Singh *Elements of Topology* Solutions

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1 Chapter 1

1.1.1 Given a set X, define $d(x,y) = 0$ if $x = y$, and $d(x,y) = 1$ if $x \neq y$. Check that d is a metric on X.

Proof. Positive definiteness follows trivially from the definition. For symmetry, if $x = y$ or $x \neq y$ then it follows that $d(x,y) = 0 = d(y,x)$ and $d(x,y) = 1 = d(y,x)$ respectively. Now let $x,y,z \in X$. If $d(x,y) = 0$ then $d(x,z) + d(z,y) \geq 0 = d(x,y)$. If $d(x,y) = 1$, then $d(x,z) + d(z,y)$ is either 1 or 2, since at least one of $x \neq z$ or $y \neq z$ holds (otherwise $x = y$ and $d(x,y) = 0$). $\square$

1.1.2 Let (X,d) be a metric space. Show that

(a) $d'(x,y) = d(x,y)/(1 + d(x,y))$ is a bounded metric on X

Proof. d' inherits symmetry from d. d' is also positive definite since if $x = y$, then $d'(x,y) = 0/1 = 0$, and if $x \neq y$, then $d(x,y) > 0$.

$d'(x,y)$ is bounded since $d(x,y) \geq 0$ implies that $d(x,y) \leq 1 + d(x,y)$, thus $d'(x,y) \leq 1$ for all choices of $x,y \in X$.

$\square$

(b) $d_1(x,y) = \min(1, d(x,y))$ is a bounded metric on X.

Proof. It is clear that d_1 is symmetric and positive definite. Now let $x,y,z \in X$. We know that $d(x,y) \leq d(x,z) + d(z,y)$. If $d(x,z) + d(z,y) < 1$, then it follows that $d_1(x,y) = d(x,y) \leq d(x,z) + d(z,y) = d_1(x,z) + d_1(z,y)$. Now if $d(x,z) + d(z,y) \geq 1$, then $d_1(x,y) \leq 1 \leq d(x,z) + d(z,y)$. If $d(x,z) + d(z,y) \geq 1$, then it follows that $d_1(x,z) + d_1(z,y) \geq 1$, thus the triangle inequality holds.

d_1 is also trivially bounded, since $\sup\{d_1(x,y) | x,y \in X\} = 1$. $\square$

1.1.3 Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and given $x \in F^n$, define $||x|| = \max_{1 \leq i \leq n} |x_i|$. Show that the function $|| \cdot ||$ satisfies the conditions (a), (b) and (d) described in Ex. 1.1.1, and hence defines a norm on F^n. This is called the Cartesian norm on F^n.

Proof. (a) $||x|| > 0$ for $x \neq 0$. Suppose $x \in F^n$ where $x \neq 0$, then for at least one x_j, $x_j \neq 0$, then it follows that $||x|| = \max_i |x_i| \geq |x_j| > 0$, thus (a) holds.
(b) $||ax|| = |a| ||x||$. Let $x \in F^n$, $a \in F$, then $||ax|| = \max_i |ax_i| = \max_i |a||x_i| = |a| \max_i |x_i| = |a| ||x||$
(d) $||x+y|| \leq ||x|| + ||y||$. Let $x,y \in F^n$, $||x+y|| = \max_i |x_i + y_i| \leq \max_i |x_i| + \max_j |y_j| = ||x|| + ||y||$. $\square$

1.1.10 Show that a subset A of a metric space (X,d) is bounded if there exists a point $x \in X$ and a real number K such that $d(x,a) \leq K$ for every $a \in A$.

Proof. Recall that a set is bounded if its diameter is bounded, i.e. $\sup\{d(a,a') : a,a' \in A\} < \infty$. Let $K \geq \operatorname{diam}(A)$, then let $x \in A$. It follows that $d(x,a) \leq \operatorname{diam}(A) \leq K$ for all $a \in A$, as required. $\square$

1.2.1

(a) Find all possible topologies on the set $X = \{a, b, c\}$.

Proof. The trivial and discrete topologies, $\{(, \{x\}, X), x \in X$
$(\emptyset, \{x, y\}, X), x, y \in X$
$(\emptyset, \{x\}, \{x, y\}, X), x, y \in X$
$(\emptyset, \{x\}, \{y\}, \{x, y\}, X), x, y \in X$ □

(b) Let $\mathscr{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathscr{T}_2 = \{\emptyset, X, \{c\}, \{b, c\}\}$ on X. Is the union of $\mathscr{T}_1$ and $\mathscr{T}_2$ a topology for X.

Proof. $\mathscr{T}_1 \cup \mathscr{T}_2 = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{b, c\}\}$. $\{a\} \cup \{c\} = \{a, c\} \notin \mathscr{T}_1 \cup \mathscr{T}_2$, thus is not a topology. □

(c) Find the smallest topology containing $\mathscr{T}_1$ and $\mathscr{T}_2$, and the largest topology contained in $\mathscr{T}_1$ and $\mathscr{T}_2$.

Proof. $\mathscr{T}_s = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ is simply the discrete topology.
$\mathscr{T}_l = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$ □

1.2.2

(a) What is the topology determined by the metric d on X given by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$

Proof. The open sets of d are the points sets, since $B_{1/2}(x) = \{x\}$ for any $x \in X$. Now let $U = \cup_\alpha \{x_\alpha\} \subset X$ be any collection of elements of X. It follows that $\cup_\alpha B_{1/2}(x_\alpha) = \cup_\alpha \{x_\alpha\} = U$ is the union of open balls, and thus is open. It follows that since any arbitrary subset of X is open, that the topology induced by d is the discrete topology. □

(b) Let X be a set containing more than one element. Can you define a metric on X so that the associated metric topology is trivial.

Proof. To see this, suppose $X = \{a, b\}$. If d is a metric on X, then $d(a, b) > 0$, let $0 < \varepsilon < d(a, b)$, then $B_\varepsilon(a) = \{a\}$, thus we again achieve the discrete topology. It follows that we cannot induce the trivial topology on X. More generally if X is a set containing more than one element, in order to induce the trivial topology, d must not distinguish points in X; however, any two distinct points $x, y \in X$ have non-zero distance by the definition of a metric, thus for any two points $x, y \in X$, $x \neq y$, there exists open balls $B_\varepsilon(x)$, $B_\delta(y)$ such that $y \notin B_\varepsilon(x)$ and vice versa, thus we cannot induce the discrete topology. (this is hinting at the fact that metric spaces are Hausdorff) □

1.2.3 Let X be an infinite set, $x_0 \in X$ a fixed point. Show that

$$\mathscr{T} = \{G | \text{either} X - G \text{ is finite or } x_0 \notin G\}$$

is a topology on X in which every point, except x_0, is both open and closed. ($(X, \mathscr{T})$ is called a *Fort space*)

Proof. X is a set with empty and thus finite compliment. $\emptyset$ is a set not containing x_0 and thus is an element of the topology.

Let $U, V \in \mathscr{T}$. Suppose $x_0 \notin U, V$, $U \cap V$, then $U \cap V$ is a set not containing x_0, and thus $U \cap V \in \mathscr{T}$. If either U or V does not contain x_0, it follows that $x_0 \notin U \cap V$, and thus $U \cap V \in \mathscr{T}$. Now assume neither U nor V contain x_0, then it follows that U^c and V^c are finite. Moreover, it follows that $U \cap V$ is an infinite set, since otherwise would imply that there exists an infinite subset of X not contained in U and likewise for V. Therefore $\mathscr{T}$ is closed under finite intersection.

Now consider the arbitrary union of sets in $\mathscr{T}$. Either they all do not contain x_0 and then it follows that their union is an element in $\mathscr{T}$, or at least one contains x_0, then it follows that at least one U is an infinite set with finite compliment, and thus the union also has finite compliment, thus $\mathscr{T}$ is contained under finite union. $\square$

1.2.4 Decide the openness and closedness of the following subsets in $\mathbb{R}$.
(a) $\{x : 1/2 < |x| \leq 1\}$, neither
(b) $\{x : 1/2 \leq |x| < 1\}$, neither
(c) $\{x : 1/2 \leq |x| \leq 1\}$, closed but not open
(d) $\{x : 0 < |x| < 1\}$ and $(1/x) \notin \mathbb{N}$, open but not closed

1.2.5 Find a topology on $\mathbb{R}$, different from the trivial topology and the discrete topology, so that every open set is closed and vice versa.

Proof. Let $\mathscr{T} = \{\emptyset, \mathbb{R}, (-\infty, 0), [0, \infty)\}$ be a topology. It is clearly neither the trivial nor discrete topology, and $\mathbb{R}$ and $\emptyset$ are both open and closed, and since $(-\infty, 0)^c = [0, \infty)$, it follows that both $(-\infty, 0)$ and $[0, \infty)$ are closed (in addition to being open sets) as required. $\square$

1.2.11 If every countable subset of a space is closed, is the topology necessarily discrete?

Proof. In order to induce the discrete topology, singletons need to be open, thus the compliment of points are closed. We should ask if we can always construct a compliment of a singleton set out of the finite union of countable sets. The answer is clearly no since take as an example $\mathbb{R}$, the complimenet of a point $\mathbb{R}\backslash\{p\}$ cannot be written as the countable union of countable sets, thus the discrete topology is not induced. $\square$

1.3.4 Let X be a space and $A \subseteq X$. Prove:

(a) $X - \bar{A} = (X - A)^o$

Proof. Let U be open in $X - \bar{A}$. Since $\bar{A}$ is closed and X is open, it follows that $X - \bar{A}$ is open and moreover $X - \bar{A} \subseteq X - A$. It follows that $X - \bar{A} = (X - \bar{A})^\circ \subseteq (X - A)^o$.

Note that $B \subset C$ implies $C^c \subset B^c$ and thus since $A \subset \bar{A}$ implies $\bar{A}^c \subset A^c$. It follows that $(X - A)^\circ = (X \cap A^c)^\circ = X^\circ \cap (A^c)^\circ = X \cap (\bar{A})^c = X - \bar{A}$.

It remains to show the reverse inclusion. $\square$

1.3.5 Let X be a space and $A \subseteq X$. Prove that A is clopen if and only if $\partial A = \emptyset$.

Proof. Let A be clopen, then it is the largest open set contained in A, thus $A = A^\circ$ and also the smallest closed set containing A, thus $A = \bar{A}$, and by 1.3.4(c), $\partial A = \bar{A} - A^\circ = A - A = \emptyset$. If $\partial A = \emptyset$ then again it follows that $\bar{A} = A^\circ = A$, and thus A is clopen. $\square$

1.3.9 Let X be an infinite set with the cofinite topology and $A \subseteq X$. Prove that if A is infinite, then every point of X is a limit point of A and if A is finite then it has no limit points.

Proof. Let A be an infinite subset of X. Let $x \in X$, and let N be a neighborhood of x. First note that N is infinite since there exists an open set $x \in U \subset N$, and U is necessarily infinite. Since N is infinite, it follows that $A \cap N \neq 0$, otherwise U^c would be infinite. Therefore indeed $x \in X$ is a limit point of A.

Now let A be a finite set. Let $x \in X$, it follows that $A\backslash\{x\}$ is a finite set. Let $U = X\backslash A \cup \{x\}$, it follows that U is an open neighborhood of x that does not intersect A, thus x is not a limit point of A. $\square$

1.3.17 Prove that the union of two nowhere dense sets is nowhere dense.

Proof. □

Theorem 1.4.4 A collection $\mathscr{B}$ of open subsets of a space X is a basis if and only if for each open subset U of X and each point $x \in U$, there exists a $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

Proof. Suppose $\mathscr{B}$ is a basis. Let U be an open set. Let $x \in U$. Since U is open it follows that $U = \cup_\alpha B_\alpha$ for $B_\alpha \in \mathscr{B}$. It follows that $x \in B_\alpha$ for at least one alpha, and thus $x \in B_\alpha \subseteq U$.
Now suppose for each open set U and every point $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \subset U$. It follows that for all x_α in U there exists such B_α, $U = \bigcup_\alpha x_\alpha \subseteq \bigcup_\alpha B_\alpha \subseteq U \Rightarrow U = \cup_\alpha B_\alpha$, thus $\mathscr{B}$ is a basis. □

Theorem 1.4.8 Two metrics d and d' on the set X are equivalent if and only if for each $x \in X$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that $B_d(x;\delta) \subseteq B_{d'}(x;\varepsilon)$ and $B_{d'}(x;\delta) \subseteq B_d(x;\varepsilon)$

Proof. Let $\varepsilon > 0$ be given. Suppose d and d' are equivalent, then it follows that □

1.4.1

(a) What is the order topology on the set $\mathbb{N}$ with the usual order relation.

Proof. Note that $x < 1$ for all $x \in \mathbb{N}$ implies that $x = 0$, in other words 0 is the least element of $\mathbb{N}$ with the usual order relation. We then note that 0 is open under the order topology. Moreover let $x \in \ltimes$, $x \neq 0$. It follows that $[0, x+1)$ and $(x-1, \infty)$ are open in the order topology, and thus $[0, x+1) \cap (x-1, \infty) = \{x\}$ is open in the order topology. Thus the order topology induces the discrete topology. □

(b) Is the order topology on $\{1,2\} \times \mathbb{N}$ in the dictionary order relation discrete?

Proof. □

1.4.11 Show that the rationals are dense in the Sorgenfrey line $\mathbb{R}_l$.

Proof. We know that $\mathbb{R} - \bar{\mathbb{Q}} = (\mathbb{R} - \mathbb{Q})^\circ$. Let $x \in (\mathbb{R} - \mathbb{Q})^\circ$, then it follows that there exists an open set $U \subseteq (\mathbb{R} - \mathbb{Q})^\circ$ such that $x \in U$. Since U is open, it follows that it can be generated by the lower limit topology as the union of sets of the form $[a, b)$, thus $x \in [a, b)$ for some $a \neq b \in \mathbb{R}$. However, we can apply the Archimedean principle to find a rational q such that $a < q < b$, thus $q \in [a, b)$. But this contradicts the fact that $[a, b) \subseteq U \subseteq (R - \mathbb{Q})^\circ$, thus $(R - \mathbb{Q})^\circ = \emptyset$. Therefore $\mathbb{R} = \bar{\mathbb{Q}}$. □

1.5.1 A subset Y of a space X is called discrete if the relative topology for Y is discrete. Prove that every subspace of a discrete space is discrete and every subspace of an indiscrete space is indiscrete.

Proof. Let X have the discrete topology, and let $A \subseteq X$. Let $U \subseteq A$. It follows that $U = U \cap A \subseteq X$ is open in A, thus A inherits the discrete topology from X.

Let X have the indiscrete topology, let $A \subseteq X$. It follows that the only open sets of A are $A = A \cap X$ and $\emptyset = A \cap \emptyset$, thus A inherits the indiscrete topology from X. □

1.5.10 Give an example of a space X which has a dense subset D and a subset Y such that $D \cap Y$ is not dense in Y.

Proof. Let $X = \mathbb{R}$ with the usual topology, $D = \mathbb{Q}$, and $Y = \{\{1\}, \{\pi\}\}$. Note that X induces the discrete on Y. It follows that $D \cap Y = \{1\}$, but that D is not dense in Y. □