Last time

- Potential energy

This time

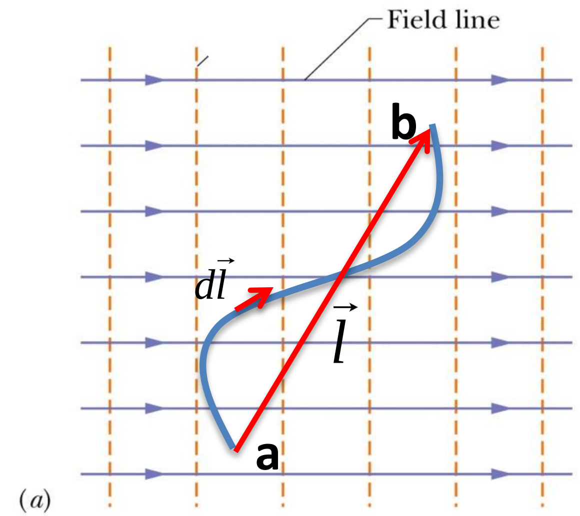
- Potential energy of a point charge in a constant electric field.
- Potential energy between two point charges, one moving along the electric field lines of the other
- Potential energy between two point charges, one moving along an arbitrary path in electric field of the other
- General results for potential energy of a static electric field.

The simplest possible case:

$$\vec{E} = \text{constant}$$

$$W = -q_0 \vec{E} \bullet \int_a^b d\vec{l}$$

$$W = -q_0 \vec{E} \bullet \vec{l}$$



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$\vec{l}$ is the displacement vector with its tail and tip at points a and b, respectively.

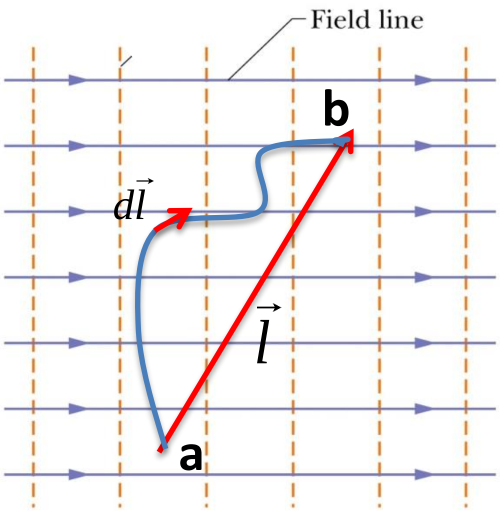
How about a different path?

The simplest possible case (a different path):

$$\vec{E} = \text{constant}$$

$$W = -q_0 \vec{E} \bullet \int_a^b d\vec{l}$$

$$W = -q_0 \vec{E} \bullet \vec{l}$$



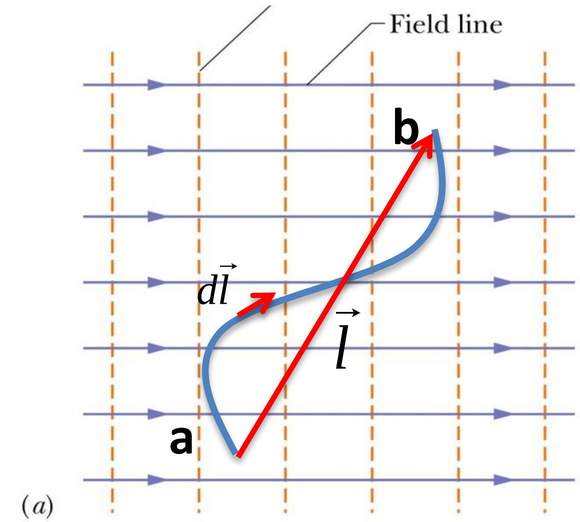
(a)

halliday_10e_fig_24_05a

Work done is the same!

$$W = -q_0 \vec{E} \bullet \vec{l}$$

This implies that the work done is independent of the path chosen as long as start and end points are the same.



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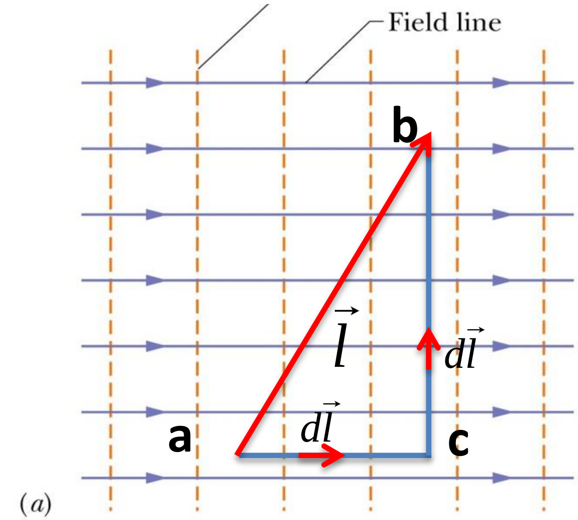
What if we chose the path from **a** to **c** and then from **c** to **b**? Work done is the same as any other path connecting a to b.

This is a very convenient path. Why?

$$W = -q_0 \int_{acb} \vec{E} \bullet d\vec{l}$$

$$W = -q_0 \int_{ac} \vec{E} \bullet d\vec{l} - q_0 \int_{cb} \vec{E} \bullet d\vec{l}$$

For the first integral electric field is parallel to the displacement vector.



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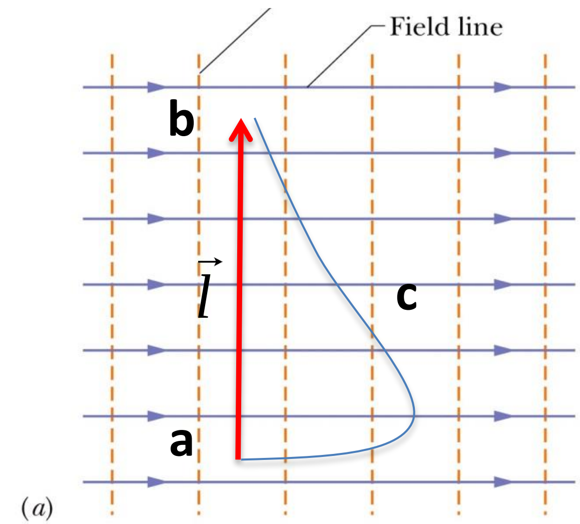
For the second integral electric field is perpendicular to the displacement vector and does not contribute to the overall work.

Only the component of $\vec{l}$ which is parallel to the electric field direction will result in non-zero work. The component which is perpendicular to electric field results in no work.

What if we chose the path from **a** to **c** and then from **c** to **b** with no net displacement parallel to electric field.

$$W = -q_0 \int_{acb} \vec{E} \bullet d\vec{l} = \vec{E} \bullet \vec{l} = 0$$



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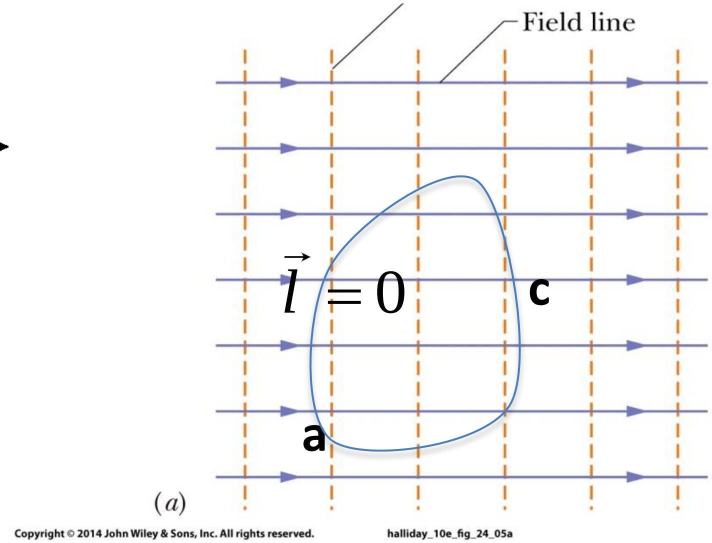
What if we chose the path from **a** to **c** and back to **a**?

$$W = -q_0 \int_{acb} \vec{E} \bullet d\vec{l} = -q_0 \vec{E} \bullet \int_{aca} d\vec{l}$$



$$\int_{aca} d\vec{l} = 0$$

$$W = 0$$

Here net displacement is zero.

If work is independent of how we go from a point **a** to a point **b,** we should then chose a path that is most convenient to calculate work done.

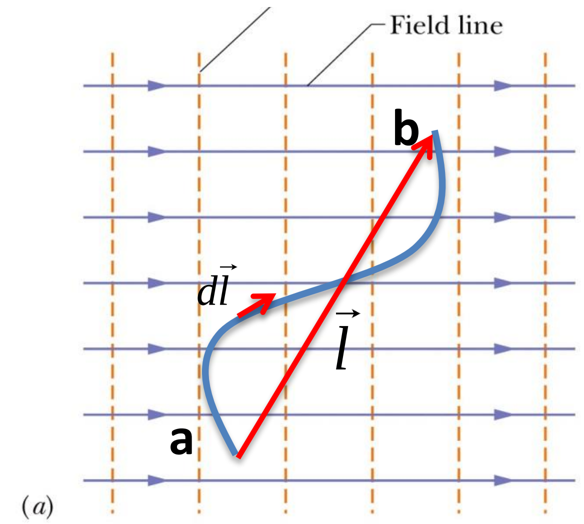
$$W_{a \to b} = -q_0 \int_{\text{Preferably a convenient path}} \vec{E} \bullet d\vec{l}$$

Potential energy

$$W = -q_0 \vec{E} \bullet \vec{l}$$

$$\vec{l} = \vec{r}_b - \vec{r}_a$$

$\vec{r}_a$ and $\vec{r}_b$ are vectors which define the position for points a and b.



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$$W = -q_0 \vec{E} \bullet (\vec{r}_b - \vec{r}_a)$$

Define the potential energy as

$$U = -q_0 \vec{E} \bullet \vec{r}$$

$$W = (U_b - U_a)$$

The work required to move a + point charge q_0 from point a to b equals the change in potential energy of q_0.

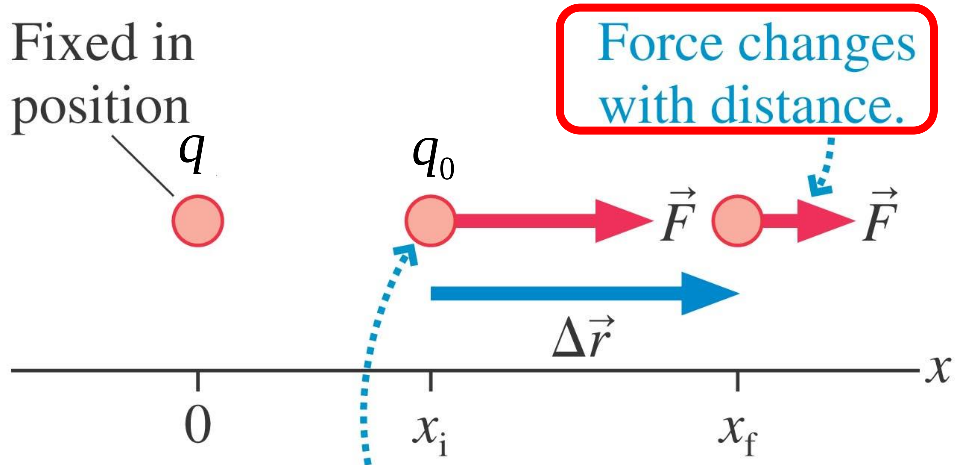
A more complicated case

Finding potential energy of a $+q_0$ in electric field of another point charge q (more building blocks)

Moving along the field lines.

$$\vec{F} = q_0 \vec{E}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

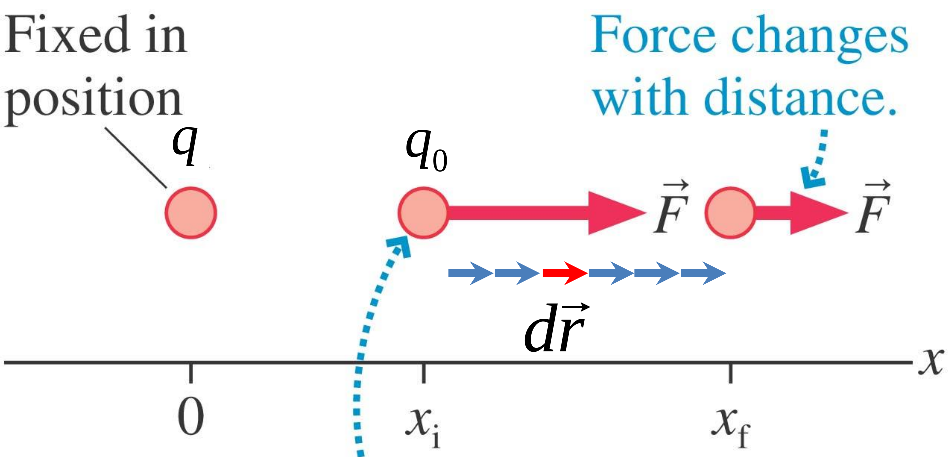


q_1 does work on q_0 as q_0 moves from x_i to x_f.

The field is **not** uniform, therefore the force is **not** constant over the displacement $\Delta\vec{r}$. We cannot simply multiply force by displacement to obtain the work done.

Finding Potential Energy of two point charges (more building blocks)

Moving along the field lines.



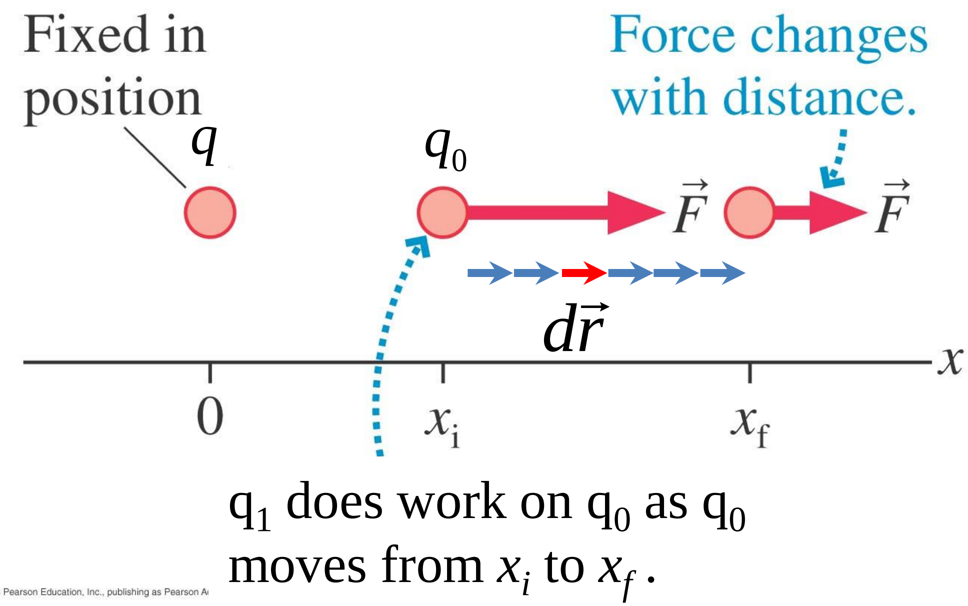
q_1 does work on q_0 as q_0 moves from x_i to x_f.

Break the displacement $\Delta\vec{r}$ into many tiny $d\vec{r} = dr\hat{r}$ displacements.

$\vec{F}$ is essentially constant over such a small displacement, so the work done on q_0 in **each** displacement is $\vec{F} \bullet d\vec{r}$.

Finding Potential Energy of two point charges (more building blocks)

The total work is the sum of all the little bits of work:



q_1 does work on q_0 as q_0 moves from x_i to x_f .

$$W_{i\to f} = -\int_{r_i}^{r_f} \vec{F} \bullet d\vec{r}$$

$$W_{i\to f} = -\int_{r_i}^{r_f} \left(F\hat{r}\right)\bullet\left(dr\hat{r}\right) = -\int_{r_i}^{r_f} F dr$$

$$W_{i\to f} = -\int_{r_i}^{r_f} \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r^2} dr$$

Finding Potential Energy of two point charges (more building blocks)

Work done **by electric force**:

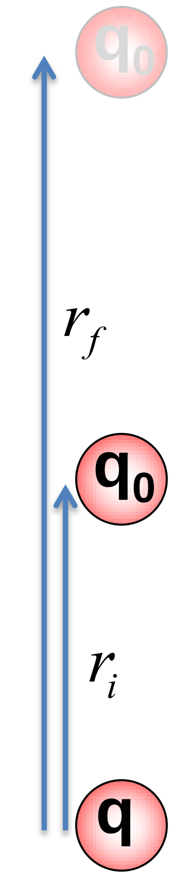
$$W_{i\to f} = -\int_{r_i}^{r_f} \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r^2}\,dr = -\frac{1}{4\pi\varepsilon_0} q_0 q \int_{r_i}^{r_f} r^{-2} dr$$

Recall from integral calculus

$$\int_{x_i}^{x_f} x^n\,dx = \frac{1}{n+1}x^{n+1}\Big|_{x_i}^{x_f} = \frac{1}{n+1}\left(x_f^{n+1} - x_i^{n+1}\right)$$

In our case, let x → r, then we have

$$W_{i\to f} = -\frac{1}{4\pi\varepsilon_0} q_0 q \int_{r_i}^{r_f} r^{-2} dr \;= -\frac{1}{4\pi\varepsilon_0} q_0 q \left(\frac{1}{-2+1} r^{-2+1}\right)\Big|_{r_i}^{r_f}$$



q held fixed

Finding Potential Energy of two point charges (more building blocks)

$$W_{i\to f} = \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r}\Bigg|_{r_i}^{r_f}$$

$$W_{i\to f} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r_f} - \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r_i}\right)$$

$$W_{i\to f} = (U_f - U_i)$$

Again, the work done is independent of the path chosen as long as start and end points are the same.

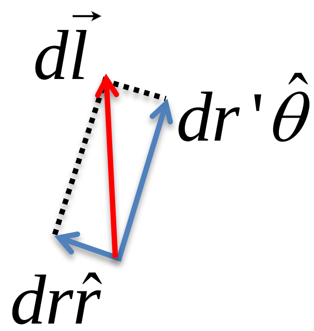
Then the potential energy of two point charges a distance r apart is

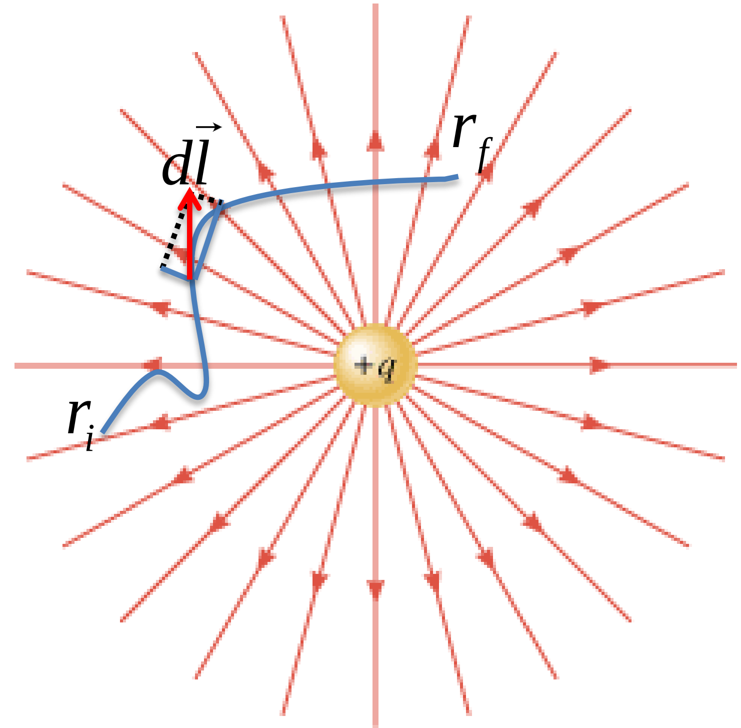
$$\boxed{U_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} \cancel{+U_0}}$$

(1) There is a U_0, but we normally set it to zero.
(2) The potential energy of two charges an infinite distance apart ($r = \infty$) is zero.

An even more complicated case!

Moving along an arbitrary path.





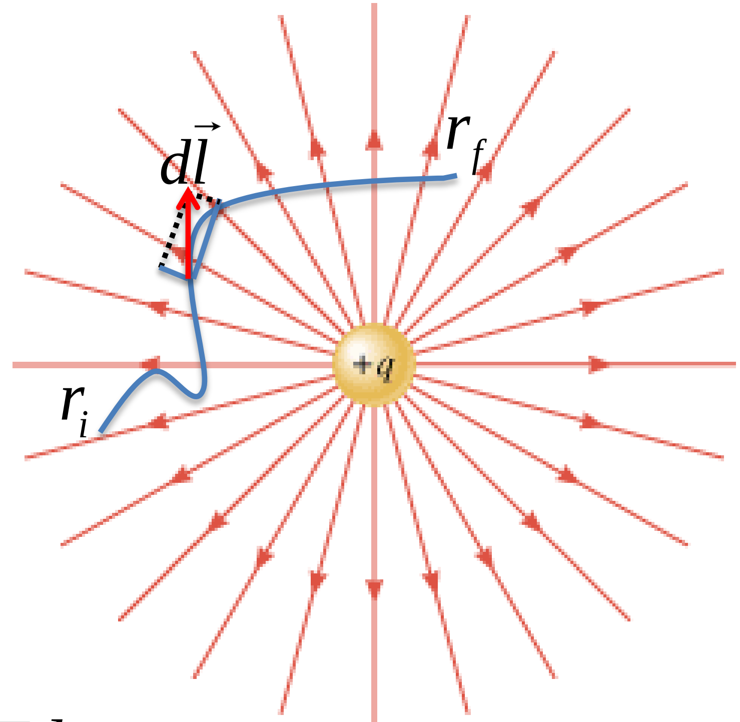
(b)

Find the components of the infinitesimal displacement vector in the direction of the field lines (radial) and the direction which is perpendicular to the radial direction.

$$d\vec{l} = dr\hat{r} + dr'\hat{\theta} \qquad \hat{r} \bullet \hat{\theta} = 0 \qquad dl = \sqrt{(dr)^2 + (dr')^2}$$

Finding Potential Energy of two point charges (more building blocks)

The total work is the sum of all the little bits of work:



(b)

$$W_{i\to f} = -\int_{r_i}^{r_f} \vec{F}\bullet d\vec{r}$$

$$W_{i\to f} = -\int_{r_i}^{r_f} \left(F\hat{r}\right)\bullet\left(dr\hat{r} + dr'\hat{\theta}\right) = -\int_{r_i}^{r_f} F dr$$

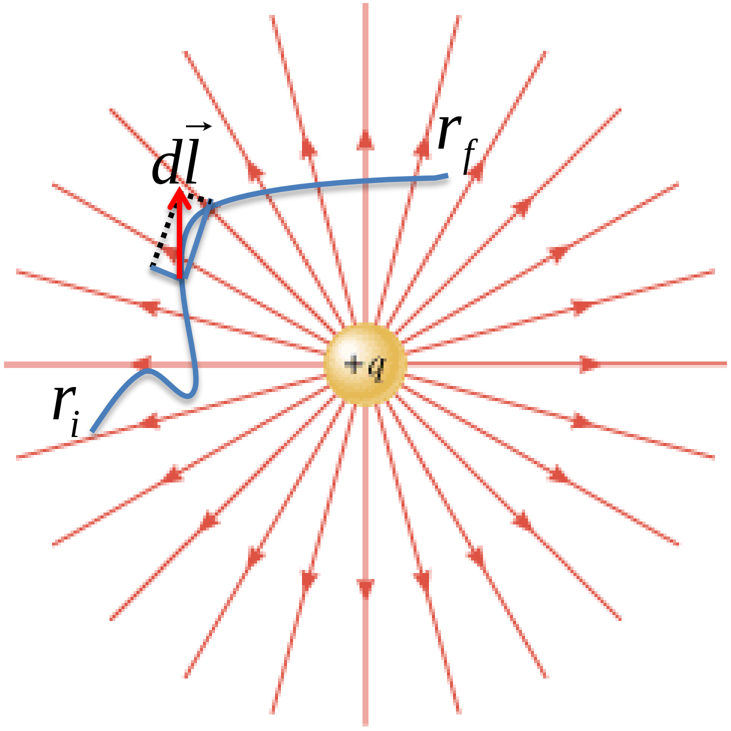
$$W_{i\to f} = -\int_{r_i}^{r_f} \overbrace{\frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r^2}} dr$$

Finding Potential Energy of two point charges (more building blocks)

$$W_{i\to f} = \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r}\bigg|_{r_i}^{r_f}$$

$$W_{i\to f} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r_f} - \frac{1}{4\pi\varepsilon_0}\frac{q_0 q}{r_i}\right)$$

$$W_{i\to f} = (U_f - U_i)$$



(b)

Again, the work done is independent of the path chosen as long as start and end points are the same.

The Coulomb force due to a point is a conservative force (independent of path taken).

Therefore the electric field due to a point charge is a conservative field like the gravitational field.

By superposition principle the electric field due to a collection of point charges is also a conservative field.

Because an arbitrary charges distribution is considered to be a collection of a large number of infinitesimal point charges, by superposition principle the electric field for an arbitrary charge distribution is also a conservative field.