ME140A: Numerical Analysis in Engineering

Lecture Notes

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9/22/22 - 9/22/22

Course Organization

- Goal: **Numerical solution of integrals and differential equations**
- Homework will rely significantly on you programming these methods you learn. MATLAB recommended, alternatives welcome
- Collaborate on the homework! You learn more that way. Just make sure that you are, in fact, learning. ☺
- HW: 10% of grade. Exams: 30%/30%/30%.
- Submit homework via email
- Office hours by appointment or Zoom, but my schedule is very open!
- Full syllabus available here
- These notes will be continually updated here

Numerical Integration

Recall: **Differentiation** systematically lets you take a function $F(x)$ and find its derivative $f(x) = F'(x)$.

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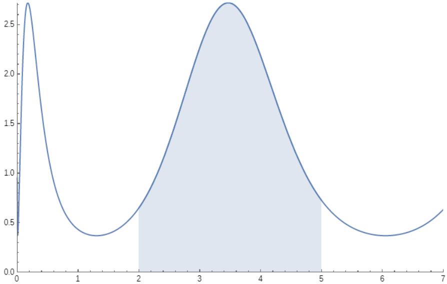

Numerical Integration

Example: Computing the position of an object after some movement.

$$y(t) = \text{Position as a function of time}$$

$$v(t) = \text{Velocity}$$

$$v = \frac{dy(t)}{dt}, \quad y(t) = \int_0^t v(t)\, dt$$

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May also have y as an integral over *several* variables, not just one. *e.g.* Dust accumulating on a surface varies with x, y, and t. Can do three integrals in a row (analytically), or one 3D integral (numerically).

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But consider:

t	y(t)
0	5
1	6.1
2	7.3
3	8.4
4	9.8
7	15.3
8	17.4
9	59.8
10	138.7
11	138.8

Issues such as irregular data, or gaps in time too large to understand what happened. Big question in its own right, Week 2!

Newton-Cotes

Problem: given $f(t)$, find $F(t) = \int_0^t f(t)\, dt$. If $f(t)$ is too complicated, let's find something simpler we can integrate. What's simple? Polynomials!

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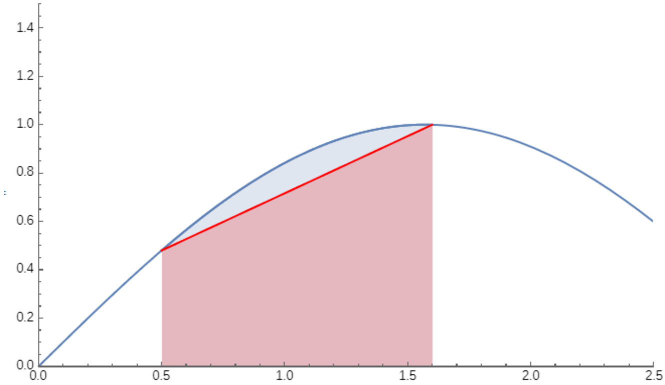
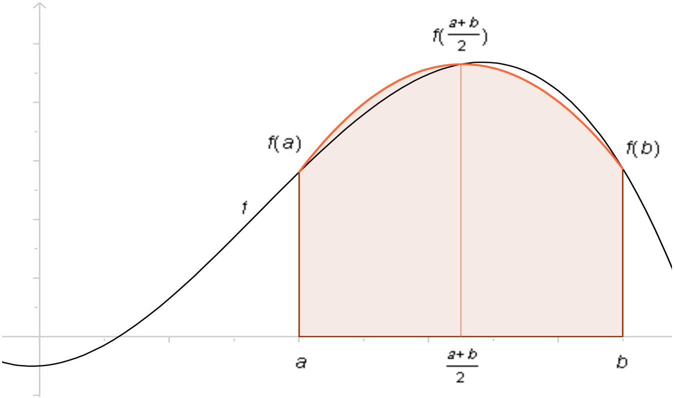
Simplest: Linear fit through two points. ("Trapezoidal rule")



Newton-Cotes

Fit quadratic ("Simpson's rule"):



Credit: Wikimedia

Newton-Cotes

In general, find

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$$

and integrate

$$\int_a^b f_n(x)\, dx$$

Turns out: a_i depend linearly on the $f(x_i)$, so the result is some weighted sum of the $f(x_i)$.

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Trapezoidal:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}(f(b) + f(a))$$

Simpson's:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(b) + 4f\left(\frac{a+b}{2}\right) + f(a)\right)$$

Newton-Cotes

If our function is too complicated over $[a, b]$, then subdivide and do each separately.

$$\int_{x=a}^{b} f(x) = \int_{x=a}^{(a+b)/2} f(x) \quad + \quad \int_{x=(a+b)/2}^{b} f(x)$$

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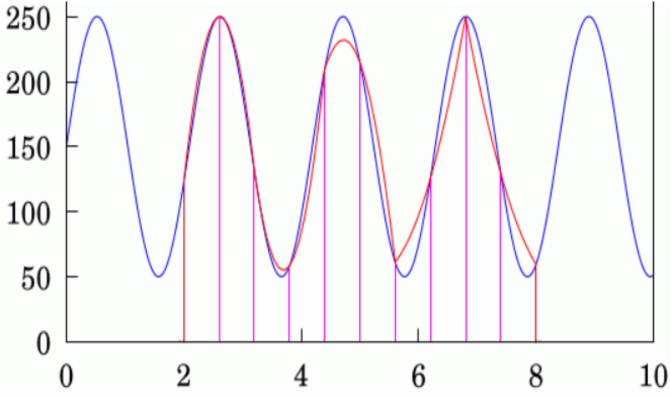

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Error Analysis

Write

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \dots$$

First and second terms accurate, third isn't. Fitting gives

$$E_{trap} \approx \frac{1}{12}|f''(\xi)|(b - a)^2$$

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Refine this with subintervals