PMAT 319 Winter 2016.
Questions For Midterm

1. Give the definitions of a translation, a halfturn, a transformation, a collineation, a dilatation, a reflection in a line, a rotation, a glide reflection.
2. Prove that a translation is a transformation.
3. Prove that a translation is a collineation.
4. Prove that a halfturn is a transformation.
5. Prove that a halfturn is a collineation.
6. Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation defined by $\tau(x, y) = (x + 1, y + 2)$. Let $P = (1, 1)$. Find the point $Q \in \mathbb{R}^2$ so that $\tau = \sigma_Q \sigma_P$.
7. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $\alpha(x, y) = (x + y, x - y)$.
(a) Is α a transformation? Prove your answer.
(b) Let l be the line with equation $2x - y = 0$. Is $\alpha(l)$ a line? Prove your answer.
(m) Let m be the line with equation $y = 5$. Is $\alpha(m)$ a line? Prove your answer.
8. Let $A = (1, 1)$, $B = (3, 2)$ and $C = (4, 4)$.
(a) Is $\tau_{AB}\sigma_C$ a halfturn? if so, find the point P so that $\tau_{AB}\sigma_C = \sigma_P$.
(b) Is $\sigma_C\tau_{AB}$ a halfturn? if so, find the point Q so that $\sigma_C\tau_{AB} = \sigma_Q$.
(c) Compute $\sigma_C\tau_{AB}\sigma_C(x, y)$ and $\tau_{\sigma_C(A)\sigma_C(B)}(x, y)$.

9. Make sure you can prove "star" theorems: Theorem 5.1, Theorem 5.2 and Theorem 7.2.
10. Prove that if m, n, l are parralell lines then $\sigma_l\sigma_n\sigma_m$ is a reflection in a line.
11. Prove that if m, n, l are lines that are concurrent at a point then $\sigma_l\sigma_n\sigma_m$ is a reflection in a line.
12. Prove that if m, n, l are lines that are neither concurrent at a point nor mutually parallel then $\sigma_l\sigma_n\sigma_m$ is a glide reflection.
13. Let $ABCD$ be a square. Let $a = \overleftrightarrow{AB}$, $b = \overleftrightarrow{BC}$, $c = \overleftrightarrow{CD}$, $d = \overleftrightarrow{DA}$, $m = \overleftrightarrow{BD}$ and $l = \overleftrightarrow{AC}$.
(a) Show that $\alpha = \sigma_d\sigma_c\sigma_b\sigma_a$ is a product of two reflections. Thus, α is either a rotation or a translation. Describe α.
(b) Repeat (a) but with $\alpha = \sigma_c\sigma_d\sigma_a\sigma_b$.
(c) Repeat (a) but with $\alpha = \sigma_d\sigma_b\sigma_c\sigma_a$.
(d) Repeat (a) but with $\alpha = \sigma_a\sigma_b\sigma_c\sigma_a$.
(e) Repeat (a) but with $\alpha = \sigma_c\sigma_m\sigma_l\sigma_a$.
(f) Repeat (a) but with $\alpha = \sigma_m\sigma_l\sigma_a\sigma_m$.
(g) Show that $\beta = \sigma_l\sigma_m\sigma_l$ is a reflection. Describe the line of reflection of β.
(h) Show that $\zeta = \sigma_b\sigma_m\sigma_a$ is a reflection. Describe the line of reflection of ζ.
(i) Show that $\varphi = \sigma_a\sigma_m\sigma_a$ is a reflection. Describe the line of reflection of φ.
14. Let ABC be an equilateral triangle. Let M be the midpoint of $\overline{AB}$. Let $a = \overleftrightarrow{BC}$, $b = \overleftrightarrow{AC}$, $c = \overleftrightarrow{AB}$ and $m = \overleftrightarrow{AM}$.
(a) Show that $\alpha = \sigma_a\sigma_m\sigma_c$ is a glide reflection. What is the axis of α?
(b) Show that $\beta = \sigma_c\sigma_b\sigma_a$ is a glide reflection. What is the axis of β?

(c) Let $\delta = \rho_{B,120^\circ}\rho_{A,120^\circ}$. Is δ a rotation or a translation? Describe δ.
(d) Let $\eta = \rho_{B,120^\circ}\rho_{A,60^\circ}$. Is η a rotation or a translation? Describe η.
(e) Repeat (a) but with $\alpha = \sigma_c\sigma_m\sigma_l\sigma_a$.
(f) Repeat (a) but with $\alpha = \sigma_m\sigma_l\sigma_a\sigma_m$.

15. Let $ABCD$ be a square. Let $a = \overleftrightarrow{AD}$, b be the perpendicular bisector of $\overline{AB}$, $c = \overleftrightarrow{CD}$ and d be the perpendicular bisector of $\overline{AD}$. We know that $\gamma_1 = \sigma_c\sigma_b\sigma_a$ and $\gamma_2 = S_aS_dS_c$ are glide reflections.
(a) What is $\sigma_b\gamma_1$? Draw a sketch and explain.
(b) What is $\sigma_{\overleftrightarrow{BC}}\gamma_1$? Draw a sketch and explain.
(c) What is $\gamma_2\gamma_1$? Draw a sketch and explain.

16. Suppose that G is a subgroup of $\mathcal{I}(\mathbb{R}^2)$. Let $ABCD$ be a square in $\mathbb{R}^2$. Let m be the line through B and D. Let n be the line through B and is perpendicular to m. Prove that if $\rho_{A,90^\circ}$ and σ_m are elements of G then $\rho_{C,90^\circ}$, $\rho_{B,180^\circ}$ and σ_n are also elements of G.

5. For each of the following statements, determine whether the statement is TRUE or FALSE and give a brief explanation.
(a) The product of two glide-reflections is always a rotation.

(b) If α and β are isometries os $\mathbb{R}^2$ so that $\alpha^2 = \beta^2$ then $\alpha = \beta$.

(c) The product of two rotations is always a rotation.

(d) The product of two translations is always a translation.

(e) The product of a translation and a glide-reflection is always a glide-reflection.

(f) The product of a translation and a line reflection is always a glide-reflection.

(g) If a frieze group contains a glide reflection γ then γ^2 generates the subgroup of all translations in the frieze group.

(h) The group generated by a glide-reflection is always infinite.

(i) The group generated by a halfturn and a line reflection is always infinite.

(j) The group generated by a rotation is always finite.