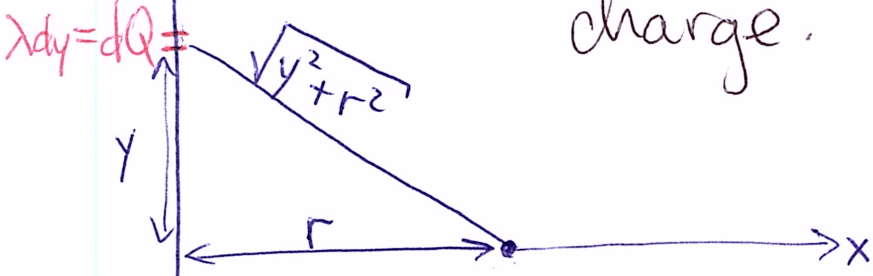
Potential of a line of charge.



$$dV = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dy}{\sqrt{y^2+r^2}}$$

(Potential due to dQ. Note: this assumes V=0 at infinity)

$$V(r) = \int_{-\infty}^{\infty} \frac{1}{4\pi\epsilon_0}\frac{\lambda\, dy}{\sqrt{y^2+r^2}}$$

(The integrand is symmetric, so integral from -infinity to +infinity is twice the integral from 0 to +infinity)

$$= \frac{2\lambda}{4\pi\epsilon_0}\int_0^{\infty}\frac{dy}{\sqrt{y^2+r^2}} = \frac{2\lambda}{4\pi\epsilon_0}\int_0^{\infty}\frac{dy}{r\sqrt{1+(y/r)^2}}$$

from calculus: $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} = \ln\left(\sqrt{1+x^2}+x\right)$

$$\left(x = \frac{y}{r} \Rightarrow dx = \frac{dy}{r}\right)$$

(substitute X in place of y/r)

$$V(r) = \frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{\sqrt{1+(y/r)^2}+y/r}{1}\right)$$

(limit as y goes to infinity)

$$V(r) = \frac{2\lambda}{4\pi\epsilon_0}\ln\left(\lim_{y\to\infty}\left(\sqrt{1+(y/r)^2}+y/r\right)\right)$$

(This gives an infinite answer regardless of the value of r. Fix: change the zero of V by adding constant)

$$V(r) = \frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{y}{r}\sqrt{1+(r/y)^2}+y/r\right)+V_0$$

(Factor out the divergent piece (y/r) and for the well behaved piece in square brackets, set y = infinity)

$$V(r) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{y}{r}\left[1+\sqrt{1+(r/y)^2}\right]\right)+V_0$$

$$V(r) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{2y}{r}\right) \overbrace{- \frac{\lambda}{2\pi\epsilon_0}\ln(2y)}^{V_o}$$

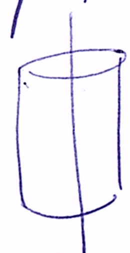
(choose the constant Vo such that the infinity cancels out)

$$V(r) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\cancel{2y}}{r}\frac{1}{\cancel{2y}}\right)$$

(After this convoluted process, we arrive at a useful expression.)

$$\boxed{V(r) = \frac{-\lambda}{2\pi\epsilon_0}\ln(r)} \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 r} = -\frac{\partial V}{\partial r}$$

This was horrible. Instead: easy to use Gauss' Law to find $E = \frac{\lambda}{2\pi\epsilon_0 r}$



$$\oint \vec{E}\cdot d\vec{A} = E\underbrace{A}_{2\pi r L} = \frac{\overbrace{q_{enc}}^{\lambda L}}{\epsilon_0}$$

(After using Gauss' Law to find E, use the link between E and V to find ΔV)

$$\Delta V_{AB} = -\int_A^B \vec{E}\cdot d\vec{r}$$ radially away from the line.

$$= -\int_A^B \frac{\lambda}{2\pi\epsilon_0 r}\,dr = \frac{-\lambda}{2\pi\epsilon_0}\ln r\Big|_A^B$$

$$\boxed{\Delta V_{AB} = \frac{-\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_B}{r_A}\right)}$$

(Since this gives the potential DIFFERENCE between points A and B, we did not have to worry about where V=0. This was a three-line calculation)