Computer Assisted Mathematics

As. PhD. Eng. Raul-Cristian ROMAN

E-mail addresses: raul.roman@aut.upt.ro
raul-cristian.roman@upt.ro

Course: Wednesday, 16:00-18:00, zoom

Laboratory activities: zoom
34% of the final grade
As. PhD. Eng. Raul-Cristian ROMAN
As. PhD. Eng. Elena-Lorena HEDREA
lorena.hedrea@aut.upt.ro

Distributed Evaluation: 1 problem 1^{st} part – 40 min
1 problem 2^{nd} part – 40 min
66% of the final grade
Week 11 – 1^{st} evaluation 1^{st} part
Week 12 – 1^{st} evaluation 2^{nd} part
Week 13 – 2^{nd} evaluation 1^{st} part
Week 14 – 2^{nd} evaluation 2^{nd} part

Computer Assisted Mathematics

Part 1:

Ch. 1: Error Theory

Ch. 2: Elements regarding numerical matrix calculation

Ch. 3: Numerical solving of a system of linear algebraic equations

Ch. 4: Numerical calculation of eigenvalues and eigenvectors

Part 2:

Ch. 5: Numerical solving of equations and systems of nonlinear algebraic equations

Ch. 6: Numerical solving of equations and systems of ordinary differential equations

Ch. 7: Numerical approximation of functions

Computer Assisted Mathematics. Bibliography

R.-E. Precup, *Matematici asistate de calculator. Algoritmuri*, Editura Orizonturi Universitare, Timişoara, 2007.

A. Kovács, A., R.-E. Precup, B. Paláncz, L. Kovács: *Modern numerical methods in engineering*, Editura Politehnica, Timişoara, 2012.

R.-E. Precup, L. Dragomir, I. Bulaviţchi: *Matematici asistate de calculator. Aplicaţii*, Editura Politehnica, Timişoara, 2002.

L. Dragomir, *Aplicaţii de matematică asistată de calculator*, Editura Politehnica, Timişoara, 2010.

S. Kilyeni, *Metode numerice*, vol. 1 şi 2, ediţiile 1, 2, … Editura Orizonturi Universitare, Timişoara, 1997, 2000 şi alte ediţii.

P. Năslău, *Metode numerice*, Editura Politehnica, Timişoara, 1999 şi alte ediţii.

G. Babescu, A. Kovacs, I. Stan, Gh. Tudor, R. Anghelescu, A. Filipescu, *Analiză numerică*, Editura Politehnica, Timişoara, 2000.

V. Ionescu, C. Popeea, *Optimizarea sistemelor*, Editura Didactică şi Pedagogică, Bucureşti, 1981.

I. Dumitrache, C. Buiu, *Algoritmi genetici. Principii fundamentale şi aplicaţii în automatică*, Editura Mediamira, Cluj-Napoca, 2000.

J. Penny, G. Lindfield, *Numerical Methods Using MATLAB, Second edition*, Prentice Hall, Upper Saddle River, NJ, 2000.

M. Ghinea, V. Fireţeanu, *MATLAB. Calcul numeric. Grafică. Aplicaţii*, Editura Teora, Bucureşti, 1997 şi alte ediţii.

T. A. Beu, *Calcul numeric în C*, Editura Albastră, Cluj-Napoca, 2000.

Chapter 1. Introductory aspects regarding error theory

The solving of some scientific and technical **problems** involves the application of **numerical methods**. A numerical method is considered **effective** when the **precision of the numerical computations** is good, which translates into **reduced errors**.

1.1. Error. Approximation

Having a real numerical unit with **the exact value** x, for which the approximate value $x^{\sim}$ is known (resulting from a numerical calculation or an experiment). $x^{\sim}$ = an ***approximation*** of the exact value x.

The Error ε of the approximation $x^{\sim}$ of x – definition:

$$\varepsilon = x - x^{\sim} . \qquad (1.1)$$

$\varepsilon > 0$: $x^{\sim}$ **approximates** x **through lacking**;

$\varepsilon < 0$: $x^{\sim}$ **approximates** x **through addition**.

Error. Approximation. Absolute error

(1.1) $\Rightarrow$ **approximation formula**:

$$x = x^{\sim} + \varepsilon \, . \tag{1.2}$$

The sign of the error is of no interest $\Rightarrow$ ***absolute error*** ε_a ($\varepsilon_{a\,x}$):

$$\varepsilon_a = | \, \varepsilon \, | = | \, x - x^{\sim} | \, . \tag{1.3}$$

ε_a is not sufficient to characterize the degree of precision of an approximation. Example:

Absolute error. Relative error

$x = 4$, $\tilde{x} = 5$, $y = 499$, $\tilde{y} = 500$ $\Rightarrow$ $\varepsilon_{a\,x} = \varepsilon_{a\,y} = 1$.

$\Rightarrow$ it cannot be determined that $\tilde{y}$ approximates much better y than $\tilde{x}$ does x.

$\Rightarrow$ it is necessary to insert another unit that accurately expresses the degree of precision of an approximation: ***the relative error*** of the approximation $\tilde{x}$ of x, ε_r ($\varepsilon_{r\,x}$):

$$\varepsilon_r = \frac{|\,\varepsilon_a\,|}{|\,\tilde{x}\,|} = \frac{|\,x - \tilde{x}\,|}{|\,\tilde{x}\,|} \,. \qquad (1.4)$$

Relative error

In practice: Percentage relative error:

$$\varepsilon_r^{\%} = 100 \cdot \varepsilon_r \,. \qquad (1.5)$$

Previous example:

$\varepsilon_{r\,x} = 1/5 = 0.2 = 20\,\%$, $\varepsilon_{r\,y} = 1/500 = 0.002 = 0.2\,\%$,

$\Rightarrow$ the second approximation is more accurate than the first one.

From the point of view of the origin of the errors: classification in **three categories**:

Error Classification

1) Inherent errors (initial) – occurrence: the mathematical model associated with the practical problem does not fully correspond to this problem; cannot be influenced by the calculation method.

2) Method errors – occurrence: using a certain numeric method to solve the problem; can be reduced by choosing the most appropriate method of solving.

3) Calculation errors – related exclusively to the numerical calculation methods, the way in which the calculation was made and the computing technique used.

Calculation errors

Calculation errors – of three ***types***:

A) Truncation errors – related solely to the numerical methods used. Occurrence: numerical calculation processes with infinite theoretical convergence are replaced by the same processes but with practically finite convergence – to all iterative or approximation methods.

Example: calculation of e^x values for various values of the argument x with MacLaurin series:

Calculation errors (cont'd 1)

$$e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \dots + \frac{x^k}{k\,!} + \dots \,. \qquad (1.6)$$

The number of terms is infinite. In practical calculations, however, a finite, reasonable number of terms (5, 6, 7, 8, ...) are used, depending on the value of x. The omitted terms cause the truncation error to occur.

Cannot be calculated exactly, but can be estimated.

Calculation errors (cont'd 2)

B) Rounding errors – all calculations can be made only by considering a finite number of digits for numerical values, although some numerical values have more digits or even an infinite number of digits (irrational numbers).

Example: calculations are made with numerical values having 6 significant digits. Numerical value x = 842.78425 can be approximated:

Calculation errors (cont'd 2)

- through lacking, by value: $x^{\sim}= 842.784$, with $\varepsilon_{r\ x}= 0.000030\ \%$;
- through addition, by value: $x^{\sim}= 842.785$, with $\varepsilon_{r\ x}= 0.000089\ \%$.

Rounding errors can also be caused by conversions from one counting system to another with another base.

C) Propagation errors – occur due to errors in the previous steps of an iterative computational process.

1.2. Propagation of errors

Arithmetic operations with approximated values $\Rightarrow$ the need to know the effect of operand errors on the result error and on subsequent operations.

Having two operands with the exact values x and y and the approximate values $x^{\sim}$, respectively $y^{\sim}$. The approximation formulas are:

$$x = x^{\sim} + \varepsilon_x \,, \; y = y^{\sim} + \varepsilon_y \,. \tag{2.1}$$

Elementary arithmetic operations

A) Addition

$x + y = x^{\sim} + y^{\sim} + \varepsilon_x + \varepsilon_y$ – from (2.1),

Propagation of errors for elementary arithmetic operations /1

but (1.2) $\Rightarrow x + y = x^{\sim} + y^{\sim} + \varepsilon_{x+y}$.

$$\Rightarrow \varepsilon_{x+y} = \varepsilon_x + \varepsilon_y \,. \tag{2.2}$$

The absolute value properties are applied $\Rightarrow$

$$\varepsilon_{a\ x+y} \le \varepsilon_{a\ x} + \varepsilon_{a\ y} \,. \tag{2.3}$$

Expression of the relative error:

$$\varepsilon_{r\ x+y} = \frac{\varepsilon_{a\ x+y}}{|\,x^{\sim} + y^{\sim}\,|} \le \frac{\varepsilon_{a\ x}}{|\,x^{\sim} + y^{\sim}\,|} + \frac{\varepsilon_{a\ y}}{|\,x^{\sim} + y^{\sim}\,|} =$$

Propagation of errors for elementary arithmetic operations /2

$$= \frac{\varepsilon_{a\,x}}{|\,x^{\sim}\,|} \cdot \frac{|\,x^{\sim}\,|}{|\,x^{\sim} + y^{\sim}\,|} + \frac{\varepsilon_{a\,y}}{|\,y^{\sim}\,|} \cdot \frac{|\,y^{\sim}\,|}{|\,x^{\sim} + y^{\sim}\,|} =$$

$$= \varepsilon_{r\,x} \cdot \frac{|\,x^{\sim}\,|}{|\,x^{\sim} + y^{\sim}\,|} + \varepsilon_{r\,y} \cdot \frac{|\,y^{\sim}\,|}{|\,x^{\sim} + y^{\sim}\,|} \,. \qquad (2.4)$$

Observations:

1. The absolute error of the sum does not exceed the sum of the absolute errors of the terms.

Propagation of errors for elementary arithmetic operations /3

2. If the operands have same sign, the relative error of the sum does not exceed the sum of the relative errors of the terms.

B) Subtraction. Calculations similar to addition :

$x - y = x^{\sim} - y^{\sim} + \varepsilon_x - \varepsilon_y$ – from (2.1),

but (1.2) $\Rightarrow x - y = x^{\sim} - y^{\sim} + \varepsilon_{x-y}$.

$$\Rightarrow \varepsilon_{x-y} = \varepsilon_x - \varepsilon_y \,. \qquad (2.5)$$

Again the properties of the absolute value are applied $\Rightarrow$

$$\varepsilon_{a\ x-y} \leq \varepsilon_{a\ x} + \varepsilon_{a\ y} \,. \qquad (2.6)$$

Propagation of errors for elementary arithmetic operations /4

Expression of the relative error:

$$\varepsilon_{r\ x-y} = \frac{\varepsilon_{a\ x-y}}{|\ \tilde{x} - \tilde{y}\ |} \le \frac{\varepsilon_{a\ x}}{|\ \tilde{x} - \tilde{y}\ |} + \frac{\varepsilon_{a\ y}}{|\ \tilde{x} - \tilde{y}\ |} =$$

$$= \frac{\varepsilon_{a\ x}}{|\ \tilde{x}\ |} \cdot \frac{|\ \tilde{x}\ |}{|\ \tilde{x} - \tilde{y}\ |} + \frac{\varepsilon_{a\ y}}{|\ \tilde{y}\ |} \cdot \frac{|\ \tilde{y}\ |}{|\ \tilde{x} - \tilde{y}\ |} =$$

$$= \varepsilon_{r\ x} \cdot \frac{|\ \tilde{x}\ |}{|\ \tilde{x} - \tilde{y}\ |} + \varepsilon_{r\ y} \cdot \frac{|\ \tilde{y}\ |}{|\ \tilde{x} - \tilde{y}\ |}\ . \qquad (2.7)$$

Propagation of errors for elementary arithmetic operations /5

Observations: 1. The absolute error of the difference does not exceed the sum of the absolute errors of the terms.

2. Precision is strongly affected when the two operands **have the same sign and are of close value** $\Rightarrow$ such subtractions should be avoided.

C) Multiplication

$xy = (x^{\sim} + \varepsilon_x)(y^{\sim} + \varepsilon_y) = x^{\sim}y^{\sim} + x^{\sim}\varepsilon_y + y^{\sim}\varepsilon_x + \varepsilon_x\varepsilon_y$ – from (2.1).

The term $\varepsilon_x\varepsilon_y$ is negligible $\Rightarrow xy \approx x^{\sim}y^{\sim} + x^{\sim}\varepsilon_y + y^{\sim}\varepsilon_x$,

Propagation of errors for elementary arithmetic operations /6

$$\Rightarrow \varepsilon_{xy} = x^{\sim}\varepsilon_y + y^{\sim}\varepsilon_x \,. \tag{2.8}$$

Absolute error majorant – from (2.8):

$$\varepsilon_{a\ xy} \le |\, x^{\sim} \,|\, \varepsilon_{a\ y} + |\, y^{\sim} \,|\, \varepsilon_{a\ x} \,. \tag{2.9}$$

Relative error majorant – from (2.9):

$$\varepsilon_{r\ xy} = \frac{\varepsilon_{a\ xy}}{|\, x^{\sim}y^{\sim} \,|} \le \frac{|\, x^{\sim} \,|\, \varepsilon_{a\ y} + |\, y^{\sim} \,|\, \varepsilon_{a\ x}}{|\, x^{\sim}y^{\sim} \,|} = \frac{\varepsilon_{a\ y}}{|\, y^{\sim} \,|} + \frac{\varepsilon_{a\ x}}{|\, x^{\sim} \,|} = \varepsilon_{r\ x} + \varepsilon_{r\ y} \,. \tag{2.10}$$

Non-zero operands! $\Rightarrow$ similar conclusion to addition.

Propagation of errors for elementary arithmetic operations /7

D) Division. Calculations similar to multiplication:

$$\frac{x}{y} = \frac{x^{\sim} + \varepsilon_x}{y^{\sim} + \varepsilon_y} = \frac{(x^{\sim} + \varepsilon_x)(y^{\sim} - \varepsilon_y)}{y^{\sim 2} - \varepsilon_y^2} = \frac{x^{\sim}y^{\sim} + y^{\sim}\varepsilon_x - x^{\sim}\varepsilon_y - \varepsilon_x\varepsilon_y}{y^{\sim 2} - \varepsilon_y^2}.$$

The terms $\varepsilon_x\varepsilon_y$ and ε_y^2 are negligible $\Rightarrow$

$$\frac{x}{y} \approx \frac{x^{\sim}}{y^{\sim}} + \frac{\varepsilon_x}{y^{\sim}} - \frac{x^{\sim}\varepsilon_y}{y^{\sim 2}} \Rightarrow$$

$$\varepsilon_{x/y} = \frac{\varepsilon_x}{y^{\sim}} - \frac{x^{\sim}\varepsilon_y}{y^{\sim 2}}. \qquad (2.11)$$

Propagation of errors for elementary arithmetic operations /8

Absolute error majorant – from (2.11):

$$\varepsilon_{a\ x/y} \le \frac{\varepsilon_{a\ x}}{|\ y^{\sim}\ |} + \frac{|\ x^{\sim}\ |\ \varepsilon_{a\ y}}{y^{\sim 2}} \qquad (2.12)$$

Relative error majorant – from (2.12):

$$\varepsilon_{r\ x/y} = \frac{\varepsilon_{a\ x/y}}{|\ x^{\sim}/y^{\sim}\ |} \le \frac{\varepsilon_{a\ x} /\ |\ y^{\sim}\ |\ +\ |\ x^{\sim}|\varepsilon_{a\ y} /\ |\ y^{\sim}\ |^2}{|\ x^{\sim}\ |\ /\ |\ y^{\sim}|} =$$

Propagation of errors for elementary arithmetic operations /9

$$= \frac{\varepsilon_{a\,x}}{|\,x^{\sim}\,|} + \frac{\varepsilon_{a\,y}}{|\,y^{\sim}\,|} = \varepsilon_{r\,x} + \varepsilon_{r\,y}. \qquad (2.13)$$

Non-zero operands! $\Rightarrow$ similar conclusion to addition.

Example Explain an absolute error majorant according to the absolute errors of terms and also a relative error majorant according to the relative errors of the terms in the calculation of the expressions:

a) $u= (x+y)+z$;
b) $u=x\cdot(y\cdot z)$
c) $u=(x/z)\cdot y$
d) $u=x\cdot(y+z)$

Chapter. 2. Elements regarding numerical matrix calculation

A **matrix** $\underline{A}$ **with a dimension of m** $\times$ **n** – rectangular array with real or complex elements arranged on *m* lines and *n* columns:

$$\underline{A} = \left[a_{ij}\right]_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{1}$$

2.0. Matrix

The transposed of matrix $\underline{A}$ is matrix $\underline{A}^T$, with a dimension $n \times m$, obtained by changing the lines with the columns:

$$\underline{A}^T = \left[a_{ji}\right]_{\substack{j=\overline{1,n} \\ i=\overline{1,m}}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}. \tag{2}$$

Properties for the transposed of the sum and of the product:

$$(\underline{A} + \underline{B})^T = \underline{A}^T + \underline{B}^T, \tag{3}$$

Definitions

$$(\underline{A} \cdot \underline{B})^T = \underline{B}^T \cdot \underline{A}^T . \quad (4)$$

The conjugated matrix of $\underline{A}$ is the matrix $\bar{\underline{A}}$, which is obtained by replacing the elements of $\underline{A}$ with their conjugates:

$$\overline{\underline{A}} = \left[\overline{a}_{ij}\right]_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}} . \quad (5)$$

if $n = 1$, then the matrix has the dimension $m \times 1$ and is a **column** type matrix:

Definitions (cont'd 1)

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}. \qquad (6)$$

if $m = 1$, then the matrix has the dimension $1 \times n$ and is a **line** type matrix (transposed of a column):

$$\underline{b}^T = [b_1 \;\; b_2 \;\cdots\; b_n]\,. \qquad (7)$$

If $m=n$, then matrix $\underline{A}$ is ***a square matrix of order n***.

Definitions (cont'd 2)

A square matrix $\underline{A}$ is ***symmetric*** if $\underline{A}^T = \underline{A} \Leftrightarrow$

$a_{ij} = a_{ji}$, $i, j = \overline{1, n}$.

Example of a symmetric matrix: $\underline{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & -5 \end{bmatrix}$.

Definitions (cont'd 3)

A square matrix $\underline{A}$ is ***antisymmetric*** if $\underline{A}^T=-\underline{A}$, namely $a_{ij}=-a_{ji}$, $i,j=\overline{1,n}$. It can be observed: $a_{ii}=0$, $i=\overline{1,n}$.

Example of a antisymmetric matrix: $$\underline{A}=\begin{bmatrix} 0 & 2 & -5 \\ -2 & 0 & 3 \\ 5 & -3 & 0 \end{bmatrix}.$$

Definitions (cont'd 4)

A square matrix $\underline{A}$ is ***upper triangular*** if

$$a_{ij} = 0, \ \forall i > j.$$

Example of a upper triangular matrix: $\underline{A} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$

Definitions (cont'd 5)

A square matrix $\underline{A}$ is ***lower triangular*** if

$a_{ij} = 0$, $\forall i < j$.

Example of a lower triangular matrix: $\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 9 & 7 \end{bmatrix}$.

Definitions (cont'd 6)

A square matrix $\underline{A}$ is ***diagonal*** if

$a_{ij} = 0$, $\forall i \neq j$, namely

$$\underline{A} = diag(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}. \qquad (8)$$

Definitions (cont'd 7)

If for a diagonal matrix the elements on the main diagonal = 1, then the matrix is the ***identity matrix***:

$$\underline{I} = diag(1,1,\ldots,1) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \qquad (9)$$

Definitions (cont'd 8)

Tridiagonal matrix

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{bmatrix}. \quad (10)$$

Definitions (cont'd 9)

Hessenberg matrix

$$\underline{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3,n} \\ 0 & 0 & a_{43} & \cdots & a_{4,n-1} & a_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}. \tag{11}$$

Definitions (cont'd 10)

Bloc diagonal matrix

$$\underline{A} = \begin{bmatrix} \underline{A}_1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \underline{A}_p \end{bmatrix}, \qquad (12)$$

with the blocs $\underline{A}_1,\ldots,\underline{A}_p$ - square matrices, not necessarily of the same order.

Definitions (cont'd 11)

A square matrix $\underline{A}$ is ***diagonally dominant*** if

$$\left|a_{ii}\right| > \sum_{j \neq i} \left|a_{ij}\right|, \forall i = \overline{1, n}.$$

Example of a diagonally dominant matrix: $\underline{A} = \begin{bmatrix} -5 & 1 & 2 \\ -1 & 6 & -3 \\ 4 & 1 & 7 \end{bmatrix}$.

Definitions (cont'd 12)

A square matrix $\underline{A}$ is ***orthogonal*** if

$\underline{A}^{-1} = \underline{A}^{T}$, namely $\underline{A} \cdot \underline{A}^{T} = \underline{I}$.

A square matrix $\underline{A}$ is ***Hermitian*** if

$\underline{A} = (\bar{\underline{A}})^{T}$. Any real symmetric matrix is a Hermitian matrix.

A real square matrix $\underline{A}$ is ***singular*** if det $\underline{A} = 0$.

If det $\underline{A} \neq 0$, matrix $\underline{A}$ is ***nonsingular***.

2.1. Definition and properties of the inverse matrix

Considering the real square nonsingular matrix $\underline{A}$ of order *n*.

The inverse matrix $\underline{A}^{-1}$ of matrix $\underline{A}$ is defined as being the matrix (real square of order n) that satisfies

$$\underline{A} \cdot \underline{A}^{-1} = \underline{A}^{-1} \cdot \underline{A} = \underline{I} \quad , \qquad (1.1)$$

where $\underline{I}$ is the identity matrix of order *n*.

Expression $\underline{A}^{-1}$:

$$\underline{A}^{-1} = \frac{1}{\det \underline{A}} \cdot \underline{A}^{+} \quad , \qquad (1.2)$$

Inverse

where: det $\underline{A}$ (real nonzero number) – ***the determinant*** of matrix $\underline{A}$ and

$\underline{A}^{+}$ – **adjunct matrix**.

$\underline{A}^{+}$ = the transposed of the matrix obtained by replacing the elements of $\underline{A}$ with **their algebraic complements** (their **cofactors**); the cofactor of

a_{ij} ($i,j=\overline{1,n}$) is the minor multiplied with $(-1)^{i+j}$.

(1.2) $\Rightarrow$ way of calculating the inverse, **hardly algorithmizable**

+ requires a **large amount of calculations**.

Inverse (cont'd 1)

Example: Let $\underline{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$. Calculate $\underline{A}^{-1}$.

Solution: It is analyzed whether $\underline{A}$ is invertible:

$$\det \underline{A} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} \Leftrightarrow \det \underline{A} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 5 \end{vmatrix} \Leftrightarrow \det \underline{A} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \Leftrightarrow$$

det $\underline{A}$ = 1 $\neq$ 0 $\Rightarrow$ $\underline{A}$ is invertible.

Inverse (cont'd 2)

The transposed matrix $\underline{A}$: $\underline{A^T} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$. The adjunct matrix:

$$\underline{A}^{+} = \begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}, \quad \underline{A}^{+} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

Inverse. Properties

$$(1.2) \Rightarrow \quad \underline{A}^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

Properties of the inversion operation:

$$\det(\underline{A}^{-1}) = \frac{1}{\det \underline{A}}, \tag{1.3}$$

$$(\underline{A} \cdot \underline{B})^{-1} = \underline{B}^{-1} \cdot \underline{A}^{-1}, \tag{1.4}$$

$$(\underline{A}^{-1})^T = (\underline{A}^T)^{-1}. \tag{1.5}$$