4d null coordinates

Tim Bates

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1 Null Coordinates

When working with the divergencelessness of the stress tensor, it becomes convenient to consider a coordinate system which we call 'null coordinates'. Let $u = t - z$, $v = t + z$, $x = x$, $y = y$. u and v are null lines which on a spacetime diagram correspond to the rightward and leftward lightrays depicted in Figure 1.

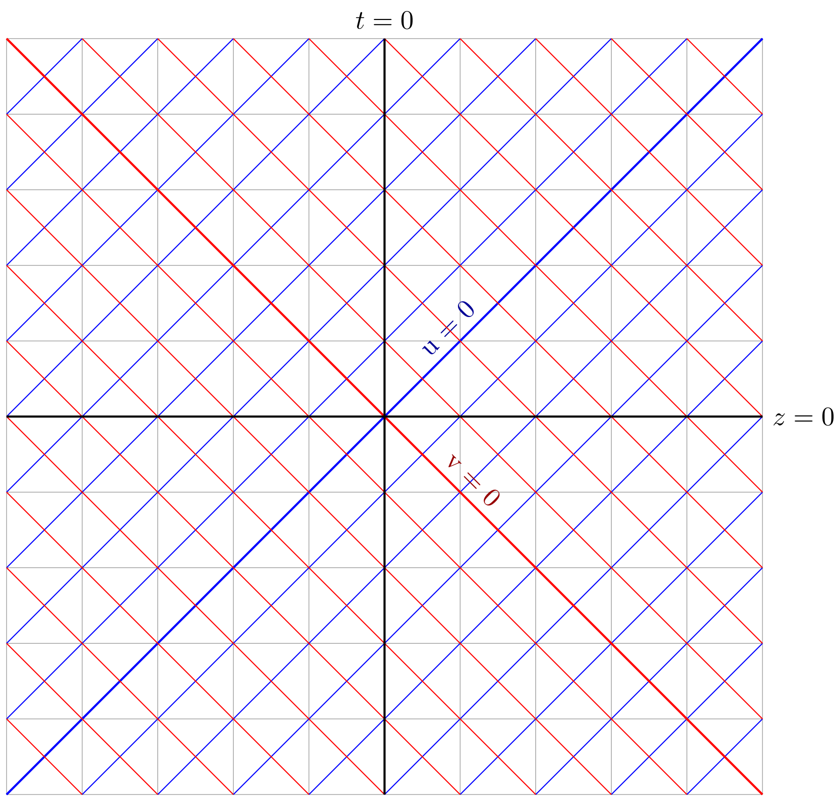

Figure 1: Null Coordinates

Transforming back to usual Minkowski coordinates is $t = \frac{u+v}{2}$, $z = \frac{v-u}{2}$. We then have that $dt = \frac{1}{2}(du + dv)$, $dz = \frac{1}{2}(dv - du)$. We can note that $z^2 - t^2 = -uv$, thus the line element for null coordinates is given by

$$ds^2 = -dudv + dx^2 + dy^2 \qquad \text{(Eq. A.1)}$$

The metric tensor then takes the following covariant and contravariant forms.

$$g_{\mu\nu} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(Eq. A.2)}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{(Eq. A.3)}$$

We remark that raising and lowering indices switches the place of u and v.

2 What can we say about these coordinates in 4d spacetime

As in 2d, $T_{uu} = \partial_u\phi\partial_u\phi$, and $T_{vv} = \partial_v\phi\partial_v\phi$. One result in 2d is that $T_{uv} = T_{vu} = 0$, this simplifies checking the trace condition. Does this hold in 4d?

$$\begin{aligned}
T_{uv} =&\partial_u\phi\partial_v\phi - \frac{1}{2}g_{uv}((-4)(\partial_v\phi\partial_u\phi)) - \frac{1}{2}g_{uv}((\partial_x\phi)^2 + (\partial_y\phi)^2)\\
=&\partial_u\phi\partial_v\phi - \partial_v\phi\partial_u\phi - \frac{1}{2}\left(\frac{-1}{2}\right)((\partial_x\phi)^2 + (\partial_y\phi)^2)\\
=&\frac{1}{4}((\partial_x\phi)^2 + (\partial_y\phi)^2)
\end{aligned}$$

So this is nonzero and is in fact dependent on the flow in the transverse directions. The conservation law then requires us to take one more derivative,

$$\begin{aligned}
\partial^v T_{vu} =& -2\partial_u T_{vu}\\
=& -\frac{1}{2}(2\partial_x\phi\partial_u\partial_x\phi + 2\partial_y\phi(\partial_u\partial_y\phi))\\
=& -(\partial_x\phi\partial_u\partial_x\phi + \partial_y\phi\partial_u\partial_y\phi)
\end{aligned}$$

$$\begin{aligned}
\partial^u T_{uv} =& -2\partial_v T_{uv}\\
=& -(\partial_x\phi\partial_v\partial_x\phi + \partial_y\phi\partial_v\partial_y\phi)
\end{aligned}$$

So

$$\begin{aligned}
\partial^\alpha T_{\alpha u} =&\partial^u(\partial_u\phi\partial_u\phi) - (\partial_x\phi\partial_u\partial_x\phi + \partial_y\phi\partial_u\partial_y\phi) + \partial_x(\partial_x\phi\partial_u\phi) + \partial_y(\partial_y\phi\partial_u\phi)\\
=&\partial^u(\partial_u\phi\partial_u\phi) - (\partial_x\phi\partial_u\partial_x\phi + \partial_y\phi\partial_u\partial_y\phi) + \partial_x\partial_x\phi\partial_u\phi + \partial_x\phi\partial_x\partial_u\phi + \partial_y\partial_y\phi\partial_u\phi + \partial_y\phi\partial_y\partial_u\phi\\
=&[-(\partial_x\phi(\partial_u\partial_x\phi) + \partial_y\phi(\partial_u\partial_y\phi)) + \partial_x\phi(\partial_x\partial_u\phi) + (\partial_y\phi)(\partial_y\partial_u\phi)]\\
&+ \partial^u(\partial_u\phi\partial_u\phi) + (\partial^x\partial_x\phi)(\partial_u\phi) + (\partial^y\partial_y\phi)(\partial_u\phi)\\
=&0 + (2\partial^u\partial_u\phi + \partial^x\partial_x\phi + \partial^y\partial_y\phi)(\partial_u\phi)\\
=&(\partial^\mu\partial_\mu\phi)(\partial_u\phi) \quad \text{since } \partial^u\partial_u\phi = \partial^v\partial_v\phi\\
=&0
\end{aligned}$$

and similarly

$$\partial^\alpha T_{\alpha v} = 0$$

To be complete, here are the remaining components of the stress energy tensor.

$$\begin{aligned}
T_{ux} =&\partial_u\phi\partial_x\phi\\
T_{uy} =&\partial_u\phi\partial_y\phi\\
T_{vx} =&\partial_v\phi\partial_x\phi\\
T_{vy} =&\partial_v\phi\partial_y\phi\\
T_{xx} =&\partial_x\phi\partial_x\phi - \frac{1}{2}((\partial_x\phi)^2 + (\partial_y\phi)^2 - 4\partial_u\phi\partial_v\phi)\\
T_{yy} =&\partial_y\phi\partial_y\phi - \frac{1}{2}((\partial_x\phi)^2 + (\partial_y\phi)^2 - 4\partial\phi_u\partial_v\phi)\\
T_{xy} =&\partial_x\phi\partial_y\phi
\end{aligned}$$

Divergencelessness is captured in the computation $\partial^\mu T_{\mu\nu}$, we must compute the following

$$\partial^u T_{uu} + \partial^v T_{uv} + \partial^x T_{ux} + \partial^y T_{uy}$$

$$\partial^u T_{vu} + \partial^v T_{vv} + \partial^x T_{vx} + \partial^y T_{vy}$$

$$\partial^u T_{xu} + \partial^v T_{xv} + \partial^x T_{xx} + \partial^y T_{xy}$$

$$\partial^u T_{yu} + \partial^v T_{yv} + \partial^x T_{yx} + \partial^y T_{yy}$$

and verify that each vanishes.

2.1 Divergencelessness of a uniformly accelerating source

$$\phi = \frac{q}{4\pi R}\theta(t+z) \tag{1}$$
$$\phi = \frac{q}{4\pi R}\theta(v) \tag{2}$$

Where

$$R = \frac{a}{2}[(X^2 - a^{-2})^2 + 4a^{-2}\rho^2]^{1/2} \tag{3}$$
$$X^2 = -uv + \rho^2 \tag{4}$$

Calculations: $\partial_\mu R$

$$\begin{aligned}
\partial_u R &= \frac{-a^2}{4}\frac{v(X^2 - a^{-2})}{R} \\
\partial_v R &= \frac{-a^2}{4}\frac{u(X^2 - a^{-2})}{R} \\
\partial_x R &= \frac{a^2 x}{2R}[X^2 - 3a^{-2}] \\
\partial_y R &= \frac{a^2 y}{2R}[X^2 - 3a^{-2}]
\end{aligned}$$

Calculations: $\partial_\mu \phi$

$$\begin{aligned}
\partial_u \phi &= \frac{-q}{4\pi R^2}(\partial_u R)\theta(v) \\
\partial_v \phi &= \frac{-q}{4\pi R^2}(\partial_v R)\theta(v) + \frac{q}{4\pi R}\delta(v) \\
\partial_x \phi &= \frac{-q}{4\pi R^2}(\partial_x R)\theta(v) \\
\partial_y \phi &= \frac{-q}{4\pi R^2}(\partial_y R)\theta(v)
\end{aligned}$$

Calculations: $\mathbf{T}_{\mu\nu}$

$$\begin{aligned}
T_{uu} &= \left(\frac{-q}{4\pi R^2}(\partial_u R)\theta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_u R)\theta(v)\right) \\
T_{vv} &= \left(\frac{-q}{4\pi R^2}(\partial_v R)\theta(v) + \frac{q}{4\pi R}\delta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_v R)\theta(v) + \frac{q}{4\pi R}\delta(v)\right) \\
T_{uv} &= \frac{1}{4}\left(\left(\frac{-q}{4\pi R^2}(\partial_x R)\theta(v)\right)^2 + \left(\frac{-q}{4\pi R^2}(\partial_y R)\theta(v)\right)^2\right) \\
T_{ux} &= \left(\frac{-q}{4\pi R^2}(\partial_u R)\theta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_x R)\theta(v)\right) \\
T_{uy} &= \left(\frac{-q}{4\pi R^2}(\partial_u R)\theta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_y R)\theta(v)\right) \\
T_{vx} &= \left(\frac{-q}{4\pi R^2}(\partial_v R)\theta(v) + \frac{q}{4\pi R}\delta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_x R)\theta(v)\right) \\
T_{vy} &= \left(\frac{-q}{4\pi R^2}(\partial_v R)\theta(v) + \frac{q}{4\pi R}\delta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_y R)\theta(v)\right) \\
T_{xx} &= \left(\frac{-q}{4\pi R^2}(\partial_x R)\theta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_x R)\theta(v)\right) \\
&\quad -\frac{1}{2}\left(\left(\frac{-q}{4\pi R^2}(\partial_x R)\theta(v)\right)^2 + \left(\frac{-q}{4\pi R^2}(\partial_y R)\theta(v)\right)^2 - 4\left(\frac{-q}{4\pi R^2}(\partial_u R)\theta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_v R)\theta(v) + \frac{q}{4\pi R}\delta(v)\right)\right) \\
T_{yy} &= \left(\frac{-q}{4\pi R^2}(\partial_y R)\theta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_y R)\theta(v)\right) \\
&\quad -\frac{1}{2}\left(\left(\frac{-q}{4\pi R^2}(\partial_x R)\theta(v)\right)^2 + \left(\frac{-q}{4\pi R^2}(\partial_y R)\theta(v)\right)^2 - 4\left(\frac{-q}{4\pi R^2}(\partial_u R)\theta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_v R)\theta(v) + \frac{q}{4\pi R}\delta(v)\right)\right) \\
T_{xy} &= \left(\frac{-q}{4\pi R^2}(\partial_x R)\theta(v)\right)\left(\frac{-q}{4\pi R^2}(\partial_y R)\theta(v)\right)
\end{aligned}$$

Calculation: $T_{u\mu}$ Simplified

$$
\begin{aligned}
T_{uu} =& \frac{q^2}{16\pi^2 R^4}(\partial_u R)^2\theta(v)\\
T_{uv} =& \frac{q^2((\partial_x R)^2 + (\partial_y R)^2)\theta(v)}{64\pi^2 R^4}\\
T_{ux} =& \frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_x R)\theta(v)\\
T_{uy} =& \frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_y R)\theta(v)
\end{aligned}
$$

Verified $\partial^\mu T_{u\mu}$ this morning (March 2). The theta function terms add to zero since ϕ satisfies the wave equation and we are left with

$$\frac{q^2}{16\pi^2 R^4}(\partial_u R)^2\delta(v)$$

which is zero since $\partial_u R = 0$ when $v = 0$.

Calculation: $T_{v\mu}$ Simplified

$$
\begin{aligned}
T_{vv} =& \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)^2\theta(v)}{R^4} - \frac{2(\partial_v R)\delta(v)\theta(v)}{R^3} + \frac{1}{R^2}\delta(v)\delta(v)\right]\\
T_{vu} =& \frac{q^2((\partial_x R)^2 + (\partial_y R)^2)\theta(v)}{64\pi^2 R^4} = \frac{q^2}{16\pi^2}\left[\frac{((\partial_x R)^2 + (\partial_y R)^2)}{4R^4}\right]\theta(v)\\
T_{vx} =& \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)(\partial_x R)}{R^4}\theta(v) - \frac{(\partial_x R)}{R^3}\delta(v)\theta(v)\right]\\
T_{vy} =& \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)(\partial_y R)}{R^4}\theta(v) - \frac{(\partial_y R)}{R^3}\delta(v)\theta(v)\right]
\end{aligned}
$$

Theta function terms will sum to 0 by wave equation. We are left with terms which are delta functions or products of theta and delta functions which since $\delta(v)\theta(v) = \frac{1}{2}\delta(v)$ means we can consider a grand sum of delta functions which are

$$\partial_u\left[2\frac{(\partial_v R)}{R^3}\right]\delta(v) - \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{2R^4}\right]\delta(v) + \partial_x\left[\frac{-(\partial_x R)}{2R^3}\right]\delta(v) + \partial_y\left[\frac{-(\partial_y R)}{2R^3}\right]\delta(v)$$

differentiating the terms and pulling out the delta function

$$
\begin{aligned}
&\left(-2\left[\frac{-(\partial_{uv}R)R + (\partial_v R)(\partial_u R)}{R^4}\right] - \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{2R^4}\right] - \frac{1}{2}\left[\frac{(\partial_{xx}R)R - 3(\partial_x R)^2}{R^4}\right] - \frac{1}{2}\left[\frac{(\partial_{yy}R)R - 3(\partial_y R)^2}{R^4}\right]\right)\delta(v)\\
&\frac{-1}{2}\left(4\left[\frac{-(\partial_{uv}R)R + (\partial_v R)(\partial_u R)}{R^4}\right] + \left[\frac{(\partial_x R)^2 + (\partial_y R)^2}{R^4}\right] + \left[\frac{(\partial_{xx}R)R - 3(\partial_x R)^2}{R^4}\right] + \left[\frac{(\partial_{yy}R)R - 3(\partial_y R)^2}{R^4}\right]\right)\delta(v)
\end{aligned}
$$

Evaluating this at $v = 0$, we know that $\partial_u R$ is zero so

$$\frac{-1}{2R^4}\left((-4(\partial_{uv}R) + (\partial_{xx}R) + (\partial_{yy}R))R - 2((\partial_x R)^2 + (\partial_y R)^2\right)\delta(v)$$

Since we are able to add in terms that are equal to zero,

$$
\begin{aligned}
&\frac{-1}{2R^4}\left((\partial^\mu\partial_\mu R)R - 2(-4(\partial_u R)(\partial_v R) + (\partial_x R)^2 + (\partial_y R)^2\right)\delta(v)\\
&\frac{-1}{2R^4}\left((\partial^\mu\partial_\mu R)R - 2(\partial^\mu R\partial_\mu R\right)\delta(v)
\end{aligned}
$$

It follows that this vanishes since the wave equation vanishing implies: $(\partial^\mu\partial_\mu R)R - 2\partial^\mu R\partial_\mu R = 0$

Calculation: $T_{x\mu}$ Simplified

$$
\begin{aligned}
T_{xu} =& \frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_x R)\theta(v)\\
T_{xv} =& \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)(\partial_x R)}{R^4}\theta(v) - \frac{(\partial_x R)}{R^3}\delta(v)\theta(v)\right]\\
T_{xx} =& \frac{q^2}{16\pi^2}\left(\left[\frac{(\partial_x R)^2 - (\partial_y R)^2 + 4(\partial_v R)(\partial_u R)}{2R^4}\right]\theta(v) - \frac{2(\partial_u R)}{R^3}\theta(v)\delta(v)\right)\\
T_{xy} =& \frac{q^2}{16\pi^2 R^4}(\partial_x R)(\partial_y R)\theta(v)
\end{aligned}
$$

$(-2\partial_v T_{xu})$ yields a delta function term $\frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_x R)\delta(v)$ which at $v=0$ is zero since $\partial_u R$ vanishes. T_{xy} will not give rise to any delta function terms. The only delta function terms will be

$$2\partial_u\left(\frac{(\partial_x R)}{R^3}\delta(v)\theta(v)\right) - \partial_x\left(\frac{2(\partial_u R)}{R^3}\theta(v)\delta(v)\right) = 2\left[\frac{(\partial_{ux}R - \partial_{xu}R)}{R^3} - 3\frac{(\partial_u R\partial_x R - \partial_x R\partial_u R)}{R^4}\right]\theta(v)\delta(v) = 0$$

Indeed vanishes.

Calculation: $T_{y\mu}$ Simplified

$$\begin{aligned}
T_{yu} =& \frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_y R)\theta(v) \\
T_{yv} =& \frac{q^2}{16\pi^2}\left[\frac{(\partial_v R)(\partial_y R)}{R^4}\theta(v) - \frac{(\partial_y R)}{R^3}\delta(v)\theta(v)\right] \\
T_{yx} =& \frac{q^2}{16\pi^2 R^4}(\partial_x R)(\partial_y R)\theta(v) \\
T_{yy} =& \frac{q^2}{16\pi^2}\left(\left[\frac{(\partial_y R)^2 - (\partial_x R)^2 + 4(\partial_v R)(\partial_u R)}{2R^4}\right]\theta(v) - \frac{2(\partial_u R)}{R^3}\theta(v)\delta(v)\right)
\end{aligned}$$

$(-2\partial_v T_{yu})$ yields a delta function term $\frac{q^2}{16\pi^2 R^4}(\partial_u R)(\partial_y R)\delta(v)$ which at $v=0$ is zero since $\partial_u R$ vanishes. T_{yx} will not give rise to any delta function terms. The only delta function terms will be

$$2\partial_u\left(\frac{(\partial_y R)}{R^3}\delta(v)\theta(v)\right) - \partial_y\left(\frac{2(\partial_u R)}{R^3}\theta(v)\delta(v)\right) = 2\left[\frac{(\partial_{uy}R - \partial_{yu}R)}{R^3} - 3\frac{(\partial_u R\partial_y R - \partial_x R\partial_u R)}{R^4}\right]\theta(v)\delta(v) = 0$$

Indeed vanishes.

3 Graphing Vector Fields

Our next goal is to plot the vector field of

$$S^\mu = -g^{\mu\alpha}T_{\alpha\beta}\xi^\beta \tag{5}$$

where ξ^β is a Killing vector.

Minkowski Time Killing Vector

$$\partial_t = \frac{1}{2}[\hat{u} + \hat{v}]$$

So we can compute the components of S^μ_M,

$$\begin{aligned}
S^u_M &= T_{vu} + T_{vv} \\
S^v_M &= T_{uu} + T_{uv} \\
S^x_M &= -\frac{1}{2}[T_{xv} + T_{xu}] \\
S^y_M &= -\frac{1}{2}[T_{yu} + T_{yv}]
\end{aligned}$$

In the past, moving along u, the flux diverges from the plane of the source in the transverse directions. When we get close to the future horizon (u = 0), the flux starts moving inward from the transverse directions. For the Rindler Killing vector, we obtain the following figures. When $|u| > |v|$, we have a outward flux, and when $|v| > |u|$, we have inward flux. This is because at these points the sum of components inside the S^x and S^y expressions change sign. How does this look like on z t plane? $-u = v$ in the R region is the line $z = 0$, when t is negative, we have outward flux. When t is positive, we hvae inward flux.

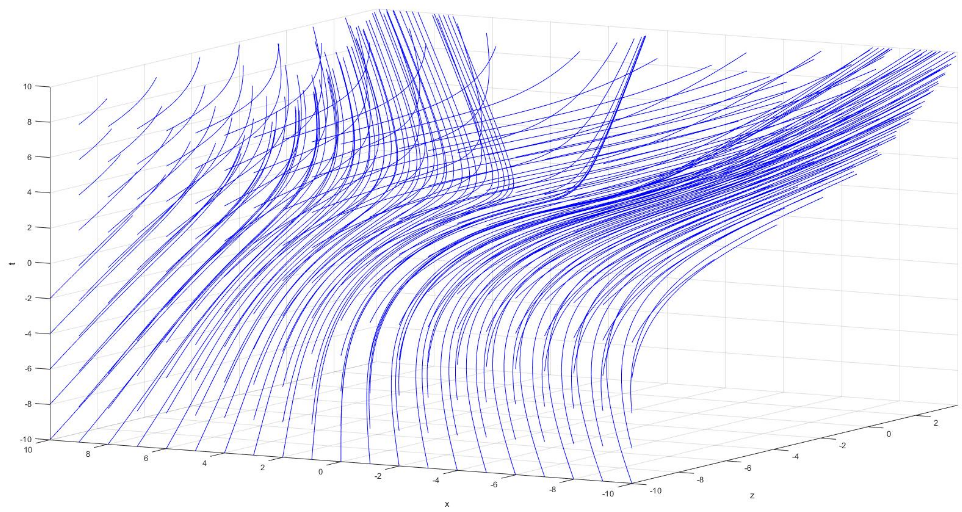

Figure 2: Minkowski Flux

Rindler Time Killing Vector The Rindler time Killing vector expressed in these coordinates is

$$\partial_\lambda = \frac{1}{2}\left[v\hat{v} - u\hat{u}\right] \tag{6}$$

Here the $\hat{u}$ and $\hat{v}$ are basis vectors in coordinates for the vector field so the coordinates for S_R^μ are

$$\begin{aligned} S_R^u &= vT_{vv} - uT_{uv} \\ S_R^v &= vT_{uv} - uT_{uu} \\ S_R^x &= -\frac{1}{2}[vT_{xv} - uT_{xu}] \\ S_R^y &= -\frac{1}{2}[vT_{yv} - uT_{yu}] \end{aligned}$$