} = \frac{q_{tot}}{C_{eq1}} = \frac{C_{eq2}V}{C_{eq1}} = \frac{C_3 V}{C_1 + C_2 + C_3}$$

We can now step down one more level to the original circuit. Since C_1 and C_2 are in parallel the potential across each must be the same, so we have found $V_1 = V_{eq1}$. This enables us to easily find the charge on C_1.

$$q_1 = C_1 V_1 = \frac{C_1 C_3 V}{C_1 + C_2 + C_3}$$

Our ultimate goal is to find the difference in q_1 and V_1 after C_3 breaks down. We therefore need to reanalyze the circuit treating C_3 as if it were not there. We begin as with the last case by treating the entire circuit as one capacitor of capacitance C_{eq1}. The potential difference and charge across this capacitor are

$$V_{eq1} = V \quad \& \quad q_{eq1} = C_{eq1} V = (C_1 + C_2) V$$

We now step down one level and treat C_1 and C_2 as separate entities. Since they are in parallel, they have the same potential difference across them, but the charge will be different.

$$q_1 = C_1 V$$

Hence, the difference in the amount of charge on C_1 is

$$\Delta q = C_1 V - \frac{C_1 C_3 V}{C_1 + C_2 + C_3} = \left(\frac{C_1(C_1 + C_2)}{C_1 + C_2 + C_3} \right) V = 789\,\mu C$$

and the change in potential is

$$\Delta V = V - \frac{C_3 V}{C_1 + C_2 + C_3} = \left(\frac{C_1 + C_2}{C_1 + C_2 + C_3} \right) V = 79.9\,V$$

Problem 25.24

Figure 18 shows a variable "air gap" capacitor for manual tuning. Alternate plates are connected together; one group of plates is fixed in position, and the other group is capable of rotation. Consider a capacitor of $n = 8$ plates of alternating polarity each plate having an area $A = 1.25\,cm^2$ and separated from adjacent plates by distance $d = 3.4\,mm$. What is the maximum capacitance of the device?

Since one group of plates is connected to the positive terminal of the battery and the other group of plates is connected to the negative terminal, in order to have a capacitor at all we must have maximum overlap of the plates. In this case, the entire circuit consists of $8 - 1 = 7$ identical

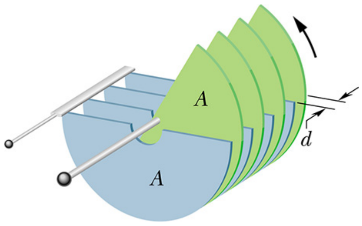

Figure 18: A Variable Capacitor

parallel-plate capacitors connected in parallel. The capacitance of one of the identical capacitors is given by

$$C = \frac{\epsilon_0 A}{d}$$

The capacitance of the entire arrangement is hence

$$C_{tot} = 7\frac{\epsilon_0 A}{d} = 2.28\,pF$$

Problem 25.48

Figure 19 shows a parallel-plate capacitor with plate area $A = 5.56\,cm^2$ and separation distance $d = 5.56\,mm$. The left half of the gap is filled with material of dielectric constant $\kappa_1 = 7.0$; the right half is filled with material of dielectric constant $\kappa_2 = 12.0$. What is the capacitance?

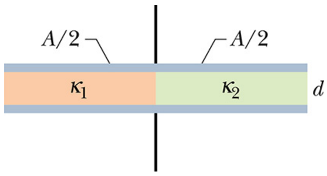

Figure 19: A Capacitor with Two Different Dielectric Slabs

Since the plates of a parallel plate capacitor are conductors and conductors are always equipotentials, the potential on each side of the capacitor must be the same. The charge therefore will not be the same on both sides unless $\kappa_1 = \kappa_2$. Since the potential is the same on both sides, we can treat this single capacitor as being made up of two capacitors in parallel. This is the reverse of taking two parallel capacitors and treating them as one capacitor. Since they are in parallel, the capacitance will be

$$C = C_1 + C_2 = \frac{\kappa_1 \epsilon_0 \frac{A}{2}}{d} + \frac{\kappa_2 \epsilon_0 \frac{A}{2}}{d} = (\kappa_1 + \kappa_2)\frac{\epsilon_0 A}{2d} = 8.41\,pF$$

Problem 25.62

In figure 20, the battery potential difference V is $10\,V$ and each of the seven capacitors has capacitance $10\,\mu F$. What is the charge on (a) capacitor 1 and (b) capacitor 2?

Capacitor 1 is very straightforward. It has one plate wired directly to the positive side of the battery and the other wired directly to the negative side of the battery. The potential difference across it must therefore be V. Hence,

$$q_1 = C_1 V = \boxed{1 \times 10^4\,C}$$

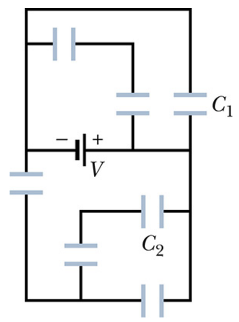

Figure 20: An Arrangement of Capacitors

Capacitor 2 is not as easy however. We will find the charge on capacitor 2 in the usual manner, by working the circuit down to its simplest form and then working back out. Before we start simplifying, we will note that the upper portion of the circuit plays no role in the bottom portion and we will therefore ignore it all together. The idea behind simplifying a circuit is to look for pairs of capacitors that are wired either in series or in parallel. The first pair that we will simplify is C_2 and the other capacitor on the same wire as C_2. Since they are in parallel,

$$K_1 = \frac{1}{\frac{1}{C} + \frac{1}{C}} = \frac{C}{2}$$

where we are using K to denote an imaginary equivalent capacitor.

We can now treat this equivalent capacitor as being in parallel with the one which is nearest to it.

$$K_2 = \frac{C}{2} + C = \frac{3C}{2}$$

And finally, this equivalent capacitor is in series with the fourth capacitor in the lower portion of the circuit.

$$K_3 = \frac{1}{\frac{1}{C} + \frac{2}{3C}} = \frac{3C}{5}$$

We are now ready to step our way back out to the answer. The charge on K_3 is

$$q = K_3 V = \frac{3C}{5} V$$

Stepping back one more level, K_2 and the other capacitor are in series and must therefore have the same charge. Hence, $q = \frac{3C}{5} V$. We can use this information to calculate the voltage across K_2.

$$V = \frac{q}{K_2} = \frac{3C}{5} \frac{2}{3C} V = \frac{2}{5} V$$

Stepping back another level, K_1 and C are now in parallel and must therefore have the same voltage difference. Using this to find the charge gives.

$$q = K_1 V = \frac{C}{2} \frac{2}{5} V = \frac{CV}{5}$$

Finally, since K_1 is made up of two capacitors in series (one of which is C_2), the charge must be that same on both. The charge on C_2 is therefore

$$q_2 = \frac{CV}{5} = \boxed{2 \times 10^{-5}\, C}$$

Problem 25.64

The capacitances of the four capacitors shown in figure 21 are given in terms of a certain quantity C. (a) If $C = 50\,\mu F$, what is the equivalent capacitance between points A and B? (b) Repeat for point A and D.

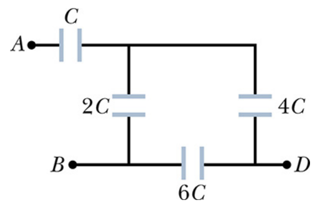

Figure 21: An Arrangement of Capacitors

Part a

We begin as usual by finding a pair of capacitors between A and B which are strictly in series or in parallel. Since the $6C$ and the $4C$ capacitor are in series, we will start with them. Their equivalent capacitance is

$$K_1 = \frac{1}{\frac{1}{6C} + \frac{1}{4C}} = \frac{12C}{5}$$

This equivalent capacitor is now in parallel with the $2C$ capacitor. Therefore,

$$K_2 = 2C + \frac{12C}{5} = \frac{22C}{5}$$

Finally, this equivalent capacitor is in series with the C capacitor.

$$K_{AB} = \frac{1}{\frac{1}{C} + \frac{5}{22C}} = \frac{22C}{27} = \boxed{46.7\,\mu F}$$

Part b

From A to D we proceed in a similar manner. The $2C$ and $6C$ capacitors are in series.

$$K_1 = \frac{1}{\frac{1}{2C} + \frac{1}{6C}} = \frac{3C}{2}$$

This equivalent capacitor is in parallel with the $4C$ capacitor.

$$K_2 = 4C + \frac{3C}{2} = \frac{11C}{2}$$

Finally, this equivalent capacitor is in series with the C capacitor.

$$K_{AD} = \frac{1}{\frac{1}{C} + \frac{2}{11C}} = \frac{11C}{13} = \boxed{42.3\,\mu F}$$

7 Chapter 26: Current and Resistance

In the preceding five chapters we have spent a lot of time learning the principles of electrostatics. The subject is called electrostatics because it is the physics of stationary charges. In this and subsequent chapters we will analyze some of what happens when charges are moving (particularly in large quantities). This subject is generally called magnetostatics, a term which will become more clear in the future.

7.1 Electric Current

The central idea of moving charges is called current. Current is defined as

$$i = \frac{dq}{dt} \tag{22}$$

Just to be clear, lets discuss what this means. Current is the *net*flow of charges through a given area per unit time. In a metal, electrons are whizzing around at extremely high speeds in random directions, so if we look at a cross-section of area through the metal billions of electrons will flow through in a given amount of time. There will be just as many electrons passing through in one direction as the other though and hence, the current in the metal will be zero. If we hook the metal up to the battery, then the potential difference in the metal will cause the electrons to travel more in one direction than the other and hence will give a non-zero current.

The direction of current is given by the direction that positive charges would move, even though we now know that electrons are the charge carriers in metals. This is due to an arbitrary choice made by Ben Franklin and has now become a matter of convention.

The way we have defined current in equation 22 is only useful when we are talking about very thin wires as in circuits. For thicker wires, the current flow may not be uniform throughout the cross-section of the wire. We therefore introduce the idea of current density, or current per unit cross-sectional area. The current is given from the current density by

$$i = \int \vec{J} \cdot d\vec{A}$$

7.2 Resistance and Resistivity

If we apply the same potential difference across a piece of metal and a piece of rubber, the amount of current that flows is clearly very different. We define the resistance of a piece of material as the ratio of this potential to the amount of current that flows through.

$$R = \frac{V}{i} \tag{23}$$

this is commonly known as Ohm's law. Ohms law states that the current is linearly proportional to the applied potential. This is not true for all materials and fails spectacularly in the case of semiconductors (the basis of modern electronics). It is very useful however for discussing most simple circuits.

The *resistance* of a piece of material is dependent not only upon the type of material, but also upon the geometrical configuration of the piece of material. As an example, consider a piece of metal which we have hooked up in two different ways. If we place our wires which have the potential across them very close together on the surface of the metal, then the resistance will be very low. If we instead place our wires very far apart on the piece of metal, the resistance will be much larger. It is therefore useful to define a size/shape independent parameter which we will call the *resistivity* and give the symbol ρ.

$$\rho = \frac{E}{J}$$

We can write this equation in vector form as

$$\vec{E} = \rho \vec{J} \tag{24}$$

Another material parameter which often comes up in application it the conductivity of a material. It is simply the inverse of the resistivity.

$$\sigma = \frac{1}{\rho}$$

For a uniform material with a uniform cross sectional current density, we can easy calculate the resistance from the resistivity. Note that for a uniform cross sectional current density, it is necessary that the potential difference be applied uniformly across the material. When this is the case, the electric field is given simply by $E = \frac{V}{L}$ where L is the length of a material and the current density is given by $J = \frac{i}{A}$ where A is the cross sectional area. Using these relations, we can see that the resistance of a uniform piece of material of length L and cross sectional area A is

$$R = \rho \frac{L}{A}$$

7.3 Power in Electric Circuits

From our everyday experiences, we should expect that powering electric circuits uses up some energy. Up until now in our discussion so circuits, we have not discussed this energy loss. We are now ready to do so because the energy lost in electrical circuits is mostly lost to the variance resistances in the circuit. The rate of electrical energy transfer in a circuit is given by

$$P = iV$$

Notice that for circuits which obey Ohm's law we can rearrange this expression to get two different forms.

$$P = \frac{V^2}{R} \quad \& \quad P = i^2 R$$

7.4 Problems

Problem 26.24

Figure 22 gives the electric potential $V(x)$ along a copper wire carrying uniform current, From a point of higher potential $V_s = 12\,\mu V$ at $x = 0$ to a point of zero potential at $x_s = 3\,m$. The wire has a radius of $2\,mm$. What is the current in the wire?

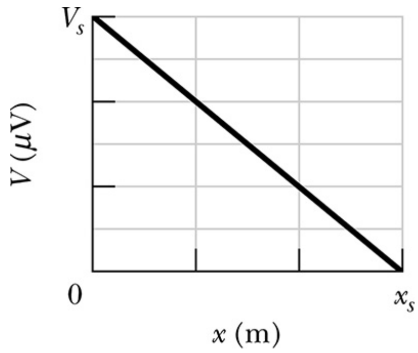

Figure 22: Plot for Problem 26.24

The problem tells us that the material is copper. From the table in the book we can see that the resistivity of copper is

$$\rho = 1.69 \times 10^{-8}\,\Omega \cdot m$$

Since we also know the length (from the plot) and the cross sectional area of the wire we can calculate the resistance.

$$R = \rho \frac{L}{A} = \rho \frac{x_s}{\pi r^2}$$

Hence, from Ohm's law the current in the wire is

$$i = \frac{V}{R} = \frac{V_s}{\rho \frac{x_s}{\pi r^2}} = \frac{V_s\,\pi r^2}{\rho\, x_s} = 3.0 \times 10^{-3}\,A$$

8 Chapter 27: Circuits

The last two chapters have, in essence, been a build up to this chapter. We have spent these last two chapters learning about three key concepts: capacitors, resistors, and energy in electric circuits. In this chapter we will bring these three ideas together and use them to talk about electric circuits as a whole. We will begin to see why electric circuits are useful and why they have become the cornerstone of the modern world.

8.1 Work, Energy, and Emf

An 'emf device' is a word from antiquity which we still use, but it means in essence, a device which produces a potential difference. Emf (given the symbol $\mathcal{E}$) stands for *electromotive force* which is a misnomer because as we will see shortly, emf is not a force but rather a potential difference. Emf is defined through the work which it does on charges

$$\mathcal{E} = \frac{dW}{dq} \tag{25}$$

When talking about emf devices, we make a distinction between an ideal and a real emf device. An **ideal emf device** lacks any internal resistance. Hence, if we were to hook up an ideal emf device to a circuit without any resistance it would drive current through the wire infinitely fast. A **real emf device** has some internal resistance and hence, when hooked up to a circuit without any resistance will have a limit to how much current it can drive. Note that all emf devices are real devices. We can (and will) treat a real emf device as an ideal one hooked up in series with a resistor whose value is set by the internal resistance of the real emf device.

8.2 Analyzing Simple Circuits

One of the best ways to relate circuits to our everyday experiences is to imagine traveling around the circuit. As we travel around the circuit, we will pass different objects like resistors, junctions, and capacitors (and later inductors) which will change the values of the current and potential in different ways. All we must do is keep a tally of how these properties change as we traverse the circuit and we will be able to learn a lot about electrical circuits. There are two analogies which will take us a long way in thinking about electrical circuits.

- Potential (Voltage) is like the height of a mountain.
- Current is like a flowing liquid through a pipe whose radius does not change.

Potential is like the height of a mountain for a couple of reasons. The first reason is that any one point in the circuit can only have one value for its potential similar to the way that a mountain only has one height at each point. The second one, which is a more useful part of the analogy, is that if we walk around a mountain and return to the same point, then that point must have the same height no matter which path we took. Potential is the same way. If we walk around the circuit starting at the battery (or anywhere else) and return to that point later on, then that point must have the same potential. To state it more formerly: *The algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.*

Current acts like a fluid for a couple of reasons. First of all, if we have a circuit which consists of only one loop with no branches, then the current must be the same at all points in the circuit. Similarly, a fluid flowing through a pipe which does not split off into separate sections must be flowing at the same rate at all points in the pipe. If the fluid were not flowing at the same rate at all points then some of the fluid would be building up somewhere even though there is nowhere for it to go. Also, if there are many pipes (or wires) which meet at a junction, then the rate of fluid (current) flowing into and out of that junction from all of the pipe must sum to zero. Otherwise, the fluid (or current) would be building up in the junction even though there is nowhere for it to go. This is sometimes know at Kirchhoff's junction rule: *The sum of the currents entering any junction must be equal to the sum of the currents leaving that junction.*

8.3 Resistors in Series and in Parallel

Figure 23 (a) shows three resistors in series. As we did with capacitors, we want to find a way to simplify this circuit down to a circuit with one resistor in it. To do so we will use two of the rules which we observed above. 1. If we traverse the entire circuit the sum of all of the potential changes must be zero. 2. For a single loop circuit, the current must be the same at all points. If we start at point b and walk around the circuit, we first pass the battery at which we gain V in potential.

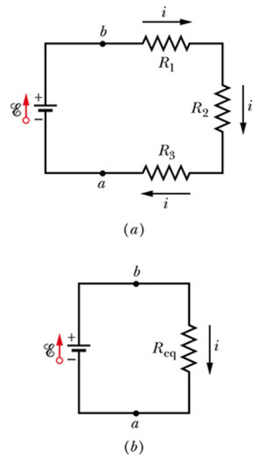

Figure 23: Three Resistors in Series with Their Simplified Equivalent Resistor

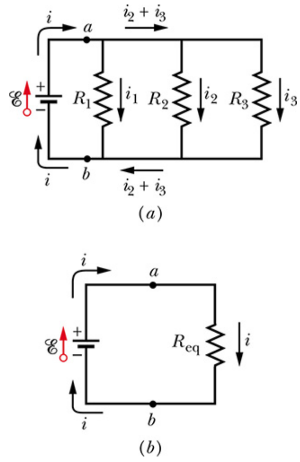

Figure 24: Three Resistors in Parallel with Their Simplified Equivalent Resistor

Next, we pass through the three resistors at each of which we lose iR_i in potential. After that, we find ourselves back at point b and therefore

$$V - iR_1 - iR_2 - iR_3 = 0 \quad \Rightarrow \quad V = iR_1 + iR_2 + iR_3 \quad \Rightarrow \quad V = i\left(R_1 + R_2 + R_3\right)$$

By comparing the last form of the expression with Ohm's law, we can see what the resistance of our simplified resistor is. For n resistors in series, the equivalent simplified resistor is

$$R_{eq} = \sum_{j=1}^{n} R_j \tag{26}$$

We now want to do the same thing, except for multiple resistors in parallel (see figure 24). In this case, we will use the fact that since opposite sides of each resistor are wired to opposite sides of the battery, the potential difference across each resistor must be the same. We will also use the fact noticed above that the sum of the currents through each resistor must be equal to the current flowing through the rest of the circuit (the battery in this case).

$$i = i_1 + i_2 + i_3 = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

Or, rearranging

$$V = i\left(\frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}\right)$$

Hence, we see that (similar to capacitors in series) n resistors in parallel can be reduced to an equivalent resistor of resistance

$$\frac{1}{R_{eq}} = \sum_{j=1}^{n} \frac{1}{R_j} \tag{27}$$

8.4 RC Circuits

Up until now we have dealt with circuits which involve either capacitors of resistors but none which involve both at the same time. Lets consider this possibility now. The two situations which we will consider are: 1. The capacitor begins charged and is then allowed to discharge through a resistor. 2. The capacitor begins uncharged and is then charged by a battery through a resistor. Figure 25 illustrates how we could see both of these situations in one circuit. When the switch is in position a then the capacitor is charging through the resistor. When the switch is in position b the capacitor is discharging through the resistor.

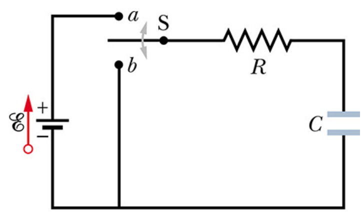

Figure 25: A circuit for charging and discharging a capacitor through a resistor.

8.4.1 Charging a Capacitor

If we set the switch to position a in figure 25 and begin walking around the circuit, then the potential differences of the battery, resistor, and capacitor must sum to zero.

$$V - iR + \frac{q}{C} = 0$$

Everything in this equation is a constant except for the current and the charge. Recall that the current is related to the charge however via $i = \frac{dq}{dt}$. Plugging this in gives

$$R\frac{dq}{dt} + \frac{q}{C} = V$$

This is a inhomogeneous, linear differential equation whose solution (though not difficult) is beyond the scope of this course. Suffice it to say that the solution to such an equation is:

$$q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right) \quad \Rightarrow \quad i = \frac{dq}{dt} = \left(\frac{V}{R}\right)e^{-\frac{t}{RC}}$$

Notice that this is a time dependent process. This is the first way we have seen in which a circuit can have a sense of time. This situation pops up often enough that we assign the thing on the bottom of the fraction in the exponential its own name. The *RC time constant* is

$$\tau = RC$$

8.4.2 Discharging a Capacitor

If we move the switch in figure 25 to position b and repeat our analysis, we find

$$R\frac{dq}{dt} + \frac{q}{C} = 0$$

The solution to this homogeneous linear differential equation is

$$q = q_0 e^{-\frac{t}{RC}} \quad \Rightarrow \quad i = -\left(\frac{q_0}{RC}\right)e^{-\frac{t}{RC}}$$

So, we see that discharging a capacitor also has an exponential time dependence with the same characteristic time constant.

8.5 Problems

Problem 27.33

In figure 26 the ideal batteries have emfs $\mathcal{E}_1 = 5\,V$ and $\mathcal{E}_2 = 12\,V$, the resistances are each $2\,\Omega$, and the potential is defined to be zero at the grounded point of the circuit. What are potentials (a) V_1 and (b) V_2 at the indicated points?

The first step in this problem is to simplify the two resistors in the upper right corner of the circuit. Since they are in parallel their equivalent resistance is

$$P_1 = \frac{1}{\frac{1}{R} + \frac{1}{R}} = \frac{R}{2}$$

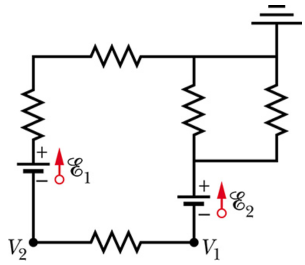

Figure 26: The circuit for Problem 27.33

(Note that I will generally use P to denote a simplified resistor instead of R_{eq}.) By simplifying these resistors, we now have a single loop circuit with 4 resistors in series and two batteries. The next step is to figure out what the current is in this single loop circuit.

Since the rest of the resistors are in series, we can simplify them to a single resistor of resistance $P_2 = 3R + \frac{R}{2} = \frac{7R}{2}$ with two voltage sources. Hence, the current flowing through the circuit is

$$i = \frac{V}{R} = \frac{\mathcal{E}_2 - \mathcal{E}_1}{P_2}$$

Now that we know the current flowing through the circuit, we can take one step back to our first simplified circuit and start traversing the circuit to find the voltages V_2 and V_2. Since the top right corner of the circuit is grounded (set at 0 potential) we will start there and work our way counter-clockwise around the circuit. We must be careful here because the current will be flowing with the direction which we are going and hence, we will lose potential as we go through each resistor. On the way to V_2 we pass through two resistors and $\mathcal{E}_1$.

$$V_2 = 0 - iR - iR - \mathcal{E}_1 = -\mathcal{E}_1 - 2\frac{\mathcal{E}_2 - \mathcal{E}_1}{P_2}R = -\mathcal{E}_1 - \frac{4}{7}(\mathcal{E}_2 - \mathcal{E}_1) = -\frac{4}{7}\mathcal{E}_2 - \frac{3}{7}\mathcal{E}_1 = \boxed{-9\,V}$$

Finding V_1 is now a simple matter of going one resistor further.

$$V_1 = V_2 - iR = -\frac{4}{7}\mathcal{E}_2 - \frac{3}{7}\mathcal{E}_1 - \frac{2}{7}(\mathcal{E}_2 - \mathcal{E}_1) = -\frac{6}{7}\mathcal{E}_2 - \frac{1}{7}\mathcal{E}_1 = \boxed{-11\,V}$$

Problem 27.35

In figure 27, $R_1 = 2\,\Omega$, $R_2 = 5\,\Omega$, and the battery is ideal. What value of R_3 maximizes the dissipation rate in resistance 3?

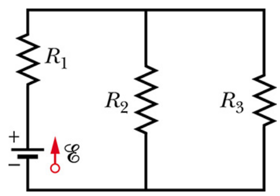

Figure 27: The circuit for Problem 27.35

The power dissipated by resistor three is given by: $P_3 = i_3 V_3$. Hence, our first job is to find the current flowing through and the potential difference across resistor three. Lets start with the potential difference. If we simplify R_2 and R_3 down to an equivalent resistor we find that their equivalent resistance is

$$P = \frac{R_2 R_3}{R_2 + R_3}$$

Simplifying the circuit one more level gives us the current flowing through the battery and R_1.

$$i_1 = \frac{\mathcal{E}}{R_1 + P}$$

Stepping back one level gives us the potential difference across P.

$$V_P = \mathcal{E} - i_1 R_1 = \mathcal{E}\left(1 - \frac{R_1}{R_1 + P}\right) = \mathcal{E}\frac{P}{R_1 + P}$$

In the end we will need to express this in terms of R_3, but we will keep it in this form for the time being.

Since R_2 and R_3 are in parallel, they must both have a potential V_P across them. Hence, we can express the current traveling through R_3 as

$$i_3 = \frac{V_P}{R_3} = \frac{\mathcal{E}}{R_3}\frac{P}{R_1 + P}$$

Finally, putting all of this together and expressing P in terms of R_1 and R_2, we see that the power dissipated by resistor 3 is

$$\begin{aligned}
P_3 = i_3 V_3 &= \frac{1}{R_3}\left(\frac{\mathcal{E}P}{R_1 + P}\right)^2 \\
&= \frac{\mathcal{E}^2}{R_3}\left(\frac{\frac{R_2 R_3}{R_2 + R_3}}{R_1 + \frac{R_2 R_3}{R_2 + R_3}}\right)^2 \\
&= \frac{\mathcal{E}^2}{R_3}\left(\frac{R_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1}\right)^2 \\
&= R_2^2 \mathcal{E}\frac{R_3}{(R_1 R_2 + R_2 R_3 + R_3 R_1)^2}
\end{aligned}$$

Now that we know the power dissipated we simply need to maximize this with respect to R_3. As usual, we do so by taking the derivative with respect to R_3 and setting it equal to zero.

$$\begin{aligned}
\frac{\partial}{\partial R_3}P_3 &= R_2^2 \mathcal{E}\left[\frac{1}{(R_1 R_2 + R_2 R_3 + R_3 R_1)^2} - \frac{2R_3(R_1 + R_2)}{(R_1 R_2 + R_2 R_3 + R_3 R_1)^3}\right] \\
&= R_2^2 \mathcal{E}\left[\frac{R_1 R_2 + R_2 R_3 + R_3 R_1 - 2R_3(R_1 + R_2)}{(R_1 R_2 + R_2 R_3 + R_3 R_1)^3}\right] \\
&= R_2^2 \mathcal{E}\left[\frac{R_1 R_2 - R_2 R_3 - R_3 R_1}{(R_1 R_2 + R_2 R_3 + R_3 R_1)^3}\right] = 0
\end{aligned}$$

For the term on the left to be zero, the numerator must go to zero. Hence,

$$R_1 R_2 - R_2 R_3 - R_3 R_1 = 0 \quad \Rightarrow \quad \boxed{R_3 = \frac{R_1 R_2}{R_1 + R_2}}$$

9 Chapter 28: Magnetic Fields

Most of us are familiar with magnetic fields from childhood, and we generally encounter permanent magnets on a daily basis. It might seem odd then that we spent the first third of the course talking about electric fields which we are much less familiar with. One of the main reasons for this ordering, as we will see, is that magnetic fields are a good deal more complicated than electric fields. Some of the reasons that magnetic fields are more complicated are:

1. There is no such thing as a charged particle for magnetic fields (at least we have never found one). Magnetic fields are generally created in the laboratory by currents.
2. The force that magnetic fields exert on particles is given by a cross product. This means that the magnetic force is always perpendicular to the field and the velocity of the particle.
3. Since the force is always perpendicular to the velocity, magnetic fields do no work.

9.1 The Definition of $\vec{B}$

Recall that we defined the electric field by first talking about the force which two charged particles exert on each other and then factoring out one of the charges. This lead us to the concept of a field which existed everywhere in space as opposed the idea of a force which only exists between two objects.

Since magnetic fields do not have the equivalent of 'charged' particles we will not be able to define the field in the same way. We instead define the field directly through its force law which is found experimentally to be

$$\vec{F_B} = q\vec{v} \times \vec{B} \tag{28}$$

Recall that we can expand the cross product to find the magnitude of the force as

$$F_B = qVB \sin\phi$$

where ϕ is the angle between the field and the velocity of the charged particle.

9.2 The Hall Effect and Circulating Charged Particles

We have been told in various courses throughout high school and college that the things which move in a conductor are electrons. We have also recently seen that although electrons are the things which move and create currents, we define a positive current flow as being in the opposite direction. One can ask: how do we know that it is electrons which are flowing within the conductor and not some sort of positively charged particles?

One way to tell (the most famous one) is by a phenomenon known as the Hall effect. Imagine putting a current carrying wire into a uniform magnetic field. We know now that the moving charges (of either sign) will experience a force due to their motion in the magnetic field. This force will tend to push the charges to one side or the other of the conductor. The key part is that the force on the charges will be in the same direction. The reason that the force is in the same direction is because the opposite signs would flow in opposite directions.

We can not watch the individual charges in a conductor, but we can look at the bulk effect which is that the charges will move to one side of the wire until they have established a potential difference which cancels the force of the magnetic field. We can calculate the number of charge carriers per unit volume by setting the magnitude of the electric and magnetic fields equal to each other. If we do so we find

$$n = \frac{Bi}{Vle}$$

One other important interaction between charged particles and magnetic fields is circulation. If a charged particle enters a uniform magnetic field, it will experience an acceleration which causes it to move in a circle. As we saw in PHY 2048, a particle moving in a circle is undergoing centripetal acceleration. We can use this face to calculate the radius of the charged particle's path in the field.

The force of the magnetic field is

$$|F_B| = |q|vB$$

Setting this equal to the centripetal acceleration force gives

$$|q|vB = \frac{mv^2}{r} \quad \Rightarrow \quad r = \frac{mv}{|q|B} \tag{29}$$

It is important to note that if the particle enters the magnetic field with a component of its velocity which is directed parallel to the field, then that component of the velocity will be unaffected. The path of the particle will therefore be a helix.

9.3 Magnetic Force on a Current Carrying Wire

We have already seen in our discussion of the Hall effect that the conduction electrons in a current carrying wire experience a net sideways force when the wire is placed in a magnetic field. Since these electrons cannot fly out of the wire, they must transfer the force that they feel to the wire itself. We therefore conclude that a current carrying wire experiences a force when placed in a magnetic field.

In order to figure out how strong this force is, we need to express the charge in the magnetic force equation in terms of the current.

$$q = it = i\frac{L}{v_d}$$

We can plug this directly into the magnetic force equation, but we must be careful to direction right since this term will cancel with the velocity. Singe the current moves in the direction of the wire, we can use the L in the above equation to express the direction.

$$\vec{F}_B = i\vec{L} \times \vec{B} \tag{30}$$

where L is the length of the wire over which the magnetic field acts and the direction is the direction of the current flow.

9.4 Torque on a Current Carrying Loop

The subject of how a a current carrying loop acts when placed in a magnetic field is very important because it is the phenomenon upon which all electric motors are built. Imagine a square loop of wire placed in a uniform magnetic field (see figure 28). The portions of the in which the current flows parallel to the field will experience no force. The portions of the wire in which the current flows perpendicular to the field will experience an equal and opposite force. These two forces will tend to rotate the loop of wire.

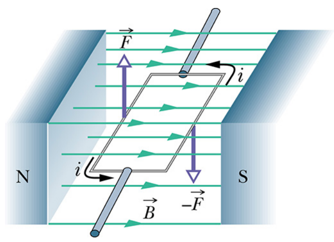

Figure 28: A current carrying loop placed in a uniform magnetic field.

We can calculate the strength of this torque exerted on the wire and it turns out to be

$$\tau = NiAB\sin\theta \tag{31}$$

One interesting feature of this equation is that the geometry of the wire does not matter, only the area enclosed by it does. If we want to use this to make the coil spin, then we have to alternate the magnetic field back and forth so that the coil is always experiencing a torque in the same direction.

9.5 The Magnetic Dipole Moment

In order to calculate the torque on a current carrying loop, we can assign to it a value called the magnetic dipole moment, μ.

$$\mu = NiA$$

This may seem silly at this moment to give a special name to this quantity, but rest assured that this shows up over and over again in physics.

We can give the magnetic dipole moment a direction by having it point normal to the patch of are that it encloses. There are of course two directions normal to the surface, but we will choose to use the right hand rule to make it unambiguous. This right hand rule is slightly different than the one that we are used to. It works like this: take your four fingers and curl them around the loop of wire in the same direction that the current is flowing; your thumb then points in the direction of the magnetic moment.

Using this direction for the magnetic moment, we can write down the torque on a magnetic dipole

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{32}$$

We can also define a magnetic potential energy associated with the direction of the dipole in the field.

$$U(\theta) = -\vec{\mu} \cdot \vec{B}$$

9.6 Problems

Problem 27.44

In figure 29, a metal wire of mass $m = 24.1\,mg$ can slide with negligible friction on two horizontal parallel rails separated by distance $d = 2.56\,cm$. The track lies in a vertical uniform magnetic field of magnitude $56.3\,mT$. At time $t = 0$, device G is connected to the rails, producing a constant current $i = 9.13\,mA$ in the wire and rails (even as the wire moves). At $t = 61.1\,ms$, what are the wire's (a) speed and (b) direction of motion (left or right)?

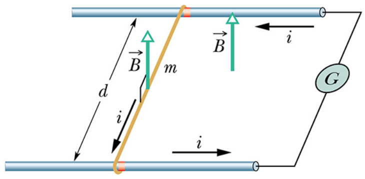

Figure 29: Figure for Problem 27.44.

Part (b) is easy enough to do that we might as well do it first. The direction of the force on the sliding piece of wire is given by the right hand rule. If we lay our hands along the direction of the current and curl our fingers into the direction of the magnetic field, our thumb is left pointing leftwards. Hence, the direction of the force (and therefore the speed) of the section of wire will be <u>leftwards</u>.

In order to calculate the speed of the wire after the given amount of time, we need to first calculate the force on the wire. Recall that the force on a current carrying wire is

$$\vec{F} = i\vec{L} \times \vec{B} \quad \Rightarrow \quad |F| = idB$$

The acceleration of the wire if therefore

$$a = \frac{idB}{m}$$

Finally, by recalling our constant acceleration formulas from last semester we can find the speed.

$$v = at = \boxed{\frac{idBt}{m} = 3.34 \times 10^{-2}\,\frac{m}{s}}$$

Problem 27.58

A magnetic dipole with a dipole moment of magnitude $0.02\,\frac{J}{T}$ is released from rest in a uniform magnetic field of magnitude $52\,mT$. The rotation of the dipole due to the magnetic force on it is unimpeded. When the dipole rotates through the orientation where its dipole moment is aligned with the magnetic field, its kinetic energy is $0.8\,mJ$. (a) What is the initial angle between the dipole moment and the magnetic field? (b) What is the angle when the dipole is next (momentarily) at rest?

Recall our expression for the potential energy of a magnetic dipole in a magnetic field.

$$U = -\vec{\mu} \cdot \vec{B} = -|\mu||B| \cos\theta$$

We can use this to calculate the work done by the applied field. The problem states that the dipole is initially at rest and that its final position is aligned with the magnetic field. We can therefore use conservation of energy to solve this problem.

$$\begin{aligned} K_f + U_f &= K_i + U_i \\ K - \mu B &= 0 - \mu B \cos\theta \\ \Rightarrow \quad \theta &= \arccos\left(1 - \frac{K}{\mu B}\right) \end{aligned}$$

For the second part of the problem, we do not even have to do any calculations. This is analogous to a ball rolling through a valley. The ball will roll down the hill, picking up kinetic energy and then on the other side it will roll back up to the same height as it started at. In our case, the magnetic dipole will return to the same angle that it started at.

$$\theta_2 = \arccos\left(1 - \frac{K}{\mu B}\right)$$

10 Chapter 29: Magnetic Fields Due to Currents

In the last chapter we discussed how charged particles interact with magnetic fields without mentioning how magnetic fields are created. When we discussed electric fields earlier in the semester, we started with the idea of a charged particle and the field which it produced. We then used the principle of superposition to calculate the electric fields of more complicated objects.

In magnetism we do not have the luxury of a 'charged' particle, so we will have to approach the building of magnetic fields from a different perspective. We will see shortly that currents give rise to magnetic fields in the same way that charges give rise to electric fields. The fundamental unit for magnetism will be a differential piece of current $id\vec{s}$ and we will use it to build complex fields.

10.1 Calculating the Magnetic Field of a Current Distribution

As previously mentioned, the fundamental unit of magnetic field production is a differential piece of current $id\vec{s}$. We can already see that calculating magnetic fields will be more complicated than calculating electric fields because charge is a scale whereas current is a vector. It is an experimental fact that the differential bit of magnetic field due to a differential piece of current is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{id\vec{s}\times\hat{r}}{r^2} \tag{33}$$

which is known as the Biot-Savart law. The new constant which shows up in this equation, μ_0, is called the permeability of free space and has the SI value of $\mu_0 = 4\pi \times 10^{-7}\frac{Tm}{A}$.

Because of the vector nature of this law and the cross product which is involved, it is much harder to work with in general than it was with electric fields. The best way to use it is to exploit symmetry to simplify the situation into a solvable integral. The book works out two examples for which I will simply quote the results here. For a long (approximately infinite) strait wire, the magnitude of the magnetic field around the wire is

$$B = \frac{\mu_0 i}{2\pi R}$$

Because of symmetry, we can see that the field must wrap around the wire, but there is some ambiguity as to which direction. This ambiguity is resolved just like all of the other directional ambiguities which we have encountered, with the right hand rule. (Note that this ambiguity is because we have not seen the calculation of the magnetic field and were only given the magnitude, in reality the Biot-Savart law leaves no ambiguity.) To determine the direction of the field, we lay our thumb along the wire in the direction of the current and the field wraps around the wire in the direction that our fingers wrap around when we curl them.

The other result derived in the book is the field for the center of a circular arc of angle ϕ.

$$B = \frac{\mu_0 i\phi}{4\pi R}$$

The direction for this case is determined in the same way as for the infinite straight wire.

10.2 Force Between Two Parallel Currents

Magnetic fields were first discovered by studying the force between two current carrying wires. We saw in the last section that when a long straight wire is carrying current, it will produce a magnetic field around itself. If we then place another current carrying wire in that magnetic field, we know from last chapter that it will feel a force due to the field. The calculations of the forces between the two wires is very straightforward. We simply start by calculating the field of either wire and then by calculating the force that that field exerts on the other wire.

As an example, consider two wires a distance d apart and carrying equal currents running in the same direction. The magnetic field of the right hand wire is

$$B = \frac{\mu_0 i}{2\pi d}$$

pointing upwards at the left hand wire. A section of length L on the left hand wire then feels an attractive force

$$F = iLB = \frac{\mu_0 i^2 L}{2\pi d}$$

Hence, the force per unit length of one wire on the other is

$$\frac{F}{L} = \frac{\mu_0 i^2}{2\pi d}$$

It should be clear that had the current in the second wire been running the other direction, then the force between the two wires would have been repulsive. This leads us to a general truth about the force between two current carrying wires. If the currents run parallel, then the force is attractive. If the currents run anti-parallel, then the force is repulsive.

10.3 Ampere's Law

Ampere's law is essentially the Gauss's Law of magnetism.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} \tag{34}$$

The left hand side of the integral says that we should pick any closed (the loop on the integral sign means closed) loop and evaluate the dot product of the magnetic field with that loop and sum up all of the contributions as we go around the loop. The right hand side of the equation tells us that the value of the integral is directly proportional to the current that the loop encloses.

It is easy to see from looking at the equation why Ampere's law is similar to Gauss's law, but lets discuss why it is different. The main difference is that the right hand side is a line integral instead of a surface integral. Hence, instead of Gaussian surfaces, we will need Amperian loops. These loops are simply imaginary paths of integration in space. The beauty of Ampere's law is that no matter how we pick our path, the integral must be equal to the current which penetrates the surface. Note that this integral is, in general, not possible to do in closed form so we must exploit symmetry in order to break open the left hand side.

10.3.1 Magnetic Field of a Long Straight Wire.

The equation we derived earlier for the magnetic field of a long straight wire was derived using the Biot-Savart law, it is much easier to derive from Ampere's law.

We first need to make a couple of symmetry arguments. Because of radial symmetry, the magnetic field must make circles around the wire. Furthermore, it must be constant at equal radii from the loop. We know form the right-hand rule which direction the field must circle the wire. Hence, if we choose to integrate in that direction

$$\vec{B} \cdot d\vec{s} = |B| ds$$

Since we just argued that B is a constant at constant radius, we can pull it out of the integral all together.

$$|B| \oint ds = \mu_0 i \quad \Rightarrow \quad |B| = \frac{\mu_0 i}{2\pi r}$$

which is exactly the equation we derived from the Biot-Savart law.

10.4 Problems

Problem 29.24

A current is set up in a wire loop consisting of a semicircle of radius $4\,cm$, a smaller concentric semicircle, and two radial straight lengths, all in the same plane. Figure 30 shows the arrangement but is not drawn to scale. The magnitude of the magnetic field produced at the center of curvature is $47.25\,\mu T$. The smaller semicircle is then flipped over (rotated) until the loop is again entirely in the same plane. The magnetic field produced at the (same) center of curvature how has magnitude $15.75\,\mu T$, and its direction is reversed. What is the radius of the smaller semicircle?

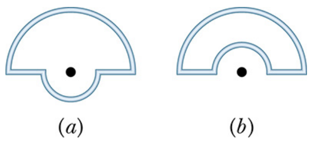

Figure 30: Figure for problem 29.24

We first need to recognize that the straight portions of wire do not contribute to the magnetic field. We can see this from the Biot-Savart law because $d\vec{s}$ and $\vec{r}$ lie along the same line and their cross product is therefore zero. Hence, the only portions of the wire which will contribute are the semicircles.

Recall the equation for the magnetic field of a section of circular wire.

$$|B| = \frac{\mu_0 i \phi}{4\pi r}$$

In our case, both circular segments take up an angle of $\phi = \pi$. We now need to write down the magnetic field in both cases. We must be careful to keep track of our signs because the field will point down in one case and up in the other.

$$B_1 = \frac{\mu_0 i}{4R} + \frac{\mu_0 i}{4r} \qquad -B_2 = \frac{\mu_0 i}{4R} - \frac{\mu_0 i}{4r}$$

Note that since the problem did not tell us which direction the current is flowing, I simply assumed a direction. Had we chosen the direction incorrectly, we would have obtained a negative sign when solving for it.

Lets now solve the system of equations for r. Subtracting the two equations gives

$$B_1 + B_2 = \frac{\mu_0 i}{2r} \quad \Rightarrow \quad r = \frac{\mu_0 i}{2(B_1 + B_2)}$$

We are not quite done yet though because we do not know the current. Adding the two equations gives

$$B_1 - B_2 = \frac{\mu_0 i}{2R} \quad \Rightarrow \quad i = \frac{2R(B_1 - B_2)}{\mu_0}$$

Finally, substituting this into the first equation gives

$$r = \frac{B_1 - B_2}{B_1 + B_2} R = 2\,cm$$

Problem 29.48

In figure 31, a long circular pipe with outside radius $R = 2.6\,cm$ carries a (uniformly distributed) current $i = 8\,mA$ into the page. A wire runs parallel to the pipe at a distance of $3R$ from center to center. Find (a) the magnitude and (b) the direction (into or out of the page of the current in the wire such that the net magnetic filed at point P has the same magnitude as the net magnetic field at the center of the pipe but is in the opposite direction.

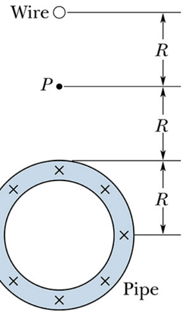

Figure 31: Figure for problem 29.48

This problem is essentially practice in applying Ampere's law. Before we start, we can draw some analogies with Gauss's law and the electric field to preemptively guess some of the features of the problem. Recall that the electric field inside of a uniformly charged spherically symmetric surface was zero. We might therefore expect that the magnetic field inside of a uniform cylindrically symmetric current distribution is zero. Similarly, we saw that for a sphere of charge, the electric field outside acted as if the charge were all concentrated at the center. We might therefore expect that the magnetic field outside of a uniform cylindrically symmetric current distribution behaves like a thin wire carrying the same current. We will see that all of these expectations are true.

Lets begin by ignoring the wire and only worrying about the cylinder. Because the cylinder is spherically symmetric, the magnetic field both inside and outside must circle it radially and must be constant at constant radii. Hence, both inside and outside, we can break open the left hand side of Ampere's law to find.

$$B = \frac{\mu_0 i_{enc}}{2\pi r}$$

Since an Amperian loop drawn inside of the tube will enclose no current, the field inside bust be zero. Likewise, an Amperian loop drawn outside of the tube will enclose all of the current and the field outside will act like an infinite wire.

If we now bring the wire back into consideration, the field inside of the tube will come solely from it.

$$B_0 = -\frac{\mu i_2}{2\pi(3R)} = \frac{\mu_0 i_2}{6\pi R}$$

where we have used i_2 to denote the current of the wire. The field is negative because we have assumed that the current flows in the same direction as the tube and have decided to call a leftward pointing field negative. If we have made the wrong assumption about the direction of the current flow, then it will show up as a minus sign at the end of the problem.

The field at point P has a contribution from the wire and from the tube.

$$B_P = -\frac{\mu_0 i_2}{2\pi R} + \frac{\mu_0 i}{4\pi R}$$

The condition that we are looking for is that $B_p = -B_0$

$$-\frac{\mu_0 i_2}{2\pi R} + \frac{\mu_0 i}{4\pi R} = \frac{\mu_0 i_2}{6\pi R} \quad \Rightarrow \quad -6i_2 + 3i = 2i_2 \quad \Rightarrow \quad \boxed{i_2 = \frac{3}{8} i}$$

Since the current came up positive, we know that we got the direction correct when we assumed that the current moved in the same direction as that of the tube.

11 Chapter 30: Induction and Inductance

In the last section we considered how magnetic fields are produced by currents. In this chapter we will see that changing magnetic fields produce currents in conducting materials. One can see how this will make for a very complicated situation. I.E. currents produce magnetic fields which produce currents which produce magnetic fields ad infinitum. We will not have to worry about such complexity in general though because we will only be considering first order effects.

Since changing magnetic fields produce currents and currents are induced by potential differences, we are lead to the conclusion that changing magnetic fields produce electric fields. This is our first encounter with what we will see is a very interesting and far reaching fact: electricity and magnetism are linked together in a very complicated way.

11.1 Faraday's Law and Lenz's Law

Faraday's law is (at least in appearance) similar to Gauss's law. Like Gauss's law, Faraday's law involves the concept of flux. Recall that the electric flux involved in Gauss's was given by

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

The magnetic flux which plays a role in Faraday's law has almost the exact same form

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \tag{35}$$

Notice the small difference between the two flux statements: the magnetic flux does not have to be over a complete surface. This is not a fundamental difference between electric and magnetic flux but rather a difference between the Gauss's and Faraday's laws. In general, the flux that we will be talking about with Faraday's law is the magnetic flux through a closed circuit, which is not a closed surface.

Although both Faraday's law and Gauss's law involve the concept of flux, they are actually very different. The main difference is that Faraday's law relates something to the rate of change of the flux whereas Gauss's law relates something simply to the flux. Without further adieu, Faraday's law is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{36}$$

This law is very interesting because there are many ways to change the flux through a loop of wire. The most obvious way is to change the strength of the magnetic field. The other not so obvious ways are to: 1. rotate the wire in the field so that the angle between the wire and the field change, 2. move the wire to a different part of the field that has different strength, and 3. change the shape of the circuit so that more of the field is enclosed.

Faraday's law tells us that a emf (and hence a current) will be induced in a closed loop when the change the magnetic flux through the loop. There is an ambiguity though because Faraday's law tells us nothing about the direction of the current flow. Lenz's law resolves this ambiguity.

Lenz's law is a simple statement: The current induced in a loop of wire by changing the magnetic flux through the loop will try to oppose the change in the magnetic field. To be more specific, if the flux is decreasing, the current will flow in the direction which tries to add more flux through the loop. If the flux is increasing, the current will try to flow in the direction that takes flux away from the loop. We will get some practice with Lenz's law later on while working the problems.

11.2 Induced Electric Fields

Since changing magnetic fields induce an emf and emf's are induced by electric fields, we are lead to the conclusion that changing magnetic fields must produce an electric field. We must be careful at this point to distinguish between an emf and a potential difference. Up until now the two have been virtually the same, but we are about to see the distinction.

Recall the definition of potential

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}$$

From the last chapter (Kirchoff's loop rule) we know that the potential is a single valued quantity and therefore if we walk around a closed circuit and come back to the original point we should get zero change in potential.

To get at the concept of emf, lets go back to the definition.

$$\mathcal{E} = \frac{dW}{dq}$$

So, emf is the amount of work done per unit charge around a closed loop in the circuit. Recall from physics 2048 the definition of work

$$W = \int \vec{F} \cdot d\vec{s} = q \int \vec{E} \cdot d\vec{s}$$

Hence, taking the work to be done around a closed loop and differentiating with respect to charge we find

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s} \tag{37}$$

We can see that this looks very similar to the definition of potential. We know however that if we integrate the electric field around a closed loop we must get zero potential difference. The moral of the story is: *potential only has meaning for electric fields produced by static charges and must be abandoned when talking about electric fields produced by induction.*

With this new concept of emf in mind, we can rewrite Faraday's law as

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{38}$$

11.3 Inductors and Inductance

As we saw in the last chapter, if we establish a current in a loop of wire, then there will be a magnetic field produced. From this chapter we now know that that magnetic field will contribute some magnetic flux to the loop. Similar to capacitance, we define a quantity called inductance which describes the ration between these two quantities.

$$L = \frac{\Phi_B}{i} \tag{39}$$

When talking about capacitance, the quintessential capacitor was the parallel plate capacitor. When talking bout inductance, the quintessential inductor will be a long solenoid. Lets calculate the inductance of a solenoid with n turns per unit length and a cross sectional area A. The magnetic field of such a solenoid is

$$B = \mu_0 i n$$

The total flux through all N turns is given by

$$\Phi_B = N\Phi_i = \mu_0 i n^2 A l$$

Hence, the inductance is

$$L_s = \mu_0 n^2 A l$$

For a normal circuit, the current will be changing. If we change the current flowing through our inductor, then we will induce a change in the magnetic flux through the wire and hence we will induce an emf. From Lenz's law, the emf induced will tend to oppose the change in the current. Hence, an inductor in a circuit will make the circuit tend to oppose changes in current flow. To see how this *self-induced* emf contributes lets go back to Faraday's law.

$$\mathcal{E}_L = -\frac{d\Phi_B}{dt} = -L\frac{di}{dt}$$

So, initially the inductor opposes changes in the current, but after the current gets flowing the inductor acts like an ordinary conducting wire.

11.4 RL Circuits

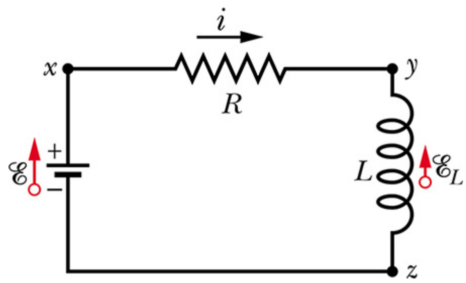

Figure 32: A simple RL circuit.

Lets take a closer look at how an inductor acts in a circuit by considering the simple case of an inductor an a resistor in a circuit series (see figure 32). We can write down the equation governing such a circuit by using Kirchoff's loop rule.

$$L\frac{di}{dt} + Ri - \mathcal{E} = 0$$

This is a simple differential equation which we have encountered previously with RC circuits. The solutions to this equation depend on the initial conditions of the circuit.

Lets consider the situation in which the circuit initially has no current flowing in it and then the voltage source is turned on. The solution in this case is

$$i(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L})$$

where

$$\tau_L = \frac{L}{R}$$

If we consider the case in which the circuit initially has a current flowing and then the voltage source is turned off, then we must modify our differential equation by dropping the $\mathcal{E}$ term. The solution in this case is

$$i = \frac{\mathcal{E}}{R}e^{-t/\tau_L} = i_0 e^{-t/\tau_L}$$

Notice that, as with capacitors, the same time constant shows up in the rising up and in the decay process. Also notice that the form of the equations is exactly the same as it was with capacitors. We can therefore draw the useful analogy: capacitors are to charge as inductors are to current.

11.5 Energy Stored in a Magnetic Field

As with capacitors, we are now in a position to talk about the energy stored in a magnetic field. We will forgo the hand-wavy derivation and simply quote the results here. The energy stored in a magnetic field within a conductor is

$$U = \frac{1}{2}Li^2 \tag{40}$$

Note that this is very similar to the energy dissipated by a resistor and to the energy stored in a capacitor.

As with capacitors, we can use the idea of energy stored in an inductive magnetic field to sneak in the back door of the much broader and more important concept of energy density. The energy density of a magnetic field (no matter how it is produced) is

$$u = \frac{1}{2\mu_0}B^2 \tag{41}$$

Notice that this is very similar to the energy density of an electric field.

11.6 Problems

Problem 30.15

In figure 33, a stiff wire bent into a semicircle of radius $a = 2, cm$ is rotated at constant angular speed $f = 40\,rev/s$ in a uniform $20\,mT$ magnetic field. What are the (a) frequency and (b) amplitude of the emf induced in the loop?

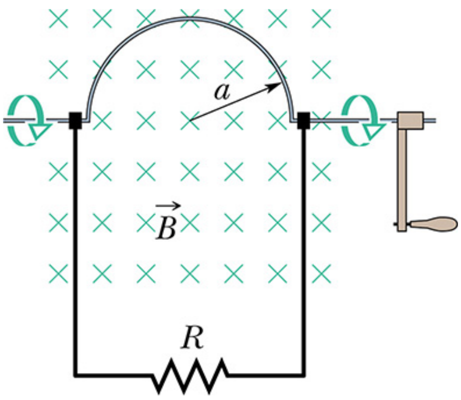

Figure 33: Figure for problem 30.15

The induced emf is generated by the changing magnetic flux through the circuit. In this case, the changing magnetic flux is not due to the magnetic field changing but rather the amount of area though which the field passes is changing. Since the area oscillates back and forth at the frequency f, the frequency of the emf must be the same.

To find the amplitude of the emf we must do some calculations. We want to write down the flux through the loop as a function of time. Lets return to the definition of flux.

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = BA_1 + \int B\, dA_2 \cos\theta$$

where θ is the angle between the magnetic field and the normal to the changing surface area (in or out will both work just fine). The first term is the magnetic flux through the square part of the loop which does not change. Since it does not change, it will not contribute to the emf and can therefore be neglected.

Integrating over dA_2 gives

$$\Phi_B = B(\frac{1}{2}\pi a^2)\cos\theta$$

Since the loop is turning, θ is a function of time. If the loop turns at a rate of f, then *theta* changes at a rate of

$$\omega = 2\pi f$$

Hence, the magnetic flux through the circuit as a function of time is

$$\Phi_B = \frac{1}{2}B\pi a^2\cos(2\pi f t)$$

Finally, from Faraday's law the induced emf will be

$$\mathcal{E}(t) = -B\pi^2 a^2 f \sin(2\pi f t)$$

from which we can read off the amplitude as

$$\boxed{|\mathcal{E}| = B\pi^2 a^2 f}$$

Problem 30.24

For the wire arrangement in figure 34, $a = 12\,cm$ and $b = 16\,cm$. The current in the long straight wire is given by $i = 4.5\,t^2 - 10\,t$, where i is in Amperes and t is in seconds. (a) Find the emf in the square loop at $t = 3\,s$. (b) What is the direction of the induced current in the loop?

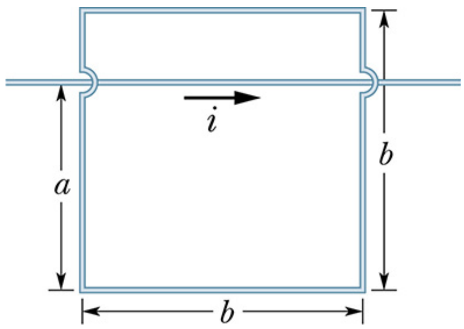

Figure 34: Figure for problem 30.24

The long straight wire is carrying a current and is therefore going to produce a magnetic field.

$$B = \frac{\mu_0 i}{2\pi r}\hat{\phi}$$

where r is the distance from the wire.

In order to calculate the emf in the square wire, we need to calculate the magnetic flux through the square as a function of time.

$$\Phi_B = \int \vec{B}(t) \cdot d\vec{A}$$

The magnetic field is not constant over the area so we can not use our usual trick of pulling B out of the integral. We will actually have to integrate. We can simplify our lives a little by noting that the magnetic field points out of the page in the upper portion of the square and into the page in the lower portion of the square. Some of the flux through the lower portion will be canceled by the upper portion and we therefore only need to consider a part of the lower portion while integrating.

$$\begin{aligned}
\Phi_B &= \int_{b-a}^{a} \left(\frac{\mu_0 i(t)}{2\pi r}\right)(b\,dr) \\
&= \frac{\mu_0 b}{2\pi} i(t) \int_{b-a}^{a} \frac{dr}{r} \\
&= \frac{\mu_0 b}{2\pi} i(t)\,[\ln(r)]_{b-a}^{a} \\
&= \frac{\mu_0 b}{2\pi} \ln\left(\frac{a}{b-a}\right) i(t)
\end{aligned}$$

Hence, from Faraday's law, the emf generated in the loop as a function of time is

$$\mathcal{E} = -\frac{d}{dt}\Phi_B = \frac{\mu_0 b}{2\pi}\ln\left(\frac{a}{b-a}\right)\frac{d}{dt}i(t) = \frac{\mu_0 b}{2\pi}\ln\left(\frac{a}{b-a}\right)(9t - 10)$$

In order to discern which direction the induced current is in, we need to consider Lenz's law. At $t = 3\,s$ the derivative of the current in the long straight wire is positive and hence the current is increasing. Since the current in the long straight wire is increasing, the strength of the magnetic field (and therefore the amount of flux) is increasing. Since the flux is increasing, the current established in the square wire will try to cancel out the other field. The net flux points into the page and therefore the established current will try to produce a field which points out of the page. From the righthand rule, this corresponds to a current flowing in the counter *counter-clockwise direction.*

12 Chapter 31: Electromagnetic Oscillations and Alternating Current

12.1 LC Circuits

We have now seen the three major components of standard analog circuits: the resistor, the capacitor, and the inductor. We have looked at the properties of the three devices in isolation as well as the properties of circuits containing the RC and LR in pairs. Lets now look at the remaining pair, the LC circuit.

Before we delve into the mathematics, lets look at this circuit from an energy point of view. In previous chapters we derived the energy stored in a capacitor and an inductor. They were, respectively,

$$U_E = \frac{q^2}{2C} \quad \& \quad U_B \frac{Li^2}{2}$$

When we go through the mathematics of this simple circuit we will see that energy will oscillate from one form of energy to the other. This situation is analogous to the simple harmonic oscillator from physics 1. We usually started the harmonic oscillator with some potential energy and when we released it the energy was transferred to kinetic energy. After that the energy was converted back to potential and the whole process started over again. These type of simple harmonic oscillations will show up in LC circuits in exactly the same way.

Recall that for a simple harmonic oscillator the frequency at which the oscillations happened was a characteristic of the system given by

$$\omega_{shm} = \sqrt{\frac{k}{m}}$$

From the preceding discussion we should expect that the LC oscillations will have a similar characteristic frequency governed by the parameters of the system. They in fact do, and we will soon see that the frequency is given by

$$\omega_{LC} = \frac{1}{\sqrt{LC}}$$

So, for LC circuits we can make the identification that L plays the role of mass and $\frac{1}{C}$ plays the role of the spring constant.

Lets delve into the formal mathematics of such a circuit. We again write down a governing equation by using Kirchoff's loop rule and taking a walk around the circuit. When we do we find

$$L\frac{di}{dt} + \frac{q}{C} = 0 \quad \Rightarrow \quad L\frac{d^2}{dt}q + \frac{1}{C}q = 0$$

Or, rearranging

$$\frac{d^2}{dt^2}q(t) = -\frac{1}{LC}q(t)$$

The solution to this equation should be fairly obvious. What functions do we know whose second derivative is equal to itself times a negative sign? Sines and Cosines.

Whether we choose the sine function of the cosine function really depends on whether we start out with the capacitor charged or uncharged. We will assume that it starts charged and that the initial current flow is zero. Hence, the charge on the capacitor as a function of time is

$$q(t) = Q\cos(\omega t + \phi) \tag{42}$$

where Q is the initial charge on the capacitor

$$\omega = \frac{1}{\sqrt{LC}} \tag{43}$$

We can easily find the current by taking a derivative.

12.2 RLC Circuits: Damped Oscillations

12.2.1 Unforced Oscillations

We saw in the last section that a simple LC circuit will oscillate just like a spring with no friction. What will happen now if we introduce a resistor to the circuit? By recognizing that a resistor acts like a frictional force in circuits we can take our spring analogy further and predict that the circuit will oscillate, but that the oscillations will eventually die out just like a spring.

We will first think about this circuit with no driving voltage. That is, we will charge up the capacitor and then let it go. As we did with the LC circuit, lets use Kirhoff's loop rule to write down a differential equation for the circuit.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} = \frac{1}{C}q = 0$$

Unlike the other differential equations which we have encountered, the solution to this differential equation is nontrivial though not impossible to discern.

$$q = Qe^{-\frac{Rt}{2L}}\cos(\omega' t + \phi) \quad \text{where} \quad \omega' = \sqrt{\omega^2 - \left(\frac{R}{2L}\right)^2} \tag{44}$$

where $\omega = \frac{1}{\sqrt{LC}}$ just like the undamped circuit. Notice that this solution has two components. The cosine term oscillates like we would expect for a mass on a spring, and the exponential term damps out the oscillations as time goes on. We have implicitly chosen $t = 0$ to be the time at which we let the charge go from the capacitor.

One other interesting property of this solution is that the natural frequency (a.k.a. resonance frequency) of the oscillations is shifted to a lower frequency. In this chapter we will only work with situations in which R is small enough that we can replace ω' with ω.

Why do the oscillations die down over time? The answer is simple, during each oscillation, current is passing through the resistor which is therefore converting the energy in the circuit into thermal energy. Hence, after each oscillation the amount of energy stored back in the capacitor for the next oscillation is a little bit less.

12.2.2 Forced Oscillations

The situation we just considered began by charging up the capacitor and then letting the circuit go. What if we instead hook the circuit up to an alternating current source which drives the current in the circuit at a particular frequency?

$$\mathcal{E} = \mathcal{E}_m \sin(\omega_d t)$$

This type of power source is called alternating current (AC) and is the way in which electricity is transmitted through power lines to our homes.

What we will see shortly is that no matter what the natural frequency of the circuit is (i.e. no matter what the value of ω' is above), the circuit will always oscillate at the driving frequency ω_d. This does not mean that the amplitude will be the same at all frequencies, because it will in fact not be. We will see that the largest amplitude occurs at or near the natural frequency of the circuit. This is the idea of resonance and is exactly the same concept as the resonance of masses on springs from physics 1.

12.3 Forced Oscillations in Various Circuits

Before we delve into a full analysis of the driven RLC circuit, lets think about how each of the three components acts under forced oscillations. What we will discover is that we can generalize the idea of resistance to a frequency dependent reactance and the analysis of the complicated RLC circuit will become much easier.

Resistive Load Lets first consider an alternating emf hooked up to a resistor. By the loop rule

$$v_R = \mathcal{E}_m \sin(\omega_d t) = V_R \sin(\omega_d t)$$

We are using the notation still that capital letters denote amplitudes and lower case letters denote time dependent quantities. Hence, the current is

$$i_R = \frac{v_R}{R} = \frac{V_R}{R}\sin(\omega_d t) = I_R \sin(\omega_d t - \phi)$$

So, the current amplitude is related to the potential amplitude in the usual Ohm's law way.

$$V_R = I_R R \tag{45}$$

We can also see by comparison that $\phi = 0$ and therefore the potential and the current in a circuit are in phase.

Capacitive Load Now, let us give the same treatment to a capacitor hooked up to a driving voltage source. The voltage across the capacitor is set by the driving emf.

$$v_C = V_C \sin\omega_d t$$

The charge and current are therefore given by

$$q = Cv_C = CV_C \sin\omega_d t \quad \Rightarrow \quad i_C = \omega_d C V_C \cos\omega_d t = I_C \sin(\omega_d t - \phi)$$

By making comparisons in the second equation above we can see that, for a capacitive load, $\phi = -90°$. We can also identify an Ohm's law type equation for the relation between the current amplitude and the potential amplitude.

$$V_C = X_C I_C \qquad \text{where} \qquad X_C = \frac{1}{\omega_d C} \tag{46}$$

The X_C in the above equation is called the capacitive reactance of the circuit. It has units of resistance (ohms), and is essentially a frequency dependent resistance. Note that the phase shift between the current and the potential is $-90°$ because this fact will become important when we put more than one component in a circuit.

Inductive Load Let us again give the same treatment to an alternating emf source hooked up to an inductor. The potential across the inductor is set by the voltage source

$$v_L = V_L \sin\omega_d t$$

Recall the defining equation of inductance

$$v_L = L\frac{di_L}{dt}$$

Hence, to get the current from the first equation above we must integrate once. The current is therefore

$$i_L = -\left(\frac{V_L}{\omega_d L}\right)\cos\omega_d t = I_L \sin(\omega_d t - \phi)$$

Making the same identifications as in the capacitive load case we see that we can define an inductive reactance which also satisfies an Ohm's law type relation.

$$V_L = X_L I_L \qquad \text{where} \qquad X_L = \omega_d L \tag{47}$$

We also notice that $\phi = +90°$.

12.4 Phasors and the RLC Circuit

Recall from last semester that a phasor was a vector rotating the the xy plane at a given angular frequency. take our RLC circuit and project the potential of each of the components onto this phasor diagram, e find an easy way to calculate the amplitude of the final current amplitude.

The current and its phase will be the same throughout the circuit, so we must project our phasors with the proper angular shifts derived above.

- Resistor: Current and voltage are in phase.
- Capacitor: The voltage *follows* the current (and hence the voltage of the resistor) by 90°.
- Inductor: The voltage *leads* the current (and hence the voltage of the resistor) by 90°.

To find the final voltage in the circuit, we simply need to add the lengths of all of these vector vectorially. The amplitude is given by

$$\mathcal{E}_m^2 = (IR)^2 + (IX_L - IX_C)^2 \quad \Rightarrow \quad I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C^2)}}$$

Or, plugging in for X_L and X_C, the amplitude of the current in the circuit is given by

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} \tag{48}$$

We can also find the overall phase shift between the current and the potential using the standard methods of vector analysis.

$$\tan\phi = \frac{X_L - X_C}{R}$$

If we plot this current amplitude versus the natural frequency, we see the phenomena of resonance emerge (see figure 35). By looking at the plot we can clearly see that the largest amplitude of the current is induced when the circuit is driven precisely at its resonance frequency.

$$\omega_d = \omega = \frac{1}{\sqrt{LC}}$$

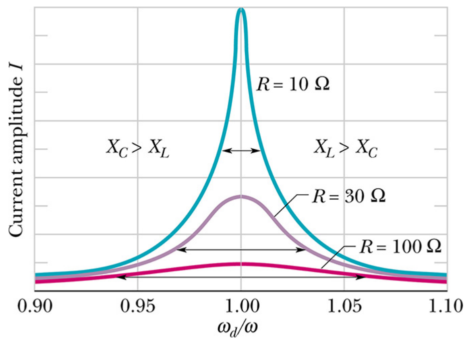

Figure 35: Resonance in an RLC circuit.

12.5 Problems

Problem 31.21

In an oscillating LC circuit, $L = 3\,mH$ and $C = 2.7\,\mu F$. At $t = 0$ the charge on the capacitor is zero and the current is $2\,A$. (a) What is the maximum charge that will appear on the capacitor? (b) At what earliest time $t > 0$ is the rate at which energy is stored in the capacitor greatest, and (c) what is that greatest rate?

Part a

In order to match the initial conditions, we will choose to use a sine instead of a cosine to describe the charge on the capacitor.

$$q(t) = Q\sin(\omega t)$$

Note that we could also have met the initial conditions by choosing the proper phase shift in the cosine term, but this will simplify our life. We do not know the initial charge on the capacitor, but we do know the initial current i_0. If we take a derivative of the charge

$$\frac{dq}{dt} = \omega Q\cos(\omega t)$$

we see that the maximum current is

$$I = i_0 = \omega Q \qquad \Rightarrow \qquad Q = \frac{i_0}{\omega} = i_0\sqrt{LC}$$

Part b

This problem asks about the rate of change of the energy on the capacitor, so we must take a derivative of the energy.

$$\frac{dU}{dt} = \frac{d}{dt}\frac{q^2}{2C} = \frac{q}{C}\frac{dq}{dt} = \frac{1}{C}q(t)\,i(t)$$

We have used the chain rule to simplify before plugging in the actual quantities. Hence, the rate of change of the energy is

$$\frac{dU}{dt} = \frac{1}{C}Q^2\omega\sin(\omega t)\cos(\omega t) = \sqrt{\frac{L}{C}}i_0^2\sin(\omega t)\cos(\omega t)$$

We want to maximize this with respect to time, so we need to take one more derivative.

$$\frac{d}{dt}\left(\frac{dU}{dt}\right) = \sqrt{\frac{L}{C}}i_0^2\left[\cos^2(\omega t) - \sin^2(\omega t)\right]$$

Setting this equal to zero and solving for t gives

$$\tan(\omega t) = 1 \qquad \Rightarrow \qquad t = \frac{1}{\omega}\frac{\pi}{4} = \frac{\pi}{4}\sqrt{LC}$$

Part c

For the final part of this question, we simply need to plug our time into our equation for $\frac{dU}{dt}$.

$$\left(\frac{dU}{dt}\right)_{max} = \sqrt{\frac{L}{C}}i_0^2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{\frac{L}{C}}i_0^2$$

Problem 31.26

A single-loop circuit consists of a $7.2\,\Omega$ resistor, a $12\,H$ inductor, and a $3.2\,\mu F$ capacitor. Initially the capacitor has a charge of $6.2\,\mu C$ and the current is zero. Calculate the charge on the capacitor N complete cycles later for (a) $N = 5$, (b) $N = 10$, and (c) $N = 100$.

We showed earlier in this chapter that the charge on the capacitor in an RLC circuit has a cosine oscillation which is exponentially damped.

$$q(t) = q_0 e^{-\frac{Rt}{2L}}\cos(\omega' t)$$

Note that we have dropped the ϕ term inside of the cosine because the problem says that the capacitor starts off fully charged. The cosine term will have completed N complete cycles when

$$\omega' t = 2\pi N \quad \Rightarrow \quad t = \frac{2\pi N}{\omega'}$$

The book says the we can generally treat ω' as if it were ω, but keeping it as ω' does not complicate this problem much so we will do so. Hence,

$$t_N = \frac{2\pi N}{\sqrt{\omega^2 - \left(\frac{R}{2L}\right)^2}} = \frac{2\pi N}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}$$

After a complete cycle the cosine term will return to 1 so we only need to worry about the exponential term when computing the magnitude of the new charge.

$$q_N = q_0 \exp\left(-\frac{R}{2L}\frac{2\pi N}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}\right) = \boxed{q_0 \exp\left(\frac{-2\pi N}{\sqrt{\frac{4L}{R^2C} - 1}}\right)}$$

If we plug in the numbers we find

$$q_5 = 5.85\,\mu C \qquad q_{10} = 5.52\,\mu C \qquad q_{100} = 1.93\,\mu C \qquad q_{10{,}000} = 1.16 \times 10^{-50}\,\mu C$$

Problem 31.34

An ac generator has emf $\mathcal{E} = \mathcal{E}_m \sin\omega_d t$, with $\mathcal{E}_m = 25\,V$ and $\omega_d = 377\,\frac{rad}{s}$. It is connected to a $12.7\,H$ inductor. (a) What is the maximum value of the current? (b) When the current is a maximum, what is the emf of the generator? (c) When the emf of the generator is $-12.5\,V$ and increasing in magnitude, what is the current?

Although we derived it above, lets rederive the current in this circuit. The potential across the inductor is set by the emf and is given by

$$v = \mathcal{E}_m \sin\omega_d t$$

The defining of an inductor was its relationship to the current

$$v = -L\frac{di}{dt} = \mathcal{E}_m \sin\omega_d t$$

This is a simple differential equation which we should all be able to solve. We want a function of i whose derivative with respect to time gives us a $-\sin\omega_d t$. Clearly, the answer is

$$i(t) = \frac{\mathcal{E}_m}{\omega_d L}\cos\omega_d t$$

For this we can easily read off the magnidtude of the current

$$\boxed{i_m = \frac{\mathcal{E}}{\omega_d L}}$$

The current is at its max when $t = 0$ ($\cos\omega_d t = 1$), but at $t = 0$ the <u>emf is zero</u>.

To solve the final part of the problem we must find the time at which $\mathcal{E}(t) = -12.5\,V$. Solving for t we find

$$t' = \frac{1}{\omega_d}\arcsin\left(\frac{-12.5\,V}{\mathcal{E}_m}\right)$$

At which time the current is

$$\boxed{i(t') = \frac{\mathcal{E}_m}{\omega_d L}\cos\left[\arcsin\left(\frac{-12.5\,V}{\mathcal{E}_m}\right)\right]}$$

Problem 31.44

An alternating emf source with a variable frequency f_d is connected in series with a $50\,\Omega$ resistor and a $20\,\mu F$ capacitor. The emf amplitude is $12\,V$. (a) Draw a phasor diagram for phasor V_R (the potential across the resistor) and phasor V_C (the potential across the capacitor). (b) At what driving frequency f_d do the two phasors have the same length? At that driving frequency, what are (c) the phase angle in degrees, (d) the angular speed at which the phasors rotate, and (e) the current amplitude?

We will not draw the figure here in the notes, but it should look exactly like figure 31-14 (b) in the textbook except without the V_L vector.

The length of the V_R and V_C vectors is given by their Ohm's law relations.

$$V_C = I\,X_C \qquad V_R = I\,R$$

Setting these equal to each other and solving for the frequency gives

$$IX_C = IR \quad \Rightarrow \quad \frac{1}{\omega_d C} = R \quad \Rightarrow \quad \omega_d = \frac{1}{RC}$$

Since the problem asks for the frequency and not the angular frequency, we need to divide by $2Pi$.

$$f_d = \frac{1}{2\pi RC}$$

The phase angle is defined as the angle between the resultant vector and the V_R vector. Note that is this case it will be a negative angle since the resultant vector will trail behind the V_R vector.

$$\phi = \arctan\left(\frac{V_R}{-V_C}\right) = \arctan\left(\frac{IR}{-IR}\right) = -45^\circ$$

Since the circuit is being driven, it oscillates at the driving frequency. Hence, the phasors rotate at the angular driving frequency $\frac{1}{RC}$.

Finally, we can find the amplitude of the current by using either of the Ohm's law style formulas mentioned above.

$$I = \frac{V}{R}$$

13 Chapter 32: Maxwell's Equations; Magnetism of Matter

In the first section of these notes I commented that the entire subject of electromagnetism can be written down in five fairly simple equations. The fifth of these equations is the Lorentz force law which we have already seen.

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

This equation describes how charged particles interact with the electric and magnetic fields.

The other four equations describe how the fields themselves are generated and how they interact with each other. We have already seen two and a half (the half will be explained in a moment) of these equations, although they did not look quite the same. The equations in the first section are written in their differential form, while we have seen them in their integral form. The two and a half equations which we have already seen are

$$\begin{array}{ll} \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon}0 & \text{Gauss's law for electricity} \\ \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} & \text{Faraday's Law} \\ \oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} & \text{Ampere's Law} \end{array}$$

In order to complete the full set of equations we will have to introduce the one and a half equations which are missing. We will begin by introducing a new equation which is exactly like Gauss's law for magnetic fields, and then we will add another term onto Ampere's law (the Maxwell term).

13.1 Gauss's Law for Magnetic Fields

Gauss's law for magnetic fields is

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{49}$$

This is essentially a statement that there is no such thing as magnetic charge. (Recall the rhs of Gauss's law for electric fields above.)

To get a feel for what it means, consider a simple magnetic with a north and a south pole. If we break the magnet into two pieces, we would find that both pieces have both a north and a south pole. If we again break one of the pieces into two we would again find that the pieces had a north and a south pole. We could carry on breaking up the magnetic until we got to the level of atoms and electrons and we would still find that the magnetic fields of these objects had both a north and a south pole. This is because the most simple magnetic structure is a magnetic dipole (a field with a north and a south pole).

13.2 Maxwell's Modification of Ampere's Law

We have talked about how there are one and a half necessary modifications to complete Maxwell's equations. The half part comes in because the part which Maxwell actually added to the equations is merely a modification of Ampere's law. Before we introduce the modification, lets try to motivate the necessity for the change.

What Maxwell did was to notice a deep inconsistency in the equations already introduced. We can see a part of the inconsistency by considering figure 36. In the figure a charging capacitor is shown with two different Amperian surfaces. Both of these surfaces have the same closed loop and hence, should give the same result if we apply Ampere's law to them. If we apply Ampere's law to $S1$ we find that their is some current enclosed and hence there is a magnetic field around the boundary. If we apply Ampere's law to $S2$ we find that their is no current piercing the surface and our conclusion is that there is no magnetic field around the boundary.

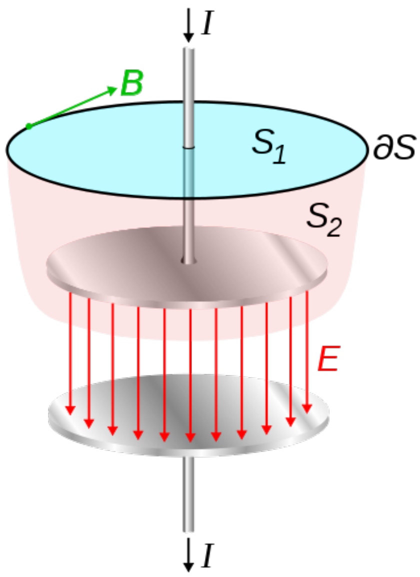

Figure 36: A Charging Capacitor with two different Amperian Surfaces.

It seems natural to assume that the necessary modification must have something to do with the changing electric field inside of the capacitor. Maxwell realized that the Ampere law needed a modification similar to the rhs of Faraday's law.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

The reason that the four equations are associated with Maxwell even though he only added one small term is because after adding this term Maxwell showed that light is an electromagnetic wave. This tied together the fields of optics and electromagnetism, which until then had been completely separate.

Maxwell's term is often called the displacement current because we can rewrite Ampere's law in a form which appears to have two currents on the rhs.

$$\oint \vec{B} \cdot d\vec{S} = \mu_0(i_d + i_{enc}) \qquad \text{where} \qquad i_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

Thinking of this term as a sort of current is also good analogy for thinking about the magnetic field between the capacitor plates.

If we apply Ampere's law to a circular parallel plate capacitor of radius R, then we find that the field inside of the capacitor looks like

$$B = \begin{cases} \left(\frac{\mu_0 i_d}{2\pi R^2}\right) r & r \leq R \\ \frac{\mu_0 i_d}{2\pi r} & r > R \end{cases}$$

So, we see that the field outside of the capacitor looks the same as it would outside a wire carrying a current i_d.

13.3 Magnetism at the Atomic Level

A reasonable question to ask at this point is: if there is no such thing as magnetic charge, then how on earth do we have permanent magnets? The answer, as alluded to in the introduction to this section, is that electrons (and many other atomic particles) have an intrinsic magnetic dipole moment. To be clear, when we say intrinsic we mean that the dipole moment of the electron is one of its fundamental properties like its mass and its charge. Any real discussion of discussion of the electron requires quantum mechanics, so we will only allude to the general features here.

A useful (though incorrect in some ways) analogy for thinking about the electron is as a small spinning ball of charge. A spinning ball of charge has moving charges and these create the magnetic dipole field of the electron. (Again, this is only an analogy and is not the actual quantum mechanical description of the electron.) It is useful to describe this dipole field in terms of the spinning angular momentum of the electron $\vec{S}$. The ration of the magnetic dipole moment to the spin angular momentum of the electron is

$$\vec{\mu}_S = -\frac{e}{m}\vec{S}$$

The spin angular momentum is therefore an intrinsic property of the electron as well.

There is also a contribution to the dipole moment of atoms from the electron orbiting around the atom. Although the quantum mechanics of atoms does not containing orbiting electrons, it does make for a good classical analogy. A moving charged particle looks like a little loop of current which is the quintessential small magnetic dipole. Similarly to the spin of the electron, we relate the dipole moment to the angular momentum of the atom via

$$\vec{\mu}_{orb} = -\frac{e}{2m}\vec{L}_{orb}$$

Dipole moments add like vectors, so even though an atom has electrons and an orbital dipole moment, they may all add up to cancel each other. For other materials, the dipole moments will not add up to zero, but when we put a bunch of the same atoms together the randomness of the orientation will give the material a net zero dipole moment. We generally break materials up into three classes

- **Diamagnetism** Diamagnetism happens when we place a material in an external field. The field will tend to align the dipoles of the electrons and atoms, even if they have a net zero dipole moment. This effect is exhibited by all materials, but it is very weak. The net dipole moment induced in the material will be opposite to the field in this case.

- **Paramagnetism:** Paramagnetism occurs when we place an external magnetic field on a material whose atoms do not have a net zero dipole moment. In this case, the dipole moments align with the field and produce a much stronger and more noticeable effect than diamagnetism does. In this case the net dipole moment of the material will be aligned with the field.
- **Ferromagnetism:** Ferromagnetism is essentially the same as paramagnetism except the structure of the materials provides some way for the dipole moments to stay aligned for some time after the field is removed. These are the magnetic materials that we are familiar with (common fridge magnets) like iron and nickel. Like paramagnetism, the dipole moment will tend to align with the field.

13.4 Problems

Problem 32.7

Suppose that a parallel-plate capacitor has circular plates with raduis $R = 30\,mm$ and a plate separation of $5\,mm$. Suppose also that a sinusoidal potential difference with a maximum value of $150\,V$ and a frequency of $60\,Hz$ is applied across the plates. (a) Find $B_{max}(R)$, the maximum value of the induced magnetic field that occurs at $r = R$. (b) Plot $B_{max}(r)$ for $0 < r < 10\,cm$.

Recall the relationship between the electric field and the potential.

$$E = \frac{dV}{dx}$$

We have dropped the negative sign because we are only concerned with magnitudes. Assuming the potential changes linearly across the capacitor, this can be written as

$$E = \frac{\Delta V}{\Delta x} = \frac{1}{d} V \sin(\omega t)$$

Now that we have the electric field in the capacitor we can calculate the displacement current within the capacitor.

$$i_d = \epsilon_0 \frac{d\Phi_E}{dt} = \epsilon_0 A \frac{dE}{dt} = \frac{\epsilon_0 A}{d} V\omega \cos(\omega t)$$

Plugging this into Ampere's law gives us the field.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 i_d \quad \Rightarrow \quad B = \frac{1}{2\pi r} \mu_0 \frac{\epsilon_0 A}{d} V\omega \cos(\omega t) = \frac{\mu_0 \epsilon \omega (\pi r^2)}{d\, 2\pi r} \cos(\omega t) = \frac{\mu_0 \epsilon_0 \omega r}{2d} \cos(\omega t)$$

Hence, the maximum field is

$$B_{max} = \frac{\mu_0 \epsilon_0 \omega}{2d} r$$

Note that this formula holds at $r = R$ and $r < R$. We could plot this as a function of r, but we can clearly see that it is linear in r.

14 Chapter 33: Electromagnetic Waves

As mentioned in the notes on chapter 32, one of the defining achievements of James Clerk Maxwell's career as a physicist was to show that light is an electromagnetic wave. This unified two fields which were both large areas of science but were previously thought to be unconnected.

Today we understand that electromagnetic waves are much more ubiquitous than was originally thought. Visible light takes up only a very narrow portion of the full range of the electromagnetic spectrum of waves. We are of course partial to visible light because it is the range that our eyes are sensitive to. In figure 37 the full electromagnetic spectrum is shown as well as some of the common uses for different parts.

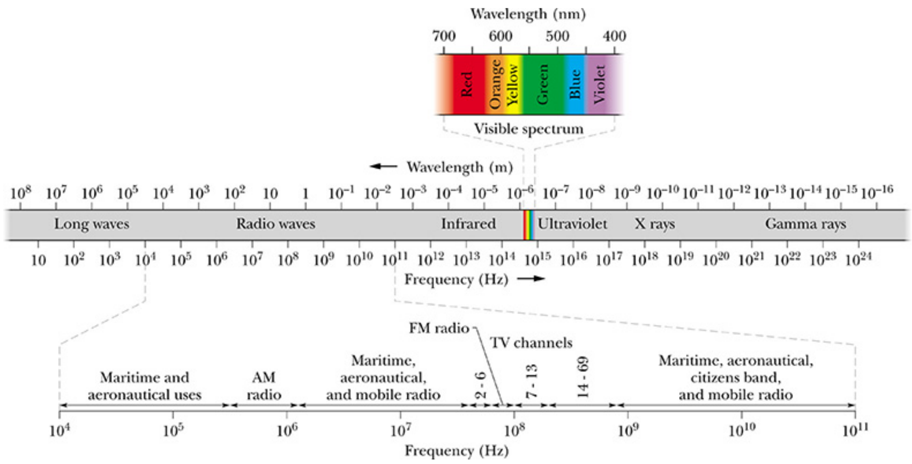

Figure 37: The Electromagnetic Spectrum.

14.1 Traveling Electromagnetic Waves

As we have seen already, changing magnetic fields induce electric fields and changing electric fields induce magnetic fields. The basic idea of an electromagnetic wave is that an oscillating field in an antenna for example, induces a changing electric field which travels off, or radiates, at the speed of light together with a changing magnetic field.

We will discuss the general properties here, but for a mathematical discussion of the specifics one can refer to the textbook. The general structure of an EM wave can be seen in figure 38. The general properties are

- The electric and magnetic waves $\vec{E}$ and $\vec{B}$ are always perpendicular to the direction of travel.
- The electric and magnetic waves are always perpendicular to each other.
- The fields vary sinusoidally in space and time, just like the waves on a string discussed earlier, and the fields are always in phase with each other.

$$E = E_m \sin(kx - \omega t)$$
$$B = B_m \sin(kx - \omega t)$$

- EM waves in vacuum always travel at the same speed, c, which is called the speed of light. This speed is given by the constants that we have been working with all semester.

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{50}$$

- The ratio of the amplitudes of the electric and magnetic waves is fixed in terms of this speed.

$$\frac{E_m}{B_m} = c \tag{51}$$

14.2 Energy and Momentum in EM Waves

Anyone who has been out in the sun for a while or stood next to a campfire at night knows that em waves carry energy. The warmth that we feel in these two cases is the energy from these waves begin deposited in our skin.

The quantity which describes the energy carried by an em wave is the Poynting vector. Poynting is the last name of the physicist who first discussed its properties, but it turns out to be aptly

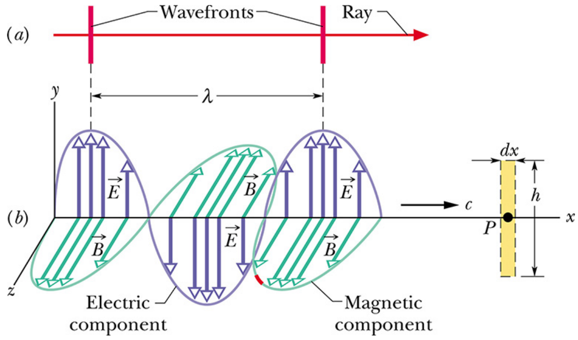

Figure 38: Two Different Representations of an Electromagnetic Wave.

named because the Poynting vector points in the direction that the wave is traveling in. The Poynting vector is defined as

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{52}$$

The direction of the Poynting vector points in the direction that the energy is carried, but the magnitude of the Poynting vector tells us how much power is carried per unit area at that instant.

$$|\vec{S}| = \left(\frac{power}{area}\right)_{inst}$$

Since the E and B fields are related in an EM wave, we can write the magnitude of the Poynting vector as

$$S = \frac{1}{\mu_0}E^2$$

As noted before, this is the instantaneous amount of energy carried by the wave. Since the frequency of em waves is typically very high, we are usually more interested in the average amount of energy carried by the wave. This is given by the rms value of the magnitude of the electric field

$$S_{avg} = \frac{1}{2\mu_0}E_m^2$$

Not only do electromagnetic waves carry energy, but just like particles they also carry momentum. This means that when they are absorbed by objects or reflected by objects, they exert a force. This force is know as radiation pressure and its strength depends on whether the wave is reflected or absorbed.

$$\Delta p = \frac{\Delta U}{c} \qquad \text{Total Absorption}$$

$$\Delta p = \frac{2\Delta U}{c} \qquad \text{Total Reflection}$$

Here ΔU is the energy change of the object which is doing the reflecting or absorbing.

One interesting application of this radiation pressure is to use a space sail to launch long term missions to the outer solar system. The basic idea is to send a small satellite out into space which can expand a very large ($\sim 1\,mile^2$ to be effective) piece of material out to use as a sail. Just like sails on sailboats this sail will catch the solar 'wind' and ride it out into the far reaches of the solar system.

14.3 Polarization

In figure 38 the electric field is shown as oscillating up and down in the plane of the page. We could just as easily have it oscillating in any other direction, and in general most light sources emit many light rays with the electric fields oscillating in all different directions. The direction that the electric field points is known as the polarization of the wave.

We can treat randomly polarized light as being the sum of two waves, one polarized in the x direction and the other polarized in the y direction. If we send this light through a special piece of material called a polarizer, then only the component of the wave which is polarized in the direction of the polarizer will be transmitted. This has two effects, it leaves the output wave polarized in the direction of the polarizer and it reduces the intensity of the wave by 1/2.

$$I = \frac{1}{2} I_0$$

If we send light which is already polarized through one of these polarizers, then only a portion of the light will be transmitted. The amount transmitted now depends on the angle between the polarizers axis and the polarization of the incoming light wave. If the axis of the polarizer is rotated 90° with respect to the light wave, then none of the light will be transmitted. For a general angle between the polarization of the light and the polarizer's axis, the amount of intensity transmitted is given by

$$I = I_0 \cos^2 \theta$$

Running light through a polarizer is not the only way to turn unpolarized light into polarized light. Light is also partially (or fully depending on the angle) polarized when it reflects off of a surface. This is the principle upon which polarized sunglasses work. In particular, at one special angle the reflected light will be completely polarized in the reflecting plane. This angle is called Brewster's angle and is given by

$$\theta_B = \tan^{-1} \frac{n_2}{n_1},$$

where n_1 is the index of refraction of the medium in which the incident and reflected waves travel, and n_2 is the index of refraction off of which the wave reflects. We will describe the meaning of the index of refraction in a minute.

14.4 Reflection and Refraction

In many cases it is convenient to forget completely about the wave nature of light and to treat it as a ray (depicted graphically in figure 38). A ray is like a straight arrow which travels along straight lines unless it intersects an interface. When one of these rays does reach an interface, some of the ray will be transmitted and some of the ray will be reflected.

We describe the the angles of reflection and refraction (the transmitted ray) by angles with respect to the normal to the surface. The angle of reflection is straightforward,

$$\theta_1 = \theta_1'$$

The angle of refraction is not quite as straightforward

$$n_2 \sin \theta_2 = n_1 \sin \theta_1$$

where n_1 and n_2 are called the indices of refraction and are properties of the materials through which the light is traveling.

Notice that this equation tells us that light traveling from a medium of index refraction less than the new medium (air to water for instance), then the outgoing ray will bend in towards the normal. For a ray traveling from a medium of higher index of refraction to a medium of lower index of refraction, the outgoing light ray will bend away from the normal. For this second case, a steep incoming angle can lead to a situation in which Snell's law predicts an angle which is greater than 90°. This situation is called total internal reflection and is given by

$$\theta_c = \sin^{-1} \frac{n_2}{n_1}$$

14.5 Problems

Problem 33.42

In figure 39, unpolarized light is sent into a system of three polarizing sheets, which transmits 0.05 of the initial light intensity. The polarizing direction of the first and third sheets are as angles $\theta_1 = 0°$ and $\theta_3 = 90°$. What are the (a) smaller and (b) larger possible values of angle θ_2 ($< 90°$) for the polarizing direction of sheet 2?

This is actually a very interesting problem because if the middle polarizer were not there, then no light would be transmitted. The first polarizer simply serves to transmit half of the original intensity of the light and to polarize the light in the y direction. The second polarizer will decrease the intensity of the light and rotate the polarization. The third polarizer will have the same effect as the second. We must be careful when using the intensity formula for a polarizer because the angle is defined as the angle between the polarization of the wave and the direction of the polarizer. The third polarizer will therefore have to be relative to the polarization leaving the second.

$$\begin{aligned} I &= I_0 \left(\frac{1}{2}\right) \left(\cos^2\theta_2\right) \left(\cos^2(\theta_3 - \theta_2)\right) \\ &= \frac{I_0}{2} \cos^2\theta_2 \cos^2(90° - \theta_2) \\ &= \frac{I_0}{2} \cos^2\theta_2 \sin^2\theta_2 \\ &= \frac{I_0}{8} \sin^2(2\theta_2) \end{aligned}$$

Hence,

$$\theta_2 = \frac{1}{2} \sin^{-1}\left(\sqrt{\frac{8I}{I_0}}\right)$$

Plugging in for I/I_0 we find that the two values of θ_2 less than 90° are $\theta_2 = 19.6°$ and $\theta_2 = 70.4°$.

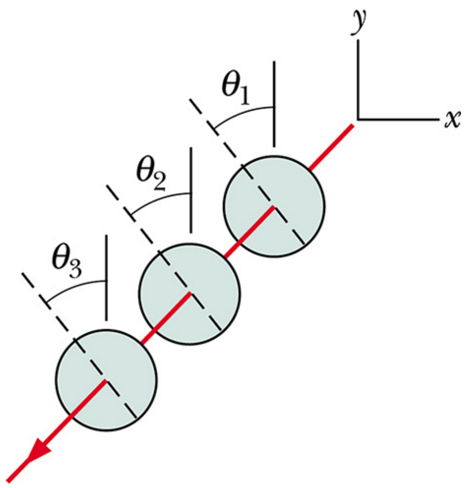

Figure 39: Figure for Problem 33.42.

Problem 33.53

In figure 40, light is incident ant an angle $\theta_1 = 40.1°$ on a boundary between two transparent materials. Some of the light travels down through the next three layers of transparent materials, while some of it reflects upward and then escapes into the air. If $n_1 = 1.30$, $n_2 = 1.40$, $n_3 = 1.32$, and $n_4 = 1.45$, what is the value of (a) θ_5 and (b) θ_4?

Lets begin by calculating θ_5. Since a ray reflects with the same angle that it was incident with, the ray will hit the other surface at the same angle of incidence, θ_1. Hence, when it leaves the surface it will refract out into the air with an angle given by Snell's law. If we assume that the index of refraction is 1, then

$$\sin\theta_5 = n_1 \sin\theta_1 \quad \Rightarrow \quad \theta_5 = \sin^{-1}(n_1 \sin\theta_1) = 56.9°$$

Although the calculation of θ_4 at first looks much more complex, it will turn out to be just as easy. Snell's law for the first interface gives

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

and for the second interface it gives

$$n_2 \sin\theta_2 = n_3 \sin\theta_2$$

The important thing to notice now is that the θ_2 is the same in bot equations because the two interfaces are parallel to each other. Hence, we can combine these two formulas as well as the one for the final interface to find

$$n_1 \sin\theta_1 = n_4 \sin\theta_4 \quad \Rightarrow \quad \theta_4 = \frac{1}{n_4}\sin^{-1}(n_1 \sin\theta_1) = 39.21°$$

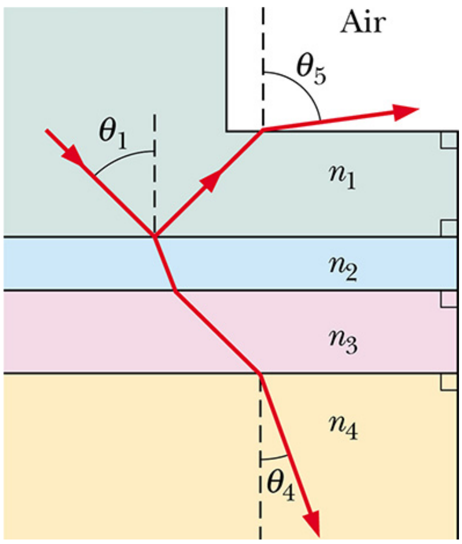

Figure 40: Figure for Problem 33.53.

Problem 33.59

In figure 41, light initially in material 1 refracts into material 2, crosses that material, and is then incident at the critical angle on the interface between materials 2 and 3. The indexes of refraction are $n_1 = 1.60$, $n_2 = 1.40$, and $n_3 = 1.20$. (a) What is angle θ? (b) If θ is increased, is there refraction of light into material 3?

Since the light is incident at the critical angle in material 2, the angle with respect to the normal in material 2 must be

$$\theta = \arcsin\left(\frac{n_3}{n_2}\right)$$

We need to relate this to the angle with respect to the normal at the interface between 1 and 2. Since these two interfaces are at right angles to each other, we can use that fact that the 3 angles of a triangle sum to 180°. Hence, the angle with respect to the normal at the other interface is

$$\theta_2 = 90° - \theta_c = 90° - \arcsin\left(\frac{n_3}{n_2}\right)$$

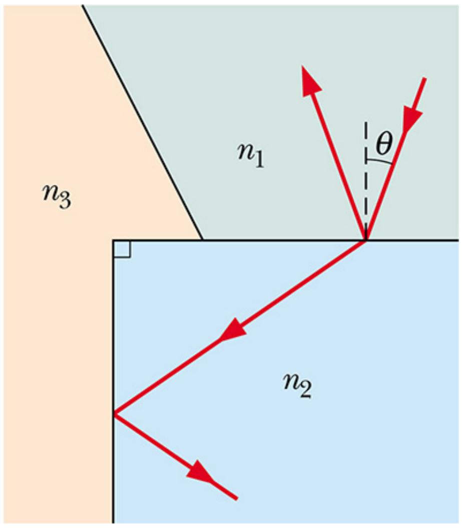

Figure 41: Figure for Problem 33.59.

Finally, applying Snell's law at this interface gives us the angle of interest.

$$n_1 \sin\theta = n_2 \sin\theta_2 \quad \Rightarrow \quad \theta = \arcsin\left[\frac{n_2}{n_1}\sin\left(90^\circ - \arcsin\left(\frac{n_3}{n_2}\right)\right)\right] = 26.8^\circ$$

For part b, if we increase the angle θ, then the ray will hit the second interface at an angle smaller than the critical angle and some of the light will be refracted through.

15 Chapter 34: Images

In the last chapter we discussed a few of the general principles of optics. In this chapter we will go a little deeper into the topic of geometrical optics and discuss the principles behind images. In discussing images, we need to make a distinction between two different types.

Virtual images are images which are perceived in the brain but which can not be projected onto a screen. An example of a virtual image is the image that we see of ourselves when we look into a mirror. It appears to us as if there is a copy of ourselves standing behind the mirror. If we placed a screen at the point where our eyes were when we saw this image, however we would not be able to see the image on the screen.

Real images are images which can be projected onto a screen. An example of a real image is the picture that we see on the screen at the movie theater. A system of lenses in the projector creates a real image which is then projected onto the screen. We perceive these images by placing a screen at the focal point of the image, but if we were to place our eyes at that point the image would seem fuzzy and incomprehensible.

15.1 Plane Mirrors

To begin exploring these ideas of objects and their images, lets consider a simple plane mirror. When dealing with mirrors our goal will be to relate the distance from the object, p, to the distance from the image, i (both are measured from the mirror). From this information we can determine the magnification of the mirror.

For a plane mirror we know already that the angle of reflection is equal to the angle of incidence. If we apply this simple principle to many rays coming from a point object O (see figure 42), then we can clearly see that the object appears to to observer seeing the reflected rays as if it were behind the mirror. To determine the image distance we choose two rays (see figure 43) and work out the required geometry. After doing so, we find

$$p = -i.$$

Note that the negative sign implies that the image is a virtual image and not a real one.

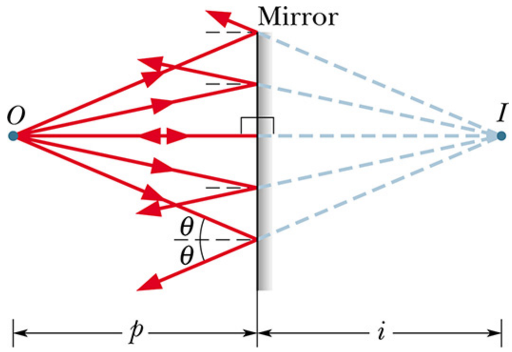

Figure 42: The Apparent Image Behind the Mirror.

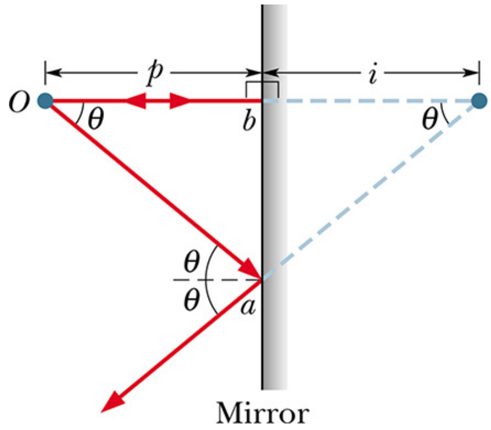

Figure 43: Using Rays to Locate the Image

15.2 Spherical Mirrors

Spherical mirrors are a little more complicated than plane mirrors. Our goal is still to relate the object distance to the image distance, but we must first talk about how to tell one spherical mirror from another. A spherical mirror is essentially a cutout of a big sphere (hence the name), so its major distinguishing characteristic is its radius of curvature. For concave mirrors, we will say that the radius of curvature is positive and for convex mirrors we will sat that the radius of curvature is negative.

The focal point of a spherical mirror is given by

$$f = \frac{1}{2}r$$

and takes the same sign as the radius of curvature. The focal point means different things for convex and concave mirrors. For concave mirrors, all rays coming in parallel to the central axis will reflect off of the mirror through the focal point. For convex mirrors, all rays coming in parallel to the central axis will reflect as if they came from the focal point. For clarification, see figure 44

For a spherical mirror, the object distance is related to the image distance by the focal length of the mirror

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f}. \tag{53}$$

One other special property of spherical mirrors is that the image will not necessarily appear the same size as the original object. This property is exploited in large telescopes to magnify distant objects. The magnification of a mirror is given by

$$m = -\frac{i}{p} \qquad \& \qquad |m| = \frac{h'}{h},$$

where h' is the height of the image and h is the height of the object. Note that a negative magnification implies that the object is inverted.

We can learn much about the properties of spherical mirrors by ray tracing. If done carefully, ray tracing can even be used to find the magnification and image distance of a spherical mirror. There are four points which one must know in order to use ray tracing on spherical mirrors.

- A ray that is initially parallel to the central axis reflects through the focal point (ray 1 in figure 44).
- A ray that reflects from the mirror after passing through the focal point emerges parallel to the central axis (ray 2 in figure 44).

- A ray that reflects from the mirror after passing through the center of curvature C returns along itself (ray 3 in figure 44).

- A ray that reflects from teh mirror at point c is reflected symmetrically about that axis (ray 4 in figure 44).

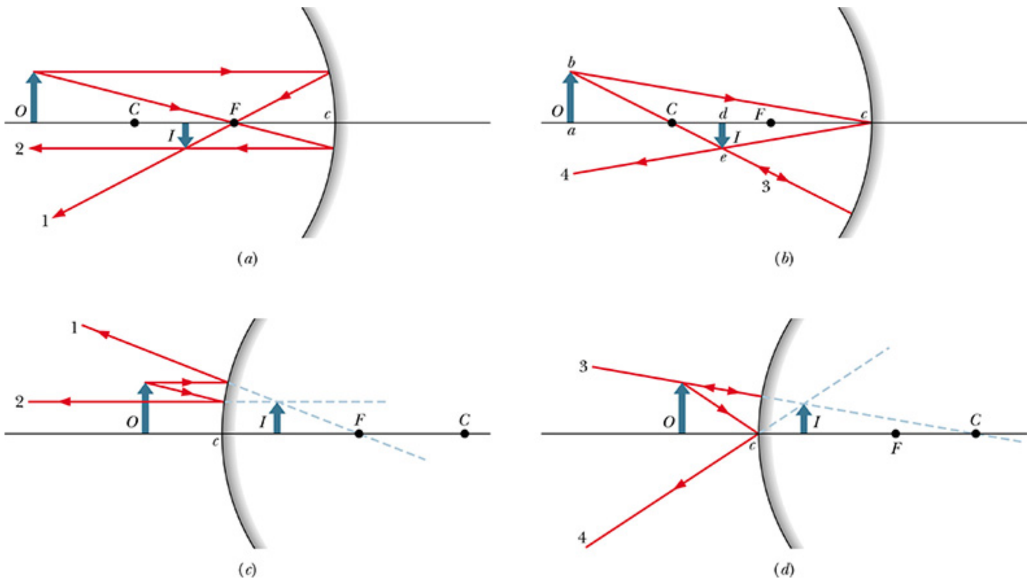

Figure 44: Tracing Rays for Spherical Mirrors

15.3 Thin Lenses

In many ways thin lenses are just like mirrors. They have the same relationship between their object and image distances

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f},$$

and most of the ray tracing procedures are exactly parallel to the spherical mirror ray tracing rules. The focal length of a lens is somewhat more complicated

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_2} - \frac{1}{r_2}\right)$$

where r_1 is the radius to the side of the lens closest to the object and r_2 is the radius of curvature further from the object. Note that the we must be careful to use the proper sign with the radii of curvature. As has been the trend with signs in the rest of this chapter, if the center of the radius is on the opposite side of the lens than the object then it is negative and vice versa if it is on the same side. Also note that a thin lens has two focal points, one on each side of the lens but both the same distance away.

Similar to the convex and concave mirrors of the last section, we will see that lenses come in a converging and a diverging variety. In figure 45 we can see why the two types of lenses act differently.

The ray tracing procedures for lenses are very similar to those for spherical mirrors and the ray tracing technique can be employed with the same utility.

- A ray that is initially parallel to the central axis of the lens will pass through (or look as if it passed through) the focal point on the far (near) side of the lens for a converging (diverging) lens.

- A ray that initially pass through (or is heading towards) the focal point on the near (far) side of the lens will emerge parallel to the central axis for a converging (diverging) lens.

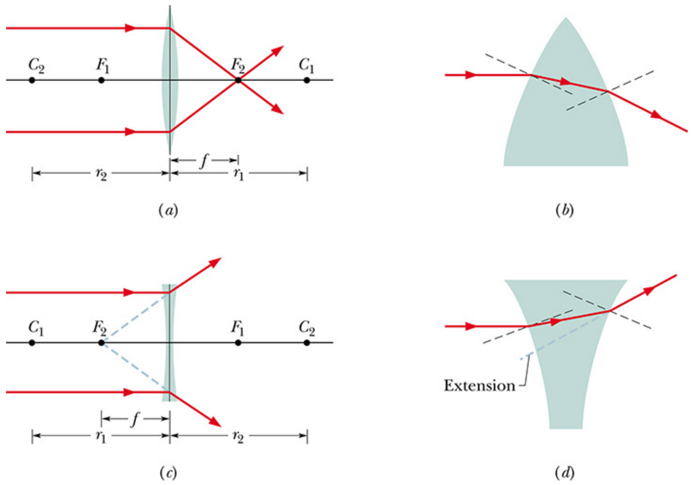

Figure 45: Converging and Diverging Lenses

- A ray that is initially heading toward the center of the lens will emerge from the lens with no change in its direction for both types of lens.

16 Chapter 35: Interference

We encountered the idea of interference in our first semester physics course when discussing the concept of waves. The basic idea behind interference is that waves, in general, obey the principle of superposition and when they combine can add up to produce a larger wave or cancel out to produce a smaller wave (or none at all). This concept was fairly simple in wave mechanics because we mostly dealt with waves traveling along one dimension like on a string. It is a bit more complicated to apply to optics because optical waves travel in three dimensions. This complication does lead to some rather amazing results however.

Before we delve into some of these interesting properties, lets discuss an interesting way of thinking about traveling light waves. *Hyugens' Principle* is a rather simple concept which gives us an helpful way of thinking about how waves spread as they travel. The basic idea is that we treat each point on the phase front of the light wave as emitting a small spherical wave at any arbitrary point in time. To find out where the wave is later on, we simply take the surface tangent to all of these secondary wavelets (see figure 46).

This concept naturally explains the idea of diffraction of light around edges and through small openings. We can also use this idea to explain Snell's law and to give a physical interpretation to the index of refraction. When applied to Snell's law, we naturally see that the index of refraction of a medium is simply proportional to the speed of light in that medium.

$$n = \frac{c}{v}$$

With this interpretation of the index of refraction we can also deduce some other properties of light traveling through a medium. If light travels more slowly in one medium than another, then the light will emerge from the medium at a later time than if it had not gone through the medium. This shows up as a phase difference between the two beams and can lead to interference.

16.1 Diffraction and the Double Slit Experiment

By applying Hyugens' principle to a wave traveling through a small opening, we can naturally see how the wave would spread out later it emerges from the opening. On the other side of the opening each wavefront emits tiny spherical wavelets, but when we go to find the tangent at a later time, the wavelets which were cutoff when the wave traveled through the opening are missing. Hence, the new wave fronts are curved and traveling outward like growing spheres.

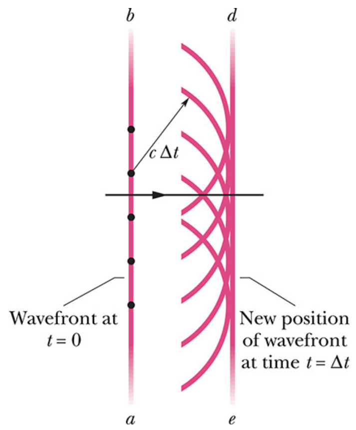

Figure 46: Hyugens' Principle

For a the double slit experiment we let a single monochromatic light wave travel through two side by side slits. The waves spread out from both slits in expanding spheres and hit a screen screen which is sitting in front of them. It takes the two waves from each slit a different amount of time to get to each point on the screen and the waves therefore come in with different phases. At some points on the screen the waves will cancel each other out, and at some points they will interfere constructively. We can calculate the spacing between the bright and dark fringes by considering figure 47.

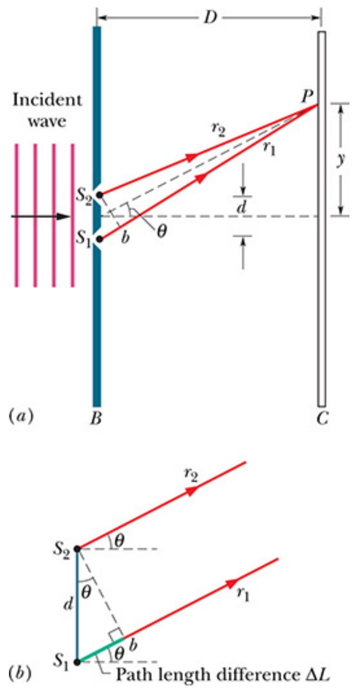

Figure 47: Young's Double Slit Experiment

Since the screen is very far away from the two slits as compared to the distance between the slits, we can approximate the rays going to a particular point as being parallel. The geometry is then fairly simple. If the spacing between the two slits is d, then the relative path length difference between the two waves as a function of angle is

$$\Delta L = d \sin \theta.$$

If the two waves reach the screen half of a wavelength apart, then they will interfere destructively and if they reach with a full wavelength difference (or no difference) then they will interfere constructively. We can use this to write down the conditions for bright fringes and dark fringes. Bright fringes must satisfy

$$d \sin\theta = m\lambda,$$

where m is any integer. Dark fringes must satisfy

$$d \sin\theta = (m + \tfrac{1}{2})\lambda$$

where m is again an integer.

The above equations are satisfied for any m, but in practice only those close to zero are important because the intensity of successive bright fringes falls off rapidly. The intensity is as a function of θ is given by

$$I = 4I_0 \cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right),$$

where I_0 is the intensity of the light before the slits.

16.2 Interference from Thin Films

One other important example of interference in optical systems is from thin films. As we saw in the previous two chapters, when light encounters a medium with a different index of refraction some of the light gets transmitted and some of the light gets reflected. If the light is traveling into and then out of the medium, then some of the light will be reflected at the first interface and some of the light will be reflected at the second interface (see figure 48). The two reflected rays can meet up at the original interface and interfere destructively or constructively.

This issue is complicated by one nuance. When light reflects off of a surface it can pick up a phase shift. The rule for this phase shift is as follows: if the light reflects off of a medium of lower index of refraction, the phase shift is zero; if the light reflects off of a medium of higher index of refraction, the phase shift is $\frac{1}{2}$ of a wavelength.

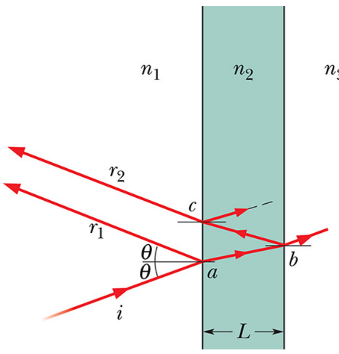

Figure 48: Interference in Thin Films

16.3 Problems

Problem 35.23: In figure 49, two isotropic point sources of light (S_1 and S_2) are separated by distance $2.7\,\mu m$ along a y axis and emit in phase at wavelength $900\,nm$ and at the same amplitude. A light detector is located at point P at coordinate x_P on the x axis. What is the greatest value of x_P at which the detected light is minimum due to destructive interference?

The path length traveled by a wave emitted by S_1 is $x_1 = x_P$ while the path length traveled by a wave from S_2 is $x_2 = \sqrt{d^2 + x_p^2}$. The difference in the path length traveled by the two waves is then obtained by taking the difference of these two equations. We can convert this to the number of wavelengths of difference by dividing by the wavelength of the light.

$$\Delta = \frac{x_2 - x_1}{\lambda} = \frac{\sqrt{x_P^2 + d^2} - x_P}{\lambda}$$

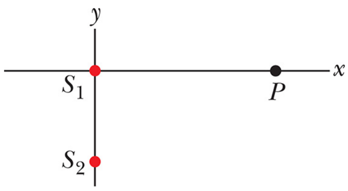

Figure 49: Figure for Problem 35.23

We now need to use some physical reasoning to figure out when the two will interfere destructively at the largest x_P. Notice that if x_P goes off to infinity, then the difference between the two goes to zero. Hence, the maximum x_P which leads to destructive interference is when the difference is $\frac{1}{2}$.

$$\frac{\sqrt{x_P^2 + d^2} - x_P}{\lambda} = \frac{1}{2}$$

This is a quadratic equation in x_P that we can easily solve.

$$\begin{aligned} x_P^2 + d^2 &= \left(x_P + \frac{\lambda}{2}\right)^2 \\ \Rightarrow \qquad x_P^2 + d^2 &= x_P^2 + \lambda x_P + \frac{\lambda^2}{4} \\ \Rightarrow \qquad \lambda x_P &= d^2 - \frac{\lambda^2}{4} \\ \Rightarrow \qquad x_P &= \frac{d^2}{\lambda} - \frac{\lambda}{4} \end{aligned}$$

Hence, our solution is $x_P = 7.87\,\mu m$.

Problem 35.75: Figure 50a shows a lens with radius of curvature R lying on a flat glass plate and illuminated from above by light with wavelength λ. Figure 50b a photograph taken from above the lens) shows that circular interference fringes (called Newton's rings) appear, associated with the variable thickness d of the air film between the lens and the plate. Find the radii r of the interference maxima assuming $r/R \ll 1$.

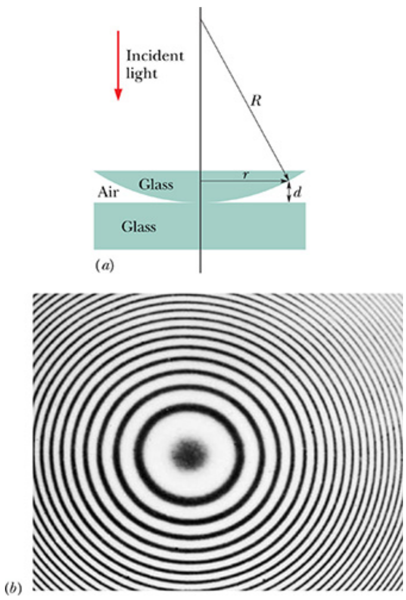

Figure 50: Figure for Problem 35.75

Our first goal is to set out finding d in terms of r and R since this will control the interference upon reflection. Using a little trigonometry we can see that the hypotenuse of the triangle formed by r and R in the figure is $h = \sqrt{R^2 - r^2}$. Hence, d is given by

$$d = R - \sqrt{R^2 - r^2} \qquad \Rightarrow \qquad r = \sqrt{2dR - d^2}$$

Since the problem tells us that $r/R \ll 1$ we can treat the light as if it were all coming straight into the lens/plate setup and reflecting straight back (i.e. we will neglect the fact that the camera is a single point). The other approximation that we will make is that since the radius of curvature of the lens is very small we can neglect any refraction effects and treat the rays as if they were reflected straight back and not at an angle.

With these approximations, the interference upon reflection is simply a function of d just like in thin films. We must be careful in this case because one of the reflections will pick up a phase shift relative to the other. The first reflection off of the air has 0 phase shift while the second reflection will pick up half of a wavelength. Hence, the condition for constructive interference is that $2d$ be a half integer multiple of λ.

$$2d = \left(m + \frac{1}{2}\right)\lambda \qquad \Rightarrow \qquad r = \sqrt{\left(m + \frac{1}{2}\right)\lambda R - \frac{1}{4}\left(m + \frac{1}{2}\right)^2 \lambda^2}$$