

4d null coordinates

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1 Null Coordinates

When working with the divergencelessness of the stress tensor, it becomes convenient to consider a coordinate system which we call ‘null coordinates’. Let $u = t - z$, $v = t + z$, $x = x$, $y = y$. u and v are null lines which on a spacetime diagram correspond to the rightward and leftward lightrays depicted in Figure 1.

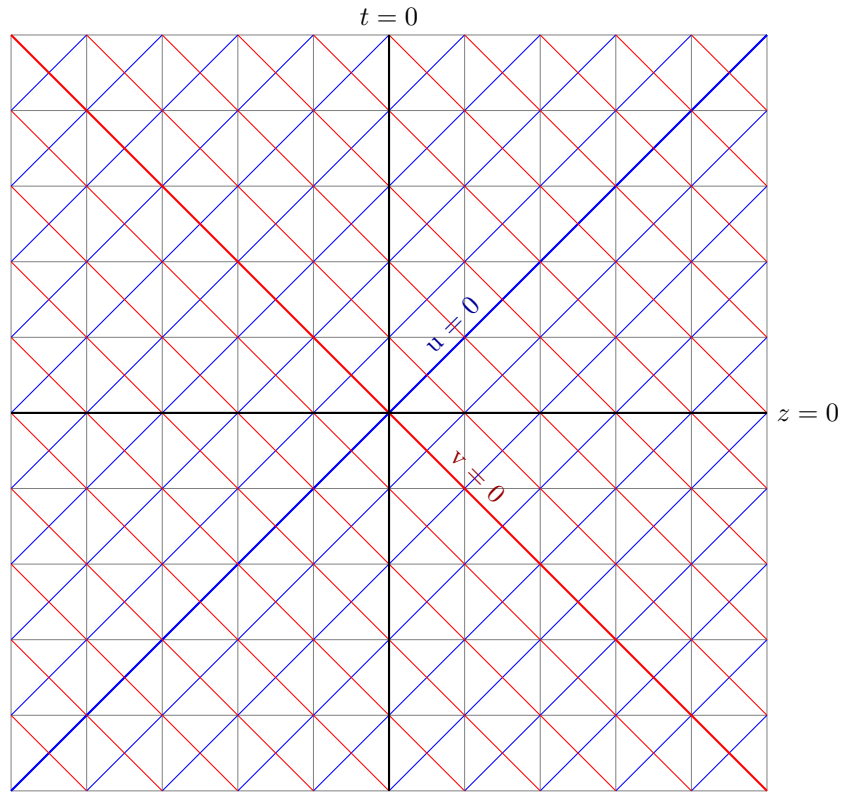


Figure 1: Null Coordinates

Transforming back to usual Minkowski coordinates is $t = \frac{u+v}{2}$, $z = \frac{v-u}{2}$. We then have that $dt = \frac{1}{2}(du + dv)$, $dz = \frac{1}{2}(dv - du)$. We can note that $z^2 - t^2 = -uv$, thus the line element for null coordinates is given by

$$ds^2 = -dudv + dx^2 + dy^2 \quad (\text{Eq. A.1})$$

The metric tensor then takes the following covariant and contravariant forms.

$$g_{\mu\nu} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{Eq. A.2})$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{Eq. A.3})$$

We remark that raising and lowering indices switches the place of u and v .

2 What can we say about these coordinates in 4d spacetime

As in 2d, $T_{uu} = \partial_u \phi \partial_u \phi$, and $T_{vv} = \partial_v \phi \partial_v \phi$. One result in 2d is that $T_{uv} = T_{vu} = 0$, this simplifies checking the trace condition. Does this hold in 4d?

$$\begin{aligned} T_{uv} &= \partial_u \phi \partial_v \phi - \frac{1}{2} g_{uv} ((-4)(\partial_v \phi \partial_u \phi)) - \frac{1}{2} g_{uv} ((\partial_x \phi)^2 + (\partial_y \phi)^2) \\ &= \partial_u \phi \partial_v \phi - \partial_v \phi \partial_u \phi - \frac{1}{2} \left(\frac{-1}{2} \right) ((\partial_x \phi)^2 + (\partial_y \phi)^2) \\ &= \frac{1}{4} ((\partial_x \phi)^2 + (\partial_y \phi)^2) \end{aligned}$$

So this is nonzero and is in fact dependent on the flow in the transverse directions. The conservation law then requires us to take one more derivative,

$$\begin{aligned} \partial^v T_{vu} &= -2 \partial_u T_{vu} \\ &= -\frac{1}{2} (2 \partial_x \phi \partial_u \partial_x \phi + 2 \partial_y \phi (\partial_u \partial_y \phi)) \\ &= -(\partial_x \phi \partial_u \partial_x \phi + \partial_y \phi \partial_u \partial_y \phi) \\ \\ \partial^u T_{uv} &= -2 \partial_v T_{uv} \\ &= -(\partial_x \phi \partial_v \partial_x \phi + \partial_y \phi \partial_v \partial_y \phi) \end{aligned}$$

So

$$\begin{aligned} \partial^\alpha T_{\alpha u} &= \partial^u (\partial_u \phi \partial_u \phi) - (\partial_x \phi \partial_u \partial_x \phi + \partial_y \phi \partial_u \partial_y \phi) + \partial_x (\partial_x \phi \partial_u \phi) + \partial_y (\partial_y \phi \partial_u \phi) \\ &= \partial^u (\partial_u \phi \partial_u \phi) - (\partial_x \phi \partial_u \partial_x \phi + \partial_y \phi \partial_u \partial_y \phi) + \partial_x \partial_x \phi \partial_u \phi + \partial_x \phi \partial_x \partial_u \phi + \partial_y \partial_y \phi \partial_u \phi + \partial_y \phi \partial_y \partial_u \phi \\ &= [-(\partial_x \phi (\partial_u \partial_x \phi) + \partial_y \phi (\partial_u \partial_y \phi)) + \partial_x \phi (\partial_x \partial_u \phi) + (\partial_y \phi) (\partial_y \partial_u \phi)] \\ &\quad + \partial^u (\partial_u \phi \partial_u \phi) + (\partial^x \partial_x \phi) (\partial_u \phi) + (\partial^y \partial_y \phi) (\partial_u \phi) \\ &= 0 + (2 \partial^u \partial_u \phi + \partial^x \partial_x \phi + \partial^y \partial_y \phi) (\partial_u \phi) \\ &= (\partial^\mu \partial_\mu \phi) (\partial_u \phi) \quad \text{since } \partial^u \partial_u \phi = \partial^v \partial_v \phi \\ &= 0 \end{aligned}$$

and similarly

$$\partial^\alpha T_{\alpha v} = 0$$

To be complete, here are the remaining components of the stress energy tensor.

$$\begin{aligned} T_{ux} &= \partial_u \phi \partial_x \phi \\ T_{uy} &= \partial_u \phi \partial_y \phi \\ T_{vx} &= \partial_v \phi \partial_x \phi \\ T_{vy} &= \partial_v \phi \partial_y \phi \\ T_{xx} &= \partial_x \phi \partial_x \phi - \frac{1}{2} ((\partial_x \phi)^2 + (\partial_y \phi)^2 - 4 \partial_u \phi \partial_v \phi) \\ T_{yy} &= \partial_y \phi \partial_y \phi - \frac{1}{2} ((\partial_x \phi)^2 + (\partial_y \phi)^2 - 4 \partial_u \phi \partial_v \phi) \\ T_{xy} &= \partial_x \phi \partial_y \phi \end{aligned}$$

Divergencelessness is captured in the computation $\partial^\mu T_{\mu\nu}$, we must compute the following

$$\partial^u T_{uu} + \partial^v T_{uv} + \partial^x T_{ux} + \partial^y T_{uy}$$

$$\partial^u T_{vu} + \partial^v T_{vv} + \partial^x T_{vx} + \partial^y T_{vy}$$

$$\partial^u T_{xu} + \partial^v T_{xv} + \partial^x T_{xx} + \partial^y T_{xy}$$

$$\partial^u T_{yu} + \partial^v T_{yv} + \partial^x T_{yx} + \partial^y T_{yy}$$

and verify that each vanishes.

2.1 Divergencelessness of a uniformly accelerating source

$$\phi = \frac{q}{4\pi R} \theta(t + z) \quad (1)$$

$$\phi = \frac{q}{4\pi R} \theta(v) \quad (2)$$

Where

$$R = \frac{a}{2} [(X^2 - a^{-2})^2 + 4a^{-2}\rho^2]^{1/2} \quad (3)$$

$$X^2 = -uv + \rho^2 \quad (4)$$

Calculations: $\partial_\mu R$

$$\partial_u R = \frac{-a^2}{4} \frac{v(X^2 - a^{-2})}{R}$$

$$\partial_v R = \frac{-a^2}{4} \frac{u(X^2 - a^{-2})}{R}$$

$$\partial_x R = \frac{a^2 x}{2R} [X^2 - 3a^{-2}]$$

$$\partial_y R = \frac{a^2 y}{2R} [X^2 - 3a^{-2}]$$

Calculations: $\partial_\mu \phi$

$$\partial_u \phi = \frac{-q}{4\pi R^2} (\partial_u R) \theta(v)$$

$$\partial_v \phi = \frac{-q}{4\pi R^2} (\partial_v R) \theta(v) + \frac{q}{4\pi R} \delta(v)$$

$$\partial_x \phi = \frac{-q}{4\pi R^2} (\partial_x R) \theta(v)$$

$$\partial_y \phi = \frac{-q}{4\pi R^2} (\partial_y R) \theta(v)$$

Calculations: $\mathbf{T}_{\mu\nu}$

$$T_{uu} = \left(\frac{-q}{4\pi R^2} (\partial_u R) \theta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_u R) \theta(v) \right)$$

$$T_{vv} = \left(\frac{-q}{4\pi R^2} (\partial_v R) \theta(v) + \frac{q}{4\pi R} \delta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_v R) \theta(v) + \frac{q}{4\pi R} \delta(v) \right)$$

$$T_{uv} = \frac{1}{4} \left(\left(\frac{-q}{4\pi R^2} (\partial_x R) \theta(v) \right)^2 + \left(\frac{-q}{4\pi R^2} (\partial_y R) \theta(v) \right)^2 \right)$$

$$T_{ux} = \left(\frac{-q}{4\pi R^2} (\partial_u R) \theta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_x R) \theta(v) \right)$$

$$T_{uy} = \left(\frac{-q}{4\pi R^2} (\partial_u R) \theta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_y R) \theta(v) \right)$$

$$T_{vx} = \left(\frac{-q}{4\pi R^2} (\partial_v R) \theta(v) + \frac{q}{4\pi R} \delta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_x R) \theta(v) \right)$$

$$T_{vy} = \left(\frac{-q}{4\pi R^2} (\partial_v R) \theta(v) + \frac{q}{4\pi R} \delta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_y R) \theta(v) \right)$$

$$T_{xx} = \left(\frac{-q}{4\pi R^2} (\partial_x R) \theta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_x R) \theta(v) \right)$$

$$- \frac{1}{2} \left(\left(\frac{-q}{4\pi R^2} (\partial_x R) \theta(v) \right)^2 + \left(\frac{-q}{4\pi R^2} (\partial_y R) \theta(v) \right)^2 - 4 \left(\frac{-q}{4\pi R^2} (\partial_u R) \theta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_v R) \theta(v) + \frac{q}{4\pi R} \delta(v) \right) \right)$$

$$T_{yy} = \left(\frac{-q}{4\pi R^2} (\partial_y R) \theta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_y R) \theta(v) \right)$$

$$- \frac{1}{2} \left(\left(\frac{-q}{4\pi R^2} (\partial_x R) \theta(v) \right)^2 + \left(\frac{-q}{4\pi R^2} (\partial_y R) \theta(v) \right)^2 - 4 \left(\frac{-q}{4\pi R^2} (\partial_u R) \theta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_v R) \theta(v) + \frac{q}{4\pi R} \delta(v) \right) \right)$$

$$T_{xy} = \left(\frac{-q}{4\pi R^2} (\partial_x R) \theta(v) \right) \left(\frac{-q}{4\pi R^2} (\partial_y R) \theta(v) \right)$$

Calculation: $T_{u\mu}$ Simplified

$$\begin{aligned} T_{uu} &= \frac{q^2}{16\pi^2 R^4} (\partial_u R)^2 \theta(v) \\ T_{uv} &= \frac{q^2 ((\partial_x R)^2 + (\partial_y R)^2) \theta(v)}{64\pi^2 R^4} \\ T_{ux} &= \frac{q^2}{16\pi^2 R^4} (\partial_u R) (\partial_x R) \theta(v) \\ T_{uy} &= \frac{q^2}{16\pi^2 R^4} (\partial_u R) (\partial_y R) \theta(v) \end{aligned}$$

Verified $\partial^\mu T_{u\mu}$ this morning (March 2). The theta function terms add to zero since ϕ satisfies the wave equation and we are left with

$$\frac{q^2}{16\pi^2 R^4} (\partial_u R)^2 \delta(v)$$

which is zero since $\partial_u R = 0$ when $v = 0$.

Calculation: $T_{v\mu}$ Simplified

$$\begin{aligned} T_{vv} &= \frac{q^2}{16\pi^2} \left[\frac{(\partial_v R)^2 \theta(v)}{R^4} - \frac{2(\partial_v R) \delta(v) \theta(v)}{R^3} + \frac{1}{R^2} \delta(v) \delta(v) \right] \\ T_{vu} &= \frac{q^2 ((\partial_x R)^2 + (\partial_y R)^2) \theta(v)}{64\pi^2 R^4} = \frac{q^2}{16\pi^2} \left[\frac{((\partial_x R)^2 + (\partial_y R)^2)}{4R^4} \right] \theta(v) \\ T_{vx} &= \frac{q^2}{16\pi^2} \left[\frac{(\partial_v R) (\partial_x R)}{R^4} \theta(v) - \frac{(\partial_x R)}{R^3} \delta(v) \theta(v) \right] \\ T_{vy} &= \frac{q^2}{16\pi^2} \left[\frac{(\partial_v R) (\partial_y R)}{R^4} \theta(v) - \frac{(\partial_y R)}{R^3} \delta(v) \theta(v) \right] \end{aligned}$$

Theta function terms will sum to 0 by wave equation. We are left with terms which are delta functions or products of theta and delta functions which since $\delta(v) \theta(v) = \frac{1}{2} \delta(v)$ means we can consider a grand sum of delta functions which are

$$\partial_u \left[2 \frac{(\partial_v R)}{R^3} \right] \delta(v) - \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{2R^4} \right] \delta(v) + \partial_x \left[\frac{-(\partial_x R)}{2R^3} \right] \delta(v) + \partial_y \left[\frac{-(\partial_y R)}{2R^3} \right] \delta(v)$$

differentiating the terms and pulling out the delta function

$$\begin{aligned} & \left(-2 \left[\frac{-(\partial_{uv} R) R + (\partial_v R) (\partial_u R)}{R^4} \right] - \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{2R^4} \right] - \frac{1}{2} \left[\frac{(\partial_{xx} R) R - 3(\partial_x R)^2}{R^4} \right] - \frac{1}{2} \left[\frac{(\partial_{yy} R) R - 3(\partial_y R)^2}{R^4} \right] \right) \delta(v) \\ & - \frac{1}{2} \left(4 \left[\frac{-(\partial_{uv} R) R + (\partial_v R) (\partial_u R)}{R^4} \right] + \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{R^4} \right] + \left[\frac{(\partial_{xx} R) R - 3(\partial_x R)^2}{R^4} \right] + \left[\frac{(\partial_{yy} R) R - 3(\partial_y R)^2}{R^4} \right] \right) \delta(v) \end{aligned}$$

Evaluating this at $v = 0$, we know that $\partial_u R$ is zero so

$$\frac{-1}{2R^4} ((-4(\partial_{uv} R) + (\partial_{xx} R) + (\partial_{yy} R)) R - 2((\partial_x R)^2 + (\partial_y R)^2)) \delta(v)$$

Since we are able to add in terms that are equal to zero,

$$\begin{aligned} & \frac{-1}{2R^4} ((\partial^\mu \partial_\mu R) R - 2(-4(\partial_u R) (\partial_v R) + (\partial_x R)^2 + (\partial_y R)^2)) \delta(v) \\ & \frac{-1}{2R^4} ((\partial^\mu \partial_\mu R) R - 2(\partial^\mu R \partial_\mu R)) \delta(v) \end{aligned}$$

It follows that this vanishes since the wave equation vanishing implies: $(\partial^\mu \partial_\mu R) R - 2\partial^\mu R \partial_\mu R = 0$

Calculation: $T_{x\mu}$ Simplified

$$\begin{aligned} T_{xu} &= \frac{q^2}{16\pi^2 R^4} (\partial_u R) (\partial_x R) \theta(v) \\ T_{xv} &= \frac{q^2}{16\pi^2} \left[\frac{(\partial_v R) (\partial_x R)}{R^4} \theta(v) - \frac{(\partial_x R)}{R^3} \delta(v) \theta(v) \right] \\ T_{xx} &= \frac{q^2}{16\pi^2} \left(\left[\frac{(\partial_x R)^2 - (\partial_y R)^2 + 4(\partial_v R) (\partial_u R)}{2R^4} \right] \theta(v) - \frac{2(\partial_u R)}{R^3} \theta(v) \delta(v) \right) \\ T_{xy} &= \frac{q^2}{16\pi^2 R^4} (\partial_x R) (\partial_y R) \theta(v) \end{aligned}$$

$(-2\partial_v T_{xu})$ yields a delta function term $\frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_x R)\delta(v)$ which at $v = 0$ is zero since $\partial_u R$ vanishes. T_{xy} will not give rise to any delta function terms. The only delta function terms will be

$$2\partial_u \left(\frac{(\partial_x R)}{R^3} \delta(v) \theta(v) \right) - \partial_x \left(\frac{2(\partial_u R)}{R^3} \theta(v) \delta(v) \right) = 2 \left[\frac{(\partial_{ux} R - \partial_{xu} R)}{R^3} - 3 \frac{(\partial_u R \partial_x R - \partial_x R \partial_u R)}{R^4} \right] \theta(v) \delta(v) = 0$$

Indeed vanishes.

Calculation: $T_{y\mu}$ Simplified

$$\begin{aligned} T_{yu} &= \frac{q^2}{16\pi^2 R^4} (\partial_u R)(\partial_y R) \theta(v) \\ T_{yv} &= \frac{q^2}{16\pi^2} \left[\frac{(\partial_v R)(\partial_y R)}{R^4} \theta(v) - \frac{(\partial_y R)}{R^3} \delta(v) \theta(v) \right] \\ T_{yx} &= \frac{q^2}{16\pi^2 R^4} (\partial_x R)(\partial_y R) \theta(v) \\ T_{yy} &= \frac{q^2}{16\pi^2} \left(\left[\frac{(\partial_y R)^2 - (\partial_x R)^2 + 4(\partial_v R)(\partial_u R)}{2R^4} \right] \theta(v) - \frac{2(\partial_u R)}{R^3} \theta(v) \delta(v) \right) \end{aligned}$$

$(-2\partial_v T_{yu})$ yields a delta function term $\frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_y R)\delta(v)$ which at $v = 0$ is zero since $\partial_u R$ vanishes. T_{yx} will not give rise to any delta function terms. The only delta function terms will be

$$2\partial_u \left(\frac{(\partial_y R)}{R^3} \delta(v) \theta(v) \right) - \partial_y \left(\frac{2(\partial_u R)}{R^3} \theta(v) \delta(v) \right) = 2 \left[\frac{(\partial_{uy} R - \partial_{yu} R)}{R^3} - 3 \frac{(\partial_u R \partial_y R - \partial_x R \partial_u R)}{R^4} \right] \theta(v) \delta(v) = 0$$

Indeed vanishes.

3 Graphing Vector Fields

Our next goal is to plot the vector field of

$$S^\mu = -g^{\mu\alpha} T_{\alpha\beta} \xi^\beta \quad (5)$$

where ξ^β is a Killing vector.

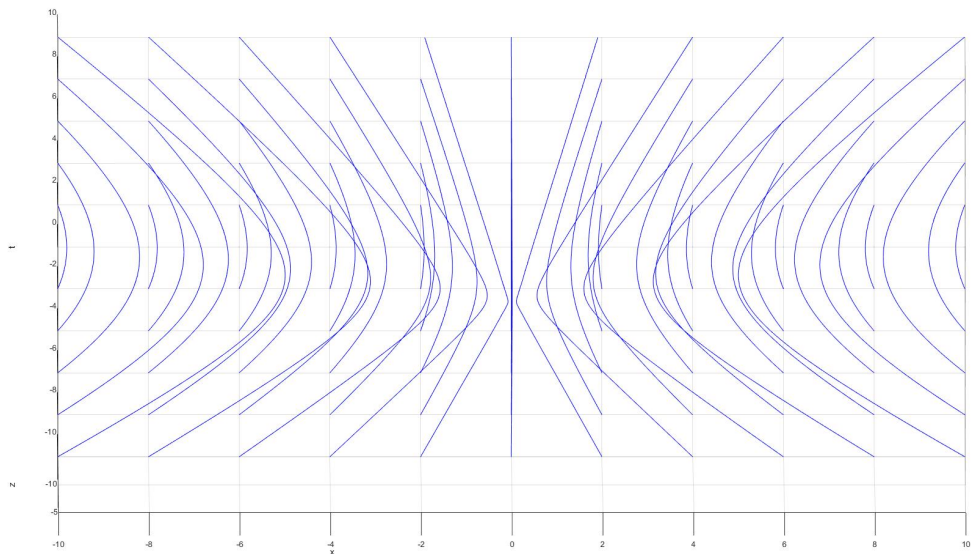
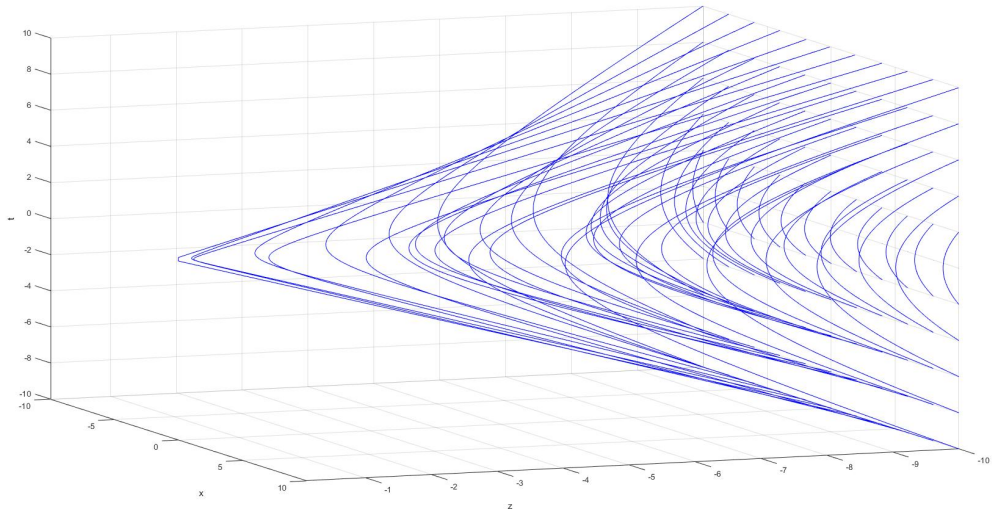
Minkowski Time Killing Vector

$$\partial_t = \frac{1}{2} [\hat{u} + \hat{v}]$$

So we can compute the components of S_M^μ ,

$$\begin{aligned} S_M^u &= T_{vu} + T_{vv} \\ S_M^v &= T_{uv} + T_{vv} \\ S_M^x &= -\frac{1}{2} [T_{xv} + T_{xu}] \\ S_M^y &= -\frac{1}{2} [T_{yu} + T_{yv}] \end{aligned}$$

In the past, moving along u , the flux diverges from the plane of the source in the transverse directions. When we get close to the future horizon ($u = 0$), the flux starts moving inward from the transverse directions. For the Rindler Killing vector, we obtain the following figures. When $|u| > |v|$, we have an outward flux, and when $|v| > |u|$, we have inward flux. This is because at these points the sum of components inside the S^x and S^y expressions change sign. How does this look like on the $z-t$ plane? $-u = v$ in the R region is the line $z = 0$, when t is negative, we have outward flux. When t is positive, we have inward flux as can be seen in the figures below.



The figures above show the flux of energy-momentum in x-z-t coordinates for $a = 1$ projected onto $y = 0$.

Question: Since this is symmetric about $t = 0$ (and under time reversal) where does the energy that is escaping into the R region come from?

Rindler Time Killing Vector The Rindler time Killing vector expressed in these coordinates is

$$\partial_\lambda = \frac{1}{2} [v\hat{v} - u\hat{u}] \quad (6)$$

Here the \hat{u} and \hat{v} are basis vectors in coordinates for the vector field so the coordinates for S_R^μ are

$$\begin{aligned} S_R^u &= vT_{vv} - uT_{uv} \\ S_R^v &= vT_{uv} - uT_{uu} \\ S_R^x &= -\frac{1}{2} [vT_{xv} - uT_{xu}] \\ S_R^y &= -\frac{1}{2} [vT_{yv} - uT_{yu}] \end{aligned}$$

4 Electromagnetic Vector Fields

We now wish to graph $S^\mu = -g^{\mu\alpha}T_{\alpha\nu}\xi^\nu$ where $T_{\alpha\nu}$ is given by an electromagnetic uniformly accelerating source.

4.1 Minkowski Flux

In Minkowski coordinates, the timelike Killing vector is given by

$$\xi^\mu = \partial_t$$

Next we need $T_{\mu\nu}$, obtained from Boulware (IV.2), ignoring the delta function terms,

$$T^{tt} = \frac{q^2}{(4\pi)^2} \left(\frac{a^2}{8R^6} [(\rho^2 + a^{-2} - Z^2)^2 + 4\rho^2(t^2 + z^2)] \right) \quad (7)$$

$$T^{tz} = \frac{q^2}{(4\pi)^2} \frac{g^2 tz \rho^2}{R^6} \quad (8)$$

$$T^{t\rho} = \frac{q^2 \rho}{(4\pi)^2} \left[\frac{a^2(\rho^2 + a^{-2} - Z^2)t}{2R^6} \right] \quad (9)$$

where $Z^2 = z^2 - t^2$ and $R = \frac{a}{2}[(a^{-2} + \rho^2 + Z^2)^2 - 4a^{-2}Z^2]^{1/2}$, so

$$T^{tt} = \frac{q^2 a^2}{8(4\pi)^2 R^6} [(\rho^2 + a^{-2} - z^2 + t^2)^2 + 4\rho^2(t^2 + z^2)] \quad (10)$$

$$T^{tz} = \frac{q^2 g^2 tz \rho^2}{(4\pi)^2 R^6} \quad (11)$$

$$T^{t\rho} = \frac{q^2 \rho a^2 (\rho^2 + a^{-2} - z^2 + t^2)t}{(4\pi)^2 2R^6} \quad (12)$$

Lastly $S^\mu = -T^{\mu\nu} g_{\nu\alpha} \xi^\alpha = T^{t\mu}$