"The 50/50/90 rule: Anytime you have a 50/50 chance of getting something right, there's a 90% probability you'll get it wrong."

Andy Rooney

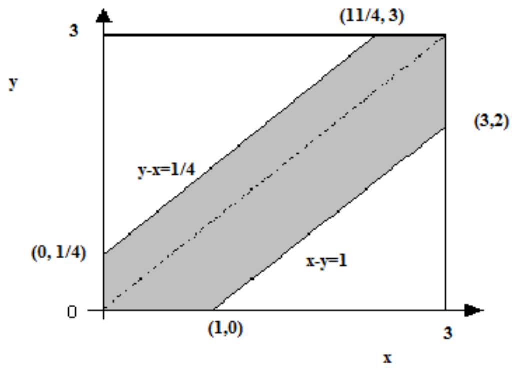
2 Classical problems in probability theory

Chance of meeting in a restaurant

A man and a woman decide to meet in a restaurant after 21 o'clock. The restaurant closes at 24 o'clock. Because of their busy schedule they decide that whoever arrives first at the restaurant will wait, for a while, for the other one. The man would be ready to wait an hour and the woman only 15 minutes!

What's the probability that they will meet?

Solution: We will model mathematically the problem in the following way: let us denote by x the time when the women arrives at the restaurant and by y the time when the man arrives. We can consider 21 o'clock to be 0 and then 24 will mean 3. Thus $x, y \in [0, 3]$. All the possibilities are represented by the points (x, y) located inside the square $[0, 3] \times [0, 3]$ drawn below.



If the man arrives first, that means $y \leq x$, then they will meet if $x - y \leq 1$ (the time when the woman arrives is at most one hour later). The arrival times which satisfy this restriction are contained in the gray region, between the first bisectrix $y = x$ and the line $x - y = 1$

If the woman arrives first, i.e. $x \leq y$, then they will meet only if $y - x \leq \frac{1}{4}$. All the possible arrival times which satisfy this restriction are contained in the gray region, between the first bisectrix $y = x$ and the line $y - x = \frac{1}{4}$

The chance that they will meet is computed using the formula

$$P = \frac{\text{number of favourable cases}}{\text{number of possible cases}}.$$

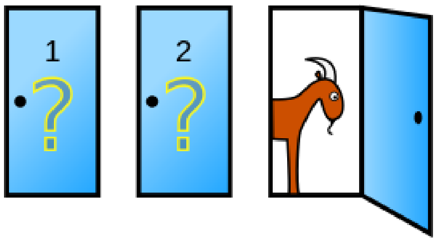
Of course, there is an infinity of favourable and possible cases. However we can estimate the probability without counting all the points (x, y) but using the areas of the regions corresponding to the favourable cases and to all the possible cases. The probability they do meet is

$$P = \frac{\text{area of the gray region}}{\text{area of the square}} = \frac{\frac{103}{32}}{3^2} \approx 35\%.$$

Remark

The chance of getting it wrong is 90% according to Andy Rooney, so read the solution again, in order to verify if you understood it.

The Monty Hall problem



The Monty Hall problem is a probability puzzle named after Monty Hall, the original host of the TV show *Let's Make a Deal*. It's a famous paradox that has a solution that is so absurd, most people refuse to believe it's true.

There are 3 doors, behind which are two goats and a car. You pick a door (call it door no.1). You're hoping for the car of course. Monty Hall, the game show host, examines the other doors, no. 1 and no. 3, and opens one with a goat. If both doors have goats, he picks randomly.

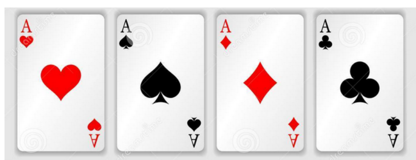
He then, plays with your mind, and says to you

"-Do you want to switch and to pick door no. 2 ?"

Is it to your advantage to switch your choice ?

Surprisingly, the odds aren't 50/50.

The next card



I shuffle a deck of cards and deal them one by one, as slowly as you need me to. You observe the sequence of cards and at any point of your choosing you say Stop. I then deal the next card: if it's **black**, you win. If it's **red**, you lose. No jokers and no sleight-of-hand.

If you fail to say Stop until the very end, the last card determines the outcome of the game.

What's your strategy ?

The boy-girl paradox

A. Mr. Smith has two children. At least one of them is a boy. What is the probability that the other child is a girl ?

B. Mr. Smith has two children. At least one of them is a boy born on Tuesday. What is the probability that other child is a girl ?

The paradox: Neither of the probabilities are 50% and the answers are not the same.

Classical schemes of probability theory

• here are a few useful tools for dealing with elementary probability problems

1. Poincaré's theorem

$$P\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} P(E_k) - \sum_{k=1}^{n-1}\sum_{j=k+1}^{n} P(E_k \cap E_j) +$$

$$+\sum_{k=1}^{n-2}\sum_{j=k+1}^{n-1}\sum_{i=j+1}^{n} P(E_k \cap E_j \cap E_i) - \ldots + (-1)^{n-1} P(E_1 \cap E_2 \ldots \cap E_n)$$

e.g. for $n = 3$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$$

$$+P(A \cap B \cap C)$$

2. The multiplication rule

$$P\left(\bigcap_{k=1}^{n} E_k\right) = P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 \cap E_2) \cdot \ldots \cdot P\left(E_n | \bigcap_{k=1}^{n-1} E_k\right)$$

if the events are independent

$$P\left(\bigcap_{k=1}^{n} E_k\right) = P(E_1) \cdot P(E_2) \cdot \ldots \cdot P(E_n).$$

3. The binomial experiment

• is a statistical experiment that has the following properties

· the experiment consists of n repeated *trials*

· each trial can result in just two possible outcomes: we call one of these outcomes a *success* and the other, a *failure*

· the probability of success, denoted by p, is the same on every trial.

· the probability of failure, denoted $q = 1 - p$, is the same on every trial

· the trials are independent: the outcome on one trial does not affect the outcome on other trials.

• the binomial probability refers to the probability that a binomial experiment results in exactly k successes out of n trials

$$P = C_n^k \cdot p^k \cdot q^{n-k}$$

e.g.: flip a coin 6 times, the probability to get 4 heads is

$$P = C_6^4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{6-4}$$

• the probability that a binomial experiment results in at least k successes is

$$P = 1 - \sum_{i=0}^{k-1} C_n^i \cdot p^i \cdot q^{n-i}$$

• the probability that the k-th success is obtained after exactly r trials is

$$P = C_{r-1}^{k-1} p^k (1-p)^{r-k}, \qquad r \geq k.$$

4. The multinomial experiment

• generalizes the binomial experiment
· now each trial can have k outcomes: $E_1, E_2, \ldots, E_k$
· each of these outcomes have the probabilities $p_1, p_2, \ldots p_k$
· the n trials are again independent.
• the multinomial probability is the probability that E_1 occurs n_1 times, E_2 occurs n_2 times,... E_k occurs n_k times

$$P = \frac{n!}{n_1! n_2! \cdot \ldots \cdot n_k!} p_1^{n_1} p_2^{n_2} \cdot \ldots \cdot p_k^{n_k}$$

where $n = n_1 + n_2 + \ldots + n_k$

5. Poisson's scheme

• let $A_1, A_2, ..., A_n$, be n independent events of an experiment.
• denote by p_i the probability of A_i to occur and by $q_i = 1 - p_i$, $i = \overline{1,n}$ the probability of the complementary event.
• the probability that k events, out of those n, occur is given by the coefficient of X^k in the expression

$$(p_1 X + q_1) \cdot (p_2 X + q_2) \cdot ... \cdot (p_n X + q_n)$$

Solved problems

Problem 1

A labourer produces n items. Let us denote by A_i, $i = \overline{1,n}$ the event: the i-th item is defective. Translate mathematically, using set theory, the following events:
a) None of the produced items is defective,
b) At least one of the items is defective,
c) Only one of the items is defective,
d) Exactly two of the items are defective,
e) At least two items are not defective,
f) At most two items are defective.

Solution: Since we denoted by A_i the event "the i-th item is defective", then the complementary event $\overline{A_i}$ means the "i-th item is not defective". All the events mentioned above can be decomposed using these "elementary" events.

a) None of the produced items is defective

$$\overline{A_1} \cap \overline{A_2} \cap ... \overline{A_n}$$

b) At least one of the items is defective

$$A_1 \cup A_2 \cup ... \cup A_n$$

c) Only one of the items is defective

$$\bigcup_{i=1}^{n} \left(\overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_{i-1}} \cap A_i \cap \overline{A_{i+1}} \cap ... \cap \overline{A_n}\right)$$

d) Exactly two of the items are defective

$$\bigcup_{\substack{i=1 \\ i<j}}^{n} \left(\overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_{i-1}} \cap A_i \cap \overline{A_{i+1}} \cap ... \cap \overline{A_{j-1}} \cap A_j \cap \overline{A_{j+1}} \cap ... \cap \overline{A_n}\right)$$

e) The event "at least two items are not defective" is the complementary event of "at most one item is not defective"

$$\overline{\left(\bigcap_{i=1}^{n} A_i\right) \cup \left[\bigcup_{i=1}^{n} A_1 \cap A_2 \cap ... A_{i-1} \cap \overline{A_i} \cap A_{i+1} \cap ... \cap A_n\right]}$$

f) At most two items are defective

$$\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \cup \left[\bigcup_{i=1}^{n} \left(\overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_{i-1}} \cap A_i \cap \overline{A_{i+1}} \cap ... \cap \overline{A_n}\right)\right]$$

$$\cup \left[\bigcup_{\substack{i=1 \\ i<j}}^{n} \left(\overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_{i-1}} \cap A_i \cap \overline{A_{i+1}} \cap ... \cap \overline{A_{j-1}} \cap A_j \cap \overline{A_{j+1}} \cap ... \cap \overline{A_n}\right)\right]$$

Problem 2

There are n couples in a dance course, the dance partners are chosen randomly. Find the probability that every man will be dancing with his wife at some chosen moment. Find the limit of this probability when $n \to \infty$.

Solution: Let us define the "elementary events":
E_1: the first man dances with his wife at that given moment
E_2: the second man dances with his wife
. . . .
E_n: the n-th man dances with his wife
It is easy to show

$$P\left(E_{i_1} \cap E_{i_2} \cap ... \cap E_{i_p}\right) = \frac{(n-p)!}{n!}$$

since if p pairs are husband-wife, the other $n-p$ are random pairs man-woman which can be formed in $(n-p)!$ possible ways.

The event E: "every man will be dancing with his wife" can be described

$$E = \overline{E}_1 \cap \overline{E}_2 \cap \ldots \overline{E}_n$$

Hence the complementary form of Poincare's theorem provides

$$P\left(\bigcap_{k=1}^{n} \bar{E}_k\right) = 1 - \sum_{i=1}^{n} P(E_i) + \sum_{1 \le i < j \le n} P(E_i \cap E_j) + \ldots$$

$$+(-1)^p \sum_{1 \le i_1 < i_2 < \ldots < i_p \le n} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_p}) + \ldots + (-1)^n P(E_1 \cap E_2 \cap \ldots E_n).$$

$$\begin{aligned} P(E) &= 1 - C_n^1 \frac{(n-1)!}{n!} + C_n^2 \frac{(n-2)!}{n!} - \ldots + (-1)^n C_n^n \frac{(n-n)!}{n!} \\ &= 1 - \frac{n!}{1!\,(n-1)!} \frac{(n-1)!}{n!} + \frac{n!}{2!\,(n-2)!} \frac{(n-2)!}{n!} - \ldots + (-1)^n \frac{1}{n!} \\ &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} \end{aligned}$$

For the second part of the problem one has

$$\lim_{n\to\infty} P(E) = \frac{1}{e}.$$

since we can use the Maclaurin expansion of e^{-x}

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^n}{n!} + \ldots$$

Problem 3

One of four different prizes, that are randomly put into each box of a cereal, is a ticket to the local zoo. A family decided to buy this cereal until they will win four tickets. What is the probability that the family has to buy 10 boxes in order to win four tickets ? The same problem for 16 boxes.

Solution: First one has to observe the binomial experiment involved here. At each try you win the ticket (the success) with a probability $p = \frac{1}{4}$ or you don't win it (the failure) with a probability $q = \frac{3}{4}$. The statement of this problem translates into finding the probability that the k-th success is obtained after exactly r trials and this can be obtained using

$$P = C_{r-1}^{k-1} p^k q^{r-k}, \qquad r \ge k.$$

There is nothing magic in this formula, it has an easy argumentation. If the k-th success is obtained in the r-th trial $\Longrightarrow$ in the previous $r-1$ trials there were exactly $k-1$ successes. According to the binomial probability formula $k-1$ successes in $r-1$ trials have the probability

$$C_{r-1}^{k-1} p^{k-1} q^{r-1-(k-1)}$$

Finally, the multiplication rule provides the above formula of P.

We are interested in the particular cases $r = 4$ and $r = 10$ and we need to register exactly $k = 4$ successes. This leads to

$$P_1 = C_{10-1}^{4-1} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{10-4} \quad \text{and} \quad P_2 = C_{16-1}^{4-1} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{16-4}$$

Problem 4

Find the probability that among 7 people:

a) No two were born on the same day of the week

b) At least two were born on the same day

c) Two were born on Sunday and two on Tuesday

Solution: a) Finding the day when each of these people is born can be interpreted as a multinomial experiment with 7 trials, and each trial has 7 possible outcomes:

E_1: the person is born on Monday

E_2: the person is born in Tuesday

. . . .

E_7: the person is born on Sunday

Obviously $P(E_1) = P(E_2) = \ldots = P(E_7) = \frac{1}{7}$.

The event "no two were born on the same day of the week" is equivalent with imposing the restrictions: E_1 will occur once, E_2 will occur once, ..., E_7 will occur once. The requested probability is thus the multinomial probability

$$P = \frac{7!}{1! \cdot 1! \cdot \ldots \cdot 1!} \left(\frac{1}{7}\right)^1 \cdot \left(\frac{1}{7}\right)^1 \cdot \ldots \cdot \left(\frac{1}{7}\right)^1 = \frac{7!}{7^7}$$

b) The event " at least two were born on the same day" is complementry to the event investigated above hence

$$P = 1 - \frac{7!}{7^7}$$

c) One needs to redefine the possible outcomes of the multinomial experiment (rearranging the information) in the following way

E_1: the person is born on Tuesday

E_2: the person is born on Sunday

E_3: the person is born on any other day of the week but Tuesday or Sunday

First of all $P(E_1) = P(E_2) = \frac{1}{7}$ but $P(E_3) = \frac{5}{7}$. Let us also observe that we want E_1 to occur $n_1 = 2$ times, E_2 to occur $n_2 = 2$ times and E_3 to occur $n_3 = 3$ times. The requested probability will be the multinomial probability

$$P = \frac{7!}{2! \cdot 2! \cdot 3!} \left(\frac{1}{7}\right)^2 \cdot \left(\frac{1}{7}\right)^2 \cdot \left(\frac{5}{7}\right)^3$$

Proposed problems

Problem 1. *Eight students are to be put up in a student residence in three rooms, two of which have three beds and one has two beds. In how many ways can the students be distributed over the three rooms ?*

Problem 2. *A worker realized* 5 *items of a product. We denote by* $E_i, i = \overline{1,5}$ *the event: the i-th item is defective. Using set theory describe the following events:*

i) None of the items are defective

ii) At least one of the items is defective

iii) Exactly one of the items is defective

iv) Exactly two are defective

v) At least two items are not defective

vi) At most two items are defective

vii) Assuming $P(E_i) = \frac{1}{10}$, $i = \overline{1,5}$ *estimate the probabilities of the above events.*

Problem 3. *John knows the answers to 1 of the 10 multiple choice questions on the Special Mathematics exam. He has skipped several of the lectures, he must take random guesses for the other nine. Assuming each question has four answers, what is the probability he will get exactly 7 of the last questions right? Every answer worths one point and he needs at least 5 points to pass the exam. What is the probability he will pass?*

Problem 4. *a) An experiment of drawing a random card from an ordinary playing cards deck is done with replacing it back. This was done ten times. Find the probability of getting* 2 *spades* $\spadesuit$, 3 *diamonds* $\diamondsuit$, 3 *clubs* $\clubsuit$ *and* 2 *hearts* $\heartsuit$.

b) The numbers $1, 2, 3, \ldots n$ *are written in random order. What is the probability of having* 1 *and* 2 *on consecutive positions ?*

Problem 5. *At some moment in a backgammon game you have to roll a* 6 *or the sum of the two numbers to be* 6 *to put your opponent's rear checker on the bar. What is the probability of hitting your opponent's checker ? What is the probability of hitting at least twice in* 4 *attempts?*

Problem 6. *a) A single card is chosen at random from a standard deck of* 52 *playing cards. What is the probability of choosing a king* (K) *or a club* (♣)*?*

b) A professor gives only two types of exams, "easy" and "hard". You will get a hard exam with probability 0.80. *The probability that the first question on the exam will be marked as difficult is* 0.90 *if the exam is hard and is* 0.15 *otherwise. What is the probability that the first question on your exam is marked as difficult ? What is the probability that your exam is hard given that the first question on the exam is marked as difficult?*

Problem 7. *Two friends decide to meet at* 21 : 00 *pm in a restaurant. They decided that who ever reaches the restaurant earlier will wait for the other person for 20 minutes. The restaurant closes at* 23 : 00 *pm. What's the probability that those friends do meet?*

Problem 8. *A person wrote 5 letters, sealed them in envelopes and wrote the different addresses randomly on each of them. Find the probability that at least one of the envelopes has the correct address.*

Problem 9. *Find the probability of drawing a king, a queen, a king and a knave in this order, from a deck of* 52 *cards, in four consecutive draws. The cards drawn are not replaced.*

Problem 10. *Twelve persons get on a train that has six cars. Each passenger may select with equal probability each of the cars. Find the probability: that*

(a) there will be two passengers in each car,

(b) there will be one car without passengers, one with one passenger, two with two passengers each and the remaining two with three and four passengers, respectively.