Chapter VI

AFFINE SPACES

1 Definitions. Properties

Consider $\wp = \{A, B, ...\}$ a set of points and the vector space $(\mathbb{R}^n, +, \cdot)$.
Definition 1.1. *The triple* $\mathbf{A^n} = (\wp, \mathbb{R}^n, \varphi)$, *where*

$$\varphi : \wp \times \wp \to \mathbb{R}^n, (A, B) \in \wp \times \wp \longrightarrow \varphi(A, B) = \overline{AB} \in \mathbb{R}^n,$$

such that the following axioms hold:
$(AS1):\ \overline{AB} + \overline{BC} = \overline{AC}, \quad \forall A, B, C \in \wp$, *see the Figure 6.1;*

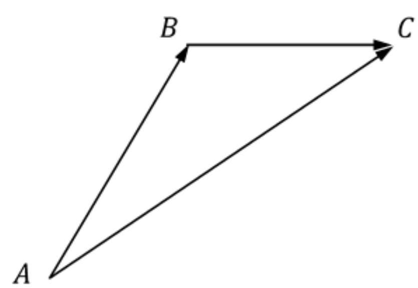

Figure 6.1

$(AS2):\ \forall O \in \wp\ \forall\, \overline{v}\ \in V\ \exists! M \in \wp$ *such that* $\varphi(O, M) = \overline{v}$,
is called **an affine space.**

We denote $\varphi(O, M) = \overline{OM}$, see the Figure 6.2.

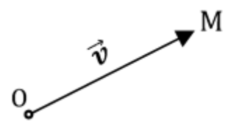

Figure 6.2

The affine space **A** that corresponds to the vector space V is denoted by $(\mathbf{A}, \mathbf{V}, \varphi)$.

Proposition 1.1. *1.* $\varphi(A, A) = 0,\ \forall A \in \wp$;
2. $\varphi(A, B) = -\varphi(B, A),\ \forall A, B \in \wp$.

To any two points $A(x_1, x_2, ..., x_n), B(y_1, y_2, ..., y_n)$ from $\mathbb{R}^n$ corresponds the vector

$$\varphi(A, B) = \overline{AB} = \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \\ \vdots \\ y_n - x_n \end{pmatrix},$$

so the triple $(\mathbb{R}^n, \mathbb{R}^n, \varphi)$ is an affine space, denoted by $\mathbf{A^n}$. This means that we can regard $\mathbb{R}^n$ as a set of points (any point $A \in \mathbb{R}^n$ can be written as $(x_1, x_2, ..., x_n)$), or a set of vectors (any vector $v \in \mathbb{R}^n$ can be written as $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$).

Example Consider the point $O(-2, 3, 1)$ and the vector $\overline{v} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ from $\mathbf{A^3}$. Find the coordinates of $M \in \mathbf{A^3}$ such that $\overline{OM} = \overline{v}$.

Solution: If we take $M(x, y, z)$, then $\overline{OM} = \begin{pmatrix} x+2 \\ y-3 \\ z-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$, so we obtain $x = 1, y = 3, z = 3$, which means that $M(1, 3, 3)$.

Remark 1.1. Each vector $\overline{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ can be identified to the vector $\overline{OM}$, where $O(0, 0, ..., 0)$ and $M(x_1, x_2, ..., x_n)$.

Definition 1.2. *The affine space* $\mathbf{A^n} = (\mathbb{R}^n, \mathbb{R}^n, \varphi)$ *embedded with the standard inner product is called* **the Euclidean space** $\mathbf{E^n} = (\mathbf{A^n}, \cdot)$.

Definition 1.3. *The pair* $\mathbf{R} = \{O,\, B = \{e_1, e_2, ..., e_n\}\}$ *with* $O \in \mathbb{R}^n$ *and* B *is an orthonormal basis for* $\mathbb{R}^n$, *is called* **an orthonormal frame** *in* E^n.

The point O is **the origin of the frame.**

Remark 1.2. To each orthonormal frame $\mathbf{R} = \{O,\, B\}$ corresponds an orthonormal system of axes $Ox_1, Ox_2, ..., Ox_n$ given by

$$Ox_i = \{M \mid \overline{OM} = te_i, t > 0\}, i = \overline{1, n},$$

called **the coordinates axes.**

To each point $M(x_1, x_2, ..., x_n)$ corresponds the vector

$$\overline{OM} = x_1\overline{e}_1 + x_2\overline{e}_2 + ... + x_n\overline{e}_n$$

called **the position vector** of $M \in \mathbf{E^n}$ (denoted by $\overline{r}$) related to the frame $\mathbf{R}$:

$$M(x_1, x_2, ..., x_n) \in \mathbf{R^n} \longleftrightarrow \overline{OM} = x_1\overline{e}_1 + x_2\overline{e}_2 + ... + x_n\overline{e}_n.$$

The coordinates of the position vector $\overline{OM}$ in the basis B are the **coordinates of M related to the frame R**, denoted by $M(x_1, x_2, ..., x_n)_{\mathbf{R}}$.

The coordinates of the frame's origin are the coordinates of the vector $\overline{OO} = (0, 0, ..., 0)^t$, so the coordinates of O are $(0, 0, ..., 0)$.

Remark 1.3. If $B_c = \{\overline{\imath} = (1, 0, 0), \overline{\jmath} = (0, 1, 0), \overline{k} = (0, 0, 1)\}$ then $\mathbf{R} = \{O(0, 0, 0),\, B_c\}$ is the **Cartesian frame,** and Ox, Oy, Oz are **the Cartesian coordinate system**.

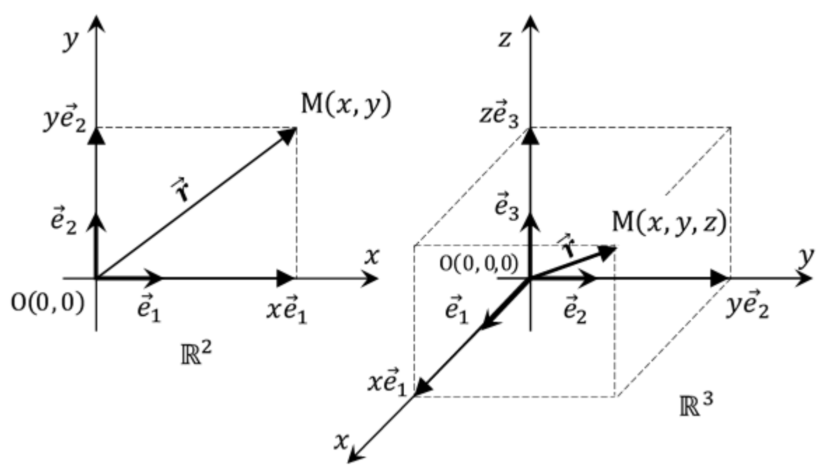

Figure 6.3: The Cartesian coordinate systems in $\mathbb{R}^2$ and $\mathbb{R}^3$

Consider the frames $\mathbf{R} = \{O,\, B = \{v_1, v_2, ..., v_n\}\}$ and $\mathbf{R}' = \{O',\, B' = \{v_1', v_2', ..., v_n'\}\}$ in $\mathbf{A^n}$ and the point $M \in \mathbf{A^n}$.

Definition 1.4. *Two basis B and B' for $\mathbb{R}^n$ have* **the same orientation** *(denoted $B \sim B'$) iff det $T_{BB'} > 0$; otherwise, B and B' have* **opposite orientation.**

Definition 1.5. *Each basis for $\mathbb{R}^n$ that has the same orientation as the standard basis is called* **right-handed basis;** *otherwise, is called* **left-handed basis.**

Proposition 1.2. *An orthonormal basis $B' = \{v_1', v_2'\}$ for $\mathbb{R}^2$ has the same orientation as the orthonormal basis $B = \{v_1, v_2\}$ iff the angles between v_1 and v_1', respectively v_2 and v_2', are equal. Otherwise, the two basis have* **opposite orientation.**

$$B \sim B' \Longleftrightarrow \angle(v_1, v_1') = \angle(v_2, v_2').$$

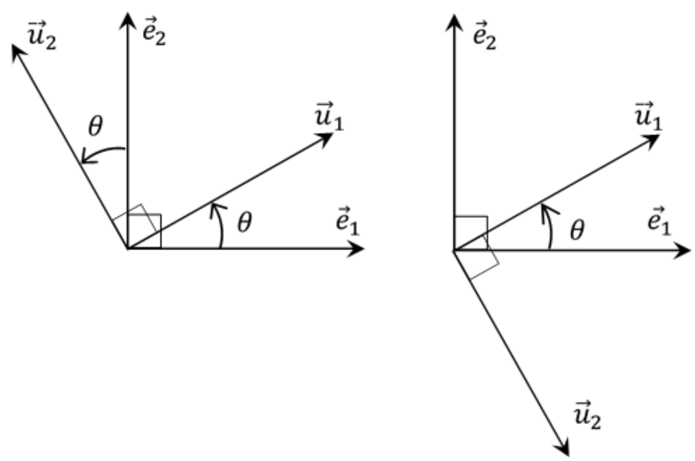

Figure 6.4: Right-handed and left-handed basis

Definition 1.6. A right-handed orthonormal frame *in $\mathbb{R}^2$ is an orthonormal frame for that the Oy axis is obtaining via an anti-clockwise rotation of 90° of the Ox axis. (see the Figure 6.5)*

Let us consider two frames $\mathbf{R} = \{O,\, B = \{v_1, v_2, ..., v_n\}\}$ and $\mathbf{R}' = \{O',\, B' = \{v_1', v_2', ..., v_n'\}\}$ in E^n and $M \in \mathbb{R}^n$.

Proposition 1.3. *If* $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ *are the coordinates of* M *related to the frame* $\mathbf{R}$ *and* $(x'_1, x'_2, ..., x'_n) \in \mathbb{R}^n$ *are the coordinates of* M *related to* $\mathbf{R}'$*, then the following formula holds:*

$$X_{\mathbf{R}} = A_{\mathbf{R}} + T_{BB'} X_{\mathbf{R}'},$$

where:

$X_{\mathbf{R}}$ *is the matrix containing the coordinates of* M *related to* $\mathbf{R}$,

$$X_R = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

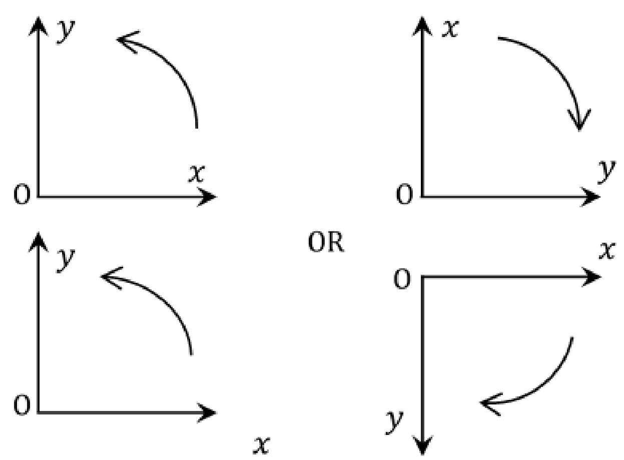

Figure 6.5: Right-handed and left-handed orthonormal frames

$X_{\mathbf{R}'}$ *is the matrix containing the coordinates of* M *related to* $\mathbf{R}'$:

$$X'_R = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix},$$

and $T_{BB'}$ *is the transition matrix from* $\mathbf{B}$ *to* $\mathbf{B}'$*, see the Figure 6.6.*

Specific cases:

1. If $B = B'$ then $T_{BB'} = I_n$, so the above formula becomes $X_{\mathbf{R}} = A_{\mathbf{R}} + X_{\mathbf{R}'}$, or equivalent

$$x_i = a_i + x_i', i = \overline{1, n},$$

which is the equation of a **translation**.

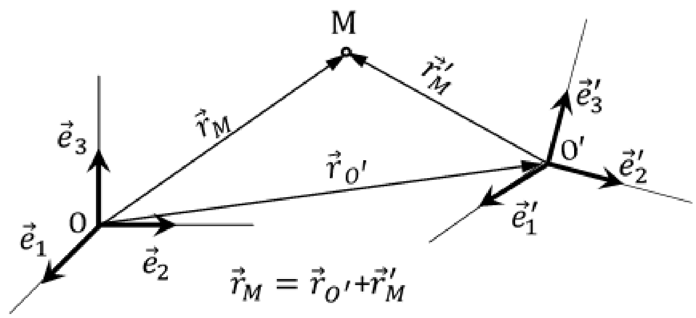

Figure 6.6

2. If $O = O'$ then we obtain **a rotation through an angle** of equations:

$$x_i = \alpha_i^1 x_1' + \alpha_i^2 x_2' + ... + \alpha_i^n x_n', i = \overline{1, n}.$$

Let us consider now the Euclidean space E^3 and the Cartesian coordinate system $\{O, \bar{\imath}, \bar{\jmath}, \bar{k}\}$.

Definition 1.7. *If $\bar{x} = x_1\bar{\imath} + x_2\bar{\jmath} + x_3\bar{k}$ and $\bar{y} = y_1\bar{\imath} + y_2\bar{\jmath} + y_3\bar{k}$, then the vector given by:*

$$\bar{x} \times \bar{y} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2)\,\bar{\imath} - (x_1y_3 - x_3y_1)\,\bar{\jmath} + (x_1y_2 - x_2y_1)\,\bar{k}$$

is called **the cross-product of** $\bar{x}, \bar{y}$.

Proposition 1.4. *The cross-product $\bar{x} \times \bar{y}$ of the vectors $\bar{x}$ and $\bar{y}$ has the following properties:*
a) $\bar{x} \times \bar{y} \perp \bar{x}$, $\bar{x} \times \bar{y} \perp \bar{y}$;
b) $\bar{x} \times \bar{y} = \bar{0} \Longleftrightarrow \bar{x} \parallel \bar{y}$;

c) If $\bar{x} \times \bar{y} \neq \bar{0}$ then $(\bar{x}, \bar{y}, \bar{x} \times \bar{y})$ is a right-handed basis.
d) If $\bar{x} \neq \bar{0}$ şi $\bar{y} \neq \bar{0}$ then:

$$|\bar{x} \times \bar{y}| = |\bar{x}| \cdot |\bar{y}| \sin \angle (\bar{x}, \bar{y}) ;$$

e) $\bar{x} \times \bar{y}$ does not depend on the basis B.
f) $\bar{x} \times \bar{y} = -\bar{y} \times \bar{x}$.

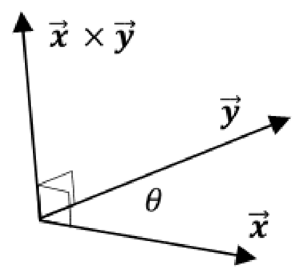

Figure 6.7: The cross-product of two vectors is orthogonal to each vector

Proposition 1.5. *The length $\| \bar{x} \times \bar{y} \|$ of the cross-product of $\bar{x}$ and $\bar{y}$ is the area of the parallelogram defined by the two vectors, (see the Figure 6.8.)*

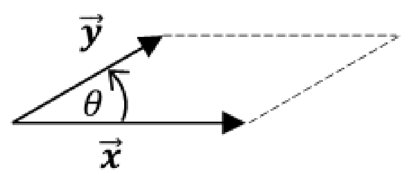

Figure 6.8

Warning: The cross-product of $\bar{x}$ and $\bar{y}$ is NOT associative!

Definition 1.8. The scalar triple product *(or* **mixed product***, or* **box product***) of $\overline{x}, \overline{y}$, and $\overline{z}$ is the scalar*

$$(\overline{x}, \overline{y}, \overline{z}) = \langle \overline{x}, \overline{y} \times \overline{z} \rangle .$$

Proposition 1.6. *For any three vectors $\overline{x}, \overline{y}, \overline{z} \in E^3$ the following relation holds:*

$$\overline{x} \times (\overline{y} \times \overline{z}) = \langle \overline{x}, \overline{z} \rangle \, \overline{y} - \langle \overline{x}, \overline{y} \rangle \, \overline{z}.$$

Proposition 1.7. *The mixed product of $\bar{x}, \bar{y}$, and $\bar{z}$ is zero iff they are coplanar vectors.*

$$(\overline{x}, \overline{y}, \overline{z}) = 0 \Longleftrightarrow \textit{coplanar vectors}$$

Proposition 1.8. *Geometrically, the triple product is the (signed) volume of the parallelepiped defined by the three given vectors* $|(\overline{x}, \overline{y}, \overline{z})|$.

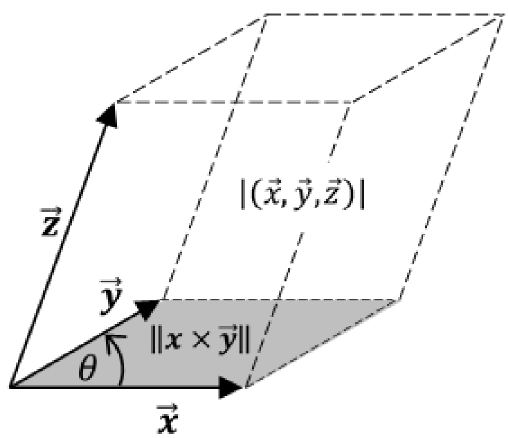

Figure 6.9

Proposition 1.9. *If*

$$\begin{aligned} \overline{x} &= x_1\overline{i} + y_1\overline{j} + z_1\overline{k}; \\ \overline{y} &= x_2\overline{i} + y_2\overline{j} + z_2\overline{k}; \\ \overline{z} &= x_3\overline{i} + y_3\overline{j} + z_3\overline{k}, \end{aligned}$$

then the mixed product is given by:

$$(\overline{x}, \overline{y}, \overline{z}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

2 Solved Problems

1. Let ABC be a triangle and A_1, B_1, C_1 the midpoints of the line segments BC, CA, and AB.
a) Prove that
$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\overrightarrow{OA} + 2\overrightarrow{AA_1} = 3\overrightarrow{OB} + 2\overrightarrow{BB_1} = 3\overrightarrow{OC} + 2\overrightarrow{CC_1}.$$
b) Prove that there exists a unique point G (the triangle's centroid) such that:
$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0}.$$
c) Prove that any point M satisfies:
$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = 3\overrightarrow{MG}.$$

Solution:
a) The following relations hold:
$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{OA} + \overrightarrow{AC}$$
$$= 3\overrightarrow{OA} + 2\overrightarrow{AA_1} + \overrightarrow{A_1B} + \overrightarrow{A_1C} = 3\overrightarrow{OA} + 2\overrightarrow{AA_1},$$
because
$$\overrightarrow{A_1B} + \overrightarrow{A_1C} = \overrightarrow{0}.$$
Using similar arguments, we can prove the rest of the statements.

b) Using the previous result a),
$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0} \Longleftrightarrow 3\overrightarrow{GA} + 2\overrightarrow{AA_1} = \overrightarrow{0} \Longleftrightarrow \overrightarrow{GA} = 2/3\overrightarrow{A_1A}$$
it follows that G is the centroid of the triangle ABC.

c) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\overrightarrow{OG} + \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 3\overrightarrow{OG}.$

2. Let ABC and $A_1B_1C_1$ two triangles (see the figure) having G and G_1 as centroids. Prove that:
$$\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = 3\overrightarrow{GG_1}$$
and find a necessary and sufficient condition that two triangles have the same centroid.

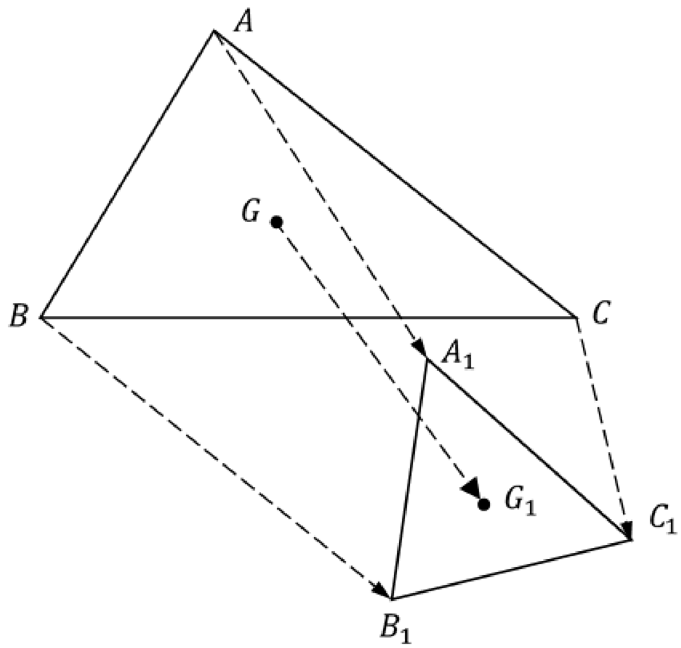

Figure 6.10

Solution: It is easy to see that:

$$\overrightarrow{AA_1} = \overrightarrow{AG} + \overrightarrow{GG_1} + \overrightarrow{A_1G_1}.$$

Adding the analogous relations for $\overrightarrow{BB_1}$ şi $\overrightarrow{CC_1}$ we obtain:

$$\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = 3\overrightarrow{GG_1} - (\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}) +$$
$$+\overrightarrow{A_1G_1} + \overrightarrow{B_1G_1} + \overrightarrow{C_1G_1} = 3\overrightarrow{GG_1}.$$

So, the necessary and sufficient condition that $G = G_1$ consists in:

$$\overrightarrow{GG_1} = \overrightarrow{0}$$

or, equivalently:

$$\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{0}.$$

3. Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ two vectors corresponding to two orthogonal chords of a circle having the center O, and let M be their intersection point. Prove that:

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} = 2\overrightarrow{MO}.$$

Solution:

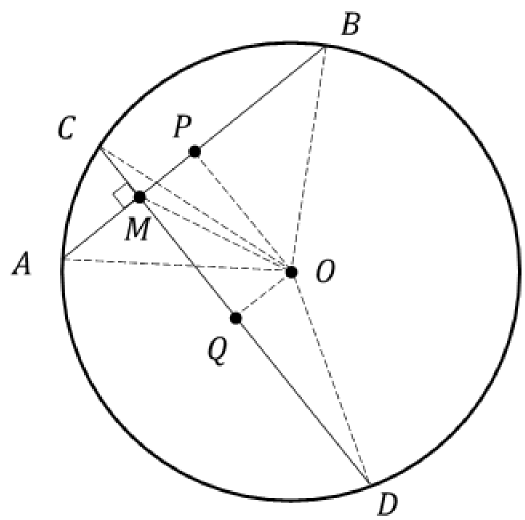

Figure 6.11

We denote by P and Q the midpoints of the chords $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Then $OQMP$ is a rectangle:

$$\overrightarrow{OM} = \overrightarrow{OQ} + \overrightarrow{OP}.$$

But:

$$\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OC} + \overrightarrow{OD}$$

and:

$$\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

hence

$$2\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}.$$

On the other hand:

$$\overrightarrow{MA} = \overrightarrow{MO} + \overrightarrow{OA}$$

and: $\overrightarrow{MB}, \overrightarrow{MC}$ şi $\overrightarrow{MD}$. So we obtain:

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} = 4\overrightarrow{MO} + 2\overrightarrow{OM} = 2\overrightarrow{MO}.$$

4. Considering the points $A(1,-1,1), B(2,1,-1), C(3,1,2),\ D(\frac{8}{3},1,1)$, and $E(4,-1,1)$, verify if the points A, B, C, D and, respectively A, B, C, E are coplanar.

Solution: Successively, we have:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

and

$$\overline{AB} = \bar{i} + 2\bar{j} - 2\bar{k}.$$

In addition:

$$\overline{BC} = \bar{i} + 3\bar{k},\ \overline{CD} = -1/3\bar{i} - \bar{k},\ \overline{CE} = \bar{i} - 2\bar{j} - \bar{k}.$$

The vectors $\overline{AB}, \overline{BC}$ and $\overline{CD}$ (respectively $\overline{AB}, \overline{BC}$ and $\overline{CE}$) are coplanar iff the points A, B, C, D (respectively A, B, C, E) are coplanar.

Due to the fact that:

$$(\overline{AB}, \overline{BC}, \overline{CD}) = \begin{vmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -\frac{1}{3} & 0 & -1 \end{vmatrix} = 0$$

it follows that the vectors $\overline{AB}, \overline{BC}$ and $\overline{CD}$ are coplanar, so the points A, B, C, D belongs to the same plane.

Computing the mixed product of the vectors $\overline{AB}, \overline{BC}, \overline{CE}$ we find:

$$(\overline{AB}, \overline{BC}, \overline{CE}) = \begin{vmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ 1 & -2 & -1 \end{vmatrix} = 18 \neq 0,$$

so the points A, B, C, E are not coplanar.

5. If A, B, C, E are the points from the previous problem, find the areas A_1 and A_2 of the triangles ABC and ABE .

Solution: We denote by σ the aria of the parallelogram determined by the vectors $\overline{AB}$ and $\overline{BC}$. Using the geometric handle of the mixed product, we find:

$$\sigma = \left\| \overline{AB} \times \overline{BC} \right\| = \left\| 6\bar{i} - 5\bar{j} - 2\bar{k} \right\| = \sqrt{65}.$$

The aria of the triangle ABC is:

$$A_1 = \frac{1}{2}\sigma = \frac{\sqrt{65}}{2}.$$

Using similar arguments, we obtain $A_2 = \sqrt{18}$.

6. Let $A(0,-5,0)$ and $B(1,-2,3)$ two points in E_3. Find:
a) A vector $\overline{v}$ parallel to the plane determined by $\overline{\imath}$ and $\overline{\jmath}$ such that $\|\overline{v}\| = \left\|\overline{AB}\right\|$ and $\overline{v} \perp \overline{AB}$.
b) An unit vector $\overline{u}$ orthogonal to $\overline{v}$ and $\overline{AB}$.

Solution:
a) We have:

$$\overline{AB} = \overline{\imath} + 3\overline{\jmath} + 3\overline{k}.$$

Let $\overline{v} = x\overline{\imath} + y\overline{\jmath}$. Due to the fact that the two vectors $\overline{v}$ and $\overline{AB}$ have the same length, we obtain

$$x^2 + y^2 = 19;$$

now, using that the two vectors are orthogonal to each other:

$$\overline{v} \cdot \overline{AB} = 0 \Leftrightarrow x + 3y = 0.$$

We obtain:

$$\overline{v} = \pm \left(3\sqrt{\frac{19}{10}}\overline{\imath} + \sqrt{\frac{19}{10}}\overline{\jmath} \right).$$

b) The unit vector $\overline{u}$ can be found using the formula:

$$\overline{u} = \pm \frac{\overline{v} \times \overline{AB}}{\left\|\overline{v} \times \overline{AB}\right\|} = \pm \left(\frac{3}{\sqrt{136}}\overline{\imath} + \frac{9}{\sqrt{136}}\overline{\jmath} - \frac{10}{\sqrt{136}}\overline{k} \right).$$

7. Consider the vectors $\overline{v} = 2\overline{\imath} + 3\overline{\jmath} - 3\overline{k}$ and $\overline{u} = -2\overline{\imath} - 3\overline{\jmath} + \overline{k}$. Find:
a) the angle between the two vectors;
b) the projection of the vector $\overline{v}$ onto $\overline{u}$;

Solution:
a) Using the formula:

$$\cos\angle\left(\overline{v},\overline{u}\right) = \frac{\overline{v}\cdot\overline{u}}{\|\overline{v}\|\cdot\|\overline{u}\|},$$

we obtain:

$$\cos\angle(\overline{v},\overline{u}) = -\frac{8}{\sqrt{77}}.$$

b) We have:

$$pr_{\overline{u}}\overline{v} = \frac{\overline{v}\cdot\overline{u}}{\|\overline{u}\|^2}\overline{u} = -\frac{8}{7}\left(-2\overline{\imath} - 3\overline{\jmath} + \overline{k}\right).$$

8. Given the points $A(1,-2,3)$ and $B(2,-1,8)$ in E_3.
a) Find all the points C that belongs to the plane xOy such that the triangle ABC is isosceles, $\|\overline{AB}\| = \|\overline{AC}\|$ and $\overline{AB}\cdot\overline{AC} = -9$.
b) Compute the aria of the triangle ABC.
c) If $D(2,-3,4)$, check if $ABCD$ is a tetrahedron and find its volume.

Solution:
a) Let $C(x,y,0)\in E^3$, so:

$$\overline{AB} = \overline{\imath} + \overline{\jmath} + 5\overline{k}$$

and

$$\overline{AC} = (x-1)\overline{\imath} + (y+2)\overline{\jmath} - 3\overline{k};$$

It follows that:

$$\|\overline{AB}\| = \sqrt{27},\ \|\overline{AC}\| = \sqrt{(x-1)^2 + (y+2)^2 + 9}.$$

The two conditions from above lead us to:

$$\begin{cases} (x-1)^2 + (y+2)^2 = 18 \\ x + y = 5. \end{cases}$$

The system has a unique solution $x = 4, y = 1$ so $C(4,1,0)$.
b) The aria of the triangle ABC is:

$$A_{\Delta ABC} = \tfrac{1}{2}\left\|\overline{AB}\times\overline{AC}\right\| = 18\sqrt{2}.$$

c) $ABCD$ is a tetrahedron iff the vectors $\overline{AB}, \overline{AC}$ and $\overline{AD}$ are not coplanar. Computing:

$$(\overline{AB}, \overline{AC}, \overline{AD}) = -32 \neq 0,$$

we find that the vectors are not coplanar, so $ABCD$ is a tetrahedron. Its volume is $V_{ABCD} = \frac{32}{3}$.

3 Exercises

1. Let us consider the vectors $\overline{u} = \overline{\imath} + 2\overline{\jmath} + \overline{k}$ and $\overline{v} = -2\overline{\imath} + \overline{\jmath} + 2\overline{k}$. Find:

   (i) The norms of the vectors $\overline{u}$ and $\overline{v}$ and the angle between $\overline{u}$ and $\overline{v}$;

   (ii) The projection of the vector $\overline{u}$ onto $\overline{v}$;

   (iii) The aria of the parallelogram defined by $\overline{u}$ and $\overline{v}$;

2. Let us consider the vectors $\overline{u} = (1, -2, 3)$ and $\overline{v} = (0, 3, 2)$. Find:

   (i) The angle between $\overline{u}$ and $\overline{v}$;

   (ii) The length of the height of the parallelogram construct on the vectors $\overline{u}$ and $\overline{v}$ corresponding to the base $\overline{u}$.

   (iii) Find a vector $\overline{w}$ having the same direction and sens as $\overline{u}$ and the length 56.

3. Let us consider the points $A(2, 2, 1)$ and $B(4, 1, 3)$. Find:

   (i) The length of the vector $\overrightarrow{AB}$;

   (ii) A vector $\overline{v}$ included in the plane xOy such that $\|v\| = \|\overrightarrow{AB}\|$ and $\overline{v} \perp \overrightarrow{AB}$.

4. Let us consider the points $A(4, -2, 2)$, $B(3, 1, 1)$, $C(4, 2, 0)$ and $D(0, 0, 9)$. Find the length of the height from D of the tetrahedron $ABCD$.

5. Let us consider the points $A(1, 2, -1)$, $B(1, 0, 3)$, $C(2, 1, 2)$ and $D(2, 3, 4)$. Find:

   (i) The aria of the triangle ABC and the length of the height from A;

   (ii) The length of the median from A of the triangle ABC, the perimeter of the triangle ABC and the measure of the angle ABC;

   (iii) The volume of the tetrahedron $ABCD$ and the length of the height from D.

6. Consider the triangle ABC and G be its centroid. Prove that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0}$.

7. Given the triangle ABC, G be its centroid and O a point that belongs to the triangle's plane, prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\overrightarrow{OG}$.

8. Find the scalar $\lambda \in \mathbb{R}$ such that the vectors:

   $$\overrightarrow{v_1} = 2\overrightarrow{i} + (\lambda - 2)\overrightarrow{j} + \overrightarrow{k},\ \overrightarrow{v_2} = -\overrightarrow{i} + \lambda\overrightarrow{j} - 2\overrightarrow{k}$$

   are coplanar.

9. Find a decomposition of the vector $\overrightarrow{v} = 2\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$ parallel to the vectors:
$$\overrightarrow{a} = \overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}, \overrightarrow{b} = \overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$$
and:
$$\overrightarrow{c} = -\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}.$$

10. Given the vectors:
$$\overrightarrow{a} = 2\overrightarrow{i} - \overrightarrow{j} - 6\overrightarrow{k}, \overrightarrow{b} = 3\overrightarrow{i} + 2\overrightarrow{j} + 5\overrightarrow{k}, \overrightarrow{c} = -\frac{8}{3}\overrightarrow{i} - \overrightarrow{j} - \frac{4}{3}\overrightarrow{k},$$
prove that $\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$ are coplanar.

11. Consider the vectors: $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \in E_3$ such that:
$$\|\overrightarrow{a}\| = 1, \left\|\overrightarrow{b}\right\| = 2, \|\overrightarrow{c}\| = 3.$$
If the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\frac{\pi}{6}$, then compute the norm of the vector $\overrightarrow{v} = \overrightarrow{a} - 2\overrightarrow{b} + 3\overrightarrow{c}$.

12. Consider the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC} \in E_3$
$\overrightarrow{OA} = 12\overrightarrow{i} - 4\overrightarrow{j} + 3\overrightarrow{k}, \overrightarrow{OB} = 3\overrightarrow{i} + 12\overrightarrow{j} - 4\overrightarrow{k}, \overrightarrow{OC} = 2\overrightarrow{i} + 3\overrightarrow{j} - 4\overrightarrow{k}.$
a) Prove that OAB is a isosceles right triangle.
b) Compute the perimeter of the triangle ABC.

13. Consider the vectors: $\overrightarrow{a}, \overrightarrow{b} \in E_3$ such that:
$$\overrightarrow{a} = 3\overrightarrow{m} - 2\overrightarrow{n}, \overrightarrow{b} = \overrightarrow{m} + 2\overrightarrow{n},$$
such that $\|\overrightarrow{m}\| = 1, \|\overrightarrow{n}\| = 2$ and the angle between $\overrightarrow{m}$ and $\overrightarrow{n}$ is $\frac{\pi}{3}$.
a) Compute the lengths of the diagonals of the parallelogram determined by the two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$;
b) Compute the angle between the diagonals of the parallelogram.

14. Consider the points $A(1,-2,1), B(2,1,-1), C(3,2,-6)$.
Find:
a) the inner product $\overrightarrow{AB} \cdot \overrightarrow{AC}$ and the measure of the angle between the vectors $\overrightarrow{AB}$ şi $\overrightarrow{AC}$;
b) the vector product $\overrightarrow{AB} \times \overrightarrow{AC}$ and the aria of the triangle ABC;
c) a vector $\overrightarrow{v}$ orthogonal to the plane determinate by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and having the magnitude equal to $3\sqrt{182}$.

15. Given the vectors: $\overrightarrow{a}, \overrightarrow{b} \in E^3$ such that:

$$\overrightarrow{a} = 3\overrightarrow{m} - \overrightarrow{n}, \overrightarrow{b} = \overrightarrow{m} + 3\overrightarrow{n},$$

such that $\|\overrightarrow{m}\| = 3, \|\overrightarrow{n}\| = 2$ and the angle between the vectors $\overrightarrow{m}$ şi $\overrightarrow{n}$ equal to $\frac{\pi}{2}$, compute the aria of triangle determined by $\overrightarrow{a}$ and $\overrightarrow{b}$.

16. Consider the vectors:

$$\overrightarrow{a} = \overrightarrow{i} + \overrightarrow{j} + 2\overrightarrow{k}, \overrightarrow{b} = 2\overrightarrow{i} - \overrightarrow{j} + \lambda\overrightarrow{k}, \overrightarrow{c} = \overrightarrow{i} - 2\overrightarrow{j} + \overrightarrow{k}.$$

Find the value of $\lambda \in \mathbb{R}$ such that the vector $\overrightarrow{v} = \overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$ is parallel to the plane xOy.

17. The points $A(-1,2,-1), B(-2,5,1), C(-1,6,0), D(2,3,-6)$ are the vortexes of a quadrilateral. Prove that $ABCD$ is a plane quadrilateral and compute its area.

18. Given the vectors:

$$\overrightarrow{a} = 2\overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k}, \overrightarrow{b} = \overrightarrow{i} + \overrightarrow{j} - 2\lambda\overrightarrow{k}, \overrightarrow{c} = \lambda\overrightarrow{i} + 2\overrightarrow{j},$$

find the scalar $\lambda \in \mathbb{R}$ such that the volume of the parallelepiped built on the vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ equals to 5.

19. The points $A(1,-5,4), B(0,-3,1), C(-2,-4,0), D(4,4,-2)$ are the vortexes of a tetrahedron. Compute the length of the height from A.

20. Consider the orthogonal coordinate system xOy that corresponds to the orthonormal frame $R(O, \bar{\imath}, \bar{\jmath})$, the point $O'(2,3)_R$ and the orthonormal frame

$$R'(O', \left\{\bar{u}_1 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \bar{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}\right\}).$$

Let $x'O'y'$ the orthogonal coordinate system that corresponds to R'. Is $x'O'y'$ a left handed or a right handed orthonormal frame? If $M(1,5)_R$ find the coordinates of M relative to R'.