Frieze Groups

<u>Definition</u> A <u>frieze group</u> is a subgroup $\mathcal{F}$ of $\mathcal{I}$ so that $\mathcal{F} \cap \mathcal{T}$ is a cyclic group generated by a non-identity translation τ; that is, $\mathcal{F} \cap \mathcal{T} = \langle \tau \rangle$ and we note that $\langle \tau \rangle$ is an infinite group.

Let $\mathcal{F}$ be a frieze group with $\mathcal{F} \cap \mathcal{T} = \langle \tau \rangle$

REMARKS.

① Let $\tau = \tau_{AB}$. Then

$\overrightarrow{AB}$ is the <u>direction</u> of τ

AB is the <u>length</u> of τ

$\tau_{AB} = \tau_{A'B'} \implies AB = A'B'$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$

② Let $\tau' \in \mathcal{F} \cap \mathcal{T}$. Then

$\tau' = \tau^n$ for some integer n.

$\tau' = \tau_{CD} \implies CD \geq AB$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$.

(we say that τ is shorter than τ' if $AB < CD$)

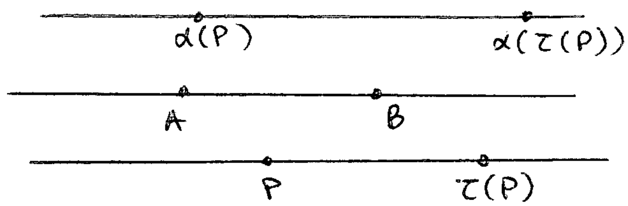
③ Let $\alpha \in \mathcal{F} \setminus \langle \tau \rangle$ and $\tau = \tau_{AB}$. (so, $B = \tau(A)$)

Then $\alpha \tau_{AB} \alpha^{-1} \in \mathcal{F}$ (because $\mathcal{F}$ is a group)

However, $\alpha \tau_{AB} \alpha^{-1} = \tau_{\alpha(A)\alpha(B)}$ is a translation

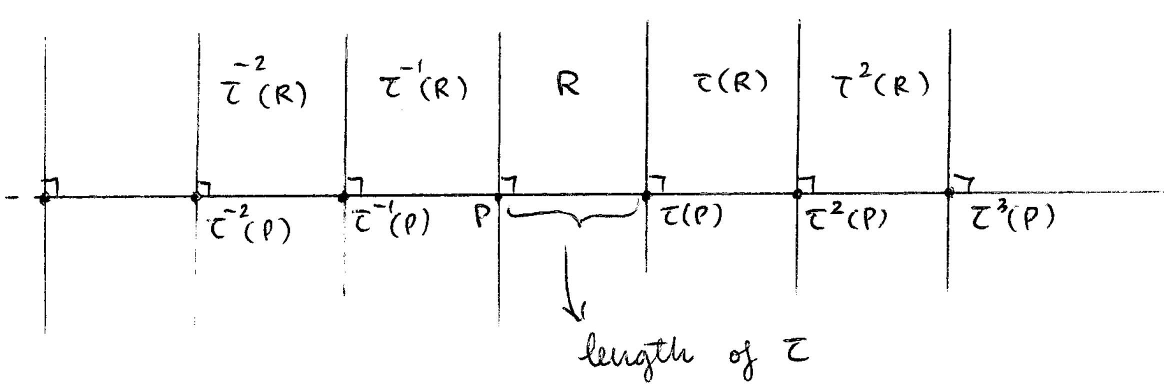
and so $\tau_{\alpha(A)\alpha(B)} \in \langle \tau \rangle$ and so $\overleftrightarrow{\alpha(A)\alpha(B)} \parallel \overleftrightarrow{AB}$.

Now, for any point $P \in \mathbb{R}^2$,
Since $\alpha\tau\alpha^{-1}(\alpha(P)) = \alpha\tau(P)$ and $\alpha\tau\alpha^{-1} = \tau_{\alpha(A)\alpha(B)}$
we have $\overleftrightarrow{\alpha(P)\,\alpha(\tau(P))} \parallel \overleftrightarrow{\alpha(A)\,\alpha(B)} \parallel \overleftrightarrow{AB} \parallel \overleftrightarrow{P\,\tau(P)}$



Thus, if α is a reflection then $\alpha = \sigma_\ell$ for some line $\ell \parallel \overleftrightarrow{AB}$ or $\ell \perp \overleftrightarrow{AB}$, and
if α is a rotation then α must be a halfturn,
if α is a glide reflection then the axis of α must be parallel to $\overleftrightarrow{AB}$ (note that $\alpha^2 \in \mathcal{F} \cap \mathcal{T}$ in this case).
For simplicity, let c be a line parallel to $\overleftrightarrow{AB}$. Then
$\tau'(\ell) = \ell$ for all line $\ell \parallel c$ and $\tau' \in \mathcal{F} \cap \mathcal{T}$.

④ Let $S \subseteq \mathbb{R}^2$ such that $\mathcal{F} = \mathcal{I}_S$.
(We call S a <u>frieze pattern</u>)
Then $\tau^n(S) = S$ for all $n \in \mathbb{Z}$, and so
$\tau^n(P) \in S$ for all $n \in \mathbb{Z}$ and $P \in S$, and
hence $\overleftrightarrow{\tau^{n-1}(P)\,\tau^n(P)} \parallel \ell$ for all $n \in \mathbb{Z}$ and $P \in S$



Consider R, part of S in the infinite strip with width equals the length of τ as shown in the above figure. Then

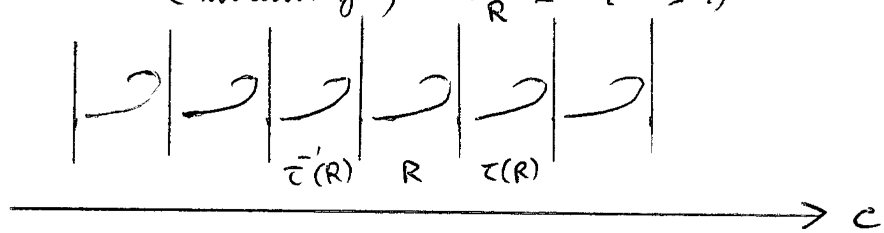
$$\tau^{-n}(R) \cong R \cong \tau^{n}(R),$$

so S is simply an infinite number of copies of R layed side by side in the direction of τ.

R is called a <u>fundamental domain</u> of $\mathcal{F}$

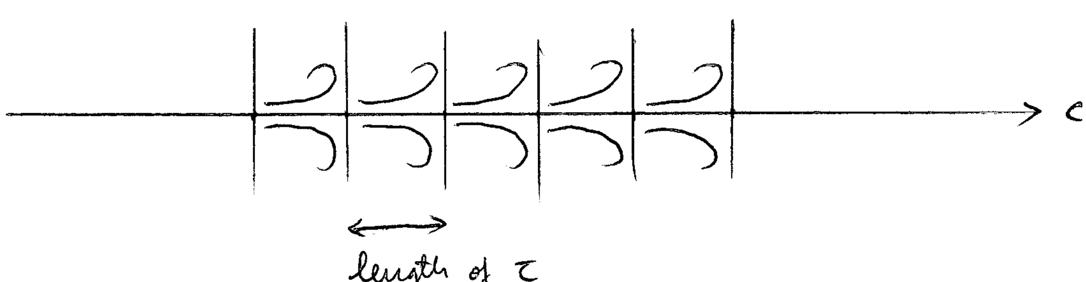
POSSIBILITIES FOR $\mathcal{F}$.

① $\mathcal{F}_1 = \langle \tau \rangle$. This is the case R has no symmetry (meaning, $\mathcal{I}_R = \{i\}$.)



In this case $\alpha(c) = c$ for all $\alpha \in \mathcal{F}_1$.

② $\mathcal{F}_1^1 = \langle \tau, \sigma_c \rangle$. This is the case R has a line of symmetry (in direction of τ)



__Note__: In this case, $\tau'\sigma_c = \sigma_c\tau'$ for all $\tau' \in \langle \tau \rangle$.
This is because $\tau' = \sigma_a\sigma_b$ where $a \parallel b \perp c$ and so

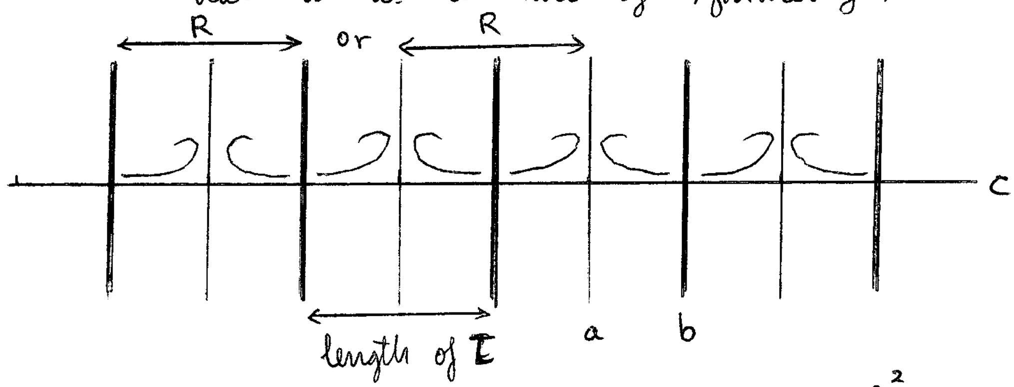
$$\tau'\sigma_c = \sigma_a(\sigma_b\sigma_c) = \sigma_a\sigma_c\sigma_b = \sigma_c\sigma_a\sigma_b = \sigma_c\tau'.$$

Thus,

$$\mathcal{F}_1^1 = \{\tau^n\sigma_c \mid n \in \mathbb{Z}\} \cup \{\tau^n \mid n \in \mathbb{Z}\} \supseteq \mathcal{F}_1^1$$

and $\alpha(c) = c$ for all $\alpha \in \mathcal{F}_1^1$.

(3) $\mathcal{F}_1^2 = \langle \tau, \sigma_a \rangle$ where $a \perp c$. This is the case R has a as a line of symmetry.



In this case, $\tau = \sigma_b\sigma_a$ and so $\sigma_b = \tau\sigma_a \in \mathcal{F}_1^2$.

Thus, $\sigma_a(R) = R$ and $\sigma_b(R) = R$.

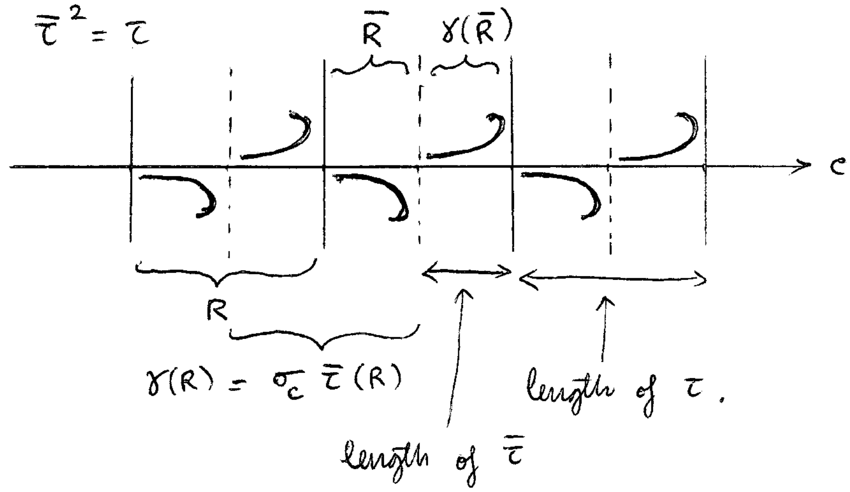
__Note__. $\tau'\sigma_a = \sigma_a\tau'^{-1}$ for all $\tau' \in \langle\tau\rangle$
$\tau'\sigma_a$ is a reflection and so $(\tau'\sigma_a)^2 = \iota$.

$$\mathcal{F}_1^2 = \{\sigma_a\tau^n \mid n \in \mathbb{Z}\} \cup \{\tau^n \mid n \in \mathbb{Z}\} \supset \mathcal{F}_1.$$

and $\alpha(c) = c$ for all $\alpha \in \mathcal{F}_1^2$.

④ $\mathcal{F}_1^3 = \langle \gamma \rangle$ where γ is a glide reflection with axis c and $\gamma^2 = \tau$.

Then $\gamma = \sigma_c \bar{\tau}$ with $\bar{\tau}^2 = \tau$



Note.

- $\mathcal{F}_1^3 = \{\gamma^n \mid n \in \mathbb{Z}\} \supset \{\gamma^{2n} \mid n \in \mathbb{Z}\} = \{\tau^n \mid n \in \mathbb{Z}\} = \mathcal{F}_1$

- $\alpha(c) = c$ for all $\alpha \in \mathcal{F}_1^3$.
- The cases $\gamma^2 = \tau^m$, $m > 1$, are already covered.

  When m is even, $m = 2n$, for some $n > 1$.

  $\gamma^2 = \tau^{2n}$ and $\gamma = \tau' \sigma_c \Rightarrow \gamma = \tau^n \sigma_c$

  $\Rightarrow \sigma_c$ is in the group, which was consider in $\mathcal{F}_1^1$.

  When m is odd, $m = 2n-1$ for some $n > 2$

  $\gamma^2 = \tau^{2n-1}$ and since γ is a product of a translation and σ_c, $\gamma = \tau^{n-1} \bar{\tau} \sigma_c \Rightarrow \bar{\tau}\sigma_c$ is in the group and $\bar{\tau}\sigma_c$ is a glide reflection so that $(\bar{\tau}\sigma_c)^2 = \tau$.

⑤ $\mathcal{F}_2 = \langle \tau, \sigma_A \rangle$ where $A \in c$. Thus, A is a point of symmetry for S.

Put $A_0 = A$ and $A_i = \tau^i(A)$. Then since $\tau \sigma_{A_{i-1}} \tau^{-1} = \sigma_{\tau(A_{i-1})} = \sigma_{A_i}$, A_i is also a point of symmetry for S.

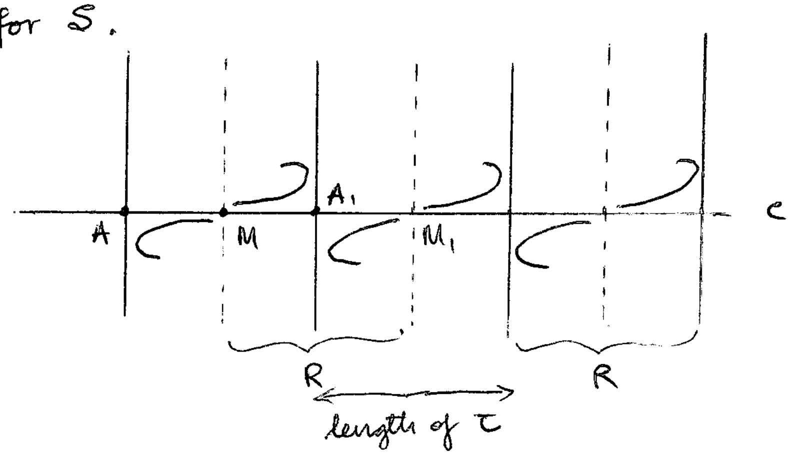
Note that $\tau\sigma_A$ is a halfturn (being the product of three halfturns), so $\tau\sigma_A = \sigma_M$ for some point M and hence with $M_0 = M$, $M_i = \tau^i(M)$, the points M_i are points of symmetry for S.

<u>Note</u>.

$\tau\sigma_A = \sigma_A \tau^{-1}$ because $(\tau\sigma_A)^2 = i$

$\mathcal{F}_2 = \mathcal{F}_1 \cup \{\tau^n \sigma_A \mid n \in \mathbb{Z}\}$

$\alpha(c) = c$ for all $\alpha \in \mathcal{F}_2$.



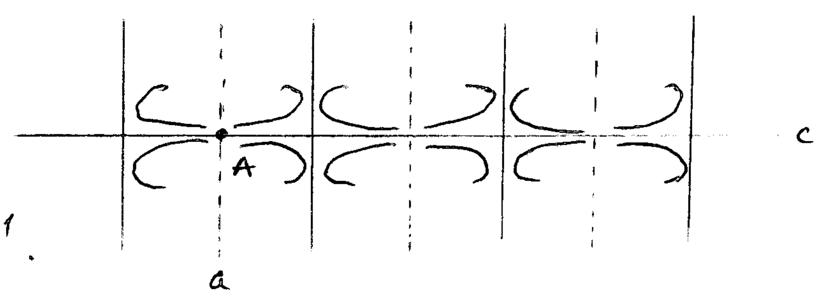
⑥ $\mathcal{F}_2' = \langle \tau, \sigma_A, \sigma_c \rangle$ where $A \in c$.

Note that $\sigma_A = \sigma_c \sigma_a$

$\Rightarrow \sigma_a = \sigma_c \sigma_A \in \mathcal{F}_2'$

and hence

$\mathcal{F}_2'$ contains $\mathcal{F}_2$, $\mathcal{F}_1^2$ and $\mathcal{F}_1'$.



There are points and lines of symmetries in this case.

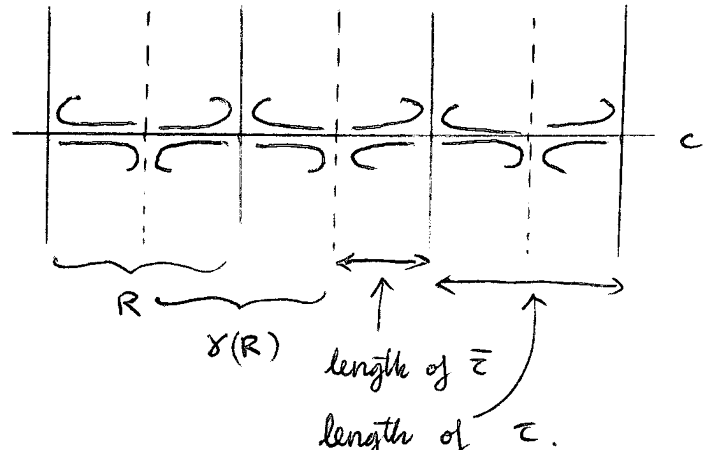
<u>Note</u>: $\mathcal{F}_2' = \{\tau^i \sigma_A^j \sigma_c^k \mid i, j, k \in \mathbb{Z}\}$

and $\alpha(c) = c$ for all $\alpha \in \mathcal{F}_2'$.

⑦ $\mathcal{F}_2^2 = \langle \gamma, \sigma_A \rangle$ where γ is a glide reflection with axis c, $\gamma^2 = \tau$ and $A \in c$.

Note that

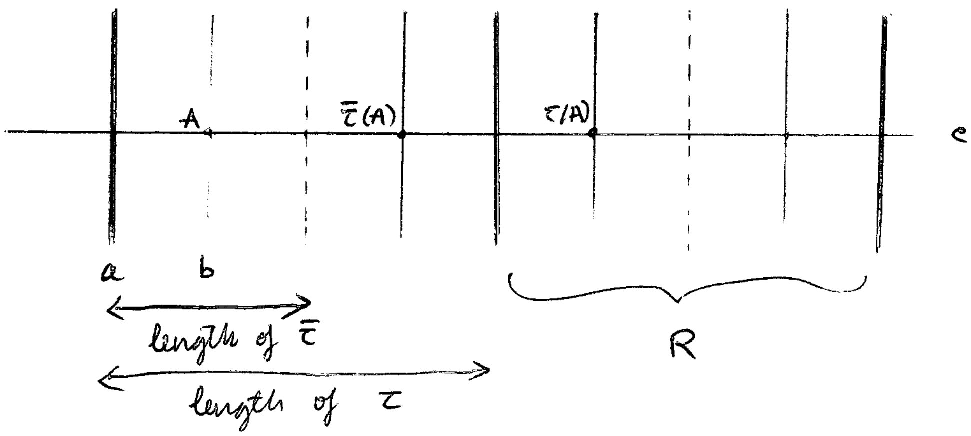
$$\mathcal{F}_2^2 \supset \langle \gamma \rangle = \mathcal{F}_1^3$$



$\gamma = \bar{\tau}\sigma_c$ where $\bar{\tau}^2 = \tau$

choose line $a \perp c$ so that $\gamma = \sigma_A \sigma_a$

(note that $\sigma_A = \sigma_c \sigma_b$ and $\bar{\tau} = \sigma_b \sigma_a$)



Note that: $\tau = \gamma^2$, $\sigma_a = \sigma_A \gamma \Rightarrow \langle \tau, \sigma_A, \sigma_a \rangle \subseteq \mathcal{F}_2^2$

and $\gamma = \sigma_A \sigma_a \Rightarrow \mathcal{F}_2^2 \subseteq \langle \tau, \sigma_A, \sigma_a \rangle$

Thus, $\mathcal{F}_2^2 = \langle \gamma, \sigma_A \rangle = \langle \tau, \sigma_A, \sigma_a \rangle$
where $A \in c$, $\gamma = \sigma_A \sigma_a$, $a \perp c$ and $\gamma^2 = \tau$.

Note that $\alpha(c) = c$ for all $\alpha \in \mathcal{F}_2^2$.

FRIEZE THEOREM: Let $\mathcal{F}$ be a frieze group with $\mathcal{F} \cap \mathcal{T} = \langle \tau \rangle$. Then there is a line c such that $\alpha(c) = c$ for all $\alpha \in \mathcal{F}$ (c is called the center of $\mathcal{F}$) and $\mathcal{F}$ is one of the following groups

$\mathcal{F}_1 = \langle \tau \rangle$

$\mathcal{F}_1^1 = \langle \tau, \sigma_c \rangle$

$\mathcal{F}_1^2 = \langle \tau, \sigma_a \rangle$, $a \perp c$

$\mathcal{F}_1^3 = \langle \gamma \rangle$,
γ is glide reflection with axis c, $\gamma^2 = \tau$

$\mathcal{F}_2 = \langle \tau, \sigma_A \rangle$, $A \in c$

$\mathcal{F}_2^1 = \langle \tau, \sigma_A, \sigma_c \rangle$, $A \in c$.

$\mathcal{F}_2^2 = \langle \gamma, \sigma_A \rangle = \langle \tau, \sigma_A, \sigma_a \rangle$

$\gamma = \sigma_A \sigma_a$, $a \perp c$, $A \in c$
and $\gamma^2 = \tau$.