NUMERICAL SOLUTION FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN 2D

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Abstract. In this report, we examine three finite-difference methods for solving incompressible Navier-Stokes equations[3]. The first method is called method of pertubation. It is a min-max problem and we want to minimize some cost function to find the unique minimizer.

The second and third methods are called projection method. These method involves seperating the equation and solves velocities and pressure individually. We first obtain the auxiliary field $u^{n+\frac{1}{3}}, u^{n+\frac{2}{3}}$, which are not divergence-free. And then we solve for p by solving the Poisson equation, and at last add the pressure p to the auxiliary field $u^{n+\frac{1}{3}}$,$u^{n+\frac{2}{3}}$ to get the divergence free u at next time step.

Key words. Navier-Stokes Equations, Finite Difference Method, Projection Method

1. Introduction.

1.1. Navier-Stokes Equation. For a bounded open subset $\Omega \subset \mathbb{R}^2$ and $T > 0$, let $u = (u_1, u_2) \in \Omega \times [0, T]$ that satisfies the incompressible fluid

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad (f\,\text{given}) \\ &\nabla u = 0 \end{aligned} \tag{1.1}$$

with initial and boundary conditions

$$\begin{aligned} u(x,t) &= 0 & x \in \partial\Omega \\ u(x,0) &= u_0(x) & (u_0\,\text{given}) \end{aligned} \tag{1.2}$$

1.2. Method of Perturbation (Method I). This method comes from control theory. Consider the problem

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p &= f \\ \frac{\partial p}{\partial t} + \Delta u &= 0 \end{aligned} \tag{1.3}$$

The associated Arrow-Hurwicz system is

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= f \\ \frac{\partial p}{\partial t} + \nabla\cdot u &= 0 \end{aligned} \tag{1.4}$$

But neither (3), (4) represents the flow before steady state. So we introduce a time scale c^2 to make the flow "little compressible".

$$\frac{\partial p}{\partial t} + c^2\nabla\cdot u = 0$$

It has been shown that the solution converge Navier-Stokes equation[1].
The discretization of this problem is by letting

$$
\begin{aligned}
u_{1}{}_{i,j}^{n} &= u(ih, jh, nk) \\
u_{2}{}_{i,j}^{n} &= v(ih, jh, nk) \\
p_{i+1/2,j+1/2} &= v((i+1/2)h, (j+1/2)h)
\end{aligned} \tag{1.5}
$$

and we discretize $\nabla \cdot u$ using a forward different and Δu using central difference discretization

$$
\begin{aligned}
\nabla \cdot u_{i,j} &= \frac{1}{2h}(u_{1i+1,j} - u_{1i,j} + u_{1i+1,j+1} - u_{1i,j+1}) + \frac{1}{2h}(u_{2i,j+1} - u_{2i,j} + u_{2i+1,j+1} - u_{2i+1,j}) \\
\Delta u_{i,j} &= \frac{1}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j})
\end{aligned} \tag{1.6}
$$

So (4) can be written as

$$
\begin{aligned}
u_{i,j}^{n+1} =& u_{i,j}^{n} + \nu\frac{k}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) \\
&- \frac{k}{2h}\left[u_{1}{}_{i,j}^{n}(u_{i+1,j}^{n} - u_{i-1,j}^{n}) + u_{2}{}_{i,j}^{n}(u_{i,j+1}^{n} - u_{i,j-1}^{n})\right] \\
&- k(\nabla p)_{i,j}^{n} + kf_{i,j}^{n}
\end{aligned} \tag{1.7}
$$

where $\nabla p_{i,j}^{n} = \begin{pmatrix} D_x p_{i,j}^{n} \\ D_y p_{i,j}^{n} \end{pmatrix}$
and

$$
\begin{aligned}
D_x p_{i,j}^{n} &= \frac{1}{2h}\left(p_{i+1/2,j+1/2}^{n} - p_{i-1/2,j+1/2}^{n} + p_{i+1/2,j-1/2}^{n} - p_{i-1/2,j-1/2}^{n}\right) \\
D_y p_{i,j}^{n} &= \frac{1}{2h}\left(p_{i+1/2,j+1/2}^{n} - p_{i+1/2,j-1/2}^{n} + p_{i-1/2,j+1/2}^{n} - p_{i-1/2,j-1/2}^{n}\right)
\end{aligned}
$$

The intermediate p can be obtained using the second equation of (4).

$$
\begin{aligned}
p_{i+1/2,j+1/2}^{n+1} =& p_{i+1/2,j+1/2}^{n} \\
&- \frac{k}{2h}c^2(u_{1i+1,j} - u_{1i,j} + u_{1i+1,j+1} - u_{1i,j+1} + u_{2i,j+1} - u_{2i,j} + u_{2i+1,j+1} - u_{2i+1,j})
\end{aligned}
$$

The grid of this discretization is shown below

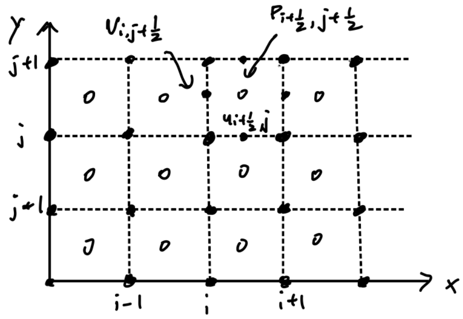

FIG. 1.1.

Stability analysis[2] shows that the system is stable when

$$k \le \nu \frac{h^2}{4},\ \frac{k}{h}(|u^n| + \sqrt{|u^n|^2 + 4c^2}) < 1 \tag{1.8}$$

1.3. Projection methods(Method II & III). Equation (1) can be written as

$$\begin{aligned} &\frac{\partial u}{\partial t} + \nabla p = \mathcal{F}(u) \\ &\nabla(\partial_t u) = 0 \end{aligned} \tag{1.9}$$

where $\mathcal{F}(u) = f + \nu\Delta u - u\nabla u$ is a function depending on f and u but not p. We can first approximate $\frac{\partial u}{\partial t}$ by $Tu := u^{n+1} - u^n$ where u^n represent u at time n. We first propose the auxilary field $u^{n+\frac{2}{3}}$ that satisfy

$$u^{n+\frac{2}{3}} - u^n = \mathcal{F}(u) \tag{1.10}$$

This auxilary field is different from u^{n+1} since the equation doesn't involve the pressure term. To include the pressure term, combine equation (3) and (4) and replace $\frac{\partial u}{\partial t}$ with Tu, we have

$$\mathcal{F}(u) = u^{n+\frac{2}{3}} - u^n = Tu + \nabla p^{n+1} \tag{1.11}$$

$$u^{n+\frac{2}{3}} = u^{n+1} + \nabla p^{n+1} \tag{1.12}$$

Notice that $u^{n+\frac{2}{3}}$ may not satisfy the divergence free condition $\nabla u = 0$, but u^{n+1} must satisfy $\nabla u^{n+1} = 0$. Also u^{n+1} have to satisfy the boundary condition $u^{n+1}\nu = 0$ at $\partial\Omega$. Therefore, take ∇ on equation(6), we get that the pressure p solves the Poisson problem with Neumann boundary condition

$$\begin{aligned}\Delta p^{n+1} &= \nabla u^{n+\frac{2}{3}} \\ \frac{\partial p^{n+1}}{\partial \nu}|_{\partial\Omega} &= u^{n+\frac{2}{3}} \cdot \nu|_{\partial\Omega}\end{aligned} \tag{1.13}$$

1.3.1. Evaluation of $u^{n+\frac{2}{3}}$(Difference between Method II & III). We want to find a method to find $u^{n+\frac{2}{3}}$ from equation (4). From the definition of $\mathcal{F}(u)$, we get equation (4) actually represent the Burger's equation

$$u_t = \frac{u^{n+\frac{2}{3}} - u^n}{k} = f + \nu\Delta u - u\nabla u \tag{1.14}$$

For method II, we find $u^{n+\frac{2}{3}}$ directly by solving

$$\frac{u^{n+\frac{2}{3}} - u^n}{k} = f^n + \nu(\Delta u)^n - (u\nabla u)^n$$

But for method III, we use the Alternating Direction Implicit(ADI) method to calculate the partial derivative in x_1, x_2 seperately. Set the intermediate field between $u^{n+\frac{2}{3}}$ and u^n to be $u^{n+\frac{1}{3}}$. Then we obtain

$$\begin{aligned}u^{n+\frac{1}{3}} - u^n &= f^{n+\frac{1}{3}} + \nu\frac{\partial^2 u^{n+\frac{1}{3}}}{\partial x_1^2} - u_1^n \frac{\partial u^{n+\frac{1}{3}}}{\partial x_1} \\ u^{n+\frac{2}{3}} - u^{n+\frac{1}{3}} &= f^{n+\frac{2}{3}} + \nu\frac{\partial^2 u^{n+\frac{2}{3}}}{\partial x_2^2} - u_2^{n+\frac{1}{3}} \frac{\partial u^{n+\frac{2}{3}}}{\partial x_2}\end{aligned} \tag{1.15}$$

Let

$$\begin{aligned}\frac{\partial^2 u}{\partial x_1^2} &= \frac{1}{h^2}(u_{j+1,k} - 2u_{j,k} + u_{j-1,k}) \\ \frac{\partial^2 u}{\partial x_2^2} &= \frac{1}{h^2}(u_{j,k+1} - 2u_{j,k} + u_{j,k-1}) \\ \frac{\partial u}{\partial x_1} &= \frac{1}{h}(u_{j+1,k} - u_{j-1,k}) \\ \frac{\partial u}{\partial x_2} &= \frac{1}{h}(u_{j,k+1} - u_{j,-1})\end{aligned} \tag{1.16}$$

Also, for the left hand side, let $T(u) := (u^{n+\frac{2}{3}} - u^n)/(k)$ instead. Then we obtain the alternating direction method

(1.17)
$$\begin{aligned}u^{n+\frac{1}{3}} &\approx u^n + f^{n+\frac{1}{3}} + \frac{\nu k}{h^2}(u_{j+1,k}^{n+\frac{1}{3}} - 2u_{j,k}^{n+\frac{1}{3}} + u_{j-1,k}^{n+\frac{1}{3}}) - \frac{k}{h}u_1^n(u_{j+1,k}^{n+\frac{1}{3}} - u_{j-1,k}^{n+\frac{1}{3}}) \\ u^{n+\frac{2}{3}} &\approx u^{n+\frac{1}{3}} + f^{n+\frac{2}{3}} + \frac{\nu k}{h^2}(u_{j,k+1}^{n+\frac{2}{3}} - 2u_{j,k}^{n+\frac{2}{3}} + u_{j,k-1}^{n+\frac{2}{3}}) - \frac{k}{h}u_2^{n+\frac{1}{3}}(u_{j,k+1}^{n+\frac{2}{3}} - u_{j,k-1}^{n+\frac{2}{3}})\end{aligned}$$

2. Numerical result.

2.1. Example 1: Lid Driven Cavity. The figure below are simulations I ran with method II and $h = 1/60, k = 2.5 * h^2$ and t stop at $t_{end} = 20,000dt$ so that we could get the stationary solution. For the boundary, I have $u = 1, v = 0$ on the top boundary and no-slip condition on the other sides.

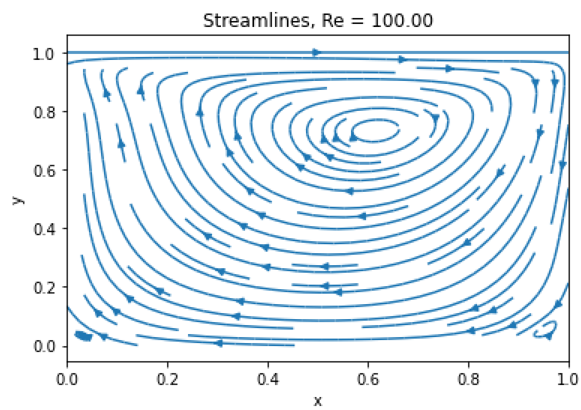

FIG. 2.1. *Re=100*

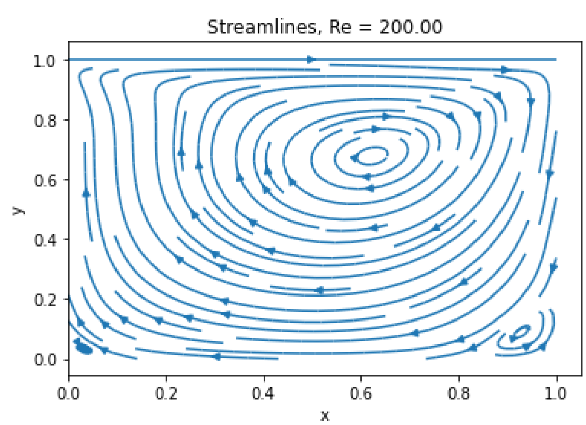

FIG. 2.2. *Re=200*

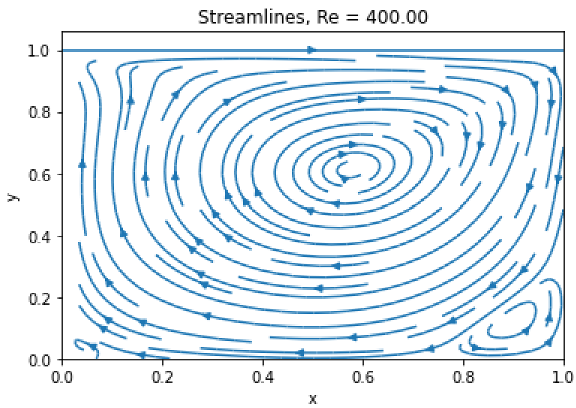

FIG. 2.3. *Re=400*

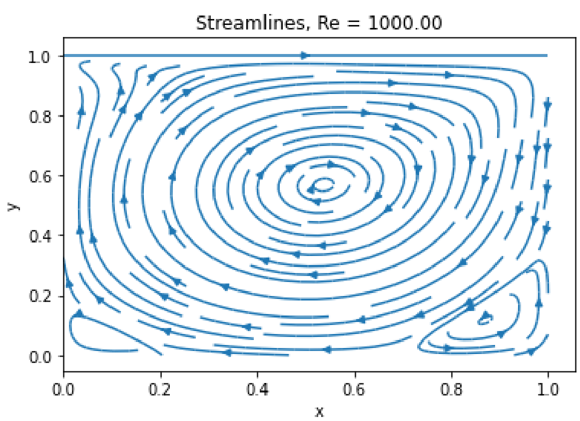

FIG. 2.4. *Re=1000*

The result is shown below

Re	Scheme	h	k	ψ	t_{end}
10	I	1/40	2.5 h^2	0.0704	500dt
10	II	1/40	2.5 h^2	0.0614	500dt
10	III	1/40	2.5 h^2	0.0703	500dt
100	I	1/40	2.5 h^2	0.0708	500dt
100	II	1/40	2.5 h^2	0.0571	500dt
100	III	1/40	2.5 h^2	0.0697	500dt

2.2. Example 2: Backward facing step. The figure below are simulations I ran with method II and $h = 1/20, k = 0.001$ and t stop at $t_{end} = 1$. The domain is a 10×1 rectangle. The step is a 4×0.4 rectangle.

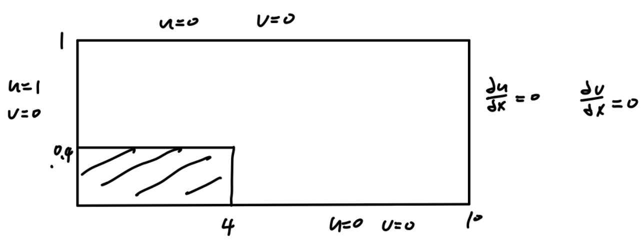

Fig. 2.5. *Setup of the problem*

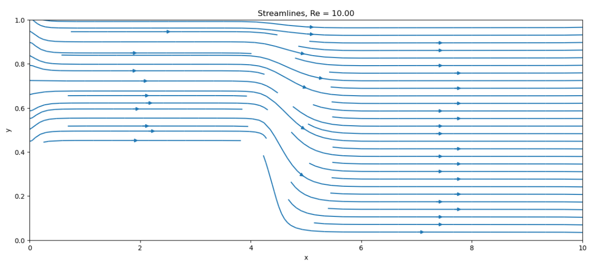

Fig. 2.6. *Re = 10*

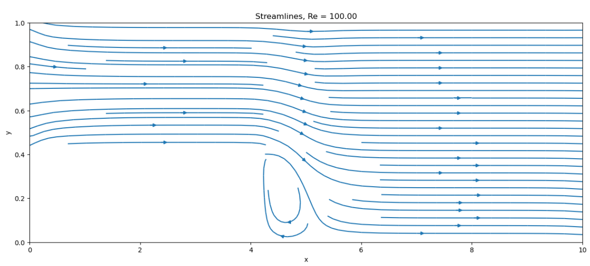

Fig. 2.7. *Re = 100*

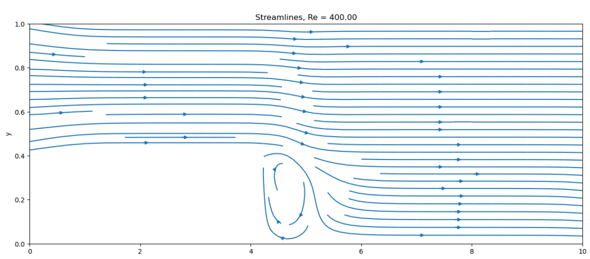

Fig. 2.8. *Re = 400*

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