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Basic Definitions

Definition. (Inner Product) An *inner product* on V is a function that takes each ordered pair (x, y) of elements of V to a number $\langle x, y\rangle \in \mathbb{F}$ and, $\forall x, y, z \in V, \lambda \in \mathbb{F}$, has the following properties:

Positivity	$\langle y, y\rangle \geq 0$
Definiteness	$\langle y, y\rangle = 0 \iff y = 0$
Additivity	$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
Homogeneity	$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$
Conjugate Symmetry	$\langle x, y\rangle = \overline{\langle y, x\rangle} = \langle y, x\rangle^*$

Definition. (Norm) Given a vector space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, a *norm* is a mapping $\|\cdot\|) : V \to [0, \infty)$ for which

(i) $\|f\| \geq 0$ and $\|f\| = 0 \iff f = 0$
(ii) $\|\lambda f\| = |\lambda|\|f\|$
(iii) $\|f + g\| \leq \|f\| + \|g\|$

for all $f, g \in V$, $\lambda \in \mathbb{F}$. If V is an inner product space, there is a trivial norm:

$$\|f\| = \sqrt{\langle f, f\rangle}$$

Normed Spaces

Theorem. (Triangle Inequality)

$$\|x + a\| \leq \|x\| + \|a\|$$

Theorem. (Reverse Triangle Inequality)

$$\big|\|x\| - \|a\|\big| \leq \|x - a\|$$

Example. (Example Normed Spaces)

$$V = \mathbb{R}^n, \quad \|x\|_2 = \sqrt{\textstyle\sum_{j=1}^n x_j^2}$$
$$V = \mathbb{R}^2, \quad \|x\|_1 = |x_1| + |x_2|$$
$$V = \mathbb{R}^2, \quad \|x\|_\infty = \max\{|x_1|, |x_2|\}$$
$$V = C([0, 1], \mathbb{R}), \quad \|f\|_p = \left(\textstyle\int_0^1 |f|^p\right)^{\frac{1}{p}}$$

Definition. (Open and Closed Balls) With $(X, \|\cdot\|)$ a normed linear space, with $x_0 \in X$, and $r > 0$:

Open Ball With centre x_0 and radius r as $\mathcal{B}(x_0, r) = \{x \in X : \|x - x_0\| < r\}$

Closed Ball With centre x_0 and radius r as $\mathcal{B}[x_0, r] = \{x \in X : \|x - x_0\| \leq r\}$

Sphere With centre x_0 and radius r as $\mathcal{S}(x_0, r) = \{x \in X : \|x - x_0\| = r\}$

Inner Product Spaces

Example. (Inner-Product Space Examples)

- ℓ^p space:

$$\ell^2 = \ell^2(\mathbb{N}, \mathbb{C}) = \left\{\{c_k\}_{k=1}^\infty : \sum_{k=1}^\infty |c_k|^2 < \infty\right\}$$

And hence for ℓ^2 (the only ℓ^p space that is a Hilbert space)

$$\langle x, y\rangle_{\ell^2} = \sum_n x_n\overline{y_n}$$

- $L^2(\mathbb{R})$ space:

$$\langle x, y\rangle_{L^2} = \int_{-\infty}^\infty f(x)\overline{g(x)}\,\mathrm{d}x$$

Theorem. (Cauchy-Schwarz Inequality) Given that V is an inner-product space over $\mathbb{F}$, then for all $f, g \in V$:

$$|\langle f, g\rangle| \leq \|f\|\|g\|$$

When $\{f, g\}$ is a linearly dependent set, we have equality.

Sequences and Convergence

Definition. (Cauchy Sequence) A sequence $\{a_n\}$ of real numbers is a *Cauchy Sequence* if

$$\forall \varepsilon > 0, \exists N > 0 : m, n > N \implies \|a_m - a_n\| < \varepsilon$$

Given that $n > N \implies \|x_n - x\| < \varepsilon$, then $\lim_{n\to\infty} x_n = x$.

Definition. (Geometric Series) A geometric series (aka "geometric progression") has sequence elements of the form $a_n = ar^n$, where r is the "*common ratio*". The partial sum is given by:

$$s_k = a\left(\frac{1 - r^k}{1 - r}\right)$$

and so, given that $|r| < 1$, the partial sum limits to:

$$\lim_{n\to\infty} \sum_{k=1}^n ar^k = \frac{a}{1 - r}$$

Operator Theory

Proposition. (Operator Continuity) Let X and Y be normed linear spaces and $T : X \to Y$ be a linear mapping. Then, the following are equivalent:

(i) T is continuous at every point in X
(ii) T is continuous at 0
(iii) T is bounded

Hilbert Spaces

Theorem. (Riesz Representation Theorem) Given a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and a linear functional $F : \mathcal{H} \to \mathbb{C}$, there exists a unique $y = y_f$, for $y_f \in \mathcal{H}$ such that

$$f(x) = \langle x, y_f\rangle, \quad \forall x \in \mathcal{H}$$

Definition. (Adjoint Operator) Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear mapping between Hilbert spaces. The mapping $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ is called the *adjoint* of the operator T and, for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, is defined:

$$\langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1$$

T is said to be *self-adjoint* if $T = T^*$.

Definition. (Unitary Operator) Let $\mathcal{H}$ be a Hilbert space and $U \in \mathcal{B}(\mathcal{H})$. We say that U is *unitary* if U is invertible and $U^{-1} = U^*$. Equivalently, $UU^* = U^*U = I$.

Definition. (Bounded Linear Operator) Let $T : X \to Y$ be a bounded linear operator between normed linear spaces. The *norm* of T is defined to be:

$$\|T\| = \inf\{M > 0 : \forall x \in X, \|Tx\| \leq M\|x\|\}$$

Note that $\|Tx\| \leq \|T\|\|x\|$ and $\|T\|$ is the smallest M such that this is valid.

Proposition. Let $T : X \to Y$ be a bounded linear operator between normed linear spaces. Then

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$$

Definition. (Normal Operator) $T \in \mathcal{L}(V)$ is *normal* if $TT^* = T^*T$. Furthermore, T is normal if and only if $\|Tv\| = \|T^*v\|$.

Example. (Orthogonal Sets)

$$\mathcal{H} = \mathbb{C}^N \text{ with } \left\{f_j(k) = \frac{1}{\sqrt{N}}e^{2\pi ijk/N}\right\}_{j=0}^{N-1}$$
$$\mathcal{H} = L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \text{ with } \left\{f_j(x) = e^{2\pi ijx}\right\}_{j=-\infty}^\infty$$
$$\mathcal{H} = L^2(\mathbb{R}) \text{ with } \left\{f_j(x) = \chi_{[j,j+1]}(x)\right\}_{j=-\infty}^\infty$$
$$\mathcal{H} = \ell^2(\mathbb{N}) \text{ with } \left\{e_j(k) = \delta_{ij}\right\}_{j=1}^\infty$$

Proposition. For $\{x_n\}$ an orthonormal set in $\mathcal{H}$:

$$\sum_{n=1}^\infty |\langle x, x_n\rangle|^2 \leq \|x\|^2$$

Spectral Theory

Theorem. (Normal Operator Eigenvectors) Suppose $T \in \mathcal{L}(V)$ is normal. Then, eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

Theorem. (Real Spectral Theorem) Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then, the following are equivalent:

(i) T is self-adjoint.
(ii) V has an orthonormal basis consisting of eigenvectors of T.
(iii) T has a diagonal matrix with respect to some orthonormal basis of V.

Theorem. (Complex Spectral Theorem) Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then, the following are equivalent:

(i) T is normal.
(ii) V has an orthonormal basis consisting of eigenvectors of T.
(iii) T has a diagonal matrix with respect to some orthonormal basis of V.

Theorem. (Diagonalisation) A square $n \times n$ matrix, A, with entries in a field $\mathbb{F}$ is called diagonalisable or non-defective if there exists an $n \times n$ invertible matrix (i.e. an element of the general linear group $\mathrm{GL}_n(\mathbb{F})$), P, such that $P^{-1}AP$ is a diagonal matrix.

Given that Λ is diagonal matrix with eigenvalue entries of A on the main diagonal, then $AP = P\Lambda$ implies that $P = V$, the matrix whose columns are eigenvectors to A, corresponding to the eigenvalue in the same column of Λ. In this case:

$$A = V\Lambda V^{-1}$$

Note that one of the *spectral theorems* may guarantee the existence of Λ.

Function Spaces

Definition. (Lebesgue Spaces) Given two measurable functions f and g, we say that $f \sim g$ if $f = g$ *almost everywhere*.

For $1 \leq p < \infty$:

$$L^p(\mathbb{R}) = \left\{f : \mathbb{R} \to \mathbb{C} : \|f\|_p = \left(\int |f|^p\right)^{\frac{1}{p}} < \infty\right\} / \sim$$

For $p = \infty$:

$$L^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : \|f\|_\infty = \operatorname{ess\,sup} |f(x)| < \infty\} / \sim$$

where

$$\operatorname{ess\,sup} |f| = \inf\{\alpha \in \mathbb{R} \mid \mu\{x \in \mathbb{R} : |f(x)| > \alpha\} = 0\}$$

Inequalities

Theorem. (Arithmetic-Geometric Mean Inequality) For positive x and y:

$$4xy \leq (x + y)^2$$

Notice that by expanding the RHS:

$$2xy \leq x^2 + y^2$$

Trigonometry

Definition.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$
$$\cosh z = \frac{e^z + e^{-z}}{2}$$

Definition. (Double Angle Formulae)

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

Definition. (Half Angle Formulae)

$$\begin{aligned}\sin^2\theta &= \frac{1 - \cos 2\theta}{2} \\ \sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 - \cos\theta}{2}} \\ \cos^2\theta &= \frac{1 + \cos 2\theta}{2} \\ \cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 + \cos\theta}{2}} \\ \tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}\end{aligned}$$

Definition. (Ptolemy's Identities (Difference Formulas))

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Definition. (Trigonometric Integrals)

$$\begin{aligned}\int \frac{1}{\sin ax}\,\mathrm{d}x &= -\frac{1}{a}\ln|\csc ax + \cot ax| + C \\ \int \frac{1}{\cos ax}\,\mathrm{d}x &= -\frac{1}{a}\ln\left|\tan\left(\frac{1}{2}ax + \frac{1}{4}\pi\right)\right| + C \\ \int \tan ax\,\mathrm{d}x &= -\frac{1}{a}\ln|\cos ax| + C = \frac{1}{a}\ln|\sec ax| + C \\ \int \tan^2 ax\,\mathrm{d}x &= \tan x - x + C \\ \int \sec ax\,\mathrm{d}x &= \frac{1}{a}\ln|\sec ax + \tan ax| + C \\ &= \frac{1}{a}\ln\left|\tan\left(\frac{1}{2}ax + \frac{1}{4}\pi\right)\right| + C \\ &= \frac{1}{a}\arctan(\sin ax) + C \\ \int \frac{1}{x^2 + a^2}\,\mathrm{d}x &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C\end{aligned}$$

Fourier Series

Definition. (Fourier Series) For $f \in \mathcal{H}$ and let $\{e_n\}_{n=-\infty}^{\infty}$ be an orthonormal basis in $\mathcal{H}$, the **Fourier series** of f is

$$f(x) = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$$

Theorem. (Parseval Equality) With $f \in \mathcal{H}$ and $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis for $\mathcal{H}$, then for $\mathcal{F} : \mathcal{H} \to \ell^2(\mathbb{N})$:

$$(\mathcal{F} f)_n = \langle f, e_n \rangle$$

is unitary. Furthermore, we have

$$\langle f, g \rangle_{\mathcal{H}} = \langle \mathcal{F} f, \mathcal{F} g \rangle_{\ell^2}$$

Theorem. (Plancherel Equality) With $f \in \mathcal{H}$ and $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis for $\mathcal{H}$:

$$\|f\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 = \|\mathcal{F} f\|_{\ell^2}^2$$

L^1 Fourier Transform

Definition. (L^1 Fourier Transform) For $f \in L^1(\mathbb{R})$, the **Fourier Transform** $\hat{f}$ of f is defined:

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t} \, \mathrm{d}t \quad (\xi \in \mathbb{R})$$

Proposition. (Properties)

- $\left\|\hat{f}\right\|_{\infty} \leq \|f\|_1$
- $\hat{f} \in C(\mathbb{R})$
- $\lim_{|\xi| \to \infty} \left|\hat{f}(\xi)\right| = 0$
- $\hat{f} \equiv 0 \iff f = 0$ almost everywhere

Theorem. (In the Schwarz Space) For $f \in \mathcal{S}(\mathbb{R})$:

- $\mathcal{F} : \mathcal{S} \to \mathcal{S}$
- If $f \in \mathcal{S}$ then, for all $t \in \mathbb{R}$:
  $$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi t} \, \mathrm{d}\xi$$
- If $f, g \in \mathcal{S}$ then $\langle f, g \rangle = \left\langle \hat{f}, \hat{g} \right\rangle$

Remark. (Differentiation)

$$\mathcal{F} \mathrm{D} f(\xi) = 2\pi i \xi \mathcal{F} f(\xi)$$
$$\mathcal{F}(-2\pi i t f) = \mathrm{D} \mathcal{F} f$$

Example. With $B(t) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(t)$:

$$\hat{B}(\xi) = \mathrm{sinc}(\xi) = \frac{\sin(\pi \xi)}{\pi \xi}$$

And for $k \in \mathbb{Z}$, $\hat{B}(k) = \delta_{k0}$.

L^2 Fourier Transform

Proposition. (Properties) With $\mathcal{F}$ and $\mathcal{F}^{-1}$ defined on $L^2(\mathbb{R})$, then:

- $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear operators.
- $\mathcal{F}$ and $\mathcal{F}^{-1}$ are unitary operator on $L^2(\mathbb{R})$, and so for all $f, g \in L^2(\mathbb{R})$
  $$\left\langle \hat{f}, \hat{g} \right\rangle = \langle f, g \rangle$$
  Equivalently, $\mathcal{F}^* = \mathcal{F}^{-1}$.

Convolution

Definition. (Convolution) For $f, g \in L^1(\mathbb{R})$, we define the **convolution**:

$$f * g(t) = \int_{-\infty}^{\infty} f(s) g(t - s) \, \mathrm{d}s$$

Theorem. If $f, g \in L^1(\mathbb{R})$ then

- $f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- $\widehat{(f * g)}(\xi) = \hat{f}(\xi) \hat{g}(\xi)$.

Example. Let $f = \chi_{[-\frac{1}{2}, \frac{1}{2}]}$. Then,

$$f * f(x) = g(x) = \max\{1 - |x|, 0\}$$
$$\hat{g}(\xi) = \hat{f}(\xi)^2 = \frac{\sin^2(\pi \xi)}{\pi^2 \xi^2}$$

Covariance

Definition. Given $t_0, \xi_0 \in \mathbb{R}$ we define the following operators on $L^2(\mathbb{R})$:

$$\tau_{t_0} f(t) = f(t - t_0)$$
$$M_{\xi_0} f(t) = e^{-2\pi i \xi_0 t} f(t)$$
$$D_a f(t) = \frac{1}{\sqrt{a}} f\left(\frac{t}{a}\right)$$

Proposition. (Properties)

- Each operator has an inverse, as follows:
  $$\tau_{t_0}^{-1} = \tau_{-t_0} \quad M_{\xi_0}^{-1} = M_{-\xi_0} \quad D_a^{-1} = D_{a^{-1}}$$
- Each operator is unitary.
- Each operator has a relationship with the Fourier transform:
  $$\mathcal{F} \tau_{t_0} = M_{t_0} \mathcal{F}$$
  $$\mathcal{F} M_{\xi_0} = \tau_{-\xi_0} \mathcal{F}$$
  $$\mathcal{F} D_a = D_{a^{-1}} \mathcal{F}$$

Sampling

Definition. (Bandlimit) Given $\Omega > 0$, let

$$B_\Omega = \left\{ f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for } \xi > \frac{\Omega}{2} \right\}$$

Functions in B_Ω are said to be **bandlimited** with **bandwidth** Ω.

Theorem. (Shannon Sampling Theorem) If $f \in B_\Omega$, then

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\Omega}\right) \mathrm{sinc}(\Omega t - n)$$

where $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

Theorem. (Poisson Summation Formula) If $f, \hat{f} \in L^1(\mathbb{R})$ and

$$\sum_{n=-\infty}^{\infty} f(t + n) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \hat{f}(\xi + n)$$

are absolutely convergent for all $t, \xi \in \mathbb{R}$, then

$$\sum_{n=-\infty}^{\infty} f(t + n) = \sum_{m=-\infty}^{\infty} \hat{f}(m) e^{2\pi i m t}$$

Remark. This can be used to show that certain infinite sums have some particular value.

Uncertainty

Theorem. (Heisenberg Uncertainty Inequality)

$$\int_{-\infty}^{\infty} (t - t_0)^2 |f(t)|^2 \, \mathrm{d}t \cdot \int_{-\infty}^{\infty} (\xi - \xi_0)^2 \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \geq \frac{\|f\|_2^4}{16\pi^2}$$

Time-Frequency Analysis

Definition. (Short-Time Fourier Transform) Let $\varphi \in L^2(\mathbb{R})$. Given $b, \xi \in \mathbb{R}$, let

$$\varphi_{b,\xi}(t) = M_{-\xi} \tau_b \varphi(t) = e^{2\pi i \xi t} \varphi(t - b)$$

Then the short-time Fourier transform $S_\varphi f$ of f with respect to φ is defined by

$$S_\varphi f(b, \xi) = \langle f, \varphi_{b,\xi} \rangle$$

Remark. The short-time Fourier transform is also known as the sliding window Fourier transform, the windowed Fourier transform and the Gabor transform (especially in the case where the window φ is a Gaussian). Note that

$$S_\varphi f(b, \xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t} \overline{\varphi(t - b)} \, \mathrm{d}t = \left(\mathcal{F} f \overline{\varphi(\cdot - b)} \right)(\xi)$$

$\varphi_{b,\xi}$ is obtained from φ by a translation in time by b and in frequency by ξ, i.e., a translation in "phase space" (the time-frequency plane) by (b, ξ).

Also, if $\varphi \equiv 1$, $S_\varphi f(b, \xi) = \hat{f}(\xi)$, but in fact since φ is assumed to have decay properties, $S_\varphi f$ represents a windowed version of the Fourier transform around $t = b$. By the Plancherel formula we may write the integral as

$$S_\varphi f(b, \xi) = \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i b(s - \xi)} \overline{\hat{\varphi}(s - \xi)} \, \mathrm{d}s$$

$S_\varphi f(b, \xi)$ represents time-frequency localised information about f, i.e., the information about f near the time b and the frequency ξ.

Theorem. If $f \in L^2(\mathbb{R})$, then $S_\varphi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ is a *multiple of an isometry*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left|S_\varphi f(b, \xi)\right|^2 \mathrm{d}b \, \mathrm{d}\xi = \|\varphi\|_2^2 \int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \|\varphi\|_2^2 \|f\|_2^2$$