Singh *Elements of Topology* Solutions

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1 Chapter 1

1.1.1 Given a set X, define $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ if $x \neq y$. Check that d is a metric on X.

Proof. Positive definiteness follows trivially from the definition. For symmetry, if $x = y$ or $x \neq y$ then it follows that $d(x, y) = 0 = d(y, x)$ and $d(x, y) = 1 = d(y, x)$ respectively. Now let $x, y, z \in X$. If $d(x, y) = 0$ then $d(x, z) + d(z, y) \geq 0 = d(x, y)$. If $d(x, y) = 1$, then $d(x, z) + d(z, y)$ is either 1 or 2, since at least one of $x \neq z$ or $y \neq z$ holds (otherwise $x = y$ and $d(x, y) = 0$). $\square$

1.1.2 Let (X, d) be a metric space. Show that

(a) $d'(x, y) = d(x, y)/(1 + d(x, y))$ is a bounded metric on X

Proof. d' inherits symmetry from d. d' is also positive definite since if $x = y$, then $d'(x, y) = 0/1 = 0$, and if $x \neq y$, then $d(x, y) > 0$.

$d'(x, y)$ is bounded since $d(x, y) \geq 0$ implies that $d(x, y) \leq 1 + d(x, y)$, thus $d'(x, y) \leq 1$ for all choices of $x, y \in X$.

$\square$

(b) $d_1(x, y) = \min(1, d(x, y))$ is a bounded metric on X.

Proof. It is clear that d_1 is symmetric and positive definite. Now let $x, y, z \in X$. We know that $d(x, y) \leq d(x, z) + d(z, y)$. If $d(x, z) + d(z, y) < 1$, then it follows that $d_1(x, y) = d(x, y) \leq d(x, z) + d(z, y) = d_1(x, z) + d_1(z, y)$. Now if $d(x, z) + d(z, y) \geq 1$, then $d_1(x, y) \leq 1 \leq d(x, z) + d(z, y)$. If $d(x, z) + d(z, y) \geq 1$, then it follows that $d_1(x, z) + d_1(z, y) \geq 1$, thus the triangle inequality holds.

d_1 is also trivially bounded, since $\sup\{d_1(x, y) | x, y \in X\} = 1$. $\square$

1.1.3 Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and given $x \in F^n$, define $||x|| = \max_{1 \leq i \leq n} |x_i|$. Show that the function $|| \cdot ||$ satisfies the conditions (a), (b) and (d) described in Ex. 1.1.1, and hence defines a norm on F^n. This is called the Cartesian norm on F^n.

Proof. (a) $||x|| > 0$ for $x \neq 0$. Suppose $x \in F^n$ where $x \neq 0$, then for at least one x_j, $x_j \neq 0$, then it follows that $||x|| = \max_i |x_i| \geq |x_j| > 0$, thus (a) holds.
(b) $||ax|| = |a|||x||$. Let $x \in F^n$, $a \in F$, then $||ax|| = \max_i |ax_i| = \max_i |a||x_i| = |a| \max_i |x_i| = |a|||x||$
(d) $||x+y|| \leq ||x||+||y||$. Let $x, y \in F^n$, $||x+y|| = \max_i |x_i+y_i| \leq \max_i |x_i|+\max_j |y_j| = ||x||+||y||$. $\square$

1.1.10 Show that a subset A of a metric space (X, d) is bounded if there exists a point $x \in X$ and a real number K such that $d(x, a) \leq K$ for every $a \in A$.

Proof. Recall that a set is bounded if its diameter is bounded, i.e. $\sup\{d(a, a') : a, a' \in A\} < \infty$. Let $K \geq \text{diam}(A)$, then let $x \in A$. It follows that $d(x, a) \leq \text{diam}(A) \leq K$ for all $a \in A$, as required. $\square$

1.2.1

(a) Find all possible topologies on the set $X = \{a, b, c\}$.

Proof. The trivial and discrete topologies, $\{(, \{x\}, X), x \in X$
$(\emptyset, \{x, y\}, X), x, y \in X$
$(\emptyset, \{x\}, \{x, y\}, X),\ x, y \in X$
$(\emptyset, \{x\}, \{y\}, \{x, y\}, X), x, y \in X$ □

(b) Let $\mathscr{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathscr{T}_2 = \{\emptyset, X, \{c\}, \{b, c\}\}$ on X. Is the union of $\mathscr{T}_1$ and $\mathscr{T}_2$ a topology for X.

Proof. $\mathscr{T}_1 \cup \mathscr{T}_2 = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{b, c\}\}$. $\{a\} \cup \{c\} = \{a, c\} \notin \mathscr{T}_1 \cup \mathscr{T}_2$, thus is not a topology. □

(c) Find the smallest topology containing $\mathscr{T}_1$ and $\mathscr{T}_2$, and the largest topology contained in $\mathscr{T}_1$ and $\mathscr{T}_2$.

Proof. $\mathscr{T}_s = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ is simply the discrete topology.
$\mathscr{T}_l = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$ □

1.2.2

(a) What is the topology determined by the metric d on X given by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$

Proof. The open sets of d are the points sets, since $B_{1/2}(x) = \{x\}$ for any $x \in X$. Now let $U = \cup_\alpha \{x_\alpha\} \subset X$ be any collection of elements of X. It follows that $\cup_\alpha B_{1/2}(x_\alpha) = \cup_\alpha \{x_\alpha\} = U$ is the union of open balls, and thus is open. It follows that since any arbitrary subset of X is open, that the topology induced by d is the discrete topology. □

(b) Let X be a set containing more than one element. Can you define a metric on X so that the associated metric topology is trivial.

Proof. To see this, suppose $X = \{a, b\}$. If d is a metric on X, then $d(a, b) > 0$, let $0 < \varepsilon < d(a, b)$, then $B_\varepsilon(a) = \{a\}$, thus we again achieve the discrete topology. It follows that we cannot induce the trivial topology on X. More generally if X is a set containing more than one element, in order to induce the trivial topology, d must not distinguish points in X; however, any two distinct points $x, y \in X$ have non-zero distance by the definition of a metric, thus for any two points $x, y \in X$, $x \neq y$, there exists open balls $B_\varepsilon(x)$, $B_\delta(y)$ such that $y \notin B_\varepsilon(x)$ and vice versa, thus we cannot induce the discrete topology. (this is hinting at the fact that metric spaces are Hausdorff) □

1.2.3 Let X be an infinite set, $x_0 \in X$ a fixed point. Show that

$$\mathscr{T} = \{G | \text{either} X - G \text{ is finite or } x_0 \notin G\}$$

is a topology on X in which every point, except x_0, is both open and closed. ($(X, \mathscr{T})$ is called a *Fort space*)

Proof. X is a set with empty and thus finite compliment. $\emptyset$ is a set not containing x_0 and thus is an element of the topology.

Let $U, V \in \mathscr{T}$. Suppose $x_0 \notin U, V$, $U \cap V$, then $U \cap V$ is a set not containing x_0, and thus $U \cap V \in \mathscr{T}$. If either U or V does not contain x_0, it follows that $x_0 \notin U \cap V$, and thus $U \cap V \in \mathscr{T}$. Now assume neither U nor V contain x_0, then it follows that U^c and V^c are finite. Moreover, it follows that $U \cap V$ is an infinite set, since otherwise would imply that there exists an infinite subset of X not contained in U and likewise for V. Therefore $\mathscr{T}$ is closed under finite intersection.

Now consider the arbitrary union of sets in $\mathscr{T}$. Either they all do not contain x_0 and then it follows that their union is an element in $\mathscr{T}$, or at least one contains x_0, then it follows that at least one U is an infinite set with finite compliment, and thus the union also has finite compliment, thus $\mathscr{T}$ is contained under finite union. $\square$

1.2.4 Decide the openness and closedness of the following subsets in $\mathbb{R}$.
(a) $\{x : 1/2 < |x| \leq 1\}$, neither
(b) $\{x : 1/2 \leq |x| < 1\}$, neither
(c) $\{x : 1/2 \leq |x| \leq 1\}$, closed but not open
(d) $\{x : 0 < |x| < 1\}$ and $(1/x) \notin \mathbb{N}$, open but not closed

1.2.5 Find a topology on $\mathbb{R}$, different from the trivial topology and the discrete topology, so that every open set is closed and vice versa.

Proof. Let $\mathscr{T} = \{\emptyset, \mathbb{R}, (-\infty, 0), [0, \infty)\}$ be a topology. It is clearly neither the trivial nor discrete topology, and $\mathbb{R}$ and $\emptyset$ are both open and closed, and since $(-\infty, 0)^c = [0, \infty)$, it follows that both $(-\infty, 0)$ and $[0, \infty)$ are closed (in addition to being open sets) as required. $\square$

1.2.11 If every countable subset of a space is closed, is the topology necessarily discrete?

Proof. In order to induce the discrete topology, singletons need to be open, thus the compliment of points are closed. We should ask if we can always construct a compliment of a singleton set out of the finite union of countable sets. The answer is clearly no since take as an example $\mathbb{R}$, the complimenet of a point $\mathbb{R}\backslash\{p\}$ cannot be written as the countable union of countable sets, thus the discrete topology is not induced. $\square$

1.3.4 Let X be a space and $A \subseteq X$. Prove:

(a) $X - \bar{A} = (X - A)^o$

Proof. Let U be open in $X - \bar{A}$. Since $\bar{A}$ is closed and X is open, it follows that $X - \bar{A}$ is open and moreover $X - \bar{A} \subseteq X - A$. It follows that $X - \bar{A} = (X - \bar{A})^\circ \subseteq (X - A)^o$.

Note that $B \subset C$ implies $C^c \subset B^c$ and thus since $A \subset \bar{A}$ implies $\bar{A}^c \subset A^c$. It follows that $(X - A)^\circ = (X \cap A^c)^\circ = X^\circ \cap (A^c)^\circ = X \cap (\bar{A})^c = X - \bar{A}$.

It remains to show the reverse inclusion. $\square$

1.3.5 Let X be a space and $A \subseteq X$. Prove that A is clopen if and only if $\partial A = \emptyset$.

Proof. Let A be clopen, then it is the largest open set contained in A, thus $A = A^\circ$ and also the smallest closed set containing A, thus $A = \bar{A}$, and by 1.3.4(c), $\partial A = \bar{A} - A^\circ = A - A = \emptyset$. If $\partial A = \emptyset$ then again it follows that $\bar{A} = A^\circ = A$, and thus A is clopen. $\square$

1.3.9 Let X be an infinite set with the cofinite topology and $A \subseteq X$. Prove that if A is infinite, then every point of X is a limit point of A and if A is finite then it has no limit points.

Proof. Let A be an infinite subset of X. Let $x \in X$, and let N be a neighborhood of x. First note that N is infinite since there exists an open set $x \in U \subset N$, and U is necessarily infinite. Since N is infinite, it follows that $A \cap N \neq 0$, otherwise U^c would be infinite. Therefore indeed $x \in X$ is a limit point of A.

Now let A be a finite set. Let $x \in X$, it follows that $A\backslash\{x\}$ is a finite set. Let $U = X\backslash A \cup \{x\}$, it follows that U is an open neighborhood of x that does not intersect A, thus x is not a limit point of A. $\square$

1.3.17 Prove that the union of two nowhere dense sets is nowhere dense.

Proof. □

Theorem 1.4.4 A collection $\mathscr{B}$ of open subsets of a space X is a basis if and only if for each open subset U of X and each point $x \in U$, there exists a $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

Proof. Suppose $\mathscr{B}$ is a basis. Let U be an open set. Let $x \in U$. Since U is open it follows that $U = \cup_\alpha B_\alpha$ for $B_\alpha \in \mathscr{B}$. It follows that $x \in B_\alpha$ for at least one alpha, and thus $x \in B_\alpha \subseteq U$.
Now suppose for each open set U and every point $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \subset U$. It follows that for all x_α in U there exists such B_α, $U = \bigcup_\alpha x_\alpha \subseteq \bigcup_\alpha B_\alpha \subseteq U \Rightarrow U = \cup_\alpha B_\alpha$, thus $\mathscr{B}$ is a basis. □

Theorem 1.4.8 Two metrics d and d' on the set X are equivalent if and only if for each $x \in X$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that $B_d(x;\delta) \subseteq B_{d'}(x;\varepsilon)$ and $B_{d'}(x;\delta) \subseteq B_d(x;\varepsilon)$

Proof. Let $\varepsilon > 0$ be given. Suppose d and d' are equivalent, then it follows that □

1.4.1

(a) What is the order topology on the set $\mathbb{N}$ with the usual order relation.

Proof. Note that $x < 1$ for all $x \in \mathbb{N}$ implies that $x = 0$, in other words 0 is the least element of $\mathbb{N}$ with the usual order relation. We then note that 0 is open under the order topology. Moreover let $x \in \ltimes$, $x \neq 0$. It follows that $[0, x+1)$ and $(x-1, \infty)$ are open in the order topology, and thus $[0, x+1) \cap (x-1, \infty) = \{x\}$ is open in the order topology. Thus the order topology induces the discrete topology. □

(b) Is the order topology on $\{1, 2\} \times \mathbb{N}$ in the dictionary order relation discrete?

Proof. □

1.4.11 Show that the rationals are dense in the Sorgenfrey line $\mathbb{R}_l$.

Proof. We know that $\mathbb{R} - \bar{\mathbb{Q}} = (\mathbb{R} - \mathbb{Q})^\circ$. Let $x \in (\mathbb{R} - \mathbb{Q})^\circ$, then it follows that there exists an open set $U \subseteq (\mathbb{R} - \mathbb{Q})^\circ$ such that $x \in U$. Since U is open, it follows that it can be generated by the lower limit topology as the union of sets of the form $[a, b)$, thus $x \in [a, b)$ for some $a \neq b \in \mathbb{R}$. However, we can apply the Archimedean principle to find a rational q such that $a < q < b$, thus $q \in [a, b)$. But this contradicts the fact that $[a, b) \subseteq U \subseteq (R - \mathbb{Q})^\circ$, thus $(R - \mathbb{Q})^\circ = \emptyset$. Therefore $\mathbb{R} = \bar{\mathbb{Q}}$. □

1.5.1 A subset Y of a space X is called discrete if the relative topology for Y is discrete. Prove that every subspace of a discrete space is discrete and every subspace of an indiscrete space is indiscrete.

Proof. Let X have the discrete topology, and let $A \subseteq X$. Let $U \subseteq A$. It follows that $U = U \cap A \subseteq X$ is open in A, thus A inherits the discrete topology from X.

Let X have the indiscrete topology, let $A \subseteq X$. It follows that the only open sets of A are $A = A \cap X$ and $\emptyset = A \cap \emptyset$, thus A inherits the indiscrete topology from X. □

1.5.10 Give an example of a space X which has a dense subset D and a subset Y such that $D \cap Y$ is not dense in Y.

Proof. Let $X = \mathbb{R}$ with the usual topology, $D = \mathbb{Q}$, and $Y = \{\{1\}, \{\pi\}\}$. Note that X induces the discrete on Y. It follows that $D \cap Y = \{1\}$, but that D is not dense in Y. □

2 Chapter 2

2.1.1 Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies on a set X. Show that

(a) The identity function $i : (X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ is continuous if and only if $\mathscr{T}_1$ is finer than $\mathscr{T}_2$

Proof. Suppose $\mathscr{T}_1$ is finer than $\mathscr{T}_2$, then it follows that every open set $U \in \mathscr{T}_2$ is open in $\mathscr{T}_1$, thus $i^{-1}(U)$ is open in $\mathscr{T}_1$, therefore i is continuous. Now suppose T_1 is not finer than $\mathscr{T}_2$, then it follows that there exists some open set U in $\mathscr{T}_2$ such that U is not open in $\mathscr{T}_1$, thus $i^{-1}(U)$ is not open and therefore i is not continuous. $\square$

(b) i is a homeomorphism if and only if $\mathscr{T}_1 = \mathscr{T}_2$.

Proof. i is clearly bijective. In order for i to be a homeomorphism, i and i^{-1} must both be continuous which according to the result in (a) is true if and only if $\mathscr{T}_1$ and $\mathscr{T}_2$ are both finer than each other, thus $\mathscr{T}_1 = \mathscr{T}_2$. $\square$

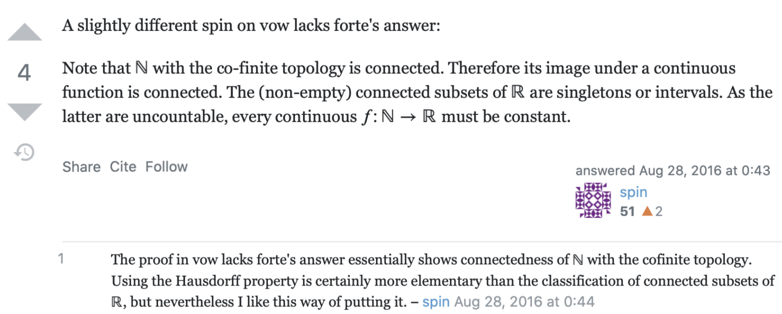
2.1.4 Prove that the inversion function $x \mapsto x^{-1}$ on $\mathbb{R} - \{0\}$ is continuous and open.

Proof. First, let $x \in \mathbb{R}\backslash\{0\}$, note that $1/x \in \mathbb{R}$, and moreover that $f^{-1}(x) = 1/x$ is unique, which together imply that f is bijective. It remains to show the continuity conditions. Assuming the usual topology for $\mathbb{R}\backslash\{0\}$, let x be a point in $\mathbb{R} - \{0\}$, let U be open in $\mathbb{R}\backslash\{0\}$. Since U is generated by open intervals (a, b), it suffices from theorem 2.1.3 to show that $f^{-1}((a, b))$ is open. Moreover we can further assume that $a, b > 0$ or that $a, b < 0$, since if $a < 0$ and $b > 0$, it follows that $(a, b) = (a, 0) \cup (0, b)$. wlog let $a, b > 0$. It follows that $f^{-1}((a, b)) = (1/b, 1/a)$ is open in $\mathbb{R}\backslash\{0\}$. Thus f is continuous, and upon substitution by $1/a$, $1/b$ it followsw immediately that f is an open map. $\square$

2.1.5 Let X be an uncountable set with the cofinite topology. Show that every continuous function $X \to \mathbb{R}$ is constant.

Proof. Let $f : X \to \mathbb{R}$ be continuous, but not constant, i.e. for some $a, b \in X$, $f(a) \neq f(b)$. It follows that there exist open sets $f(a) \in U$, $f(b) \in V$, such that $U \cap V = 0$, (by Hausdorff property of $\mathbb{R}$ with the usual topology). Moreover $f^{-1}(U) = U'$ and $f^{-1}(V) = V'$ are open, thus U' and V' are uncountable infinite subsets of X, and by a result in the previous chapter, it follows that $U' \cap V' \neq 0$, but this implies that for some $u \in U' \cap V'$, $f(u) \in U \cap V$, but since $U \cap V = 0$, this is a contradiction, thus f must be constant. $\square$

Given that the problem specifies uncountablility of X, but my solution did not require it, so I was wondering if there was something missing. My argument is correct but I found a nice post on stack proving this via connectedness rather than Hausdorffness.


A slightly different spin on vow lacks forte's answer:

4

Note that $\mathbb{N}$ with the co-finite topology is connected. Therefore its image under a continuous function is connected. The (non-empty) connected subsets of $\mathbb{R}$ are singletons or intervals. As the latter are uncountable, every continuous $f : \mathbb{N} \to \mathbb{R}$ must be constant.

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answered Aug 28, 2016 at 0:43
spin
51 ▲2

1 The proof in vow lacks forte's answer essentially shows connectedness of $\mathbb{N}$ with the cofinite topology. Using the Hausdorff property is certainly more elementary than the classification of connected subsets of $\mathbb{R}$, but nevertheless I like this way of putting it. – spin Aug 28, 2016 at 0:44

2.1.6 Give an example of a function $f : X \to Y$ between spaces and a subspace A such that $f|A$ is continuous, although f is not continuous at any point of A.

Proof. $\square$

2.1.7 Let $f : X \to Y$ be continuous and $A \subseteq X$. If x is a limit point of A, is $f(x)$ a limit point of $f(A)$?

Proof. No, consider the constant map $f : X \to Y$, $f(x) = y_0$ which is trivially continuous, it follows that $f(A) = y_0$, and that $f(A)\backslash\{y_0\} = \emptyset$, thus $f(A)$ is not a limit point of itself. $\square$

2.1.9 Let $\{U_\alpha\}$ be a family of open subsets of a space X with $X = \cup U_\alpha$. Show that a function f from X into a space Y is continuous if and only if $f|U_\alpha$ is continuous for each index α.

Proof. $\square$

Theorem If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Proof. Let D be a discreet space, and $g : f(X) \to D$ be a continuous map. If $f' : X \to f(X)$ is the map defined by f, then the composition $gf' : X \to D$ is continuous. Since X is connected, gf' is constant and so g is constant. By theorem 3.1.2, $f(X)$ is connected. $\square$

3 Chapter 3

3.1.1 Let $(X, \mathscr{T})$ be a connected space. If $\mathscr{T}'$ is a topology on X coarser than $\mathscr{T}$, show tht $(X, \mathscr{T}')$ is connected.

Proof. We know that if $f : X \to D$ where D is a 2 point discrete space is continuous for $\mathscr{T}$, then f is a constant map. Now let $g : X \to D$ be continuous on $\mathscr{T}'$. Assume towards contradiction that $(X, \mathscr{T}')$ is disconnected, and that $g : X \to D$ is a continuous non-constant map, then it follows that $g^{-1}(a) = U_a$ and $g^{-1}(b) = U_b$ are open and disjoint in $(X, \mathscr{T}')$ and moreover form a disconnection of X. Now since $\mathscr{T}$ is finer than $\mathscr{T}'$, it follows that U_a and U_b are open and disjoint in $\mathscr{T}$, and thus form a disconnection of $(X, \mathscr{T})$, but this is a contradiction of the fact that (X, T) is connected, thus $(X, \mathscr{T}')$ is connected. $\square$

3.1.2 Let A and B be separated subsets of a space and C be a connected subset of $A \cup B$. Show that $C \subset A$ or $C \subset B$.

Proof. First note that if C is a connected subset of $A \cup B$ in X, then it is also a connected subset of $A \cup B$ in the subspace $A \cup B$. Let $C \subset A \cup B$ be connected. Assume towards contradiction that $C \cap A \neq 0$ and that $C \cap B \neq 0$, and that $C = (C \cap A) \cup (C \cap B)$, then since C is connected, C is not the union of two nonempty separated sets in $A \cup B$; however, $C \cap A$ and $C \cap B$ are disconnected, since if they are connected, then it would follow from theorem 3.1.9 that A and B are connected, thus C must be either in A or in B. $\square$

3.1.4 Let A be a connected subset of a space X, are A° and ∂A connected? Does the converse hold?

Proof. Let $A = (0, 1) \subset \mathbb{R}$, then $\partial A = \{0, 1\}$ is a two point discrete space and thus disconnected. Now let A be two disks in $\mathbb{R}^2$ which intersect at a point, A is connected, but A° is disconnected.

For the converse, note that $A = (0, 1) \cup \{2\}$ is disconnected, but A° is connected, thus A° connected, does not imply that A is connected. Let $X = (0, 1) \cup (3, 4)$, let $A = (0, 3/4) \cup (3, 4)$, then $\partial A = \{3/4\}$ is connected, but A is disconnected. $\square$

3.1.5 Let A_n, $n = 1, 2, ...$ be a countable family of connected subsets of a space X such that $A_n \cap A_{n+1} \neq \emptyset$ for each n. Prove that $\cup A_n$ is connected.

Proof. Suppose A_n is disconnected, then there exists some disconnection U and V. We know from exercise 3.1.2 that every connected subset of A_n must be in either U or V. Moreover if all A_i are in either U or V, then they do not form a seperation, a contradiction, thus there exist some i for which A_i are in U and some for which A_i are in V. suppose wlog $A_i \subset U$ and $A_j \subset V$ for some index j $j > i$ (wlog), it follows however that $\cup_{k=i}^{j} A_k$ is disconnected, however this contradicts the fact that the finite union of connected sets is connected, thus A_n is connected. $\square$

3.1.8 Show that $\mathbb{R}^{n+1} - \mathbb{S}^n$ is the union of two disjoint open connected sets.

Proof. First note that $A = B_{\varepsilon=1}(0)$ is an open connected set in $\mathbb{R}^{n+1}$ and note that $B = \mathbb{R}^{n+1} - \bar{B}_{\varepsilon=1}(0)$ is open and connected in $\mathbb{R}^{n+1}$. A and B are clearly disjoint, and moreover $\mathbb{R}^{n+1} - \mathbb{S}^n = A \cup B$. $\square$

3.1.13 Give an example of two connected subsets of a space whose intersection is disconnected. Let $A = \{(x, y)|y \geq 0, x^2 + y^2 = 1\}$ and $B = \{(x, y)|y \leq 0, x^2 + y^2 = 1\}$, both are connected, but $A \cap B = \{(-1, 0), (1, 0)\}$ is disconnected.

3.2.2 Prove that a connected clopen subset of a space X is a componenet of X.

Proof. Let U be a connected clopen subset of X. Let $x \in U$, and consider $C(x)$. Clearly $U \subseteq C(x)$ as U is a connected componenet containing x. Further, suppose U is a proper subset of $C(x)$. Since U is a clopen proper subset of X, it follows that $X - U$ is clopen, and that any connected subset of X must be in either U or $X - U$, thus $C(x)$ must be in either U or $X - U$, which contradicts the assumption that U is a proper subset of $C(x)$, and it follows that $U = C(x)$ $\square$

3.2.9 Show that the subspace $A = \{0\} \cup \{1/n|n = 1, 2, ...\}$ is totally disconnected.

Proof. Notice that every $x \in A$ is also in $\mathbb{Q}$, and since $\mathbb{Q}$ is totally disconnected, it follows that A must also be totally disconnected. $\square$

3.2.10 Prove that any open subset of $\mathbb{R}^n$ can have at most countably many components. Is the same true for closed subsets?

Proof. Let $U \subset \mathbb{R}^n$ be open. It follows that $U = \cup_\alpha B_r(x)$. Suppose towards contradiction that U has uncountably many components. $\square$