

# EE4C06 Networking

## Graph Theory

### Week 1

- Adjacency matrix:  $A$ 
  - walk成为一个 path 需要所有的顶点都不相同。它是 trail 需要所有的边不相同。一个图是连通的需要对任意两个顶点, 有 path 连接它们。一个 cycle 是一个闭的 path, 一个 tree 是不含 cycle 的连通图。
  - complementary Adjacency matrix:  $A^c = u \cdot u^T - I - A$
  - subgraph Adjacency matrix,  $A = \begin{bmatrix} A_s & B \\ C & A_{G \setminus S} \end{bmatrix}, C = B^T$
  - Number of k-hop walks between node  $i$  and  $j$ :  $(A^k)_{ij}$
  - Total number of k-hop walks in G:  $N_k = u^T A^k u = \sum_{i=1}^N \sum_{j=1}^N (A^k)_{ij}$
  - Total number of closed k-hop walks in G:  $W_k = \sum_{j=1}^N (A^k)_{jj} = \text{trace}(A^k)$
- Incidence matrix:  $B, u^T \cdot B = 0$
- Laplacian matrix:  $Q = B \cdot B^T$
- degree of nodes:  $d = A \cdot u$ 
  - Regular graph: all nodes have the same degree
  - degree & #link:  $u^T d = u^T A u = \sum_{j=1}^N d_j = 2L$   
 $\rightarrow 2 - \frac{2}{N} \leq E[D] = \frac{1}{N} \sum_{j=1}^N d_j = \frac{2L}{N} \leq N - 1$  (connected graph)
  - degree & Adjacency matrix:  $d_j = (A^2)_{jj}$
  - At least two nodes in G have the same degree
  - The number of nodes with odd degree is even
- links of graph:
  - Tree:  $L = N - 1$
  - Ring:  $L = N$
  - Complete graph:  $L = N(N - 1)/2$
- Clustering coefficient:  $C_G(V) = \frac{2y}{d_v(d_v - 1)} \leq 1$ , where  $y$  is the number of links between neighbors. If  $d_v = 1$ ,  $C_G(V) = 0$ 
  - It measures the local density around node  $v$
  - The clustering coefficient of a graph G:  $C_B = \frac{1}{N} \sum_{v=1}^N C_G(v)$
  - Another definition:  $C_G = \frac{6 \times \# \text{triangles}}{N_2 - W_2} = \frac{\text{trace}(A^3)}{d^T d - L} = \frac{\sum_{j=1}^N (A^3)_{jj}}{\sum_{i=1}^N d_i(d_i - 1)}$
- Hopcount: Hopcount from node  $i$  to node  $j$ :  $H_{i \rightarrow j} = h(P_{i \rightarrow j}^*)$  where  $P_{i \rightarrow j}^*$  is the shortest hop path from  $i$  to  $j$ 
  - diameter  $\rho$  of  $G$ : hopcount of the longest shortest path in  $G$ . The average hopcount  $E[H]$  reflects "efficiency" of transport in  $G$ .
  - The shortest walk between  $i$  and  $j$  is also a shortest path,  $H_{ij} = k$
  - faster test of  $\rho$ : test till  $(1 + A)^\rho$  contains no zero
- Betweenness: The betweenness  $B_l / B_n$  of a link  $l$  / node  $n$  equals the number of shortest paths traversing link  $l$  / node  $n$  in  $G$ 
  - The average betweenness:  $H_G = \sum_{i=1}^N \sum_{j=i+1}^N H_{ij} = \sum_{l=1}^L B_l$   
 $\rightarrow E[B_l] = \frac{1}{L} \sum_{l=1}^L B_l = \frac{1}{L} \binom{N}{2} E[H_G]$ , where  $H$  is the distance matrix

- Degree Assortativity:  $\rho_D = \frac{N_1 N_3 - N_2^2}{N_1 \sum_{j=1}^N d_j^3 - N_2^2}$ 
  - A network is (degree) assortative if  $\rho_D > 0$
  - A network is (degree) disassortative if  $\rho_D < 0$
  - Degree-preserving rewiring (DPR) only changes Degree Assortativity rather than the degree itself.
- Connectivity of a Graph:  $\lambda(G)$  (or  $k(G)$ ): the minimum number of links (or nodes) whose removal disconnects  $G$ 
  - Menger's Theorem: The maximum number of Link(node)-disjoint paths between A and B is equal to the minimum number of links(nodes) disconnecting A and B
  - If the graph  $G$  is disconnected, then its complement  $G^c$  is connected

## Week 2

- Spectrum of A:
  - $d_{max} \geq \lambda_1 \geq \frac{2L}{N} = E[D]$ 
    - $\lambda_1 \geq E[D] \sqrt{1 + \frac{Var[D]}{E^2[D]}}$
  - all eigenvalues lie in the interval  $(-d_{max}, d_{max}]$
  - $\sum_{j=1}^N \lambda_j^k = trace(A^k) = \sum_{i=1}^N (A^k)_{ii} = W_k$  (total number of closed walks)
    - $\sum_{j=1}^N \lambda_j = 0$
    - $\sum_{j=1}^N \lambda_j^2 = 2L$
    - $\sum_{j=1}^N \lambda_j^3 = 6 \times triangles$
  - $\lambda_1$  and components of eigenvector  $x_1$  are non-negative (when disconnected,  $\equiv 0$ )
- Spectrum of Q:
  - any eigenvalue  $\mu_k$  is non-negative and smallest  $\mu_N = 0$
  - complexity (number of spanning tree) is  $\xi(G) = \frac{1}{N} \prod_{k=1}^{N-1} \mu_k$
  - algebraic connectivity  $a(G) = \mu_{N-1}$ . The graph  $G$  is only connected if and only if  $a(G) = \mu_{N-1} > 0$ . The graph  $G$  with larger  $a(G)$  is more difficult to disconnect
- The number of links between  $G_1$  and  $G_2$ :  $R = \frac{1}{4} y^T Q y$ ,  $y_i = 1$  if  $i \in G_1$ , else  $y_i = -1$  if  $i \in G_2 \setminus$ 
  - $R = \frac{1}{4} \sum_{j=1}^N \alpha_k^2 u_j$
  - $R \geq \frac{1}{4} (y^T z_{N-1})^2 \mu_{N-1}$
- Degree-preserving rewiring:
  - Largest eigenvalue of adjacency increases with degree assortativity  $\rho_D$
  - while algebraic connectivity decreases which implies that increasing assortativity creates more disconnected components.

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- Erdos-Renyi random graph
    - $a_{ij}$  is a bernoulli random variable with mean  $\rho$
    - $E[a_{ij}] = \rho$
    - the complement graph of  $G_p(N)$  is  $G_{1-p}(N)$
    - the average number of links:  $E[L] = \frac{N(N-1)}{2} \rho$
    - the average cluster coefficient is  $E[C_{G_p(N)}] = \rho$
    - $Pr[D = k] = \binom{N-1}{k} p^k (1-p)^{(N-1-k)} \approx \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}$ 
      - $\mu = (N-1)\rho$  and  $\sigma^2 = (N-1)\rho(1-\rho)$

- $\rho$  constant,  $N \rightarrow \infty \Rightarrow$  tendency towards a regular graph
- $E[D]$  constant,  $N \rightarrow \infty \Rightarrow Pr[D = k]$  becomes Poisson distribution
- the critical link density:  $\rho_c \sim \log N / N$
- $\rho_c$  small, compact  $f_\lambda(x)$ , high spike at zero, graph tends to be disconnected;  $\rho_c$  large, disperse  $f_\lambda(x)$ , lower spike at zero, graph tends to be connected.
- rewiring makes clustering coefficient and average hotcount lower
- $f_\lambda(-x) = f_\lambda(x)$  refers a tree graph
- Power-law graph (scale-free)
  - $Pr[D = k] = ck^{-\tau}$
  - The mean is not representative, because the variance is (very) large
  - robustness to random node failure
  - vulnerability to targeted hub attacks and cascading failures

## Electrical Networks

### Week 3

- Kirchhoff's Current Law:  $By = x$ , if no current injections, then  $By = 0$
- Ohm's Law:  $diag(\frac{1}{r_l})B^T v = y$ , if all resistance is 1  $\omega$ , then  $B^T v = y$
- Deductions:

$$B(B^T v) = x \rightarrow BB^T v = x \rightarrow Qv = x \text{ (unit resistance)}$$

$$B(diag(\frac{1}{r_l})B^T v) = x \rightarrow \tilde{Q}v = x \text{ (non-unit resistance)}$$

- Heterogeneous Resistance:

$$x = \tilde{Q}v$$

- $\tilde{Q} = Bdiag(\frac{1}{r_l})B^T$
- $\tilde{Q} = \tilde{\Delta} - \tilde{A}$  ( $\tilde{a}_{ij} = \frac{1}{r_{ij}}a_{ij}$ )
- Pseudo inverse of the Laplacian:  $Q^+ = \sum_{k=1}^{N-1} \frac{1}{\mu_k} z_k z_k^T$ 
  - $Q^+ x = v - v_{avg}u$
- effective resistance matrix:
  - $\omega = u\xi^T + \xi u^T - 2Q^+$ 
    - $\xi = diagonal(Q^+)$
  - $R_G = \frac{1}{2}u^T \omega u = N \times trace(Q^+) = N \sum_{k=1}^{N-1} \frac{1}{\mu_k}$

## Robustness of Networks

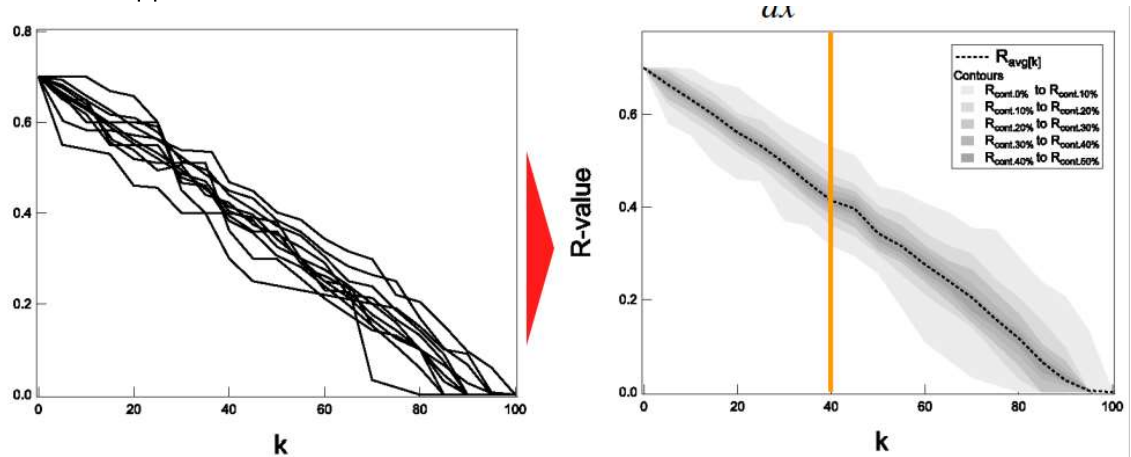
### Week 4

- R-model:  $R = \sum_{k=1}^m s_k t_k = s^T t, 0 \leq R \leq 1$

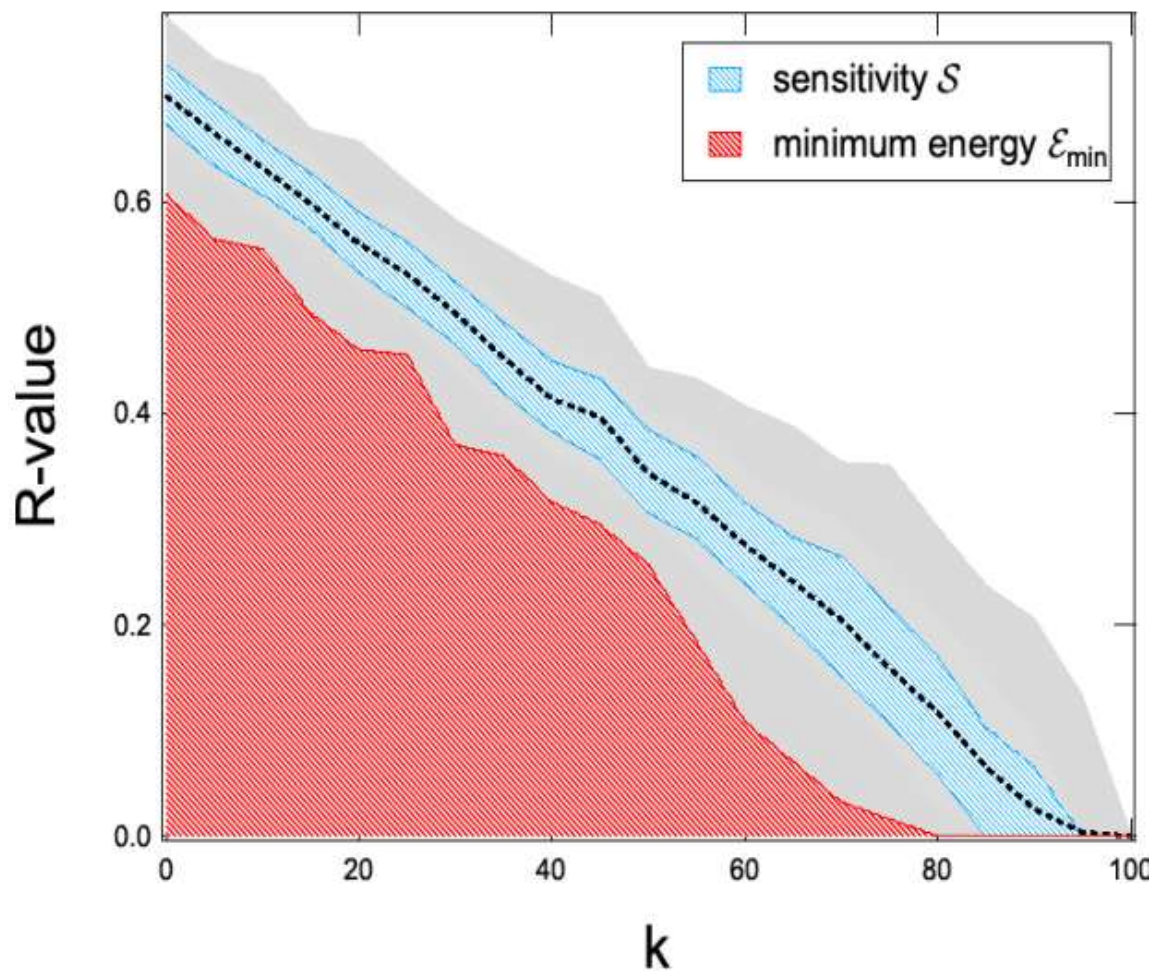
- $s$ : the service vector with  $m$  components (interpreted as weights)
- $t$ : the topology vector where each of the  $m$  components is a graph metric
- R model is linear
- Normalize  $s$  and  $t$ ,  $R = \frac{s^T t}{\sqrt{(s^T s)(t^T t)}}$
- $R = 0$  (absence of network robustness);  $R = 1$  (perfect robustness)

- Robustness Envelopes

- Stochastic approach:

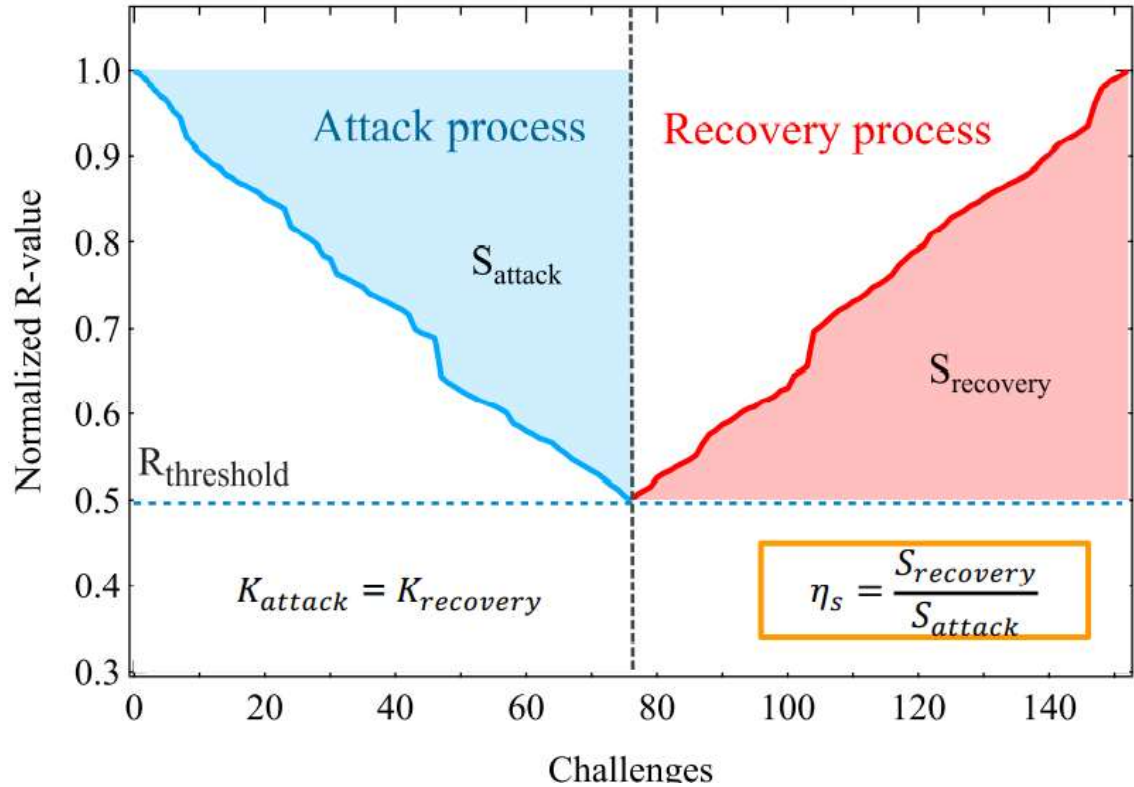


- Envelope definitions:



- Low sensitivity (blue area) results in better stability
- High energy (red area) results in better average R-value
- Different attacks (different strategies to remove links) influence differently:
  - targeted attacks: coreness, betweenness, closeness, degree, eigenvector

- betweenness, pseudo-inverse laplacian, closeness often impact most
- Failure recovery:
  - Scenario A: adding links uniformly at random in the complementary graph after attack until the normalized R-value reaches 1:  $\eta = \frac{K_{attack}}{K_{recovery}}$
  - Scenario B: adding links which are removed in the attack process until the network returns to the original:  $\eta = \frac{S_{recovery}}{S_{attack}}$

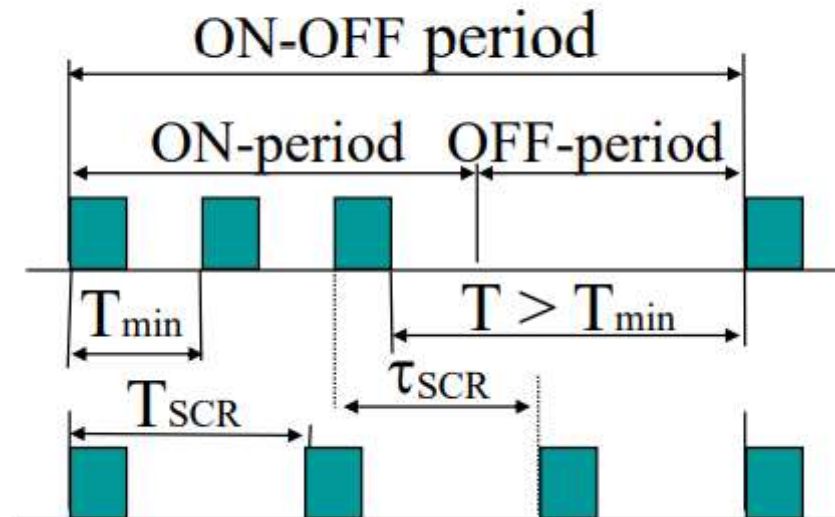


## Traffic Management

### Week 4

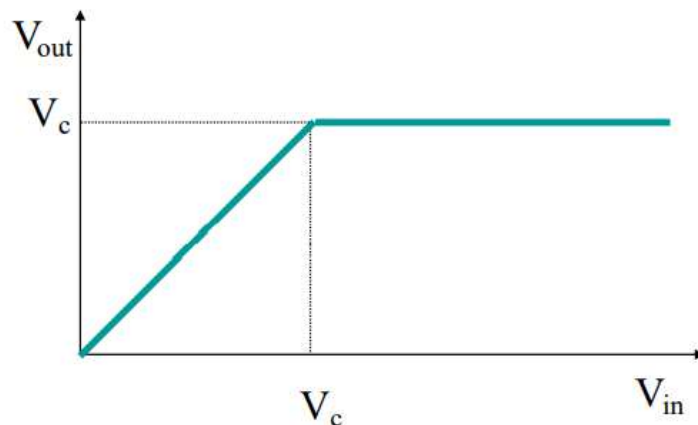
- Statistical Multiplexing: multi sources use the same bandwidth simultaneously to reduce the PCR (Peak Cell Rate)

# ON-OFF source



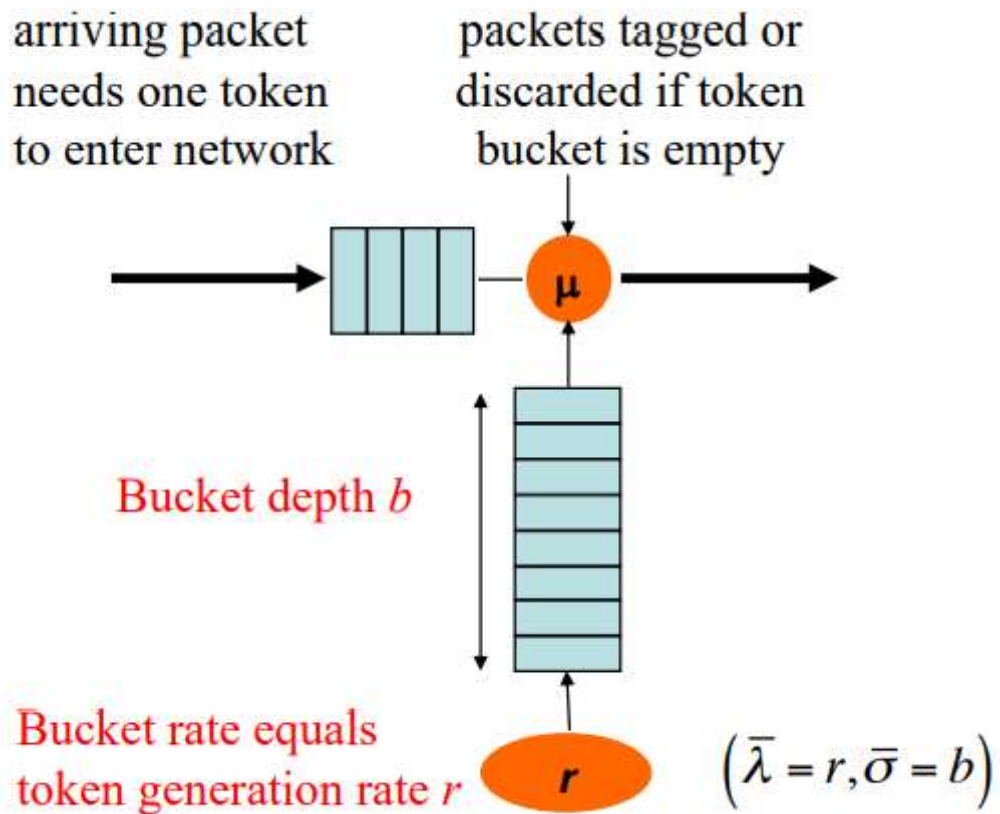
- Traffic descriptors:
  - PCR =  $1/T_{\min}$
  - SCR =  $1/T_{\text{SCR}}$
  - burst tolerance  $\tau_{\text{scr}}$

- On-Off source:
- Burstiness Constraint:  $L(u, t) = \int_u^t \lambda(\tau) d(\tau) \leq \bar{\sigma} + \bar{\lambda}(t - u)$ 
  - $L(u, t) \leq \max_{\tau \in [u, t]} \lambda(\tau)(t - u)$
  - $\lim_{t \rightarrow \infty} \frac{L(u, t)}{t - u} = \bar{\lambda} + \lim_{t \rightarrow \infty} \frac{\bar{\sigma}}{t - u}$
- Input Control:
  - This is the result of input control:



- $V_{\text{in}}$ : Received cell rate at UPC (user parameter control) input
- $V_{\text{out}}$ : Admitted cell rate at UPC output
- $V_c$ : Contracted PCR negotiated in the traffic contract

- This is the method of input control:



- QoS (quality of service)
  - Loss
  - Delay
- Connection Admission Control (CAC): Acceptance rules for new connection requests in order to guarantee the quality of service (QoS) for multimedia services in B-ISDN
- Congestion: System buffers fill up  $\rightarrow$  Loss and retransmission  $\rightarrow$  More traffic, more loss  $\rightarrow$  **Positive feedback**  $\rightarrow$  System collapse