#Class 2

Mathematical Modeling

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Outline of #class2

- Overview and recap
- Modeling for optimizing: Mathematical Programming
- Steps in modeling
- Real-world examples

Before we start…

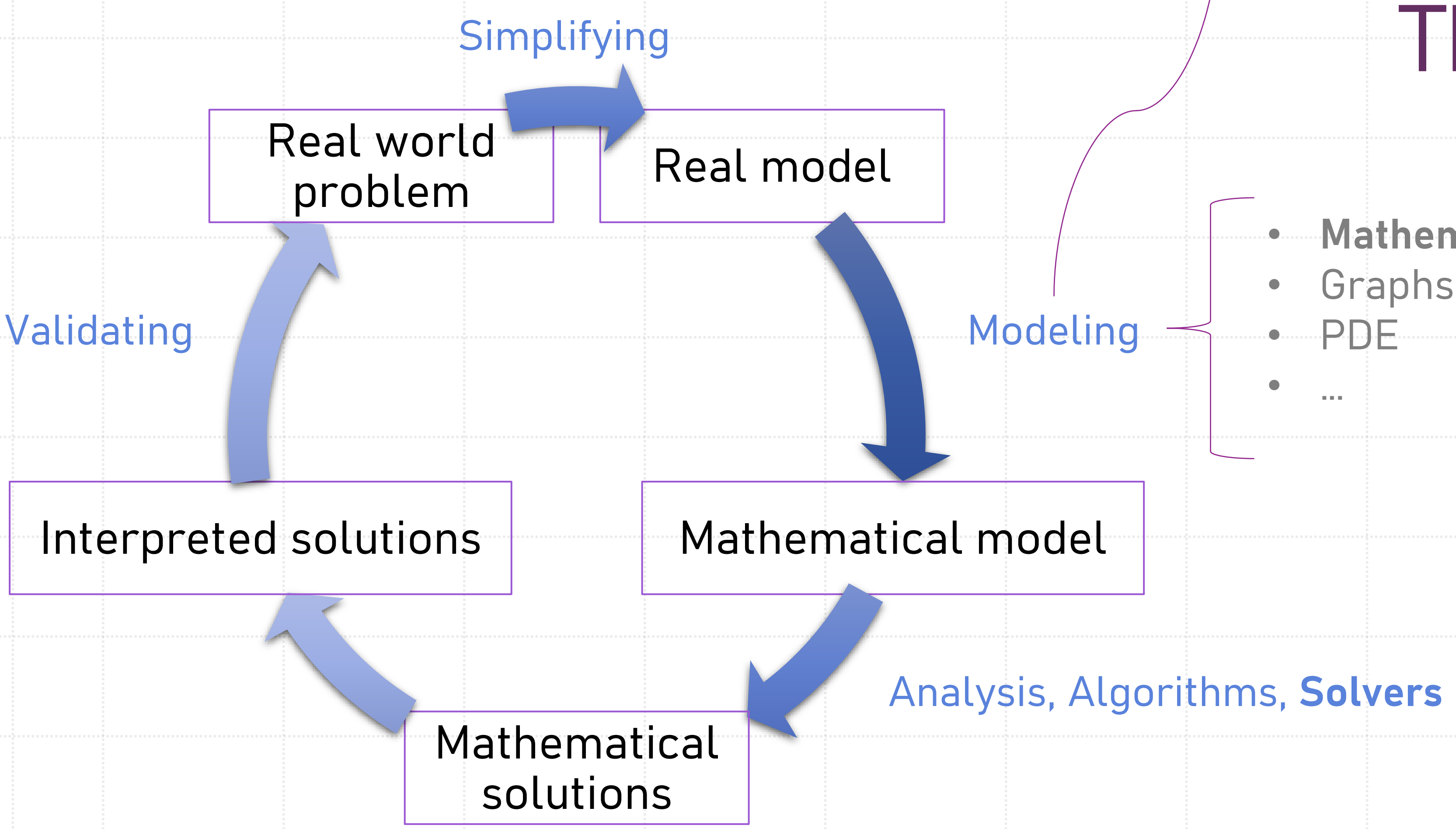
Some keywords

- Modeling;
- Optimization problem;
- Mathematical programming;
- Variables;
- Constraints;
- Objective function;
- Input data/parameters.

Overview and recap

The big picture

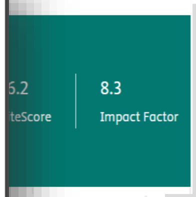
It is both a science and an art

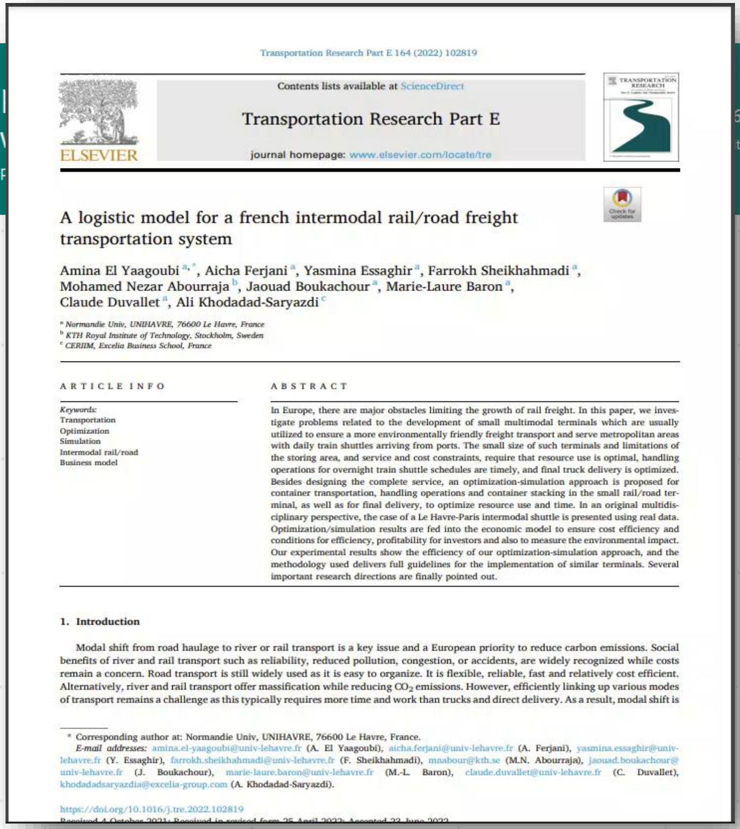


- **Mathematical programming**
- Graphs
- PDE
- ...

A real context

- Solve the container handling and storage problem in a French rail/road terminal.
- Note that all containers arriving by train must be unloaded by a specified deadline. Containers designated for early delivery should be placed on the upper levels of the stacks to minimize unnecessary loading and unloading movements during delivery operations. This must be done while adhering to structural and operational constraints.
- The optimization-simulation approach is developed using CPLEX solver for optimization and Anylogic software for simulation.





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A logistic model for a french intermodal rail/road freight transportation system

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ABSTRACT

In Europe, there are major obstacles limiting the growth of rail freight. In this paper, we investigate problems related to the development of small multimodal terminals which are usually utilized to ensure a more environmentally friendly freight transport and serve metropolitan areas with daily train shuttles arriving from ports. The small size of such terminals and limitations of the storing area, and service and cost constraints, require that resource use is optimal, handling operations for overnight train shuttle schedules are timely, and final truck delivery is optimized. Besides designing the complete service, an optimization-simulation approach is proposed for container transportation, handling operations and container stacking in the small rail/road terminal, as well as for final delivery, to optimize resource use and time. In an original multidisciplinary perspective, the case of a Le Havre-Paris intermodal shuttle is presented using real data. Optimization/simulation results are fed into the economic model to ensure cost efficiency and conditions for efficiency, profitability for investors and also to measure the environmental impact. Our experimental results show the efficiency of our optimization-simulation approach, and the methodology used delivers full guidelines for the implementation of similar terminals. Several important research directions are finally pointed out.

1. Introduction

Modal shift from road haulage to river or rail transport is a key issue and a European priority to reduce carbon emissions. Social benefits of river and rail transport such as reliability, reduced pollution, congestion, or accidents, are widely recognized while costs remain a concern. Road transport is still widely used as it is easy to organize. It is flexible, reliable, fast and relatively cost efficient. Alternatively, river and rail transport offer massification while reducing CO_2 emissions. However, efficiently linking up various modes of transport remains a challenge as this typically requires more time and work than trucks and direct delivery. As a result, modal shift is

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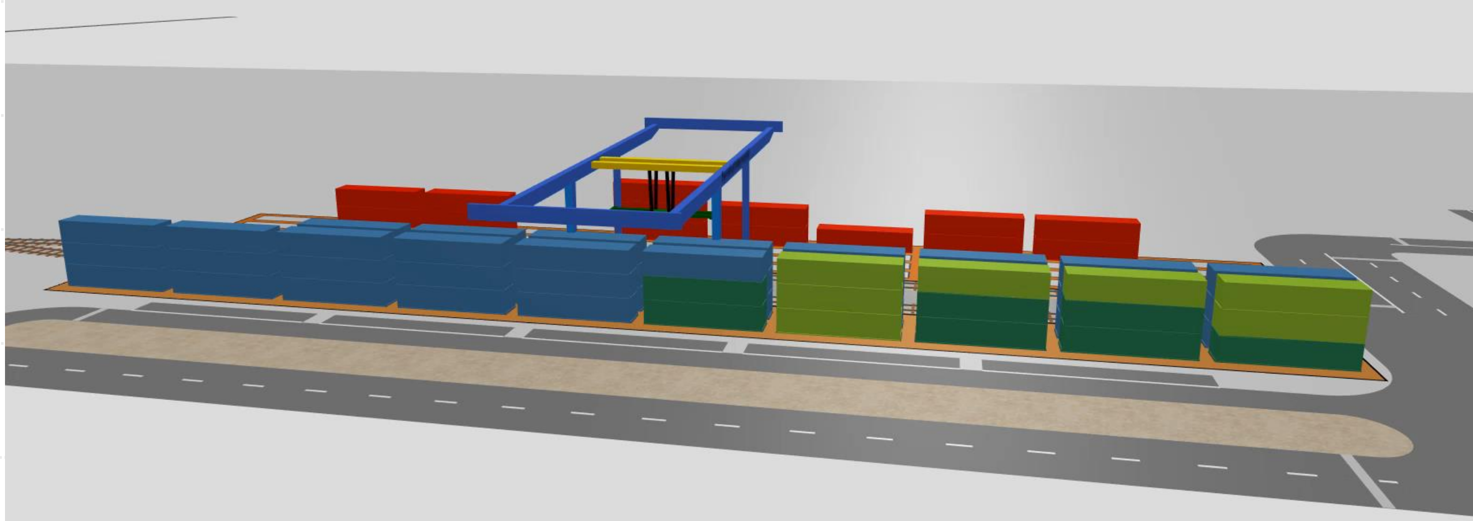


A useful paradigm : Mathematical Programming

- A model, as considered in this lecture, is a mathematical construct used to represent certain significant aspects of real-world **optimization problems**, where we wish to maximize or minimize something: **mathematical program**.

- It is probably the most commonly used standard type of models in OR.
  - Other examples of some commonly used mathematical models are simulation models, network planning models, econometric models and time series models.

Mathematical programming is very different from computer programming. Mathematical programming is ‘programming’ in the sense of ‘planning’.

Mathematical programming

A class of OR methods to mathematically model and solve **constrained optimization problems**, include:

- **Linear Programming (LP)**
- Nonlinear Programming
- **Integer (Linear) Programming (IP/ILP)**
- **Mixed Integer (Linear) Programming (MILP/MIP)**
- **Quadratic Programming (QP)**
- Dynamic Programming
- Stochastic Programming
- . . .

In this class, we confine our attention to some special types of mathematical programming models : **LP ones.**

A very simple example

- Let's propose a production plan that increases the profit of a factory!

… we need more **data** than that.

A very simple example

- Let's propose a production plan that increases the profit of a factory!

… we need more **data** than that.

- Suppose that the factory manufactures 2 products, P1 and P2, which require equipment, labor, and raw materials resources that are available in limited quantities. P1 and P2 yield sales of 6 euros and 4 euros per unit, respectively.

	P1	P2	Availability
Equipment	3	9	81
Labor	4	5	55
Raw material	2	1	20

- What quantities of products P1 and P2 should the factory produce to maximize the total profit from the sale of the two products?

Let's recap

Steps in modeling

- Get an overall idea of the problem.
- Identify the inputs data.
- Identify variables.
- Identify the objective function[2].
- Identify constraints.

What are relevant parameters or data?

Which decisions can we make? What is unknown?

What is our goal? What are we trying to achieve?

What limitations are relevant? ? What determines how a solution is valid (feasibility)?

All steps are vital to provide a valid mathematical program.

[2] If a number of conflicting objectives apply simultaneously, we talk about multiobjective optimization problems (not included in our lecture).

A very simple example: how to model it?

- What are relevant parameters or data?
  - Resources required for each product
  - Quantities of resources
  - The sale profit of each unit of each product
- Which decisions can we make? What is unknown?
  - Quantities of products P1 and P2 to produce : x_1 (for P1) and x_2 (for P2)
- What is our goal?
  - Maximize the total profit from the sale of the two products
- What limitations are relevant?
  - Limited quantities of resources/ Availability of each of the resources
  - Non-negativity of x_1 and x_2

P1 and P2 yield sales of 6 euros and 4 euros per unit, respectively.

	P1	P2	Availability
Equipment	3	9	81
Labor	4	5	55
Raw material	2	1	20

What quantities of products P1 and P2 should the factory produce

to maximize the total profit from the sale of the two products?

… In general

- Mathematical programming: is a systematic approach used for optimizing (minimizing or maximizing) the value of an objective function with respect to a set of constraints[1].
- General symbolic model:

$$
\begin{array}{ll}
\text{Maximize (or minimize):} & \left. f(x_1, x_2 \dots x_n) \right\} \text{Objective} \\[1em]
\text{Subject to:} & \left. \begin{array}{lll}
g_1(x_1, x_2 \dots x_n) & \{\leq, \geq, =\} & b_1 \\
g_2(x_1, x_2 \dots x_n) & \{\leq, \geq, =\} & b_2 \\
\vdots & & \\
g_m(x_1, x_2 \dots x_n) & \{\leq, \geq, =\} & b_m
\end{array} \right\} \text{Constraints}
\end{array}
$$

… where $x_1, x_2 \dots x_n$ are the ***decision variables***.

f and $g_i : \mathbb{R}^n \to \mathbb{R}$, $(i = \{1, \dots, m\})$ are assumed continuous (differentiable) functions.

[1] Mathematical Programming - an overview | ScienceDirect Topics

A very simple example: how to model it?

- What are relevant parameters or data?
  - Resources required for each product
  - Quantities of resources
  - The sale profit of each unit of each product
- Which decisions can we make? What is unknown?
  - Quantities of products P1 and P2 to produce : x_1 (for P1) and x_2 (for P2)
- What is our goal?
  - Maximize the total profit from the sale of the two products
- What limitations are relevant?
  - Limited quantities of ressources/ Availability of each of the resources
  - Non-negativity of x_1 and x_2

A case model

$$\begin{cases} maximize\ f(x,y) = 6x_1 + 4x_2 \\ s.t. \\ 3x_1 + 9x_2 \leq 81 \\ 4x_1 + 5x_2 \leq 55 \\ 2x_1 + x_2 \leq 20 \\ x_1, x_2 \geq 0 \end{cases}$$

Yes! Our first mathematical program…

A very simple example: Let's solve it

- Let's use one of the free open-source solvers to find the optimal solution of our first case model
- GLPK : [GLPK Online (cocoto.github.io)]

Summary Logs Output Variables Constraints

Name	Value	LB	UB
obj	65	-inf	+inf
c1	67.5	-inf	81
c2	55	-inf	55
c3	20	-inf	20

GLPK-ONLINE

```
var x1>=0;
var x2>=0;

maximize obj: 6 * x1 + 4 * x2;

s.t.
c1: 3 * x1 + 9 * x2 <= 81;
c2: 4 * x1 + 5 * x2 <= 55;
c3: 2 * x1 +     x2 <= 20;

solve;

display x1, x2, obj;
end;
```

Some real-world applications

Product Mix Problem

- An engineering factory can produce five types of products (P1, P2, … , P5) by using two production processes: grinding and drilling. After deducting raw material costs, each unit of each product yields the following contributions to profit:

P1	P2	P3	P4	P5
550€	600€	350€	400€	200€

- Each unit of each product requires a certain time on each process. These are given below (in hours). A dash indicates when a process is not needed.

	P1	P2	P3	P4	P5
Grinding	12	20	-	25	15
Drilling	10	8	16	-	-

- In addition, the final assembly of each unit of each product uses 20 hours of an employee's time. The factory has three grinding machines, and two drilling machines and works a six-day week with two shifts of 8 hours on each day. Eight workers are employed in assembly, each working one shift a day. The problem is to find how much of each product is to be manufactured so as to maximize the total profit contribution.

Let's model it together…

- In order to create a mathematical model, we introduce **variables** $x_i, i = \{1, \dots, 5\}$ representing the numbers of P1, P2, … , P5 that should be produced in a week ($x_i \in \mathbb{R}$, although this is debatable).
- As each unit of P1 yields 550€ contribution to profit and each unit of P2 yields 600€ contribution to profit, etc., our total profit contribution will be represented by the expression:

$$550x_1 + 600x_2 + 350x_3 + 400x_4 + 200x_5$$

- The **objective** of the factory is to choose x_1, x_2, … , x_5 so as to make the value of this expression as high as possible, that is, the expression (above) is the objective function that we wish to maximize.
- Clearly, our processing and labor capacities, to some extent, limit the values that the x_i can take. Given that we have only three grinding machines working for a total of 96 hours a week each, we have 288 hours of grinding capacity available. Each unit of P1 uses 12 hours grinding. x_1 units will therefore use $12x_1$ hours. Similarly, x_2 units of P2 will use $20x_2$ hours. The total amount of **grinding capacity** that we use in a week is given by the following expression:

$$12x_1 + 20x_2 + 25x_4 + 15x_5 \leq 288.$$

  - This inequality is a mathematical way of saying that we cannot use up more than the 288 hours of grinding available per week.

Let's model it together…

- The **drilling capacity** is 192 hours a week. This gives rise to the following constraint:

$$10x_1 + 8x_2 + 16x_3 \leq 192.$$

- Finally, the fact that we have only a total of eight assembly workers each working 48 hours a week gives us a **labor capacity** of 384 hours. As each unit of each product uses 20 hours of this capacity, we have the constraint:

$$20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 \leq 384$$

- We do not make negative quantities of any product. We might explicitly state these conditions by the extra constraints :

$$x_1, x_2, \ldots, x_5 \geq 0.$$

- We have now expressed our original practical problem as a mathematical model. We wish to find values for the variables $x_i, i = \{1, \ldots, 5\}$ that make the objective function as large as possible but still satisfy all constraints.
- **Write the final mathematical program (the case model).**

Blending problem

- A food is manufactured by refining raw oils and blending them together. The raw oils come in two categories:

Vegetable oils	VEG 1
	VEG 2
Non-vegetable oils	OIL 1
	OIL 2
	OIL 3

- Vegetable oils and non-vegetable oils require different production lines for refining. In any month, it is not possible to refine more than 200 tons of vegetable oils and more than 250 tons of non-vegetable oils. There is no loss of weight in the refining process and the cost of refining may be ignored.
- There is a technological restriction of hardness in the final product. In the units in which hardness is measured, this must lie between 3 and 6 per ton. It is assumed that hardness blends linearly. The costs (per ton) and hardness of the raw oils are:

	VEG 1	VEG 2	OIL 1	OIL 2	OIL 3
Cost	110€	120€	130€	110€	115€
Hardness	8.8	6.1	2.0	4.2	5.0

- The final product sells for 150€ per ton.
- How should the food manufacturer make their product in order to maximize their net profit?

Let's model it together…

- **Variables** are introduced to represent the unknown quantities. $x_i, i = \{1, \dots, 5\}$ represent the quantities (tons) of VEG 1, VEG 2, OIL 1, OIL 2 and OIL 3 that should be bought, refined and blended in a month. y represents the quantity (tons) of the product that should be made.
- Our **objective** is to maximize the net profit: $-110x_1 - 120x_2 - 130x_3 - 110x_4 - 115x_5 + 150y.$
- The refining capacities give the following two **constraints**:

$$x_1 + x_2 \leq 200,$$
$$x_3 + x_4 + x_5 \leq 250.$$

- The hardness limitations on the final product are imposed by the following two **constraints**:

$$8.8x_1 + 6.1x_2 + 2x_3 + 4.2x_4 + 5x_5 - 6y \leq 0,$$
$$8.8x_1 + 6.1x_2 + 2x_3 + 4.2x_4 + 5x_5 - 3y \geq 0.$$

- Finally, it is necessary to make sure that the weight of the final product is equal to the weight of the ingredients. This is done by a continuity **constraint**: $x_1 + x_2 + x_3 + x_4 + x_5 - y = 0.$

Write the final mathematical program (the case model). What's missing?

Now the floor is yours …

We will learn to use some solvers to solve the problems :

- **Excel solver**
  - Define and solve a problem by using Solver - Microsoft Support
  - Load the Solver Add-in in Excel - Microsoft Support
- **Linear Optimization Solver**
  - https://online-optimizer.appspot.com (real (continuous) variables)
- **GLPK**
  - GLPK Online (cocoto.github.io)
- **NEOS Server**
  - https://neos-server.org/neos/solvers/milp:CPLEX/LP.html
    (for both real and/or integer variables)
  - To be covered in the upcoming class: For integer variables, consult the CPLEX .lp file documentation (CPLEX lp files (mit.edu) and LP file format: algebraic representation - IBM Documentation) to properly define and submit the correct variable types to the solver.

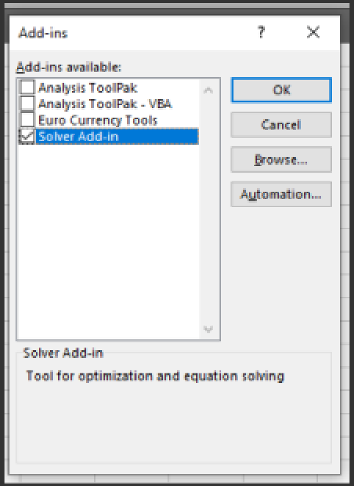
Regarding the Excel solver

Refer to the two slides from the previous class.

\>>> (TUTO) Solver: Excel

* Install Add-Ins (Data tab, Search Add-ins activate "Solver Add-in". It might also work with Online Excel).
* Official Doc:

https://support.microsoft.com/en-gb/office/define-and-solve-a-problem-by-using-solver-5d1a388f-079d-43ac-a7eb-f63e45925040



modeling to the resolution]$ _ [52/58]

\>>> (TUTO) Solver: Excel

From your table expressing the problem, implement it into excel:

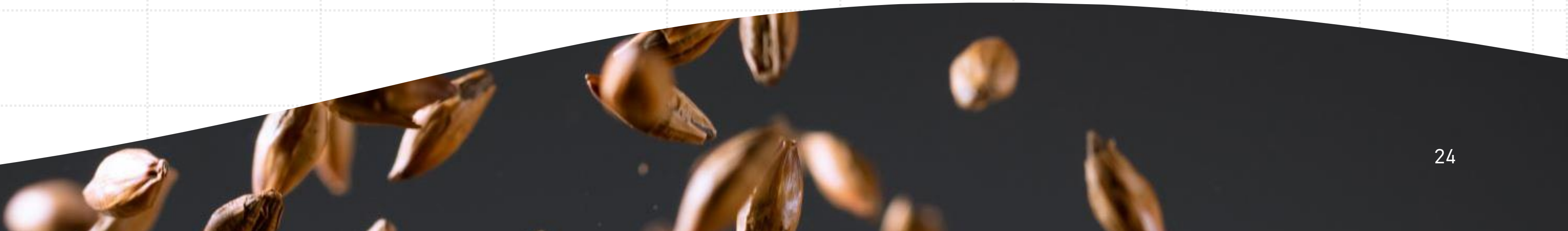
1. Write the data
2. Write the expression of the Objective Function and related the constraints
3. Configure the solver:
   * Click on "Solver" (Data Tab)
   * Implement the Objective Function
   * Objective Function
   * Choose the direction of the Objective function (Min or Max)
   * Choose the Variables ("By Changing Variable Cells")

[4. An end-to-end example: from the modeling to the resolution]$ _ [53/58]

- **File / Options**
- **Add-ins**
- Click the **Go** button
- Select **Solver Add-in**
- Finish by clicking **OK**
- Start Solver in **Data / Solver**

Baker problem

- A baker wants to produce rolls of shortcrust pastry (selling price: 4 euros per roll) and rolls of puff pastry (selling price: 5 euros per roll). The production of a roll of shortcrust pastry requires 2 kilograms of flour and 1 kilogram of butter. The production of a roll of puff pastry requires 1 kilogram of flour, 2 kilograms of butter, and 1 kilogram of salt (the pastry is salted). How can profits be maximized, knowing that there is a stock of 8 tons of flour, 7 tons of butter, and 3 tons of salt?
- Formulate this problem as a mathematical program (a case model) and solve it using Excel Solver.



Ironworks problem

- A steel mill produces metal strips and rolls [3]. It operates for 40 hours per week. The production rates are 200 strips per hour and 140 rolls per hour. The strips are sold at 25 euros each, and the rolls at 30 euros each. The market is limited: it is impossible to sell more than 6000 strips and 4000 rolls per week. How can profit be maximized?
- Formulate this problem as a mathematical program (a case model) and solve it using Excel Solver.

[3] This example is from the course of François Boulier, University of Lille.

Paper production problem

- Two factories produce paper of three different qualities. They have orders for each type of paper: the company managing the factories has contracts to supply (at least) 16 tons of lower quality paper, 5 tons of medium-quality paper, and 20 tons of high-quality paper. It costs 1000 euros per day to operate factory A and 2000 euros per day to operate factory B. Factory A produces 8 tons of lower quality paper, 1 ton of medium-quality paper, and 2 tons of high-quality paper per day. Factory B produces 2 tons of lower quality paper, 1 ton of medium-quality paper, and 7 tons of high-quality paper per day. We want to determine how many days each factory should operate to meet the demand in the most cost-effective way.
- Formulate this problem as a mathematical program (a case model) and solve it using GLPK Online (cocoto.github.io).

Another Blending problem

- The table below provides the composition and cost of 9 standard alloys made of lead, zinc, and tin. The goal is to find a mixture of these 9 alloys that allows the production of an alloy with minimal cost, containing:
- At least 30% lead
- At least 30% zinc
- At least 40% tin

Alloy	1	2	3	4	5	6	7	8	9
Lead (%)	20	50	30	30	30	60	40	10	10
Zinc (%)	30	40	20	40	30	30	50	30	10
Tin (%)	50	10	50	30	40	10	10	60	80
Unit cost	7.3	6.9	7.3	7.5	7.6	6.0	5.8	4.3	4.1

- Formulate this problem as a mathematical program.

Transportation problem 1

- Let's assume that 250 (for depot D1) and 450 (for depot D2) containers are available, and stores A, B, and C have ordered exactly 200 containers each. The transportation costs per container are as follows:

Store	A	B	C
Depot D_1	3.4	2.2	2.9
Depot D_2	3.4	2.4	2.5

- The goal is to minimize the total transportation cost of containers from depots to stores while adhering to availability and demand constraints.
- Formulate the problem as a mathematical program.

Transportation problem 2

- A company that produces Alsatian pasta stores its products in three depots, referred to as D_1, D_2, and D_3. The quantities stored in these depots are q_1, q_2, and q_3. These depots supply stores M_1, M_2, M_3, M_4, and M_5. The quantity of pasta needed by store M_j is denoted as m_j. All these transports come with a cost. Let c_{ij} represent the transportation cost per unit of product for transporting from depot D_i to store M_j. The problem is to determine how much quantity depot D_i should deliver to store M_j in order to minimize transportation costs.
- Formulate this problem as a mathematical program.

Transportation problem 3

- Consider a market with I suppliers, J buyers, and K products that are bought and sold. Supplier i has a quantity of S_{ik} of product k and sells it for a price of A_{ik} euros per unit. Buyer j demands at most D_{jk} units of product k and is willing to purchase it for a price of B_{jk} euros per unit.
- In this market, you are responsible for matching the suppliers with the buyers: you gather all the price and offer information, and then you allocate the suppliers to the buyers. Your profit is the difference between the purchase price and the selling price for each unit of product k that supplier i sells to buyer j.
- Formulate the problem of maximizing profit as a mathematical program.

… More constraints?

- Now, suppose that the suppliers and buyers are not in the same location. The transport capacity between supplier i and buyer j is at most U_{ij} units. Additionally, you are obligated to pay for the transportation of all products sold by supplier i to buyer j, at a cost of C_{ij} euros per unit.
- Modify your program to account for these new conditions.

Return to keywords

- Modeling; optimization problem; mathematical programming; variables; constraints; objective function; input data/parameters.

- **To Do:** Provide a concise explanation of the mentioned keywords.

Some references

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