Geometric Nontermination Arguments

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Introduction and Motivation

Example C-program

```
int main(){

  int a;
  int b=1;

  while(a+b>=4){
    a=3*a+b;
    b=2*b-5;
  }
}
```

- very basic C-program
- does it terminate?

  $\Rightarrow$ No!

  how can we prove this?

Integer Term Rewrite Systems (int-TRS)

int-TRS considered:

$$\begin{aligned}
&\overbrace{f_x}^{(1)} && \to \overbrace{f_y}^{(2)} (v_1, \ldots v_n) : | : cond_1 \\
&f_y(\underbrace{v_1, \ldots v_n}_{(3)}) && \to f_y (\underbrace{v'_1, \ldots v'_n}_{(3)}) : | : \underbrace{cond_2}_{(4)}
\end{aligned}$$

(1) function symbol (no variables $\Rightarrow$ start)

(3) variables v'_i as linear updates of the variables v_j

(2) function symbol

(4) a set of (in)-equations mentioning v_j and v'_i

Reading: "rewrite $f_y(v_1, \ldots, v_n)$ as $f_y(v'_1, \ldots, v'_n)$ if *cond* holds"

Geometric Nontermination Argument (GNA)

- Idea: Split program into two parts:
  - *STEM*: variable initialization and declaration

```
int a;
int b=1;
```

  - *LOOP*: linear updates and *while*-guard

```
while(a+b>=4){
  a=3*a+b;
  b=2*b-5;
}
```

- apply the definition of a *geometric nontermination argument* by J. Leike and M. Heizmann

Example

The int-TRS of the example program would be:

$$
\begin{aligned}
&f_1 \qquad\quad \rightarrow f_2(1 + 3 * v_1, -3) : | : v_1 > 2 \;\; \&\& \;\; 8 < 3 * v_1 \\
&f_2(v_1, v_2) \rightarrow f_2(3 * v_1 + v_2, v_3) : | : v_1 + v_2 > 3 \;\; \&\& \\
&v_1 > 6 \;\; \&\& \;\; 3 * v_1 > 20 \;\; \&\& \;\; 5 + v_3 = 2 * v_2 \;\; \&\& \;\; v_3 < -10
\end{aligned}
$$

The first rule represents the *STEM*

Second rule represents the *LOOP*

Definition (Geometric Non Termination Argument)

A tuple of the form:

$$(x, y_1, \dots, y_k, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_{k-1})$$

is called a *geometric nontermination argument* of size k for a program $= (STEM, LOOP)$ with n variables iff all of the following statements hold:

(domain) $x, y_1, \dots, y_k \in \mathbb{R}^n,\ \lambda_1, \dots \lambda_k, \mu_1, \dots \mu_{k-1} \geq 0$

(init) x represents the *start term* $(STEM)$

(point) $A \begin{pmatrix} x \\ x + \sum_i y_i \end{pmatrix} \leq b$

(ray) $A \begin{pmatrix} y_i \\ \lambda_i y_i + \mu_{i-1} y_{i-1} \end{pmatrix} \leq 0$ for all $1 \leq i \leq k$

Note: $y_0 = \mu_0 = 0$ instead of case distinction

Definitons: Matrices

Definition (*Guard Matrix*, *Guard Constants*)

For $1 \leq i, j \leq n$ and m the number of guards not containing "$=$": The *Guard Matrix* $G \in \mathbb{Z}^{m \times n}$ is the matrix of coefficients $a_{i,j}$ of a variable v_i within the j-th guard. The *Guard Constants* $g \in \mathbb{Z}^m$ are the constant terms c_j within the j-th guard.

Definition (*Update Matrix*, *Update Constants*)

The *Update Matrix* $U \in \mathbb{Z}^{n \times n}$ and *Update Constants* $u \in \mathbb{Z}^n$ are analogously to the *Guard Matrix* and *Guard Constants*, considering the updates (right hand side) instead of the guards.

Reminder: int-TRS

$$
\begin{array}{lll}
f_1 & \rightarrow f_2(1 + 3 * v_1, -3) : | : v_1 > 2 \ \&\& \ 8 < 3 * v_1 \\
f_2(v_1, v_2) & \rightarrow f_2(3 * v_1 + v_2, v_3) : | : v_1 + v_2 > 3 \ \&\& \\
\multicolumn{2}{l}{v_1 > 6 \ \&\& \ 3 * v_1 > 20 \ \&\& \ 5 + v_3 = 2 * v_2 \ \&\& \ v_3 < -10}
\end{array}
$$

Example (*Guard Matrix*, *Guard Constants*)

for the stated int-TRS the *Guard Constants* G and *Guard Constants* g for the loop are:

$$
G = \begin{pmatrix} -1 & -1 \\ -1 & 0 \\ -3 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } g = \begin{pmatrix} -4 \\ -7 \\ -21 \\ -6 \end{pmatrix}
$$

Reminder: int-TRS

$$
\begin{aligned}
&f_1 \quad\quad\quad\;\; \rightarrow f_2(1 + 3 * v_1, -3) : | : v_1 > 2 \;\; \&\& \;\; 8 < 3 * v_1 \\
&f_2(v_1, v_2) \rightarrow f_2(3 * v_1 + v_2, v_3) : | : v_1 + v_2 > 3 \;\; \&\& \\
&v_1 > 6 \;\; \&\& \;\; 3 * v_1 > 20 \;\; \&\& \;\; 5 + v_3 = 2 * v_2 \;\; \&\& \;\; v_3 < -10
\end{aligned}
$$

Example (*Update Matrix*, *Update Constants*)

for the stated int-TRS the *Update Matrix* U and *Update Constants* u are:

$$U = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } u = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

Definition (*Iteration Matrix*, *Iteration Constants*)

Let $\mathbf{0}$ be a matrix of the size of G with only entry's 0 and I denote the identity matrix having the same dimension as U. Then are the *Iteration Matrix* A and *Iteration Constants* b defined as:

$$A = \begin{pmatrix} G & \mathbf{0} \\ U & -I \\ -U & I \end{pmatrix} \text{ and } b = \begin{pmatrix} g \\ -u \\ u \end{pmatrix}$$

Geometric Nontermination