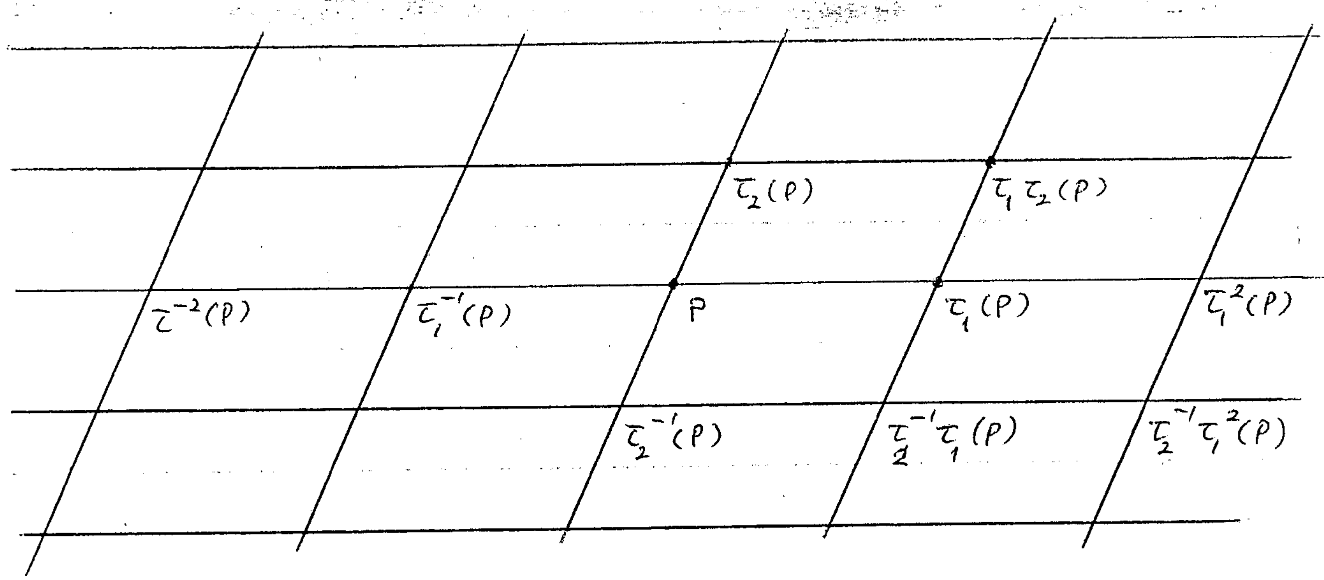
Chapter 11 Wall paper Groups

<u>Definition</u> A <u>wall paper group</u> is a subgroup W of $\mathcal{I}$ so that $W \cap \mathcal{T} = \langle \tau_1, \tau_2 \rangle$ where τ_1 and τ_2 are non-identity translations with non parallel directions.

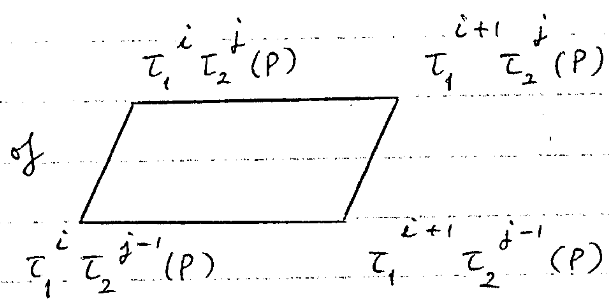
Let W be a wall paper group with $W \cap \mathcal{T} = \langle \tau_1, \tau_2 \rangle$

(1) Let $P \in \mathbb{R}^2$ and consider

$\mathcal{L} = \{ \tau_1^n \tau_2^m (P) \mid n, m \in \mathbb{Z} \}$. $\mathcal{L}$ is called the <u>translation lattice</u> of W (determined by P)



Each of



is called a <u>unit cell</u> of W

(2) A point P is called an <u>n-centre</u> of W if

$$C_n = \langle \rho_{P, \frac{360}{n}} \rangle \subset W \quad \text{where } n \in \mathbb{Z}^+.$$

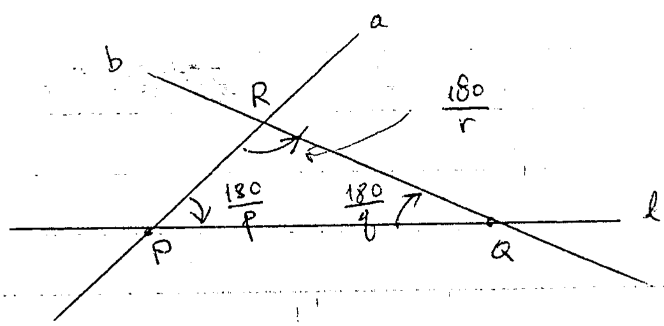
(3) Let P be a p-centre and Q be a q-centre of $\mathcal{W}$, $P \neq Q$ (Assume $p \geq 2$ and Q is closest to P)

This means $\rho_{P, \frac{360}{p}}$ and $\rho_{Q, \frac{360}{q}}$ are elements of $\mathcal{W}$

Let $l = \overleftrightarrow{PQ}$. Let a, b be lines so that

$\rho_{P, \frac{360}{p}} = \sigma_a \sigma_l$ and $\rho_{Q, \frac{360}{q}} = \sigma_l \sigma_b$

Let $R = a \cap b$.



Then $\rho_{P, \frac{360}{p}} \rho_{Q, \frac{360}{q}} = \sigma_a \sigma_l \sigma_l \sigma_b = \sigma_a \sigma_b = \rho_{R, -\frac{360}{r}} \in \mathcal{W}$

This means, R is an r-centre of $\mathcal{W}$

and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ $(p, q, r \in \mathbb{Z}^+)$

Thus the only possibilities are (assuming $p \geq 3$, $q \geq 3$

$p = q = r = 3$

$p = q = 4$, $r = 2$

$p = 3$, $q = 6$, $r = 2$

or $q = 3$, $p = 6$, $r = 2$.

Thus,

(4) If P is an n-centre for $\mathcal{W}$ then $n = 2, 3$ or 6

(5) <u>If P is an n-centre for $\mathcal{W}$ and $\alpha \in \mathcal{W}$ then $\alpha(P)$ is an n-centre for $\mathcal{W}$</u>.

This is because $\alpha \rho_{P, \frac{360}{n}} \alpha^{-1} = \rho_{\alpha(P), \pm\frac{360}{n}}$.

<u>If ℓ is a line of reflection for $\mathcal{W}$</u> ($\sigma_\ell \in \mathcal{W}$) <u>and $\alpha \in \mathcal{W}$ then $\alpha(\ell)$ is a line of reflection for $\mathcal{W}$</u>

This is because $\alpha \sigma_\ell \alpha^{-1} = \sigma_{\alpha(\ell)}$

<u>Theorem</u> (6) If $\mathcal{W}$ has a reflection σ_ℓ then the unit cell of $\mathcal{W}$ is rhombic or rectangular, where ℓ is parallel to a diagonal of the rhombus, or the sides of the rectangles.

(7) If $\mathcal{W}$ has a glide reflection γ then the translation lattice of $\mathcal{W}$ is rhombic or rectangular.

(8) If $\mathcal{W}$ has a glide reflection γ that fixes a translation lattice of $\mathcal{W}$ then $\mathcal{W}$ contains a reflection.

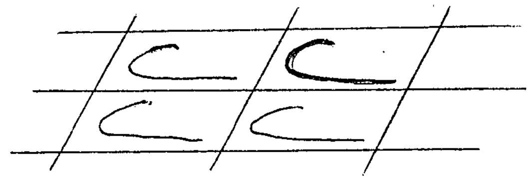
(9) Let $S \subseteq \mathbb{R}^2$ so that $\mathcal{W} = \mathcal{I}_S$ and P be an n-centre of $\mathcal{W}$. Then P is called a <u>centre of symmetry</u> for S, and P is a point of symmetry for S iff n is even.

(10) Let P and Q be n-centres for $\mathcal{W}$, and $P \neq Q$, $\tau \in \mathcal{W}$. Then $PQ \geqslant Q\tau(Q)$.

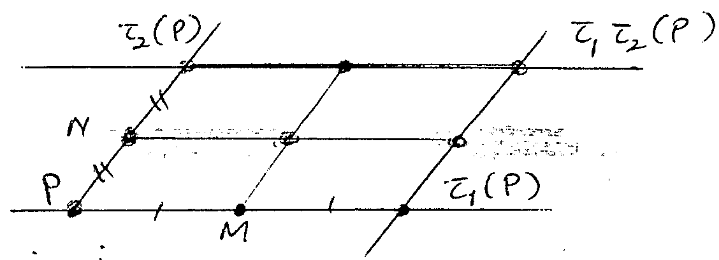
(11) Let $\mathcal{W} = \mathcal{I}_S$. A <u>base</u> of $\overset{2}{\mathcal{W}}$ is a smallest polygonal region B so that $\mathbb{R}^2$ is covered by $\{\alpha(B) \mid \alpha \in \mathcal{W}\}$. A <u>motif</u> for $\mathcal{W}$ is a subset M of a base B so that $\mathcal{I}_M = \{i\}$.

POSSIBILITIES FOR $\mathcal{W} = \mathcal{S}_S$

① $\mathcal{W}_1 = \langle \tau_1, \tau_2 \rangle$. In this case, the unit cell is base for $\mathcal{W}$



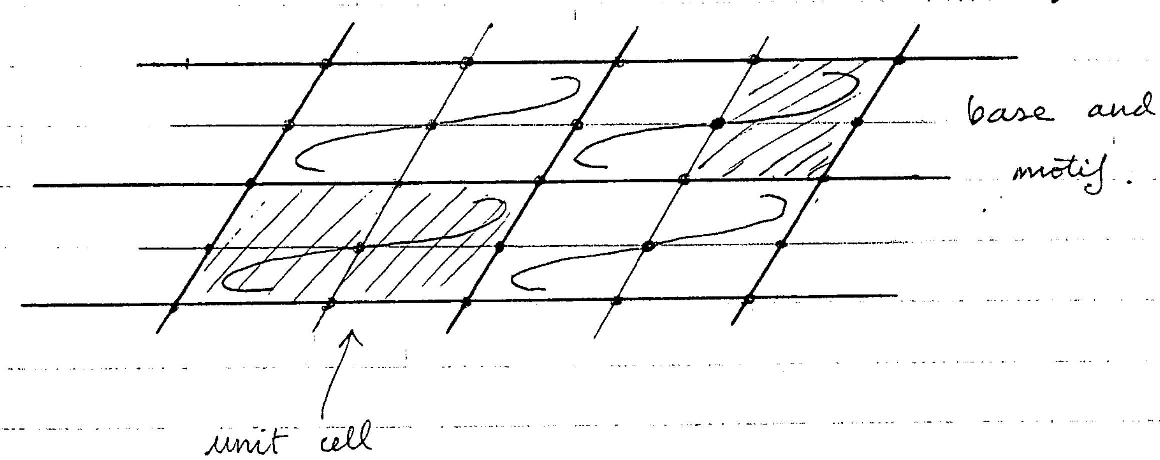
② $\mathcal{W}_2 = \langle \tau_1, \tau_2, \sigma_P \rangle$



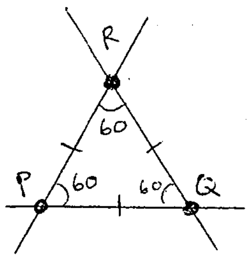
As noted for the frieze group $\mathcal{F}_2$,

$\{\tau_1^i \tau_2^j (P) \mid i,j \in \mathbb{Z}\} \cup \{\tau_1^i \tau_2^j (M) \mid i,j \in \mathbb{Z}\}$
$\cup \{\tau_1^i \tau_2^j (N) \mid i,j \in \mathbb{Z}\}$ are set of points of symmetry for $\mathcal{W}$

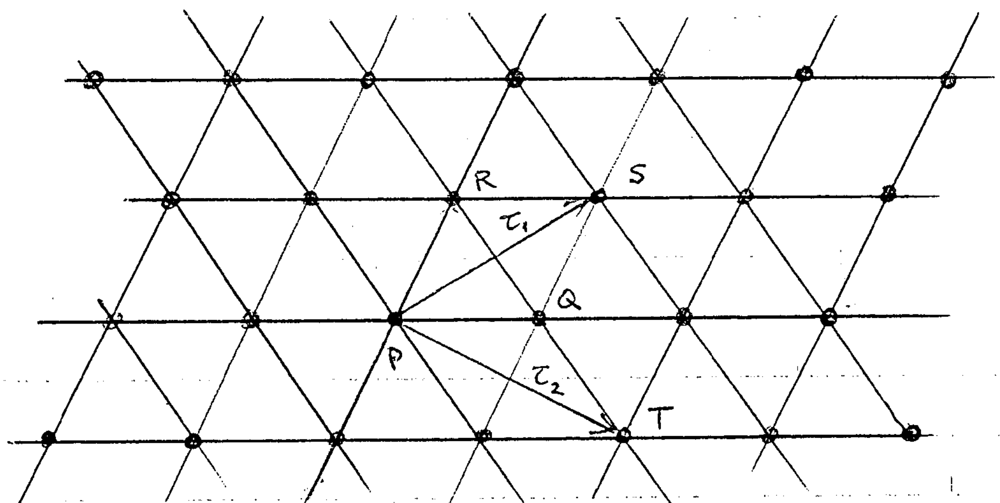
• = centre of symmetry.



(3) $W_3 = \langle \rho_{P,120}, \rho_{Q,120} \rangle = \langle \tau_1, \tau_2, \rho_{P,120} \rangle$



Assume P and Q closest possible 3-centres. Thus, there is no 3-centre inside $\triangle PQR$. Then for all $\alpha \in W_3$, $\alpha(P), \alpha(Q), \alpha(R)$ are 3-centres and $\triangle \alpha(P)\alpha(Q)\alpha(R) \cong \triangle PQR$ and contains no 3 centre inside. Thus, we obtain the <u>centre lattice</u> for W_3



We want to determine τ_1 and τ_2. Let $\tau \in W_3$. We want τ to be the shortest translations that maps 3-centres to 3-centres.

If $\tau(P) = Q$ then $\rho_{Q,120}\tau(P) = Q$, $\rho_{Q,120}\tau(Q) = R$

and $\rho_{Q,120}\tau(R) = P$ and so $\rho_{Q,120}\tau = \rho_{X,120}$ where

X is the centroid of $\triangle PQR$. This contradicts the fact that there are no 3-centre in $\triangle PQR$.

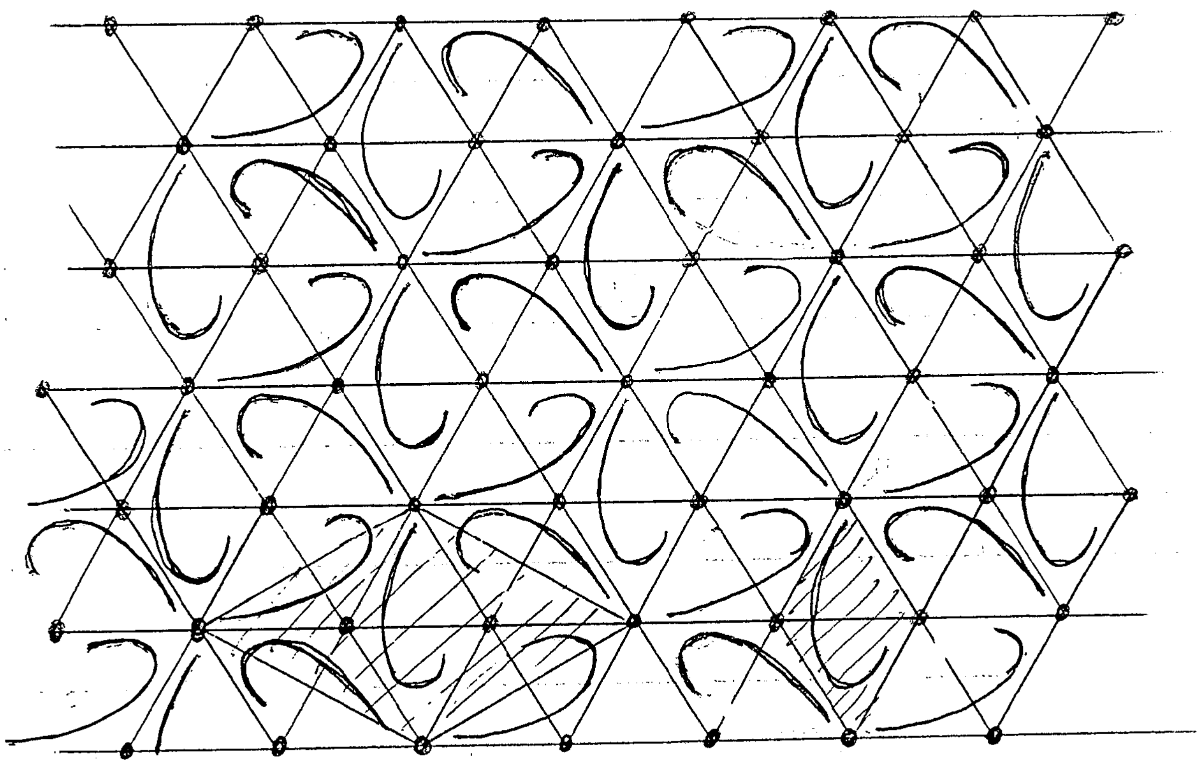
Thus, $\tau(P) \neq Q$. Similarly, $\tau(P) \neq R$.

Assuming $\tau(P)$ is a 3-centre closest to P, we conclude that $\tau(P) = S$ or $\tau(P) = T$.

Put $\tau_1 = \tau_{PS}$ and $\tau_2 = \tau_{PT}$

Then it is easy to see that $\tau_1 \rho_{Q,120} = \rho_{S,120}$

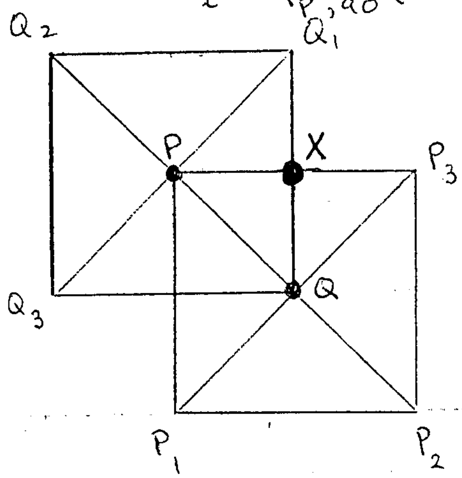
and so $\tau_1 = \rho_{S,120}\, \rho_{Q,-120}$. Similarly, $\tau_2 = \rho_{T,-120}\, \rho_{Q,120}$



unit cell base

(4) $W_4 = \langle \rho_{P,90}, \rho_{Q,90} \rangle = \langle \tau_1, \tau_2, \rho_{P,90} \rangle$

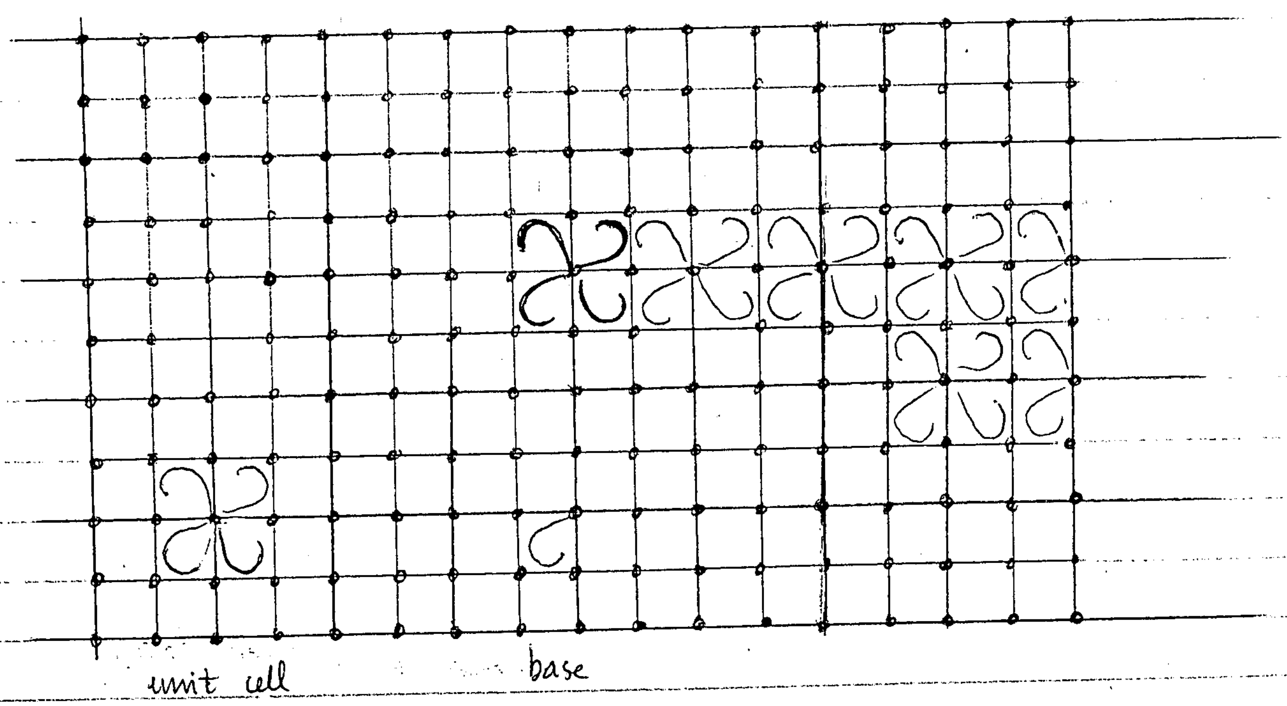
Assume that P and Q are closest 4-centres.
Note that this is the case $\frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$. So, there are 2-centres.
Suppose τ is a translation in W_4 and $\tau(P) = Q$.
Put $Q_i = \rho_{P,90}(Q)$ and $P_i = \rho_{Q,90}(P)$, $i = 1, 2, 3$.



Then

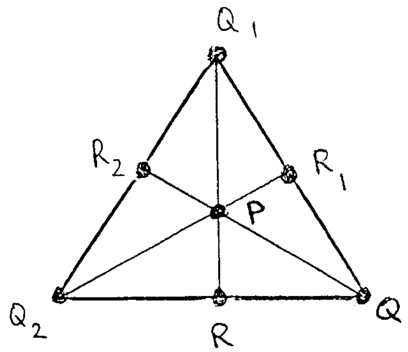
$$\left.\begin{array}{l} \rho_{Q,90}\,\tau(P) = Q \\ \rho_{Q,90}\,\tau(Q_1) = P \\ \rho_{Q,90}\,\tau(X) = X \end{array}\right\} \Rightarrow \rho_{Q,90}\,\tau = \rho_{X,90}$$

where X is the mid point of $\overline{QQ_1}$ and X is a 4-centre. This contradicts the assumption that P and Q are closest 4-centres. Thus, $\tau_1 = \tau_{PP_1}$ and $\tau_2 = \tau_{PP_3}$.
Note that $\rho_{P,90}\,\rho_{Q,90} = \rho_{X,180} = \sigma_X$, so X is a 2-centre.
• • are 4-centres • are 2 centres.



⑤ $W_6 = \langle \rho_{P,120}, \rho_{Q,60} \rangle = \langle \tau_1, \tau_2, \rho_{Q,60} \rangle$

This is the case $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$



Suppose P and Q are closest, we get

$\rho_{Q,60}\, \rho_{P,120} = \sigma_R$

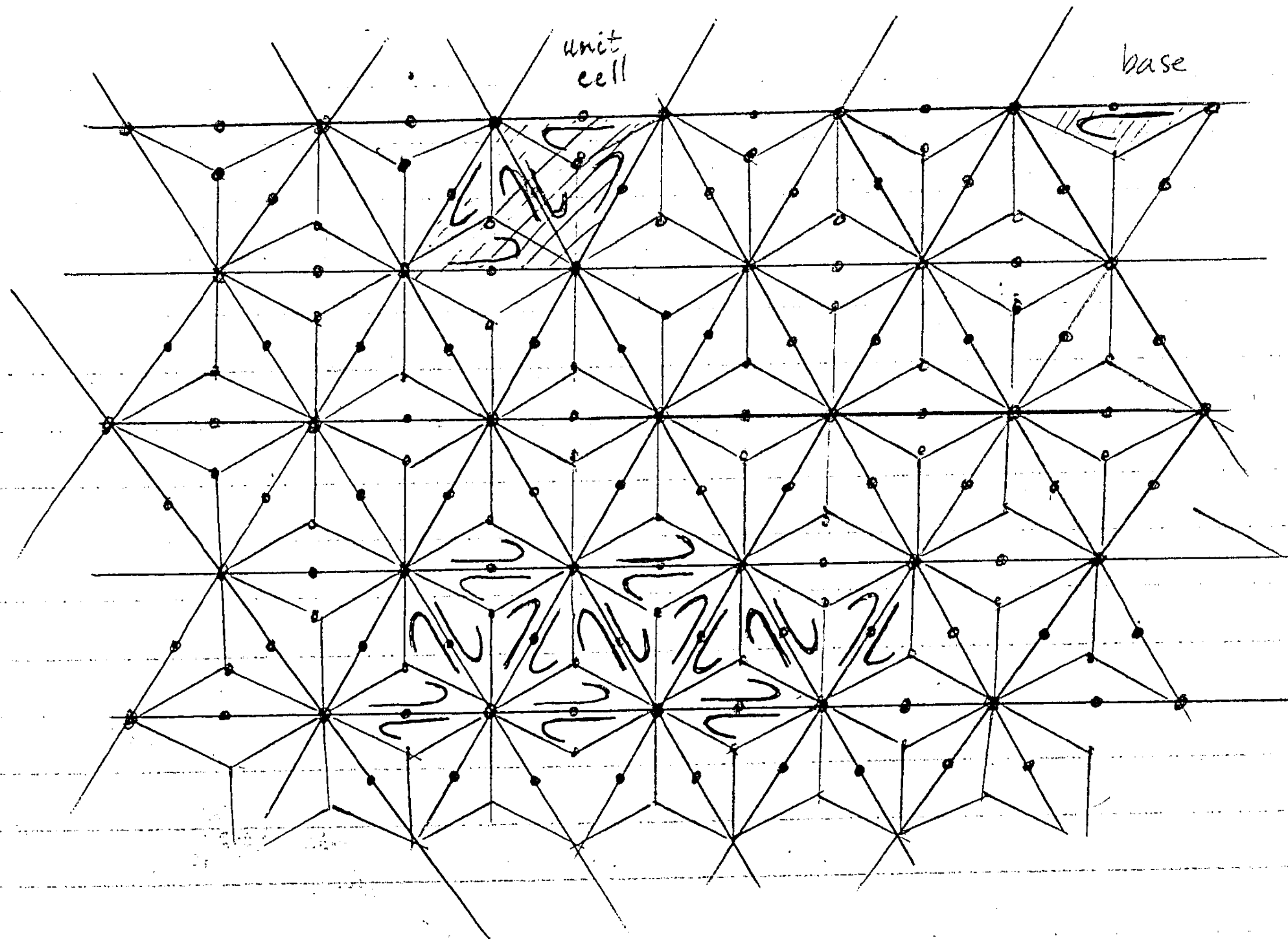
So R is a 2-centre.

$\sigma_R \sigma_Q = \sigma_R (\rho^3_{Q,60}) = \tau_{QQ_2}$

and $\sigma_{R_1} \sigma_Q = \sigma_{R_1} (\rho^3_{Q,60}) = \tau_{QQ_1}$

(Note: R_1 is a 2-centre because $R_1 = \rho_{P,120}(R)$)

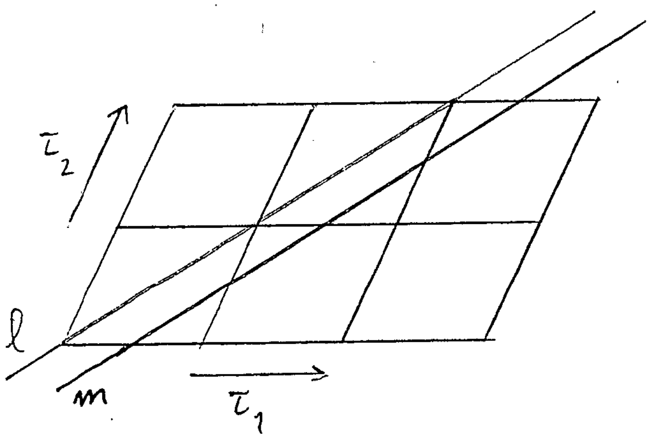
Thus, we can have $\tau_1 = \tau_{Q_2 Q}$ and $\tau_2 = \tau_{QQ_1}$.



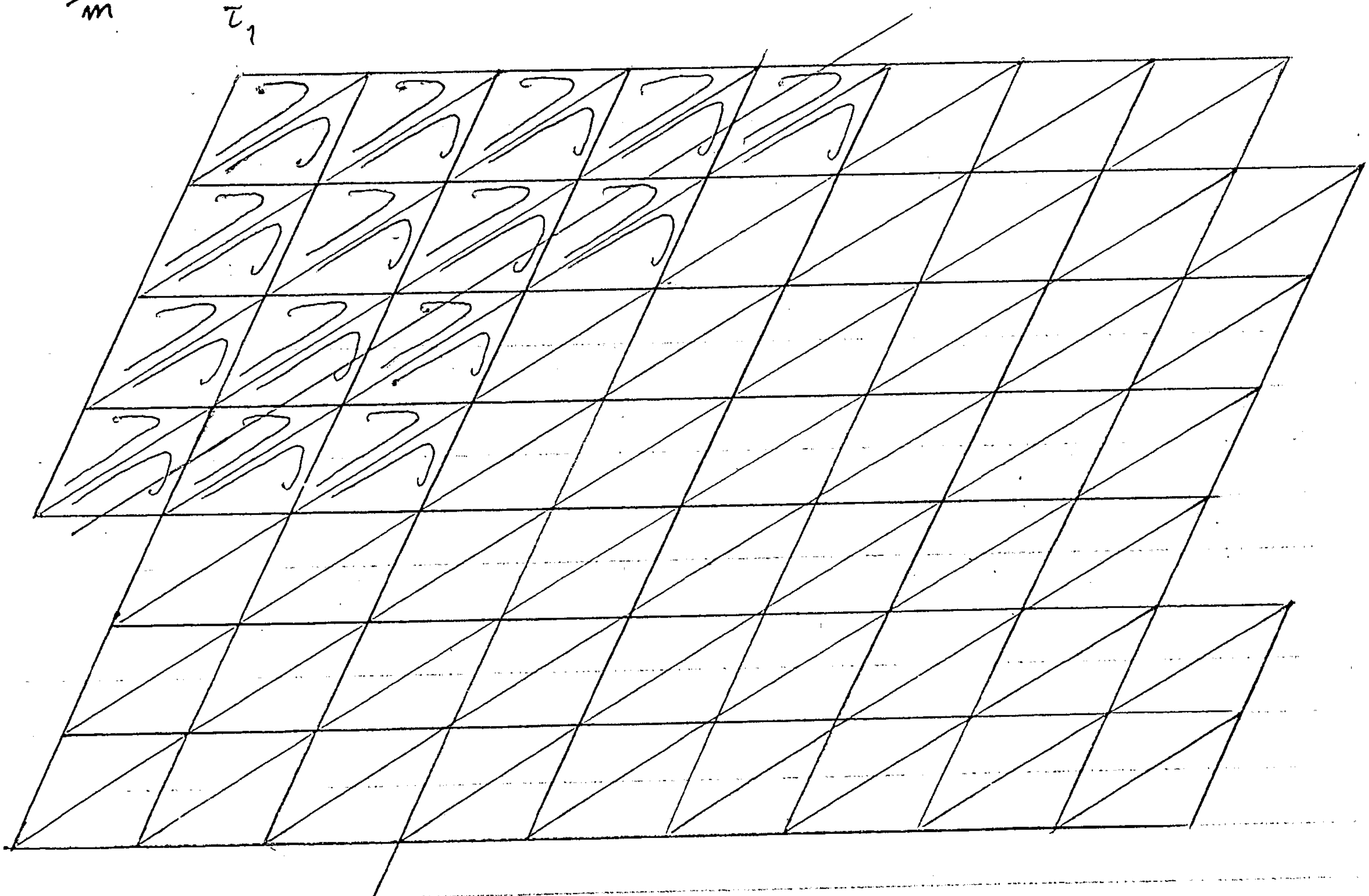
POSSIBILITIES FOR $\mathcal{W}$

① Expanding $\mathcal{W}_1'$

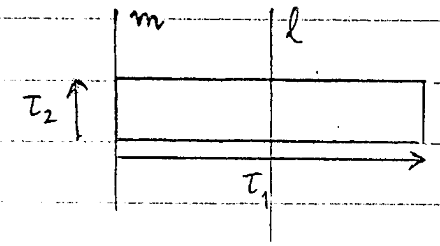
1.1 $\mathcal{W}_1' = \langle \tau_1, \tau_2, \sigma_\ell \rangle$ where the unit cells $\mathcal{U}$ are rhombic and ℓ is parallel to a diagonal of $\mathcal{U}$.



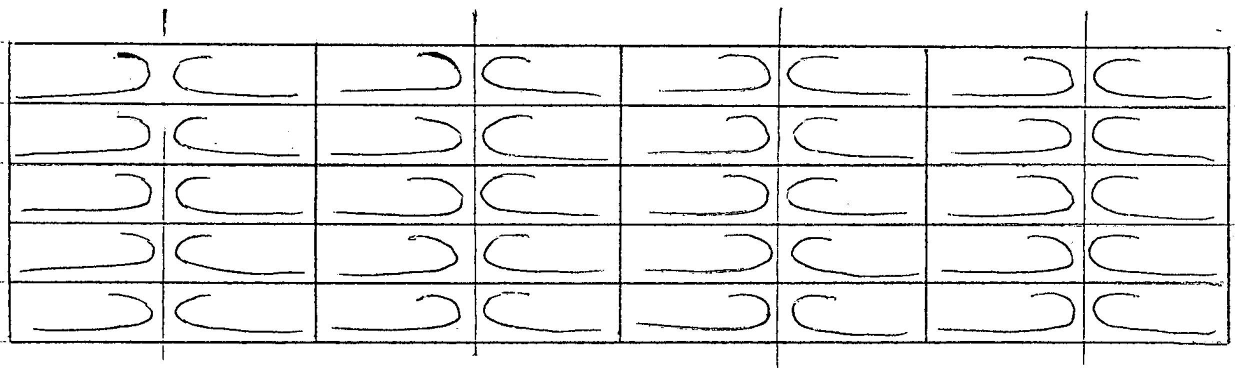
$\gamma = \tau_1 \sigma_\ell$ is a glide reflection with axis m and $\gamma^2 = \tau_2 \tau_1 = \tau_1 \tau_2$.



1.2 $\mathcal{W}_1^2 = \langle \tau_1, \tau_2, \sigma_\ell \rangle$ with unit cell $\mathcal{U}$ rectangular and ℓ is parallel to one side of $\mathcal{U}$.

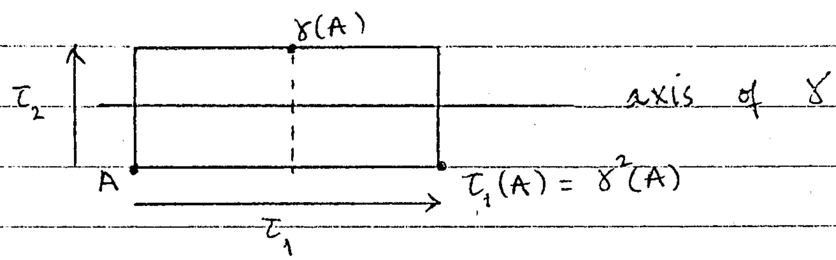


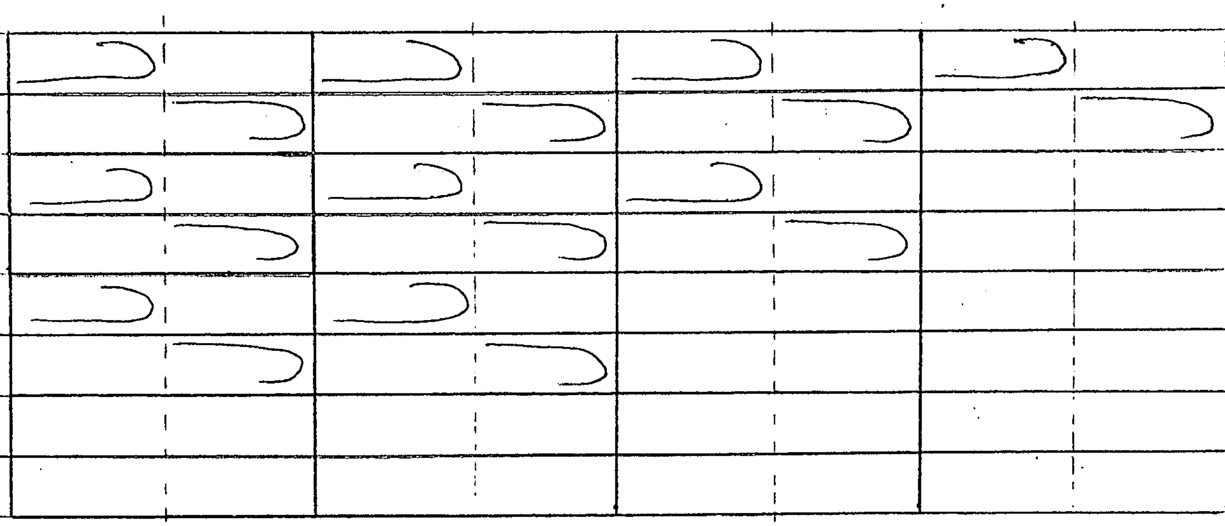
$\tau_1 = \sigma_\ell \, \sigma_m \Rightarrow \sigma_m \in \mathcal{W}_1^2$



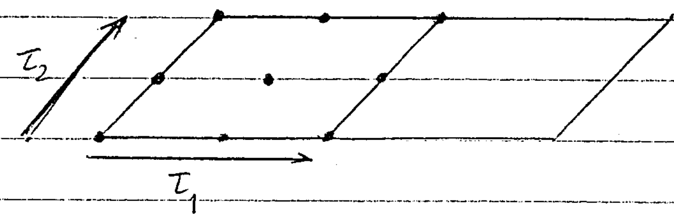
1.3 $\mathcal{W}_1^3 = \langle \tau_1, \tau_2, \gamma \rangle = \langle \gamma, \tau_2 \rangle$

where $\gamma^2 = \tau_1$ and the unit cell $\mathcal{U}$ rectangular



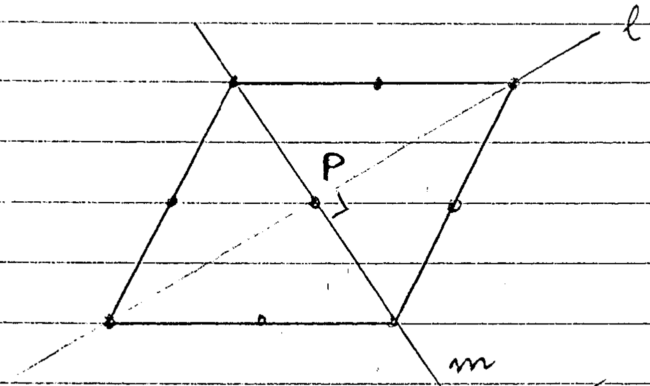


Expanding $\mathcal{W}_2 = \langle \tau_1, \tau_2, \sigma_P \rangle$



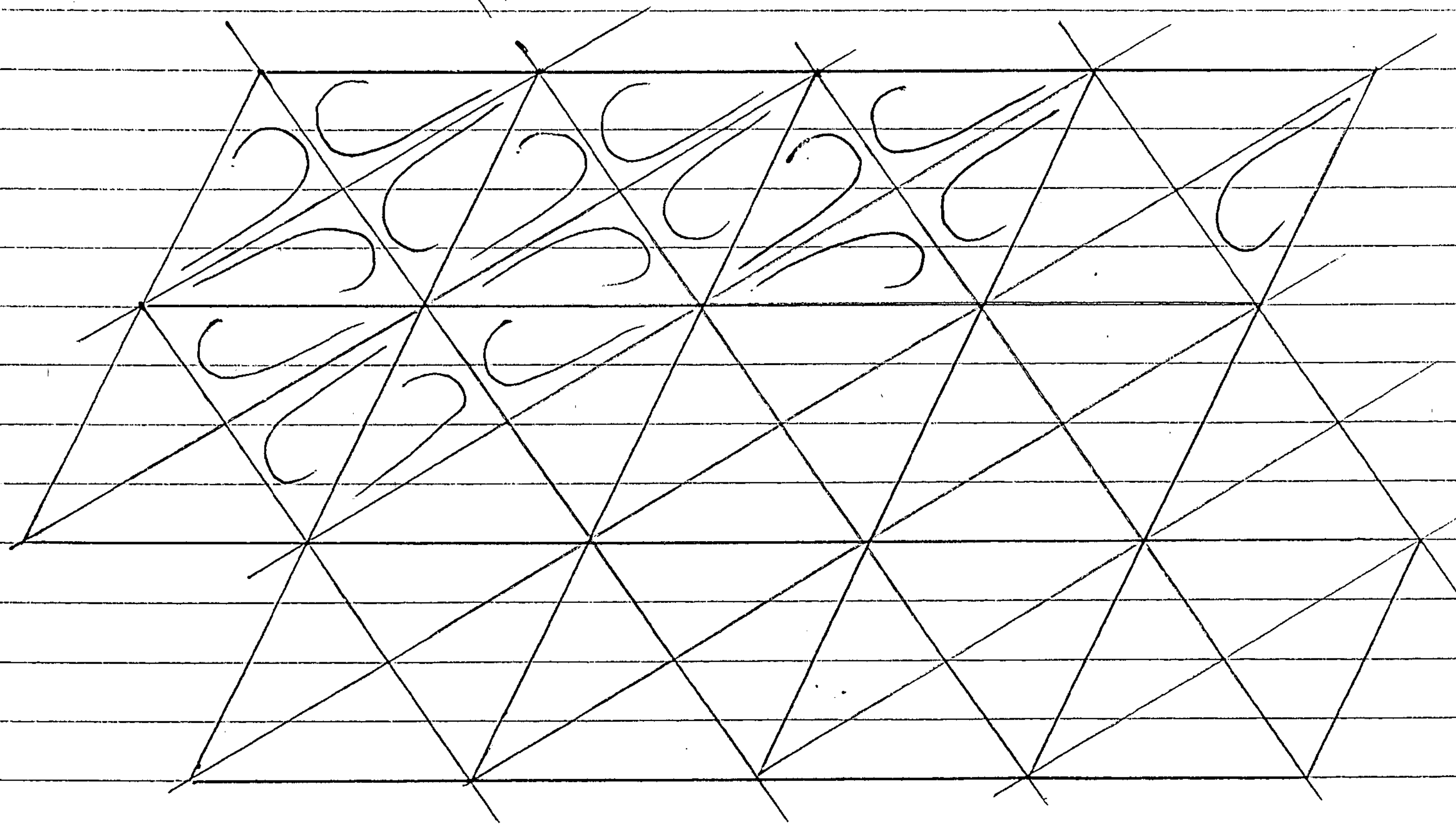
2.1 $\mathcal{W}_2' = \langle \tau_1, \tau_2, \sigma_P, \sigma_\ell \rangle$ with unit cell $\mathcal{U}$ rhombic

$P \in \ell$ and ℓ is a diagonal of $\mathcal{U}$.



Note that $\sigma_P = \sigma_m \sigma_\ell$,

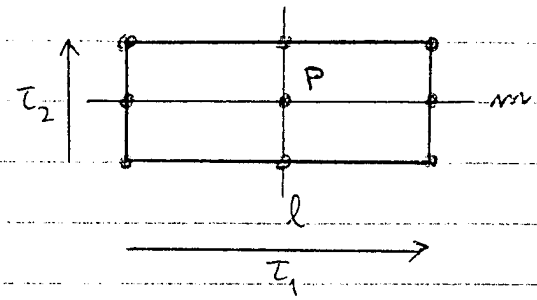
so $\sigma_m \in \mathcal{W}_2'$.

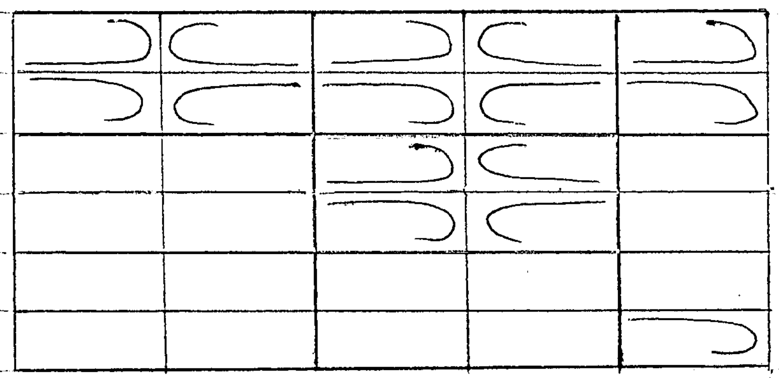


2.2 $W_2^2 = \langle \tau_1, \tau_2, \sigma_p, \sigma_\ell \rangle$ with unit cell U rectangular

$p \in \ell$ and ℓ is parallel to one side of U.

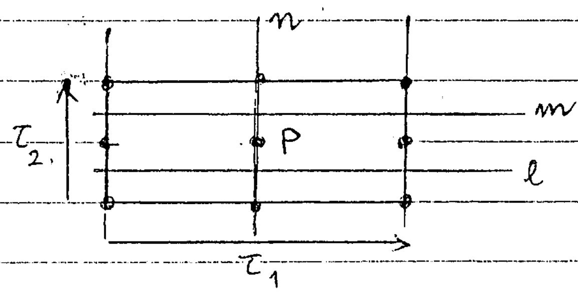
$\sigma_p = \sigma_m \sigma_\ell$ so $\sigma_m \in W_2^2$



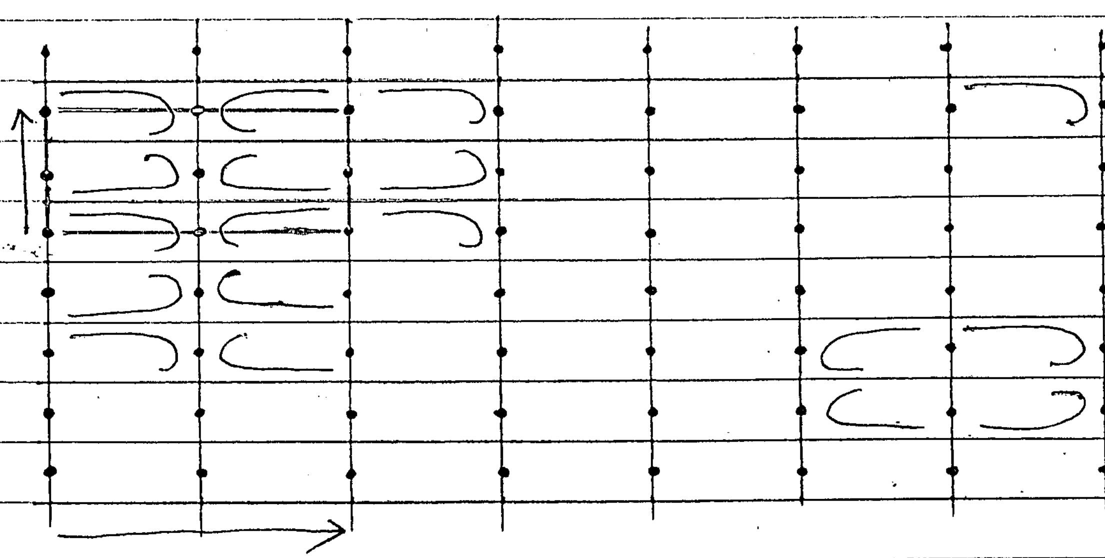


2.3 $W_2^3 = \langle \tau_1, \tau_2, \sigma_p, \sigma_\ell \rangle$ with unit cell U rectangular
ℓ is parallel to one side of U and ℓ does not contain any 2-centre. But for any 2-centre X, $\sigma_\ell(X)$ is a 2-centre. Thus we have

$\tau_2 = \sigma_m \sigma_\ell \Rightarrow \sigma_m \in W_2^3$



Consider the glide reflection $\gamma = \sigma_\ell \sigma_p$ Then the axis of γ is n and $\gamma^2 = \tau_2$.



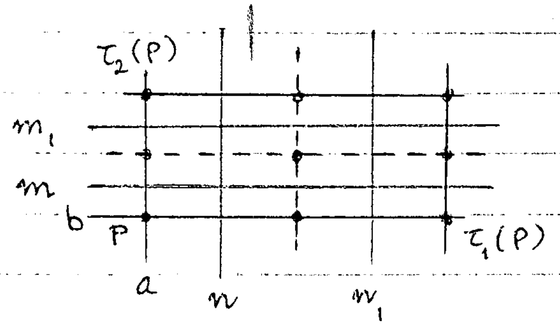
Base

Wrong!

unit cell.

2.4 $\mathcal{W}_2^{4} = \langle \tau_1, \tau_2, \sigma_P, \gamma \rangle$ with unit cell $\mathcal{U}$ rectangular

γ is a glide reflection with $\gamma^2 = \tau_2$ and with axis n which does not contain any 2-centre

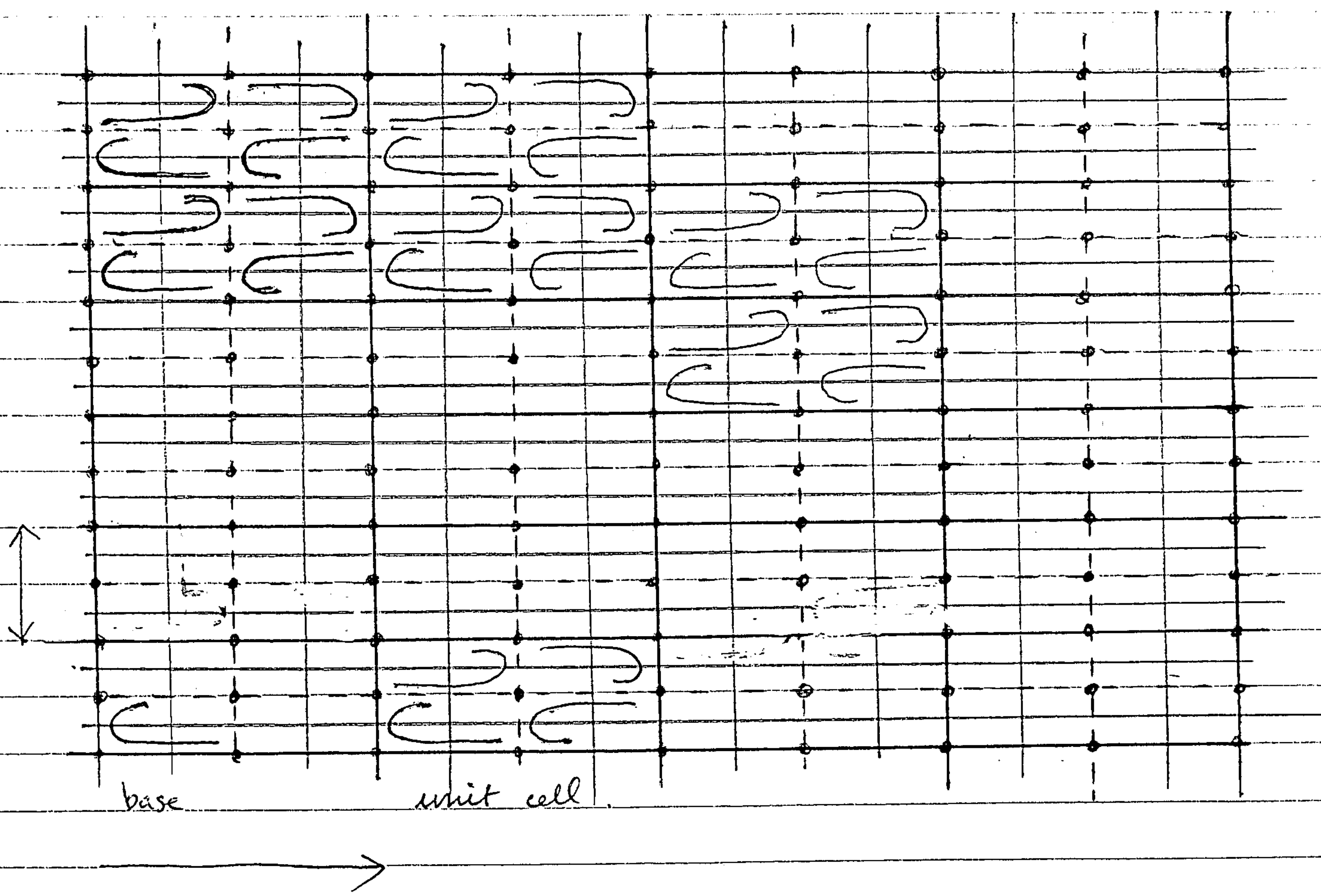


$$\gamma = \sigma_m \sigma_b \sigma_n = \sigma_n \sigma_m \sigma_b$$

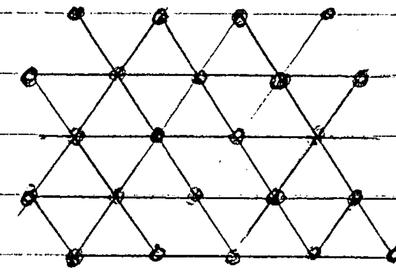
$$\gamma^2 = \tau_2 = \sigma_{m_1} \sigma_m$$

Put $\delta = \gamma \sigma_P = (\sigma_n \sigma_m \sigma_b)(\sigma_b \sigma_a) = \sigma_n \sigma_m \sigma_a = \sigma_n \sigma_a \sigma_m$,

so δ is a glide reflection with axis m and $\delta^2 = \tau_1$.

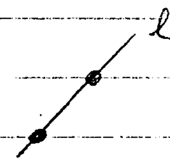


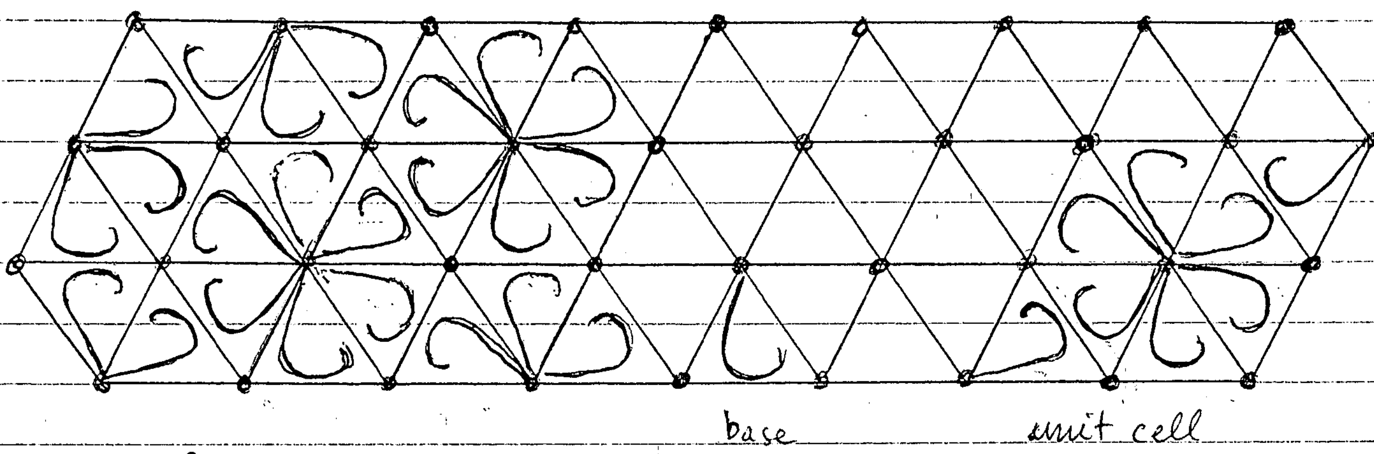
Expanding $W_3 = \langle \tau_1, \tau_2, \rho_{P_1 120} \rangle$ which have 3 type of 3-centres. It can be shown that need only to consider σ_l and there are only two possibilit for l:



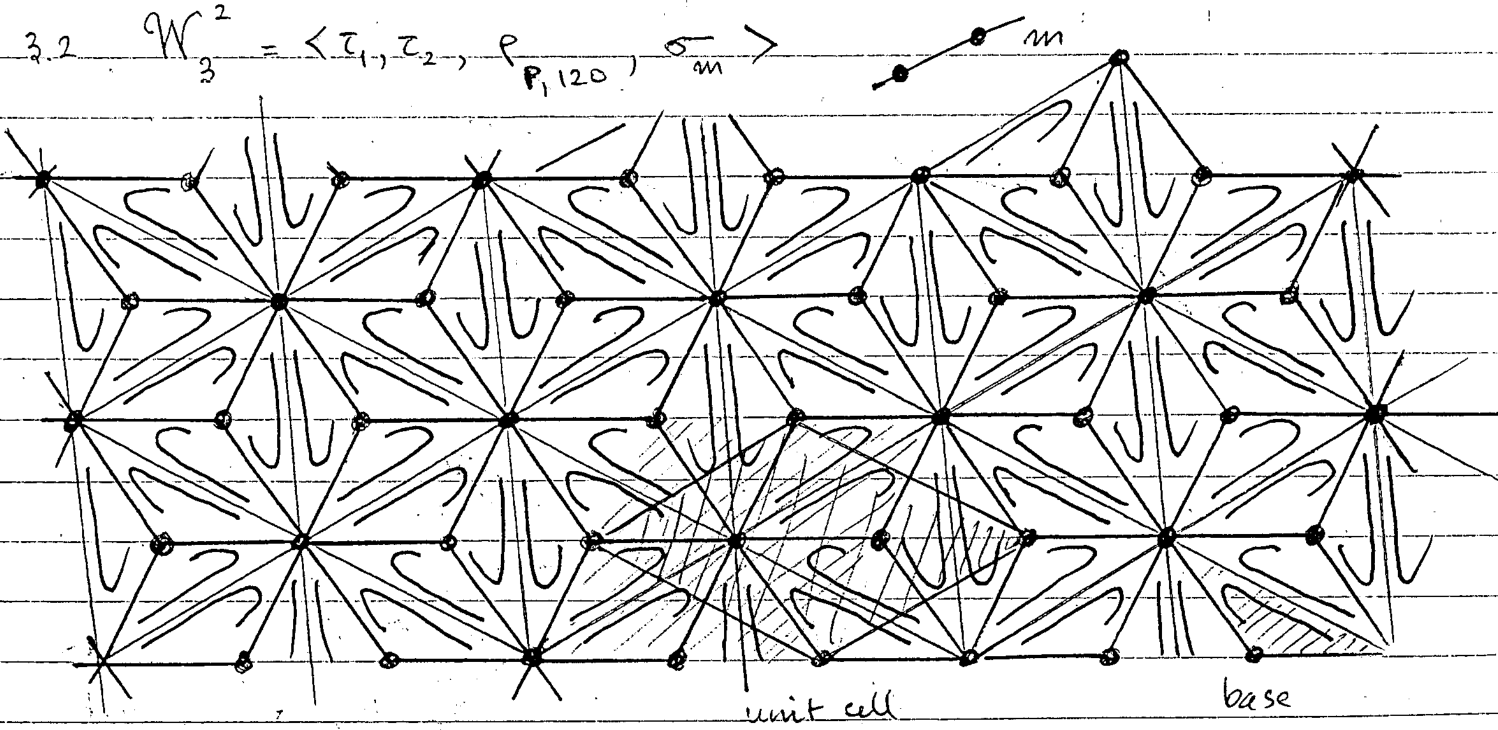
① l passes through 2 different types of 3 centre, or
② l passes through two 3-centres of the same type.

3.1 $W_3^1 = \langle \tau_1, \tau_2, \rho_{P_1 120}, \sigma_l \rangle$

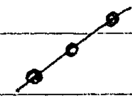
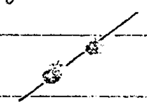


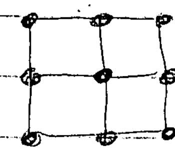


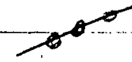
3.2 $W_3^2 = \langle \tau_1, \tau_2, \rho_{P_1 120}, \sigma_m \rangle$

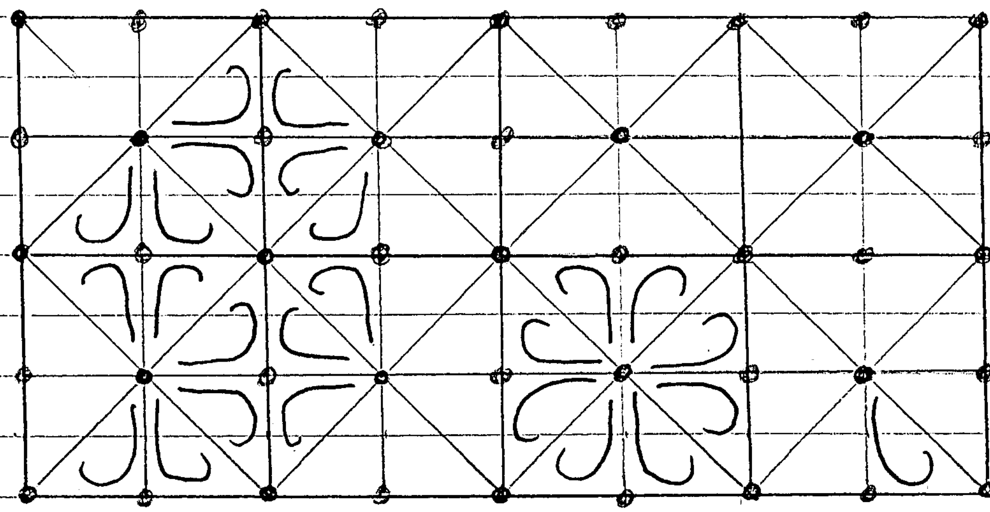


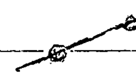
Expanding $W_4 = \langle \tau_1, \tau_2, \rho_{P,90} \rangle$

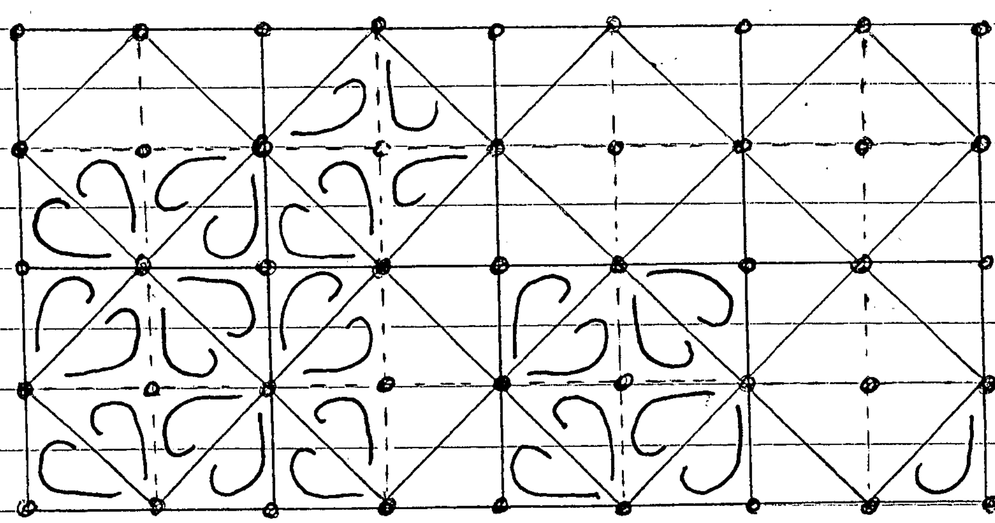
- Two type of centres (• = 4-centre and • = 2-centre)
- Can be shown that there can only be reflections, either  or 



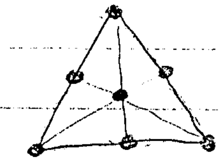
4.1 $W_4^1 = \langle \tau_1, \tau_2, \rho_{P,90}, \sigma_\ell \rangle$ where  ℓ



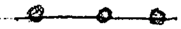
4.2 $W_4^2 = \langle \tau_1, \tau_2, \rho_{P,90}, \sigma_m \rangle$ where  m



Expanding $W_6 = \langle \tau_1, \tau_2, \rho_{Q,60} \rangle$



It can be shown that only one choice of reflection can work : σ_ℓ with



ℓ

• 6 - centre • 3-centre • 2 - centre .

6.1 $W_6' = \langle \tau_1, \tau_2, \rho_{Q,60}, \sigma_\ell \rangle$