PMAT 319 Winter 2016.
Chapter 5: Isometries.

★**Theorem 5.1:** Let $\alpha \neq i$ be an isometry of $\mathbb{R}^2$.
(a) If α fixes two distinct points of a line then α fixes that line pointwise.
(b) α fixes at most two of any three non-collinear points.
(c) α is uniquely determined by three non-collinear points and their images.
(d) If α fixes two distinct points then α is a reflection (in the line through these two points).
(e) If α fixes exactly one point then α is a product of two reflections.

Proof:
(a) Suppose that $\alpha(P) = P$ and $\alpha(Q) = Q$ where $P \neq Q$ and $l = \overleftrightarrow{PQ}$. Let $R \in l$ and $\alpha(R) = R'$. Since α is an isometry, $R'P = RP$ and $R'Q = RQ$ which implies that $R' = R$ and therefore, $\alpha(R) = R$.

(b) We prove this by contradiction. Suppose that α fixes three non-collinear points A, B and C. Then by (a), α fixes points on the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$. Let P be any point of $\mathbb{R}^2$. Choose a point Q in the interior of the triangle ABC so that $Q \neq P$, and let $l = \overleftrightarrow{PQ}$. Then l intersects the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ at at least two points. Since α fixes these two points of l, by (a), α fixes l pointwise. In particular, $\alpha(P) = P$. Thus, we have prove that $\alpha(P) = P$ for all $P \in \mathbb{R}^2$; that is, $\alpha = i$ which contradicts $\alpha \neq i$. Thus, α fixes at most two of any three non-collinear points.

(c) Let A, B, C be three non-collinear points. We prove that if β is an isometry of $\mathbb{R}^2$ so that $\beta(A) = \alpha(A)$, $\beta(B) = \alpha(B)$ and $\beta(C) = \alpha(C)$ then $\alpha = \beta$.

Suppose that β is an isometry of $\mathbb{R}^2$ so that $\beta(A) = \alpha(A)$, $\beta(B) = \alpha(B)$ and $\beta(C) = \alpha(C)$. Then $\beta^{-1}\alpha$ is an isometry that fixes three non-collinear points A, B and C. By part (b), $\beta^{-1}\alpha = i$ and so $\alpha = \beta$.

(d) Suppose that α fixes two distinct points P and Q. Let $m = \overleftrightarrow{PQ}$. By (a), α fixes every point on m. Let $A \notin m$. By (b), $\alpha(A) \neq A$. Since α is an isometry, $AP = \alpha(A)P$ and $AQ = \alpha(A)Q$. It follows that m is the perpendicular bisector of $\overline{A\alpha(A)}$ and so $\alpha = \sigma_m$.

(e) Suppose that α fixes exactly the point P. Let $Q \neq P$. Then $Q \neq \alpha(Q)$ and let m be the perpendicular bisector of $\overline{Q\alpha(Q)}$. Since α is an isometry, $PQ = P\alpha(Q)$ and hence $P \in m$. Then $\sigma_m\alpha$ fixes the points P and Q and so by (d), $\sigma_m\alpha = \alpha_l$. Now, $\alpha = \sigma_m^{-1}\alpha_l = \sigma_l\sigma_m$.

★**Theorem 5.2 (Reflection Theorem):**
(a) A product of reflections is an isometry, and conversely,
(b) Each isometry is a product of at most three reflections.
Proof:
(a) This is clear from the fact that each reflection is an isometry, and $\mathcal{I}$ is a group.
(b) Let α be an isometry. If $\alpha = i$ then $\alpha = \sigma_m^2$ for any line m, so in this case α is the product of two reflections. Suppose that $\alpha \neq i$. If α has a fixed point then by Theorem 5.1, α is the product of at most two reflections. Now, suppose that α has no fixed points.

Let $P \in \mathbb{R}^2$ and let m be the perpendicular bisector of $\overline{P\alpha(P)}$. Then $\sigma_m \alpha(P) = P$, and so has a fixed point and by Theorem 5.1, $\sigma_m \alpha$ is the product of at most two reflections and so α is the product of at most three reflections.

How to see that an isometry is the product of at most three reflections.

Definition: Two subsets S_1 and S_2 of $\mathbb{R}^2$ are *congruent* if and only if there exists an isometry α so that $\alpha(S_1) = \alpha(S_2)$.

Rotations: We denote by $\rho_{C,\theta}$ the rotation centred at C with directed angle θ.

It is easy to see that

A rotation is an isometry.

$\rho_{C,180^\circ} = \sigma_C$ and $\rho_{C,180^\circ}^{-1} = \rho_{C,180^\circ}$

When θ is not a multiple of 360°, $\rho_{C,\theta}$ has exactly one fixed point which is C.

$\rho_{C,\theta}^{-1} = \rho_{C,-\theta}$.

$\rho_{C,\theta}\rho_{C,\varphi} = \rho_{C,\theta+\varphi} = \rho_{C,\varphi}\rho_{C,\theta}$.

Fix a point C, the set $\{\rho_{C,\theta} \mid \theta \in \mathbb{R}\}$ is a group.

Note that $\rho_{O,\theta}(x, y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$ where $O = (0, 0)$ is the origin of $\mathbb{R}^2$.

In the case $C = (a, b)$. Let $\tau = \tau_{CO}$. We note that $\rho_{C,\theta} = \tau^{-1}\rho_{O,\theta}\tau$ where $\tau(x, y) = (x - a,\ y - b)$ and so

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\begin{aligned}
\rho_{C,\theta}(x, y) &= \tau^{-1}\rho_{O,\theta}\tau(x, y) \\
&= \tau^{-1}\rho_{O,\theta}(x - a,\ y - b) \\
&= \tau^{-1}((x - a)\cos\theta - (y - b)\sin\theta,\ (x - a)\sin\theta + (y - b)\cos\theta) \\
&= ((x - a)\cos\theta - (y - b)\sin\theta + a,\ (x - a)\sin\theta + (y - b)\cos\theta + b) \\
&= (x\cos\theta - y\sin\theta + b\sin\theta - a\cos\theta + a,\ x\sin\theta + y\cos\theta + b - a\sin\theta - b\cos\theta)
\end{aligned}
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