Discrete Mathematics and Graph Theory

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Topic : Logic

Warning : The following content is not verified. There might be some typos or grammatical errors, or even technical errors. Please bring to my notice if you find any. Thanks in advance.

Proposition or Statement :

A declarative sentence which is either true or false is called a **proposition or statement**.

Examples:

1. It rained yesterday.
2. Mr. X is a good singer.
3. The sky is not cloudy.
4. Oh my God.
5. Can I have your pen ?
6. Pass on that book please ?
7. So sweet.

Examples 1 to 3 are statements, and 4 to 7 are not statements.

Atomic : A statement that can't be further divided into statements.

Combined statement : A statement that is a combination of atomic statements.

Example :

1. The sky is cloudy.
2. The apples are red and sweet.
3. This topic is interesting, and so is the lecture.
4. The sky is blue and cloudy.

5. Online lectures are not interactive.

The statements 1 and 5 are atomic, and the rest are combined.

The English alphabets P, Q, R, etc., are used to denote statements and are called statement variables. The statement variables can be used to represent combined statements using some connectives such as AND, OR, NOT, etc. For example, sometimes we write, Rama is a good dancer and a singer. This statement is a combined statement. This can be denoted by a variable P. We can represent the same using two atomic statements Q: Ramu is a dancer, and R: Rama is a singer, as $Q \wedge R$ (Q and R).

Connectives :

The operators used to join two or more statements are called connectives. The following are a few connectives we frequently use.

Disjunction (OR) :

Let P and Q be two statements. Then the disjunction of P and Q is denoted by $P \vee Q$ or $P + Q$ or P OR Q, etc. The disjunction statement of P and Q takes the truth value *True* (T) if one of the variables P or Q is T, and *false* (F) only when both P and Q are false.

Truth Table :

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

1. P : Ramu is a good painter.
   Q : Ramu has good coding skill.

   $P \vee Q$: Ramu is a good painter or has good coding skills.

2. P : x is an odd integer.
   Q : x is a prime integer.

   $P \vee Q$: x is an odd integer or a prime.

3. P : An employee whose age is ≥ 40 is eligible for an executive position.
   Q : An employee whose experience is more than 6 year is eligible for an executive position.

   $P \vee Q$: The employee whose age is more than 40 or experience more than 6 years is eligible for executive position.

Conjunction (AND) :

Let P and Q be two statements. Then the conjunction of P and Q is denoted by $P \wedge Q$ or $P \cdot Q$ or P AND Q, etc. The truth value of the conjunctive statement is *true* (T) only when both variables are true, and takes *false* (F), otherwise.

Truth Table :

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

1. P : Ramu is a good painter
   Q : Ramu has good coding skill.

   $P \wedge Q$: Ramu is a good painter and has good coding skills.

2. P : x is an odd integer.
   Q : x is a prime integer.

   $P \wedge Q$: x is an odd integer and a prime.

3. P : An employee whose age is ≥ 40 is eligible for an executive position.
   Q : An employee whose experience is more than 6 years is eligible for an executive position.

   $P \wedge Q$: The employee whose age is more than 40 and whose experience more than 6 years is eligible for an executive position.

Conditional Statement or Implication :

Let P and Q be two statements. Then the conditional (implication) statement is denoted by $P \rightarrow Q$, and it is read as if P, then Q. The conditional statement is considered *false* only when Q is false and P is true. When P is false, irrespective of the value of Q, the conditional statement is considered true.

Truth Table :

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The implication can be interpreted as follows:

- Q is necessary for P and P is sufficient for Q.
- P, only if Q, etc.
- If a condition is true, then its consequence can never be false for a complete statement to be true.

- P is the antecedent, and Q is the consequent.

Examples :

1. P : Mr. X is a human.
   Q: Mr. X is a mortal.

   $P \to Q$: If Mr. X is a human, then Mr. X is mortal.

2. P : $x \neq 2$ is a prime number.
   Q : x is an odd integer.

   $P \to Q$: If x is a prime, then x is an odd integer.

3. P : Mr. X is an executive.
   Q : Mr. X is more than 40 years old.

   $P \to Q$: If Mr. X is an executive, then Mr. X is 40 or more years old.

4. (Fever $\wedge$ Cough) $\to$ Flu.

Biconditional Statement (Bi-implication) :

The Biconditional statement of P and Q is denoted as $P \leftrightarrow Q$, we read it as P if and only if Q, (we write it as P *iff* Q). The biconditional statement is *true* only when both P and Q have the same truth value.

Truth Table :

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

We will see later that $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$.

Example:

1. P : x is an odd number.
   Q : x is not divisible by 2.

   $P \leftrightarrow Q$: x is an odd *iff* x is not divisible by 2.

2. P : Mr. X is an executive.
   Q : Mr. X is 40+ age or 6+ year of experience.

   $P \leftrightarrow Q$: Mr. X is an executive *iff* Mr. X is 40 + or 6+ year experience.

Negation or NOT P :

The opposite statement of a given statement is called *NOT P* (or *Negation P*), and is denoted by $\neg P$.

Truth Table:

P	$\neg P$
T	F
F	T

$\neg P$: it is not the case of P.

Examples :

1. P : Mr. X is a human.

   $\neg P$: It is not the case that Mr. X is a human (or) Mr. X is not a human.

2. P : x is prime number.

   $\neg P$: x is not a prime.

3. P : Mr. X is an executive.

   $\neg P$: Mr. X is not an executive.

4. P : $x \neq 2$.

   $\neg P$: $x = 2$.

Exclusive OR (XOR) : Let P and Q be two statements. Then the Exclusive OR of P and Q is denoted by $P \oplus Q$ and is defined as either P or Q, i.e., $(P \wedge \neg Q) \vee (\neg P \wedge Q)$.

The difference between OR ($P \vee Q$) and XOR ($P \oplus Q$) is that OR includes AND but not XOR; i.e, $P \vee Q$ includes $P \wedge Q$ but not in $P \oplus Q$.

1. P : Ramu is a dancer.
   Q : Ramu is a Singer.

   $P \oplus Q$: Either Ramu is a dancer or a Singer, but not the both.

2. P : x is an odd integer.
   Q : x is a prime integer.

   $P \oplus Q$: Either x is an odd integer or a prime but not both.

Well Formed Formulas :

A statement formula is said to be a *well-formed formula (wff)* if the statement variables P and Q are connected by logical connectives according to the following rules.

Rule 1: The statement variables P, Q, etc, are *wff*, including $\neg P$

Rule 2: The basic connective statements, such as $P \vee Q$, $P \wedge Q$, $\neg P \vee Q$, $P \rightarrow Q$, $P \rightarrow \neg Q$, etc. are *wff*.

Rule 3: Any statement formula formed using the rules 1, 2 and parenthesis is *wff*.

Examples : The following are *wff*.

1. $P \wedge (P \rightarrow Q)$
2. $(P \vee Q) \wedge (\neg Q)$
3. $P \rightarrow (P \vee Q)$
4. $P \wedge Q \rightarrow P$ is not a *wff* as $(P \wedge Q) \rightarrow Q$ can be considered as $(P \wedge Q) \rightarrow P$ (or) $P \wedge (Q \rightarrow P)$

Tautology :

A statement formula that is always true T (for all truth values of its statement variables) is called a *tautology*. In simple words, the last column has all T in the truth table.

Example :

1. $P \vee \neg P \equiv T$
2. $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \equiv T$
3. $P \rightarrow (P \vee Q) \equiv T$

Contradiction (Fallacy) :

A statement formula that always has a truth value of False (F) for all truth values of its statement variables. In simple words, the last column has all F in the truth table.

Examples :

1. $P \wedge (\neg P) \equiv F$
2. $(P \wedge Q) \wedge (\neg P) \equiv F$
   Solution :

P	Q	$P \wedge Q$	$\neg P$	$(P \wedge Q) \wedge (\neg P)$
T	T	T	F	F
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Equivalence of Formula :

Two statement formulas P and Q are said to be equivalent, $P \equiv Q$, if the biconditional statement $P \leftrightarrow Q$ is a tautology. In the truth table, the last two columns corresponding to statement formulas P and Q, respectively, are the same.

Examples:

1. $\neg(\neg P) \equiv P$
2. $(P \rightarrow Q) \equiv (\neg P \vee Q)$
3. $(P \vee \neg P) \vee Q \equiv T$
4. $P \rightarrow (Q \rightarrow R) \equiv P \rightarrow (\neg Q \vee R) \equiv (P \wedge Q) \rightarrow R$
5. $P \rightarrow (\neg Q \vee R) \equiv P \rightarrow (Q \rightarrow R)$
6. $P \rightarrow (Q \rightarrow R) \equiv P \wedge Q) \rightarrow R$
7. $P \wedge Q) \rightarrow R \equiv P \rightarrow (\neg Q \vee R)$

Solution of 4. Consider,

$$\begin{aligned} P \rightarrow (Q \rightarrow R) &\equiv P \rightarrow (\neg Q \vee R) \\ &\equiv \neg P \vee (\neg Q \vee R) \\ &\equiv (\neg P \vee \neg Q) \vee R \qquad (\because DistributiveLaw) \\ &\equiv \neg(P \wedge Q) \vee R \qquad (\because De - Morgan'sLaw) \\ &\equiv (P \wedge Q) \rightarrow R \end{aligned}$$

One can even verify the same using a truth table as follows:

P	Q	R	$\neg Q$	$P \wedge Q$	$Q \rightarrow R$	$\neg Q \vee R$	$P \rightarrow (Q \rightarrow R)$	$(P \wedge Q) \rightarrow R$	$P \rightarrow (\neg Q \vee R)$
T	T	T	F	T	T	T	T	T	T
T	T	F	F	T	F	F	F	F	F
T	F	T	T	F	T	T	T	T	T
F	T	T	F	F	T	T	T	T	T
T	F	F	T	F	T	T	T	T	T
F	T	F	F	F	F	F	T	T	T
F	F	T	T	F	T	T	T	T	T
F	F	F	T	F	T	T	T	T	T

The last three columns are the same. Hence, the three statement formulas corresponding to the columns are equivalent.

Laws :

1. $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ (Associative law)
2. $P \vee Q \equiv Q \vee P$
   $P \wedge Q \equiv Q \wedge P$ (Commutative law)

3. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (Distributive law)

4. $P \vee (P \wedge Q) \equiv P$
   $P \wedge (P \vee Q) \equiv P$ (Absorption law)

5. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
   $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ (De'Morgan law's)

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Problem 1 : $(\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R)$
Solution : Consider,

$$\begin{aligned}
(\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)) &= (\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)) \\
&= ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)) \\
&= (\neg(P \vee Q) \wedge R) \vee ((P \vee Q) \wedge R)) \\
&= (\neg(P \vee Q) \vee (P \vee Q)) \wedge R \\
&= T \wedge R \\
&= R.
\end{aligned}$$

Problem 2 : Mr. Ramu has made the following two statements -
(1) I love the topic of logic.
(2) If I love the topic of logic, then I love Discrete Mathematics.

Find whether or not he loves discrete Mathematics, assuming that he always lies or never lies.

Solution : Let us denote the statements as follows :
P : Mr. Ramu loves the topic of logic.
Q : Mr. Ramu loves Discrete Mathematics.

Assume that Mr. Ramu never lies.
Then both statements are true.
i.e., P is true and $P \rightarrow Q$ is also true. Therefore Q must be true.

Assuming that Mr. Ramu is a liar.
Then P is false (F) and $P \rightarrow Q$ is also false (F).
However, $P \rightarrow Q$ can never be false when P is false.
Therefore Mr. Ramu never lies.
Hence, Mr. Ramu loves Discrete Mathematics.
Problem 3 : In an Island, there are two natives of tribes. One of them always lies, and the other never lies. You happen to arrive at the Island in search of a treasure of Gold. You asked a random person whom you met "Is there Gold on the Island ?" He replied, saying, "There is Gold iff I always tell the truth."

Find out whether Gold is there or not ? Also, find out which native tribe he is ?

Solution : Assume that
P : There is Gold on the Island.

Q : A native is a truth-teller.

Case I : Assume that the native is a truth-teller.
Then the statement made by him is true.
i.e., $P \leftrightarrow Q$ is true T.
Since $P \leftrightarrow Q$ is T, both P and Q are T or F together.
But Q is T, so P is T.

Hence, there is gold on the island.

Case II : Assume that the native is not a truth-teller.
Then the statement made by him is false.
i.e., $P \leftrightarrow Q$ is false F.
Since $P \leftrightarrow Q$ is false (F), either P or Q is false.
But Q is false, so P is true.

Hence, Gold on the Island.

He could be any native. We can't determine the type of native.

Tautological Implication :

A statement P is said to tautologically imply Q if $P \rightarrow Q$ is a tautology.
i.e., if P is true, then so does Q. It is denoted by $P \Rightarrow Q$.

Note : $\Rightarrow$ is not a logical connective.

Examples :

1. $P \wedge Q \Rightarrow P$:
2. $P \Rightarrow P \vee Q$
3. $(P \wedge (P \rightarrow Q)) \Rightarrow Q$

P	Q	$P \wedge Q$	$P \rightarrow Q$	$P \wedge (P \rightarrow Q)$	$P \wedge Q \rightarrow P$	$(P \wedge (P \rightarrow Q)) \rightarrow Q$	$P \rightarrow P \vee Q$
T	T	T	T	T	T	T	T
T	F	F	F	F	T	T	T
F	T	F	T	F	T	T	T
F	F	F	T	F	T	T	T

Methods of proofs :

Let P and Q be two statements. Then $P \rightarrow Q$ is a conditional statement. The other implications that can be drawn from $P \rightarrow Q$ are:

1. $Q \rightarrow P$, called converse.

2. $\neg P \to \neg Q$, called Inverse.

3. $\neg Q \to \neg P$, called Contrapositive.

Example :

Let P: A query uses an index, and Q: A query runs faster, be two statements. Then

Statement ($P \to Q$): If a query uses an index, then the query runs faster.

Converse ($Q \to P$): If a query runs faster, then it uses an index.

Inverse ($\neg P \to \neg Q$): If a query does not use an index, then it does not run faster.

Contrapositive ($\neg Q \to \neg P$): If a query does not run faster, then it does not use an index. This is logically equivalent to the original statement.

NOTE : One can verify that a statement is logically equivalent to its contrapositive. Similarly, the inverse of a statement is equivalent to the converse of the statement. i.e.,

$$P \to Q \equiv \neg Q \to \neg P \text{ and } Q \to P \equiv \neg P \to \neg Q,$$

can be verified as follows:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$Q \to P$	$\neg P \to \neg Q$	$\neg Q \to \neg P$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

There are two types of proofs -

1. Direct Proof : In a direct proof of $p(H_1, H_2, \ldots, H_n) \Rightarrow C$, we consider premises/hypothesis $H_i's$. Then we derive C logically.

2. Indirect Proof : Sometimes the hypothesis may not directly lead a conclusion. In such a case we use indirect methods.

There are two indirect methods -

1. Method of Contrapositive : $P \to Q \equiv \neg Q \to \neg P$

2. Method of Contradiction : $P(H_1, H_2, \ldots, H_n) \wedge (\neg C) \to F$.
   In the method of Contrapositive, instead of premises $P \to Q$, we prove $\neg Q \to \neg P$.

   In the method of Contradiction, we assume that $\neg C$ and $P(H_1, H_2, \ldots, H_n)$ are ture. Arrive at a contradiction F. Hence C is true.

Theory of Inference :

Rules of Inference (Principle of Reasoning) :

The Theory of deriving a conclusion from given premises or facts or axioms is called *rules of inference or inference theory.* The rules are expressed in the form of statements involved in a conclusion, which is derived from these rules and is called a valid conclusion.

1. $P \wedge (P \to Q) \Rightarrow Q$: Modus Ponens
2. $\neg Q \wedge (\neg Q \to \neg P) \Rightarrow \neg P$: Modus Tollens

Rules of inference :

Rule P : A premise(fact) can be inserted at any point in the derivation.

Rule T : If the formula q is a tautology implied by any one or more of the premise formulas, then q can be inferred.

Rule CP : If q is derived from p and a set of other premises, then we can derive $p \to q$ from the set of premises alone, i.e., Rule CP is used when the conclusion is $p \to q$.

Examples :

1. If it rains today, then I shall carry an umbrella.
It rained today.

Conclusion: I shall carry an umbrella.

2. P, T, Q is a valid conclusion from the premises $P \to Q$; $P \vee Q$; $\neg Q$

Solution :

$\{1\}$	(1)	$P \to Q$	Rule P
$\{1\}$	(2)	$\neg Q \to \neg P$	Rule T
$\{3\}$	(3)	$\neg Q$	Rule P
$\{1,3\}$	(4)	$\neg P$	Rule T
$\{5\}$	(5)	$P \vee Q$	Rule P
$\{1,3,5\}$	(6)	Q	Rule T

Problem : Prove that $\neg(P \wedge \neg Q), (\neg Q \vee R), \neg R \Rightarrow \neg P$
Solution :
<u>Direct Method</u>

$\{1\}$	(1)	$\neg Q \vee R$	Rule P
$\{1\}$	(2)	$Q \to R$	Rule T
$\{1\}$	(3)	$\neg R \to \neg Q$	Rule T
$\{4\}$	(4)	$\neg R$	Rule P
$\{1,4\}$	(5)	$\neg Q$	Rule T
$\{6\}$	(6)	$\neg(P \wedge \neg Q)$	Rule P
$\{6\}$	(7)	$\neg P \vee Q$	Rule T
$\{6\}$	(8)	$P \to Q$	Rule T
$\{6\}$	(9)	$\neg Q \to \neg P$	Rule T
$\{1,4,6\}$	(10)	$\neg P$	Rule T

Indirect Method

1.	P	$Rule P$	$(Assumption)$
2.	$\neg(P \wedge Q)$	$Rule P$	
3.	$(\neg P) \vee (\neg(\neg Q))$	$Rule T$	
4.	$\neg P \vee Q$	$Rule T$	
5.	$\neg P \vee Q \equiv P \to Q$	$Rule T$	
6.	Q	$Rule P$	
7.	$\neg Q \vee R$	$Rule P$	
8.	R	$Rule T$	
9.	$\neg R$	$Rule P$	
10.	$R \wedge \neg R = F$	$Rule T$	
11.	$\neg P$		

Problem : $(\neg P \vee Q, \neg Q \vee R, R \to S) \to (P \to S)$

Solution :

$\{1\}$	(1)	$\neg P \vee Q$	Rule P
$\{1\}$	(2)	$P \to Q$	Rule T
$\{3\}$	(3)	$\neg Q \vee R$	Rule P
$\{3\}$	(4)	$Q \to R$	Rule T
$\{5\}$	(5)	$R \to S$	Rule P
$\{3,5\}$	(6)	$Q \to S$	Rule T
$\{1,3,5\}$	(7)	$P \to S$	Rule CP

Problem : Validate the following conclusion from the given premises: If I study, then I will pass. If I don't go to a movie, then I will study. I failed. Therefore, I went to a movie.

Solution : Assume that

P: I study.
Q: I will pass.
R: I go to a movie.

The given statement can be expressed as -

If I study, then I will pass: $P \to Q$.
If I don't go to a movie, then I will study: $\neg R \to P$.
I failed: $\neg Q$.

Proof :

$\{1\}$	(1)	$\neg Q$	Rule P
$\{2\}$	(2)	$P \to Q$	Rule P
$\{2\}$	(3)	$\neg Q \to \neg P$	Rule T
$\{1, 2\}$	(4)	$\neg P$	Rule T .
$\{5\}$	(5)	$\neg R \to P$	Rule P
$\{5\}$	(6)	$\neg P \to R$	Rule T
$\{1, 2, 5\}$	(7)	R	Rule T

Problem : If a function terminates, then it releases memory. If memory is released, then no leak occurs.

Fact : The function terminates. **Conclusion :** No leakage.

Solution : Let
P : Function terminates
Q : Memory is released.
R : There is no leak.

Then we have $P \to Q$; $Q \to R$. So the fact P concludes R.

Problem : If the user is an admin, then access is granted.
Fact : Access is not granted .
Conclusion : The user is not an admin.

Problem: A query is processed either with an index or with a full table scan; If a query uses an index, then runtime ¡ 1 sec. If runtime ¡ 1 sec, then the system load is low.

Fact: The query didn't use the index.
What do you conclude from the given premises?

Solution: Let P: The query is processed,
Q: The query uses the index,
R: The query uses full table scan,
S: Runtime ¡ 1 sec,
T: System load is low.

Then we have $[(Q \wedge \neg R) \vee (\neg Q \wedge R)] \to P$; $Q \to S$; $S \to T$; $\neg Q$;
Proof :
1. $\neg Q$
2. $(Q \wedge \neg R) \vee (\neg Q \wedge R)$
3. R

Since $\neg Q$, so $\neg \to [(Q \wedge \neg R) \vee (\neg Q \wedge R)]$ (please check yourself). Therefore $\neg P$. Hence, the query is processed without knowing anything about the system load.

Normal Forms :

A statement formula $\mathbf{P}(P_1, P_2, \ldots, P_n)$, where P_i are statement variables. To find the truth value of the statement formula $\mathbf{P}$, we consider all possible truth values of $P_i's$, the number of which is 2^n. For all 2^n values of $P_i's$, if $\mathbf{P}$ is true, then $\mathbf{P}$ is a Tautology. If $\mathbf{P}$ is false for all variables $P_i's$, then $\mathbf{P}$ is a contradiction (Fallacy statement). However, sometimes, a statement formula **P** may not be a tautology nor a contradiction. In such cases, we may be interested in knowing truth values for which **P** is true. These problems are called *decision problems*. The truth table approach for the decision problem is laborious, and may not be simple. Therefore another approach called *Normal Forms*.

We now use Conjunction - Product - $\bigwedge$; Disjunctive - Sum - $\bigvee$ interchangeably.

A product of variables and their negation is called an *elementary product*.
Let P and Q be atomic variables, then P, $P\wedge\neg Q$, $\neg P\wedge Q$, $\neg P\wedge\neg Q$, $P\wedge\neg P$, etc. are elementary products.

Similarly, a sum is called an *elementary sum* if it contains variables and their negations.
Such as P, $P\vee\neg Q$, $\neg P\vee Q$, $\neg P\vee\neg Q$, $P\vee\neg P$, etc.

1. A necessary and sufficient condition for an elementary product to be false is that it contains atleast one pair a variable and its negation with product, i.e.,

$$P\wedge\neg P \text{ or } Q\wedge\neg Q \text{ or } R\wedge\neg R, \text{ etc.}$$

2. A necessary and sufficient condition for an elementary sum to be true is that it contain atleast one pair of variable and its negation with sum, i.e.,

$$P\vee\neg P \text{ or } Q\vee\neg Q \text{ or } R\vee\neg R, \text{ etc.}$$

Disjunctive Normal Form (DNF) or Sum of Product (SOP) :

A statement formula which contains the sum of products is called disjunctive normal form **DNF**.
To convert a statement formula into a DNF, the connectives $\rightarrow$ and $\leftrightarrow$ must be converted to formulas having $\vee$, $\wedge$, $\neg$ only, using $P\rightarrow Q\equiv\neg P\vee Q$, $P\leftrightarrow Q\equiv(P\rightarrow Q)\wedge(Q\rightarrow P)\equiv(P\wedge Q)\vee(\neg(P\vee Q))$.

Problem: Find a DNF of $P\wedge(P\rightarrow Q)$?
Solution:

$$\begin{aligned} P\wedge(P\rightarrow Q) &\equiv P\wedge(\neg P\vee Q) \\ &\equiv (P\wedge\neg P)\vee(P\wedge Q). \end{aligned}$$

Problem: Find a DNF of $\neg(P\vee Q)\leftrightarrow(P\wedge Q)$
Solution:

$$\begin{aligned} \neg(P\vee Q)\leftrightarrow(P\wedge Q) &\equiv (\neg(P\vee Q)\rightarrow(P\wedge Q))\wedge((P\wedge Q)\rightarrow\neg(P\vee Q)) \\ &\equiv ((P\vee Q)\vee(P\wedge Q))\wedge(\neg(P\wedge Q)\vee\neg(P\vee Q)) \\ &\equiv ((P\wedge Q)\vee(P\vee Q)\wedge\neg((P\wedge Q)\wedge(P\vee Q)) \\ &\equiv (P\vee P\vee Q)\wedge(Q\vee P\vee Q)\wedge\neg(P\wedge Q\wedge P)\vee(P\wedge Q\wedge Q) \\ &\equiv (P\vee Q)\wedge(P\vee Q)\wedge\neg((P\wedge Q)\vee(P\wedge Q)) \\ &\equiv (P\vee Q)\wedge(P\vee Q)\wedge\neg(P\wedge Q)\wedge\neg(P\wedge Q) \\ &\equiv (P\vee Q)\wedge\neg(P\wedge Q) \\ &\equiv (P\wedge\neg(P\wedge Q))\vee(Q\wedge\neg(P\wedge Q)) \\ &\equiv (P\wedge\neg P)\vee(P\wedge\neg Q))\vee(Q\wedge\neg P). \end{aligned}$$

Note : DNF a formula need not be unique.

Example : $P \vee (Q \wedge R)$ is a SOP, and can also be written as

$$\begin{aligned}
P \vee (Q \wedge R) &\equiv (P \vee Q) \wedge (P \vee R) \\
&\equiv ((P \vee Q) \wedge P) \vee ((P \vee Q) \wedge R) \\
&\equiv (P \wedge P) \vee (P \wedge Q) \vee (P \wedge R) \vee (Q \wedge R).
\end{aligned}$$

A given statement formula is false if every elementary product is false in the SOP. So the assumption to be false, the SOP or DNF shall contain a product $(P \wedge \neg P)$, a variable conjuction with its negation.

Conjunctive Normal Form (CNF) or Product of Sums (POS) :

A statement formula which contains the product of sums is called conjunctive normal form **CNF**.

Examples:

1. $P \wedge (P \to Q) \equiv P \wedge (\neg P \vee Q)$.
2. $\neg(P \vee Q) \leftrightarrow (P \wedge Q) \equiv (P \vee Q) \wedge (\neg P \vee \neg Q)$ as

$$\begin{aligned}
\neg(P \vee Q) \leftrightarrow (P \wedge Q) &\equiv (\neg(P \vee Q) \to (P \wedge Q)) \wedge ((Q \wedge P) \to \neg(P \vee Q)) \\
&\equiv ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(Q \wedge P) \vee \neg(P \vee Q)) \\
&\equiv (P \vee Q \vee \ P) \wedge (Q \vee P \vee Q) \wedge \neg((P \wedge Q) \wedge (P \vee Q)) \\
&\equiv (P \vee Q) \wedge (P \vee Q) \wedge \neg((P \wedge Q) \vee (P \wedge Q)) \\
&\equiv (P \vee Q) \wedge (\neg P \vee \neg Q)
\end{aligned}$$

3. $P \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) = T$ as

$$\begin{aligned}
P \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) &\equiv (P \vee Q) \wedge (Q \vee \neg Q) \vee \neg(P \vee Q) \\
&\equiv (P \vee Q) \wedge T \vee \neg(P \vee Q) \\
&\equiv (P \vee Q) \wedge \neg(P \vee Q) = T
\end{aligned}$$

NOTE : CNF also need to be unique.

Principal DNF or Canonical DNF or Sum of product canonical form :

Let P and Q be two statements. Then the products $P \wedge Q$, $\neg P \wedge Q$, $p \wedge \neg Q$, $\neg P \wedge \neg Q$ are called *min-terms*. These are the only four min-terms of two variables P and Q.
If there are three variables, then there are 2^3 min-terms $P \wedge Q \wedge R$, $\neg P \wedge Q \wedge R$, $P \wedge \neg Q \wedge R$, $P \wedge Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$, $\neg P \wedge Q \wedge \neg R$ and $\neg P \wedge \neg Q \wedge \neg R$.

A statement formula can be represented as a sum of products of min-terms. Such a representation is called *principal or canonical sum of products*, and is unique up to the ordering of the min-terms.

Examples:

1. $P \to Q \equiv (\neg P \vee Q)$

2. Find the Canonical sum of products of the statement formula $P \vee Q$?
Solution :

$$\begin{aligned} P \vee Q &\equiv (P \vee Q) \wedge T \\ &\equiv (P \vee Q) \wedge (P \vee \neg P) \\ &\equiv P \vee (Q \wedge \neg P) \\ &\equiv (P \wedge T) \vee (Q \wedge \neg P) \\ &\equiv [P \wedge (Q \vee \neg Q)] \vee (Q \wedge \neg P) \\ &\equiv (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \end{aligned}$$

Note: To obtain the Canonical DNF, apply $\wedge T$ to the variable not having the other variable as a pair.

3. Find the Canonical sum of products of the statement formula $P \to [(P \to Q) \wedge (\neg(\neg Q \vee \neg P))]$?
Solution :

$$\begin{aligned} P \to [(P \to Q) \wedge (\neg(\neg Q \vee \neg P))] &\equiv \neg P \vee ((\neg P \wedge P \wedge Q) \vee (Q \wedge P \wedge Q) \\ &\equiv \neg P \vee (\neg P \wedge P \wedge Q) \vee (Q \wedge P \wedge Q) \\ &\equiv \neg P \vee (F) \vee (P \wedge Q) \\ &\equiv (\neg P \wedge T) \vee (P \wedge Q) \\ &\equiv (\neg P \wedge T) \vee (P \wedge Q) \\ &\equiv [\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q) \\ &\equiv (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \end{aligned}$$

Connection of min-terms and Truth Table:

P	Q	$P \wedge Q$	$P \to Q$	$(P \to Q) \wedge (P \wedge Q)$	$P \to [(P \to Q) \wedge (P \wedge Q)]$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	F	T	F	T
F	F	F	T	F	T

In the truth table, the rows corresponding to T in the last column are min-terms formed by taking the variable in place of 'T' and negation of the variable in place of 'F'. For example, the first row is corresponding to the min-term $P \wedge Q$. Similarly, the third and fourth rows correspond to $\neg P \wedge Q$ and $\neg P \wedge \neg Q$, respectively. The sun of these min-term is the given statement formula.

4. Find the PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$?
Solution :

$$\begin{aligned} (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R) &\equiv ((P \wedge Q) \wedge T) \vee ((\neg P \wedge R) \wedge T) \vee (T \wedge (Q \wedge R) \\ &\equiv [(P \wedge Q) \wedge (R \vee \neg R)] \vee [(\neg P \wedge R) \wedge (Q \vee \neg Q)] \\ &\vee [(P \vee \neg P) \wedge (Q \wedge R)] \\ &\equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q) \\ &\vee (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R) \\ &\equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \end{aligned}$$

5. $P \vee (P \wedge Q) \leftrightarrow P$
Solution :

$$\begin{aligned}
P \vee (P \wedge Q) &\equiv (P \wedge T) \vee (P \wedge Q) \\
&\equiv (P \wedge (Q \vee \neg Q)) \vee (P \wedge Q) \\
&\equiv (P \wedge Q) \vee (P \wedge \neg Q) \vee (P \wedge Q) \\
&\equiv (P \wedge Q) \vee (P \wedge \neg Q) \\
&\equiv P \wedge T \\
&\equiv P \wedge (Q \vee \neg Q) \\
&\equiv (P \wedge Q) \vee (P \wedge \neg Q).
\end{aligned}$$

Truth Table :

P	Q	$P \wedge Q$	$P \wedge (P \wedge Q)$	min-term
T	F	F	T	$P \wedge \neg Q$
T	T	T	T	$P \wedge Q$
F	T	F	F	-
F	F	F	F	-

Principal Product of sums or Canonical CNF :

Similar to the min-terms, we have max terms (which are the sums) of P and Q as $(\neg P \vee \neg Q)$, $(P \vee \neg Q)$, $(\neg P \vee Q)$ and $(P \vee Q)$. So, the canonical CNF is the product of max terms.

Problem: Find the PCNF of $(\neg P \rightarrow R) \wedge (P \leftrightarrow Q)$?

Solution :

$$\begin{aligned}
(\neg P \rightarrow R) \wedge (P \leftrightarrow Q) &\equiv (\neg P \rightarrow R) \wedge (P \rightarrow Q) \wedge (Q \rightarrow P) \\
&\equiv (P \vee R) \wedge (\neg P \vee Q) \wedge (\neg Q \vee P) \\
&\equiv ((P \vee R) \vee F) \wedge (\neg P \vee Q \vee F) \wedge (P \vee \neg Q \vee F) \\
&\equiv [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [\neg P \vee Q \vee (R \wedge \neg R)] \vee [(P \vee \neg Q) \vee (R \wedge \neg R)] \\
&\equiv (P \vee Q \vee R) \wedge (P \vee R \vee \neg Q) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \\
&\wedge \;\; (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)
\end{aligned}$$

Truth Table :

P	Q	R	$\neg P$	$\neg P \rightarrow R$	$P \leftrightarrow Q$	$(\neg P \rightarrow R) \wedge (P \leftrightarrow Q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
F	T	T	T	T	F	F
T	F	F	F	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

In the above truth table, the rows corresponding to the F in the last column are max terms, which are formed by taking variables in the place 'F' and negation of the variables in place 'T', i.e., the third row is corresponding to the third row $(\neg P \vee Q \vee \neg R)$, and similarly, $(P \vee \neg Q \vee \neg R)$, $(\neg P \vee Q \vee R)$, $(P \vee \neg Q \vee R)$ and $(P \vee Q \vee R)$ correspond to the fourth, fifth, sixth and eighth rows, respectively.

Uniqueness of PDNF and PCNF :

Let P, Q, R be three variables. Then there are 8 min-terms and 8 max-terms. The min-terms are ordered as $m_0, m_1, \cdots, m_7$ and the max-terms are ordered as $M_0, M_1, \cdots, M_7$, where m_i and M_i are the binary representations of i in 3 bits. The corresponding min-term is obtained by replacing 1 by the variable and 0 by the negation of the corresponding variable. For example, m_3 is the binary representation of 3, i.e., 011. The corresponding min-term is $\neg P \wedge Q \wedge R$. Similarly, the max-term is obtained by interchanging the roles of 0 and 1, i.e., the max-term corresponding to $M_3 = 011$ is $P \vee \neg Q \vee \neg R$.

m_i	binary i	min-term	M_i	binary i	max-term
m_7	111	$P \wedge Q \wedge R$	M_7	111	$\neg P \vee \neg Q \vee \neg R$
m_6	110	$P \wedge Q \wedge \neg R$	M_6	110	$\neg P \vee \neg Q \vee R$
m_5	101	$P \wedge \neg Q \wedge R$	M_5	101	$\neg P \vee Q \vee \neg R$
m_4	100	$P \wedge \neg Q \wedge \neg R$	M_4	100	$\neg P \vee Q \vee R$
m_3	011	$\neg P \wedge Q \wedge R$	M_3	011	$P \vee \neg Q \vee \neg R$
m_2	010	$\neg P \wedge Q \wedge \neg R$	M_2	010	$P \vee \neg Q \vee R$
m_1	001	$\neg P \wedge \neg Q \wedge R$	M_1	001	$P \vee Q \vee \neg R$
m_0	000	$\neg P \wedge \neg Q \wedge \neg R$	M_0	000	$P \vee Q \vee R$

We can notice here that the negation of a min-term is a max-term and vice-versa, i.e., $\neg m_i = M_i$.

A statement formula can be represented in both PDNF and PCNF. So, PCNF can be obtained from PDNF, and vice versa, as follows:

first represent the given formula in PDNF. To find the PCNF, write the min-terms which are not present in the PDNF as a PDNF and apply the negation. Similarly, to obtain PDNF from PCNF, write the missing max-terms as a PCNF and apply the negation.

Example : The PCNF of $(\neg P \to R) \wedge (P \leftrightarrow Q)$ is $(P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) = M_0 \wedge M_2 \wedge M_3 \wedge M_4 \wedge M_5$. The missing max-terms are M_1, M_6 and M_7, so apply the negation on $M_1 \wedge M_6 \wedge M_7$, i.e.,

$$\begin{aligned}
\neg(M_1 \wedge M_6 \wedge M_7) &= \neg M_1 \vee \neg M_6 \vee \neg M_7 \\
&= m_1 \vee m_6 \vee m_7 \\
&= (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R).
\end{aligned}$$

We can check that

$$(\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R) \quad = \quad (\neg P \wedge \neg Q \wedge R) \vee [(P \wedge Q) \vee (\neg P \wedge \neg Q)] \wedge R$$

Predicate Calculas :

We have discussed so far the compound statements with connectives, but we have not considered any common feature of statements. The logic based upon the analysis of prediction in any statement is called *Predicate logic*

Predicates : A feature that is common among all the statements.
Example :

1. Rama is a student.
   Ravi is a student
   " is student " is called predicate.

2. All human are mortal.
   John is human being.
   Therefore John is mortal.
   "is mortal" is a predicate.
   x is greater than y, x is taller than y, are **2-place** predicates.
   a boy, is red, an animal, etc., are **1-place** predicates.

Predicate are denoted by capital letters. Predicate is not a statement, so it can't have a truth value. Predicate is a feature/characteristic of an object. A predicate with an individual object describes the feature/characteristic of that object.

Example:

1. This is a cat.
2. This fruit is an apple.
3. Ramu is a boy.
4. 5 is greater than 2.
5. Ramu is taller than Sonu.

The predicate of the above examples is :

1. $C(x)$: Being a cat (or) x is cat.
2. $A(x)$: Apple $|x$ is an apple.
3. $B(x)$: Being boy (boys) (or) x is a boy.
4. $G(x, y)$: x is greater than y.
5. $T(x, y)$: x is taller than y,

– So $C(x)$ is not a statement and so $C(x)$ has no truth value. However, $C(a)$, a is a cat is a statement. So $C(a)$ has a truth value. Similarly, A(Ramu): Ramu is a boy; $G(5, 2)$: 5 is greater than 2. T(Ramu, Sonu): Ramu is taller than Sonu; these are statements.

Predicates with connectives and Quantifiers :

Two or more predicates can be connected by connectives, such as conjunction, disjunction, implication, bi-implication, not, etc.

Quantifiers: The predicates are common features of objectives. When two or more predicates are connected through connectives, the common features are applicable to some objects or all the objects under consideration.

If a predicate is referring to all the objects under consideration, we use *universal quantifier*. It is denoted by $\forall$. The universal quantifiers are *for all*, *each*, *any* or *every*, etc.

If a predicate is referring to some specific objects under consideration, we use *existential quantifier*. It is denoted by $\exists$. The existential quantifiers are *for some*, *there are*, *a* or *a few*, etc.

Examples: Consider the following predicates.

- $S(x)$: x is a student
- $GM(x)$: x is good in mathematics
- $AI(x)$: x is an AI system
- $H(x)$: x is a human
- $M(x)$: x is mortal
- $P(x)$: x is being perceived
- $A(x)$: x acts
- $LA(x)$: x has learning ability

All students are good in mathematics : $\forall x\ (S(x) \to GM(x))$.

Some students are good in mathematics : $\exists x\ (S(x) \wedge GM(x))$.

Some students are not good in mathematics : $\exists x\ (S(x) \wedge \neg GM(x))$.

No student is good in mathematics : $\forall\, x\ (S(x) \to \neg GM(x))$.

Not all students are good in mathematics : $\neg\forall\, x\ (S(x) \to GM(x)) \equiv \exists x\ (S(x) \wedge \neg GM(x))$.

If someone is human, then they are mortal : $\forall\, x\ (H(x) \to M(x))$.

An AI system perceives and acts : $\forall\, x\ (AI(x) \to (P(x) \wedge A(x)))$.

An entity is AI if and only if it has learning ability : $\forall\, x\ (AI(x) \leftrightarrow LA(x))$.

Implication vs Conjunction :

In the first example, all students are good in mathematics is written as $\forall\, x\ (S(x) \to GM(x))$ but not $\forall x(S(x) \wedge GM(x))$. Because $\forall x(S(x) \wedge GM(x))$ is true only when both $S(x)$ and $GM(x))$ are true. So $\forall x(S(x) \wedge GM(x))$ is only true for students. It is not saying anything about the persons who are not students but are good at mathematics. This is exclusive. But $\forall x\ (S(x) \to GM(x))$ is true even if $S(x)$ is false. This is inclusive.

Similarly, in the second example, some students are good in mathematics is presented as $\exists x\ (S(x) \wedge GM(x))$ but not $\exists x\ (S(x) \to GM(x))$. Because $\exists x\ (S(x) \wedge GM(x))$ is false only when both $S(\text{dog})$ and $GM(\text{dog}))$ are false. But $\exists x\ (S(x) \to GM(x))$ is true even if both $S(\text{dog})$ and $GM(\text{dog}))$ are false, this might not infer the actual meaning in cases. For example, if we assume x as a dog, then $S(\text{dog}) \to GM(\text{dog}))$ is read as a dog is good in mathematics. This is true as per the truth values of $S(\text{dog})$ and $GM(\text{dog}))$, which are false.

We compare the same using truth tables

$$\exists x\, (S(x) \to GM(x)) \quad \text{vs} \quad \exists x\, (S(x) \wedge GM(x)).$$

Truth Table for $S(x) \to GM(x)$

$S(x)$	$GM(x)$	$S(x) \to GM(x)$
T	T	T
T	F	F
F	T	T
F	F	T

We can see that $S(x) \to GM(x)$ is true even when $S(x)$ and $GM(x)$ are false. Truth Table for $S(x) \wedge GM(x)$ is

$S(x)$	$GM(x)$	$S(x) \wedge GM(x)$
T	T	T
T	F	F
F	T	F
F	F	F

$S(x) \wedge GM(x)$ is true only when $S(x)$ and $GM(x)$ are true, but not in any other case.

Another reasoning we can describe to justify the above is using verifying the negation of the given predicates, i.e., the negation of *some students are good in mathematics* is *it is not true that some students are good in mathematics*. This is equivalent to saying *no student is good in mathematics*. Let's convert them to predicates:

It is not true that some students are good in mathematics : $\neg(\exists x\ (S(x) \wedge GM(x)))$.

No student is good at mathematics : $\forall x(S(x) \to \neg GM(x))$.

We can see that $\neg(\exists x\ (S(x) \wedge GM(x))) \equiv \forall x(\neg(S(x) \wedge GM(x))) \equiv \forall x(\neg S(x) \vee \neg GM(x)) \equiv \forall x(S(x) \to \neg GM(x))$.

However, if we look at the negation of $\exists x(S(x) \to GM(x))$, we have $\neg(\exists x(S(x) \to GM(x))) \equiv \neg(\exists x(\neg S(x) \vee GM(x))) \equiv \forall x(\neg(\neg S(x) \vee GM(x))) \equiv \forall x(S(x) \wedge \neg GM(x))$, which says only students are not good at mathematics, but no others.

Universe of discourse : If the objects or predicates are restricted to a specific class, then such a class is called *universe of discourse*. Usually, in many practical problems, the objects under consideration will always be of a specific class rather than generic. In the first and second examples, considering x as a human is more appropriate than x as a generic variable such as animals, rocks, etc. The class of humans is called the universe of discourse.

If the domain includes non-students, then $\exists x(S(x) \to GM(x))$ may be satisfied by a *non-student*, which is not intended. So the correct form is $\exists x(S(x) \wedge GM(x))$. However, if the domain is restricted to only students, then both formulas become equivalent, i.e., if the universe of discourse is students, then $S(x)$ can never be false, so we don't need a predicate to represent a student. Hence $\exists x(S(x) \to \neg GM(x)) \equiv \exists x(S(x) \wedge \neg GM(x)) \equiv \neg GM(x)$.

Exercises :

Translate each natural language sentence into predicate logic defining universe of discourse.

1. All students are good in mathematics.
   $\forall\, x\, GM(x)$, $\quad$ $x \in$ universe of discourse: Students.

2. Some students are not good in mathematics.
   $\exists\, x\, \neg GM(x)$, $\quad$ $x \in$ universe of discourse: Students.

3. No student is good in mathematics.
   $\forall\, x \neg GM(x)$, $\quad$ $x \in$ universe of discourse: Students.

4. Every human is mortal: $\forall\, x\, M(x)$, x is a human.

5. Some AI systems are not good in mathematics.
   $\exists\, x\, \neg GM(x)$, x is being only an AI system, i.e., universe of discourse: AI Systems.

2-Predicates :

- $Student(x)$: x is a student
- $GoodInMath(x)$: x is good in mathematics
- $AI(x)$: x is an AI system
- $Human(x)$: x is a human
- $Mortal(x)$: x is mortal
- $Likes(x, y)$: x likes y
- $Teaches(x, y)$: x teaches y

Basic Exercises :

1. All students are good in mathematics.
2. Some students are not good in mathematics.
3. No student is good in mathematics.
4. Not all students are good in mathematics.
5. Every human is mortal.
6. Some AI systems are not good in mathematics.
7. If someone is a student, then they are human.
8. An entity is an AI if and only if it can learn.

Advanced Exercises (Nested Quantifiers) :

9. Every student likes some AI system.
10. There exists an AI system that all students like.
11. Every student is taught by at least one human.
12. There is a human who teaches all students.
13. Every AI system is disliked by some student.
14. Not every student likes every AI system.

Solutions :

Basic Solutions :

1. $\forall x\ (Student(x) \rightarrow GoodInMath(x))$
2. $\exists x\ (Student(x) \land \neg GoodInMath(x))$
3. $\forall x\ (Student(x) \rightarrow \neg GoodInMath(x))$
4. $\neg\forall x\ (Student(x) \rightarrow GoodInMath(x)) \equiv \exists x\ (Student(x) \land \neg GoodInMath(x))$
5. $\forall x\ (Human(x) \rightarrow Mortal(x))$
6. $\exists x\ (AI(x) \land \neg GoodInMath(x))$
7. $\forall x\ (Student(x) \rightarrow Human(x))$
8. $\forall x\ (AI(x) \leftrightarrow CanLearn(x))$

Advanced Solutions :

9. $\forall x\ (Student(x) \to \exists y\ (AI(y) \wedge Likes(x,y)))$

10. $\exists y\ (AI(y) \wedge \forall x\ (Student(x) \to Likes(x,y)))$

11. $\forall x\ (Student(x) \to \exists y\ (Human(y) \wedge Teaches(y,x)))$

12. $\exists y\ (Human(y) \wedge \forall x\ (Student(x) \to Teaches(y,x)))$

13. $\forall y\ (AI(y) \to \exists x\ (Student(x) \wedge \neg Likes(x,y)))$

14. $\neg \forall x\ \forall y\ ((Student(x) \wedge AI(y)) \to Likes(x,y))$

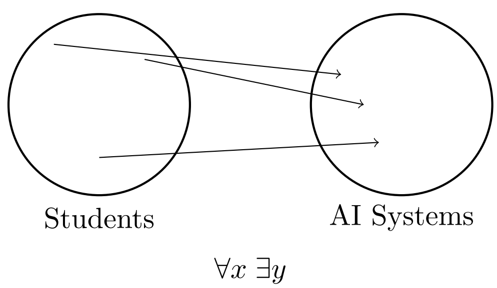
Visualizing Quantifier Order : $\forall\exists$ vs $\exists\forall$

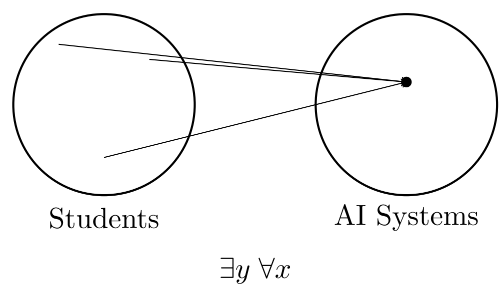
Example Statements :

- $\forall x\,(Student(x) \to \exists y\,(AI(y) \wedge Likes(x,y)))$
  "Every student likes some AI system."

- $\exists y\,(AI(y) \wedge \forall x\,(Student(x) \to Likes(x,y)))$
  "There exists an AI system that all students like."

Diagrams :





Observation :

- $\forall\, x\ \exists y$: Each student may like a *different* AI system.

- $\exists y\ \forall\, x$: There is *one special AI system* that all students like.

- For any positive integer, there is a greater positive integer.
  $\forall\, x(p(x) \to \exists\, y(p(x) \wedge G(x,y)) \equiv \forall\ \ x\,\exists\, y\, G(x,y),\ x,y \in \mathbb{Z}$

- Some student of this class are taller than all other students.
  $\exists\, x(C(x) \wedge \forall\ y\ (C(y) \to\ G(x,y)) \equiv \exists\, x\,\forall\, y, x, y \in$

- Each student is good in exactly one subject.
  $\forall\, x(c(x) \to (\exists y G(x,y) \to \forall z(z \neq y) \neg G(x,y) \equiv \forall\, x(\exists y G(x,y) \to \forall\, z(z \neq y) \neg G(x,z) x \in \{cse\}$

Write the equivalent English statement.
Let $x, y \in IIITNR$ such that Q(x,y) :x has sent a letter to y.

1. $\exists\, x \exists\, y Q(x,y)$ Some student of IIITNR have sent letter to some other students.

2. $\exists\, x \forall\, y Q(x, y)$ Some student have sent letter to all the students.

3. $\forall\, x \exists\, y$Q(x,y)All the student have sent letter to some students.

4. $\exists\, y \exists\, x$Q(x,y) Some student has sent letter to some students.

5. $\exists\, y \exists\, x$Q(x,y) same as (1).

6. $\exists y \forall x$G(x,y) some student have sent letter to all students.

$\neg(\exists\, y \forall\, x p(x, y)) \Leftrightarrow \forall\, x \exists\, y \neg p(x, y)$

Let us consider x is student and y be the course offered by institution then T(x,y) : x has taken course y.

1. x: students

2. y: Courses

3. T(x,y): x takes y.

All possible cases.

1. $\exists x\, \forall\, y, T(x, y)$: Some student takes all courses.

2. $\forall\, x \exists\, y, T(x, y)$: Each student take some course.

3. $\exists\, x \forall\, y \neg T(x, y)$ Some student are not taking all course.

4. $\exists\, y \forall\, x \neg T(x, y)$:Some course are not being taken any student.

5. $\neg(\exists\, y \forall\, x, T(x, y)) \equiv \forall\, y \exists\, x \neg T(x, y)$: Its is not the case that some course are being taken by all student $\equiv$ All course are not being taken by some students.

Rule of inference for Predicate Calculus :

- **Rule P**: Assume any premise at any time.

- **Rule US**(Universal Specification): For a given premise $\forall x$ P(x), P(a) is true $\forall a$.

- **Rule ES**(Existential Specification):For a given premise $\exists x P(x)$, P(a) is true for some a in the universe.

- **Rule UG**(Universal Generalizaton): if a premise is true for arbitrary a in the universe of discourse, then $\forall x\ P(x)$.

- **Rule EG** (Existential Generalization): If $P(a)$ is true for some a, then $\exists x\ P(x)$.

Demorgan Law :

Let $S = \{a_1, a_2, \ldots, a_n\}$ be the universe of disclosure.Then we can see that :

$$\forall\, x P(x) \equiv P(a_1) \wedge P(a_2) ... \wedge P(a_n) \tag{1}$$
$$\exists\, x P(x) \equiv P(a_1) \vee P(a_2) ... \vee P(a_n) \tag{2}$$

Now the negation of (1) and (2) is.

$$\neg(\wedge_{i=1}^{n} P(a_i)) \equiv \vee_{i=1}^{n} \neg P(a_i) \tag{3}$$
$$\Leftrightarrow \neg(\forall\, x\ P(x)) \equiv \exists x\ P(x) \tag{4}$$

Similarly,

$$\neg(\exists x P(x)) \equiv \forall x P(x) \tag{5}$$

Existential Specification (ES) : $\exists\, xP(x) \Rightarrow P(a)$ for some 'a'
Universal Generalization (UG) : P(a)for any a $\Rightarrow \forall x P(x)$
Note :

1. $\exists\, x(A(x) \vee B(x)) \equiv \exists A(x) \vee \exists\, xB(x)$
   $\exists\, x(A(x) \wedge B(x)) \equiv \exists A(x) \wedge \exists\, xB(x)$
2. $\forall\, x(A(x) \vee B(x)) \equiv \forall\, xA(x) \vee \forall\, xB(x)$
   $\forall\, x(A(x) \wedge B(x)) \equiv \forall\, xA(x) \wedge \forall\, xB(x)$
3. $\forall\, xA(x) \rightarrow B(x) \equiv \exists\, xA(x) \rightarrow B$
4. $\exists\, xA(x) \rightarrow B \equiv \forall\, xA(x) \rightarrow B$
5. $A \rightarrow \forall\, xB(x) \equiv \forall\, x(A \rightarrow B(x))$
6. $A \rightarrow \exists\, xB(x) \equiv \exists\, x(A \rightarrow B(x))$

Tautology		**Inference**
$P(a) \rightarrow \forall x P(x)$	: a is arbitrary	**UG**
$P(a) \rightarrow \exists P(x)$	: a is not arbitary	**EG**
$\forall x P(x) \rightarrow P(a)$	: a is arbitary	**US**
$\exists x P(x) \rightarrow P(a)$	:some a	**ES**

Problem : $\neg(\exists\, xp(x) \wedge Q(a)) \Rightarrow \exists\, x \quad p(x) \rightarrow \neg Q(a)$.
Solution :

1	$\neg(\exists\, xp(x) \wedge Q(a))$	Rule P
2	$\forall\, x(\neg p(x) \vee \neg Q(a))$	Rule T
3	$\neg p(a) \vee \neg Q(a)$	Rule US
4	$p(a) \rightarrow \neg Q(a)$	Rule T
5	$\exists\, xp(x) \rightarrow \neg Q(a)$	Rule EG

Problem :$\forall x(\neg P(x) \rightarrow Q(x)), \forall x(\neg Q(x) \Rightarrow P(a)$
Solution :

1	$\forall x(\neg Q(x))$	Rule P
2	$\neg Q(x)$	Rule US
3	$\forall(\neg P(x) \rightarrow Q(x)$	Rule P
4	$\forall x(\neg Q(x) \rightarrow P(x))$	Rule T
5	$\neg Q(a) \rightarrow P(a)$	Rule US
6	$P(a)$	Rule T.

Problem : All computer science students are good programmers. Mr. Ramu is not a good programmer. Therefore, Mr. Ramu is not a computer science student.
Solution :

1. $C(x)$: x is computer science student .
2. $GP(x)$: x is a good programmer.

$\forall x\, C(x) \to GP(x)$
$\neg GP(Ramu)$
$- -$
$\therefore C(Ramu)$.

OR

$\{1\}$	1	$\forall x(c(x) \to G(x)$	Rule P
$\{1\}$	2	$C(Ramu) \to Gp(Ramu)$	Rule US
$\{1\}$	3	$\neg GP(Ramu) \to \neg C(Ramu)$	Rule T
$\{2\}$	4	$\neg GP(Ramu)$	Rule P
$\{1, 2\}$	5	$\neg C(Ramu)$	Rule T

Problem : One student in this class knows how to write a programme in JAVA.Everyone who knows how to write a JAVA programme gets a job.Someone in this class can get a high paying job.
Solution : let us define two functions J(x) and H(x) such that ,

- J(x) : x write JAVA programme.
- H(x) : x gets high paying Job.

$\exists x J(x)$	Rule P
$J(a)$	Rule ES
$\forall\, x(J(x) \to H(x))$	Rule P
$J(a) \to H(a)$	Rule US
$H(a)$	Rule T
$\exists\, x H(x)$	

OR

$$\exists x(H(x) \wedge J(x)) \tag{6}$$
$$\forall\, x(J(x) \to H(x)) \tag{7}$$
$$\Rightarrow \exists\, x(H(x) \wedge H(x) \tag{8}$$

Problem : $\exists\, x(F(x) \wedge S(x)) \to \forall(M(y) \to W(y)) \Rightarrow \exists\, y(M(y) \wedge W(y))$.
Solution :

$\{1\}$	1	$\exists\, y(M(y) \wedge \neg W(y)$	Rule P
$\{1\}$	2	$M(z) \wedge \neg W(z)$	Rule ES
$\{1\}$	3	$\neg(\neg M(z) \vee W(z))$	Rule GT
$\{1\}$	4	$\neg(M(z) \to W(z))$	Rule GT
$\{1\}$	5	$\exists\, y(M(y) \to W(y))$	Rule EG
$\{1\}$	6	$\neg(\forall y(M(y) \to W(y))$	Rule T
$\{2\}$	7	$\exists\, x(F(x) \wedge S(x)) \to \forall y(M(y) \to W(y)$	Rule P
$\{2\}$	8	$\neg(\forall y(M(y) \to W(y)) \to \neg(\exists x F(x) \wedge S(x))$	Rule T
$\{1\}$	9	$\neg\exists\, x(F(x) \wedge S(x))$	Rule T