PMAT 319 Winter 2016.
Chapter 8: Glide-Reflections.

We recall that each isometry α is a product of at most three reflections. We know that a product of two reflections is either a rotation or a translation.

As for the case of the product of three reflections $\sigma_n\sigma_m\sigma_l$, we have considered the case $l \parallel m \parallel n$ (in this case, $\sigma_n\sigma_m\sigma_l$ is a reflection in a line parallel to l, m and n), and the case l, m and n are concurrent (in this case, $\sigma_n\sigma_m\sigma_l$ is a reflection in a line concurrent to l, m and n). We shall prove that for the other cases, $\sigma_n\sigma_m\sigma_l$ is a glide-reflection.

Definition: γ is a *glide-reflection with axis c* if and only if $\gamma = \sigma_c\sigma_b\sigma_a$ where $a \parallel b$, $a \neq b$ and $a \perp c \perp b$.

Theorem 8.1: Glide-reflections have no fixed points.

Proof: Let $\gamma = \sigma_c\sigma_b\sigma_a$ where $a \parallel b$, $a \neq b$ and $a \perp c \perp b$, and $P \in \mathbb{R}^2$.

Case 1: $P \in C$. Then $\sigma_b\sigma_a(P) \neq P$ because $\sigma_b\sigma_a$ is a non-identity translation, and $\sigma_b\sigma_a(P) \in c$ because $\sigma_b\sigma_a$ fixes c. Thus, $\sigma_c\sigma_b\sigma_a(P) = \sigma_b\sigma_a(P) \neq P$.

Case 2: $P \notin C$. let l be the line through P and is perpendicular to c. Let $m \perp c$ so that $\sigma_m\sigma_l = \sigma_b\sigma_a$. Put $M = m \cap c$. Then $P \neq M$ and $\gamma(P) = \sigma_c\sigma_b\sigma_a(P) = \sigma_c\sigma_m\sigma_l(P) = \sigma_c\sigma_m(P) = \sigma_M(P) \neq P$. Note that M is the midpoint of the line segment between $\gamma(P)$ and P. Thus, we have

Remark 8.2: The midpoint of any point and its image under a glide-reflection lies on the axis of the glide-reflection.

Remark 8.3: A glide-reflection fixes only one line, its axis.

★**Theorem 8.4:** A glide reflection is a composition of a reflection and a halfturn about a point not on the reflection line.

Proof: Let $\gamma = \sigma_c\sigma_b\sigma_a$ where $a \parallel b$, $a \neq b$ and $a \perp c \perp b$. Let $A = a \cap c$ and $B = b \cap c$. Since $\sigma_A = \sigma_c\sigma_a = \sigma_a\sigma_c$ and $\sigma_B = \sigma_b\sigma_c = \sigma_c\sigma_b$, we see that $\gamma = \sigma_c\sigma_b\sigma_a = \sigma_B\sigma_a$ and $\gamma = \sigma_c\sigma_b\sigma_a = \sigma_b\sigma_c\sigma_a = \sigma_b\sigma_A$.

The converse of Theorem 8.4 is also true.

Theorem 8.5: Let $A \notin l$ and $A \in a \perp l$. Then $\sigma_A\sigma_l$ and $\sigma_l\sigma_A$ are glide-reflections with axis a.

Proof: Easy.

Theorem 8.6: The inverse of a glide-reflection is a glide reflection.

Proof: Let γ be a glide-reflection then $\gamma = \sigma_A\sigma_l$ for some point A and line l where $A \notin l$. Then $\gamma^{-1} = (\sigma_A\sigma_l)^{-1} = \sigma_l\sigma_A$ which is a glide-reflection by Theorem 8.5.

Theorem 8.7: Let τ be a translation that fixes a line c. Then $\tau\gamma = \gamma\tau$ for any dlide-reflection γ with axis c.

Proof: Let $\gamma = \sigma_c\sigma_b\sigma_a$ where $a \parallel b$, $a \neq b$ and $a \perp c \perp b$. Then $\sigma_b\sigma_a$ is a translation and so $\sigma_b\sigma_a$ commutes with τ. Thus, we only need to show that $\tau\sigma_c = \sigma_c\tau$. Now, note that $\tau\sigma_c\tau^{-1} = \sigma_{\tau(c)} = \sigma_c$ because $\tau(c) = c$. Thus, $\tau\sigma_c = \sigma_c\tau$.

Theorem 8.8: The square of a glide-reflection is a non-identity translation.

Let $\gamma = \sigma_c\sigma_b\sigma_a$ where $a \parallel b$, $a \neq b$ and $a \perp c \perp b$. Then $\sigma_b\sigma_a$ is a translation which fixes line c and so by Theorem 8.7, $\sigma_b\sigma_a\gamma = \gamma\sigma_b\sigma_a$. It follows that

$\gamma^2 = \sigma_c\sigma_b\sigma_a\gamma = \sigma_c\gamma\sigma_b\sigma_a = \sigma_c\sigma_c\sigma_b\sigma_a\sigma_b\sigma_a = (\sigma_b\sigma_a)^2$ which is a translation.

★**Theorem 8.9:** Let p, q, r be lines which are neither concurrent nor mutually parallel. Then $\sigma_r\sigma_q\sigma_p$ is a glide reflection.

Proof: We have two cases.

Case 1: The line p and q intersect at a point Q. Let m be the line through Q and is perpendicular to r. Choose line l through Q with proper angle to m so that $\sigma_m\sigma_l = \sigma_q\sigma_p$. Let $P = m \cap r$. Then $\sigma_r\sigma_q\sigma_p = \sigma_r\sigma_m\sigma_l = \sigma_P\sigma_l$ where $P \notin l$, and so it is a glide-reflection.

Case 2: The line p and q are parallel. Then r and q intersect, and so from case 1, we have $\sigma_p\sigma_q\sigma_r = \sigma_P\sigma_l$ where $P \notin l$. Then $\sigma_r\sigma_q\sigma_p = (\sigma_p\sigma_q\sigma_r)^{-1} = (\sigma_P\sigma_l)^{-1} = \sigma_P\sigma_l$ which is again a glide-reflection.

Theorem 8.10: If γ is a glide-reflection with axis c and $\alpha \in I$ then $\alpha\gamma\alpha^{-1}$ is a glide-reflection with axis $\alpha(c)$.

Proof: Since γ is a glide-reflection, it is an odd isometry, and so $\gamma_1 = \alpha\gamma\alpha^{-1}$ is an odd isometry. Thus, γ_1 is either a reflection or a glide-reflection. We show that γ_1 is not a reflection by showing $\gamma_1^2 \neq i$. Suppose $\gamma_1^2 = (\alpha\gamma\alpha^{-1})^2 = \alpha\gamma^2\alpha^{-1} = i$. Then $\gamma^2 = \alpha^{-1}\alpha = i$ which contradicts Theorem 8.8. Thus, γ_1 is a glide-reflection