①

$$\Gamma\left(\frac{1}{n}\right) = \int_0^\infty x^{\frac{1}{n}-1} e^{-x}\, dx$$

$$n = 3 \Rightarrow \Gamma\left(\frac{1}{3}\right) = \int_0^\infty x^{\frac{1}{3}-1} \cdot e^{-x}\, dx$$

$$= \int_0^\infty x^{-\frac{2}{3}} \cdot e^{-x}\, dx = \int_0^\infty \frac{e^{-x}}{x^{\frac{2}{3}}}\, dx$$

$$u = \sqrt[3]{x} \Rightarrow x = u^3 \Rightarrow \frac{du}{dx} = \frac{1}{3x^{\frac{2}{3}}} \Rightarrow dx = 3x^{\frac{2}{3}} du \Rightarrow 3\int e^{-u^3} du$$

$$\Rightarrow -\Gamma\left(\frac{1}{3}, u^3\right) \Rightarrow -\Gamma\left(\frac{1}{3}, x\right)$$

Funcție Gamma incompletă

$\Gamma\left(\frac{1}{3}\right) \rightarrow$ aproximare prin metoda Lanczos: $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$

$$\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin\left(\frac{2\pi}{3}\right)} = \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi\sqrt{3}}{3}$$

$$\Rightarrow \Gamma\left(\frac{1}{3}\right) = \frac{\frac{2\pi\sqrt{3}}{3}}{\Gamma\left(\frac{2}{3}\right)} = \frac{2\pi\sqrt{3}}{3 \cdot \Gamma\left(\frac{2}{3}\right)}$$

② $J = \int_0^\infty \frac{1}{x^a + x^{\frac{1}{a}}}\,dx \qquad a \in \mathbb{R}_+ \setminus \{1\}$

$J = \int_0^1 \frac{1}{x^a + x^{\frac{1}{a}}}\,dx + \int_1^\infty \frac{1}{x^a + x^{\frac{1}{a}}}\,dx$

Pt $a = 2 \Rightarrow J = \int_0^1 \frac{1}{x^2 + x^{\frac{1}{2}}} + \int_1^\infty \frac{1}{x^2 + x^{\frac{1}{2}}}\,dx$

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{\frac{1}{x^2 + x^{\frac{1}{2}}}}{\frac{1}{x^2}} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{x^2}{x^2 + x^{\frac{1}{2}}} = 1 \qquad (1)$$

$$\lim_{x \to \infty} \frac{x^2}{x^{\frac{1}{2}} + x^2} = 1 \qquad (2)$$

(1) $\Rightarrow \int_0^1 \frac{1}{x^2 + x^{\frac{1}{2}}}\,dx$

$\int_0^1 \frac{1}{x^a}\,dx$ $\Big\} \Rightarrow \int_0^1 \frac{1}{x^2 + x^{\frac{1}{2}}} \Rightarrow$ divergent

(2) $\Rightarrow \int_1^\infty \frac{1}{x^2 + x^{\frac{1}{2}}}\,dx$

$\int_1^\infty \frac{1}{x^2}\,dx$ $\Big\} \Rightarrow \int_1^\infty \frac{1}{x^2 + x^{\frac{1}{2}}}\,dx \Rightarrow$ convergent

Suma dintre o integrală convergentă și una divergentă $\Rightarrow$ integrală divergentă.

③ $J = \int_0^\infty e^{-ax} \cos(bx)\,dx$

$a=2,\ b=3 \Rightarrow J = \int_0^\infty e^{-2x} \cos(3x)\,dx.$

$J = \int_0^\infty \left(\frac{e^{-2x}}{-2}\right)' \cos 3x\,dx$

$J = -\frac{1}{2} e^{-2x} \cos(3x) \Big/_0^\infty - \frac{1}{2} \int_0^\infty e^{-2x} \cdot 3 \sin(3x)\,dx$

$J = -\frac{1}{2} e^{-2x} \cos(3x) \Big/_0^\infty - \frac{3}{2} \int_0^\infty \left(\frac{e^{-2x}}{-2}\right)' \sin(3x)\,dx$

$J = -\frac{1}{2} e^{-2x} \cos(3x) \Big/_0^\infty - \frac{3}{2}\left(-\frac{1}{2} \cdot e^{-2x} \sin(3x)\Big/_0^\infty + \frac{3}{2} \underbrace{\int_0^\infty e^{-2x} \cdot \cos(3x)\,dx}_{J}\right)$

$J = -\frac{1}{2} e^{-2x} \cdot \cos(3x) \Big/_0^\infty + \frac{3}{4} \cdot e^{-2x} \cdot \sin(3x) \Big/_0^\infty - \frac{9}{4} J$

$J + \frac{9}{4} J = -\frac{1}{2} e^{-2x} \cdot \cos(3x) \Big/_0^\infty + \frac{3}{4} \cdot e^{-2x} \cdot \sin(3x) \Big/_0^\infty$

$\Rightarrow J = \frac{4}{13}\left(-\frac{1}{2} \cdot e^{-2x} \cdot \cos(3x) \Big/_0^\infty + \frac{3}{4} \cdot e^{-2x} \cdot \sin(3x) \Big/_0^\infty\right.$

$J = \frac{3e^{-2x} \sin(3x) - 2e^{-2x} \cos(3x)}{13} = \frac{e^{-2x} \cdot (3\sin(3x) - 2\cos(3x))}{13}$

$\Rightarrow \int_0^\infty e^{-2x} \cos 3x\,dx = F(+\infty) - F(0)$

$F(0) = -\frac{2}{13}$

$F(\infty) = \lim_{x \to +\infty} \frac{3\sin(3x) - 2\cos(3x)}{e^{2x} \cdot 13} = \frac{1}{13} \lim_{x \to +\infty} \frac{3\sin(3x) - 2\cos(3x)}{e^{2x}} = 0$

$J = F(\infty) - F(0) = 0 + \frac{2}{13} = \frac{2}{13}$