PMAT 319 Winter 2016.
Chapter 6: Translations and Rotations.

We recall that each isometry α is a product of at most three reflections. Thus, $\alpha = \sigma_l$ or $\alpha = \sigma_m\sigma_l$ or $\alpha = \sigma_n\sigma_m\sigma_l$ where l, m and n are lines.

Product of two reflections: $\alpha = \sigma_m\sigma_l$.

Case 1: l and m are parallel. If $m = l$ then $\alpha = i$. In the case $l \neq m$, we prove that α is a translation.

Let b be a line that is perpendicular to both l and m. Let $L = b \cap l$ and $M = b \cap m$. Let $L' = \sigma_m(L)$ and $M' = \sigma_l(M)$. Let $\tau = \tau_{LL'}$ which is the translation with direction $\overrightarrow{LM}$ and with distance twice the distance from l to m. We will show that $\alpha = \tau$.

Choose $N \in l \setminus b$. Then $M'LN$ is a triangle, that is M', L and N are non-collinear. Put $N' = \tau(N)$. Then $\overleftrightarrow{NN'} \parallel \overleftrightarrow{LL'}$ and m is the perpendicular bisector of both $\overline{NN'}$ and $\overline{LL'}$. Now,

$\alpha(M') = \sigma_m\sigma_l(M') = \sigma_m\sigma_l\sigma_l(M) = \sigma_m(M) = M' = \tau(M')$,

$\alpha(L) = \sigma_m\sigma_l(L) = \sigma_m(L) = L' = \tau(L)$, and

$\alpha(N) = \sigma_m\sigma_l(N) = \sigma_m(N) = N' = \tau(N)$.

Since α and τ agree on three non-collinear points M', L and N, by Theorem 5.1, $\alpha = \tau$.

Conversely, it is easy to prove that each translation is a product of two reflections in parallel lines. Note that these lines are not unique.

Case 2: l and m are not parallel. Let $C = l \cap m$ and let $\frac{\theta}{2}$ be the directed angle from l to m. Let $\rho = \rho_{C,\theta}$. We prove that $\alpha = \rho$.

Choose $L \in l$ and $M \in m$ so that $m(\angle LCM) = \frac{\theta}{2}$. Put $M' = \sigma_l(M)$ and $L' = \sigma_m(L)$. We note that L, C

and M' are non-collinear. Now,

$\alpha(L) = \sigma_m\sigma_l(L) = \sigma_m(L) = L' = \rho(L)$,

$\alpha(C) = \sigma_m\sigma_l(C) = \sigma_m(C) = C = \rho(C)$, and

$\alpha(M') = \sigma_m\sigma_l(M') = \sigma_m(M) = M = \tau(M')$.

Since α and ρ agree on three non-collinear points M', L and N, by Theorem 5.1, $\alpha = \rho$.

Conversely, it is easy to prove that each rotation is a product of two reflections in intersecting lines (whose intersection is the centre of the rotation). Note that these lines are not unique. In particular, $\sigma_P = \rho_{P,180^\circ} = \sigma_a\sigma_b$ where a and b are any two lines that are perpendicular at P.

Product of three reflections: $\alpha = \sigma_n\sigma_m\sigma_l$ where $l \parallel m \parallel n$ or l, m and n are concurrent.

Case 1: $l \parallel m \parallel n$.
Let p be the line that is parallel to l and the directed distance from p to n equals the directed distance from l to m. Then $\sigma_m\sigma_l = \sigma_n\sigma_p$ and so $\alpha = \sigma_n\sigma_m\sigma_l = \sigma_n\sigma_n\sigma_p = \sigma_p$, that is, α is a reflection in a line parallel to $l,\ m,\ n$

Case 2: l, m and n are concurrent.
Let $C = l \cap m \cap n$ and let p be the line through C so that the directed angle from p to n equals the directed angle from l to m. Then $\sigma_m\sigma_l = \sigma_n\sigma_p$ and so $\alpha = \sigma_n\sigma_m\sigma_l = \sigma_n\sigma_n\sigma_p = \sigma_p$, that is, α is a reflection in a line concurrent to $l,\ m,\ n$.

Isometries with fixed points.

If an isometry α has exactly one fixed point then α is a product of two refletions and so α is a rotation.

If an isometry α has two fixed points then $\alpha = i$ or α is a reflection.

Involutary Isometries.

Let $\alpha \in I$ be an involution; that is, $\alpha \neq i$ and $\alpha^2 = i$. Since $\alpha \neq i$, there exist points $P \neq Q$ so that $\alpha(P) = Q$ and $\alpha(Q) = P$. Let $R = \frac{1}{2}(P+Q)$. Then $\alpha(R) = R$. Thus, α has a fixed point and hence, α is either a halfturn or a reflection.

We recall that the halfturn σ_P fixes every line through P. The converse is true.

Theorem 6.1: If the non-identity rotation $\rho_{C,\theta}$ fixes a line through C then $\rho_{C,\theta} = \sigma_C$.
Proof: Suppose that $\rho_{C,\theta}$ fixes a line l where $C \in l$. Let m be the line perpendicular to l at the point C. Then there is a line n through C so that $\rho_{C,\theta} = \sigma_n \sigma_m$. Now, $l = \rho_{C,\theta}(l) = \sigma_n \sigma_m(l) = \sigma_n(l)$. Thus, l is fixed by the reflection σ_m, and so either $l = n$ or $l \perp n$. Since $\rho_{C,\theta} \neq i$, $l \neq n$ and therefore $l \perp n$ and $\rho_{C,\theta} = \sigma_C$.