EE4C06 Networking

Graph Theory

Week 1

- Adjacency matrix: A
  - walk成为一个 path 需要所有的顶点都不相同。它是 trail 需要所有的边不相同。一个图是连通的需要对任意两个顶点，有 path 连接它们。一个 cycle 是一个闭的 path，一个 tree 是不含 cycle 的连通图。
  - complementary Adjacency matrix: $A^c = u \cdot u^T - I - A$
  - subgraph Adjacency matrix, $A = \begin{bmatrix} A_s & B \\ C & A_{G\backslash S} \end{bmatrix}$, $C = B^T$
  - Number of k-hop walks between node i and j: $(A^k)_{ij}$
  - Total number of k-hop walks in G: $N_k = u^T A^k u = \Sigma_{i=1}^N \Sigma_{j=1}^N (A^k)_{ij}$
  - Total number of closed k-hop walks in G: $W_k = \Sigma_{j=1}^N (A^k)_{ij} = trace(A^k)$
- Incidence matrix: B, $u^T \cdot B = 0$
- Laplacian matrix: $Q = B \cdot B^T$
- degree of nodes: $d = A \cdot u$
  - Regular graph: all nodes have the same degree
  - degree & #link: $u^T d = u^T A u = \Sigma_{j=1}^N d_j = 2L$
    $\rightarrow 2 - \frac{2}{N} \le E[D] = \frac{1}{N}\Sigma_{j=1}^N d_j = \frac{2L}{N} \le N - 1$ (connected graph)
  - degree $ Adjacency matrix: $d_j = (A^2)_{jj}$
  - At least two nodes in G have the same degree
  - The number of nodes with odd degree is even
- links of graph:
  - Tree: $L = N - 1$
  - Ring: $L = N$
  - Complete graphL $L = N(N-1)/2$
- Clustering coefficient: $C_G(V) = \frac{2y}{dv(dv-1)} \le 1$, where y is the number of links between neighbors. If $d_v = 1$, $C_G(V) = 0$
  - It measures the local density around node v
  - The clustering coefficient of a graph G: $C_B = \frac{1}{N}\Sigma_{v=1}^N C_G(v)$
  - Another definition: $C_G = \frac{6 \times \#triangles}{N2 - W2} = \frac{trace(A^3)}{d^T d - L} = \frac{\Sigma_{j=1}^N (A^k)jj}{\Sigma_{i=1}^N di(di-1)}$
- Hopcount: Hopcount from node i to node j: $H_{i\rightarrow j} = h(P^*_{i\rightarrow j})$ where $P^*_{i\rightarrow j}$ is the shortest hop path from i to j
  - diameter ρ of G: hopcount of the longest shortest path in G. The average hopcount $E[H]$ reflects "efficiency" of transport in G.
  - The shortest walk between i and j is also a shortest path, $H_{ij} = k$
  - faster test of ρ: test till $(1 + A)^\rho$ contains no zero
- Betweenness: The betweenness B_l / B_n of a link l / node n equals the number of shortest paths traversing link l / node n in G
  - The average betweenness: $H_G = \Sigma_{i=1}^N \Sigma_{j=i+1}^N H_{ij} = \Sigma_{l=1}^L B_l$
    $\rightarrow E[B_l] = \frac{1}{L}\Sigma_{l=1}^L B_l = \frac{1}{L}\binom{N}{2} E[H_G]$, where H is the distance matrix

- Degree Assortativity: $\rho_D = \frac{N1 N3 - N_2^2}{N1 \Sigma_{j=1}^{N} d_j^3 - N_2^2}$
  - A network is (degree) assortative if $\rho_D > 0$
  - A network is (degree) disassortative if $\rho_D < 0$
  - Degree-preserving rewiring (DPR) only changes Degree Assortativity rather than the degree itself.
- Connectivity of a Graph: $\lambda(G)$ (or $k(G)$): the minimum number of links (or nodes) whose removal disconnects G
  - Menger's Theorem: The maximum number of Link(node)-disjoint paths between A and B is equal to the minimum number of links(nodes) disconnecting A and B
  - If the graph G is disconnected, then its complement G^c is connected

Week 2

- Spectrum of A:
  - $d_{max} \geq \lambda_1 \geq \frac{2L}{N} = E[D]$
    - $\lambda_1 \geq E[D]\sqrt{1 + \frac{Var[D]}{E^2[D]}}$
  - all eigenvalues lie in the interval $(-d_{max}, d_{max}]$
  - $\Sigma_{j=1}^{N} \lambda_k^k = trace(A^k) = \Sigma_{i=1}^{N} (A^k)_{ii} = W_k$ (total number of closed walks)
    - $\Sigma_{j=1}^{N} \lambda_j = 0$
    - $\Sigma_{j=1}^{N} \lambda_j^2 = 2L$
    - $\Sigma_{j=1}^{N} \lambda_j^3 = 6 \times triangles$
  - λ_1 and components of eigenvector x_1 are non-negative (when disconnected, $\equiv 0$)
- Spectrum of Q:
  - any eigenvalue μ_k is non-negative and smallest $\mu_N = 0$
  - complexity (number of spanning tree) is $\xi(G) = \frac{1}{N} \prod_{k=1}^{N-1} \mu_k$
  - algebraic connectivity $a(G) = \mu_{N-1}$. The graph G is only connected if and only if $a(G) = \mu_{N-1} > 0$. The graph G with larger $a(G)$ is more difficult to disconnect
- The number of links between G_1 and G_2: $R = \frac{1}{4} y^T Q y$, $y_i = 1$ if $i \in G_1$, else $y_i = -1$ if $i \in G_2 \backslash$
  - $R = \frac{1}{4} \Sigma_{j=1}^{N} \alpha_k^2 u_j$
  - $R \geq \frac{1}{4} (y^T z_{N-1})^2 \mu_{N-1}$
- Degree-preserving rewiring:
  - Largest eigenvalue of adjacency increases with degree assortativity ρ_D
  - while algebraic connectivity decreases which implies that increasing assortativity creates more disconnected components.

- Erdos-Renyi random graph
  - a_{ij} is a bernoulli random variable with mean ρ
  - $E[a_{ij}] = \rho$
  - the complement graph of $G_p(N)$ is $G_{1-p}(N)$
  - the average number of links: $E[L] = \frac{N(N-1)}{2} \rho$
  - the average cluster coefficient is $E[C_{Gp(N)}] = \rho$
  - $Pr[D = k] = \binom{N-1}{k} p^k (1-p)^{(N-1-k)} \approx \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}$
    - $\mu = (N-1)\rho$ and $\sigma^2 = (N-1)\rho(1-\rho)$

        - ρ constant, $N \to \infty$ => tendency towards a regular graph
        - $E[D]$ constant, $N \to \infty$ => $Pr[D = k]$ becomes Poisson distribution
        - the critical link density: $\rho_c \sim logN/N$
        - ρ_c small, compact $f_\lambda(x)$, high spike at zero, graph tends to be disconnected; ρ_c large, disperse $f_\lambda(x)$, lower spike at zero, graph tends to be connected.
    - rewiring makes clustering coefficient and average hotcount lower
    - $f_\lambda(-x) = f_\lambda(x)$ refers a tree graph

- Power-law graph (scale-free)

    - $Pr[D = k] = ck^{-\tau}$
    - The mean is not representative, because the variance is (very) large
    - robustness to random node failure
    - vulnerability to targeted hub attacks and cascading failures