PMAT 319 Winter 2016.
Chapter 7: Even Isometries.

We recall that each isometry α is a product of at most three reflections. Thus, $\alpha = \sigma_l$ or $\alpha = \sigma_m\sigma_l$ or $\alpha = \sigma_n\sigma_m\sigma_l$ where l, m and n are lines.
Also, if $\alpha \neq i$ is a product of two reflections then α is either a translation (no fixed points) or a rotation (exactly one fixed point).
We shall show that a product of four reflections is a product of two reflections.

Lemma 7.1: For any point P and any lines a and b, there are lines c and d so that $P \in c$ and $\sigma_b\sigma_a = \sigma_d\sigma_c$.
Proof:
Let P be a point, and a and b are lines.
Case 1: a and b are parallel. Let c be the line through P and parallel to a and b. Let d be the line parallel to c so that the directed distance from c to d equals the directed distance from a to d. Then $P \in c$ and $\sigma_b\sigma_a = \sigma_d\sigma_c$.
Case 2: a and b are not parallel. Let $C = a \cap b$. Let c be a line through P and C. Let d be the line through C so that the directed angle from c to d equals the directed angle from a to d. Then $P \in c$ and $\sigma_b\sigma_a = \sigma_d\sigma_c$.

★**Theorem 7.2 (Reduction Theorem)**: A product of four reflections is a product of two reflections.
Proof: Let $\alpha = \sigma_s\sigma_r\sigma_q\sigma_p$. Let $P \in p$. By Lemma 7.1, given point P and lines r and q, there exist lines l and m so that so that $P \in m$ and $\sigma_r\sigma_q = \sigma_l\sigma_m$. Now, given point P and lines s and l, there exist lines u and v so that so that $P \in v$ and $\sigma_s\sigma_l = \sigma_u\sigma_v$. Then, since $m,\ p,\ v$ are concurrent, $\sigma_v\sigma_m\sigma_p = \sigma_t$ for some line t, and hence
$\alpha = \sigma_s\sigma_r\sigma_q\sigma_p = \sigma_s\sigma_l\sigma_m\sigma_p = \sigma_u\sigma_v\sigma_m\sigma_p = \sigma_u\sigma_t$.

Theorem 7.3: A product of three reflections cannot be a product of two reflections.
Proof: We prove this by contradiction. Suppose that $\sigma_r\sigma_q\sigma_p = \sigma_s\sigma_t$. Then $\sigma_s\sigma_r\sigma_q\sigma_p = \sigma_t$. By Theorem 7.2, $\sigma_s\sigma_r\sigma_q\sigma_p = \sigma_m\sigma_l$ for some lines m amd l. Thus, $\sigma_m\sigma_l = \sigma_t$ which contradicts the fact that a product of two reflections cannot be reflection. Thus, a product of three reflections cannot be a product of two reflections.

Definition: Let α be an isometry. We say that α is an *even* (*odd*) isometry if it is a product of even (odd) number of reflections.
From the above theorems, we note that:
No isometry can be both even and odd.
An even isometry is a product of two reflections and hence it is either a rotation or a translation.
An odd isometry is either a reflection or a product of three reflections.
If α is an involutary isometry then α is a halfturn (the case α is even) or α is a reflection (the case α is odd).
Let $\mathcal{E}$ be the set of all even isometries of $\mathbb{R}^2$.

Theorem 7.4: $\mathcal{E}$ is a subgroup of $\mathcal{I}$.

Proof: Easy.

Theorem 7.5: For all $\alpha \in \mathcal{I}$ and $\sigma_P \in \mathcal{E}$, $\alpha\sigma_P\alpha^{-1} \in \mathcal{E}$ and $\alpha\sigma_P\alpha^{-1} = \sigma_{\alpha(P)}$.
Proof: Let $\alpha \in \mathcal{I}$ and $\sigma_P \in \mathcal{E}$. Then α is a product of k reflections and α^{-1} is a product of k reflections (the same k reflections in reverse order). Since σ_P is a halfturn, it is a product of two reflections and so $\alpha\sigma_P\alpha^{-1}$ is a product of $2k+2$ reflections and so $\alpha\sigma_P\alpha^{-1}$ is even. Now, $\left(\alpha\sigma_P\alpha^{-1}\right)^2 = \alpha\sigma_P\alpha^{-1}\alpha\sigma_P\alpha^{-1} = i$, so $\alpha\sigma_P\alpha^{-1}$ is an even involution and therefore, $\alpha\sigma_P\alpha^{-1} = \sigma_Q$ for some point Q where Q is the only fixed point of $\alpha\sigma_P\alpha^{-1}$. Now, $\alpha\sigma_P\alpha^{-1}(\alpha(P)) = \alpha(P)$, so $\alpha(P) = Q$ and $\alpha\sigma_P\alpha^{-1} = \sigma_{\alpha(P)}$.

Theorem 7.6: For all $\alpha \in \mathcal{I}$ and $\sigma_m \in \mathcal{I}$, $\alpha\sigma_m\alpha^{-1} \in \mathcal{I} \setminus \mathcal{E}$ and $\alpha\sigma_m\alpha^{-1} = \sigma_{\alpha(m)}$.
Proof: Similar to the proof of Theorem 7.5.

Theorem 7.7: For all $\alpha \in \mathcal{I}$ and $\tau_{AB},\ \rho_{C,\theta} \in \mathcal{E}$,

$\alpha\tau_{AB}\alpha^{-1} = \tau_{\alpha(A)\alpha(B)}$ and

$$\alpha\rho_{C,\theta}\alpha^{-1} = \begin{cases} \rho_{\alpha(C),\theta} & \text{if } \alpha \text{ is even, and} \\ \rho_{\alpha(C),-\theta} & \text{if } \alpha \text{ is odd.} \end{cases}$$

Thus, $\mathcal{E}$ is a normal subgroup of $\mathcal{I}$.

Proof: Let $\alpha \in \mathcal{I}$ and $\tau_{AB},\ \rho_{C,\theta} \in \mathcal{E}$.

Let M be the midpoint of $\overline{AB}$. Then $\tau_{AB} = \sigma_M\sigma_A = \sigma_B\sigma_M$ and so, by Theorem 7.6, $\alpha\tau_{AB}\alpha^{-1} = \alpha\sigma_M\alpha^{-1}\alpha\sigma_A\alpha^{-1} = \sigma_{\alpha(M)}\sigma_{\alpha(A)} = \tau_{\alpha(A)\alpha(B)}$.

Note that since α is an isometry, it preserves midpoints, and hence $\alpha(M)$ is the midpoint of $\overline{\alpha(A)\,\alpha(B)}$ and so $\sigma_{\alpha(M)}\sigma_{\alpha(A)} = \tau_{\alpha(A)\alpha(B)}$.

Next, we want to show that for any line l,

$$\sigma_l\rho_{C,\theta}\sigma_l^{-1} = \rho_{\sigma_l(C),-\theta} \qquad (1)$$

Let l be a line, let m be the line through C and perpendicular to l. Let n be the line through C so that $\rho_{C,\theta} = \sigma_n\sigma_m$. Then the angle from m to $\sigma_l(n)$ is $-\theta$, and

$$\begin{aligned} \sigma_l\rho_{C,\theta}\sigma_l^{-1} &= \sigma_l\sigma_n\sigma_m\sigma_l^{-1} \\ &= \left(\sigma_l\sigma_n\sigma_l^{-1}\right)\left(\sigma_l\sigma_m\sigma_l^{-1}\right) \\ &= \sigma_{\sigma_l(n)}\sigma_{\sigma_l(m)} \qquad \text{by Theorm 7.6} \\ &= \sigma_{\sigma_l(n)}\sigma_m \\ &= \rho_{\sigma_l(C),-\theta}. \end{aligned}$$

Thus, we proved (1).

Now, since α is an isometry, α is a product of at most 3 reflections, that is, $\alpha = \sigma_t$ or $\alpha = \sigma_t\sigma_s$ or $\alpha = \sigma_t\sigma_s\sigma_r$, and from (1), we can easily prove that

$$\alpha\rho_{C,\theta}\alpha^{-1} = \begin{cases} \rho_{\alpha(C),\theta} & \text{if } \alpha \text{ is even, and} \\ \rho_{\alpha(C),-\theta} & \text{if } \alpha \text{ is odd.} \end{cases}$$

Exercises: Prove the following:

1. Non-identity rotations with different centre do not commute.
2. $\sigma_n\sigma_m = \sigma_m\sigma_n$ if and only if $m = n$ or $m \perp n$.

3. The product of two rotations is a rotation or a translation.
4. The product of a rotation and a translation is a rotation or a translation.

Theorem 7.8: For points $A \neq B$ and $\phi,\ \theta \in \mathbb{R}$, $-180 < \phi,\ \theta \leq 180$ so that $\phi\theta \geq 0$ (the angles have same direction), there are lines a through A, and b through B so that

$$\rho_{B,\phi}\rho_{A,\theta} = \begin{cases} \text{a translation} & \text{if } \phi + \theta = 360^\circ \\ \rho_{C,\phi+\theta} & \text{if } \phi + \theta \neq 360^\circ, \end{cases}$$

where $C = a \cap b$.

Proof: Let $c = \overleftrightarrow{AB}$, let a be the line through A so that $\rho_{A,\phi} = \sigma_c\sigma_a$ and let b be the line through B so that $\rho_{B,\phi} = \sigma_b\sigma_c$. Then $\rho_{B,\phi}\rho_{A,\theta} = \sigma_b\sigma_c\sigma_c\sigma_a = \sigma_b\sigma_a$. Now, if $\phi + \theta = 360^\circ$ then a and b are paralell and so $\sigma_b\sigma_a$ is a translation. If $\phi + \theta \neq 360^\circ$ then a and b are not paralell. In this case, put $C = a \cap b$. Then $\sigma_b\sigma_a = \rho_{C,\phi+\theta}$.

Exercise: Prove that for any translation τ and any non-identity rotation $\rho = \rho_{C,\theta}$, there exist different points A, B and C so that $\tau = \rho_{A,180}\rho_{C,180} = \rho_{C,180}\rho_{B,180}$, and hence both $\tau\rho$ and $\rho\tau$ are rotations of θ degrees.