ME140A - Midterm 2 - Open Book

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1 1D Fixed Points - 20%

Consider the system,

$$x'(t) = (x^2 + x)(x^2 - x - 2)(x^2 - 3x + 2)^2$$

Find all the fixed points. (You shouldn't need a calculator!) Classify them by their stability. Don't answer "indeterminate" – a particular *test* of stability may be indeterminate if the test isn't enough to tell, but I want you to tell me which are stable or not!

The fixed points are the roots of $x' = f(x)$, which are $\{-1, 0, 1, 2\}$. At $x = 0$ we have $f'(0) < 0$, so it's stable. At the other points, $f' = 0$, so we have to look at the values of f on either side of the fixed point. It's positive on either side of -1, so it's half-stable (stable on left side, unstable on the right). It's negative on either side of 1, so it's half-stable (stable on the right, unstable on the left). At $x = 2$ it switches from negative to positive, so it's unstable.

8 points for getting the fixed points right, 2 points each for classifying each correctly.

2 Initial Value Problems - 40%

Consider the following system:

$$x'' = -x - 3x^3 - t^2$$

with the initial conditions $x(0) = 1$ and $x'(0) = 0$. This system has a zero, i.e. a point t_0 where $x(t_0) = 0$, with t_0 in the interval $[0.5, 1]$. Finding the points where a differential equation crosses certain thresholds is called *event detection*, and is often the metric of interest.

(a) Use two steps of the predictor-correct method with $h = 0.5$, to estimate $x(0.5)$ and $x(1.0)$. The first step gives $x(0.5) = 0.5$ and $x'(0.5) = -2.25$.

The second step gives $x(1.0) = \boxed{-0.9063}$. Note that we didn't need to compute $x'(1.0)$.

(b) Drawing a line between your two points from part (a), estimate time t_0 where $x(t_0) = 0$.

$$t_0 = \frac{0.5}{0.5 + 0.9063} 0.5 + 0.5 = \boxed{0.6778}$$

(c) From your halfway point of part (a) where you have $x(0.5)$, instead of doing a second predictor-corrector step, do a step of the Runge-Kutta 3th order (RK3) method with $h = 0.5$ to estimate $x(1.0)$. As a reminder, the RK3 method is:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1)$$

$$k_3 = f(t_n + h, y_n + 2hk_2 - hk_1)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3)$$

What is $x(1.0)$ with this alternate step? We get $k_1 = -0.90625, -2.25$, $k_2 = -1.46875, -1.99774$, $k_{3,x} = -1.77899$, and finally $x(1.0) = \boxed{-0.21335}$. Note that we didn't need to compute $k_{3,x'}$ or $x'(1.0)$.

(d) Compare your estimates of $x(1.0)$ from parts (d) and (a). How large is the error? Treating this error as your uncertainty in $x(1.0)$, what is your uncertainty in t_0? Explain why this would or wouldn't be a good be a uncertainty to report. The two values of $x(1.0)$ differ by $(-0.21335) - (-0.9063) = 0.6925$. We can see how this changes our estimated value of t_0:

$$t_0 = \frac{0.5}{0.5 + 0.21335} 0.5 + 0.5 = 0.8504$$

So that our uncertainty in t_0 is about $\boxed{0.172}$. This is a $\boxed{\text{bad}}$ estimate! Because we have error from our first step, where we stepped to $x(0.5)$, and just redoing the last step with the RK3 method won't tell us anything about the error from the first step. (It turns out the true value is $t_0 \approx 0.761$, so in this case the uncertainty was okay, but this would be very bad practice in general.)

Grading: Broke down the points as 5 points for correctly understanding this system in terms of x and x', 10 points for part a, 5 points for part b, 10 points for part c, 5 points for part c.

3 High-dimensional Fixed Points - 40%

(a) The system

$$a''(t) = -6 - 2a + a^2 + 3b$$

$$b'(t) = -2 + a + b - 2ab^2 - a'$$

has several fixed points. Find all of them. (Two of them will have neat numerical forms.) You're allowed to use Wolfram Alpha to solve equations. A fixed point requires $a'' = 0$, $b' = 0$, and of course that $a' = 0$ as well. So we solve

$$-6 - 2a + a^2 + 3b = 0$$

$$-2 + a + b - 2ab^2 = 0$$

and Wolfram Alpha gives us 5 solutions. It erroneously reports three of them as having very small imaginary values, when they're actually real.

$$a = -1, b = 1, \quad a = 0, b = 2, \quad a = -2.208, b = -1.096$$

$$a = 3.327, b = 0.528, \quad a = 3.880, b = -0.432$$

(b) Write down the Jacobian of this system. To have a Jacobian, we need to write it down in terms of first-order ODEs. Using v to stand in for a',

$$a' = v$$

$$v' = -6 - 2a + a^2 + 3b$$

$$b'(t) = -2 + a + b - 2ab^2 - v$$

The expressions are all pretty quick to differentiate.

$$\begin{bmatrix} 0 & 1 & 0 \\ -2 + 2a & 0 & 3 \\ 1 - 2b^2 & -1 & 1 - 4ab \end{bmatrix}$$

For example, the bottom right corner is $\frac{\partial(-2+a+b-2ab^2-v)}{\partial b} = 1 - 4ab$. I didn't take points off if you ordered the variables differently (for instance, (v, a, b), but it's critical that you use the same order in the rows and the columns.

(c) Remember that the trace is the sum of the diagonal of the Jacobian, and that it is equal to the sum of the eigenvalues. What can the trace be for a stable fixed point? What is the trace of this matrix? Which fixed points can you immediately check are *not* stable, based on just the trace? Since we have two zeros on the diagonal, the trace is just $1 - 4ab$. For it to be stable, all the eigenvalues have to negative, which means the trace must negative. Looking at our fixed points from part (a), we can immediately rule out $(-1, 1)$ and $(0, 2)$, since they have traces of 5 and 1 respectively. Testing the other three, $(3.880, -0.432)$ also has a positive trace. (The last two points have a negative trace). So only the points $(-2.208, -1.096)$ and $(3.327, 0.528)$ could possibly be stable.

(d) For a 3x3 matrix

$$\begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix}$$

the (real parts of the) eigenvalues are all negative if and only if:

$$p \leq 0$$

$$pt - qs \leq 0$$

$$ptx + quv + rsw - puw - qsx - rtv + \leq 0$$

Evaluate these numerically at all the fixed points you *didn't* rule out in part (c). What are all the stable fixed points of the system? We just have the two points from the last part to check. In our Jacobian, $p = 0$, so that part's already done. With our Jacobian, $pt - qs \leq 0$ is just $2 - 2a \leq 0$, or $a \geq 1$. This rules out $(-2.208, -1.096)$ but leaves $(3.327, 0.528)$ as an option. The last line becomes

$$5 - 2a + 8ab(a-1) - 6b^2 \leq 0$$

$$\implies -4.65445 \leq 0$$

which checks out, so $(3.327, 0.528)$ is indeed stable, and $\boxed{(3.327, 0.528)}$ is the only fixed point.

10 points for finding the fixed points (2pts each) in part a. 10 points for writing down the Jacobian. 5 points for getting the trace and using it correctly in part b, then 5 points for correctly classifying the five fixed points based on the trace (1pt each) in part c. 5 points in part d for ruling out one point with the second inequality, and 5 points for checking the other one works and is stable. 40 points total.