So far:

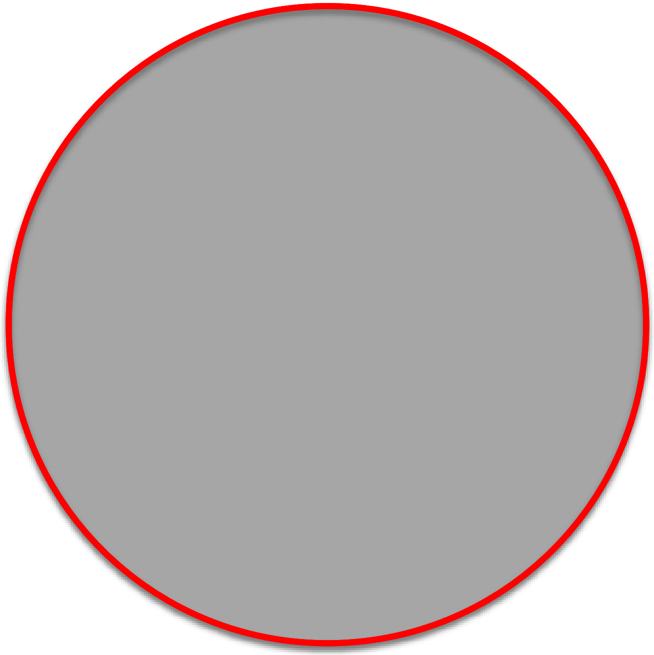
- Electric potential of a line of charge
- Calculating E using V
- Using symmetries to simplify V in some cases

To be continued:

- The power of useful models: insulating spherical shell
- Potential between two parallel charged plates
- Capacitance as a geometric quantity

Model of a charged insulating shell

Solid ball conductor with excess charge +Q evenly distributed on its surface



Hollow insulating shell with charge +Q uniformly distributed on its surface

Both objects have the exact same distribution of charges

TopHat Question

Consider a uniformly charged insulating shell with a diameter of 1.0 m and a total charge of –300 nC. What is the **magnitude** of the **electric field** at point P a distance 30 cm outside the surface?

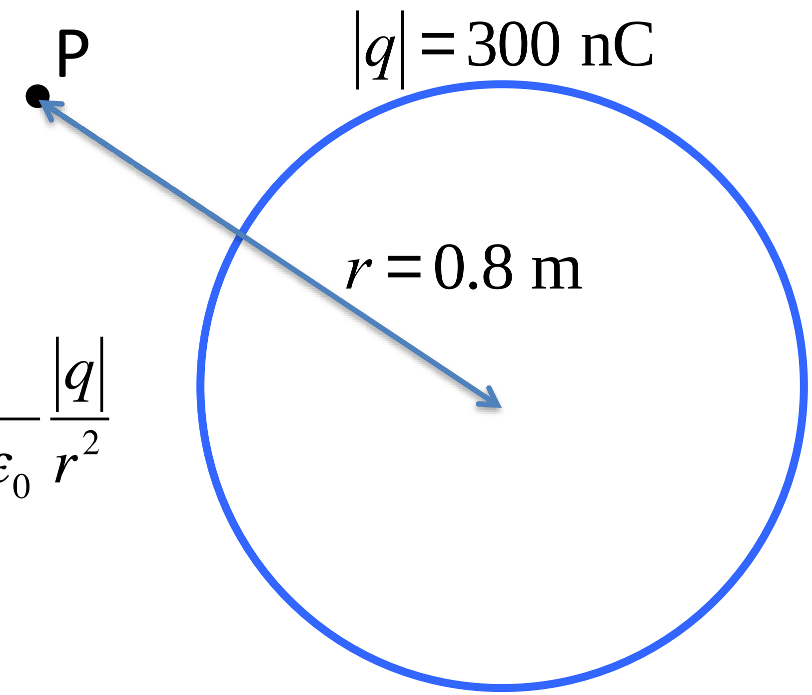
A. 3400 V/m

B. 4200 V/m

C. 9000 V/m

D. 30000 V/m

$$\left|\vec{E}\right| = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{r^2}$$



TopHat Question

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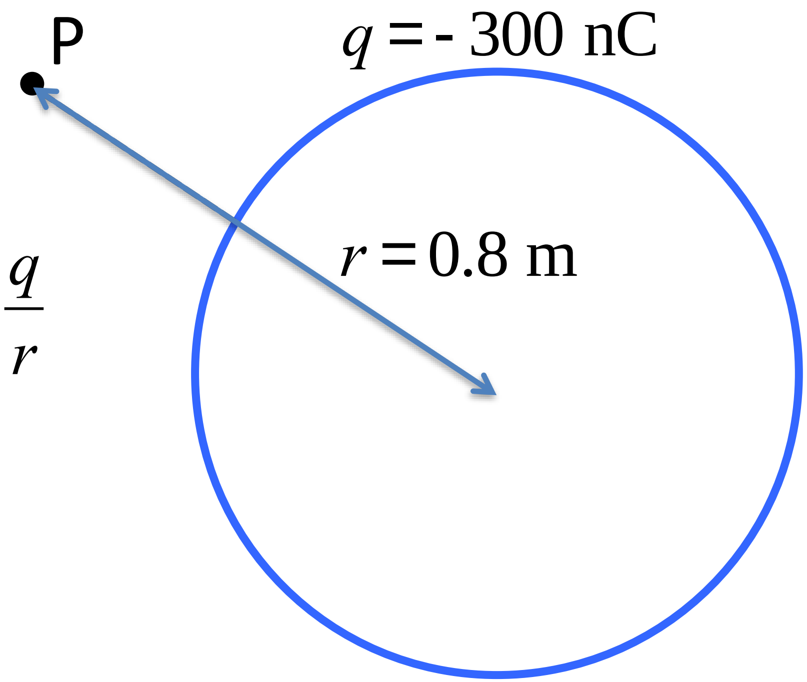
A. 3400 V

B. –9000 V

C. 9000 V

D. –3400 V

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$



TopHat Question

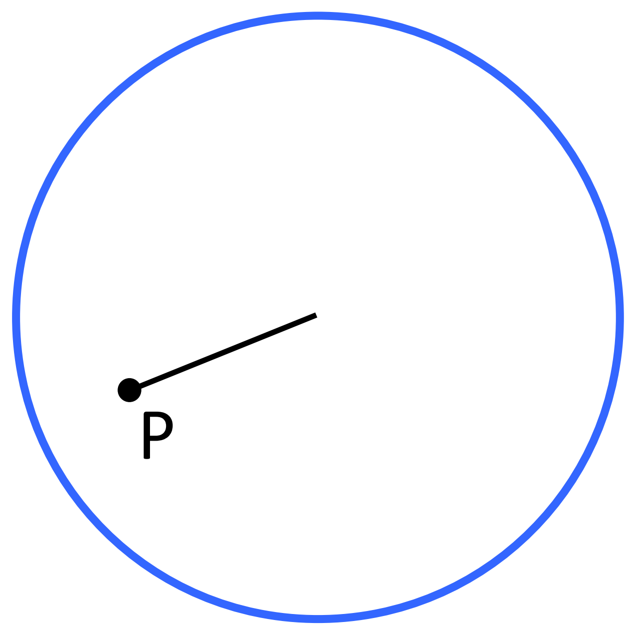
Consider a uniformly charged insulating shell with a diameter of 1.0 m and a total charge of –300 nC. What is the **magnitude** of the **electric field** at point P a distance 30 cm from the centre?

A. 30000 V/m

B. 11000 V/m

C. 9000 V/m

D. 0 V/m



TopHat Question

Consider a uniformly charged insulating shell with a diameter of 1.0 m and a total charge of –300 nC. What is the **electric potential** at point P a distance 30 cm from the centre?

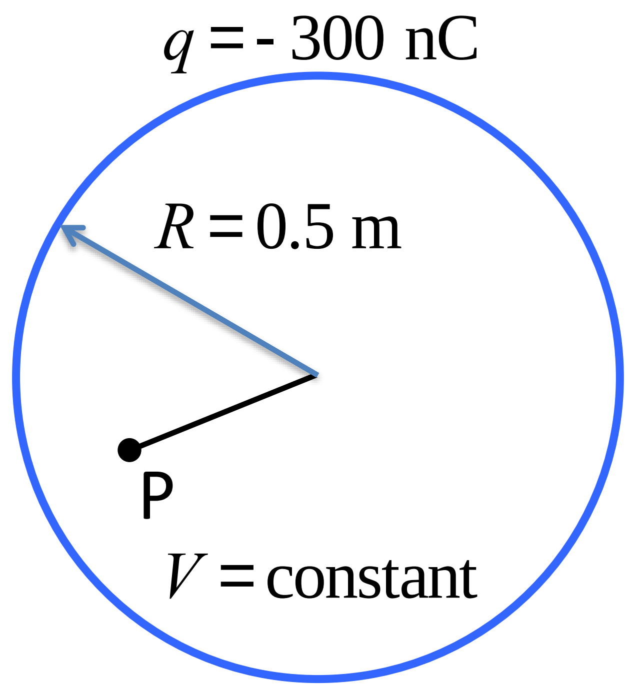
A. –9000 V

B. –5400 V

C. –3400 V

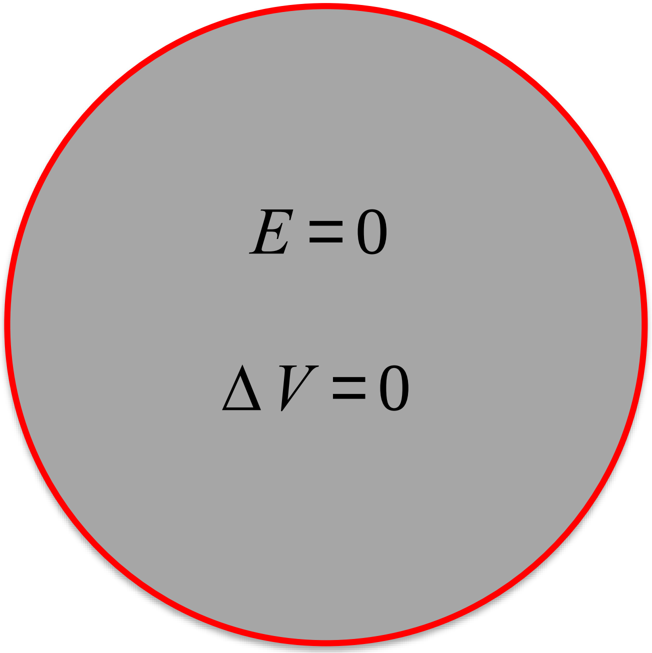
D. 0 V

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

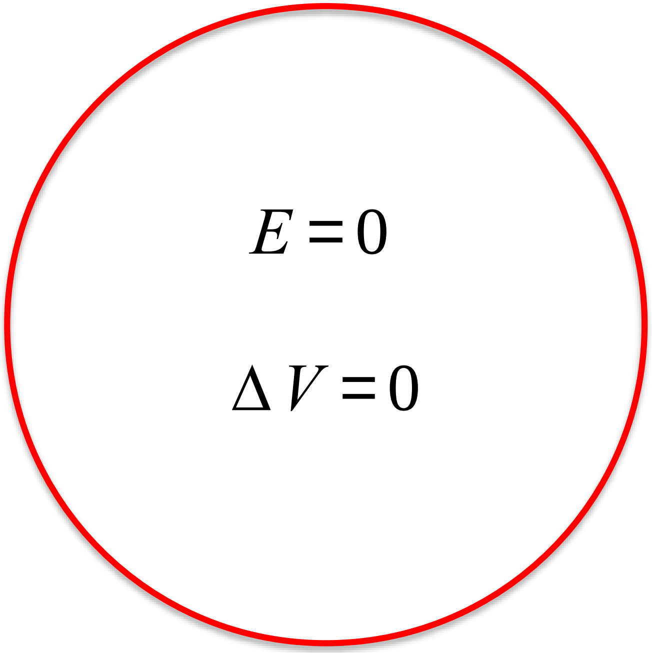


Model of a charged insulating shell

Solid ball conductor with excess charge +Q evenly distributed on its surface



Hollow insulating shell with charge +Q uniformly distributed on its surface



Both objects have the exact same distribution of charges
E and V should be the same for both!

- Look at my notes called:
- Mar_Appendix1_Potential of Insulating Sphere

Uniform Electric fields

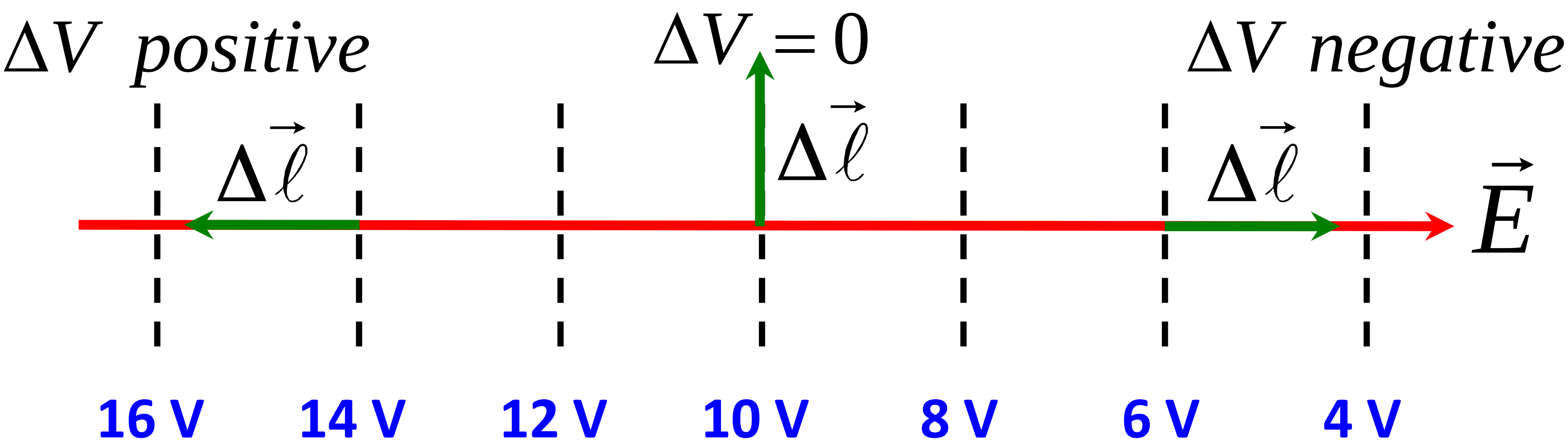
Potential and uniform E-field:

$$\Delta V = -\int_{x_i}^{x_f} \vec{E} \cdot d\vec{\ell}$$

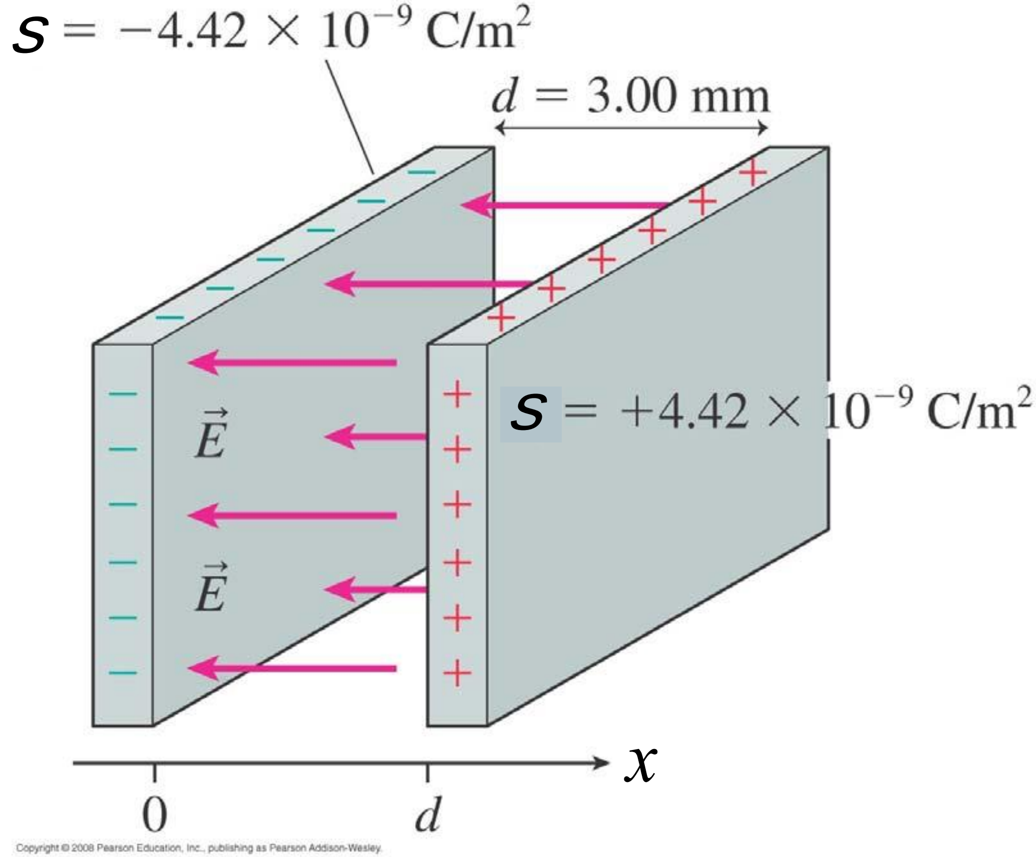
$$= -\vec{E} \cdot \Delta\vec{\ell}$$

Equipotential surfaces are **perpendicular** to $\vec{E}$, and the electric potential decreases **in the direction of** $\vec{E}$.

($\Delta\vec{\ell}$ lies along an equipotential.)

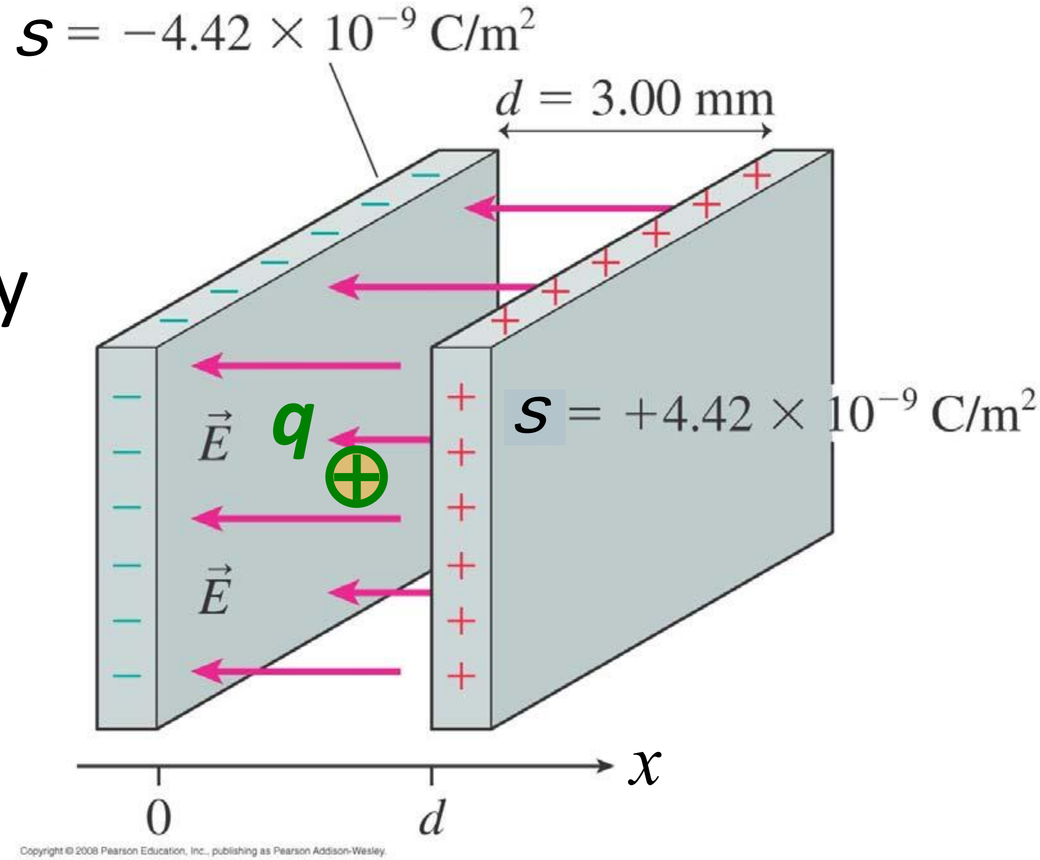


The source charges on the capacitor plates create a uniform electric field between the plates of



$$\vec{E} = \frac{\sigma}{\varepsilon_0} \text{ from positive to negative}$$

Electric potential energy of a charge q inside this uniform electric field is



$$U(x) = -q\vec{E} \cdot \vec{x}$$

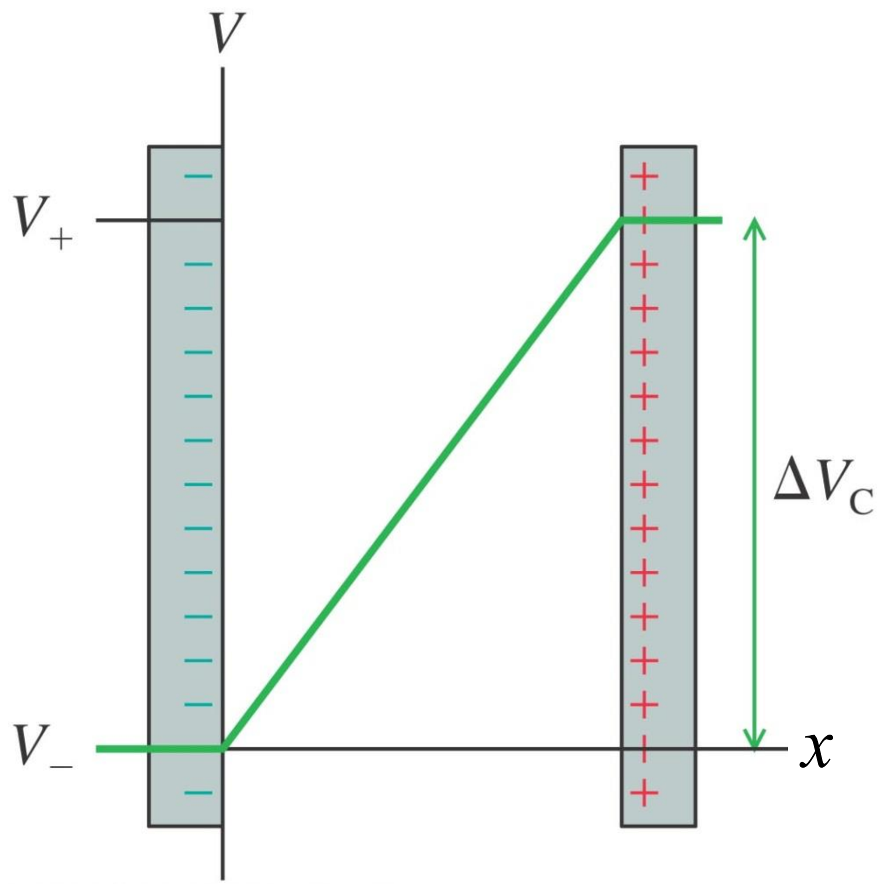
Therefore,

The electric potential inside the capacitor is

$$V = \frac{U}{q} = \frac{-q\vec{E} \cdot \vec{x}}{q} = -\vec{E} \cdot \vec{x}$$

$$\boxed{V = -\vec{E} \cdot \vec{x}}$$

The electric potential inside a charged capacitor increases linearly from the negative to the positive plate.



TopHat Question

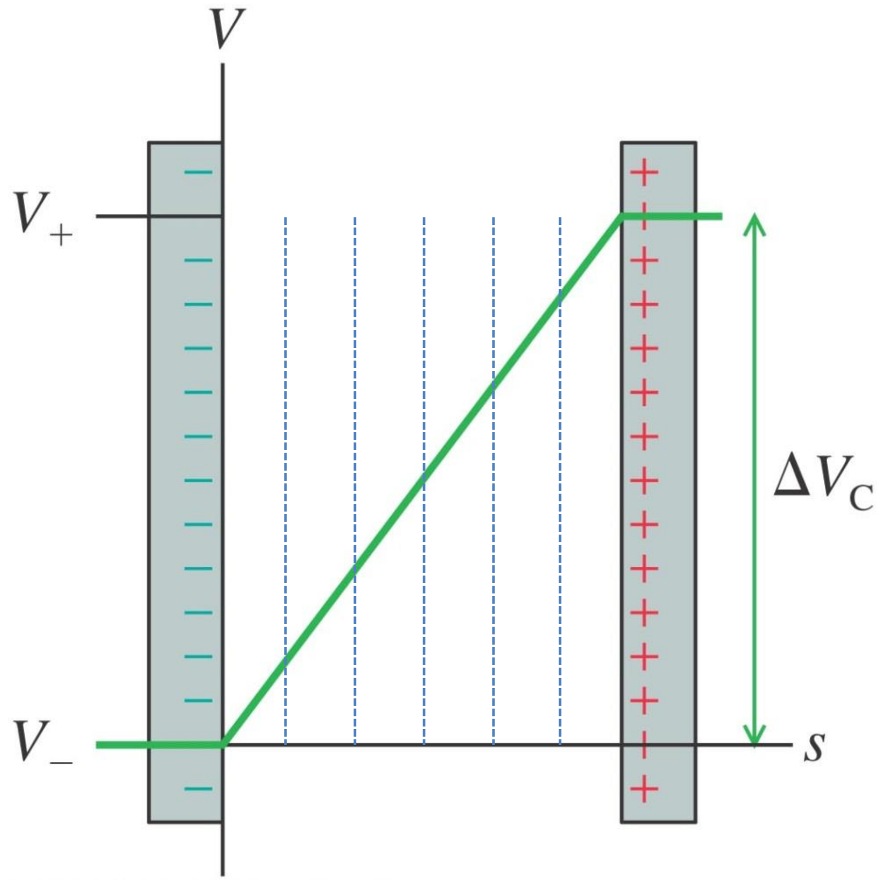
What are the lines of equipotential inside the parallel plate capacitor?

A. Vertical lines

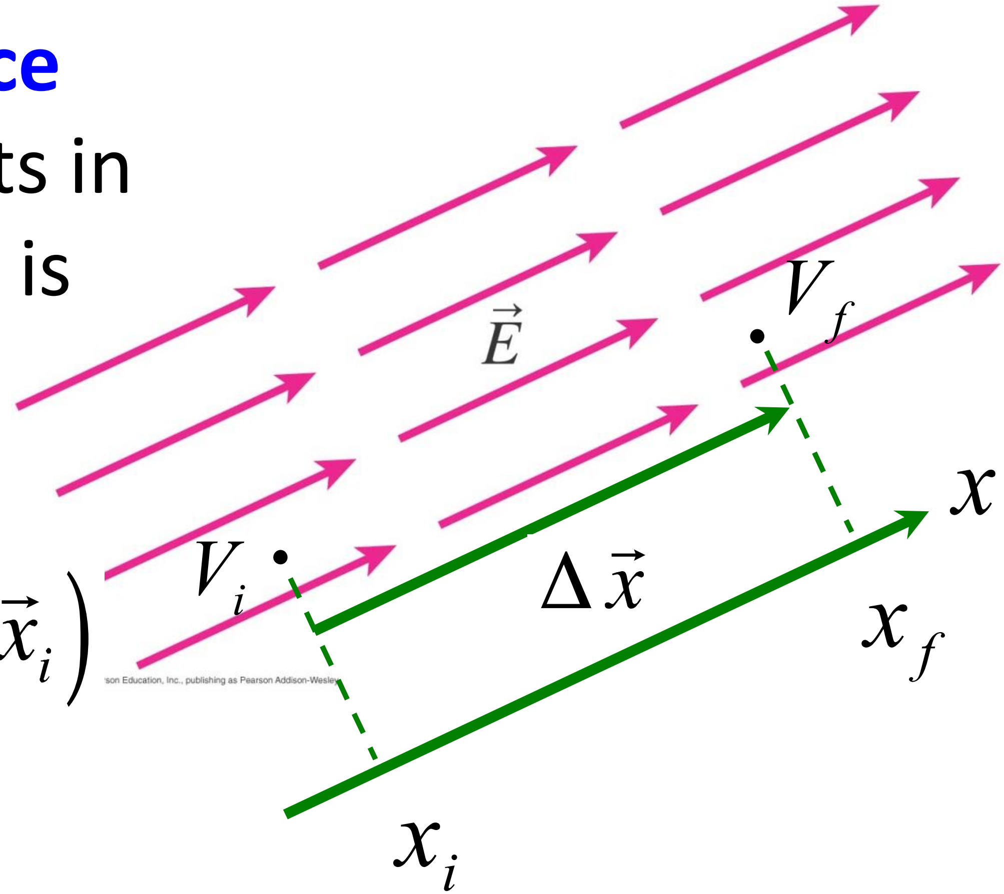
A. Horizontal lines

A. Diagonal lines slanting to the right

B. Not enough info



The **potential difference** between any two points in a **uniform** electric field is



$$\Delta V = V_f - V_i$$
$$= \left(-\vec{E}\cdot\vec{x}_f\right) - \left(-\vec{E}\cdot\vec{x}_i\right)$$
$$= -\vec{E}\cdot\left(\vec{x}_f - \vec{x}_i\right)$$
$$= -\vec{E}\cdot\Delta\vec{x}$$

Then $E_x = \dfrac{-\Delta V}{\Delta x}$

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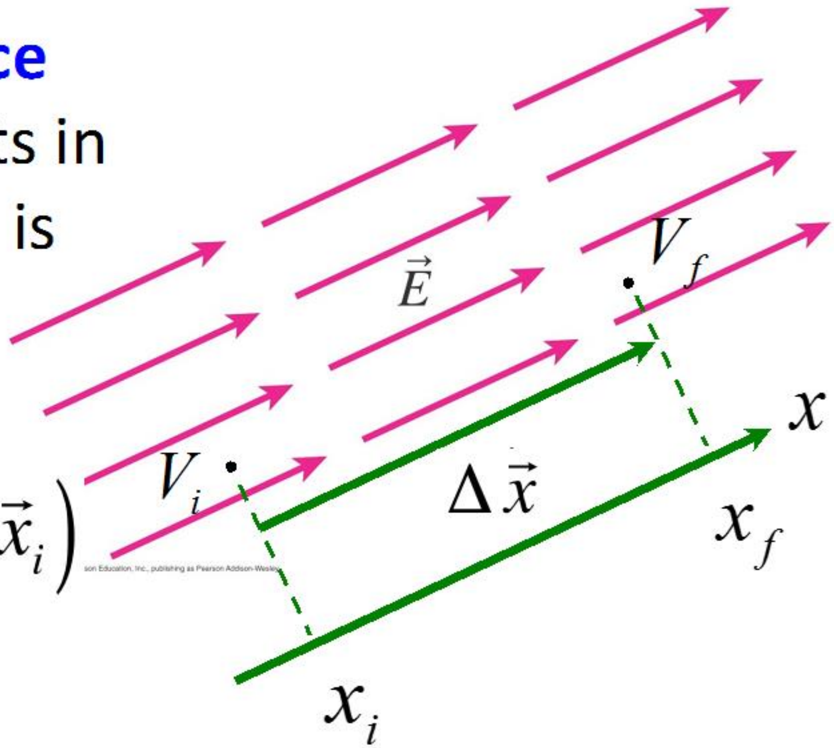

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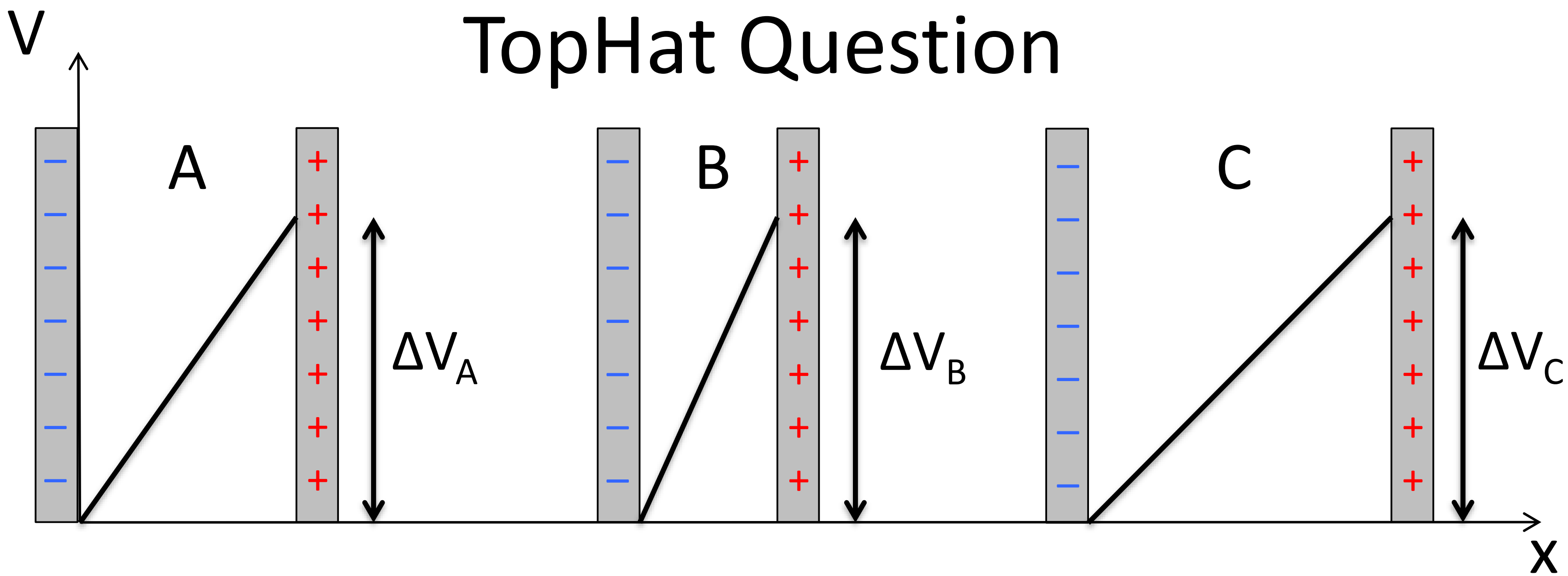
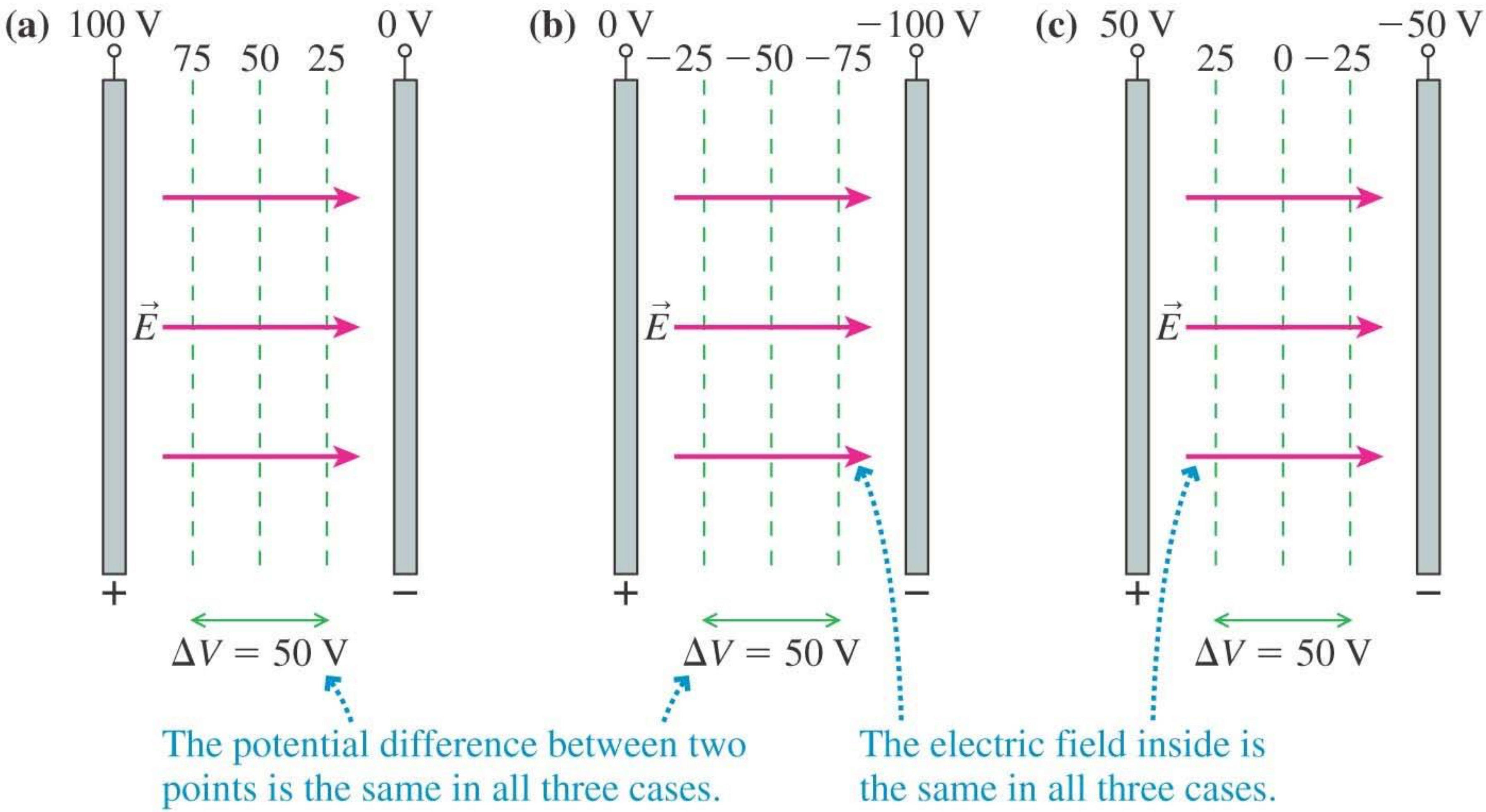
TopHat Question



Which parallel plate capacitor above has the highest E-field strength between its plates?

A.
B.
C.
D. All E-fields are equal

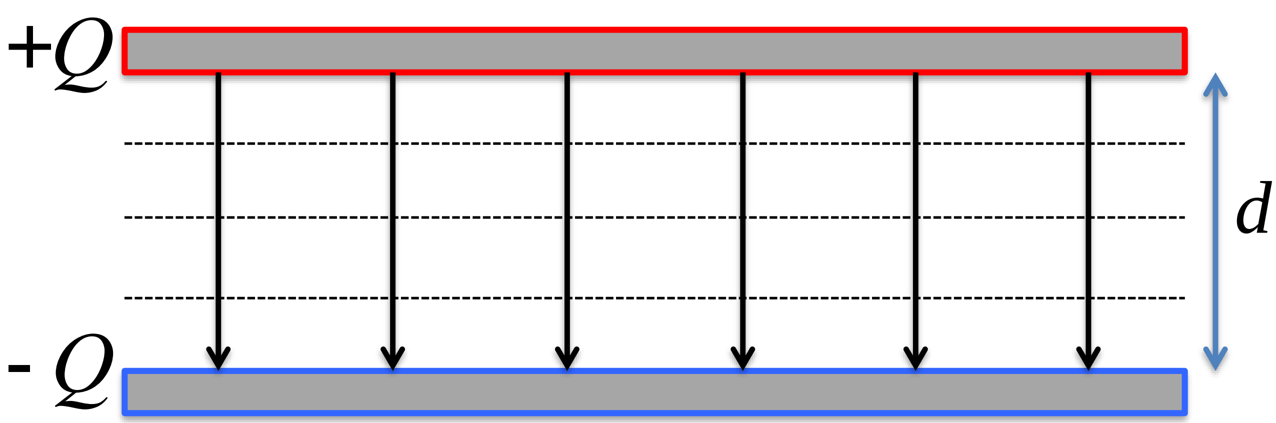


We can define V = 0 anywhere we want. Our choice of V = 0 does not affect any potential differences or the electric field.

Parallel Plate Capacitors

- One plate carries a charge $+Q$, the other plate carries a charge $-Q$.
- This creates a uniform E-field between the plates.
- This E-field can be written as a potential difference.

$$E = \frac{\sigma}{\varepsilon_o} = \frac{\Delta V_C}{d} \qquad \sigma = \frac{Q}{A} \qquad Q = \left(\frac{\varepsilon_o A}{d}\right)\Delta V_C$$



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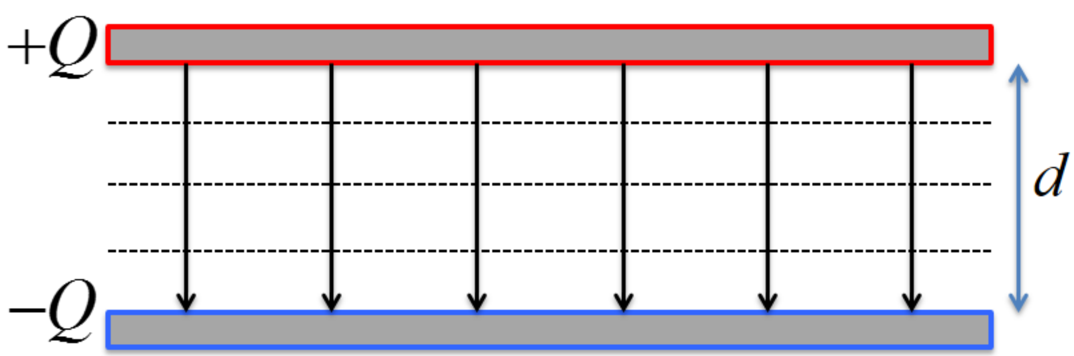

Capacitors and Capacitance

We find it useful to shorten that constant to just the letter *C*. This is a **geometric property** of the specific capacitor (not necessarily parallel plates)

$$Q = \left(\frac{\varepsilon_o A}{d} \right) \Delta V = C \Delta V$$

$$\boxed{C = \frac{\varepsilon_o A}{d}}$$

C is called the **capacitance** and it represents the "**capacity to store charge**". For any capacitor, the relationship between its stored charge and the voltage across its electrodes is given by

$$Q = C \Delta V$$

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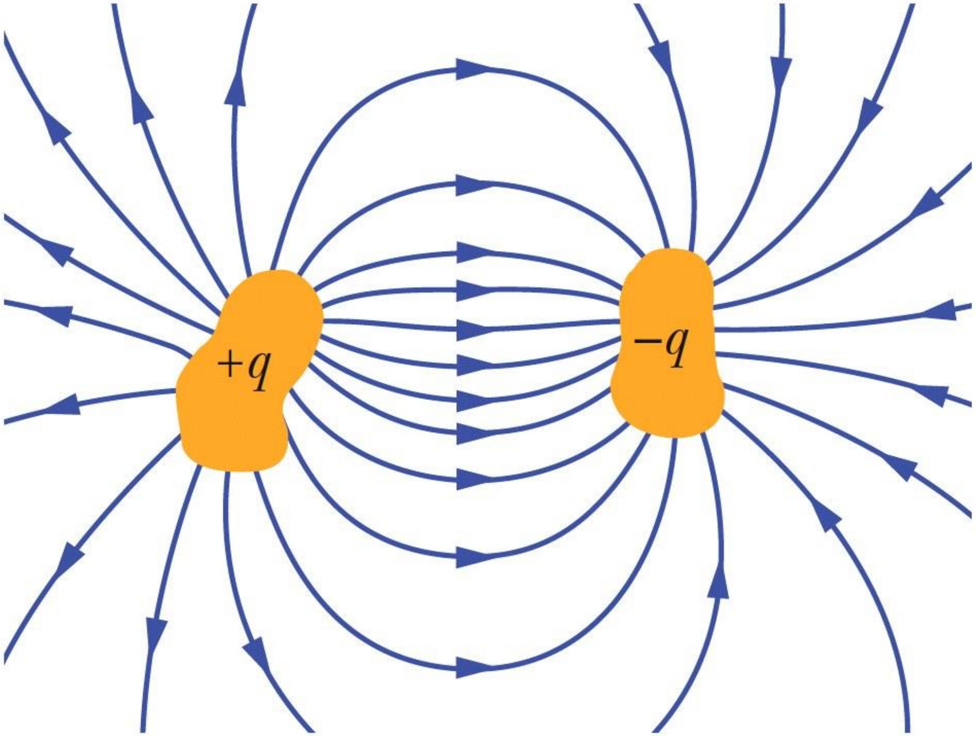
Capacitors in General

A capacitor is any two electrodes separated by some distance. Regardless of the geometry, we call the electrodes "plates".



Need to be electrodes (metal) in order to charge and discharge freely by the flow of charges.

By convention, a capacitor has equal and opposite charges on its plates, although this technically does not have to be true.

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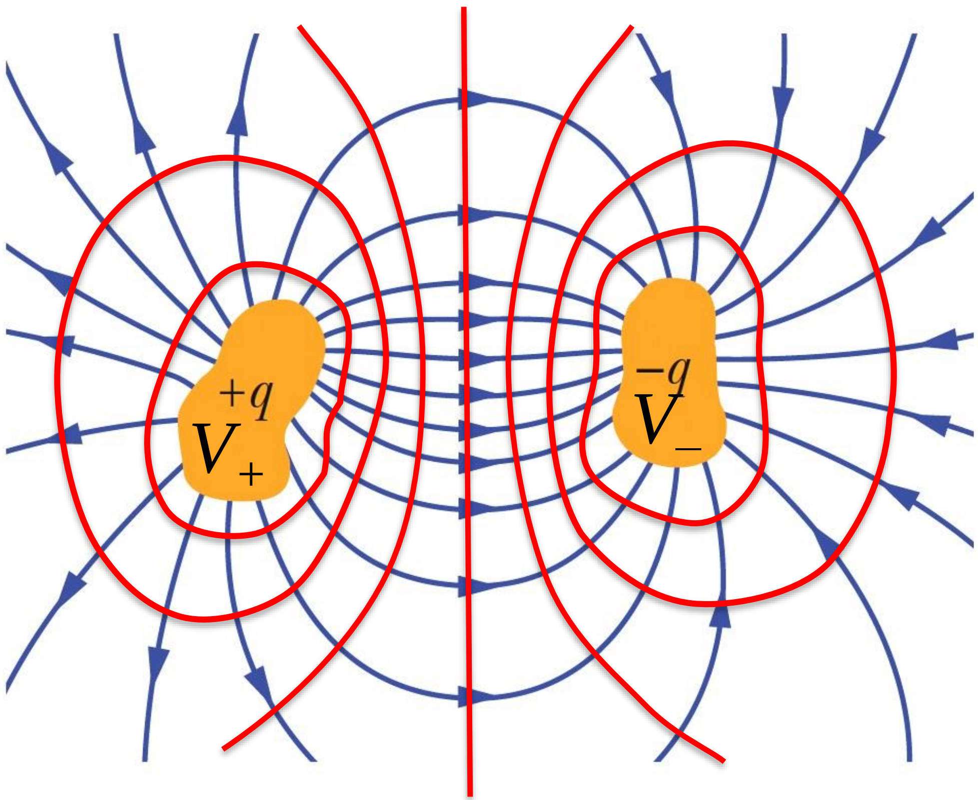
Capacitors in General

For equal but opposite charges one the plates, this arbitrary set of electrodes creates an electric field. What are the equipotentials?



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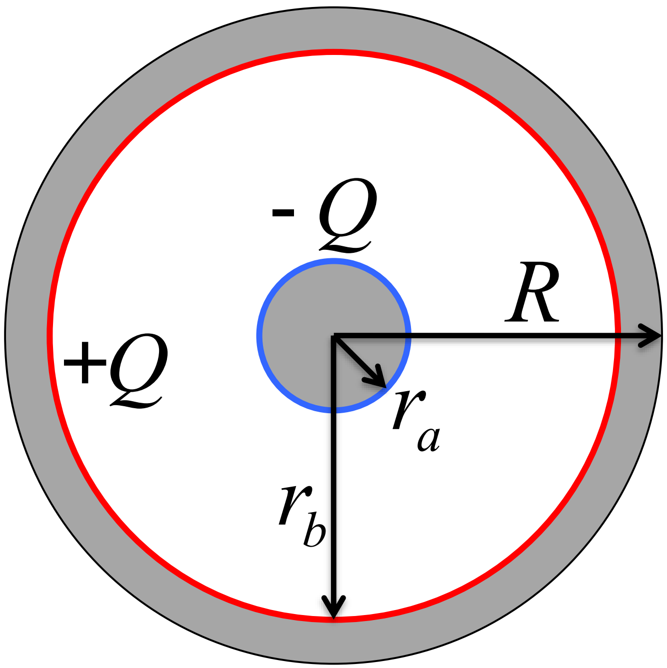
The potential changes from V_+ on the positive plate to $V\_$ on the negative plate.
This is not as simple as $\Delta V = Ed$, but the charge is still related to ΔV

$$Q = C\Delta V$$

For some geometric quantity C

Spherical Capacitor



1) What is the E-field everywhere?
2) What is V everywhere?
3) What is ΔV between the plates?
4) How can we relate ΔV to the charge on the plates?

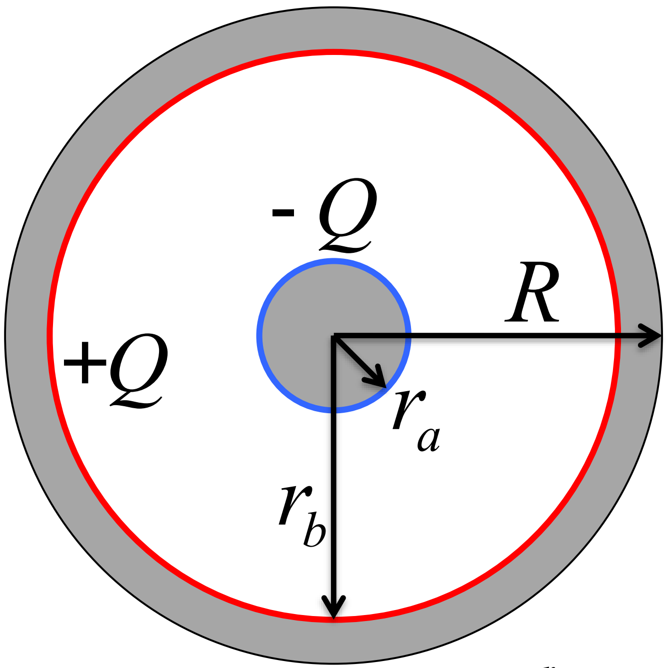
$$r > R, \quad \vec{E} = 0 \quad \text{(from Gauss' Law)}$$

$$R > r > r_b, \quad \vec{E} = 0 \quad \text{(inside a conductor)}$$

$$r < r_a, \quad \vec{E} = 0 \quad \text{(inside a conductor)}$$

$$\boxed{r_a > r > r_b, \quad \vec{E} = \frac{-Q}{4\pi\varepsilon_0 r^2}\hat{r}}$$

Spherical Capacitor



1) What is the E-field everywhere?
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3) What is ΔV between the plates?

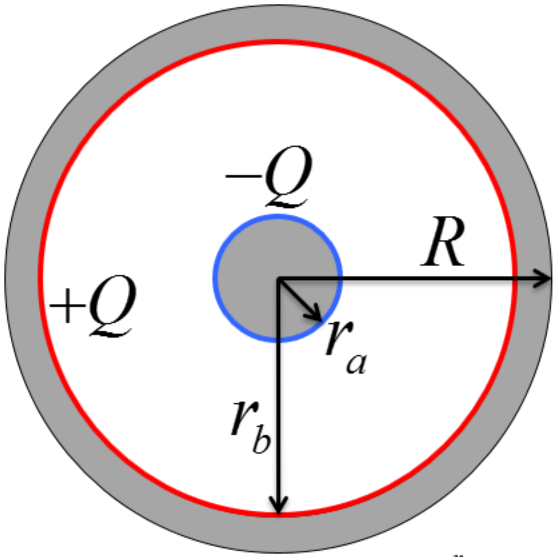
$$\boxed{V = 0 \quad \text{at infinity}}$$

$$r > r_b, \qquad \Delta V = -\int_{\infty}^{R} \vec{E} \cdot d\vec{r} = 0 \qquad \boxed{V_{r>r_b} = 0}$$

$$r_b > r > r_a, \qquad \Delta V = -\int_{r_b}^{r_a} \vec{E} \cdot d\vec{r} = -\int_{r_b}^{r_a} \frac{-Q}{4\pi\varepsilon_0 r^2} dr \boxed{= \frac{-Q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)}$$

$$r < r_a, \qquad \Delta V = -\int_{\infty}^{R} \vec{E} \cdot d\vec{r} = 0 \qquad \boxed{V_{r<r_a} = \frac{-Q}{4\pi\varepsilon_0 r_a}}$$

Spherical Capacitor



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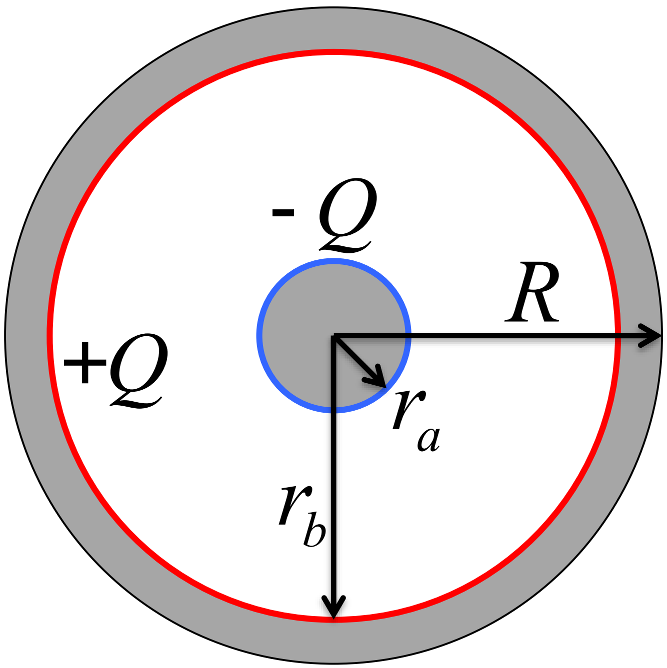

1) What is the E-field everywhere?
2) What is V everywhere?
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4) How can we relate ΔV to the charge on the plates?

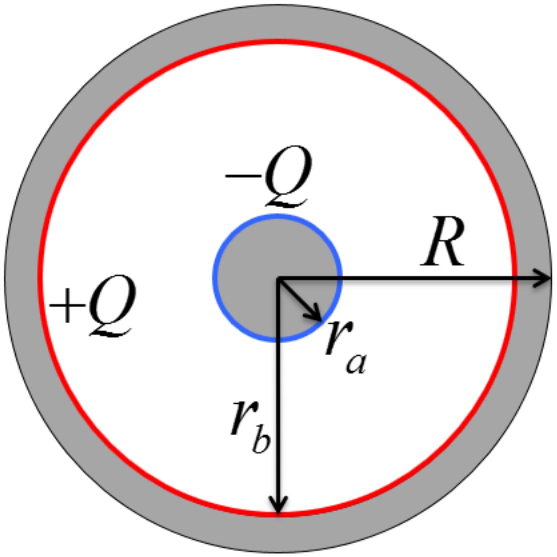
$$\Delta V_C = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{r_b - r_a}{r_b r_a}\right)$$

Rewrite this relation as

$$Q = \left(\frac{4\pi\varepsilon_0 r_b r_a}{r_b - r_a}\right)\Delta V_C$$

$$\boxed{Q = C\Delta V_C}$$

Spherical Capacitor



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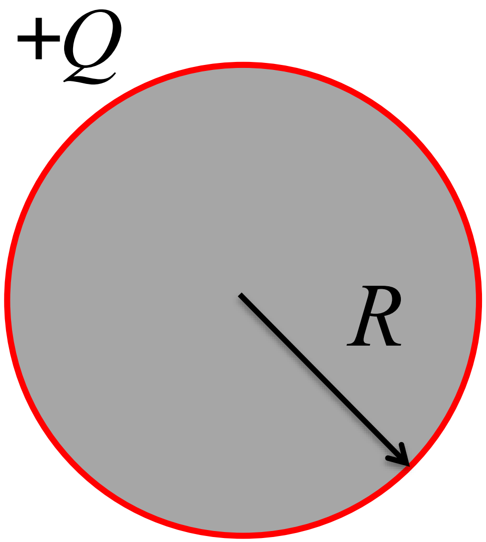
Isolated Sphere as a Capacitor

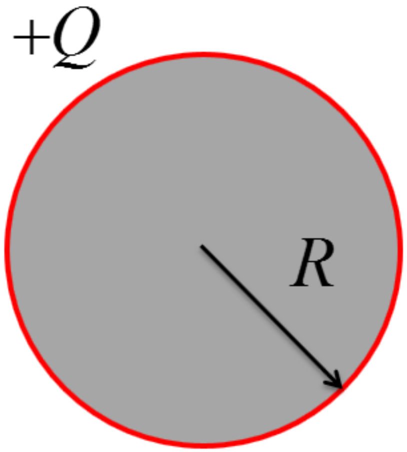


Capacitors need two plates in general for the field lines to end. In the case of a sphere, we can consider the other plate to be at infinity and define the capacitance of an isolated sphere with charge Q. This will not work for an infinite cylinder as we will see later.

Start with expression for spherical capacitor with $r_a = R$, $r_b = \infty$:

$$Q = \left(\frac{4\pi\epsilon_0 r_b r_a}{r_b - r_a} \right) \Delta V_C \rightarrow \left(4\pi\epsilon_0 R \right) \Delta V_C \qquad \boxed{C = 4\pi\epsilon_0 R}$$

Isolated Sphere as a Capacitor



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$$\boxed{C = 4\pi\varepsilon_0 R}$$

TopHat Question

How big does the radius of an isolated sphere have to be in order for it to have a capacitance of 1 F? Choose the closest answer.

A. 10^9 km

B. 10^8 km

C. 10^7 km

D. 10^6 km

For comparison, the orbit of Mercury around the sun is 5×10^7 km!
Spheres make bad capacitors; a 1 F capacitor used in circuits can fit in your hand.

Last time:

- Modeling an insulating spherical shell
- Potential between two parallel charged plates
- Capacitance as a geometric quantity
- General Capacitors, relating Q to ΔV
- Spherical capacitors (setting up a process to find C)

Today:

- Cylindrical capacitors: return of the coax cable
- Energy stored in parallel plate, spherical, and cylindrical capacitors
- Potential energy stored in the electric field itself.

V for a Line of Charge

Calculating the potential at point P due to an infinite line of charge using the potential due to dQ and integrating over the entire line produced a problem: V_P was infinite for any location of point P.

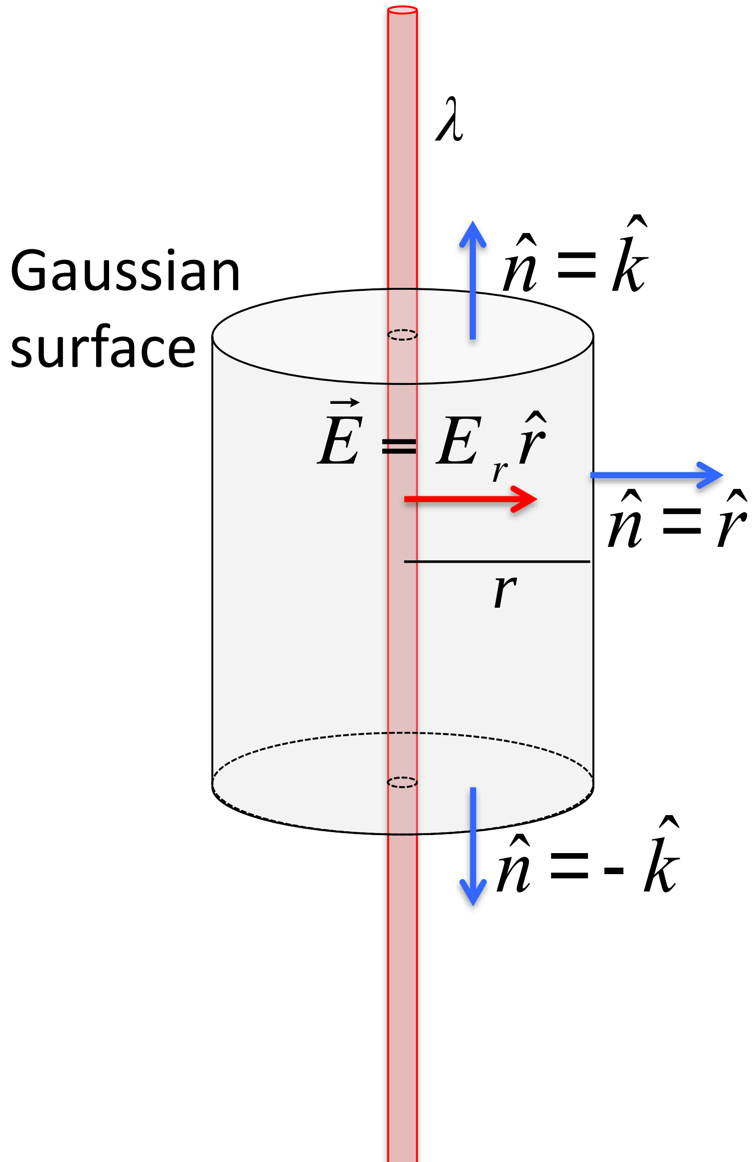
This is actually true because the way we were implicitly measuring the potential was setting V=0 at infinity.

To get around this, we had to subtract an infinite constant to cancel out the infinity in the potential, leaving a piece that is finite and gives a good measure of the potential difference between two points at finite distance away from the wire.

Whiteboard calculation (if time)

Look at my notes called:
Mar_Appendix2_Potential of an infinite line of charge

Calculate V from E



Gauss' Law: $\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$

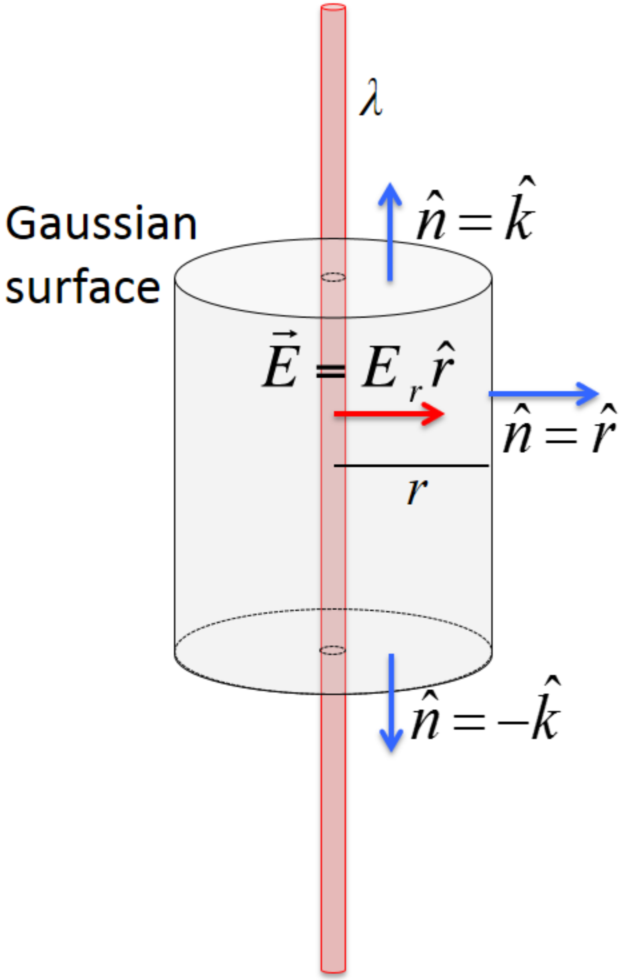
$$\oint \vec{E} \cdot d\vec{A} = \cancel{\int_{\text{top}} \vec{E} \cdot \hat{n} dA} + \cancel{\int_{\text{bottom}} \vec{E} \cdot \hat{n} dA}$$

$$+ \int_{\text{tube}} \vec{E} \cdot \hat{n} dA$$

$$E_r \oint dA = E_r 2\pi r L = \frac{\lambda L}{\varepsilon_0}$$

$$E_r = \frac{l}{2 p e_0 r}$$

Calculate V from E



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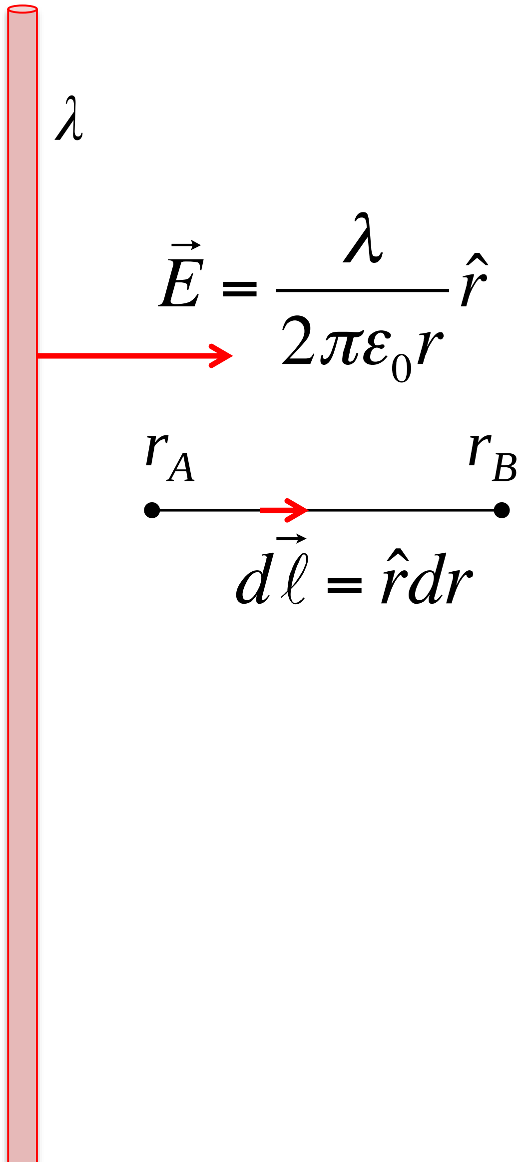
$$E_r \oint dA = E_r 2\pi r L = \frac{\lambda L}{\varepsilon_0}$$

$$E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Calculate V from E



$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

$$d\vec{\ell} = \hat{r}dr$$

$$\Delta V_{AB} = -\int_A^B \vec{E} \cdot d\vec{\ell}$$

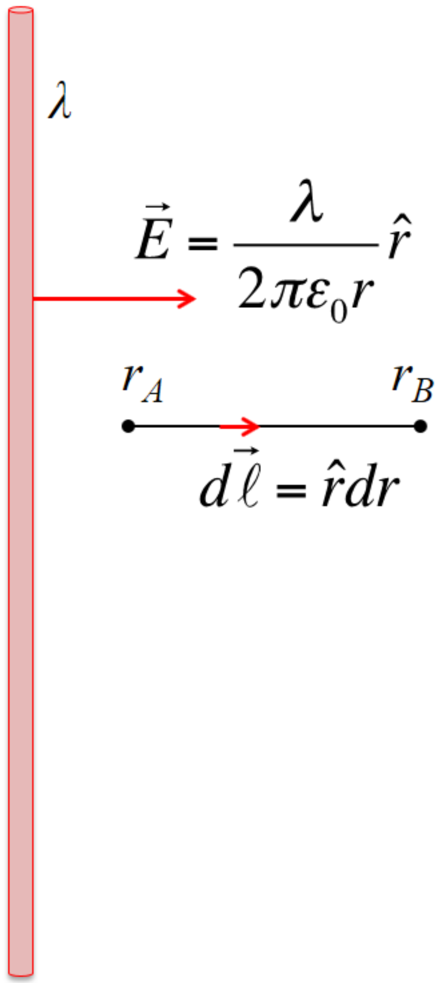
$$\Delta V_{AB} = -\int_A^B \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} \cdot \hat{r}dr$$

$$\Delta V_{AB} = -\frac{\lambda}{2\pi\varepsilon_0}\int_A^B \frac{dr}{r}$$

$$\Delta V_{AB} = -\frac{\lambda}{2\pi\varepsilon_0}\left(\ln(r_B) - \ln(r_A)\right)$$

$$\boxed{\Delta V_{AB} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_B}{r_A}\right)}$$

Calculate V from E



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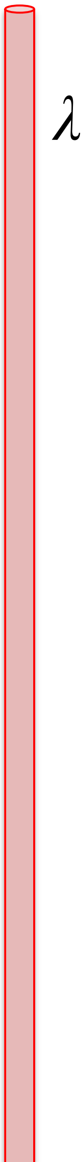
$$\Delta V_{AB} = -\int_A^B \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}\cdot\hat{r}dr$$

$$\Delta V_{AB} = -\frac{\lambda}{2\pi\varepsilon_0}\int_A^B \frac{dr}{r}$$

$$\Delta V_{AB} = -\frac{\lambda}{2\pi\varepsilon_0}\left(\ln\left(r_B\right) - \ln\left(r_A\right)\right)$$

$$\boxed{\Delta V_{AB} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_B}{r_A}\right)}$$

Calculate V from E



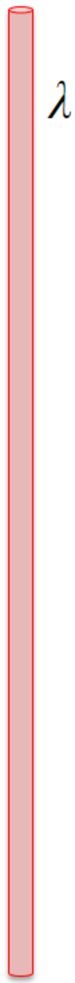
Difference in potential between any two points r_A and r_B.

$$\Delta V_{AB} = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_B}{r_A}\right)$$

If we send r_B to infinity, we get the difference in potential between infinity and any finite point r_A, but this limit gives ΔV = infinity. This is why we had to subtract infinity in the other approach.

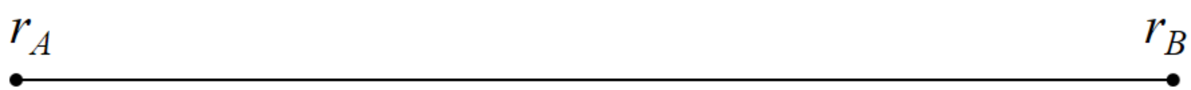
Take home message: for an infinite line of charge, we cannot set V = 0 at infinity, so we choose a more convenient location based on the problem

Calculate V from E



Difference in potential between any two points r_A and r_B.

$$\Delta V_{AB} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_B}{r_A}\right)$$



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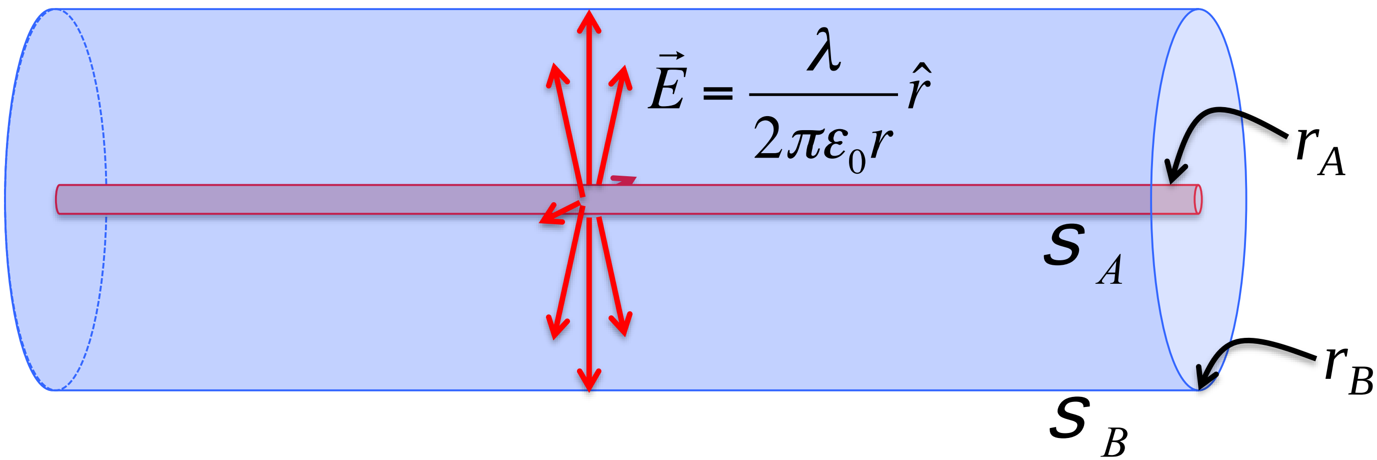
Application: Cylindrical Capacitor



Inside, E is due to central cylinder

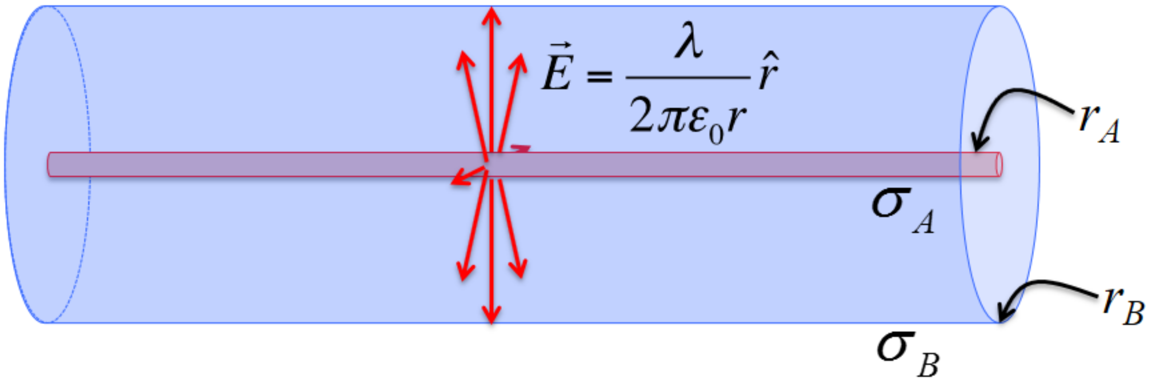
$$\Delta V_{12} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_2}{r_1}\right)$$

For some points r_1 and r_2 inside the bigger cylinder

Outside the cylinder, $E = 0$ because $q_{enc} = 0$

$$\lambda = \sigma_A 2\pi r_A = \frac{Q}{2\pi r_A L}2\pi r_A \qquad -\lambda = \sigma_B 2\pi r_B = \frac{-Q}{2\pi r_B L}2\pi r_B$$

Application: Cylindrical Capacitor



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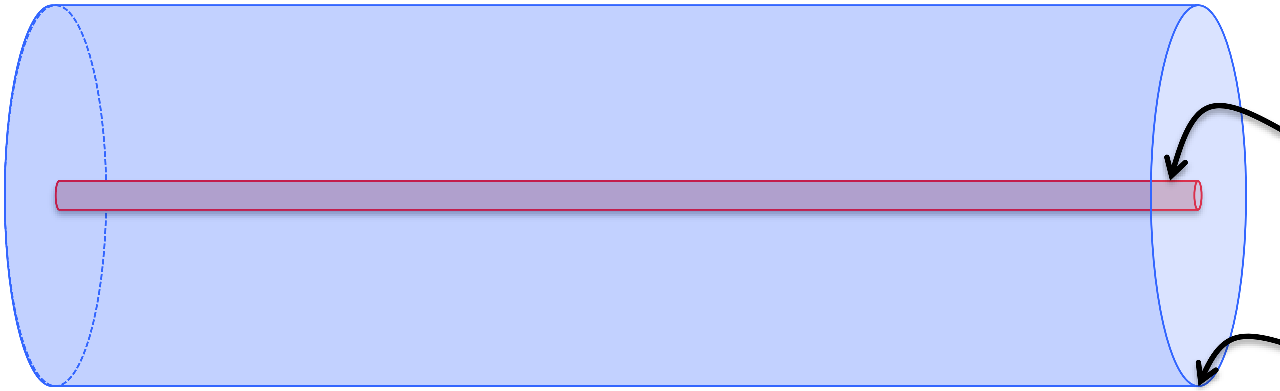
Application: Cylindrical Capacitor



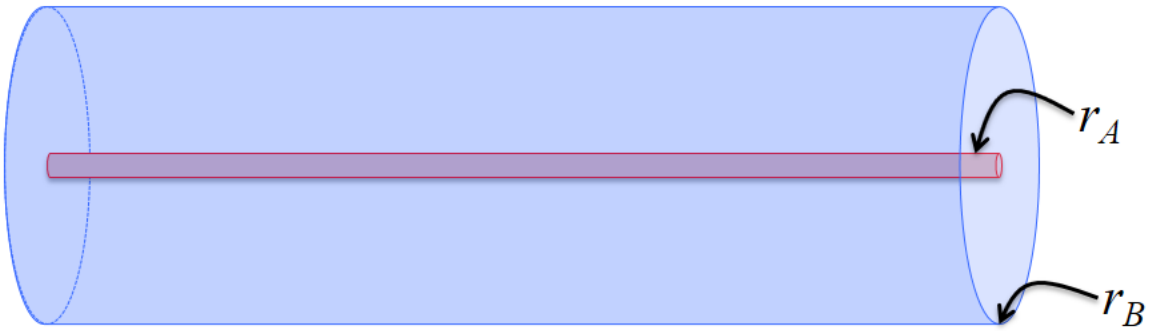
Voltage difference across the capacitor plates is obtained by taking $r_1 = r_A$ and $r_2 = r_B$:

$$\Delta V_C = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_B}{r_A}\right) = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{r_B}{r_A}\right) \qquad Q = \left(\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{r_B}{r_A}\right)}\right)\Delta V_C$$

Define capacitance per unit length via:

$$\lambda = \boxed{c}\,\Delta V_C \qquad \lambda = \boxed{\left(\frac{2\pi\varepsilon_0}{\ln\left(\frac{r_B}{r_A}\right)}\right)}\Delta V_C$$

Application: Cylindrical Capacitor



Voltage difference across the capacitor plates is obtained by taking $r_1 = r_A$ and $r_2 = r_B$:

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Define capacitance per unit length via:

$$\lambda = c\Delta V_C \qquad \lambda = \left(\frac{2\pi\varepsilon_0}{\ln\left(\frac{r_B}{r_A}\right)}\right)\Delta V_C$$

Capacitors

General relationship: $Q = C\Delta V_C$

Parallel plate capacitor: $Q = \left(\dfrac{\varepsilon_o A}{d}\right)\Delta V_C$

Spherical capacitor: $Q = \left(\dfrac{4\pi\varepsilon_0 r_b r_a}{r_b - r_a}\right)\Delta V_C$

Isolated sphere: $Q = \left(4\pi\varepsilon_0 R\right)\Delta V_C$

Cylindrical capacitor: $Q = \left(\dfrac{2\pi\varepsilon_0 L}{\ln\left(\frac{r_B}{r_A}\right)}\right)\Delta V_C$

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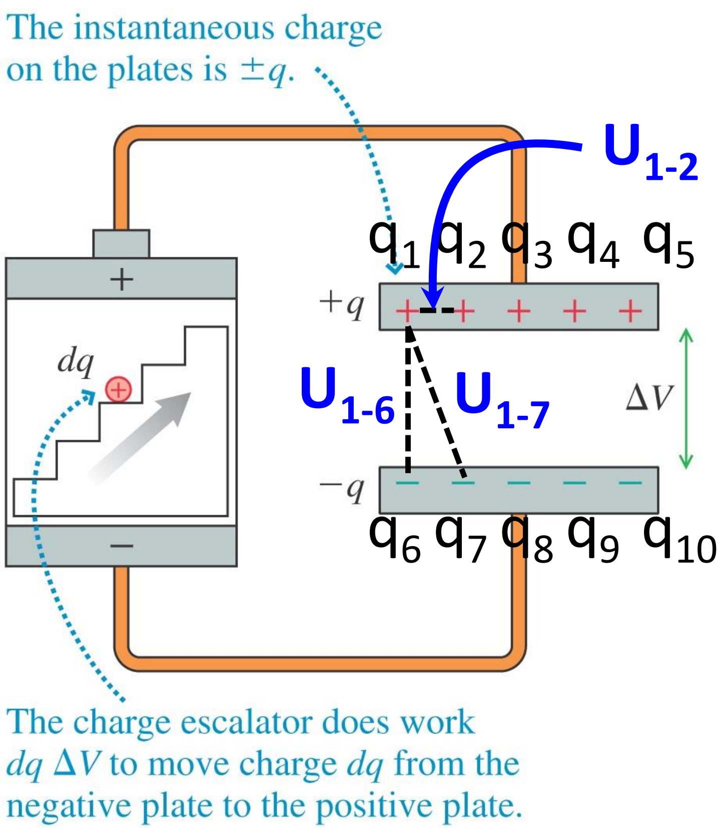
Cylindrical capacitor: $$Q = \left(\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{r_B}{r_A}\right)}\right)\Delta V_C$$

Energy Storage in Capacitors

We want to calculate this potential energy stored in the capacitor.

It is **waaaaay** too hard to add up all the potential energies of every pair of charges in the capacitor:



$$U = U_{1-2} + U_{1-3} + \ldots + U_{1-10} + U_{2-1}$$ **+ $U_{i\text{-}j}$ of every other pair**

Easier way!

Move a tiny charge, *dq*, from the negative plate to the positive plate.

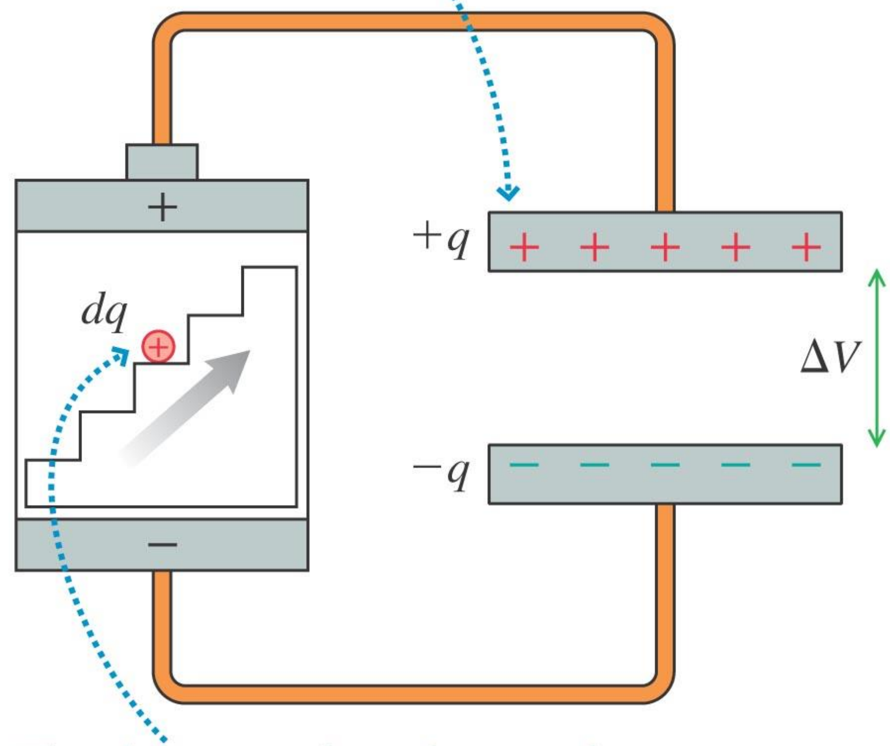
It moves through a potential difference ΔV.

So its potential energy increases by an amount

$$dU = dq\,\Delta V_C$$

But we also know $\Delta V_C = \dfrac{q}{C}$

$$dU = dq\frac{q}{C} = \frac{q\,dq}{C}$$



$$\boxed{U = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{2}\frac{Q^2}{C}}$$

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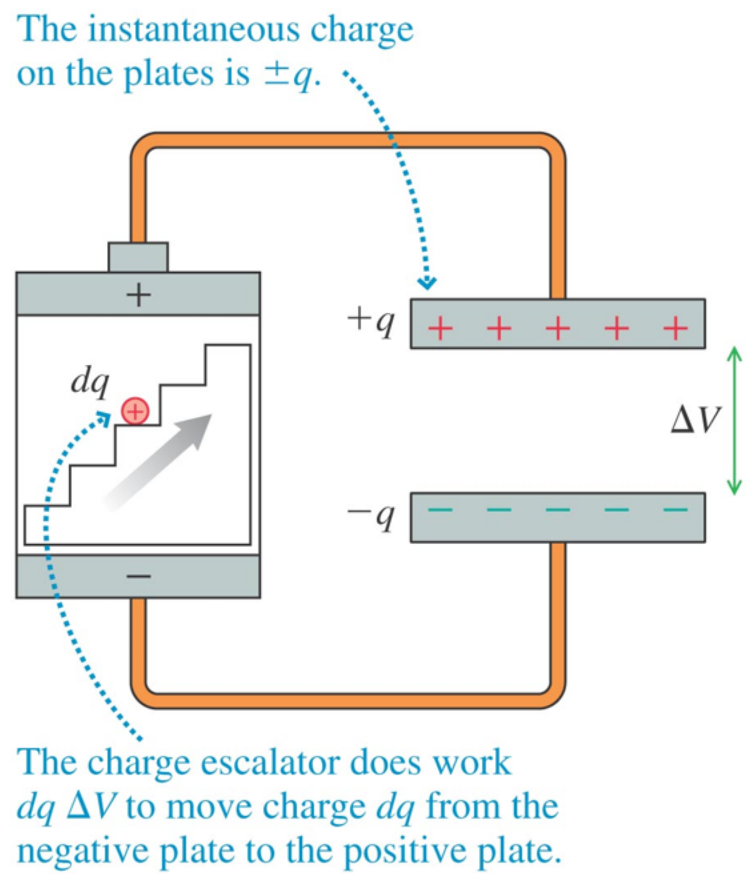


$$\boxed{U = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{2}\frac{Q^2}{C}}$$

Potential Energy in a Capacitor

Energy storage in terms of the charge on the plates:

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C}}$$

Use the general relation for a capacitor to swap charge for voltage

$$Q = C\mathrm{D}V_C$$

Energy storage in terms of the voltage across the plates:

$$U = \frac{1}{2}\frac{\left(C\mathrm{D}V_C\right)^2}{C}$$

$$\boxed{= \frac{1}{2}C\left(\mathrm{D}V_C\right)^2}$$

Potential Energy in a Capacitor

Energy storage in terms of the charge on the plates:

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C}}$$

Use the general relation for a capacitor to swap charge for voltage

$$Q = C\Delta V_C$$

Energy storage in terms of the voltage across the plates:

$$U = \frac{1}{2}\frac{\left(C\Delta V_C\right)^2}{C}$$

$$\boxed{= \frac{1}{2}C\left(\Delta V_C\right)^2}$$

Where is the Energy Stored?

$$U = \frac{1}{2}C\left(\Delta V_C\right)^2$$

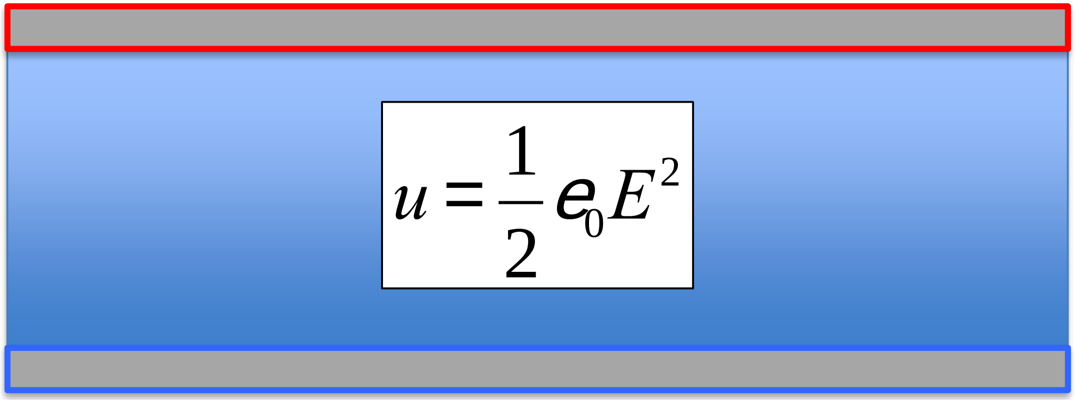
$$= \frac{1}{2}CE^2d^2$$

$$= \frac{1}{2}\frac{\varepsilon_0 A}{d}E^2d^2 = \frac{1}{2}\varepsilon_0 E^2(Ad)$$

$$\Delta V = Ed$$

$$C = \frac{\varepsilon_0 A}{d}$$

$$u = \frac{U}{Ad}$$



The capacitor’s energy is stored in the electric field between the plates!

Where is the Energy Stored?

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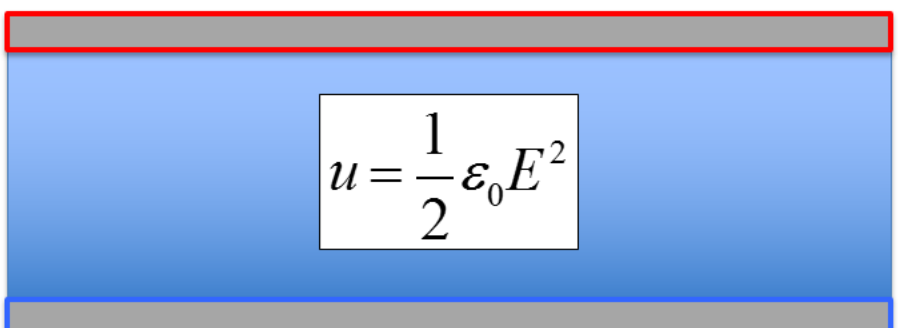

The capacitor's energy is stored in the electric field between the plates!

The following two slides you do **NOT** need to know how to reproduce for this course. They simply illustrate that the result from the previous slide applies more generally than for just a parallel plate capacitor.

Spherical Capacitor

Start with integrating $dU = udV$ over the volume between the plates

$$U = \iiint_{Vol} u\, dV = \iiint_{Vol} \frac{\varepsilon_0}{2} E^2\, dV \qquad \text{where} \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

In spherical coordinates, $dV = r^2 sin\theta dr d\theta d\phi$. Integrals over angles give.

$$\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi$$

Then U becomes

$$U = \frac{\varepsilon_0}{2} \int_{r_a}^{r_b} \frac{Q^2}{16\pi^2\varepsilon_0^2 r^4} 4\pi r^2\, dr = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0} \int_{r_a}^{r_b} \frac{dr}{r^2}$$

Performing the integral and rewriting, we indeed get U

$$U = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0\left(\frac{r_b r_a}{r_b - r_a}\right)} \boxed{= \frac{1}{2}\frac{Q^2}{C}}$$

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Cylindrical Capacitor

Start with integrating $dU = udV$ over the volume between the plates

$$U = \iiint_{Vol} u\, dV = \iiint_{Vol} \frac{\varepsilon_0}{2} E^2\, dV \qquad \text{where} \qquad \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

In cylindrical coordinates, $dV = rdrd\theta dl$.
Integrals over angle and l give

$$\int_0^{2\pi} d\theta \int_0^{L} d\ell = 2\pi L$$

Then U becomes

$$U = \frac{\varepsilon_0}{2}\int_{r_a}^{r_b} \frac{\lambda^2}{4\pi^2\varepsilon_0^2 r^2} 2\pi L r\, dr = \frac{1}{2}\frac{\lambda^2 L}{2\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{dr}{r}$$

Performing the integral and rewriting, we
indeed get U

$$U = \frac{1}{2}\frac{\lambda^2 L}{2\pi\varepsilon_0}\ln\left(\frac{r_b}{r_a}\right) = \frac{1}{2}\frac{\ln\left(\frac{r_b}{r_a}\right)}{2\pi\varepsilon_0 L}Q^2 = \boxed{\frac{1}{2}\frac{Q^2}{C}}$$

Cylindrical Capacitor

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Performing the integral and rewriting, we indeed get U

$$U = \frac{1}{2}\frac{\lambda^2 L}{2\pi\varepsilon_0}\ln\left(\frac{r_b}{r_a}\right) = \frac{1}{2}\frac{\ln\left(\frac{r_b}{r_a}\right)}{2\pi\varepsilon_0 L}Q^2 = \boxed{\frac{1}{2}\frac{Q^2}{C}}$$

Last time:

- Cylindrical capacitors
- Energy stored in parallel plate, spherical, and cylindrical capacitors
- Potential energy stored in the electric field itself

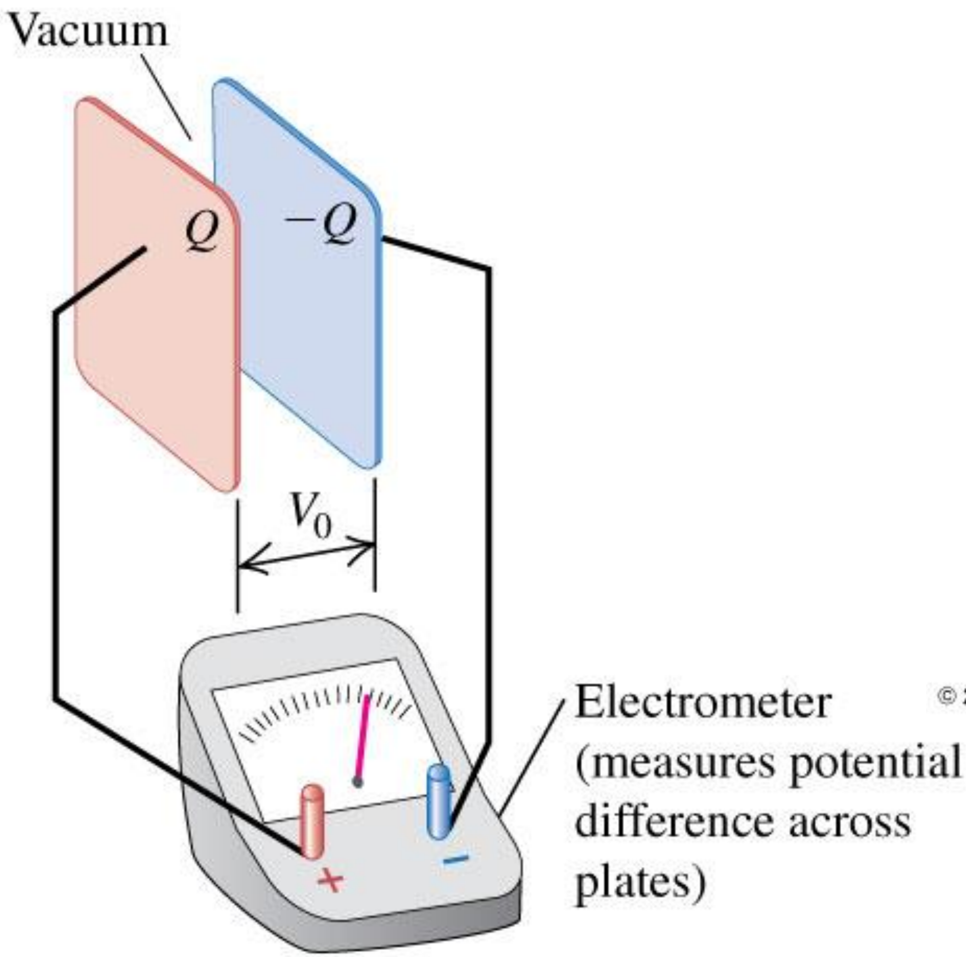
Today:

- Subtlety with capacitors: series or parallel?
- Linear dielectric materials: an atomic perspective
- Effect of dielectrics on capacitance
- Applications of dielectrics and capacitors

$$Q = C_0 V_0$$

$$V_0 = E_0 d$$

(a)

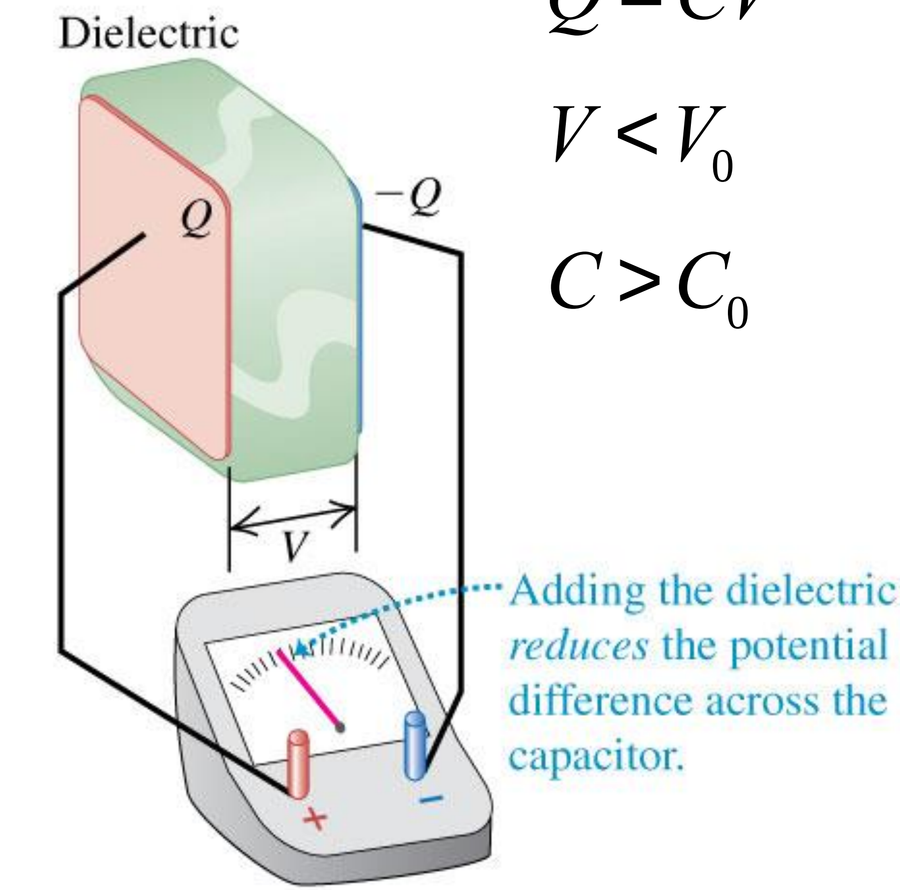


(b)

$$Q = CV$$

$$V < V_0$$

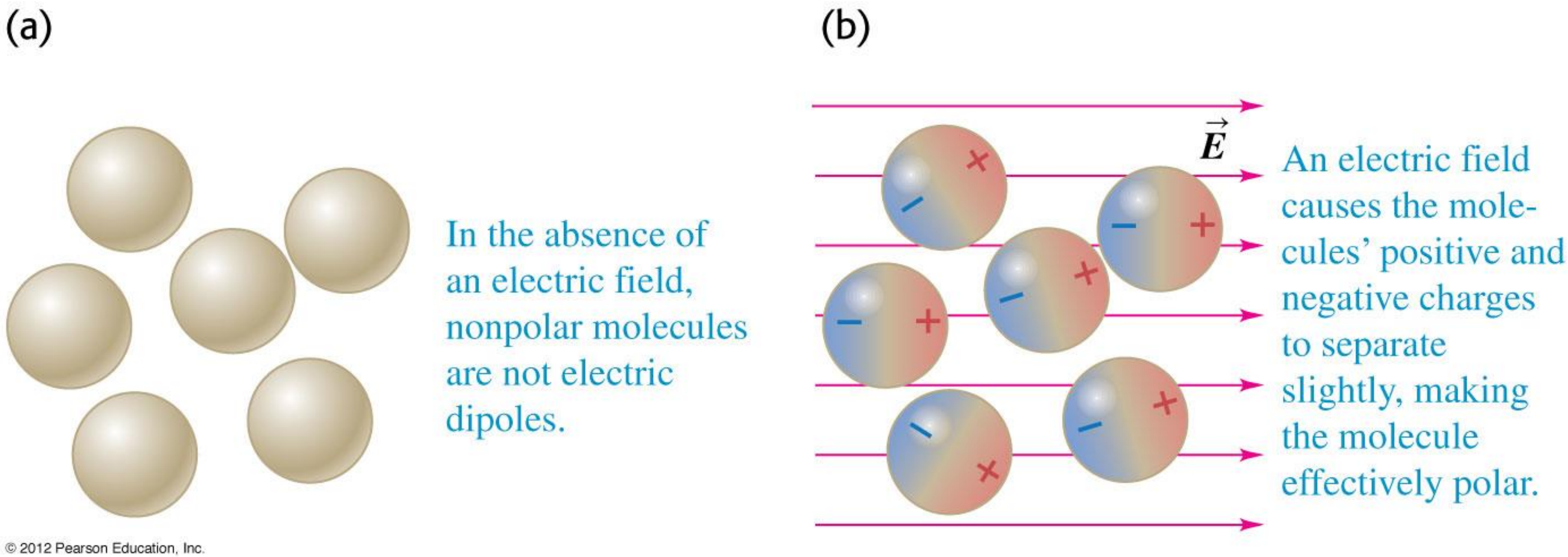
$$C > C_0$$



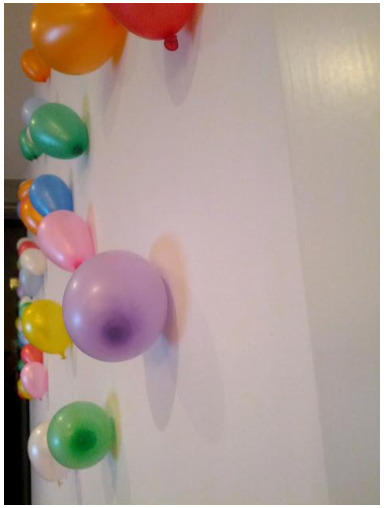
$$V = Ed$$

$$E < E_0$$

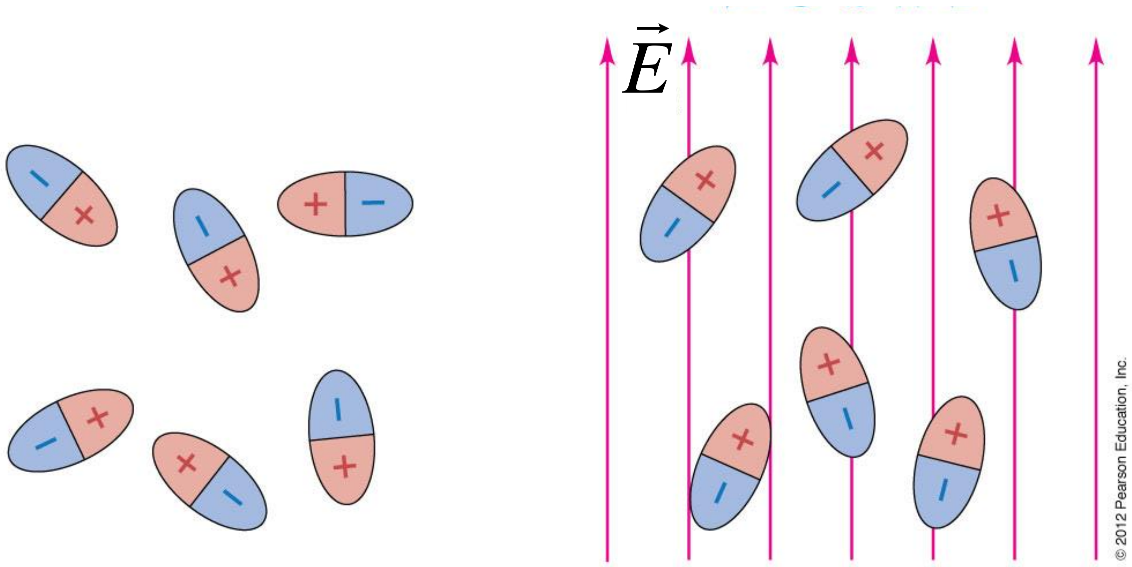
non-polar molecules

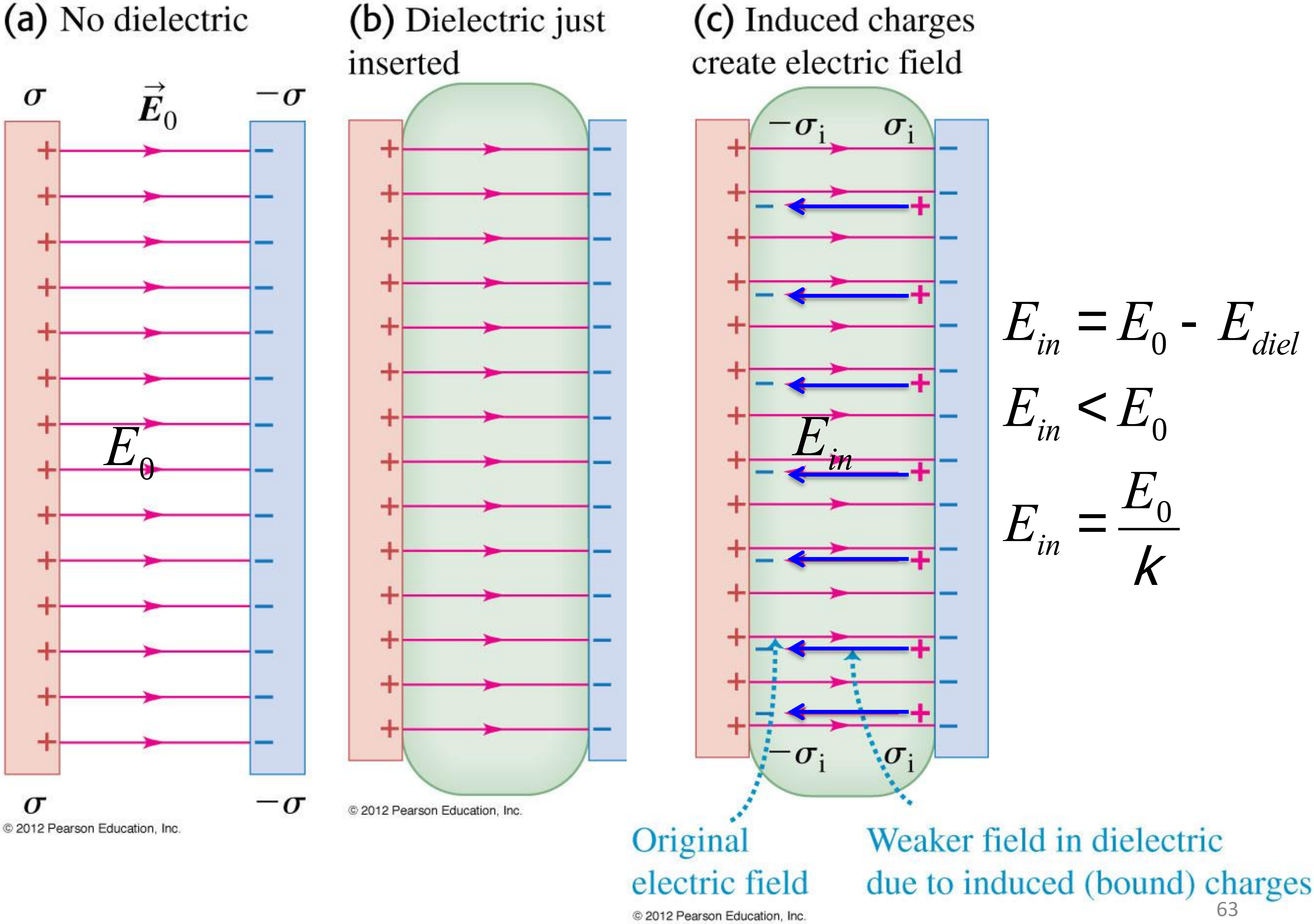


Recall the balloon on the wall example from week 1



polar molecules

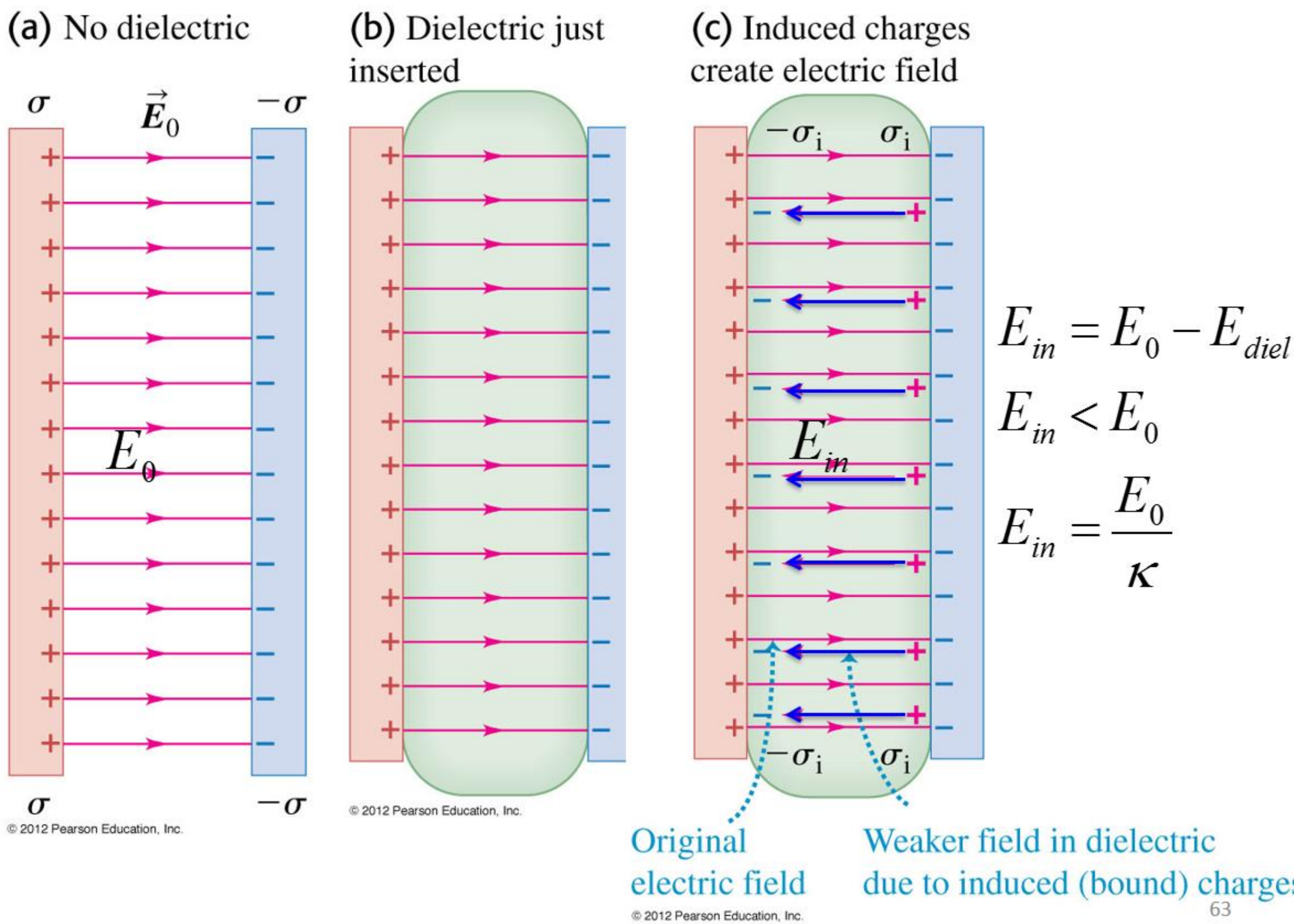




$$E_{in} = E_0 - E_{diel}$$

$$E_{in} < E_0$$

$$E_{in} = \frac{E_0}{k}$$

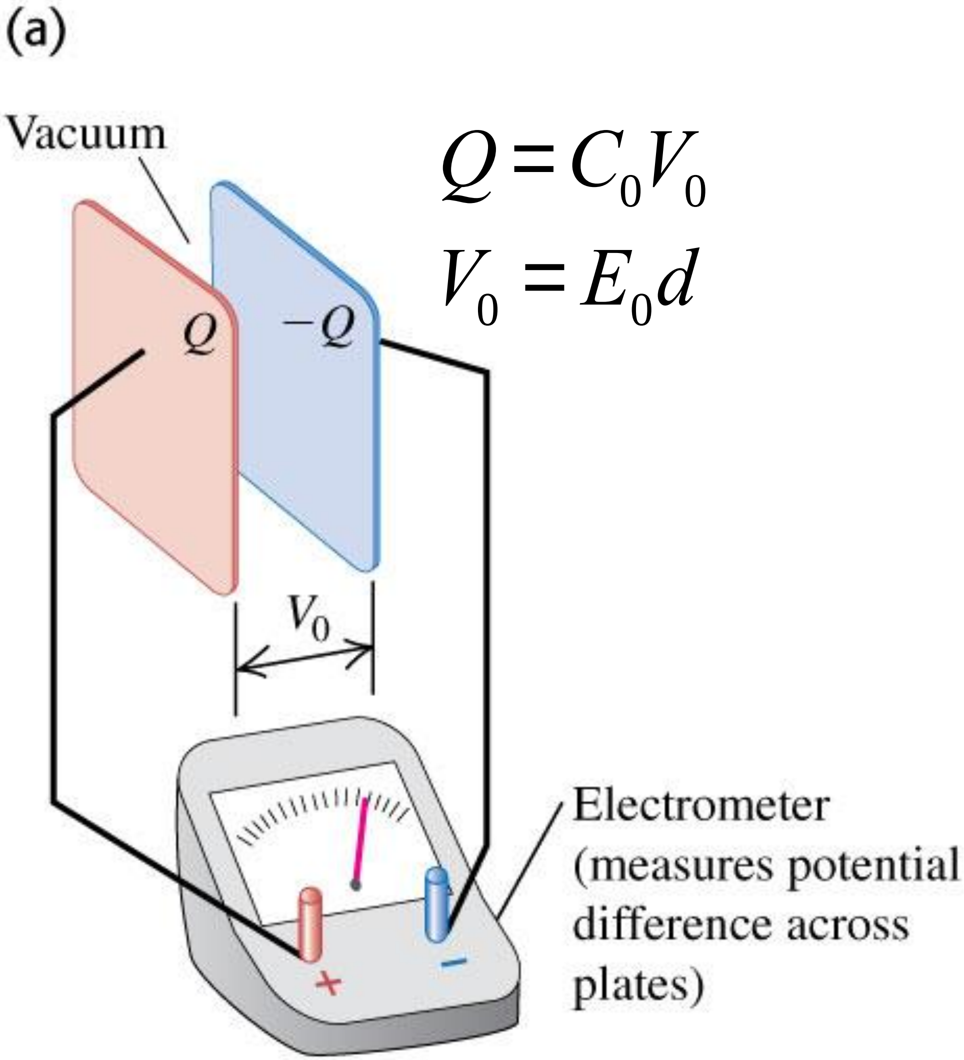


$$E_{in} = E_0 - E_{diel}$$

$$E_{in} < E_0$$

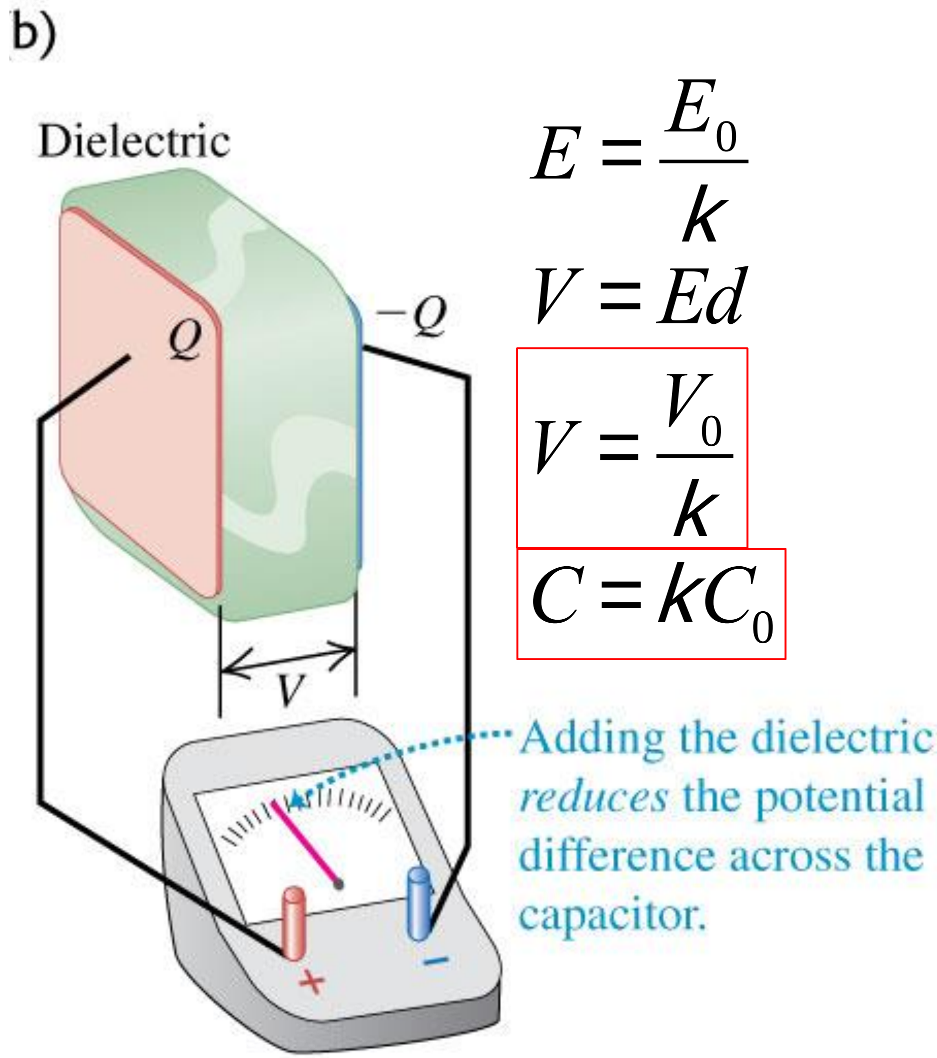
$$E_{in} = \frac{E_0}{\kappa}$$

Linear Dielectrics



$$Q = C_0 V_0$$

$$V_0 = E_0 d$$



$$E = \frac{E_0}{k}$$

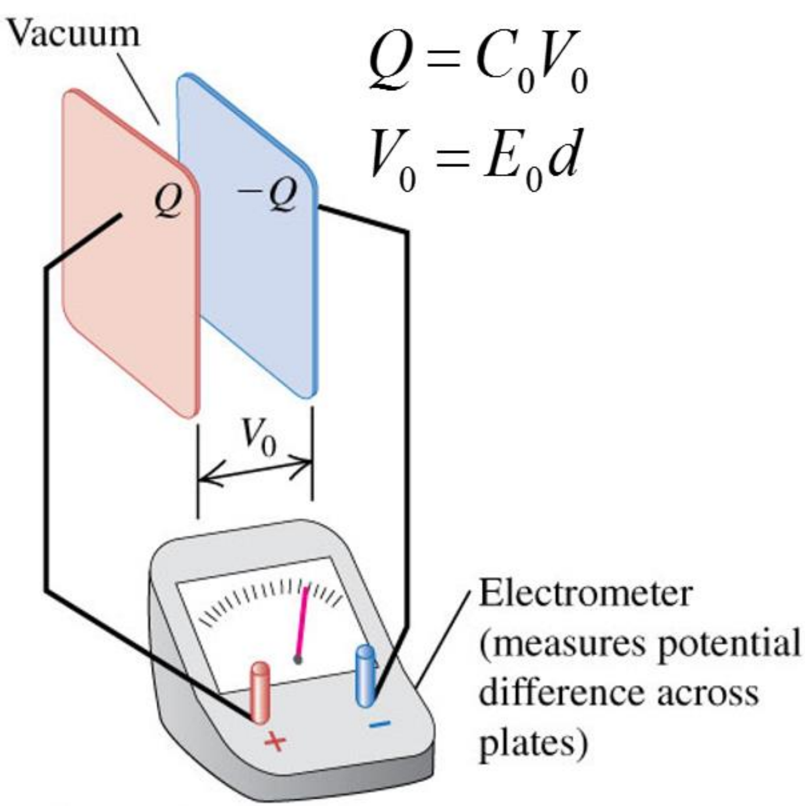
$$V = Ed$$

$$\boxed{V = \frac{V_0}{k}}$$

$$\boxed{C = kC_0}$$

Linear Dielectrics

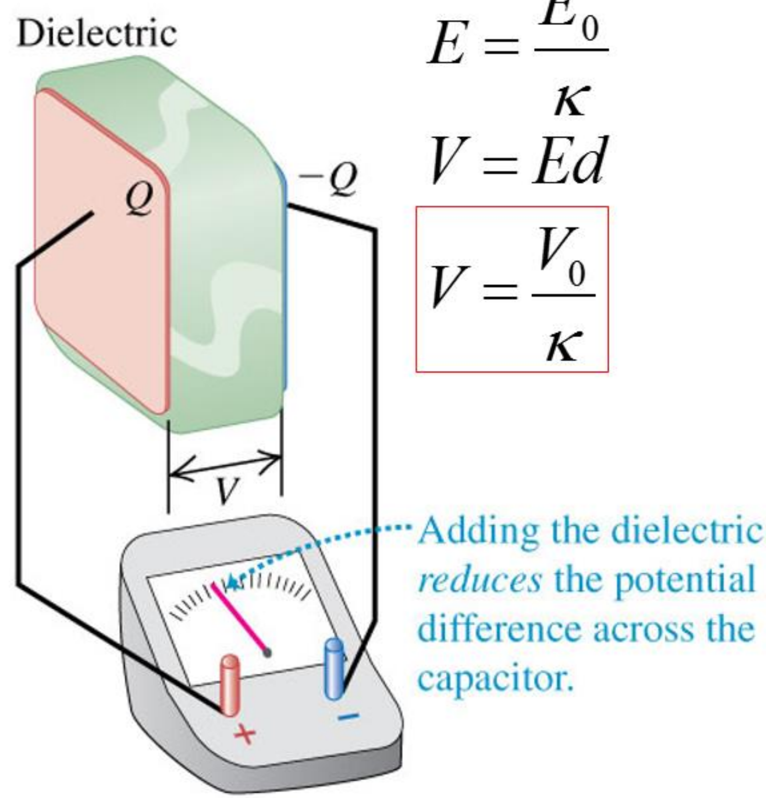
(a)



$$Q = C_0 V_0$$

$$V_0 = E_0 d$$

b)



$$E = \frac{E_0}{\kappa}$$

$$V = Ed$$

$$\boxed{V = \frac{V_0}{\kappa}}$$

Parallel Plate Capacitors With and Without Dielectrics

$$E_0 = \frac{s}{e_0} \qquad\qquad E = \frac{E_0}{k} = \frac{s}{ke_0}$$

$$V_0 = E_0 d = \frac{s\,d}{e_0} \qquad\qquad V = \frac{V_0}{k} = \frac{s\,d}{ke_0}$$

$$C_0 = \frac{Q}{V_0} = \frac{s\,A}{s\,d}\,e_0 = \frac{Ae_0}{d} \qquad\qquad C = kC_0 = \frac{Ake_0}{d}$$

Conclusion: for a linear dielectric, all the regular electrostatic equations hold if $e_0 \rightarrow e \equiv ke_0$

Now you see why e_0 is called "permittivity of free space" (i.e. in vacuum)

Parallel Plate Capacitors With and Without Dielectrics

$$E_0 = \frac{\sigma}{\varepsilon_0} \qquad E = \frac{E_0}{\kappa} = \frac{\sigma}{\kappa\varepsilon_0}$$

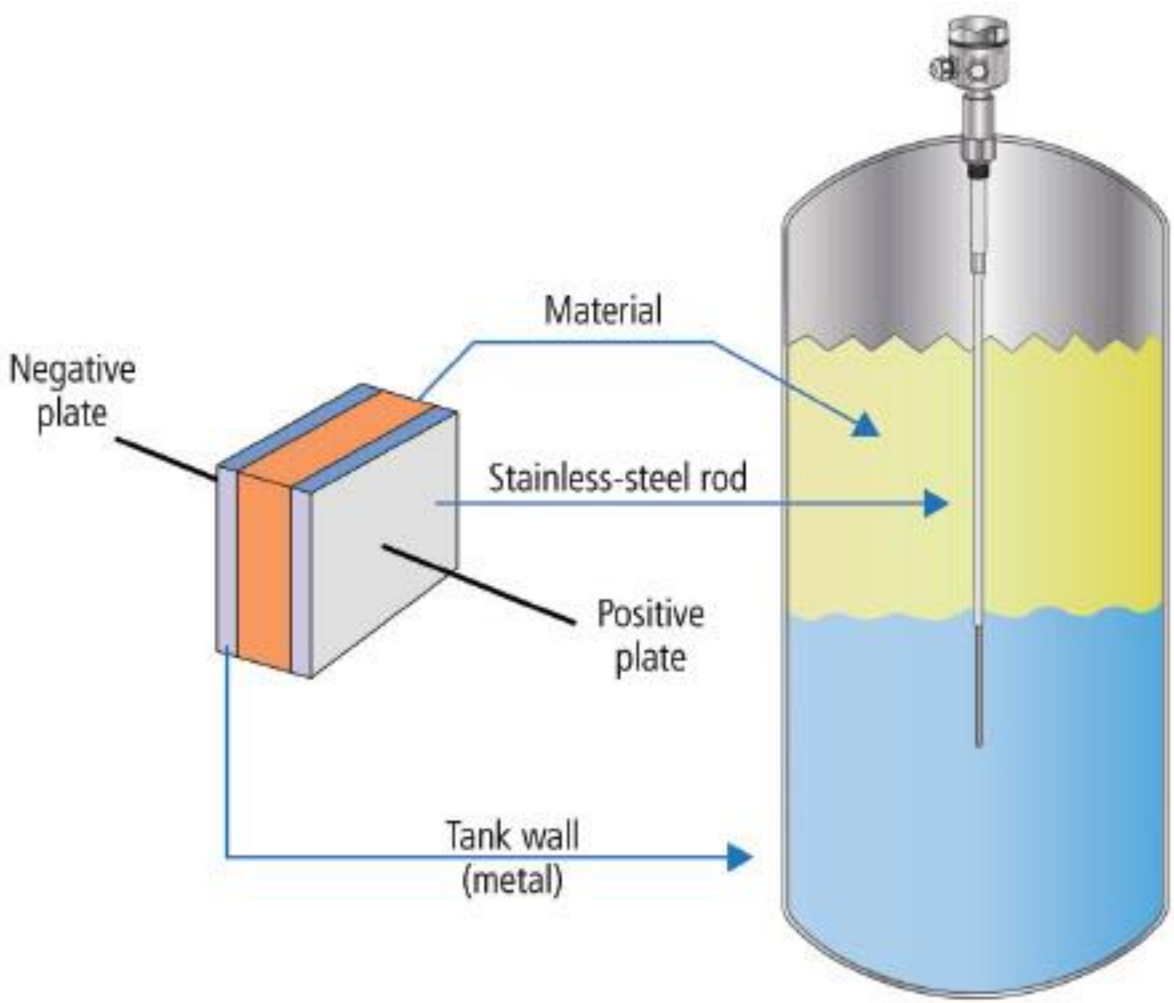
$$V_0 = E_0 d = \frac{\sigma d}{\varepsilon_0} \qquad V = \frac{V_0}{\kappa} = \frac{\sigma d}{\kappa\varepsilon_0}$$

$$C_0 = \frac{Q}{V_0} = \frac{\sigma A}{\sigma d}\varepsilon_0 = \frac{A\varepsilon_0}{d} \qquad C = \kappa C_0 = \frac{A\kappa\varepsilon_0}{d}$$

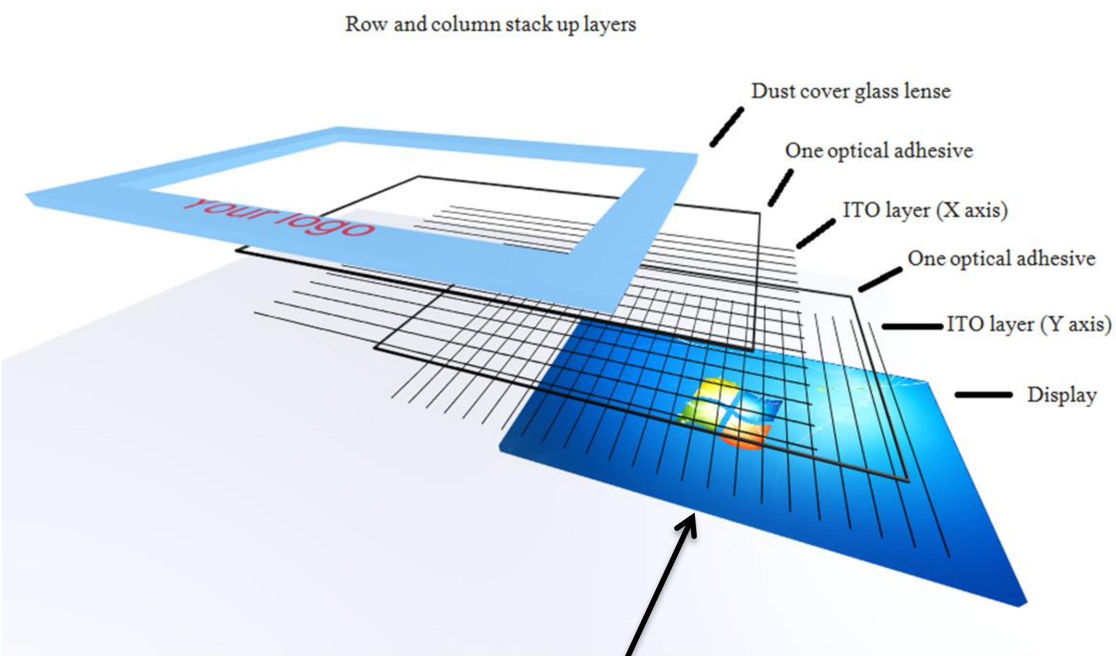
Conclusion: for a linear dielectric, all the regular electrostatic equations hold if $\varepsilon_0 \to \varepsilon \equiv \kappa\varepsilon_0$

Now you see why ε_0 is called "permittivity of free space" (i.e. in vacuum)

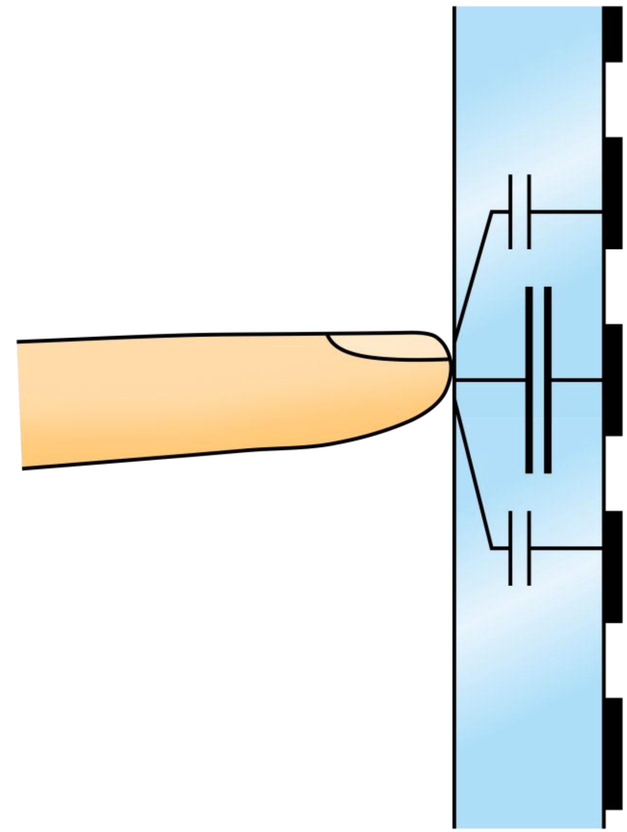
Application: Capacitive Fuel Gauge



Application: Capacitive Touch Screen



Should be



TopHat Question

A capacitor without a dielectric is charged up so that it stores potential energy U_0, and it is then disconnected so that **its charge remains the same**. A dielectric with constant $\kappa = 2$ is then inserted between the plates. What is the new potential energy stored in the capacitor **with the dielectric**?

$$U_C = \frac{Q^2}{2C} = \frac{V_C^2 C}{2}$$

A. $4U_0$

B. $2U_0$

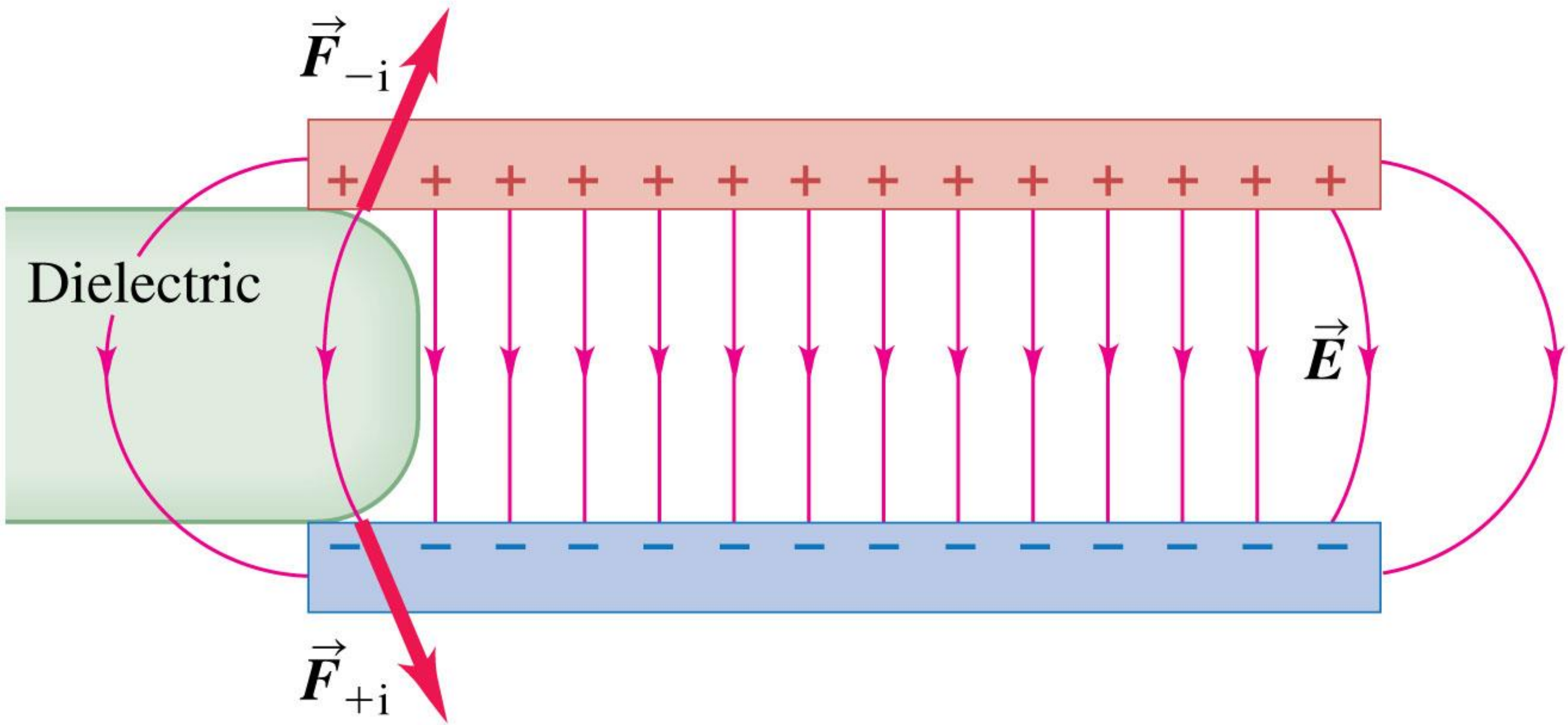
C. $\frac{1}{2}U_0$

D. $\frac{1}{4}U_0$

The **potential energy lowers** when the dielectric is added, so it will feel a **force sucking it into the gap** between the plates.

Can find the force using $F_x = -dU/dx$ and

$$W = \int dq \left[\frac{q}{C}\right] = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$



Fringe Electric Field pulls the dielectric into the gap

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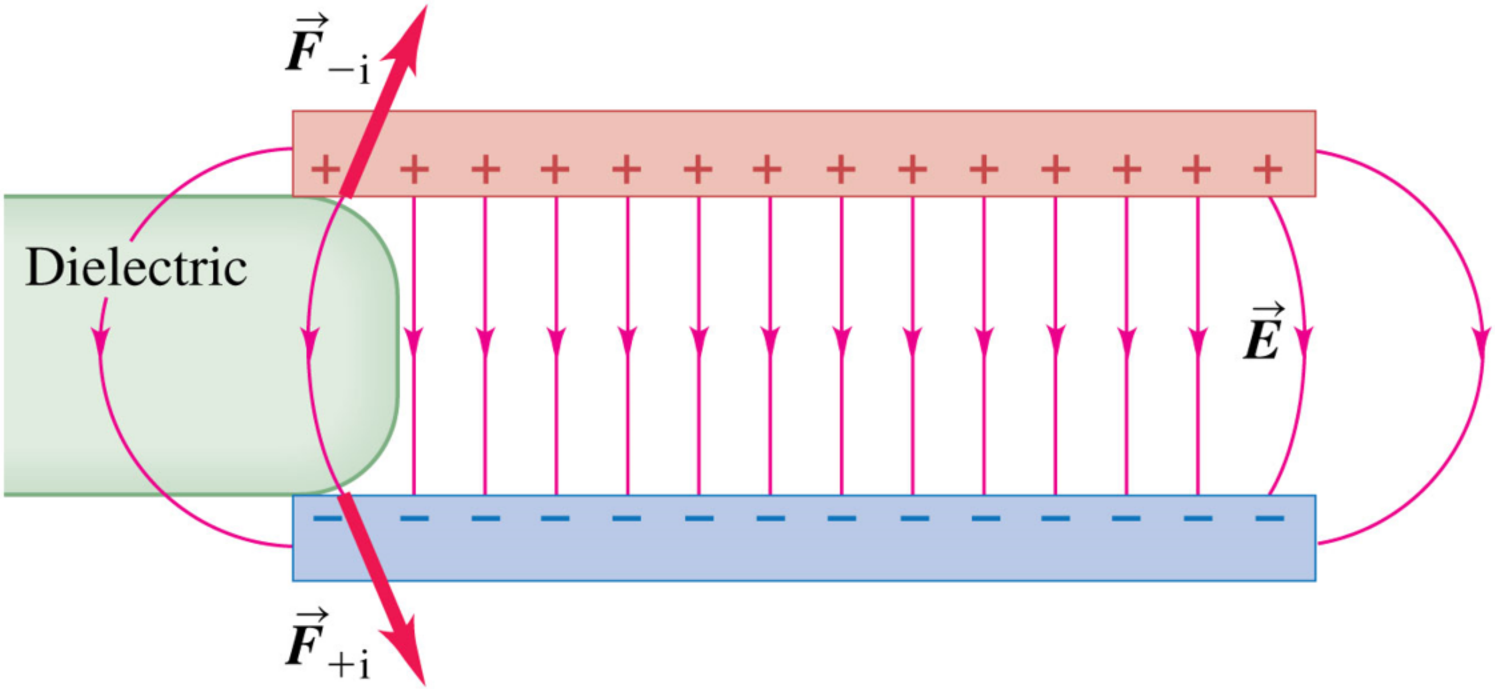


Fringe Electric Field pulls the dielectric into the gap

TopHat Question

A capacitor without a dielectric is charged up so that it stores potential energy U_0, and is kept connected **at constant voltage**. A dielectric with constant $\kappa = 2$ is then inserted between the plates. What is the new potential energy stored in the capacitor **with the dielectric**?

$$U_C = \frac{Q^2}{2C} = \frac{V_C^2 C}{2}$$

A. $4U_0$

B. $2U_0$

C. $\frac{1}{2}U_0$

D. $\frac{1}{4}U_0$

The **potential energy raises** when the dielectric is added, so it will feel a **force pushing it out of the gap** between the plates.

Wall Climbing robots

http://www.sri.com/engage/products-solutions/electroadhesive-surface-climbing-robots

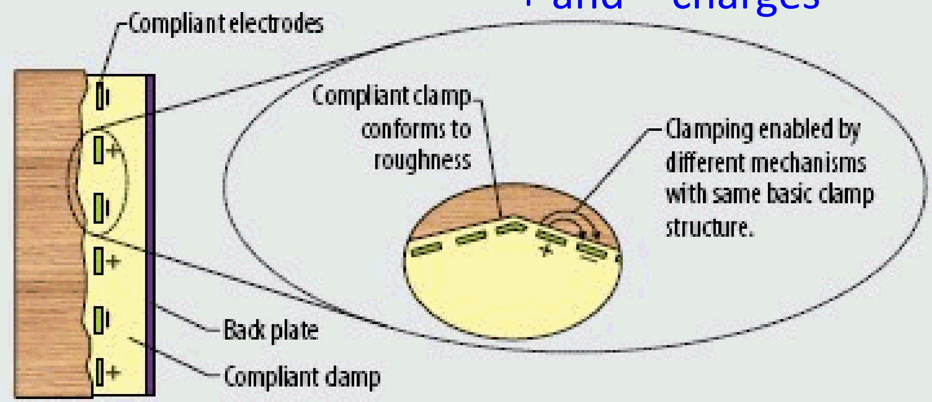
Wall Climbing robots

https://www.youtube.com/watch?v=I4DHfNtZGts&lr=1

0.5 to 1.5 N per square cm of clamp

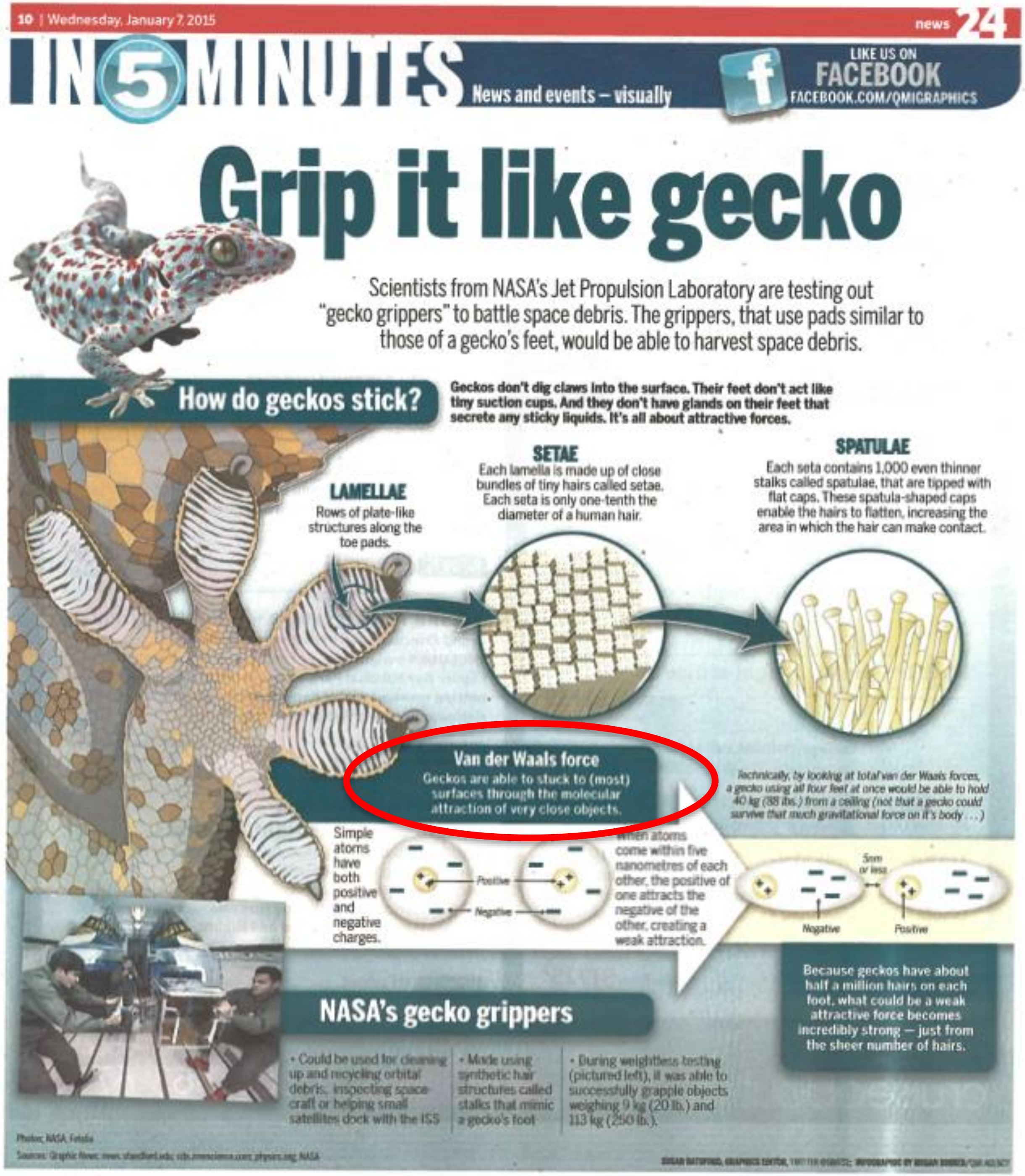
What is Electroadhesion?

Alternating
+ and – charges



- Clamps using induced electrostatic charges from on-board battery power (as opposed to passive approaches using Van der Waals forces)
- Can be switched on or off quickly (<50 ms)
- Clamps to both conductive and non-conductive substrates using the same clamp geometry
- Conforms around surface features / roughness due to compliant materials and electrodes

http://www.sri.com/engage/products-solutions/electroadhesive-surface-climbing-robots



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Grip it like gecko

Scientists from NASA's Jet Propulsion Laboratory are testing out "gecko grippers" to battle space debris. The grippers, that use pads similar to those of a gecko's feet, would be able to harvest space debris.

How do geckos stick?

Geckos don't dig claws into the surface. Their feet don't act like tiny suction cups. And they don't have glands on their feet that secrete any sticky liquids. It's all about attractive forces.

LAMELLAE
Rows of plate-like structures along the toe pads.

SETAE
Each lamella is made up of close bundles of tiny hairs called setae. Each seta is only one-tenth the diameter of a human hair.

SPATULAE
Each seta contains 1,000 even thinner stalks called spatulae, that are tipped with flat caps. These spatula-shaped caps enable the hairs to flatten, increasing the area in which the hair can make contact.

Van der Waals force
Geckos are able to stuck to (most) surfaces through the molecular attraction of very close objects.

Simple atoms have both positive and negative charges.

Positive
Negative

When atoms come within five nanometres of each other, the positive of one attracts the negative of the other, creating a weak attraction.

Technically, by looking at total van der Waals forces, a gecko using all four feet at once would be able to hold 40 kg (88 lbs.) from a ceiling (not that a gecko could survive that much gravitational force on it's body . . .)

5nm or less
Negative
Positive

Because geckos have about half a million hairs on each foot, what could be a weak attractive force becomes incredibly strong — just from the sheer number of hairs.

NASA's gecko grippers

- Could be used for cleaning up and recycling orbital debris, inspecting space craft or helping small satellites dock with the ISS
- Made using synthetic hair structures called stalks that mimic a gecko's foot
- During weightless testing (pictured left), it was able to successfully grapple objects weighing 9 kg (20 lb.) and 113 kg (250 lb.).

Photos: NASA, Fotolia

So Far

- Subtlety with capacitors: series or parallel?
- Linear dielectric materials: an atomic perspective
- Effect of dielectrics on capacitance
- Applications of dielectrics and capacitors

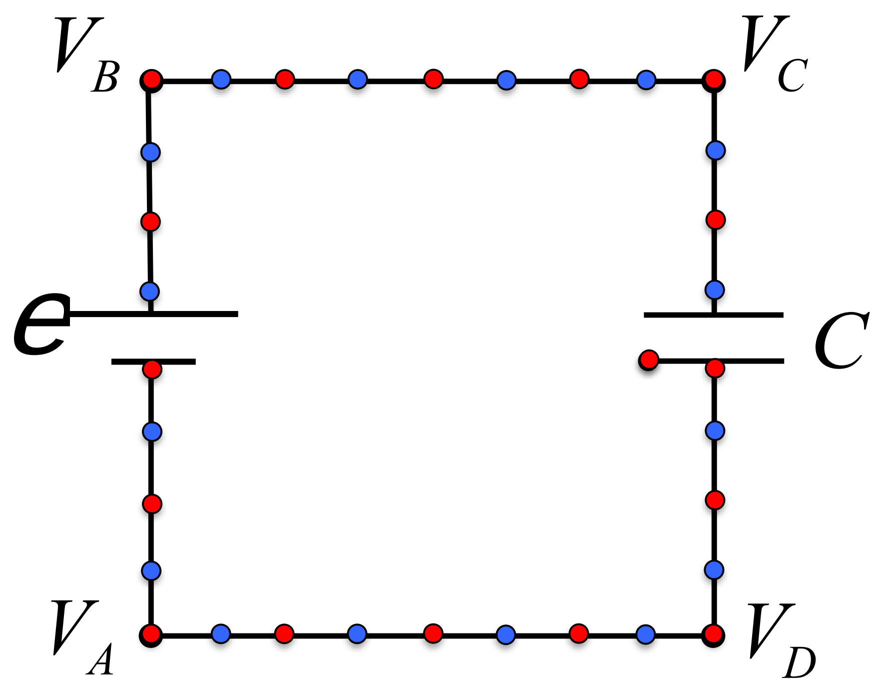
To be continued

- Capacitors in electric circuits: how charges move
- Kirchhoff's loop rule with capacitors
- Capacitors in series and parallel
- More complicated capacitor circuit

A Basic Circuit with Capacitor

The simplest capacitor circuit has an ideal battery, ideal wires, and a single capacitor.

The battery causes charges to flow from the bottom plate to the top plate. This creates a potential ΔV_C between the two plates. Remember charges never "jump the gap" between the two plates of a capacitor.



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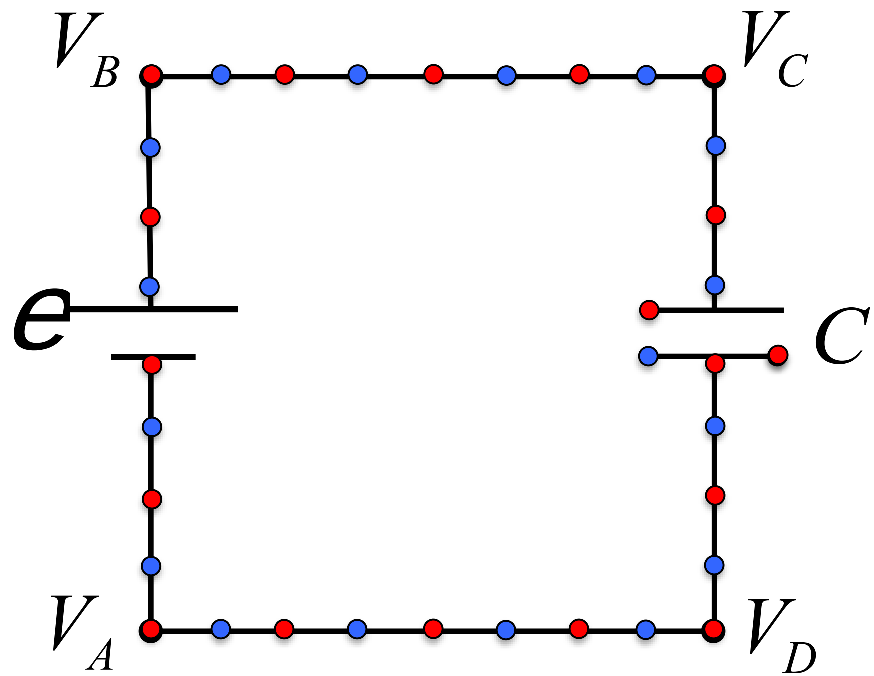

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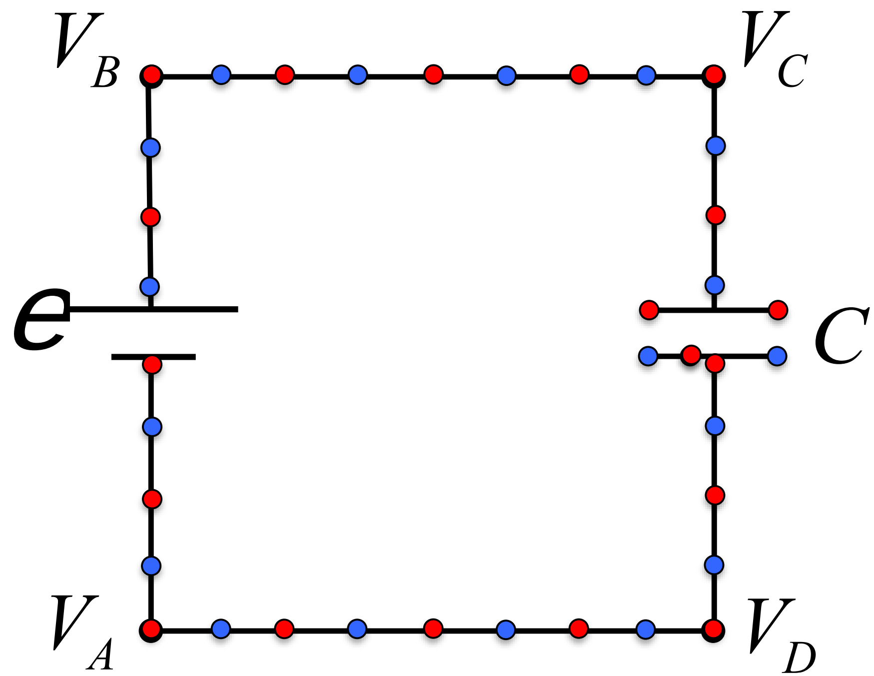

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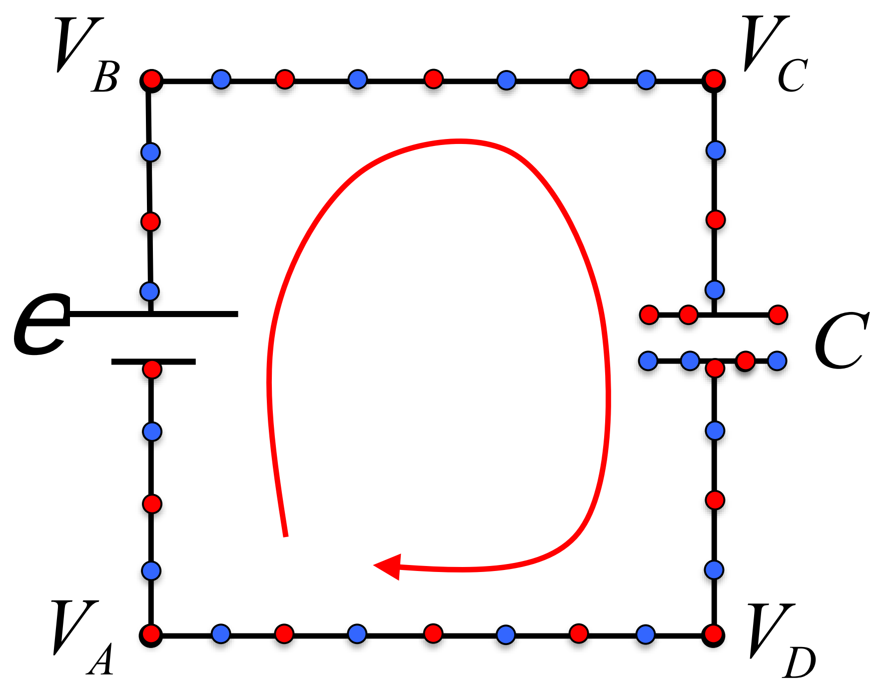
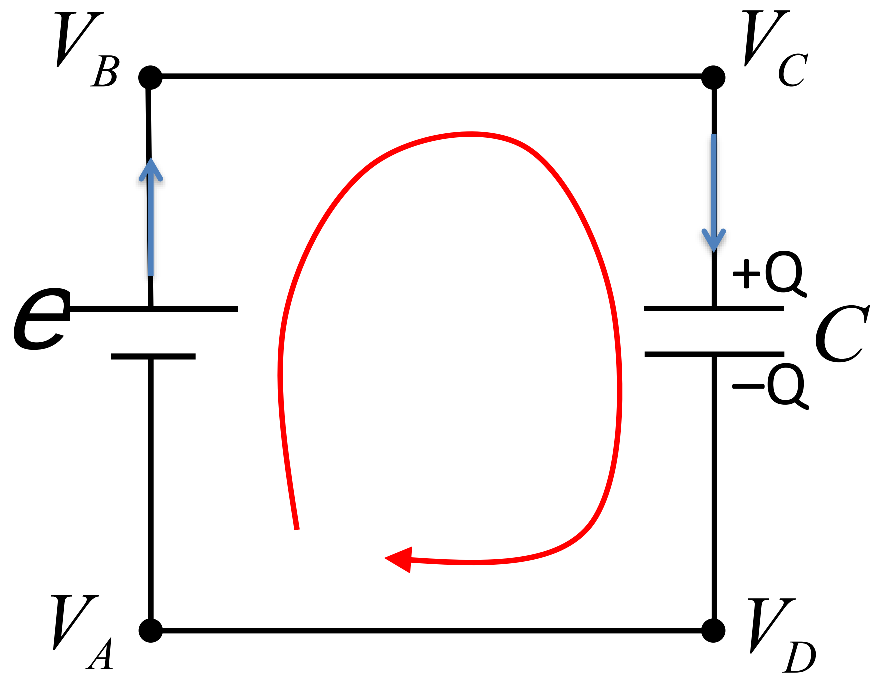


$$\boxed{\Delta V_{AB} + \Delta V_{BC} + \Delta V_{CD} + \Delta V_{DA} = 0}$$

A Basic Circuit

The voltage across a capacitor is **negative** if you are going around the loop in the direction **from the + plate to the – plate**. Current flows **from the negative** terminal **to the positive** terminal



ideal wires

$$\Delta V_{AB} + \cancel{\Delta V_{BC}} + \Delta V_{CD} + \cancel{\Delta V_{DA}} = 0$$

$$\mathcal{E} - \frac{Q}{C} = 0$$

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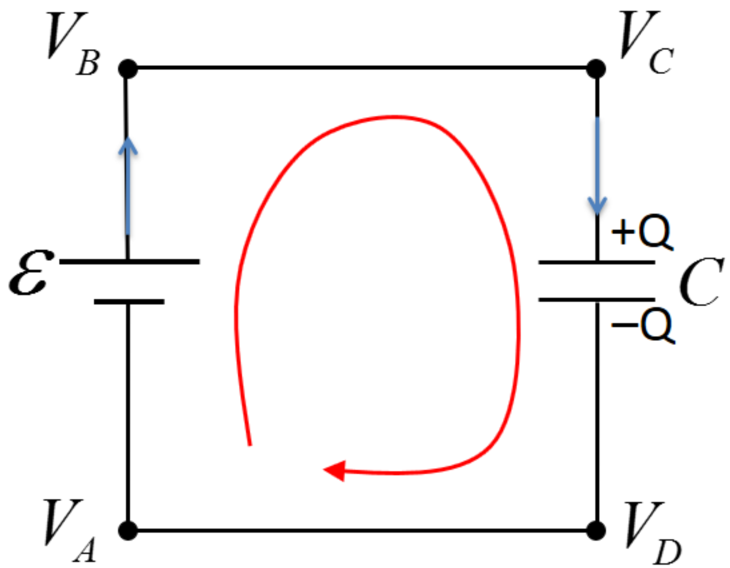


ideal wires

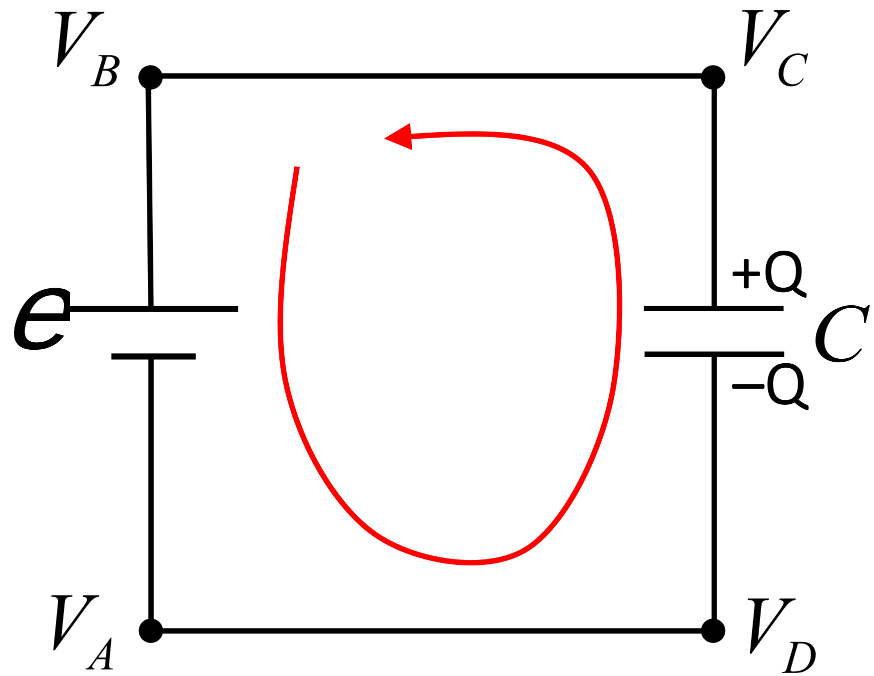
$$\Delta V_{AB} + \cancel{\Delta V_{BC}} + \Delta V_{CD} + \cancel{\Delta V_{DA}} = 0$$

$$\mathcal{E} - \frac{Q}{C} = 0$$

A Basic Circuit

The voltage across a capacitor is **positive** if you are going around the loop in the direction **from – plate to + plate**.

Voltage across a battery is **negative** going **from positive to negative**



ideal wires

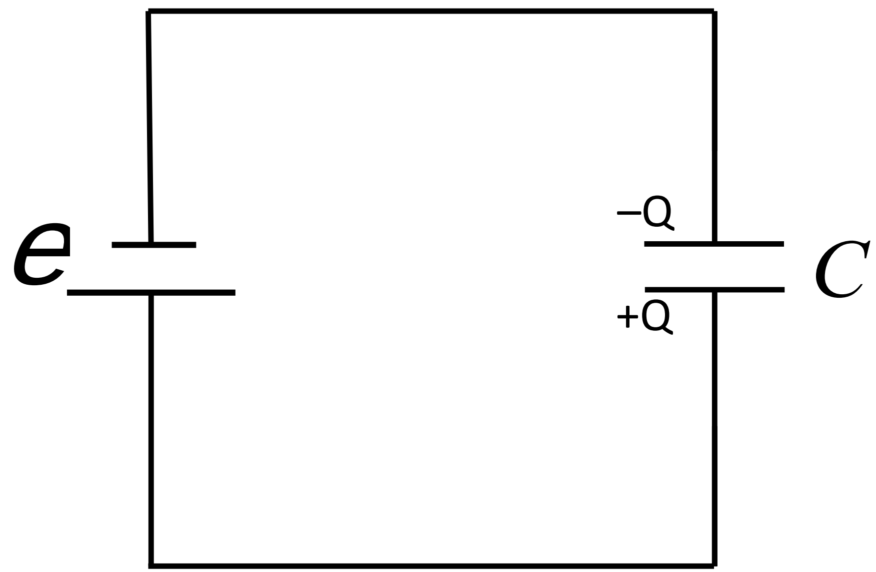
$$\Delta V_{BA} + \cancel{\Delta V_{AD}} + \Delta V_{DC} + \cancel{\Delta V_{CB}} = 0$$

$$-e + \frac{Q}{C} = 0$$

Same as before

TopHat Question

What is the charge on the top plate of the capacitor in the circuit shown?
$\mathcal{E}$ = 12 V and C = 0.25 µF.



A. Q = 3.0 µC

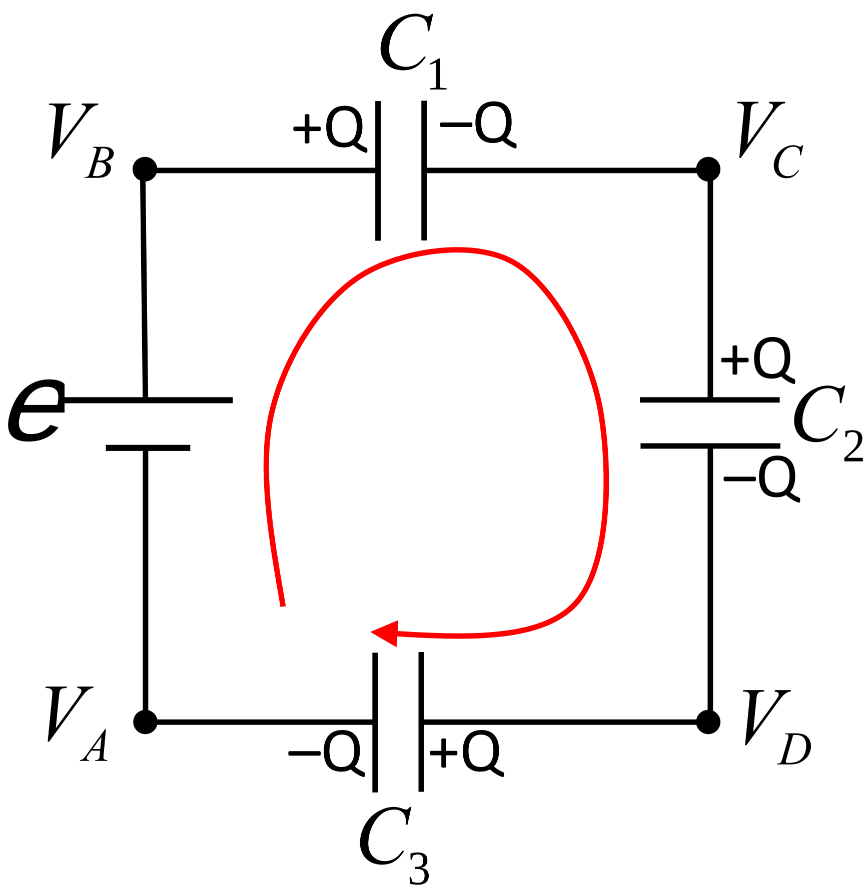
A. Q = 48 µC

C. Q = 21 nC

D. Q = –3.0 µC

Capacitors in Series

A slightly more complicated circuit has multiple capacitors in series



Kirchhoff's Loop Rule:

$$\Delta V_{AB} + \Delta V_{BC} + \Delta V_{CD} + \Delta V_{DA} = 0$$

Charge on each plate is the same

$$\varepsilon - \frac{Q}{C_1} - \frac{Q}{C_2} - \frac{Q}{C_3} = 0$$

Rewrite this as

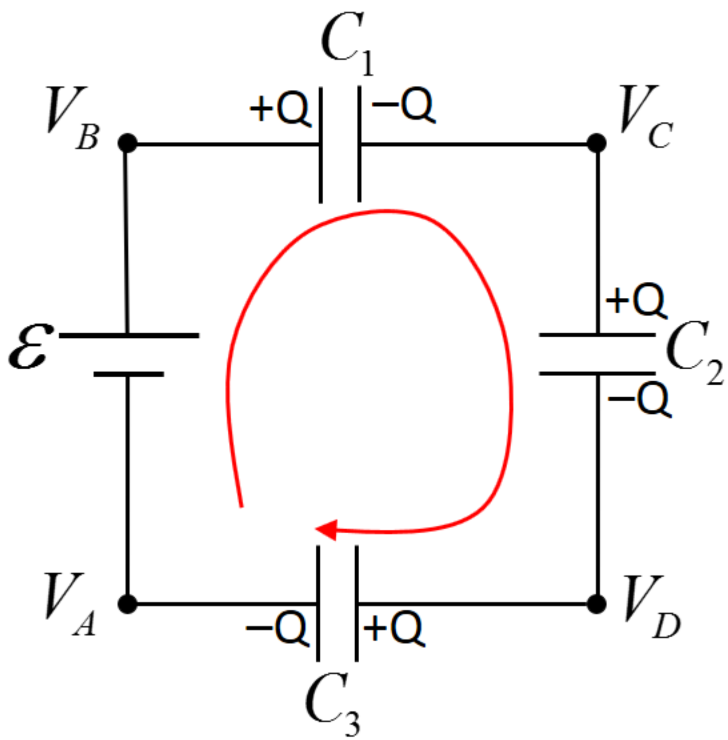
$$\varepsilon - Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) = 0$$

Define an equivalent capacitance

$$\varepsilon - \frac{Q}{C_{eq}} = 0$$

Capacitors in Series

A slightly more complicated circuit has multiple capacitors in series



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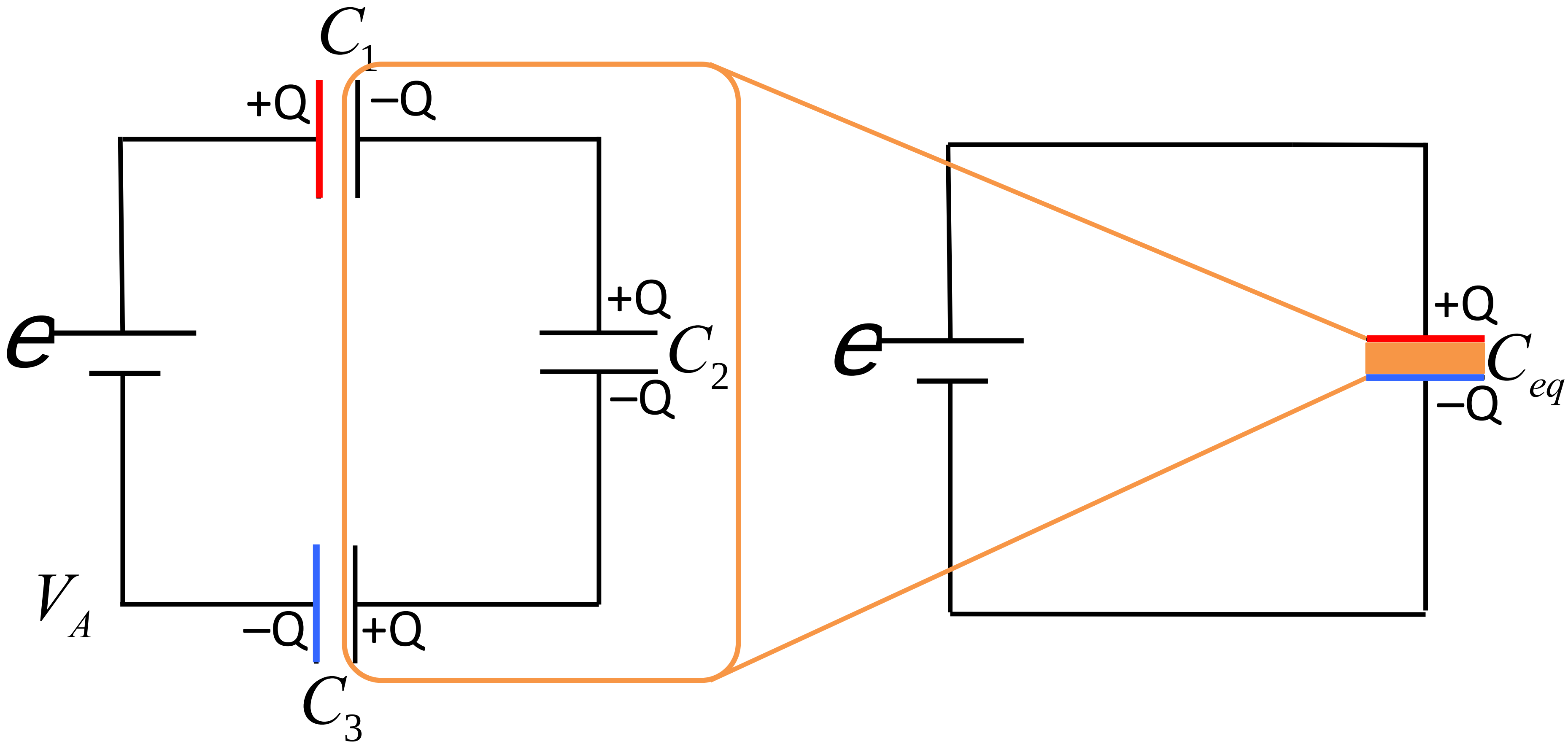
Capacitors in Series

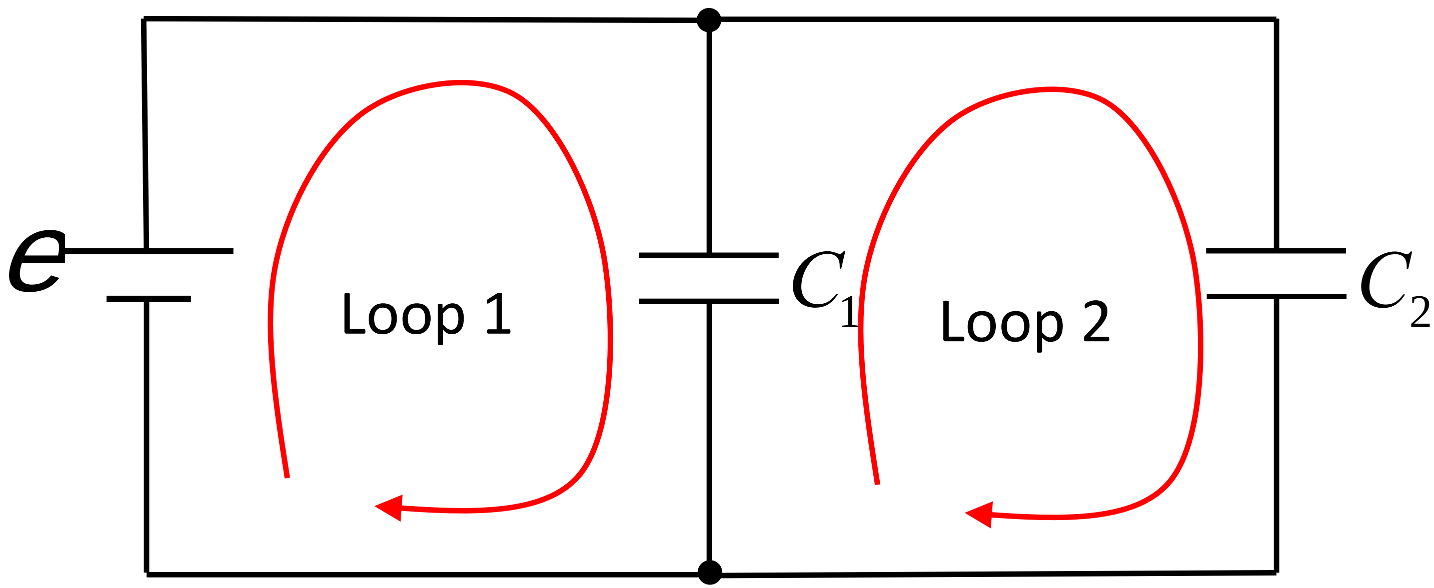


Capacitors in series act like a single equivalent capacitor:

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right)^{-1}$$

Capacitors in Parallel

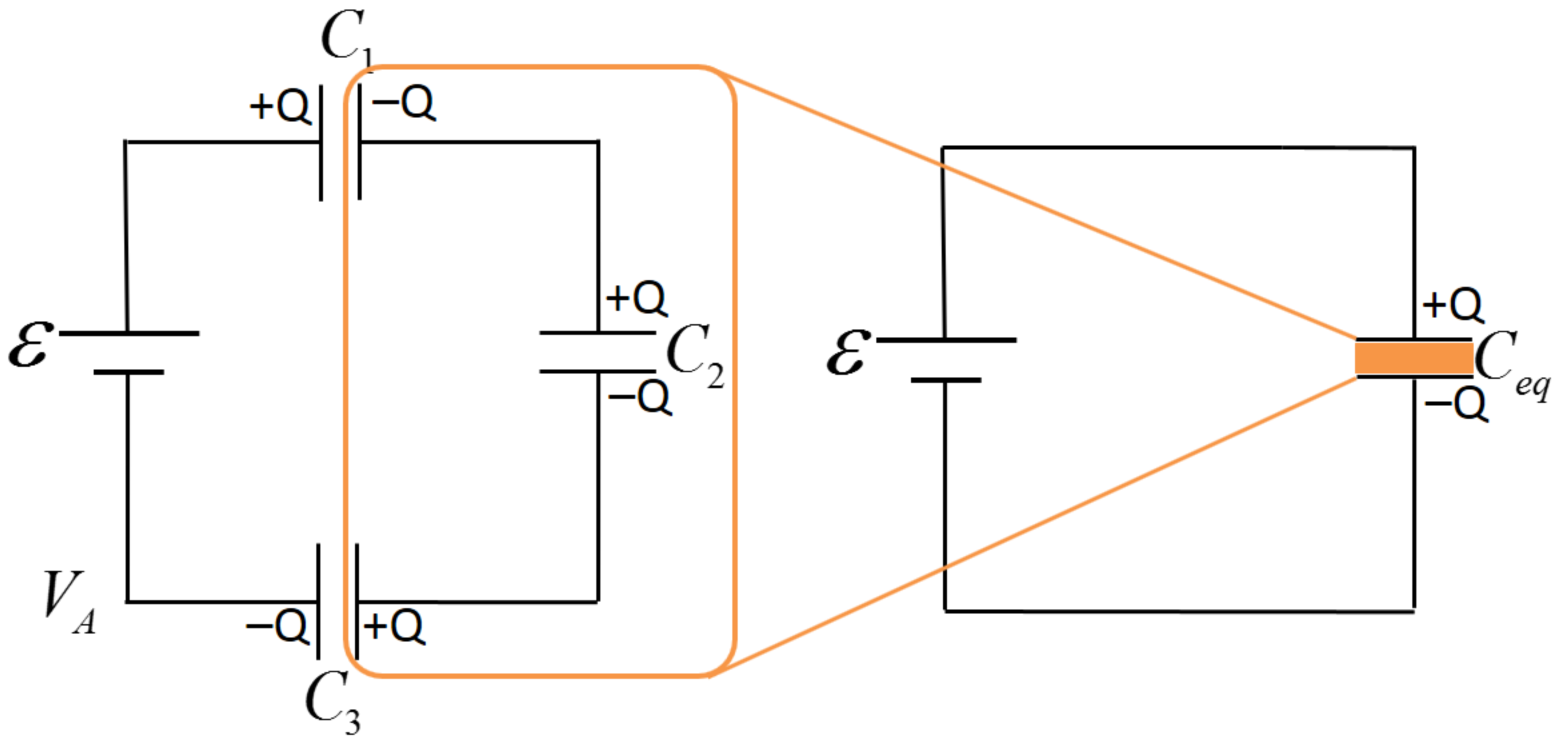
A slightly more complicated circuit has multiple branches with capacitors in parallel



Capacitors in parallel have the same voltage across their plates

Loop 1: $\varepsilon - \Delta V_{C_1} = 0$ Loop 2: $\Delta V_{C_1} - \Delta V_{C_2} = 0$

Capacitors in Series



Capacitors in series act like a single equivalent capacitor:

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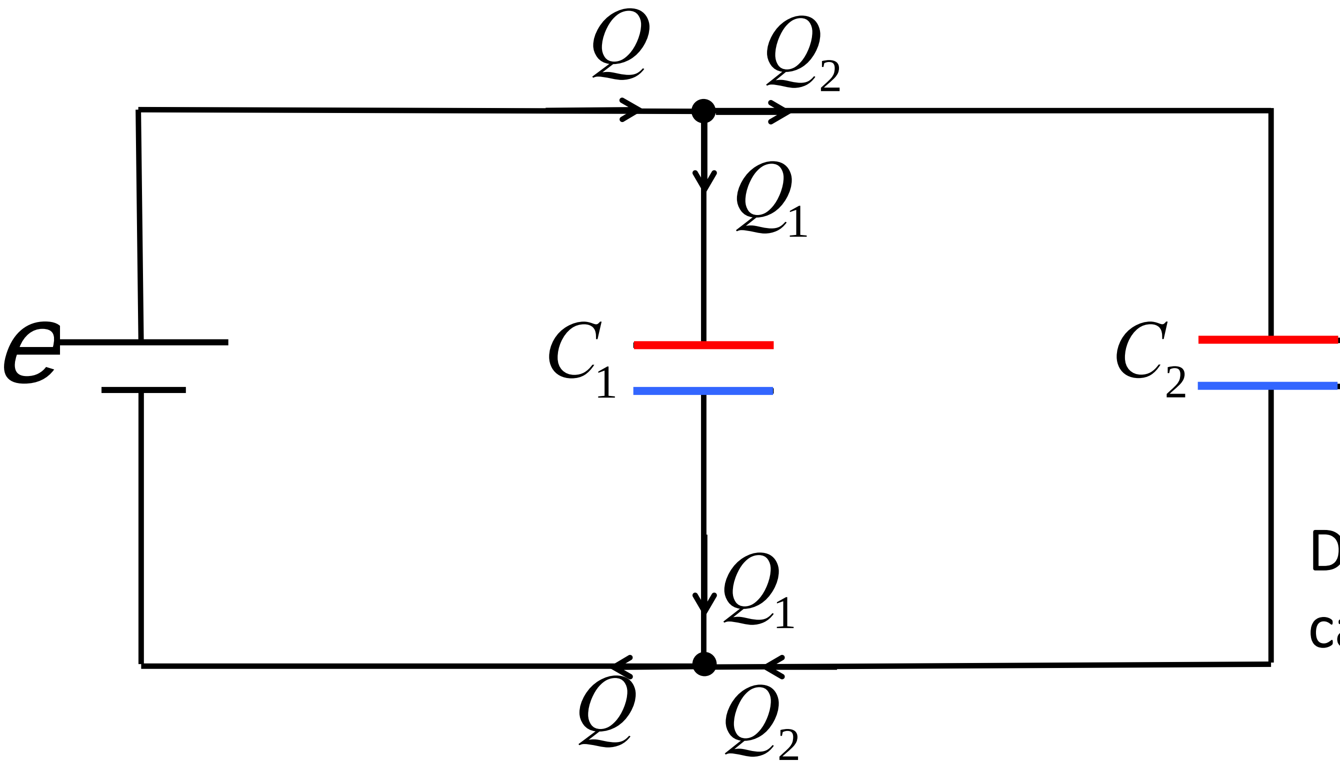
Capacitors in Parallel



$$Q = Q_1 + Q_2$$

$$Q_1 = \Delta V_{C_1} C_1$$

$$Q_2 = \Delta V_{C_2} C_2$$

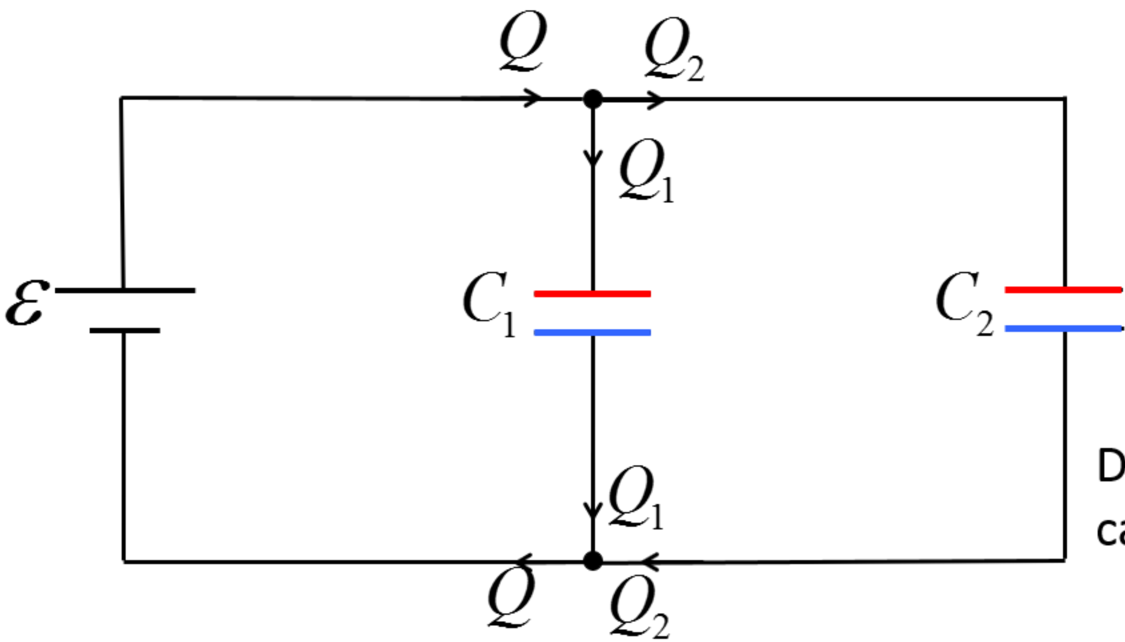
Define an equivalent capacitance:

$$Q = \mathcal{E} C_{eq}$$

From conservation of charge: $\mathcal{E} C_{eq} = \mathcal{E} C_1 + \mathcal{E} C_2$

For capacitors in parallel: $C_{eq} = C_1 + C_2$

Capacitors in Parallel



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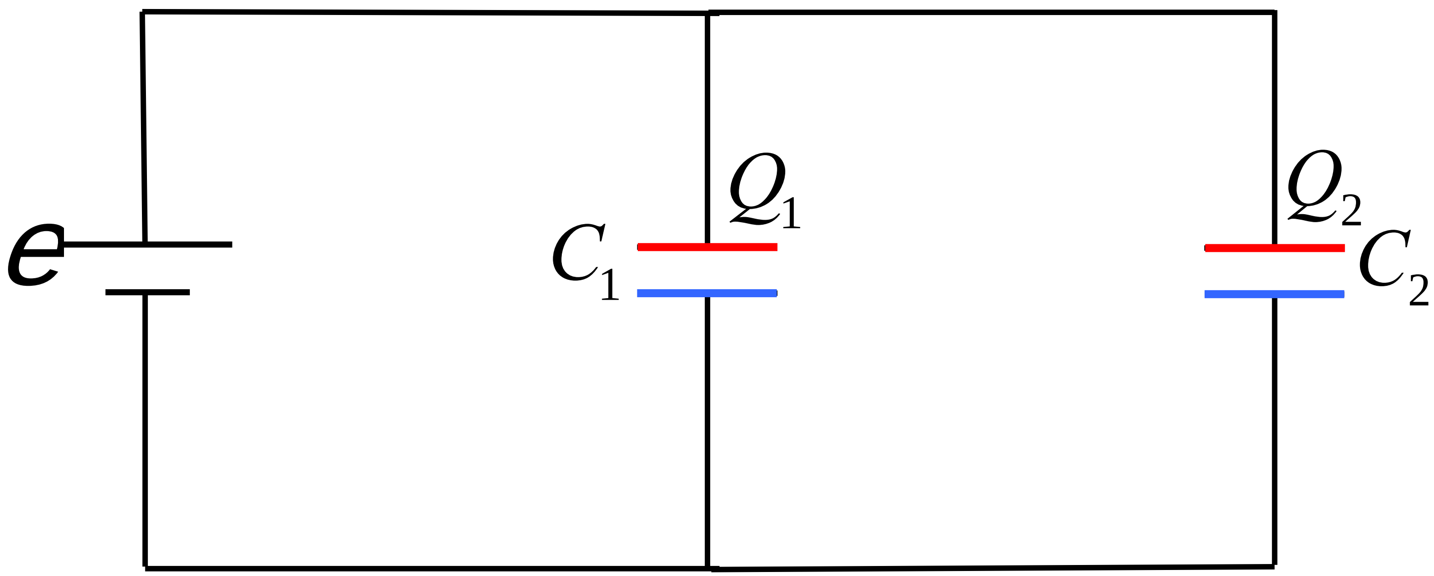


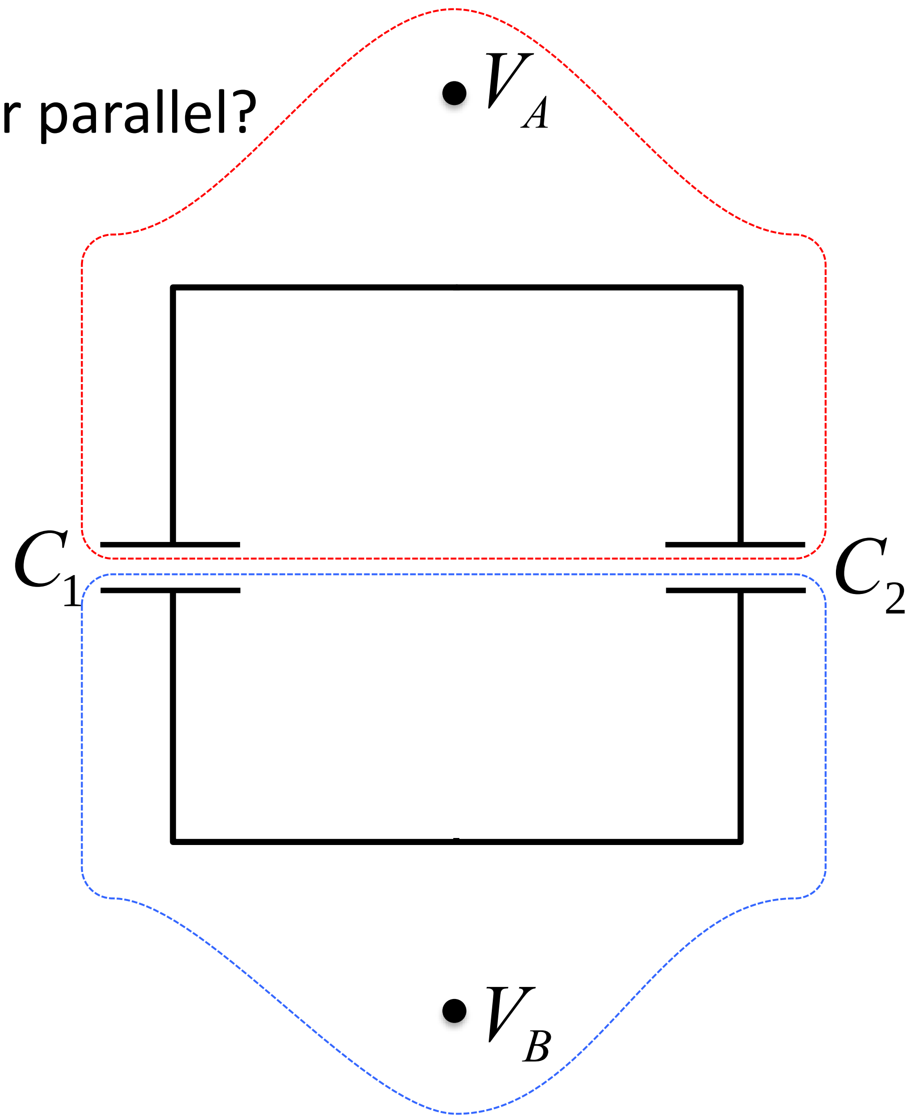
$$Q = Q_1 + Q_2$$

$$C_{eq} = C_1 + C_2$$

Capacitor Subtlety

Are these capacitors in series or parallel?

The plates are conductors, and so are the wires.
The top half of the circuit is electrically disconnected from the bottom half.
Top half is at potential V_A and the bottom half is at potential V_B.
These capacitors are in parallel! They have the same voltage across them



Summary of Capacitors

Relation between charge and voltage across plates

$$V_C = \frac{Q}{C}$$

Capacitors in Series: store the same amount of charge

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N} \right)^{-1}$$

Capacitors in Parallel: have the same voltage across them

$$C_{eq} = C_1 + C_2 + \ldots + C_N$$

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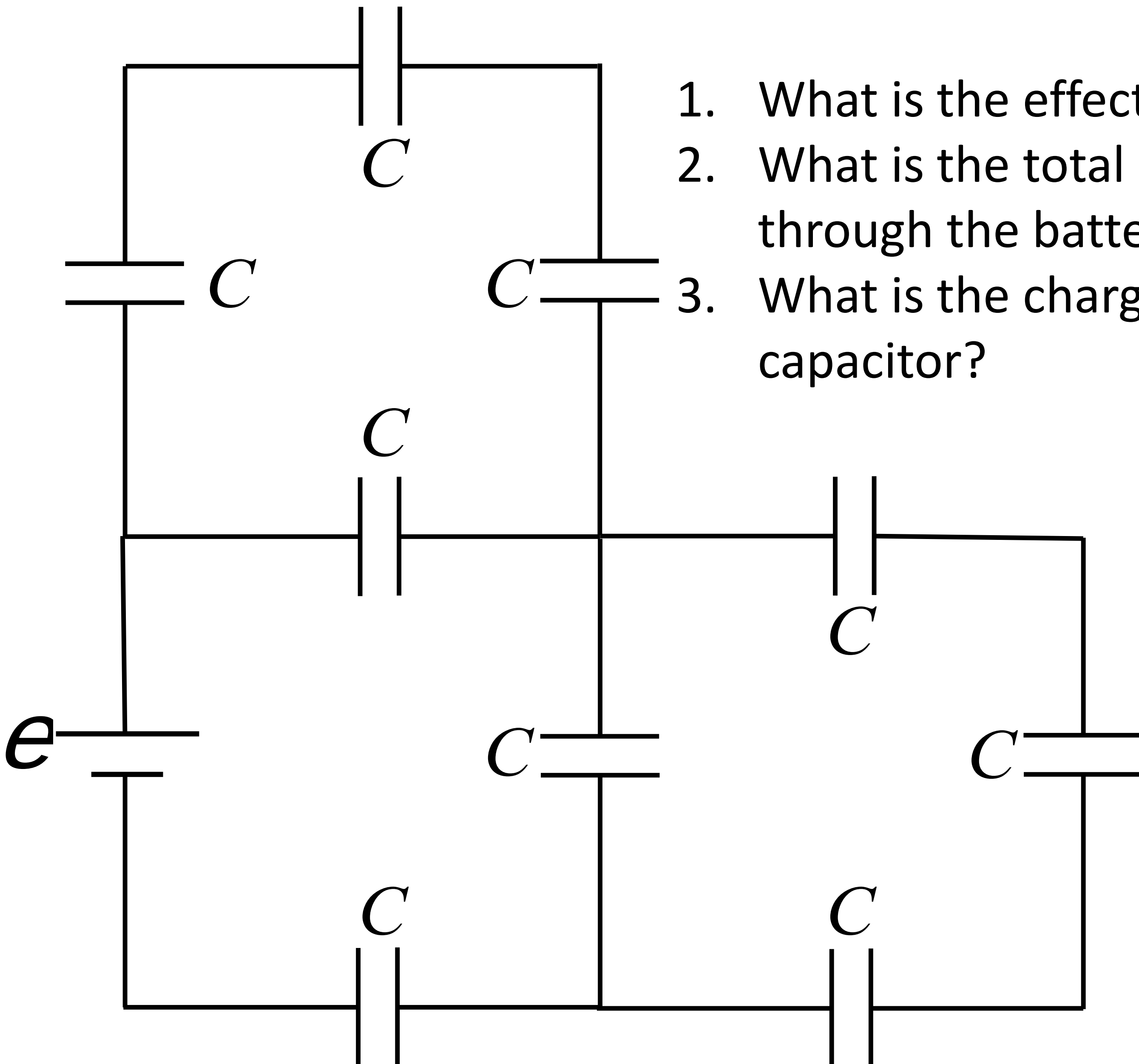
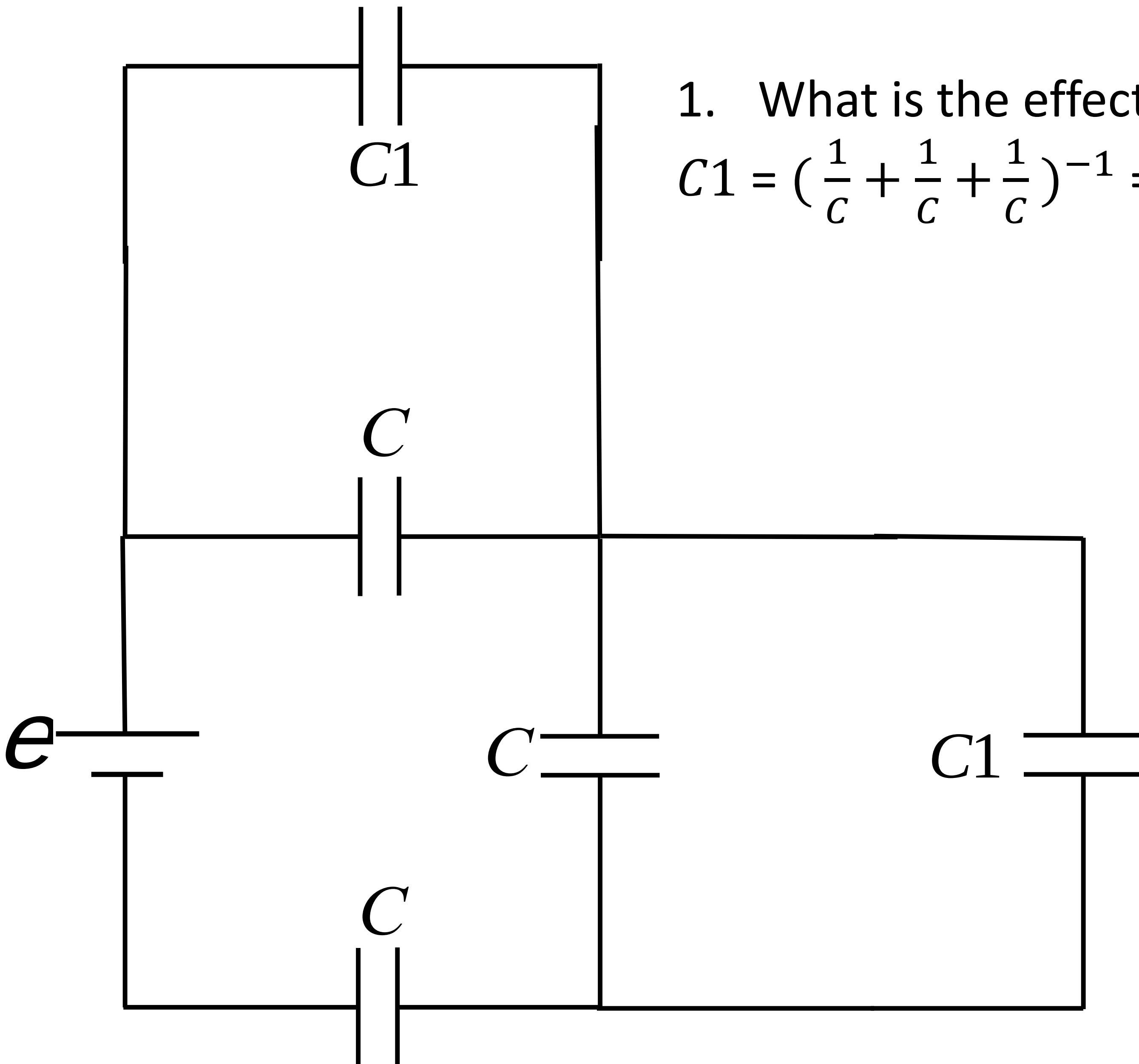
Example



1. What is the effective capacitance?
2. What is the total charge moved through the battery?
3. What is the charge on each capacitor?

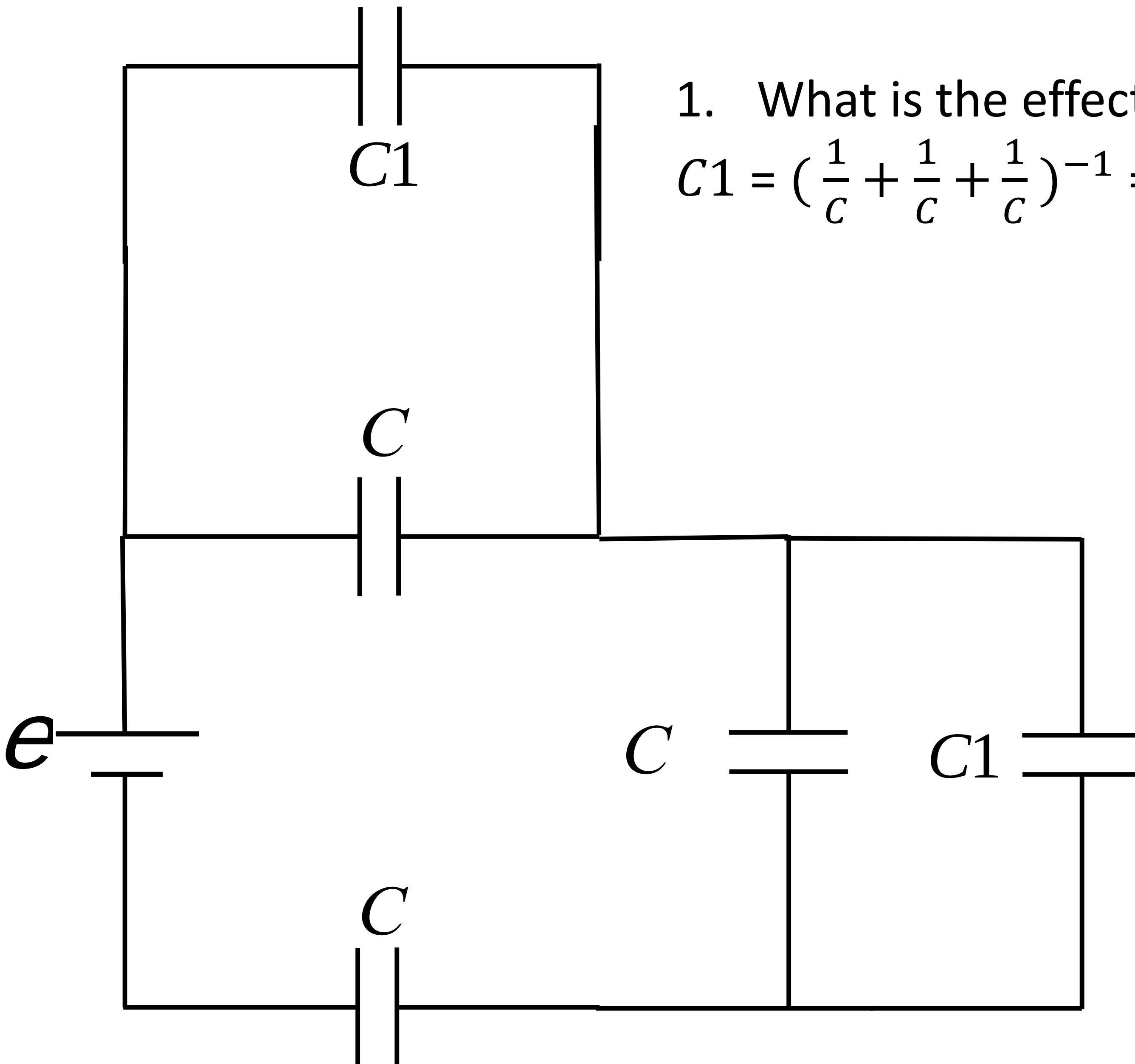
Example



1. What is the effective capacitance?

$C1 = \left(\frac{1}{C}+\frac{1}{C}+\frac{1}{C}\right)^{-1} = \frac{1}{3}\ C$

Example



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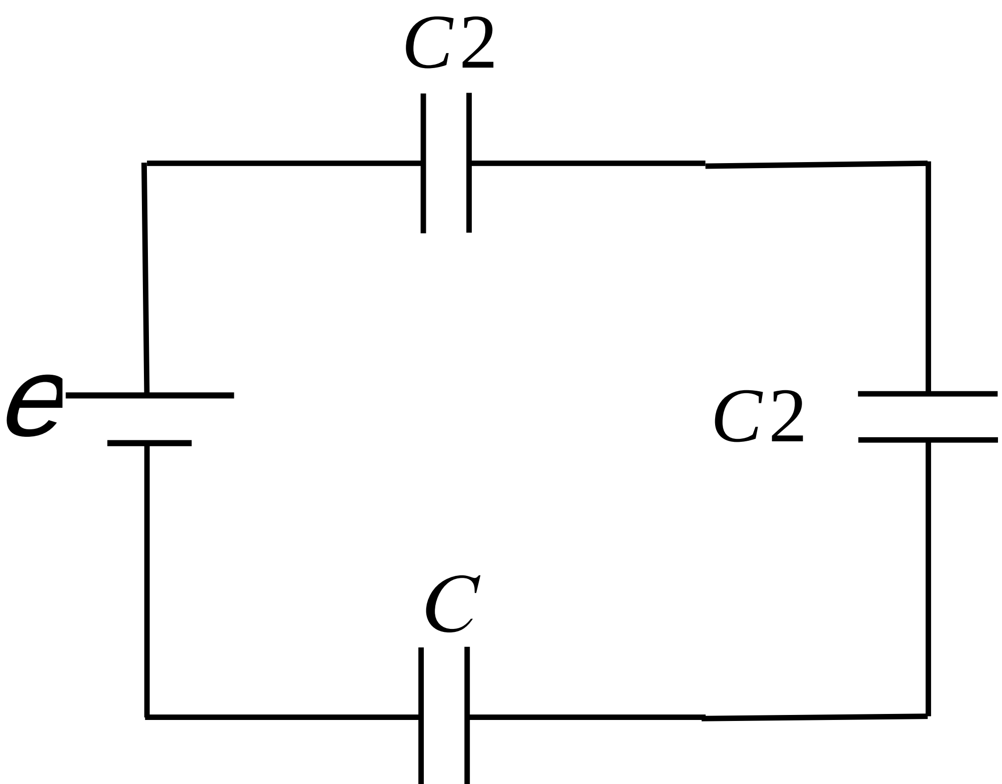
Example

1. What is the effective capacitance?

$$C1 = \left(\frac{1}{C} + \frac{1}{C} + \frac{1}{C}\right)^{-1} = \frac{1}{3}\ C$$

$$C2 = C1 + C = \frac{C}{3} + C = \frac{4C}{3}$$



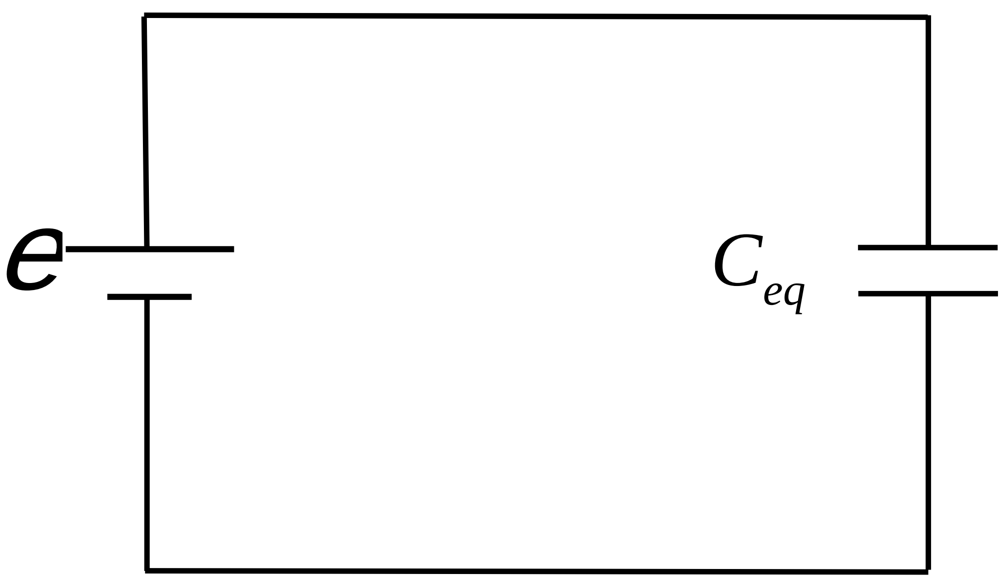
Example

1. What is the effective capacitance?

$C1 = (\frac{1}{C} + \frac{1}{C} + \frac{1}{C})^{-1} = \frac{1}{3}\ C$

$$C2 = C1 + C = \frac{C}{3} + C = \frac{4C}{3}$$

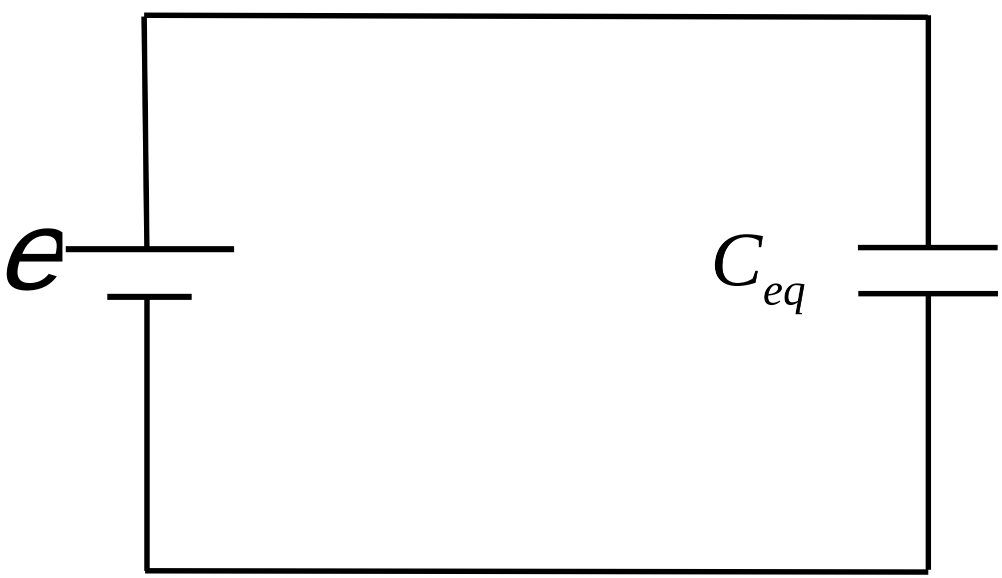
$$C_{eq} = (\frac{1}{C} + \frac{1}{C2} + \frac{1}{C2})^{-1} = (\frac{4}{4C} + \frac{3}{4C} + \frac{3}{4C})^{-1} = (\frac{10}{4C})^{-1} = 0.4C$$



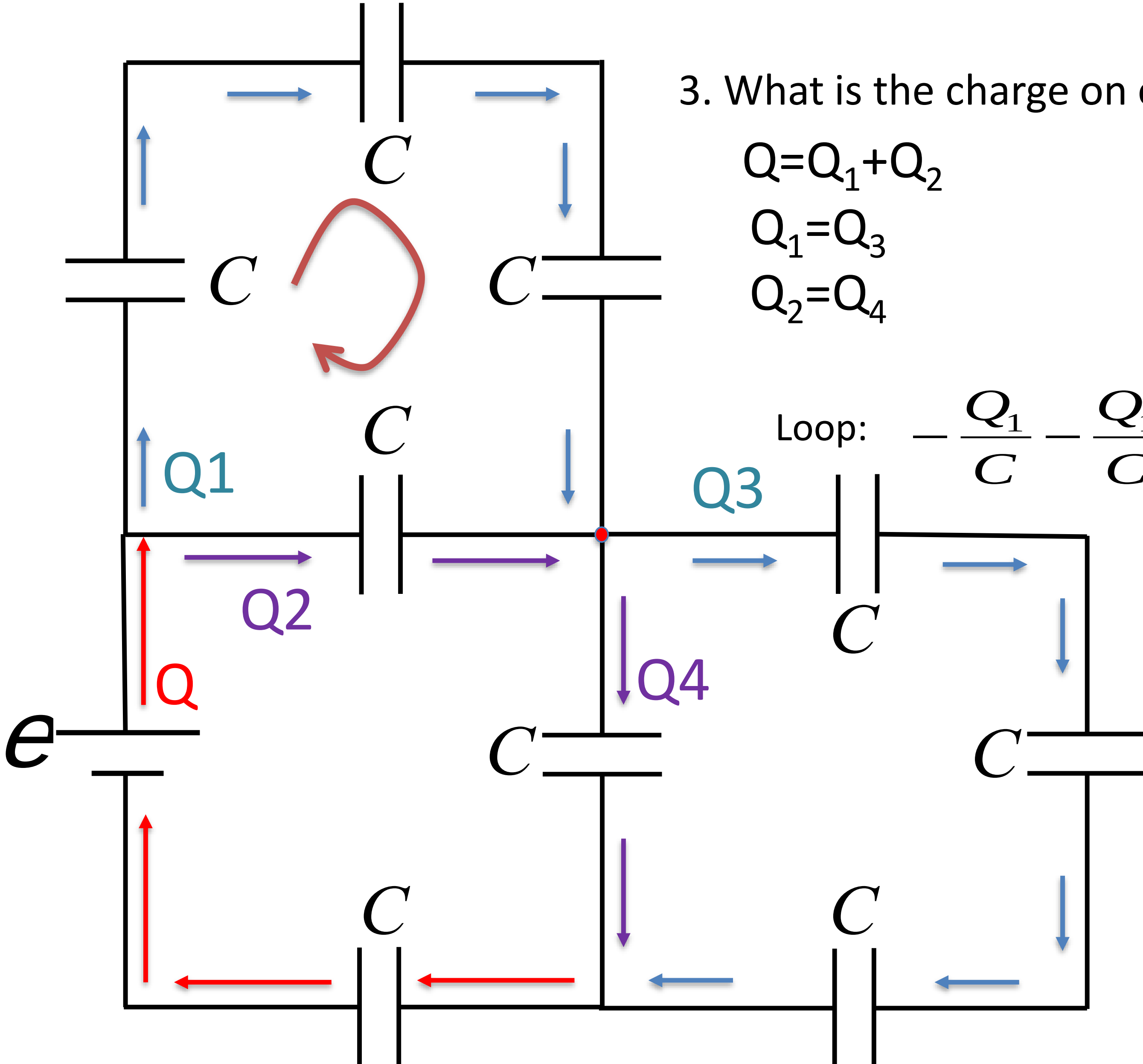
Example

2. What is the total charge moved through the battery?

$$\varepsilon - \frac{Q}{C_{eq}} = 0 \Rightarrow Q = \varepsilon C_{eq}$$



Example



3. What is the charge on each capacitor?

$Q = Q_1 + Q_2$

$Q_1 = Q_3$

$Q_2 = Q_4$

Loop: $$-\frac{Q_1}{C} - \frac{Q_1}{C} - \frac{Q_1}{C} + \frac{Q_2}{C} = 0$$

$Q_2 = 3Q_1$

$Q_1 = Q/4$

&

$Q_2 = 3Q/4$

Gauss Law for Dielectric Materials