\>>> Operations Research

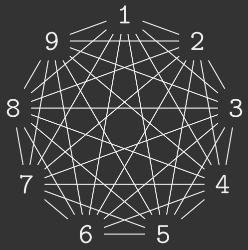
\>>> Introduction

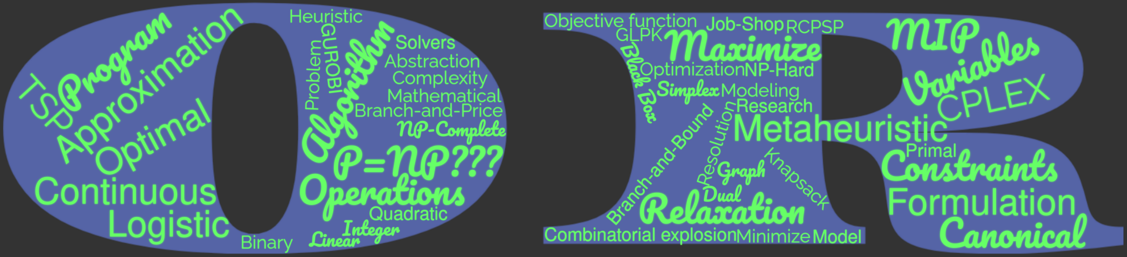
Name: Samuel Deleplanque and Amina El Yaagoubi

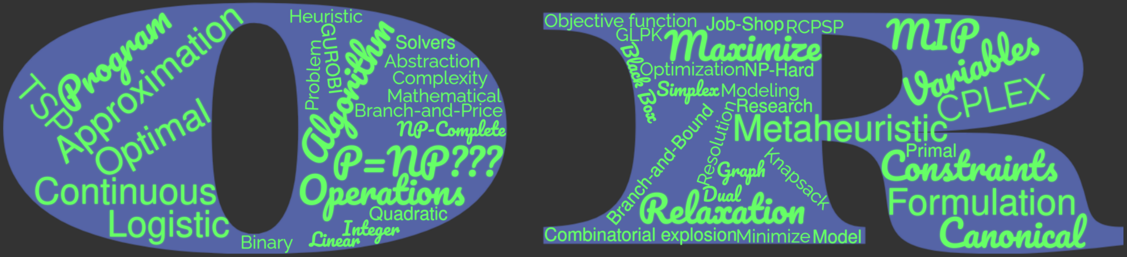
samuel.deleplanque@junia.com

amina.el-yaagoubi@junia.com

Date: September 13, 2024




TSP
Program
Approximation
Optimal
Continuous
Logistic
Binary
Heuristic
GUROBI
Problem
Algorithm
Solvers
Abstraction
Complexity
Mathematical
Branch-and-Price
NP-Complete
P=NP???
Operations
Quadratic
Integer
Linear
Objective function
GLPK
Job-Shop
RCPSP
Maximize
Black Box
Optimization
NP-Hard
Simplex
Modeling
Research
Branch-and-Bound
Resolution
Metaheuristic
Graph
Knapsack
Dual
Relaxation
Combinatorial explosion
Minimize
Model
MIP
Variables
CPLEX
Primal
Constraints
Formulation
Canonical



Your first homework

--

Try to define all these words in the context of Operations Research.

7 classes

1. Introduction

7 classes

1. Introduction
2. Mathematical Modeling

7 classes

1. Introduction
2. Mathematical Modeling
3. Linear Programming

7 classes

1. Introduction
2. Mathematical Modeling
3. Linear Programming
4. Mixed Integer Programming

7 classes

1. Introduction
2. Mathematical Modeling
3. Linear Programming
4. Mixed Integer Programming
5. Lab $\longrightarrow$ $\longrightarrow$ $\longrightarrow$ $\longrightarrow$ $\longrightarrow$ $\longrightarrow$ $\longrightarrow$

>>> Outline

7 classes

1. Introduction
2. Mathematical Modeling
3. Linear Programming
4. Mixed Integer Programming
5. Lab ⟶ ⟶ ⟶ ⟶ ⟶ ⟶ ⟶

6. Complex systems and modeling tools I
7. Complex systems and modeling tools II

Operations Research in Data Management

(M1)
Practical lab
(max: **3** people per group)

Data Storage

Company Management

pic: chatGPT 4o

Abstract. Data management and storage are critical components of any enterprise's IT infrastructure. This project focuses on optimizing operations related to **disk packing**, **data center location**, and **data transport** problems for a hypothetical large-scale cloud service provider. The aim is to minimize operational costs while maintaining high efficiency in data handling and storage. By solving the continuous relaxation and then the MILP for several instances, you will explore the complexities and interdependencies of real-world optimization problems.

JUNIA ISEN

amina.el-yaagoubi / samuel.deleplanque at junia.com

>>> Outline of the first class

1. Short Overview of the class

2. Key Concepts and Their Relevance

3. Scientific Modeling

4. An end-to-end example: from the modeling to the resolution

>>> Reading Advice

A nearly exhaustive free preprint book written by the foremost researchers in operations research: https://arxiv.org/abs/2303.14217.

Key Concepts
and
Their Relevance

>>> Two Words

Operations ***Research***

Operations Research

* Pertains to processes or activities in businesses or organizations.
* Reflects the real-world activities OR seeks to optimize, such as logistics or inventory management.

Operations *Research*

* Pertains to processes or activities in businesses or organizations.
* Reflects the real-world activities OR seeks to optimize, such as logistics or inventory management.

* Highlights the discipline's analytical rigor.
* Indicates the study of methods to discover and implement problem solutions, often via mathematical models or statistical analyses.

>>> "Operations Research" (OR)

* "Research on operations" addresses coordination of activities within various organizations like manufacturing, transportation, finance, etc. In short: Organize using mathematics.

>>> "Operations Research" (OR)

* "Research on operations" addresses coordination of activities within various organizations like manufacturing, transportation, finance, etc. In short: Organize using mathematics.
* OR, combined with the computer revolution, has widespread presence across sectors, often unnoticed.

>>> "Operations Research" (OR)

* "Research on operations" addresses coordination of activities within various organizations like manufacturing, transportation, finance, etc. In short: Organize using mathematics.
* OR, combined with the computer revolution, has widespread presence across sectors, often unnoticed.
* Research process involves observing the problem, collecting data, and formulating it.

>>> "Operations Research" (OR)

* "Research on operations" addresses coordination of activities within various organizations like manufacturing, transportation, finance, etc. In short: Organize using mathematics.
* OR, combined with the computer revolution, has widespread presence across sectors, often unnoticed.
* Research process involves observing the problem, collecting data, and formulating it.
* Scientific model constructed to abstract essence of real problem; must capture essential characteristics.

>>> "Operations Research" (OR)

* "Research on operations" addresses coordination of activities within various organizations like manufacturing, transportation, finance, etc. In short: Organize using mathematics.
* OR, combined with the computer revolution, has widespread presence across sectors, often unnoticed.
* Research process involves observing the problem, collecting data, and formulating it.
* Scientific model constructed to abstract essence of real problem; must capture essential characteristics.
* OR often seeks optimal solution, which may not be unique.

>>> "Operations Research" (OR)

* "Research on operations" addresses coordination of activities within various organizations like manufacturing, transportation, finance, etc. In short: Organize using mathematics.
* OR, combined with the computer revolution, has widespread presence across sectors, often unnoticed.
* Research process involves observing the problem, collecting data, and formulating it.
* Scientific model constructed to abstract essence of real problem; must capture essential characteristics.
* OR often seeks optimal solution, which may not be unique.
* Expertise includes mathematics, statistics, engineering, economics, management, computer science, physics, behavioral sciences, and specific OR techniques.

>>> Once upon a time... Operations Research

* Origins of operations research (OR) not clearly defined

\>>> Once upon a time... Operations Research

* Origins of operations research (OR) not clearly defined
* Charles Babbage (1791-1871) often considered father due to work on transportation costs and mail sorting, leading to Penny Post introduction in England in 1840

>>> Once upon a time... Operations Research

* Origins of operations research (OR) not clearly defined
* Charles Babbage (1791-1871) often considered father due to work on transportation costs and mail sorting, leading to Penny Post introduction in England in 1840
* World War II created urgent need for efficient resource allocation in various military operations (British and American)

>>> Once upon a time... Operations Research

* Origins of operations research (OR) not clearly defined
* Charles Babbage (1791-1871) often considered father due to work on transportation costs and mail sorting, leading to Penny Post introduction in England in 1840
* World War II created urgent need for efficient resource allocation in various military operations (British and American)
* OR teams formed, led by scientists like Patrick Blackett, Nobel Prize winner; efforts played significant role in victory (radars location).

>>> Once upon a time... Operations Research

* Origins of operations research (OR) not clearly defined
* Charles Babbage (1791-1871) often considered father due to work on transportation costs and mail sorting, leading to Penny Post introduction in England in 1840
* World War II created urgent need for efficient resource allocation in various military operations (British and American)
* OR teams formed, led by scientists like Patrick Blackett, Nobel Prize winner; efforts played significant role in victory (radars location).
* Post-war industrial growth led to similar complexity and specialization problems; OR applied to commercial, industrial, and gov. in the 1950s.

\>>> Once upon a time... Operations Research

* Origins of operations research (OR) not clearly defined
* Charles Babbage (1791-1871) often considered father due to work on transportation costs and mail sorting, leading to Penny Post introduction in England in 1840
* World War II created urgent need for efficient resource allocation in various military operations (British and American)
* OR teams formed, led by scientists like Patrick Blackett, Nobel Prize winner; efforts played significant role in victory (radars location).
* Post-war industrial growth led to similar complexity and specialization problems; OR applied to commercial, industrial, and gov. in the 1950s.
* OR Rapid growth influenced by improved techniques and rise of IT.

John Forbes Nash Jr.

* Mathematician known for his work in game theory.
* Introduced the concept of *Nash Equilibrium*.
* His insights are foundational in operations research, especially in decision-making processes and strategy optimization.
* Nash's work has applications in economics, computer science, and various fields that involve decision-making and strategy formulation.
* Watch the "A Beautiful Mind" movie!

>>> Operations Research, Decision Support, and Optimization

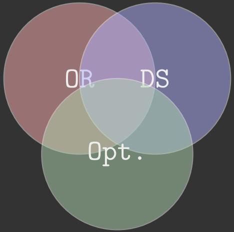
* Operations Research: The application of advanced analytical methods to help make better decisions.

>>> Operations Research, Decision Support, and Optimization

* **Operations Research:** The application of advanced analytical methods to help make better decisions.
* **Decision Support:** Systems and tools designed to aid in the decision-making process.

>>> Operations Research, Decision Support, and Optimization

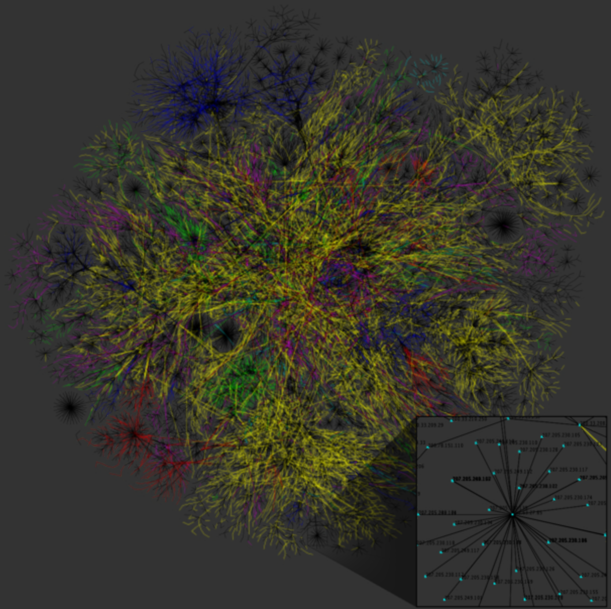
* **Operations Research:** The application of advanced analytical methods to help make better decisions.
* **Decision Support:** Systems and tools designed to aid in the decision-making process.
* **Optimization:** The action of making the best or most effective use of a situation or resource. Optimization Problems consist of maximizing, or minimizing, a quantity under constraints.



Interrelation of Fields

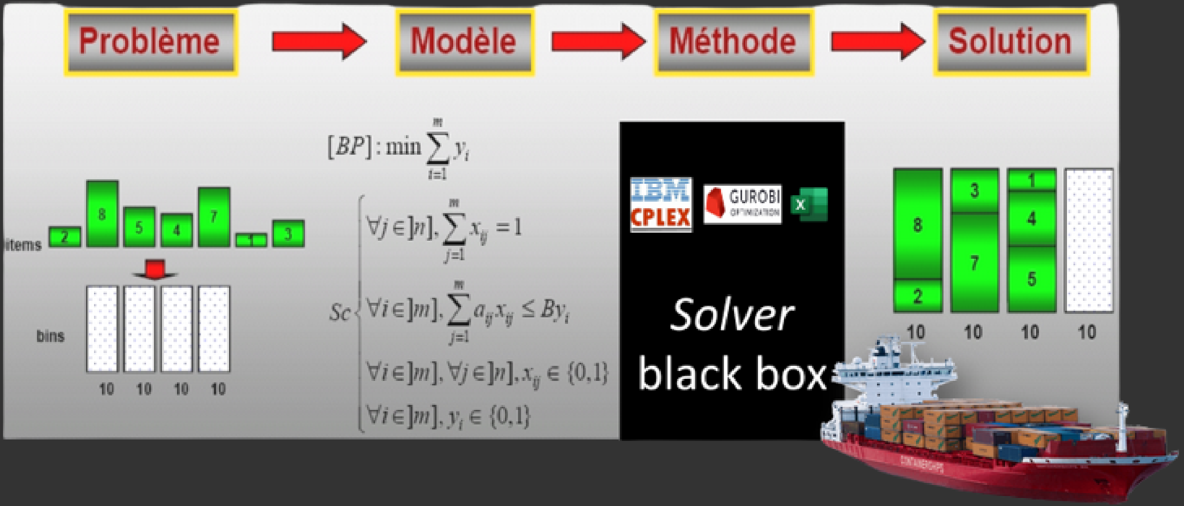
>>> Applications of OR: Supply Chain Management

\>>> Any network (here: internet map)



\>>> Applications of OR: Transportation Logistics

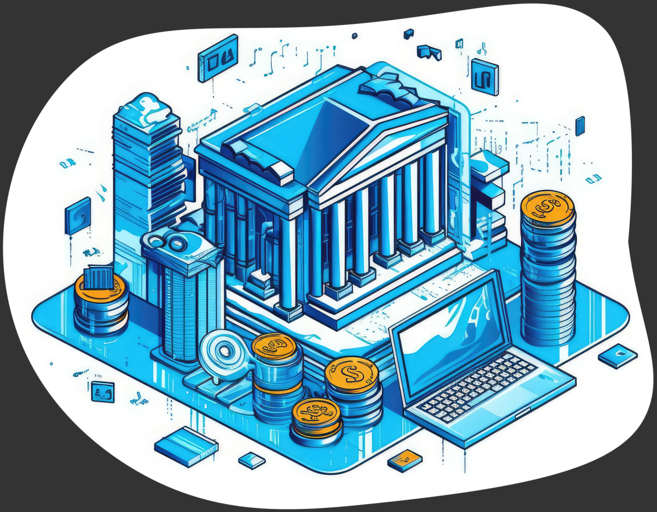
>>> Applications of OR: Transportation Logistics



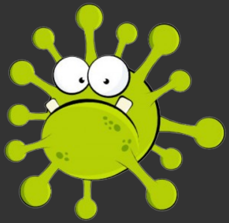
>>> Applications of OR: Energy Management

>>> Applications of OR: Humanitarian and Military

>>> Applications of OR: Financial Engineering

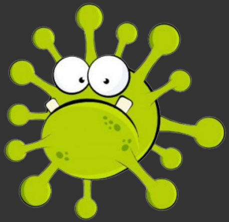


>>> Applications of OR: Healthcare, the COVID crisis example



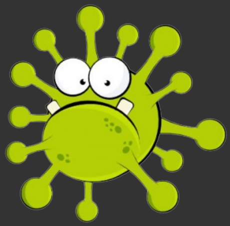
* Optimization of patient and medical equipment transport

>>> Applications of OR: Healthcare, the COVID crisis example



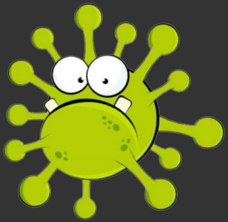
* Optimization of patient and medical equipment transport
* Supply of medical equipment

>>> Applications of OR: Healthcare, the COVID crisis example



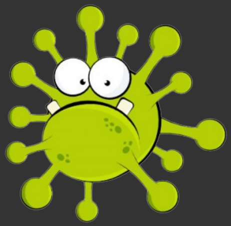
* Optimization of patient and medical equipment transport
* Supply of medical equipment
* Deployment of field hospitals

>>> Applications of OR: Healthcare, the COVID crisis example



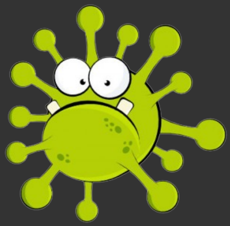
* Optimization of patient and medical equipment transport
* Supply of medical equipment
* Deployment of field hospitals
* Sizing and management of human resources

>>> Applications of OR: Healthcare, the COVID crisis example



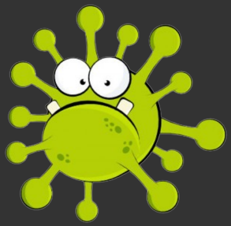
* Optimization of patient and medical equipment transport
* Supply of medical equipment
* Deployment of field hospitals
* Sizing and management of human resources
* Assignment and route calculation for home care staff

>>> Applications of OR: Healthcare, the COVID crisis example



* Optimization of patient and medical equipment transport
* Supply of medical equipment
* Deployment of field hospitals
* Sizing and management of human resources
* Assignment and route calculation for home care staff
* Optimization of production (e.g., medicines and vaccines)

>>> Applications of OR: Healthcare, the COVID crisis example



* Optimization of patient and medical equipment transport
* Supply of medical equipment
* Deployment of field hospitals
* Sizing and management of human resources
* Assignment and route calculation for home care staff
* Optimization of production (e.g., medicines and vaccines)
* Patient pathway planning

>>> Significant Yearly Savings from Operations Research

Industry	Application	Yearly Savings
Airlines	Route optimization	$300 million
Manufacturing	Production scheduling	$50 million
Retail	Inventory management	$100 million
Healthcare	Patient scheduling	$20 million
Transportation	Fleet routing	$60 million
Telecommunications	Network design	$40 million

Note: Figures are illustrative and based on various historical and aggregated industry data.

Some other cases: http://www.phpsimplex.com/en/real_cases.htm

Other optimization criteria are directly related to climate change, such as minimizing the energy consumption of any system.

\>>> Operations Research includes many topics

Core Topics

* Graph Theory
* Linear Programming
* Integer Programming
* Quadratic Programming
* Algorithms and Complexity Theory
* Programming

>>> Operations Research includes many topics

Core Topics

* Graph Theory
* Linear Programming
* Integer Programming
* Quadratic Programming
* Algorithms and Complexity Theory
* Programming

Course Philosophy

* Our aim isn't to reinvent the wheel.
* We will focus on:

>>> Operations Research includes many topics

Core Topics

* Graph Theory
* Linear Programming
* Integer Programming
* Quadratic Programming
* Algorithms and Complexity Theory
* Programming

Course Philosophy

* Our aim isn't to reinvent the wheel.
* We will focus on:
  * modeling

\>>> Operations Research includes many topics

Core Topics

* Graph Theory
* Linear Programming
* Integer Programming
* Quadratic Programming
* Algorithms and Complexity Theory
* Programming

Course Philosophy

* Our aim isn't to reinvent the wheel.
* We will focus on:
  * modeling
  * the usage of "Black Box" solvers

>>> Operations Research includes many topics

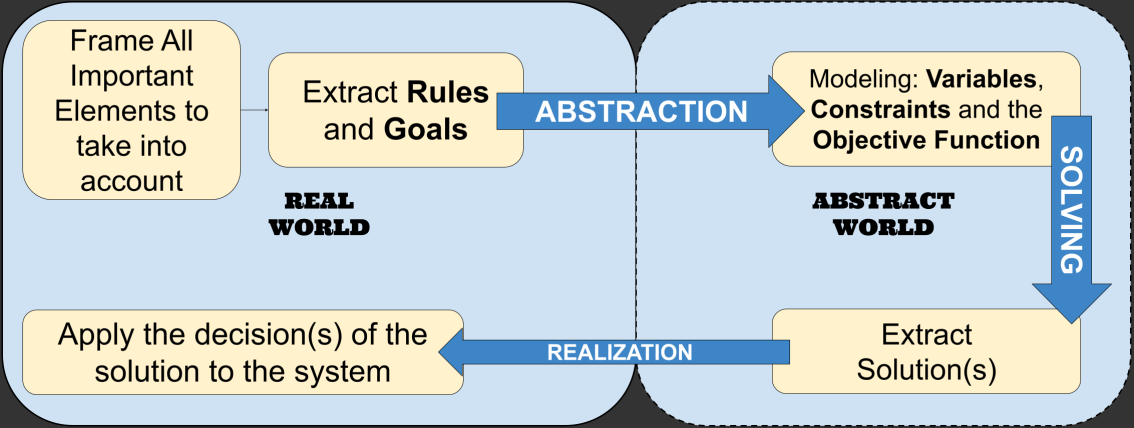
Core Topics

* Graph Theory
* Linear Programming
* Integer Programming
* Quadratic Programming
* Algorithms and Complexity Theory
* Programming

Course Philosophy

* Our aim isn't to reinvent the wheel.
* We will focus on:
  * modeling
  * the usage of "Black Box" solvers
  * learning the main solvers algorithms
* We will try to focus on real-world problems.

>>> The stages of an operations research project



Scientific Modeling

Generic Model:

$$
\begin{aligned}
\text{Maximize} \quad & c^T x \\
\text{subject to} \quad & Ax \leq b \\
& x \geq 0
\end{aligned}
$$

where:

* $x \in \mathbb{R}^n$ (or $\mathbb{N}^n$) is the vector of decision variables
* $c \in \mathbb{R}^n$ (or $\mathbb{N}^n$) is the vector of coefficients for the objective function
* $A \in \mathbb{R}^{m \times n}$ (or $\mathbb{N}^{m \times n}$) is the matrix of constraint coefficients
* $b \in \mathbb{R}^m$ (or $\mathbb{N}^m$) is the vector of constraint bounds

>>> Vocabulary: Generic and case models

Generic Model:

$$\begin{aligned} \text{Maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

where:

* $x \in \mathbb{R}^n$ (or $\mathbb{N}^n$) is the vector of decision variables
* $c \in \mathbb{R}^n$ (or $\mathbb{N}^n$) is the vector of coefficients for the objective function
* $A \in \mathbb{R}^{m \times n}$ (or $\mathbb{N}^{m \times n}$) is the matrix of constraint coefficients
* $b \in \mathbb{R}^m$ (or $\mathbb{N}^m$) is the vector of constraint bounds

Case Model:

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 5x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \leq 12 \\ & x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

where:

* x_1: Number of product A units
* x_2: Number of product B units
* The objective is to maximize profit $(3x_1 + 5x_2)$
* Constraints represent available resources
* The constraints define the region of feasible solutions

>>> Modeling with a mathematical program

* Such model is a mathematical construction used to represent significant aspects of real-world problems.

>>> Modeling with a mathematical program

* Such model is a mathematical construction used to represent significant aspects of real-world problems.
* Three main components in an optimization model:
  1. Variables: Represent modifiable components creating different configurations.

>>> Modeling with a mathematical program

* Such model is a mathematical construction used to represent significant aspects of real-world problems.
* Three main components in an optimization model:
  1. Variables: Represent modifiable components creating different configurations.
  2. Constraints: Represent limitations on variables.

>>> Modeling with a mathematical program

* Such model is a mathematical construction used to represent significant aspects of real-world problems.
* Three main components in an optimization model:
  1. Variables: Represent modifiable components creating different configurations.
  2. Constraints: Represent limitations on variables.
  3. Objective function: Assigns values to configurations; goal is optimization (minimizing or maximizing the Objective function).

>>> Types of Decision Variables

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.

>>> Types of Decision Variables

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.
* A model with (only) continuous (real) variables can be solved in a short time.

\>>> Types of Decision Variables

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.
* A model with (only) continuous (real) variables can be solved in a short time.
* => Continuous optimization

>>> Types of Decision Variables

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.
* A model with (only) continuous (real) variables can be solved in a short time.
* => Continuous optimization

Binary Variables

* Switch: ON (1) or OFF (0).
* Decision: Accept (1) or Reject (0) a proposal.
* State: Open (1) or Closed (0) status of a gate.

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.
* A model with (only) continuous (real) variables can be solved in a short time.
* => Continuous optimization

Binary Variables

* Switch: ON (1) or OFF (0).
* Decision: Accept (1) or Reject (0) a proposal.
* State: Open (1) or Closed (0) status of a gate.
* A model with binary variables CANNOT be solved in a short time.

>>> Types of Decision Variables

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.
* A model with (only) continuous (real) variables can be solved in a short time.
* => Continuous optimization

Binary Variables

* Switch: ON (1) or OFF (0).
* Decision: Accept (1) or Reject (0) a proposal.
* State: Open (1) or Closed (0) status of a gate.
* A model with binary variables CANNOT be solved in a short time.
* => Discrete optimization

>>> Types of Decision Variables

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.
* A model with (only) continuous (real) variables can be solved in a short time.
* => Continuous optimization

Binary Variables

* Switch: ON (1) or OFF (0).
* Decision: Accept (1) or Reject (0) a proposal.
* State: Open (1) or Closed (0) status of a gate.
* A model with binary variables CANNOT be solved in a short time.
* => Discrete optimization

Integer Variables

* Units Produced: Manufacture 100 units of a product.
* Assignments: Assign 3 tasks to an employee.
* Trips: 4 delivery trips per day.

>>> Types of Decision Variables

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.
* A model with (only) continuous (real) variables can be solved in a short time.
* => Continuous optimization

Binary Variables

* Switch: ON (1) or OFF (0).
* Decision: Accept (1) or Reject (0) a proposal.
* State: Open (1) or Closed (0) status of a gate.
* A model with binary variables CANNOT be solved in a short time.
* => Discrete optimization

Integer Variables

* Units Produced: Manufacture 100 units of a product.
* Assignments: Assign 3 tasks to an employee.
* Trips: 4 delivery trips per day.
* A model with integer variables CANNOT be solved in a short time.

\>>> Types of Decision Variables

Continuous Variables

* Time: 3.5 hours spent on a task.
* Temp': Maintain a room at 22.5°C.
* Concentration: 7.8% salt solution.
* A model with (only) continuous (real) variables can be solved in a short time.
* => Continuous optimization

Binary Variables

* Switch: ON (1) or OFF (0).
* Decision: Accept (1) or Reject (0) a proposal.
* State: Open (1) or Closed (0) status of a gate.
* A model with binary variables CANNOT be solved in a short time.
* => Discrete optimization

Integer Variables

* Units Produced: Manufacture 100 units of a product.
* Assignments: Assign 3 tasks to an employee.
* Trips: 4 delivery trips per day.
* A model with integer variables CANNOT be solved in a short time.
* => Discrete optimization

>>> Types of Constraints

Lower or Equal Constraints ($\leq$)

* Budget Limitation: Can't spend more than X.
* Maximum Capacity: A venue can host $\leq$ 500 people.
* Speed Limit: Vehicles can't drive faster than 60 mph.
* Storage Limit: A warehouse can store $\leq$ 1000 boxes.

>>> Types of Constraints

Lower or Equal Constraints ($\leq$)

* Budget Limitation: Can't spend more than X.
* Maximum Capacity: A venue can host $\leq$ 500 people.
* Speed Limit: Vehicles can't drive faster than 60 mph.
* Storage Limit: A warehouse can store $\leq$ 1000 boxes.

Equality Constraints (=)

* Balance: Inputs = Outputs in a system.
* Reservation: Book exactly 10 seats.
* Requirement: Produce = Demand.
* Allocation: Each person gets 1 ticket.

>>> Types of Constraints

Lower or Equal Constraints ($\leq$)

* Budget Limitation: Can't spend more than X.
* Maximum Capacity: A venue can host $\leq$ 500 people.
* Speed Limit: Vehicles can't drive faster than 60 mph.
* Storage Limit: A warehouse can store $\leq$ 1000 boxes.

Equality Constraints ($=$)

* Balance: Inputs = Outputs in a system.
* Reservation: Book exactly 10 seats.
* Requirement: Produce = Demand.
* Allocation: Each person gets 1 ticket.

Greater or Equal Constraints ($\geq$)

* Minimum Production: Produce $\geq$ 100 units.
* Safety Standards: Maintain a reserve $\geq$ 50 liters.
* Minimum Attendance: At least 30 students in a class.
* Quorum: A meeting needs $\geq$ 10 members to proceed.

>>> Modeling. Objective Function: Cost vs. Profit

Cost

* Energy: The amount of electricity consumed by a device.
* Distances: Fuel consumed due to a long commute.
* Time: Hours spent on a non-productive activity.
* ...

Anything we would want to minimize.

>>> Modeling. Objective Function: Cost vs. Profit

Cost

* Energy: The amount of electricity consumed by a device.
* Distances: Fuel consumed due to a long commute.
* Time: Hours spent on a non-productive activity.
* ...

Anything we would want to minimize.

Profit

* Revenue: Money earned from selling a product.
* Yield: Quantity of crops produced per acre.
* Investment Returns: Profit from stocks or bonds.
* ...

Anything we would want to maximize.

Optimal Solutions

* Sometimes there's a *unique* optimal solution.

Optimal Solutions

* Sometimes there's a *unique* optimal solution.
* Other times, multiple solutions offer the same optimal value.

>>> Optimal solution, Exact and Approximate Methods

Optimal Solutions

* Sometimes there's a *unique* optimal solution.
* Other times, multiple solutions offer the same optimal value.
* Exact methods guarantee reaching this optimality.

>>> Optimal solution, Exact and Approximate Methods

Optimal Solutions

* Sometimes there's a *unique* optimal solution.
* Other times, multiple solutions offer the same optimal value.
* Exact methods guarantee reaching this optimality.
* An optimal solution is one of the best feasible solutions.

Optimal Solutions

* Sometimes there's a *unique* optimal solution.
* Other times, multiple solutions offer the same optimal value.
* Exact methods guarantee reaching this optimality.
* An optimal solution is one of the best feasible solutions.
* Their Objective value gives the Min Cost or the Max Profit.

>>> Optimal solution, Exact and Approximate Methods

Optimal Solutions

* Sometimes there's a *unique* optimal solution.
* Other times, multiple solutions offer the same optimal value.
* Exact methods guarantee reaching this optimality.
* An optimal solution is one of the best feasible solutions.
* Their Objective value gives the Min Cost or the Max Profit.

Exact vs. Approximate Methods (Heuristics)

* Exact Methods: Target the true optimal solution. Time-consuming for complex problems.

>>> Optimal solution, Exact and Approximate Methods

Optimal Solutions

* Sometimes there's a *unique* optimal solution.
* Other times, multiple solutions offer the same optimal value.
* Exact methods guarantee reaching this optimality.
* An optimal solution is one of the best feasible solutions.
* Their Objective value gives the Min Cost or the Max Profit.

Exact vs. Approximate Methods (Heuristics)

* Exact Methods: Target the true optimal solution. Time-consuming for complex problems.
* Heuristics: Provide "good enough" solutions when exact methods are impractical.

>>> Optimal solution, Exact and Approximate Methods

Optimal Solutions

* Sometimes there's a *unique* optimal solution.
* Other times, multiple solutions offer the same optimal value.
* Exact methods guarantee reaching this optimality.
* An optimal solution is one of the best feasible solutions.
* Their Objective value gives the Min Cost or the Max Profit.

Exact vs. Approximate Methods (Heuristics)

* Exact Methods: Target the true optimal solution. Time-consuming for complex problems.
* Heuristics: Provide "good enough" solutions when exact methods are impractical.
* With heuristics, sometimes we obtain the optimal solution, but without a guarantee of its optimality.

>>> Optimal solution, Exact and Approximate Methods

Optimal Solutions

* Sometimes there's a *unique* optimal solution.
* Other times, multiple solutions offer the same optimal value.
* Exact methods guarantee reaching this optimality.
* An optimal solution is one of the best feasible solutions.
* Their Objective value gives the Min Cost or the Max Profit.

Exact vs. Approximate Methods (Heuristics)

* Exact Methods: Target the true optimal solution. Time-consuming for complex problems.
* Heuristics: Provide "good enough" solutions when exact methods are impractical.
* With heuristics, sometimes we obtain the optimal solution, but without a guarantee of its optimality.

In this course we will focus on the search of optimal solutions with the help of mathematical programming techniques which start with modeling the system to optimize with a mathematical program.

>>> Duality: Minimization vs. Maximization with Matrix formulation

* Primal (Minimization Problem):

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

>>> Duality: Minimization vs. Maximization with Matrix formulation

* Primal (Minimization Problem):

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

* Dual (Maximization Problem):

$$\begin{aligned} \text{maximize} \quad & b^T y \\ \text{subject to} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

\>>> Optimal Objective Value

* If the primal problem has an optimal solution x^* with objective value z^*,

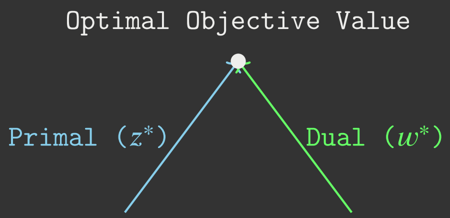
>>> Optimal Objective Value

* If the primal problem has an optimal solution x^* with objective value z^*,
* And the dual problem has an optimal solution y^* with objective value w^*,

>>> Optimal Objective Value

* If the primal problem has an optimal solution x^* with objective value z^*,
* And the dual problem has an optimal solution y^* with objective value w^*,

Then, $z^* = w^*$ under certain conditions.



>>> Canonical Formulation of a Mathematical Program

Definition

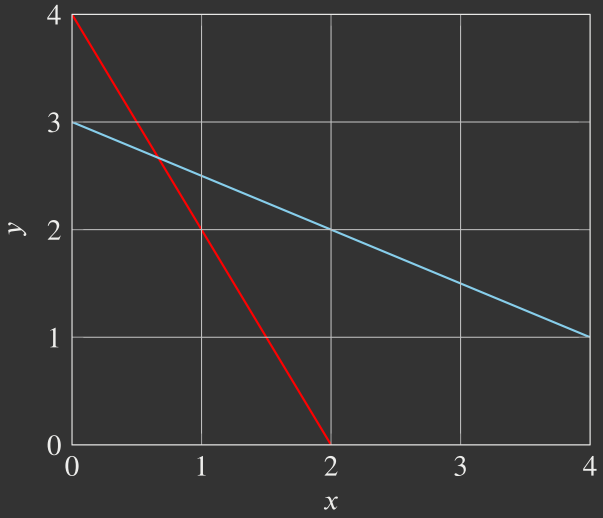
The canonical formulation of a mathematical program is a standardized way of representing optimization problems using mathematical expressions.

$$
\begin{aligned}
\text{minimize} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

Where:

* c, x are vectors of coefficients and decision variables, respectively.
* a is a matrix of constraints coefficients.
* b is a vector representing the right-hand side of the constraints.

>>> Examples. A Mathematical Program for a pretty useless case



Case Model

Objective:

Max or Min: $f(x, y) = x + y$

Constraints and *Variables*:

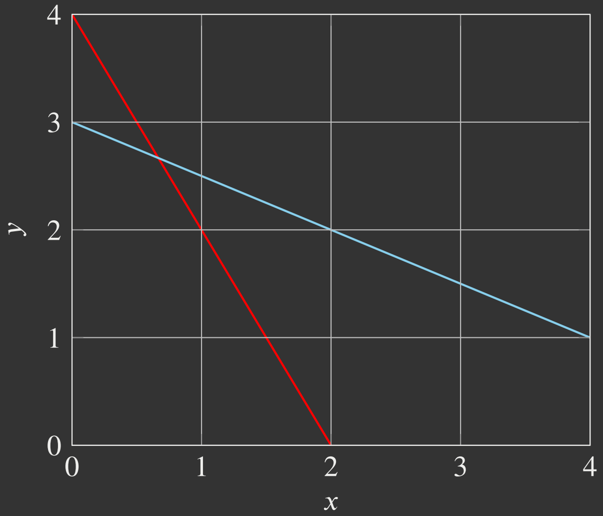
$$2x + y \leq 4$$

$$x + 2y \leq 6$$

$$x, y \geq 0$$

$x, y \in \mathbb{N}$ or $\in \mathbb{R}^+$

>>> Examples. A Mathematical Program for a pretty useless case



Case Model

Objective:

Max or Min: $f(x,y) = x + y$

Constraints and *Variables*:

$$2x + y \leq 4$$

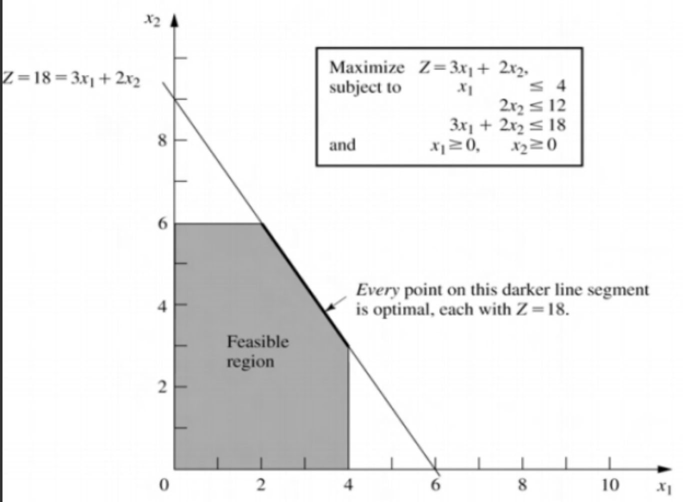
$$x + 2y \leq 6$$

$$x, y \geq 0$$

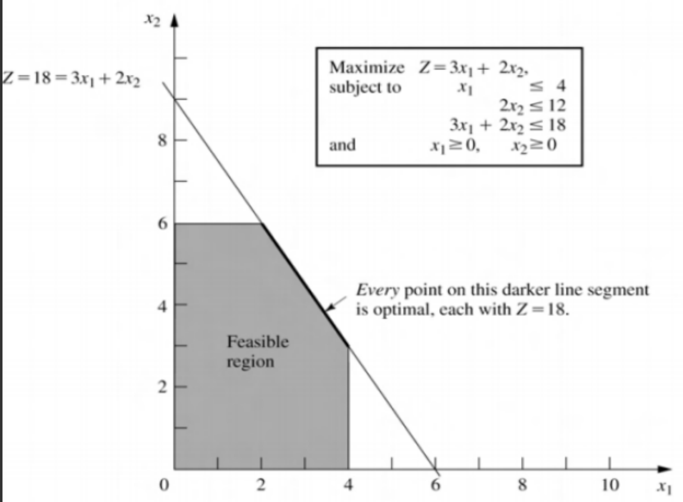
$x, y \in \mathbb{N}$ or $\in \mathbb{R}^+$

Try to find the feasible region and the optimal solutions for the minimization, and the maximization of $f(x,y) = x + y$ for continuous and then integer variables.

>>> Examples of feasible region defined by the constraints.



>>> Examples of feasible region defined by the constraints.



Remark: the graphical resolution can be done with two variables. Majority of the problems we will tackle have many variables and many constraints.

Which system have you already optimized?

Which system have you already optimized?

Let's focus on an easy problem:

The shortest path problem.

>>> Shortest Path Problem Mathematical Program (generic model)

The Shortest Path Problem (SPP) is a classical optimization problem that can be described as:

The Shortest Path Problem (SPP) is a classical optimization problem that can be described as:

* Given a directed graph $G = (V, E)$, with V the set of vertices, and E the set of edges. The source node is noted s, the target node t, and the edge weights w_{ij} for $(i, j) \in E$.

The Shortest Path Problem (SPP) is a classical optimization problem that can be described as:

* Given a directed graph $G = (V, E)$, with V the set of vertices, and E the set of edges. The source node is noted s, the target node t, and the edge weights w_{ij} for $(i, j) \in E$.
* The goal is to find the shortest path from s to t which minimizes the total weight of the path.

>>> Shortest Path Problem Mathematical Program (generic model)

The Shortest Path Problem (SPP) is a classical optimization problem that can be described as:

* Given a directed graph $G = (V, E)$, with V the set of vertices, and E the set of edges. The source node is noted s, the target node t, and the edge weights w_{ij} for $(i, j) \in E$.
* The goal is to find the shortest path from s to t which minimizes the total weight of the path.

Which type of variables for modeling the SPP?

>>> Shortest Path Problem Mathematical Program (generic model)

We need here binary variables:

$$x_{ij} = \begin{cases} 1 & \text{if edge } (i,j) \text{ is on the shortest path} \\ 0 & \text{otherwise} \end{cases}$$

>>> Shortest Path Problem Mathematical Program (generic model)

* The objective function minimizes the total weight of the selected edges.

>>> Shortest Path Problem Mathematical Program (generic model)

* The objective function minimizes the total weight of the selected edges.
* The first constraint ensures that there's exactly one outgoing edge from the source.

>>> Shortest Path Problem Mathematical Program (generic model)

* The objective function minimizes the total weight of the selected edges.
* The first constraint ensures that there's exactly one outgoing edge from the source.
* The second constraint ensures there's exactly one incoming edge to the target.

>>> Shortest Path Problem Mathematical Program (generic model)

* The objective function minimizes the total weight of the selected edges.
* The first constraint ensures that there's exactly one outgoing edge from the source.
* The second constraint ensures there's exactly one incoming edge to the target.
* The third set of constraints ensure flow conservation for all nodes except source and target.

>>> Shortest Path Problem Mathematical Program (generic model)

* The objective function minimizes the total weight of the selected edges.
* The first constraint ensures that there's exactly one outgoing edge from the source.
* The second constraint ensures there's exactly one incoming edge to the target.
* The third set of constraints ensure flow conservation for all nodes except source and target.
* The last constraint restricts the decision variable to binary values.

>>> Shortest Path Problem Mathematical Program (generic model)

* The objective function minimizes the total weight of the selected edges.
* The first constraint ensures that there's exactly one outgoing edge from the source.
* The second constraint ensures there's exactly one incoming edge to the target.
* The third set of constraints ensure flow conservation for all nodes except source and target.
* The last constraint restricts the decision variable to binary values.

The Shortest Path Problem can be efficiently solved using various algorithms like Dijkstra's or Bellman-Ford. However, its linear programming formulation provides a mathematical representation and allows integration into more complex optimization models (e.g., new constraints).

>>> Shortest Path Problem Mathematical Program

The linear programming formulation (here: a generic model) is:

Objective Function:

$$\text{Minimize} \sum_{(i,j)\in E} w_{ij} x_{ij}$$

Constraints:

$$\sum_{(s,j)\in E} x_{sj} - \sum_{(i,s)\in E} x_{is} = 1$$

$$\sum_{(t,j)\in E} x_{tj} - \sum_{(i,t)\in E} x_{it} = -1$$

$$\sum_{(k,j)\in E} x_{kj} - \sum_{(i,k)\in E} x_{ik} = 0, \quad \forall k \in V \setminus \{s,t\}$$

$$x_{ij} \in \{0,1\} \quad \forall (i,j) \in E$$

>>> Shortest Path Problem Mathematical Program

The linear programming formulation (here: a generic model) is:

Objective Function:

$$\text{Minimize} \sum_{(i,j)\in E} w_{ij}x_{ij}$$

Constraints:

$$\sum_{(s,j)\in E} x_{sj} - \sum_{(i,s)\in E} x_{is} = 1$$

$$\sum_{(t,j)\in E} x_{tj} - \sum_{(i,t)\in E} x_{it} = -1$$

$$\sum_{(k,j)\in E} x_{kj} - \sum_{(i,k)\in E} x_{ik} = 0, \quad \forall k \in V \setminus \{s,t\}$$

$$x_{ij} \in \{0,1\} \quad \forall (i,j) \in E$$

Let's do a small exercise where we look for a case model!

>>> Mathematical Programming Algorithms

Algorithms solving mathematical programming:

* For solving continuous (with real variables) problems
    * Simplex
    * Interior-point method
    * Column Generation and Benders decomposition

>>> Mathematical Programming Algorithms

Algorithms solving mathematical programming:

* For solving continuous (with real variables) problems
    * Simplex
    * Interior-point method
    * Column Generation and Benders decomposition
* For solving mixed integer problem
    * Branch-and-Bound (Mixed integer problems)
    * Branch-and-Price (Mixed integer problems)
    * Branch-and-Cut

>>> Optimization Solvers

All the techniques previously referenced have been integrated in tools called solvers. Here are the main ones:

Open Source Solvers:

* COIN-OR Linear Programming (CLP) and COIN-OR Branch and Cut (CBC)
* SCIP (Solving Constraint Integer Programs)
* GLPK (GNU Linear Programming Kit)
* OSQP (Operator Splitting Quadratic Program solver)
* Google OR-Tools

>>> Optimization Solvers

All the techniques previously referenced have been integrated in tools called solvers. Here are the main ones:

Open Source Solvers:

* COIN-OR Linear Programming (CLP) and COIN-OR Branch and Cut (CBC)
* SCIP (Solving Constraint Integer Programs)
* GLPK (GNU Linear Programming Kit)
* OSQP (Operator Splitting Quadratic Program solver)
* Google OR-Tools

Commercial Solvers:

* CPLEX (IBM)
* Gurobi
* Xpress
* (Excel)

>>> Mathematical programming Languages

Several languages and modeling tools have been developed to help express and solve optimization problems based on mathematical programs. Some of the most famous languages and tools include:

* AMPL (A Mathematical Programming Language): A popular modeling language for mathematical optimization. You can define an optimization model and then solve it using various solvers (like CPLEX, Gurobi, etc.).
* GAMS (General Algebraic Modeling System): A high-level modeling system for mathematical optimization. GAMS allows you to build large structured models and is known for its scalability.

>>> Mathematical programming Languages

Here an example of a AMPL program:

Sets:

PRODUCTS

Parameters:

profit(PRODUCTS)

amount(PRODUCTS) ≥ 0

Variables:

x(PRODUCTS) ≥ 0

Maximize

Total_Profit: $\sum$ profit$(p) \times x(p)$

>>> Mathematical programming Languages

Here an example of a GAMS program:

```
Set i "products" / A, B /;

Parameter profit(i) "profit per product"
/ A 50
B 60 /;

Variable x(i) "amount of product to produce";
Variable total_profit "total profit";

Equations profit_eq "objective function",
constraint1 "production constraint";

profit_eq..   total_profit =e= sum(i, profit(i)*x(i));

constraint1.. sum(i, x(i)) =l= 100;

Model prod_model /all/;

Solve prod_model using lp maximizing total_profit;
```

(We will continue with mathematical programs - generic and case models)

An end-to-end example
Model and resolution

>>> An end-to-end example: the outline

1. Understand the system to optimize and its limits
    * We will analyze a text describing the situation.

>>> An end-to-end example: the outline

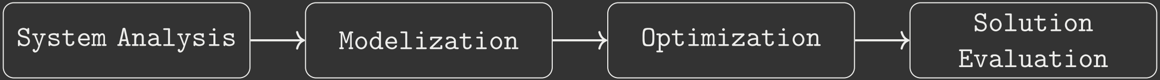
1. Understand the system to optimize and its limits
   * We will analyze a text describing the situation.
2. Model the problem with a linear program
   * We will define the variables, the Objective function and the constraints.

>>> An end-to-end example: the outline

1. Understand the system to optimize and its limits
   * We will analyze a text describing the situation.
2. Model the problem with a linear program
   * We will define the variables, the Objective function and the constraints.
3. Search for a convenient solver
   * We will start for this example with light solvers: two online solvers.

>>> An end-to-end example: the outline

1. Understand the system to optimize and its limits
   * We will analyze a text describing the situation.
2. Model the problem with a linear program
   * We will define the variables, the Objective function and the constraints.
3. Search for a convenient solver
   * We will start for this example with light solvers: two online solvers.
4. Solve and read the solution
   * We will analyze the results.



>>> An end-to-end example: Understand the system to optimize and its limits

The company "JUNISEN le lait" produces dairy products and wants to maximize its profit from its two flagship products: yogurt and pudding. One ton of yogurt brings in 500€ for the company, while a ton of pudding only brings in 450€. The law limits JUNISEN to producing a maximum of 8 tons of yogurt, but places no restrictions on pudding production. The total available production time is 60 hours, given that it takes 6 hours to produce one ton of yogurt and 5 hours for a ton of pudding. The entire production can be stored in 15 square meters, with one ton of yogurt consuming 1 square meter and pudding requiring 2 square meters.

>>> An end-to-end example: Understand the system to optimize and its limits

The company "JUNISEN le lait" produces dairy products and wants to maximize its profit from its two flagship products: yogurt and pudding. One ton of yogurt brings in 500€ for the company, while a ton of pudding only brings in 450€. The law limits JUNISEN to producing a maximum of 8 tons of yogurt, but places no restrictions on pudding production. The total available production time is 60 hours, given that it takes 6 hours to produce one ton of yogurt and 5 hours for a ton of pudding. The entire production can be stored in 15 square meters, with one ton of yogurt consuming 1 square meter and pudding requiring 2 square meters.

What refers to the variables, the objective function and what corresponds to the constraints?

>>> An end-to-end example: Understand the system to optimize and its limits

Variables Objectif function Constraints

The company "JUNISEN le lait" produces dairy products and wants to maximize its profit from its two flagship products: yogurt and pudding. One ton of yogurt brings in 500€ for the company, while a ton of pudding only brings in 450€. The law limits JUNISEN to producing a maximum of 8 tons of yogurt, but places no restrictions on pudding production. The total available production time is 60 hours, given that it takes 6 hours to produce one ton of yogurt and 5 hours for a ton of pudding. The entire production can be stored in 15 square meters, with one ton of yogurt consuming 1 square meter and pudding requiring 2 square meters.

>>> An end-to-end example: Understand the system to optimize and its limits

Variables Objectif function Constraints

The company "JUNISEN le lait" produces dairy products and wants to maximize its profit from its two flagship products: yogurt and pudding. One ton of yogurt brings in 500€ for the company, while a ton of pudding only brings in 450€. The law limits JUNISEN to producing a maximum of 8 tons of yogurt, but places no restrictions on pudding production. The total available production time is 60 hours, given that it takes 6 hours to produce one ton of yogurt and 5 hours for a ton of pudding. The entire production can be stored in 15 square meters, with one ton of yogurt consuming 1 square meter and pudding requiring 2 square meters.

Try to express the case model.

>>> Model the problem with a linear program

* We optimize the production by maximizing the following Objective function:

>>> Model the problem with a linear program

* We optimize the production by maximizing the following Objective function:

$$f(x, y) = 500x + 450y \tag{1}$$

>>> Model the problem with a linear program

* We optimize the production by maximizing the following Objective function:

$$f(x, y) = 500x + 450y \tag{1}$$

* The time resource constraint is given by:

>>> Model the problem with a linear program

* We optimize the production by maximizing the following Objective function:

$$f(x, y) = 500x + 450y \tag{1}$$

* The time resource constraint is given by:

$$6x + 5y \leq 60 \tag{2}$$

>>> Model the problem with a linear program

* We optimize the production by maximizing the following Objective function:

$$f(x, y) = 500x + 450y \tag{1}$$

* The time resource constraint is given by:

$$6x + 5y \leq 60 \tag{2}$$

* The capacity storage constraint is given by:

>>> Model the problem with a linear program

* We optimize the production by maximizing the following Objective function:

$$f(x, y) = 500x + 450y \tag{1}$$

* The time resource constraint is given by:

$$6x + 5y \leq 60 \tag{2}$$

* The capacity storage constraint is given by:

$$1x + 2y \leq 15 \tag{3}$$

>>> Model the problem with a linear program

* We optimize the production by maximizing the following Objective function:

$$f(x, y) = 500x + 450y \tag{1}$$

* The time resource constraint is given by:

$$6x + 5y \leq 60 \tag{2}$$

* The capacity storage constraint is given by:

$$1x + 2y \leq 15 \tag{3}$$

* The constraint limited the number of yogurt to be produced is:

>>> Model the problem with a linear program

* We optimize the production by maximizing the following Objective function:

$$f(x, y) = 500x + 450y \tag{1}$$

* The time resource constraint is given by:

$$6x + 5y \leq 60 \tag{2}$$

* The capacity storage constraint is given by:

$$1x + 2y \leq 15 \tag{3}$$

* The constraint limited the number of yogurt to be produced is:

$$x \leq 8 \tag{4}$$

>>> Model the problem with a linear program

We denote x and y two positive decision variables related to the number of yogurt and pudding to be produced, respectively.

$$Max \quad f(x, y) = 500x + 450y \tag{5}$$

$$6x + 5y \leq 60 \tag{6}$$

$$1x + 2y \leq 15 \tag{7}$$

$$x \leq 8 \tag{8}$$

1- TRY to solve the program graphically.

(then we will check your result with a solver).

(You've a tuto for Excel but we will focus on 2 online solvers).

2- Real (continuous) Variables: https://online-optimizer.appspot.com

3- Integer: https://neos-server.org/neos/solvers/milp:CPLEX/LP.html

For the integer variables, refer to the CPLEX .lp file documentation to properly define and submit the correct variable types to the solver.

>>> Model the problem with a linear program

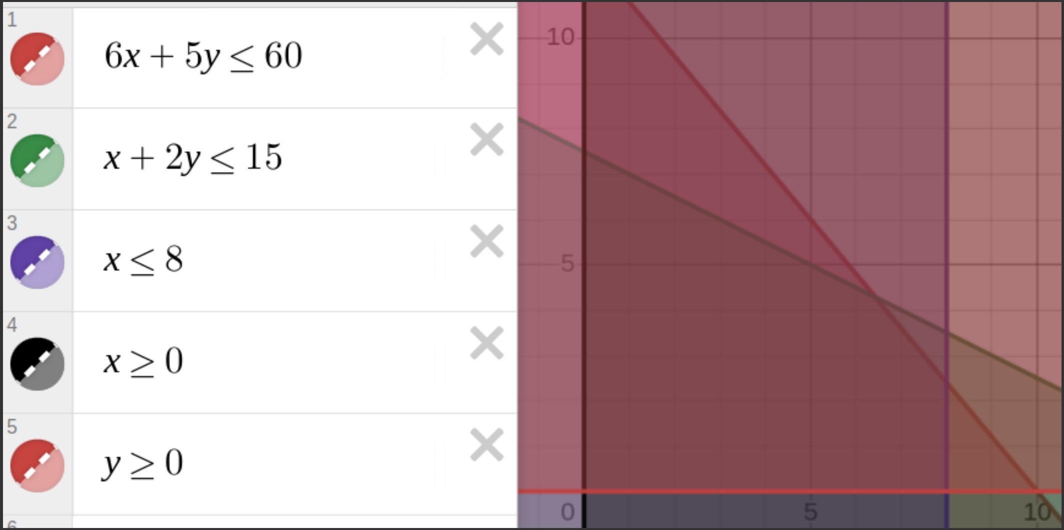
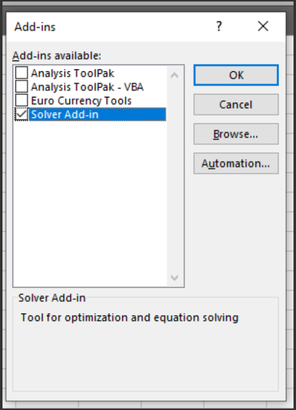


Figure 1: Graphical representation of the solution space of the linear program.

>>> (TUTO) Solver: Excel

* Install Add-Ins (Data tab, Search Add-ins activate "Solver Add-in". It might also work with Online Excel).
* Official Doc:

https://support.microsoft.com/en-gb/office/define-and-solve-a-problem-by-using-solver-5d1a388f-079d-43ac-a7eb-f63e45925040



>>> (TUTO) Solver: Excel

From your table expressing the problem, implement it into excel:

1. Write the data
2. Write the expression of the Objective Function and related the constraints
3. Configure the solver:
   * Click on "Solver" (Data Tab)
   * Implement the Objective Function
   * Objective Function
   * Choose the direction of the Objective function (Min or Max)
   * Choose the Variables ("By Changing Variable Cells")

	Yogurt	Pudding	Used	Availability/Capacity
Integer Variables (nomber of)	6,428571	4,285714		
Profit	500	450		
Time (hour for cooking a ton)	6	5	60	60
Storage Capacity Consumption	1	2	15	15
Maximum Quantity	8			
Total Profit (Objective Function)	5142,857143			

Figure 2: Excel Solver Solution Found

>>> (TUTO) Solver: Excel

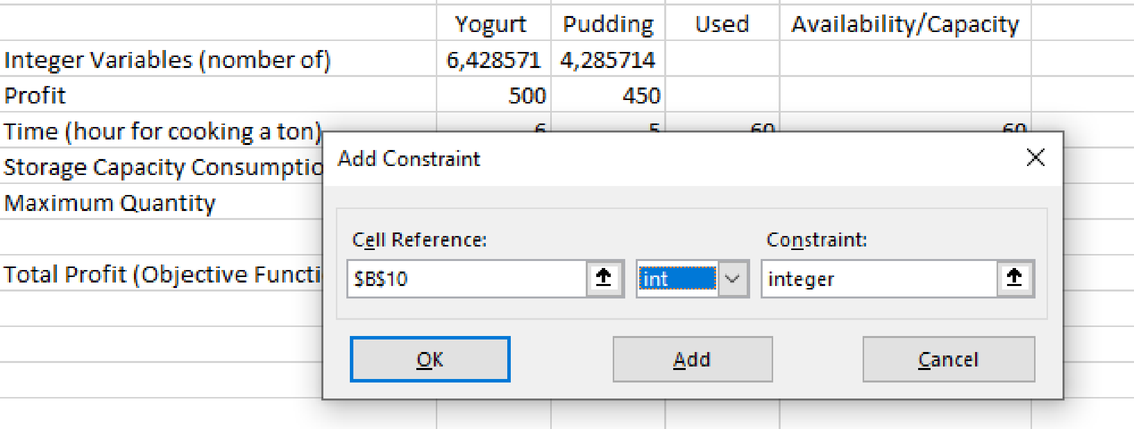
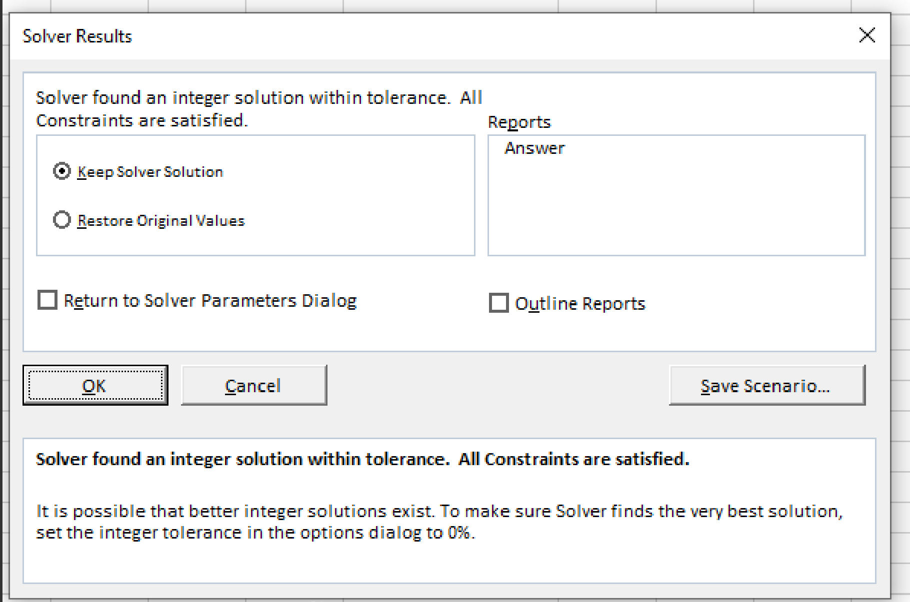


Figure 3: A new constraint per variables for obtaining integer values

>>> (TUTO) Solver: Excel



>>> (TUTO) Solver: Excel

	Yogurt	Pudding	Used	Availability/Capacity
Integer Variables (nomber of)	8	2		
Profit	500	450		
Time (hour for cooking a ton)	6	5	58	60
Storage Capacity Consumption	1	2	12	15
Maximum Quantity	8			
Total Profit (Objective Function)			4900	

>>> The last exercise of the day! Let's take some videos for the week-end!

We have a USB flash drive that is already quite full, with only 5 GB of free space remaining. We want to copy video files onto this flash drive to take with us on a trip. Each file has a size, and each video has a duration. The duration is not proportional to the size because the files have different formats; some videos are of high quality, while others are highly compressed. The following table presents the available files with durations given in minutes."

File Name	Duration (Min)	Size
Video 1	114	4,57 Go
Video 2	32	630 Mo
Video 3	20	1,65 Go
Video 4	4	85 Mo
Video 5	18	2,15 Go
Video 6	80	2,71 Go
Video 7	5	320 Mo

Table 1: Which videos should be copied to have the maximum viewing time possible?.