The Standard Model

Let Ω be the set of what can happen in the world, T be the allowed trading times, $(\mathcal{A}_t)_{t\in T}$ a collection of increasingly fine partitions of Ω representing the information available at each time, and P a positive measure with mass 1 on the algebra generated by the partitions.

The *standard model* specifies *prices* $X_t\colon\mathcal{A}_t \to \mathbb{R}^I$, and *cash flows* $C_t\colon\mathcal{A}_t \to \mathbb{R}^I$, where I are the available market instruments. Instrument prices are assumed to be perfectly liquid: they can be bought and sold at the same price in any amount. Cash flows are associated with owning an instrument: stocks have dividends, bonds have coupons, futures have margin adjustments.

A *trading strategy* is a finite collection of strictly increasing stopping times, τ_j, and trades, $\Gamma_j\colon\mathcal{A}_{\tau_j} \to \mathbb{R}^I$ indicating the number of shares to trade in each instrument. Trades accumulate to a *position*, $\Delta_t = \sum_{\tau_j<t}\Gamma_j = \sum_{s<t}\Gamma_s$ where $\Gamma_s = \Gamma_j$ when $s = \tau_j$.

The *value* of a position at time t is $V_t = (\Delta_t + \Gamma_t)\cdot X_t$: also called *marked-to-market*, is how much you would get from liquidating your position and the trades just executed assuming you could do that. The *amount* generated by the trading strategy at time t is $A_t = \Delta_t\cdot C_t - \Gamma_t\cdot X_t$: you receive the cash flows associated with your existing position and pay for the trades you just executed.

A process $M_t\colon\mathcal{A}_t \to \mathbb{R}^I$ is a martingale if $M_tP = M_uP|_{\mathcal{A}_t}$. This is defined for $A\in\mathcal{A}_t$ by $M_tP(A) = \sum_{B\subset A} M_u(B)P(B)$ where $B\in\mathcal{A}_u$. If P is understood we write this as $M_t = M_u|_{\mathcal{A}_t}$. The usual notation is $M_t = E[M_u|\mathcal{A}_t]$.

A model has *strict arbitrage* if there is no trading strategy with $\sum_j\Gamma_j = 0$, $A_{\tau_0} > 0$, and $A_t \geq 0$ for $t > \tau_0$. The Fundamental Theorem of Asset Pricing states there is not strict arbitrage if and only if there exists a positive adapted process, $D_t\colon\mathcal{A}_t \to (0,\infty)$, with

$$X_tD_t = (X_uD_u + \sum_{t<s\leq u} C_sD_s)|_{\mathcal{A}_t} \tag{1}$$

Note that if $C_t = 0$ for all $t\in T$ this says X_tD_t is a martingale.

A simple corollary using the definition of value and amount shows

$$V_tD_t = (V_uD_u + \sum_{t<s\leq u} A_sD_s)\,|_{\mathcal{A}_t} \tag{2}$$

For strategy with $\sum_j\Gamma_j = 0$, and $A_t \geq 0$ for $t > \tau_0$, $V_{\tau_0}D_{\tau_0} = (\sum_{t>\tau_0} A_t\, D_t)|_{\mathcal{A}_{\tau_0}} \geq 0$. Since $V_0 = \Gamma_0\cdot X_0$, $A_0 = -\Gamma_0\cdot X_0$, and $D_0 > 0$ we have $A_0 \leq 0$, where the 0 subscript denote time τ_0.

Every model of the form $X_tD_t = M_t - \sum_{s\leq t} C_sD_s$ where $M_t\colon\mathcal{A}_t \to \mathbb{R}^I$ is a martingale and $D_t\colon\mathcal{A}_t \to (0,\infty)$ is a positive adapted process is arbitrage-free. This is immediate by substituting X_uD_u in equation (1).

Derivative Securities

Equation (2) is the skeleton key to understanding derivative securities. A *derivative* is a contract between a buyer and a seller for exchanges of instruments over time. Assuming the payments can be cash settled, a contract specifies a sequence of times, τ_j, and cash amounts, A_j, to be given to the buyer by the seller.

Define the stopping time $\nu(\omega) = \inf\{u > t : A_u(\omega) \neq 0\}$ then $V_t D_t = (A_\nu + V_\nu) D_\nu \,|_{\mathcal{A}_\nu}$