PMAT 319 Winter 2016.
Chapter 3: Translations and Halfturns.

A function $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is a translation if there are real numbers r and s so that

$$\alpha(x, y) = (x + r, y + s) \text{ for all } (x, y) \in \mathbb{R}^2.$$

Given points $P = (a, b)$ and $Q = (c, d)$. There is a unique translation that maps P to Q. We denote this translation by τ_{PQ}. In fact, $\tau_{PQ}(x, y) = (x + c - a, y + d - b)$ for any $(x, y) \in \mathbb{R}^2$. It is clear that $\tau_{PP} = i$.

Theorem 3.1: Let A, B, C be non-collinear points. Then $\tau_{AB} = \tau_{CD}$ for some point D if and only if $ABDC$ is a parallelogram.

Proof:

Theorem 3.1 can also be written as: $\tau_{AB} = \tau_{CD} \Leftrightarrow AB = CD$ and $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ are parallel.

Theorem 3.2: Translations are transformations.

Proof:

Theorem 3.3: Translations are collineations.

Proof:

Definition: A *dilatation* is a collineation which maps a line to a parallel line.

Theorem 3.4: Translations are dilatations.

Proof:

Theorem 3.5: If $A \neq B$ and $\tau_{AB}(m) = m$ for some line m then m and $\overleftrightarrow{AB}$ are parallel.

Proof:

Thus, a non-identity translation τ_{AB} fixes exactly those lines that are parallel to $\overleftrightarrow{AB}$.

Let $\mathcal{D}$ be the set of all dilatations and $\mathcal{T}$ be the set of all translations.

Theorem 3.6: Collineations maps parallel lines to parallel lines.

Proof:

Theorem 3.7: $\mathcal{D}$ is a subgroup of $\mathcal{C}$.
Proof:

Theorem 3.8: $\mathcal{T}$ is a subgroup of $\mathcal{D}$.
Proof:

Definition: Let $P = (a, b) \in \mathbb{R}^2$. A *halfturn* about P is the transformation $\sigma_P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\sigma_P(x, y) = (-x + 2a, -y + 2b) \text{ for any } (x, y) \in \mathbb{R}^2.$$

We note that for any point $Q = (c, d)$, let $Q' = \sigma_P(Q)$. We see that $\frac{1}{2}(Q + Q') = P$ and so p is the midpoint of the line segment QQ' and hence, P, Q and $\sigma_P(Q)$ are collinear.

Theorem 3.9: Halfturns are transformations. In fact, halfturns are involutions.
Proof:

Theorem 3.10: Halfturns are dilatations.
Proof:

It is easy to see that $\sigma_P(A) = A$ if and only if $A = P$, and for any line l, $\sigma_P(l) = l$ if and only if $P \in l$. In other words, σ_P only fixes the point P, and σ_P only fixes the lines through P.

Theorem 3.11: The product of two halfturns is a translation. In fact, if $P = (a, b)$ and $P = (c, d)$ then $\sigma_P\sigma_Q = \tau_{QR} = \sigma_R\sigma_P$ where R is the point so that P is the midpoint of QR.
In other words, $\sigma_P\sigma_Q$ is the translation twice the distance from P to Q.
Proof:

The converse of Theorem 3.11 is also true.
Theorem 3.12: Each translation is a product of two halfturns.
Note that the halfturns are not unique.
Proof:

Theorem 3.13: The product of three halfturns is a halfturn. In fact, if P, Q, R are non-collinear points then $\sigma_R \sigma_Q \sigma_P = \sigma_S$ where S is the point so that $PQRS$ is a parallelogram.

Proof:

Remarks:

(1) Given any three of the four points A, B, C, D, the condition $\tau_{AB} = \sigma_D \sigma_C$ uniquely determines the fourth.

(2) For any points A, B, C, $\sigma_C \sigma_B \sigma_A = \sigma_A \sigma_B \sigma_C$. Thus, if P, Q, R are non-collinear points then $\sigma_R \sigma_Q \sigma_P = \sigma_P \sigma_Q \sigma_R = \sigma_S$ where S is the point so that $PQRS$ is a parallelogram.

(3) Let $\mathcal{H}$ be the group generated by halfturns. Then $\mathcal{H}$ is a subgroup of $\mathcal{D}$.