EE4C06 Networking

Graph Theory

Week 1

- Adjacency matrix: A
  - walk成为一个 path 需要所有的顶点都不相同。它是 trail 需要所有的边不相同。一个图是连通的需要对任意两个顶点，有 path 连接它们。一个 cycle 是一个闭的 path，一个 tree 是不含 cycle 的连通图。
  - complementary Adjacency matrix: $A^c = u \cdot u^T - I - A$
  - subgraph Adjacency matrix, $A = \begin{bmatrix} A_s & B \\ C & A_{G\backslash S} \end{bmatrix}$, $C = B^T$
  - Number of k-hop walks between node i and j: $(A^k)_{ij}$
  - Total number of k-hop walks in G: $N_k = u^T A^k u = \Sigma_{i=1}^N \Sigma_{j=1}^N (A^k)_{ij}$
  - Total number of closed k-hop walks in G: $W_k = \Sigma_{j=1}^N (A^k)_{jj} = trace(A^k)$
- Incidence matrix: B, $u^T \cdot B = 0$
- Laplacian matrix: $Q = B \cdot B^T$
- degree of nodes: $d = A \cdot u$
  - Regular graph: all nodes have the same degree
  - degree & #link: $u^T d = u^T A u = \Sigma_{j=1}^N d_j = 2L$
    $\rightarrow 2 - \frac{2}{N} \leq E[D] = \frac{1}{N} \Sigma_{j=1}^N d_j = \frac{2L}{N} \leq N - 1$ (connected graph)
  - degree of Adjacency matrix: $d_j = (A^2)_{jj}$
  - At least two nodes in G have the same degree
  - The number of nodes with odd degree is even
- links of graph:
  - Tree: $L = N - 1$
  - Ring: $L = N$
  - Complete graphL $L = N(N-1)/2$
- Clustering coefficient: $C_G(V) = \frac{2y}{dv(dv-1)} \leq 1$, where y is the number of links between neighbors. If $d_v = 1$, $C_G(V) = 0$
  - It measures the local density around node v
  - The clustering coefficient of a graph G: $C_G = \frac{1}{N} \Sigma_{v=1}^N C_G(v)$
  - Another definition: $C_G = \frac{6 \times \#triangles}{N2 - W2} = \frac{trace(A^3)}{d^T d - 2L} = \frac{\Sigma_{j=1}^N (A^3)jj}{\Sigma_{i=1}^N di(di-1)}$
- Hopcount: Hopcount from node i to node j: $H_{i \rightarrow j} = h(P^*_{i \rightarrow j})$ where $P^*_{i \rightarrow j}$ is the shortest hop path from i to j
  - diameter ρ of G: hopcount of the longest shortest path in G. The average hopcount $E[H]$ reflects "efficiency" of transport in G.
  - The shortest walk between i and j is also a shortest path, $H_{ij} = k$
  - faster test of ρ: test till $(1 + A)^\rho$ contains no zero
- Betweenness: The betweenness B_l / B_n of a link l / node n equals the number of shortest paths traversing link l / node n in G
  - The average betweenness: $H_G = \Sigma_{i=1}^N \Sigma_{j=i+1}^N H_{ij} = \Sigma_{l=1}^L B_l$
    $\rightarrow E[B_l] = \frac{1}{L} \Sigma_{l=1}^L B_l = \frac{1}{L} \begin{pmatrix} N \\ 2 \end{pmatrix} E[H_G]$, where H is the distance matrix

- Degree Assortativity: $\rho_D = \frac{N1N3 - N_2^2}{N1\Sigma_{j=1}^{N} d_j^3 - N_2^2}$
  - A network is (degree) assortative if $\rho_D > 0$
  - A network is (degree) disassortative if $\rho_D < 0$
  - Degree-preserving rewiring (DPR) only changes Degree Assortativity rather than the degree itself.
- Connectivity of a Graph: $\lambda(G)$ (or $k(G)$): the minimum number of links (or nodes) whose removal disconnects G
  - Menger's Theorem: The maximum number of Link(node)-disjoint paths between A and B is equal to the minimum number of links(nodes) disconnecting A and B
  - If the graph G is disconnected, then its complement G^c is connected

Week 2

- Spectrum of A:
  - $d_{max} \geq \lambda_1 \geq \frac{2L}{N} = E[D]$
    - $\lambda_1 \geq E[D]\sqrt{1 + \frac{Var[D]}{E^2[D]}}$
  - all eigenvalues lie in the interval $(-d_{max}, d_{max}]$
  - $\Sigma_{j=1}^{N}\lambda_k^k = trace(A^k) = \Sigma_{i=1}^{N}(A^k)_{ii} = W_k$ (total number of closed walks)
    - $\Sigma_{j=1}^{N}\lambda_j = 0$
    - $\Sigma_{j=1}^{N}\lambda_j^2 = 2L$
    - $\Sigma_{j=1}^{N}\lambda_j^3 = 6 \times triangles$
  - λ_1 and components of eigenvector x_1 are non-negative (when reducible, $\equiv 0$)
  - The radius is bounded: $E[D]\sqrt{1 + \frac{Var[D]}{E^2[D]}} \leq \lambda_1 \leq d_{\max}$
- Spectrum of Q:
  - any eigenvalue μ_k is non-negative and smallest $\mu_N = 0$
  - complexity (number of spanning tree) is $\xi(G) = \frac{1}{N}\prod_{k=1}^{N-1}\mu_k$
  - algebraic connectivity $a(G) = \mu_{N-1}$. The graph G is only connected if and only if $a(G) = \mu_{N-1} > 0$. The graph G with larger $a(G)$ is more difficult to disconnect
- The number of links between G_1 and G_2: $R = \frac{1}{4}y^T Qy$, $y_i = 1$ if $i \in G_1$, else $y_i = -1$ if $i \in G_2 \backslash$
  - $R = \frac{1}{4}\Sigma_{j=1}^{N}\alpha_k^2 u_j$
  - $R \geq \frac{1}{4}(y^T z_{N-1})^2 \mu_{N-1}$
- Degree-preserving rewiring:
  - Largest eigenvalue of adjacency increases with degree assortativity ρ_D
  - while algebraic connectivity decreases which implies that increasing assortativity creates more disconnected components.

- Erdos-Renyi random graph
  - a_{ij} is a bernoulli random variable with mean ρ
  - $E[a_{ij}] = \rho$
  - the complement graph of $G_p(N)$ is $G_{1-p}(N)$
  - the average number of links: $E[L] = \frac{N(N-1)}{2}\rho$
  - the average cluster coefficient is $E[C_{G_p(N)}] = \rho$
  - $Pr[D = k] = \binom{N-1}{k} p^k (1-p)^{(N-1-k)} \approx \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}$

- $\mu = (N-1)\rho$ and $\sigma^2 = (N-1)\rho(1-\rho)$
- ρ constant, $N \to \infty$ => tendency towards a regular graph
- $E[D]$ constant, $N \to \infty$ => $Pr[D=k]$ becomes Poisson distribution
- the critical link density: $\rho_c \sim logN/N$
- ρ_c small, compact $f_\lambda(x)$, high spike at zero, graph tends to be disconnected; ρ_c large, disperse $f_\lambda(x)$, lower spike at zero, graph tends to be connected.

  - rewiring makes clustering coefficient and average hotcount lower
  - $f_\lambda(-x) = f_\lambda(x)$ refers a tree graph

- Power-law graph (scale-free)
  - $Pr[D=k] = ck^{-\tau}$
  - The mean is not representative, because the variance is (very) large
  - robustness to random node failure
  - vulnerability to targeted hub attacks and cascading failures

Electrical Networks

Week 3

- Kirchhoff's Current Law: $By = x$, if no current injections, then $By = 0$
- Ohm's Law: $diag(\frac{1}{rl})B^Tv = y$, if all resistance is 1 ω, then $B^Tv = y$
- Deductions:

$$B(B^Tv) = x \to BB^Tv = x \to Qv = x \text{ (unit resistance)}$$

$$B(diag(\frac{1}{r_l})B^Tv) = x \to \widetilde{Q}v = x \text{ (non-unit resistance)}$$

- Heterogeneous Resistance:

$$x = \widetilde{Q}v$$

  - $\widetilde{Q} = Bdiag(\frac{1}{rl})B^T$
  - $\widetilde{Q} = \widetilde{\Delta} - \widetilde{A}$ ($\widetilde{a}_{ij} = \frac{1}{rij}a_{ij}$)

- Pseudo inverse of the Laplacian: $Q^+ = \Sigma_{k=1}^{N-1}\frac{1}{\mu k}z_kz_k^T$
  - $Q^+x = v - v_{avg}u$

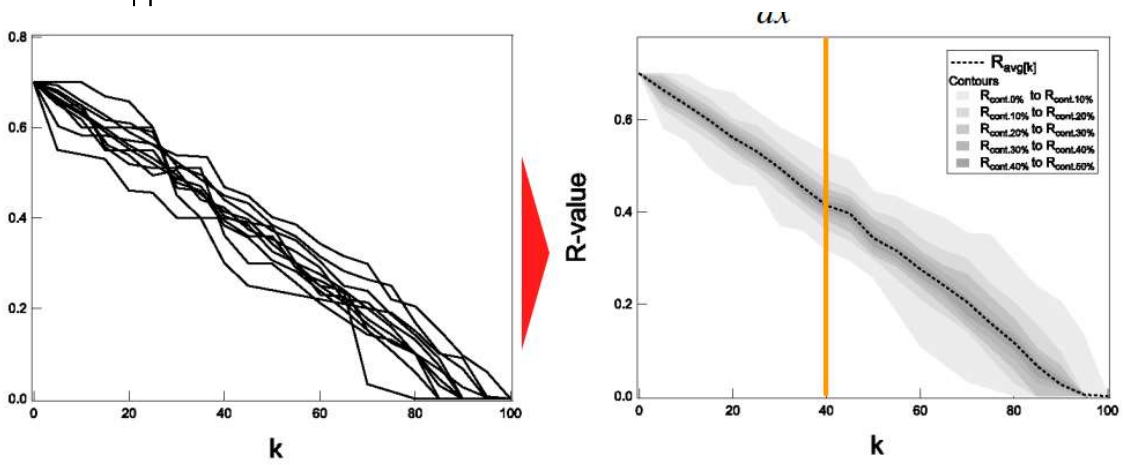
- effective resistance matrix:
  - $\omega = u\xi^T + \xi u^t - 2Q^+$
    - $\xi = diagonal(Q^+)$
  - $R_G = \frac{1}{2}u^T\omega u = N \times trace(Q^+) = N\Sigma_{k=1}^{N-1}\frac{1}{uk}$

Robustness of Networks

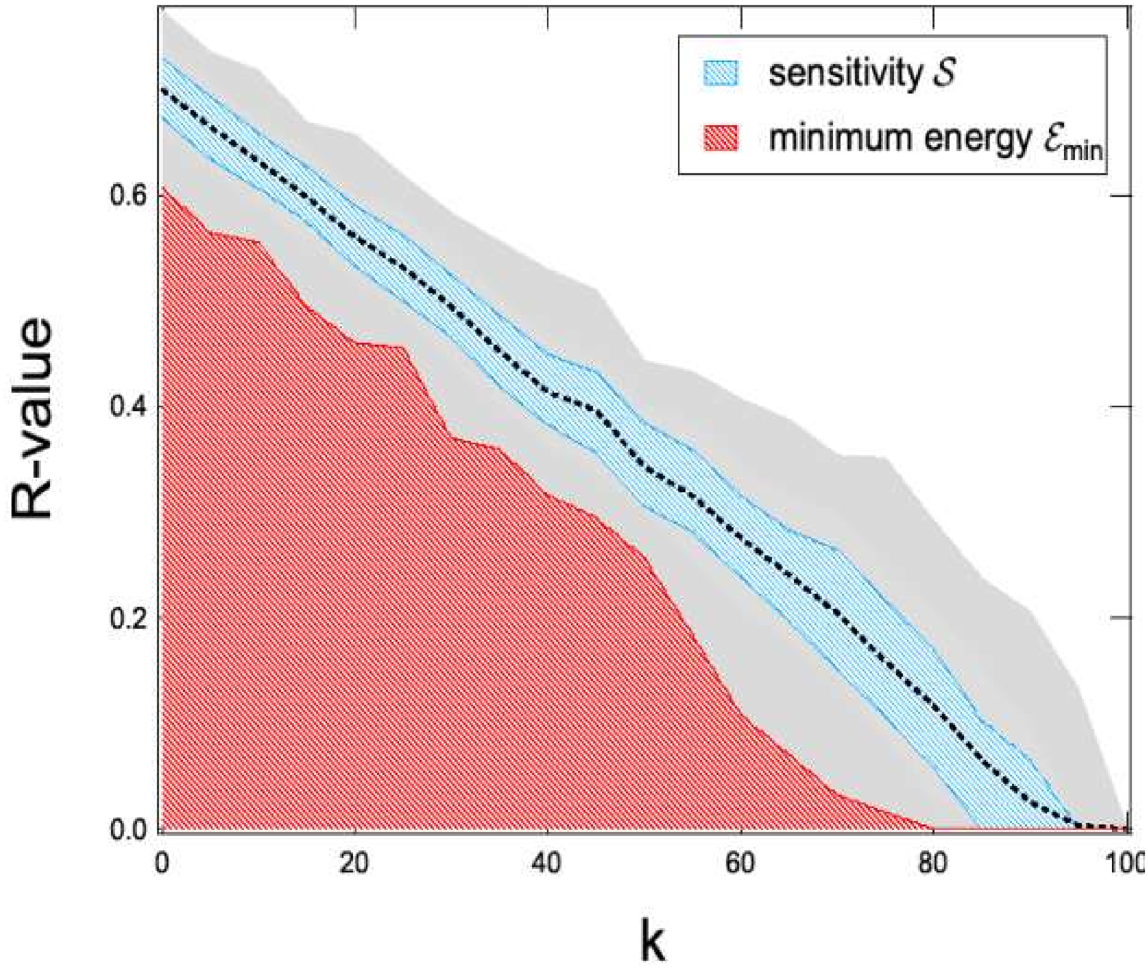
Week 4

- R-model: $R = \Sigma_{k=1}^{m} s_k t_k = s^T t$, $0 \leq R \leq 1$
  - s: the service vector with m components (interpreted as weights)
  - t: the topology vector where each of the m components is a graph metric
  - R model is linear
  - Normalize s and t, $R = \frac{s^T s}{\sqrt{(s^T s)(t^T t)}}$
  - R = 0 (absence of network robustness); R = 1 (perfect robustness)

- Robustness Envelopes
  - Stochastic approach:



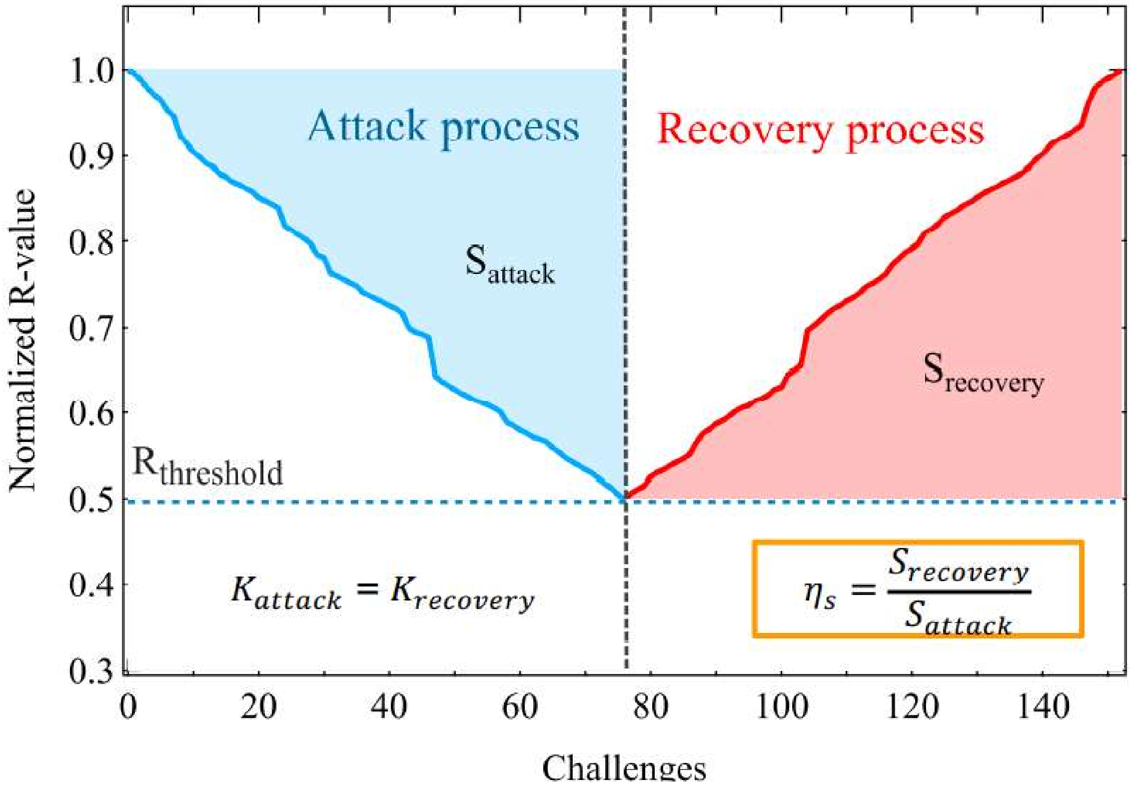
  - Envelope definitions:



    - Low sensitivity (blue area) results in better stability
    - High energy (red area) results in better average R-value

- Different attacks (different strategies to remove links) influence differently:
  - targeted attacks: coreness, betweenness, closeness, degree, elgenvector
  - betweenness, pseudo-inverse laplacian, closeness often impact most

- Failure recovery:
  - Scenario A: adding links uniformly at random in the complementary graph after attack until the normalized R-value reaches 1: $\eta = \frac{K_{attack}}{K_{recovery}}$
  - Scenario B: adding links which are removed in the attack process until the network returns to the original: $\eta = \frac{S_{recovery}}{S_{attack}}$

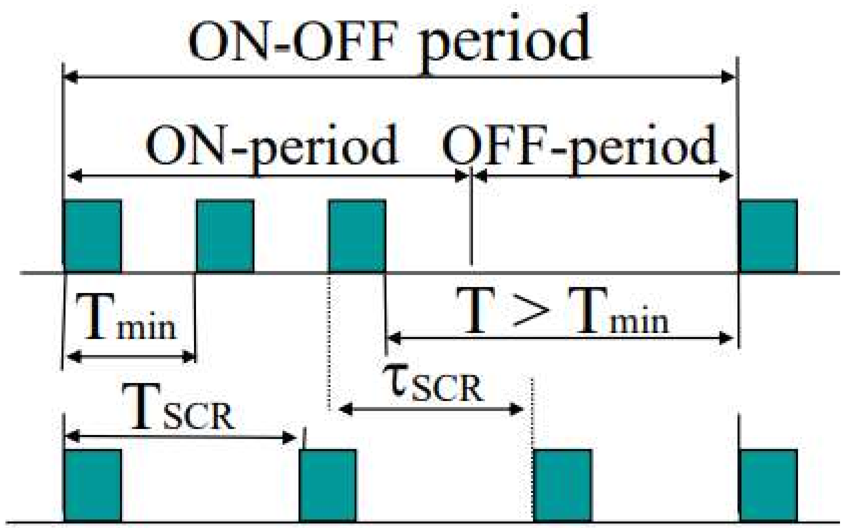


Traffic Management

Week 4

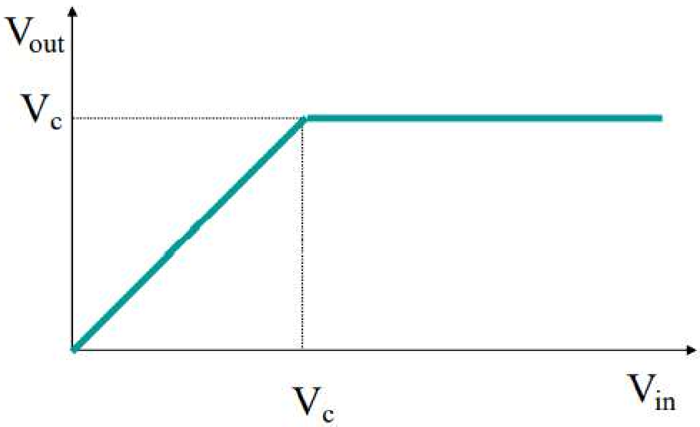
- Statistical Multiplexing: multi sources use the same bandwidth simultaneously to reduce the PCR (Peak Cell Rate)

ON-OFF source



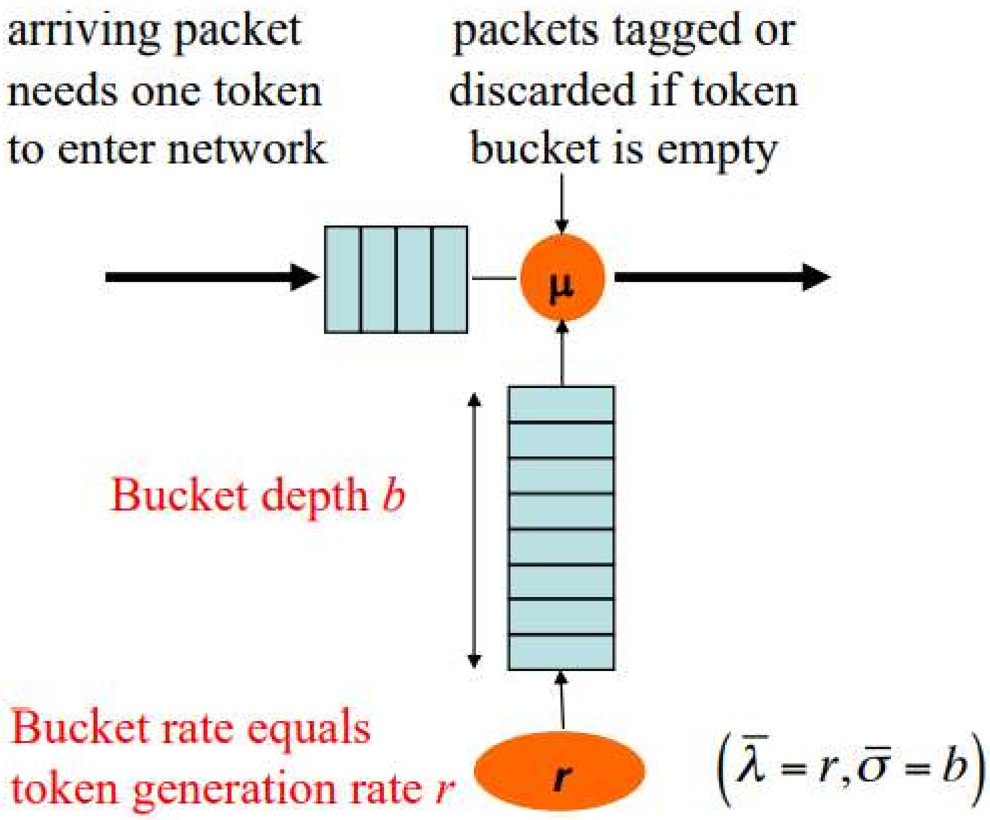
- Traffic descriptors:
  - PCR = $1/T_{min}$
  - SCR = $1/T_{SCR}$
  - burst tolerance τ_{scr}

- On-Off source:
- Burstiness Constraint: $L(u,t) = \int_u^t \lambda(\tau)d(\tau) \leq \overline{\sigma} + \overline{\lambda}(t-u)$
  - $L(u,t) \leq max_{\tau\in[u,t]}\lambda(\tau)(t-u)$
  - $\lim_{t\to\infty}\frac{L(u,t)}{t-u} = \overline{\lambda} + \lim_{t\to\infty}\frac{\overline{\sigma}}{t-u}$
- Input Control:
  - This is the result of input control:



V_{in} : Received cell rate at UPC (user parameter control) input
V_{out} : Admitted cell rate at UPC output
V_c : Contracted PCR negotiated in the traffic contract

- This is the method of input control:



- QoS (quality of service)
  - Loss
  - Delay
- Connection Admission Control (CAC): Acceptance rules for new connection requests in order to guarantee the quality of service (QoS) for multimedia services in B-ISDN
- Congestion: System buffers fill up $\rightarrow$ Loss and retransmission $\rightarrow$ More traffic, more loss $\rightarrow$ **Positive feedback** $\rightarrow$ System collapse

Scheduling

Week 6

- Head of the Line (HoL) vs. Partial Buffer Sharing (PBS) vs. Push-Out Buffer (POB)

  - HoL: always serves high priorities in the buffer before low priorities
  - PBS: below the threshold T: identical to FIFO and sequence order is preserved; above threshold T: only a high priority accepted until buffer is full
  - POB: when an arrival of high priority cell,
    - LIFO POB: the last entered low priority cell is discard
    - Random POB: a randomly chosen low priority cell is pushed out
    - FIFO POB: the first entered low priority cell is discard

- Multiplexing of regulated flows:

$$Q_t = \sup_{s<t}[\sum_{k=l}^{K} L_k(s,t) - \mu(t-s)] \leq \sup_{s<t>}[\sum_{k=l}^{K} L_k(s,t) - \lambda_k(t-s)] \leq \sum_{k=l}^{K} \sigma_k = G]$$

- N*D/D/1 queue:

$$Pr(N_s > G) = clr$$

$$Pr[N_s > x] = \exp(-\frac{2x^2}{N})\exp(-\frac{2x(1-\rho)}{\rho})$$

In the worst case scenario $\rho \to 1$, then $clr \approx \exp(-\frac{2x^2}{N})$

If the buffer size x = G, then the number Ns of flows is approximately,

$$N_s = \frac{2G^2}{-\ln(clr)}$$