PMAT 319 Winter 2016.
Chapter 4: Reflections.

Definition: Let m be a line in $\mathbb{R}^2$. The *reflection* in the line m is the function $\sigma_m : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\sigma_m(P) = \begin{cases} P & \text{if } P \in m, \\ Q & \text{if } P \notin m \text{ and } Q \text{ is the point so that } m \text{ is the perpendicular bisector of } \overline{PQ}. \end{cases}$$

We note that:
(i) σ_m is a non-identity involution; that is, $\sigma_m \in G$, $\sigma_m \neq i$ and $\sigma_m^2 = i$.
(ii) σ_m fixes exactly the points on m; that is, $\sigma_m(P) = P \Longleftrightarrow P \in m$.
(iii) For any line l, $\sigma_m(l) = l \Longleftrightarrow l = m$ or $l \perp m$.
(iv) If the line m has the equation $ax + by + c = 0$ then
$\sigma_m(x, y) = \left(x - \frac{2a(ax+by+c)}{a^2+b^2}, x - \frac{2b(ax+by+c)}{a^2+b^2}\right)$.

Definition: An *isometry* of $\mathbb{R}^2$ is a transformation α of $\mathbb{R}^2$ which preserves distance; that is,

$$\alpha(P)\alpha(Q) = PQ \text{ for all } P, Q \in \mathbb{R}^2.$$

It can be shown that isometries preserve collinearity, betweeness, midpoints, angle measure, perpendicularity.

Theorem: The set $\mathcal{I}$ of all isometries of $\mathbb{R}^2$ is a group.
Proof:

Definition: Let S be a subset of $\mathbb{R}^2$.
A *point of symmetry* of S is a point P so that $\sigma_P(S) = S$.
A *line of symmetry* of S is a line m so that $\sigma_m(S) = S$.
If P is a point of symmetry of S, we say that S is symmetrical about the point P.
If m is a line of symmetry of S, we say that S is symmetrical about the line m.
A *symmetry* of S is an isometry α so that $\alpha(S) = S$.
The set $\mathcal{I}_S$ of all symmetries of S is a group, which is called the *symmetry group* of S.