ME140A: Numerical Analysis in Engineering

Lecture Notes

Alexander Meiburg

9/22/22 - 9/22/22

Course Organization

- Goal: **Numerical solution of integrals and differential equations**
- Homework will rely significantly on you programming these methods you learn. MATLAB recommended, alternatives welcome
- Collaborate on the homework! You learn more that way. Just make sure that you are, in fact, learning. ☺
- HW: 10% of grade. Exams: 30%/30%/30%.
- Submit homework via email
- Office hours by appointment or Zoom, but my schedule is very open!
- Full syllabus available here
- These notes will be continually updated here

Numerical Integration

Recall: **Differentiation** systematically lets you take a function $F(x)$ and find its derivative $f(x) = F'(x)$.

$$\frac{d}{dx}\left(e^{\sin(x+\log x)}\right) = e^{\sin(x+\log x)}\cos(x+\log x)\,(1+1/x)$$

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Integration asks for the opposite. You have a handful of rules(!), but they can't cover every case. Often impossible, and we resort to defining new functions or using the computer

$$\int e^{-x^2}\,dx := \Phi(x)$$

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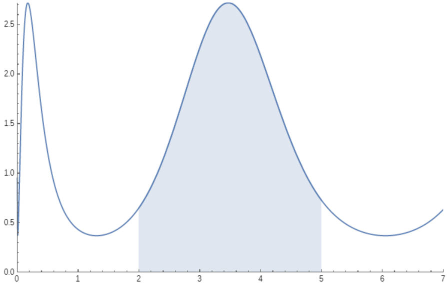

Numerical Integration

Example: Computing the position of an object after some movement.

$$y(t) = \text{Position as a function of time}$$

$$v(t) = \text{Velocity}$$

$$v = \frac{dy(t)}{dt}, \quad y(t) = \int_0^t v(t)\, dt$$

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May also have y as an integral over *several* variables, not just one. *e.g.* Dust accumulating on a surface varies with x, y, and t. Can do three integrals in a row (analytically), or one 3D integral (numerically).

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But consider:

t	y(t)
0	5
1	6.1
2	7.3
3	8.4
4	9.8
7	15.3
8	17.4
9	59.8
10	138.7
11	138.8

Issues such as irregular data, or gaps in time too large to understand what happened. Big question in its own right, Week 2!

Newton-Cotes

Problem: given $f(t)$, find $F(t) = \int_0^t f(t)\, dt$. If $f(t)$ is too complicated, let's find something simpler we can integrate. What's simple? Polynomials!

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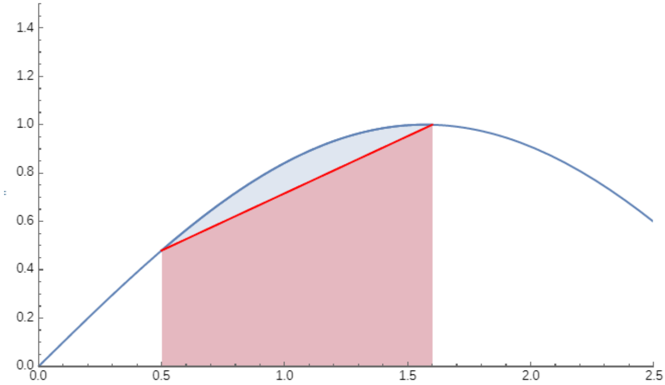
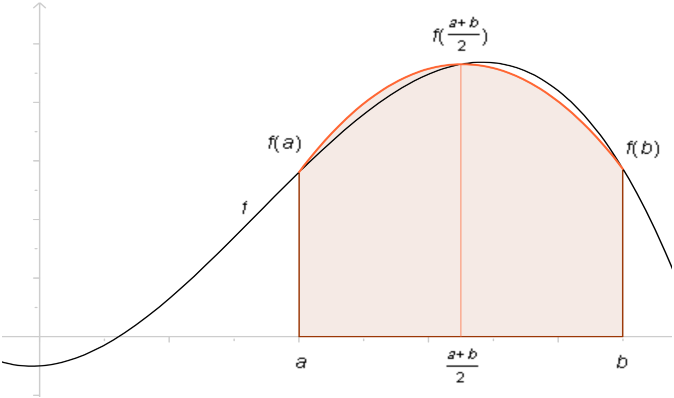
Simplest: Linear fit through two points. ("Trapezoidal rule")



Newton-Cotes

Fit quadratic ("Simpson's rule"):



Credit: Wikimedia

Newton-Cotes

In general, find

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$$

and integrate

$$\int_a^b f_n(x)\, dx$$

Turns out: a_i depend linearly on the $f(x_i)$, so the result is some weighted sum of the $f(x_i)$.

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Trapezoidal:

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2}(f(b) + f(a))$$

Simpson's:

$$\int_a^b f(x)\, dx \approx \frac{b-a}{6}\left(f(b) + 4f\left(\frac{a+b}{2}\right) + f(a)\right)$$

Newton-Cotes

If our function is too complicated over $[a, b]$, then subdivide and do each separately.

$$\int_{x=a}^{b} f(x) = \int_{x=a}^{(a+b)/2} f(x) \quad + \quad \int_{x=(a+b)/2}^{b} f(x)$$

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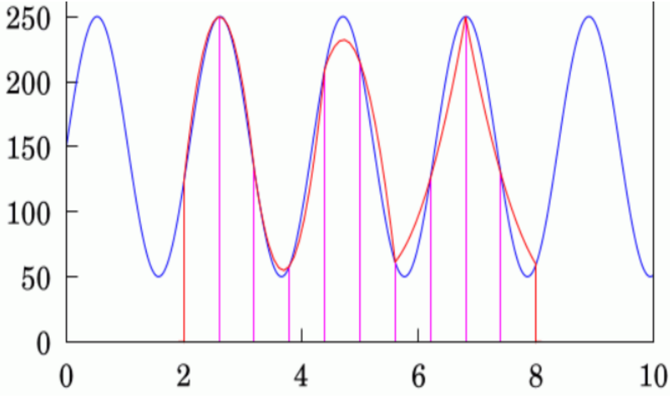

Error Analysis

Write

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$

First and second terms accurate, third isn't. Fitting gives

$$E_{trap} \approx \frac{1}{12}|f''(\xi)|(b-a)^2$$

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Refine this with subintervals

Simple Integration Rules

- Left endpoint rule:

$$F = \int_a^b f(x) \approx \frac{(a-b)}{2} f(a)$$

$$\mathrm{Err} \leq |f'| \frac{(b-a)^2}{2}$$

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- Midpoint rule:

$$F = \int_a^b f(x) \approx \frac{(a-b)}{2} f\left(\frac{a+b}{2}\right)$$

$$\mathrm{Err} \leq |f''| \frac{(b-a)^3}{24}$$

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- Trapezoid rule:

$$F = \int_a^b f(x) \approx \frac{(a-b)}{2} \frac{f(a)+f(b)}{2}$$

$$\text{Err} \leq |f''| \frac{(b-a)^3}{12}$$

Simple Integration Rules

- Simpson's "1/3" rule:

$$F = \int_a^b f(x) \approx \frac{(a-b)}{2} \frac{f(a) + 4f((a+b)/2) + f(b)}{3}$$

$$\mathrm{Err} \le |f^4| \frac{(b-a)^5}{180}$$

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- Simpson's "3/8" rule:

$$F = \int_a^b f(x) \approx \frac{(a-b)}{2} \frac{f(a) + 3f((2a+b)/3) + 3f((a+2b)/3) + f(b)}{8}$$

$$\mathrm{Err} \leq |f^4| \frac{(b-a)^5}{6480}$$

Composite Integration Rules

Subdivide into intervals of size $h = (b - a)/n$.

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$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{j=2}^{n}\left[f(x_{j-1}) + f(x_j)\right] \tag{1}$$

$$= \frac{h}{2}\left[f(x_0) + 2\sum_{j=2}^{n-1} f(x_j) + f(x_n)\right] \tag{2}$$

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Error changes from

$$\mathrm{Err} \leq |f''|\frac{(b-a)^3}{12}$$

into

$$\mathrm{Err} \leq n\cdot|f''|\frac{((b-a)/n)^3}{12} = |f''|\frac{(b-a)^3}{12n^2}$$

Scaling like $1/n^2$, so this has a *second order* approximation error. (It is a *first order* rule, because it fits a first order polynomial – a line segment.)

Composite Integration Rules

Simpson's 1/3 Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\sum_{j=1}^{n/2}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right] \tag{3}$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + f(x_n)\right] \tag{4}$$

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Error changes from

$$\mathrm{Err} \leq |f^4|\frac{(b-a)^5}{180}$$

into

$$\mathrm{Err} \leq n\cdot|f^4|\frac{((b-a)/n)^5}{180} = |f^4|\frac{(b-a)^5}{180n^4}$$

Scaling like $1/n^4$, so this has a *fourth order* approximation error. (It is a *second order* rule, because it fits a second order polynomial.)

Composite Integration Rules

...and beyond?

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...and beyond? These have been extended to use 4th, 5th, 6th... order polynomials, and get higher-order methods. In practice, the $1/n^k$ is not the limiting factor if $k > 4$, and the integral will only improve with smaller intervals.

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Plot[Sin[1 / (Sin[x] + 1.04)], {x, 2, 13}, PlotRange → All]
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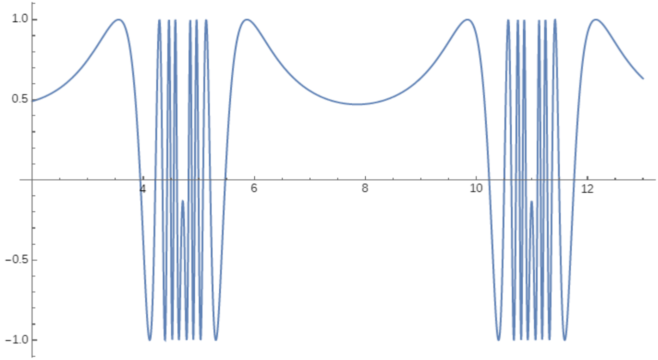
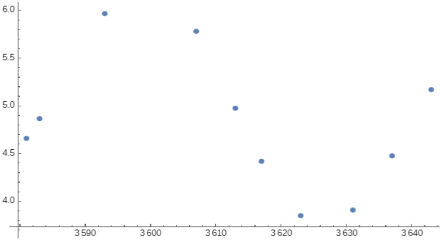


You just can't get this accurate, without having small intervals! And once you get small enough, the function will be roughly quadratic anyway.

Irregular Integration



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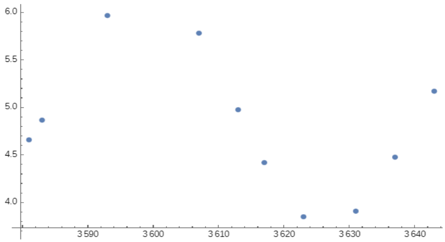
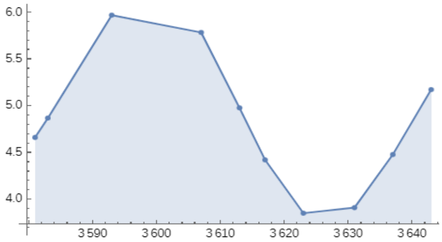
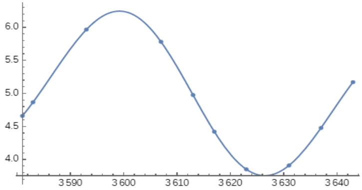


Trapezoidal integration on each part:



Irregular Integration

In principle, we can fit higher-order polynomials as well



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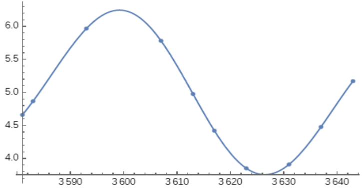
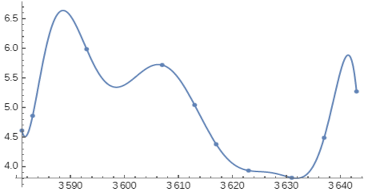


But this can again be very sensitive and unstable:



In practice, also expensive to compute. Need to recompute the "weights" each time, which requires solving a linear system.

Multidimensional Integration

Computing

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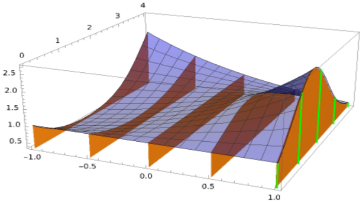
One option: do each integral with its own 1D algorithm.

$$F = \int_{x_1}^{x_2} G(x) \, dx$$

$$G(x) = \int_{y_1}^{y_2} e^{x \sin(y)} + \frac{\ln(y-x)}{\ln(y)} \, dy$$



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Or, custom multi-dimensional versions of integration rules. Simple functions integrated over squares, triangles, etc.