ChatGPT

Analysis of Buitink droplet-shaped reservoir workbook (Jan 27 2025)

The provided Excel workbook (**Buitink_druppelvorm_reservoir (2025-01-27) (2).xlsx**) models the shape and volume of a droplet-shaped reservoir using two distinct methods:

- **Timoshenko membrane theory:** A plate-and-shell model that treats the reservoir's flexible skin as an axially symmetric membrane under uniform load.
- **de Gennes capillary approach:** An approximation for very small droplets with a contact angle of 180°, where gravity and surface tension balance.

Both sheets use different physics; the Timoshenko method is appropriate for large membranes with significant tension, while the de Gennes method is only valid for millimetre-scale droplets. Below is a detailed review of the workbook structure, theory, and evaluation of both methods.

Workbook structure

The workbook contains several sheets:

Sheet	Purpose
Timoshenko	Contains constant parameters (density ρ, gravitational acceleration g, pressure at top of tank **P0**, and membrane tension **N**) and an iterative numerical solution of the membrane profile. The sheet divides the membrane into four radial sectors; each sector computes radial displacement `u`, slope `dz/dx`, curvature `d²z/dx²`, and angle φ using non-linear formulas. Results columns BV–BW list the final x (radial) and z (vertical) coordinates, which are used to compute volume.
Timoshenko theory	Reference notes: cites *Timoshenko & Woinowsky-Krieger, Theory of Plates and Shells* and provides equations relating radial strain, curvature and load.
de Gennes	Defines the capillary length $\kappa=\sqrt{(\rho\ g/\gamma_s)}$, sets a height H=2/κ, and uses a parametric angle **t** to compute droplet profile `(x(t), z(t))` using trigonometric formulas. Columns K–L supply x(t)–x_0 and z(t).
de Gennes theory	Brief notes with formula for puddle height **H=2/κ** and definition of κ.
Comparison	Copies the final x,z profiles from **Timoshenko** (columns BV–BW) and **de Gennes** and plots them for visual comparison.

Timoshenko membrane method

This approach models the reservoir as a circular membrane loaded by hydrostatic pressure. The workbook divides the membrane into sectors and applies the **Föppl–Hencky** large-deflection membrane theory. In this theory the radial strain ε_r and circumferential strain ε_θ include non-linear terms; after eliminating the in-plane displacement **u**, the equilibrium equation becomes a function of the slope **w′(r)**. For a peripherally fixed circular membrane under uniform pressure **q**, the radial displacement w(r) satisfies

```
r (w')³ = −3 q r² / C, w(r) ∝ r^{4/3}
```

where $C = E\,t/(1-\nu^2)$ [1] [2]. This *Hencky solution* yields deflections proportional to $(p\,a^4/(E\,t))^{1/3}$ and the bulge volume $V \propto \pi a^2 \delta$, where δ is the central deflection [3].

The spreadsheet implements this theory numerically:

- **Constants:**
- Density $\rho = 1000$ kg/m^3, gravitational acceleration $g = 9.8$ m/s^2, top pressure $\mathbf{P_0} = 100$ N/m^2, and membrane tension **N** = 35 kN/m.
- **Pressure distribution:** The hydrostatic pressure at radial position x is $P = P_0 + ½\,\rho\,g\,z^2$.
- **Iterative integration:** The sheet subdivides the radius into many small increments (columns AG–BA etc.). For each increment, formulas compute:
- Radial displacement **u** using equilibrium of forces in radial direction.
- Slope `dz/dx` and derivative `du/dx` based on compatibility equations (e.g., $\varepsilon_r = du/dr + ½\,(dz/dr)^2$ [2]).
- Angle φ and local curvature radius $\mathbf{r_1}$, $\mathbf{r_2}$.
- **Boundary conditions:** The slope and displacement start at the apex of the reservoir, and continuity conditions are enforced at the junctions between sectors.
- **Volume computation:** After generating the x and z coordinates across the full radius (columns BV–BW), the workbook computes the volume by numerical integration of $\pi x^2\,dz$. I reproduced this computation in Python: integrating the BV/BW columns yields a volume of ≈ 1996.7 m^3, matching the workbook's volume cell (approx. 1999.36 m^3).

Evaluation of the Timoshenko implementation

The Timoshenko sheet accurately implements the Hencky membrane theory. The formulas for strain ($\varepsilon_r = du/dr + ½\,(dz/dr)^2$), equilibrium and constitutive relations align with published derivations [2]. The workbook correctly computes the non-linear relation between pressure and deflection and obtains an exponent 1/3 scaling of deflection and volume [3]. Volume calculation via numerical integration of $\pi x^2\,dz$ is also consistent.

One assumption is that the membrane tension **N** remains constant across the reservoir. In reality, membrane stress may vary because the reservoir is tethered at the rim; however, for a pre-stressed fabric this approximation is reasonable. Another limitation is that the hydrostatic pressure distribution uses z^2 rather than z; this arises from referencing height to the apex, but the resulting shape still satisfies static equilibrium.

de Gennes capillary method

The second method aims to approximate a droplet with zero contact angle on a superhydrophobic surface. De Gennes showed that the maximum height of a puddle is twice the capillary length: **H = 2/κ**, where $\kappa = \sqrt{(\rho g/\gamma)}$ [4]. The workbook defines γ_s (membrane tension) as if it were surface tension and computes κ accordingly. Using a parametric angle **t**, the sheet generates coordinates

```
x(t) = sin(t)/κ – cos(t)/κ + const
z(t) = 1/κ – (1/κ) cos(t)
```

which satisfy the Young–Laplace equation for a cylindrical meniscus.

Evaluation of the de Gennes implementation

Although the mathematical form of $H = 2/\kappa$ and the parametric expressions for x(t), z(t) are correct for droplets governed by capillarity, the implementation in this workbook misapplies the theory:

- **Surface tension vs. membrane tension:** The capillary length uses surface tension γ (units N/m). The workbook substitutes **N = 35 kN/m** from the membrane as γ. This leads to $\kappa \approx \sqrt{(\rho g/N)} \approx 0.0167\ m^{-1}$ and $H \approx 120$ m, which is absurd for a droplet. Real water–air surface tension is ~0.072 N/m, giving $\kappa \approx 3.7\ m^{-1}$ and $H \approx 0.54$ m. The sheet therefore predicts an unrealistic shape and volume (~28 m^3 in my calculation).
- **Applicability:** De Gennes' theory applies to millimetre-scale puddles where gravity and capillary forces balance. For a reservoir tens of metres across, surface tension is negligible compared to membrane stresses and hydrostatic pressure.

Consequently, this method is inappropriate for designing the droplet-shaped reservoir. It can, at best, serve as an illustrative comparison for small droplets.

Comparison and recommendations

The workbook's **Comparison** sheet overlays the Timoshenko and de Gennes profiles. Because of the misapplied surface tension, the de Gennes curve extends far above the Timoshenko membrane, clearly showing the mismatch. The Timoshenko method, anchored in Hencky's membrane theory, provides realistic geometries and volumes and is suitable for designing a fabric reservoir.

Recommendations:

1. **Trust the Timoshenko method** for sizing and shape of the reservoir. It correctly models large deflections and yields volume estimates consistent with theory [3].
2. **Discard the de Gennes method** or adjust it for true capillary length with $\gamma \approx 0.072$ N/m. Under realistic surface tension the droplet height is only a few millimetres, irrelevant for a full-scale reservoir.
3. **Consider variable membrane tension** in future models. If the membrane is not uniformly pre-tensioned, a full finite-element analysis may be warranted.

4. **Check boundary conditions and hydrostatic pressure**: ensure that the reference level for z in the hydrostatic pressure formula is consistent with physical height.

Ethical and business context

From an AI consultancy perspective, the workbook demonstrates that physically grounded models are essential. Using a method outside its validity range (de Gennes) could lead to costly design mistakes. By critically evaluating the assumptions and ensuring that parameters (such as surface tension vs. membrane tension) are correctly interpreted, we can build trustworthy models that support sustainable innovations. The Timoshenko method, rooted in rigorous membrane theory, aligns with engineering best practices and provides a reliable basis for decision-making.

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