Geometric Nontermination Arguments

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Introduction and Motivation

Example C-program

```
int main(){

  int a;
  int b=1;

  while(a+b>=4){
    a=3*a+b;
    b=2*b-5;
  }
}
```

- very basic C-program
- does it terminate?

  $\Rightarrow$ No!

  how can we prove this?

Integer Term Rewrite Systems (int-TRS)

int-TRS considered:

$$\begin{array}{lcl} \overbrace{f_x}^{(1)} & \to & \overbrace{f_y}^{(2)}\ (v_1, \dots v_n) : | : cond_1 \\ f_y(\underbrace{v_1, \dots v_n}_{(3)}) & \to & f_y\ (\underbrace{v'_1, \dots v'_n}_{(3)}) : | : \underbrace{cond_2}_{(4)} \end{array}$$

(1) function symbol (no variables $\Rightarrow$ start)

(3) variables v'_i as linear updates of the variables v_j

(2) function symbol

(4) a set of (in)-equations mentioning v_j and v'_i

Reading: "rewrite $f_y(v_1, \dots, v_n)$ as $f_y(v'_1, \dots, v'_n)$ if *cond* holds"

Geometric Nontermination Argument (GNA)

- Idea: Split program into two parts:
  - *STEM*: variable initialization and declaration

```
int a;
int b=1;
```

  - *LOOP*: linear updates and *while*-guard

```
while(a+b>=4){
  a=3*a+b;
  b=2*b-5;
}
```

- apply the definition of a *geometric nontermination argument* by J. Leike and M. Heizmann

Example

The int-TRS of the example program would be:

$$
\begin{aligned}
&f_1 \quad \rightarrow f_2(1 + 3 * v_1, -3) : | : v_1 > 2 \ \&\& \ 8 < 3 * v_1 \\
&f_2(v_1, v_2) \rightarrow f_2(3 * v_1 + v_2, v_3) : | : v_1 + v_2 > 3 \ \&\& \\
&v_1 > 6 \ \&\& \ 3 * v_1 > 20 \ \&\& \ 5 + v_3 = 2 * v_2 \ \&\& \ v_3 < -10
\end{aligned}
$$

The first rule represents the *STEM*
Second rule represents the *LOOP*

Definition (Geometric Non Termination Argument)

A tuple of the form:

$$(x, y_1, \ldots, y_k, \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_{k-1})$$

is called a *geometric nontermination argument* of size k for a program $= (STEM, LOOP)$ with n variables iff all of the following statements hold:

$$\begin{aligned} \text{(domain)} &\quad x, y_1, \ldots, y_k \in \mathbb{R}^n,\ \lambda_1, \ldots \lambda_k, \mu_1, \ldots \mu_{k-1} \geq 0 \\ \text{(init)} &\quad \text{x represents the } \textit{start term } (STEM) \\ \text{(point)} &\quad A \begin{pmatrix} x \\ x + \sum_i y_i \end{pmatrix} \leq b \\ \text{(ray)} &\quad A \begin{pmatrix} y_i \\ \lambda_i y_i + \mu_{i-1} y_{i-1} \end{pmatrix} \leq 0 \text{ for all } 1 \leq i \leq k \end{aligned}$$

Note: $y_0 = \mu_0 = 0$ instead of case distinction

Definitons: Matrices

Definition (*Guard Matrix*, *Guard Constants*)

For $1 \leq i, j \leq n$ and m the number of guards not containing "$=$": The *Guard Matrix* $G \in \mathbb{Z}^{m \times n}$ is the matrix of coefficients $a_{i,j}$ of a variable v_i within the j-th guard. The *Guard Constants* $g \in \mathbb{Z}^m$ are the constant terms c_j within the j-th guard.

Definition (*Update Matrix*, *Update Constants*)

The *Update Matrix* $U \in \mathbb{Z}^{n \times n}$ and *Update Constants* $u \in \mathbb{Z}^n$ are analogously to the *Guard Matrix* and *Guard Constants*, considering the updates (right hand side) instead of the guards.

Reminder: int-TRS

$$f_1 \rightarrow f_2(1+3*v_1, -3) : | : v_1 > 2 \;\&\&\; 8 < 3*v_1$$
$$f_2(v_1, v_2) \rightarrow f_2(3*v_1+v_2, v_3) : | : v_1+v_2 > 3 \;\&\&\; v_1 > 6 \;\&\&\; 3*v_1 > 20 \;\&\&\; 5+v_3 = 2*v_2 \;\&\&\; v_3 < -10$$

Example (*Guard Matrix*, *Guard Constants*)

for the stated int-TRS the *Guard Constants* G and *Guard Constants* g for the loop are:

$$G = \begin{pmatrix} -1 & -1 \\ -1 & 0 \\ -3 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } g = \begin{pmatrix} -4 \\ -7 \\ -21 \\ -6 \end{pmatrix}$$

Reminder: int-TRS

$$
\begin{aligned}
&f_1 \quad\quad\quad\;\; \to f_2(1+3*v_1,-3) : | : v_1 > 2 \;\;\&\&\;\; 8 < 3*v_1 \\
&f_2(v_1,v_2) \to f_2(3*v_1+v_2,v_3) : | : v_1+v_2 > 3 \;\;\&\& \\
&v_1 > 6 \;\;\&\&\;\; 3*v_1 > 20 \;\;\&\&\;\; 5+v_3 = 2*v_2 \;\;\&\&\;\; v_3 < -10
\end{aligned}
$$

Example (*Update Matrix*, *Update Constants*)

for the stated int-TRS the *Update Matrix* U and *Update Constants* u are:

$$U = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } u = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

Definition (*Iteration Matrix*, *Iteration Constants*)

Let $\mathbf{0}$ be a matrix of the size of G with only entry's 0 and I denote the identity matrix having the same dimension as U. Then are the *Iteration Matrix* A and *Iteration Constants* b defined as:

$$A = \begin{pmatrix} G & \mathbf{0} \\ U & -I \\ -U & I \end{pmatrix} \text{ and } b = \begin{pmatrix} g \\ -u \\ u \end{pmatrix}$$

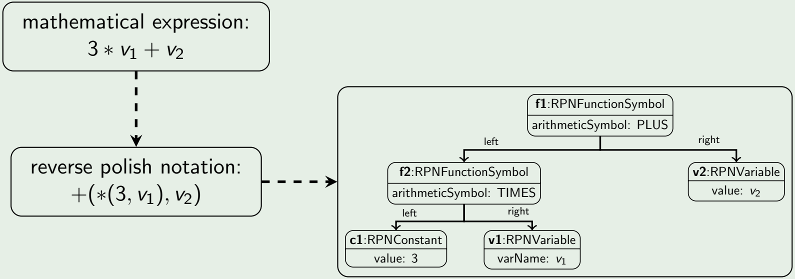
Sentence

If a geometric nontermination argument a for a program p exists, then p does not terminate.

Reverse Polish Notation Tree (RPNTree)

- simple tree structure to handle only considered terms
- classes for variables, constants and arith. operations

Example



Sat. Modulo Theorie (SMT)

- Basic idea:

  set of assertions: (in)-equations with variables

  $\xrightarrow{\text{SMT-solver}}$ a sat. model or unsat. core
- sat. model: a value for every variable s.t. all assertions hold
- unsat. core: a (minimal) set of assertions that can't hold simultaneously

Example

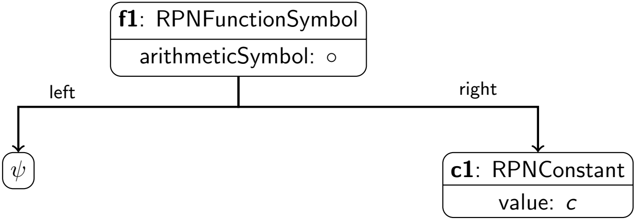
Considering the following assertions:

$$x \leq y \quad x > 5 \quad x + y \leq 20 \quad y \neq 10$$

A possible model would be $m_1 = \{x = 6, y = 6\}$.

changing the third assertion to $x + y \leq 10$:
no possible solution with unsat. core $\{x \leq y, x > 5, x + y \leq 10\}$

- assertions can be generated using *SMTFactory*
- if generated it ensures the following property:



where $\circ \in \{\leq, =\}$, $cons \in \mathbb{Z}$ and a linear update $\psi = \sum_{i=1}^{n} a_{i,j} v_i$ for variables v_i

Geometric Nontermination

Necessary steps for the derivation of a GNA:

1. derive the *STEM*
2. derive the *Guard Matrix*/*Constants*
3. derive the *Update Matrix*/*Constants*
4. compute the *Iteration Matrix*/*Constants*
5. add the criteria of a GNA as assertions to an *SMT-solver*
6. read of GNA (if exists)

Derivation: *STEM*

Consider two different possibilities:

constant stem: $f_x \to f_y(c_1, \ldots, c_n) : | : TRUE$

$\Rightarrow$ read of values

Example

$$f_1 \to f_2(10, -3) \Rightarrow STEM = (10, -3)^T$$

variable stem: $f_x \to f_y(c_1 + \sum_{i=1}^{n} a_{1,i} v_i, \ldots, c_n + \sum_{i=1}^{n} a_{n,i} v_i) : | :$
$\bigwedge_{\text{guard } g} \sum_{i=1}^{n} g_{n,i} v_i \leq c_m$

$\Rightarrow$ create assertions and derive a model

Example

$$f_1 \to f_2(1 + 3v_1, -3) : | : v_1 > 2 \;\&\&\; 8 < 3v_1$$
$$\Rightarrow \text{model } m_1 = \{v = 3\} \Rightarrow STEM = (10, -3)^T$$

Derivation: Guard Matrix/Constants

conditional term given by the *Symbolic Execution Graph*

$$r = \&\&(g_1, (\&\&(\dots, (\&\&(g_{n-1}, g_n))\dots)))$$

Algorithm 1 derive set of guards

```
function COMPUTEGUARDSET(Rule r)
    Stack stack ← r
    Set guards
    while !stack.isEmpty() do
        item ← stack.pop
        if item is of the form &&(x1, x2) then
            add x1 and x2 to stack
        else
            add item to guards
    return guards
```

- now we have $G = \{g \mid g \text{ is a guard}\}$
- Problem: g could not be in the desired $\varphi \leq c$ form.
- Even worse: g could declare new variables using "$=$"
- Solution: bring every g in the desired form, by:

1.	filter equalities	by substituting "new" variables
2.	normalizing ($\leq$)	rewrite $<, >, \geq$ to $\leq$
3.	normalizing (c)	transfer only constant term to r.h.s.

function $\textsc{filterEqualities}(G)$
    $V_{left} = \{v \mid$ the left hand side of the rule contains $v\}$
    $V_{right} = \{v \mid$ the right hand side of the rule contains $v\}$
    $V_{sub} = V_{right} - V_{left}$
    define substitution $\theta = \{\}$
    while $V_{sub} \neq \emptyset$ **do**
        select $s \in V_{sub}$
        select $g_s \in \{g \in G \mid g$ contains "$=$"$\}$
        remove g_s from G
        rewrite g_s to the form $s = \psi$
        $\theta = \theta\{s/\psi\}$
        for all $g \in G$ **do**
            $g = \theta g$
        remove s from V_{sub}
    return G

Example

From the example int-TRS we get using the decat. algorithm:
$\{v_1 + v_2 > 3,\ v_1 > 6,\ 3 * v_1 > 20,\ 5 + v_3 = 2 * v_2,\ v_3 < -10\}$

1. We compute $V_{left} = \{v_1, v_2\}$, $V_{right} = \{v_1, v_2, v_3\}$ so $V_{sub} = \{v_3\}$
2. Begin with $\theta = \{\}$
3. Since obviously $V_{sub} \neq \emptyset$ we select $s = v_3$ and select $g_s \Leftrightarrow 5 + v_3 = 2 * v_2$
4. g_s rewritten to the form $s = \psi$ then follows with $v_3 = 2 * v_2 - 5$
5. $\theta = \theta\{s/2 * v_2 - 5\} = \{s/2 * v_2 - 5\}$
6. $G = \{v_1 + v_2 > 3,\ v_1 > 6,\ 3 * v_1 > 20,\ 2 * v_2 - 5 < -10\}$
7. Since $V_{sub} = \emptyset$ return G

normalization ($\leq$)

rewrite a guard g_i of the form $g_i \Leftrightarrow \psi + c_\psi \circ c$, where $\circ \in \{<, >, \leq, \geq\}$ to the form $\eta * \psi + \eta * c_\psi \leq \eta * c - \tau$ depending on $\circ$.

$\circ$	η	τ	$\eta * \psi + \eta * c_\psi \leq \eta * c - \tau$
$<$	1	1	$\psi + c_\psi \leq c - 1$
$>$	-1	1	$-\psi - c_\psi \leq -c - 1$
$\leq$	1	0	$\psi + c_\psi \leq c$
$\geq$	-1	0	$-\psi - c_\psi \leq -c$

η is the indicator of inverting the guard to convert $\geq$ ($>$) to $\leq$ ($<$)
τ is the possible subtraction of 1 to receive the $\leq$ instead of a $<$.

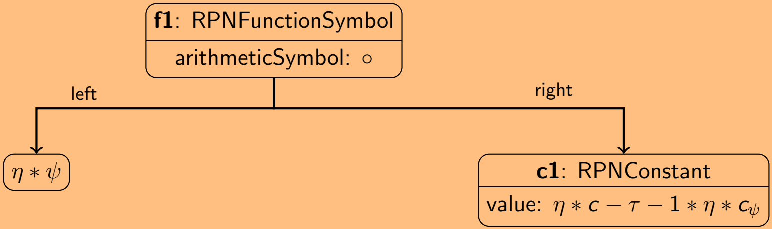
normalization (c)

Subtract the term $\eta * c_\psi$ on both sides:

final form: $\eta * \psi \leq \underbrace{\eta * c - \tau - 1 * \eta * c_\psi}_{\text{constant term}}$

Reminder: int-TRS structure

Can derive constant factors very simple using the stated structure property:



Example

Normalizing the guard $g \Leftrightarrow 3 * v_1 > 20 \Leftrightarrow \underbrace{3 * v_1}_{\psi} + \underbrace{0}_{c_\psi} > \underbrace{20}_{c}$

Looking up the row for $\circ \Leftrightarrow >$:

$\circ$	η	τ	$\eta * \psi + \eta * c_\psi \leq \eta * c - \tau$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$>$	-1	1	$-\psi - c_\psi \leq -c - 1$

Result with $\eta = -1$, $\tau = 1$ in:
$-(3 * v_1) - (0) \leq -20 - 1 \Leftrightarrow -3 * v_1 \leq -21$

- now every guard has the form $\varphi \leq c$
- deriving *Guard Constants* is very simple
- deriving *Guard Matrix* is read off the coefficients.
  (more detailed within the *Update Matrix*)

$\Rightarrow$ *Update Matrix/Constants* derived ✓

Derivation: Update Matrix/Constants

- do not contain any (in-)equalities
- follow the form: $c + \sum_{i=1}^{n} a_i v_i$ (not $v_i a_i$)
- Problem: can still contain new variables from the guards

  Solution: apply substitutions to the updates
- Problem: constant term can occur anywhere

  Solution: perform recursive search under certainty of two aspects:

  1. at most one constant term exists
  2. the constant term is not multiplied

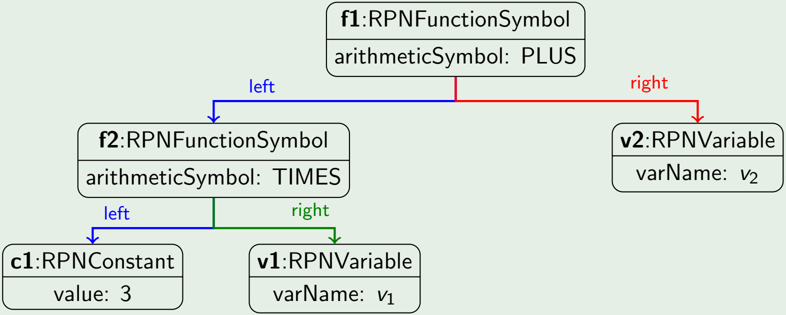
Algorithm 2 Derivation of a coefficient

```
function GETCOEFFICIENT(query)
    if this == query then
        return 1
    else if this does not contain query then
        return 0

    if this represents PLUS then
        if left side contains query then
            return getCoefficient(query)
        else
            return getCoefficient(query)
    if this represents TIMES then
        if this.right == query then
            return this.left.value
```

Example

An example of coefficient derivation for query v_1:



Update Matrix/Constants & *Iteration Matrix/Constants*

- derived a coefficient for every variable and constant per update

$\Rightarrow$ *Iteration Matrix/Constants* ✓

- *Iteration Matrix/Constants* given by:

$$A = \begin{pmatrix} G & \mathbf{0} \\ M & -I \\ -M & I \end{pmatrix} \text{ and } b = \begin{pmatrix} g \\ -u \\ u \end{pmatrix}$$

can be computed

$\Rightarrow$ *Iteration Matrix/Constants* ✓

SMT-Problem

- given A and b use a *SMT-solver* to dis-/prove existence of a GNA
- Problem: $\lambda_i * y_i$ is non-linear

  Solution: compute λ_i as the i-th eigenvalue of U
- Problem: $\mu_i * y_i$ is non-linear

  can approach this problem in 2 ways:

  1. use *quantifier free non-linear integer arithmetic*
  2. iterate over all μ's

Reminder: Domain Criteria

(domain) $x, y_1, \ldots, y_k \in \mathbb{R}^n,\ \lambda_1, \ldots \lambda_k, \mu_1, \ldots \mu_{k-1} \geq 0$

$\Rightarrow$ adds no assertions

Reminder: Initiation Criteria

(init) x represents the *start term* (*STEM*)

$\Rightarrow$ adds no assertions

Reminder: Point Criteria

(point) $A \begin{pmatrix} x \\ x + \sum_i y_i \end{pmatrix} \leq b$

- y_i unknown $\Rightarrow$ create $s_i = x_i + \sum_{j=1}^{n} y_{i,j}$
- $A \begin{pmatrix} x \\ s \end{pmatrix} \leq b$

$$\Leftrightarrow \begin{pmatrix} & G & & 0 & \dots & 0 \\ a_{1,1} & \dots & a_{1,n} & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} & 0 & \dots & -1 \\ -a_{1,1} & \dots & -a_{1,n} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & \dots & -a_{n,n} & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_n \end{pmatrix} \leq \begin{pmatrix} g \\ -u_1 \\ \vdots \\ -u_n \\ u_1 \\ \vdots \\ u_n \end{pmatrix}$$

- y_i unknown $\Rightarrow$ create $s_i = x_i + \sum_{j=1}^{n} y_{i,j}$

$$\begin{pmatrix} \boxed{a_{1,1} * x_1 & \dots & a_{1,n} * x_n & -1 * s_1 & \dots & 0 * s_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} * x_1 & \dots & a_{n,n} * x_n & 0 * s_1 & \dots & -1 * s_n \\ \boxed{-a_{1,1} * x_1 & \dots & -a_{1,n} * x_n & 1 * s_1 & \dots & 0 * s_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} * x_1 & \dots & -a_{n,n} * x_n & 0 * s_1 & \dots & 1 * s_n \end{pmatrix} \leq \begin{pmatrix} \boxed{-u_1} \\ \vdots \\ -u_n \\ \boxed{u_1} \\ \vdots \\ u_n \end{pmatrix}$$

$\Rightarrow$ add guard assertion, n assertions of equality and addition assertion for every s_i

Reminder: Ray Criteria

(ray) $A \begin{pmatrix} y_i \\ \lambda_i y_i + \mu_{i-1} y_{i-1} \end{pmatrix} \leq 0$ for all $1 \leq i \leq k$

$i = 1$: $\Rightarrow \mu_{i-1} y_{i-1} = 0 \Rightarrow A \begin{pmatrix} y_1 \\ \lambda_1 y_1 \end{pmatrix} \leq 0$

add assertion with $y_{1,j}$ $1 \leq j \leq n$ as new variables

$i > 1$: with λ_i as the i-th eigenvalue

add assertion with $y_{i,j}$ $1 \leq j \leq n$ and μ_i as new variables

$\Rightarrow$ all necessary assertions stated ✓

let *SMT-solver* derive a GNA (if exists)

Reminder: Derived matrices and values

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -3 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} -4 \\ -7 \\ -21 \\ -6 \\ 0 \\ 5 \\ 0 \\ -5 \end{pmatrix}$$

$$\lambda_1 = 3, \quad \lambda_2 = 2, \quad x = (10, -3)^T$$

Example (Assertions I: Point Crit.)

- add guards. I.e.: $-10 - (-3) \leq -4$
- add sum rules. I.e.: $30 - 3 - s_1 = 0$
- add $s_1 = y_{1,1} + y_{2,1}$ and $s_2 = y_{1,2} + y_{2,2}$

Reminder: Derived matrices and values

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -3 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} -4 \\ -7 \\ -21 \\ -6 \\ 0 \\ 5 \\ 0 \\ -5 \end{pmatrix} \qquad \begin{array}{c} \lambda_1 = 3, \\ \lambda_2 = 2 \\ x = (10, -3)^T \end{array}$$

Example (Assertions II: Ray Crit.)

i=1: add for instance $-y_{1,1} - y_{1,2} \leq 4$, $3 * y_{1,1} + y_{1,2} - 3 * y_{1,1} \leq 0$

i>1: add for instance $-y_{2,1} - y_{2,2} \leq 4$,

$3 * y_{2,1} + y_{2,2} - 1 * (2 * y_{2,1} + \mu * y_{1,1}) \leq 0$

Verification of a GNA

- received a GNA form the *SMT-solver*
- want to verify the correctness of the *SMT-solver*'s model

$\Rightarrow$ recalculating of a GNA with the matrices and given values

Example (Validating a GNA I)

The *SMT-solver* gave us: $y_1 = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$, $y_2 = \begin{pmatrix} 8 \\ -8 \end{pmatrix}$, $\mu_1 = 0$

(domain) obviously true ✓

(init) checked against the *STEM* ✓

(point) $A \begin{pmatrix} 10 \\ -3 \\ 10+9+8 \\ -3+0+(-8) \end{pmatrix} \leq b \Leftrightarrow A \begin{pmatrix} 10 \\ -3 \\ 27 \\ -11 \end{pmatrix} \leq b$ ✓

Example (Validating a GNA II)

The *SMT-solver* gave us: $y_1 = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$, $y_2 = \begin{pmatrix} 8 \\ -8 \end{pmatrix}$, $\mu_1 = 0$

(ray)

$$i = 1:\ A \begin{pmatrix} 9 \\ 0 \\ 3 * 9 \\ 3 * 0 \end{pmatrix} \leq 0 \Leftrightarrow A \begin{pmatrix} 9 \\ 0 \\ 27 \\ 0 \end{pmatrix} \leq 0 \checkmark$$

$$i > 1:\ A \begin{pmatrix} 8 \\ -8 \\ 2 * 8 + 0 * 9 \\ 2 * (-8) + 0 * 0 \end{pmatrix} \leq 0 \Leftrightarrow A \begin{pmatrix} 8 \\ -8 \\ 16 \\ -16 \end{pmatrix} \leq 0 \checkmark$$

$\Rightarrow$ the derived GNA is applicable $\Rightarrow$ nontermination is proven