PURE MATHEMATICS 319 L02 WINTER 2016
QUIZ 1 SOLUTIONS.

1. Give the *definition* of each of the following.
(a) A tranformation of $\mathbb{R}^2$.
Solution: A tranformation of $\mathbb{R}^2$ is a one-to-one and onto function from $\mathbb{R}^2$ to $\mathbb{R}^2$.
(b) A collineation of $\mathbb{R}^2$.
Solution: A collineation of $\mathbb{R}^2$ is a transformation of $\mathbb{R}^2$ that maps every line to a line.
(c) A translation.
Solution: A translation is a function $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ so that for every $(x, y) \in \mathbb{R}^2$, $\tau(x, y) = (x + r, y + s)$ for some $r, s \in \mathbb{R}$.
(d) A halfturn.
Solution: A halfturn centred at $P = (a, b)$ is the function $\sigma_P : \mathbb{R}^2 \to \mathbb{R}^2$ so that for every $(x, y) \in \mathbb{R}^2$, $\sigma_P(x, y) = (-x + 2a, -y + 2b)$.

2. Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation defined by $\tau(x, y) = (x - 1, y - 2)$. Let $P = (1, 1)$. Find the point $Q \in \mathbb{R}^2$ so that $\sigma_Q\sigma_P = \tau$. Make sure that you prove that $\sigma_Q\sigma_P = \tau$.
Solution: Let $Q = \left(\frac{1}{2}, 0\right)$. We prove that $\sigma_Q\sigma_P = \tau$. We note that $\sigma_P(x, y) = (-x + 2, -y + 2)$ and $\sigma_Q(x, y) = (-x + 1, -y)$ for each $(x, y) \in \mathbb{R}^2$. Now, for any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
\sigma_Q\sigma_P(x, y) &= \sigma_Q(\sigma_P(x, y)) \\
&= \sigma_Q(-x + 2, -y + 2) \\
&= (-(-x + 2) + 1, -(-y + 2)) \\
&= (x - 1, y - 2) \\
&= \tau(x, y).
\end{aligned}$$

Thus, $\sigma_Q\sigma_P = \tau$.

3. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $\alpha(x, y) = (x + 2y, x + y)$ for each $(x, y) \in \mathbb{R}^2$.
(a) Prove that α is a transformation.
Solution: First, we prove that α is one-to-one. Suppose that $(a, b), (c, d)$ are points in $\mathbb{R}^2$ so that $\alpha(a, b) = \alpha(c, d)$, that is $(a + 2b, a + b) = (c + 2d, c + d)$. Then

$a = 2(a + b) - (a + 2b) = 2(c + d) - (c + 2d) = c$ and $b = (a + 2b) - (a + b) = (c + 2d) - (c + d) = d$. Thus, $(a, b) = (c, d)$.

Thus, α is one-to-one.

Next, we prove that α is onto. Suppose that $P = (a, b)$ is a point in $\mathbb{R}^2$. Let $Q = (2b - a, a - b)$. Then $\alpha(Q) = ((2b - a) + 2(a - b), (2b - a) + (a - b)) = (a, b) = P$. Thus, α is onto.

(b) Let l is the line with equation $2x + y = 1$. Prove that $\alpha(l)$ is a line.
Solution: In fact, we prove that $\alpha(l) = m$ where m is the line with equation $x - 3y = -1$.

First, we prove that $\alpha(l) \subseteq m$. Let $P \in \alpha(l)$, that is, $P = \alpha(Q)$ for some point $Q = (a, b) \in l$. Since $(a, b) \in l$, we know that $2a + b = 1$, and from $P = \alpha(Q)$, we know $P = (a + 2b, a + b)$. Now, $(a + 2b) - 3(a + b) = -2a - b = -(2a + b) = -1$. This means the coordinates of P satisfy the equation of m, so $P \in m$.

Next, we prove that $m \subseteq \alpha(l)$. Let $R = (s,t) \in m$, that is, $s - 3t = -1$. Let $S = (2t - s, s - t)$. We prove that $S \in l$ and $\alpha(S) = R$.

Now,$2(2t - s) + (s - t) = -s + 3t = -(s - 3t) = -1$, which says that the coordinates of S satisfy the equation of l, so $S \in l$.

Next, $\alpha(S) = R$ as shown in (a). Since $S \in l$ and $\alpha(S) = R$, we get $R \in \alpha(l)$.

Thus, $\alpha(l) = m$.