3.0 Quadratic Surfaces (Chapter 10.5)

Def: A surface defined by

$$(*) \quad Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$$

with $A, B, C, D, E, F, G, H, I, J \in \mathbb{R}$ is called a <u>quadratic surface</u>

If $(*)$ can be re-written as the product of two planes i.e.

$$(*) = (A_1x+B_1y+C_1z-D_1)(A_2x+B_2y+C_2z-D_2)=0$$

We say that the surface is <u>degenerate</u> and <u>nondegenerate</u> otherwise.

Nondegenerate quadratic surfaces falls into 6 categories.

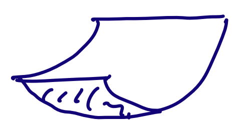
1) <u>Spheres</u>: The equation $x^2+y^2+z^2=a^2$ represents a sphere of radius a centered at the origin.

2) <u>Cylinders</u>

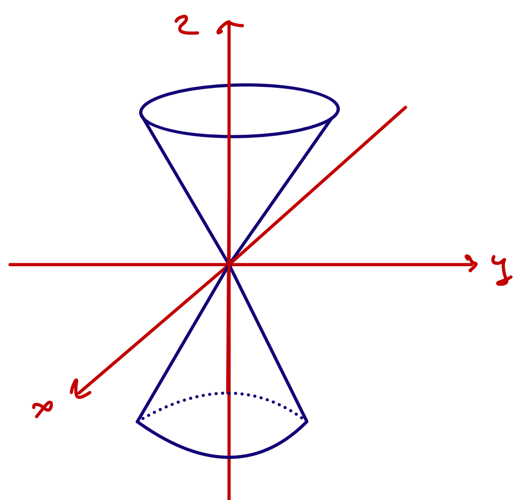
   i) The equation $x^2+y^2=a^2$ represents a right circular cylinder.

   ii) Quadric cylinder can also be elliptic, parabolic, hyperbolic.

   <u>Ex.</u> Parabolic $z=x^2$

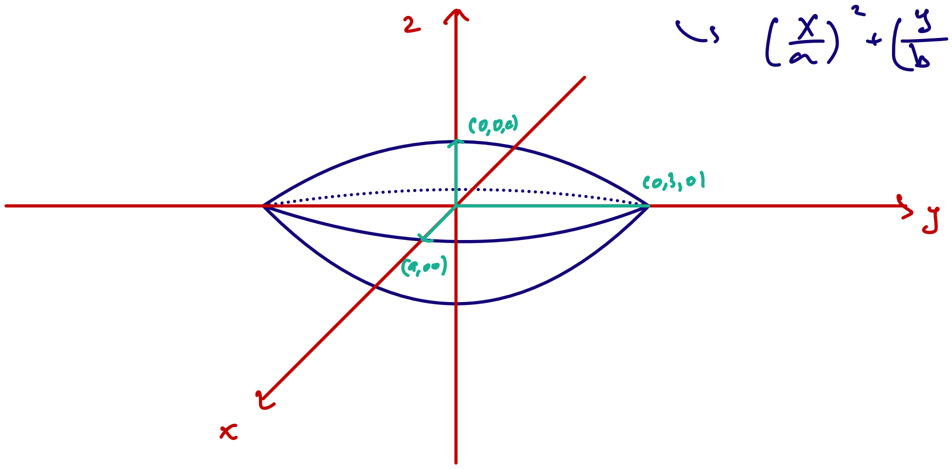


3) <u>Cones</u>: The equation $z^2=x^2+y^2$ represents a right-circular cone.



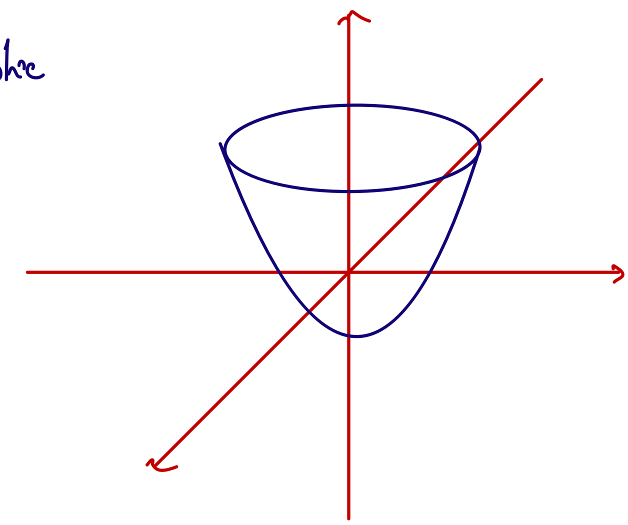
4) Ellipsoids: The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ represents an ellipsoid.

↳ $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

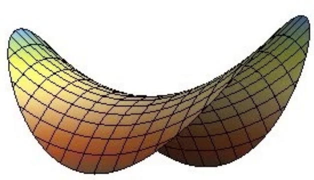


5) Paraboloids: The equations $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ and $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ both represent respectively, an elliptic paraboloid and a hyperbolic paraboloid.

1) Elliptic

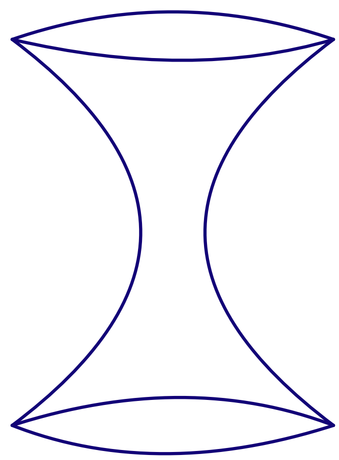


2) hyperbolic

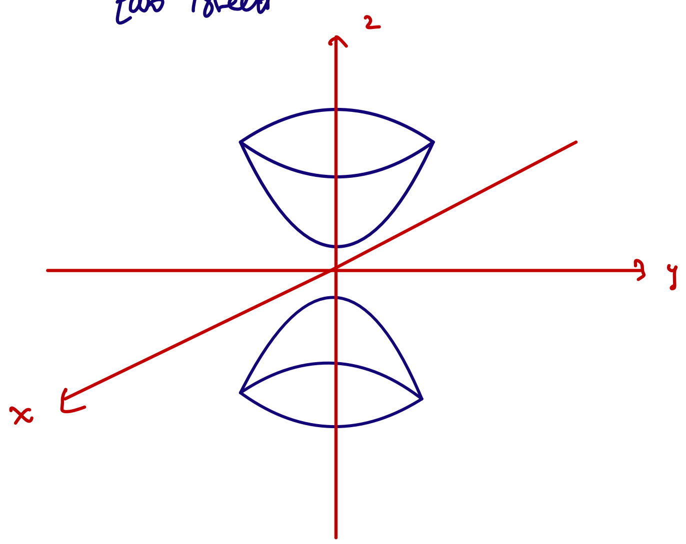


$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2},$$

6) Hyperboloid: The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ represents a hyperboloid of one sheet.



The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$ represents a hyperboloid of two sheets



Def: A ruled surface is a surface such that through each point of it passes a straight line that is fully contained in the surface.

Ex: Show that if the point (a,b,c) lies on the hyperbolic paraboloid $z=y^2-x^2$, then the line parametrized by $x=a+t$, $y=b+t$, and $z=c+2(b-a)t$ lies entirely on the surface.

We know that since the point (a,b,c) is on the surface then $c=b^2-a^2$

Also, $z = y^2 - x^2$

$$c + 2(b-a)t = (b+t)^2 - (a+t)^2$$
$$= (b^2 - 2bt + \cancel{t^2}) + (-a^2 - 2at - \cancel{t^2})$$
$$c + 2(b-a)t = b^2 - a^2 + 2(b-a)t$$
$$c + 2(b-a)t = c + 2(b-a)t$$
$$\checkmark.$$

Thm: Cones, cylinders, hyperbolic, paraboloid, and hyperboloid of one sheet are the only quadric surfaces that are also ruled surfaces