Bachelor Thesis

GEOMETRIC NON-TERMINATION ARGUMENTS FOR INTEGER PROGRAMS

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Erklärung Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie Zitate kenntlich gemacht habe.

Aachen, den September 3, 2017

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Abstract

The topic of program termination analysis undergoes a significantly importance increase owed to the expansion of software usage throughout everyday life. Since fixing problems caused by software bugs lead to an overhead of support the initial guarantee of correctness can save time spend on fixing these problems. Therefore the research of automated assisting during the engineering of large programs is a growing field. One major point of a correct program is determined by the termination, which means the resulting in a final state after finitely many steps. Even though such a tool can never provide correctness and soundness in every condition since it would have to solve the *halting problem*, which is proven by Touring to be undecidable, a variety of tools addressing this problematic exist. For example AProVE. Restricting the underlying program to certain circumstances tools like AProVE are able to decide termination.

In this thesis we extend the possibilities of proofing nontermination using AProVE with a special set of programs based on the Geometric Non-Termination approach of Jan Leike and Matthias Heizmann. Altering the underlying structure from linear loop programs to integer term rewrite systems we prove nontermination using a *geometric nontermination argument* derived from the program itself. Through the usage of linear algebra and *Satisfiability Modulo Theorie* solvers we are able to prove the existence of a *geometric nontermination argument* which results in a proof of nontermination of the integer term rewrite system. All this implemented as one additional approach within AProVE leads to a more applicable tool. We restrict ourselves to only have linear updates of the variables in order to be able to apply the underlying approach, which provides correctness and soundness for this particular set of programs.

As a result we will see that the implemented technique provides the desired mechanism of proving nontermination for the considered programs under certain limitations, which are reasoned by the complexity of integer term rewrite systems in general. One restriction is the existence of a start term within the system, which is mandatory to apply the definition of a *geometric nontermination argument*. Further the handling of newly introduced variables within the systems are only very basic, since correctness of using division on integers is not generally given.

In summary we can derive that *Geometric Nontermination* is a promising topic for integer term rewrite system, like it is for linear lasso programs. Such a termination analysis obviously is only applicable for programs computing mathematical procedures, which are restricted to linear updates. The restriction to only use linear updates and it’s consequences regarding in modern programs need further investigation to evaluate the applicability in real industrial software.

Contents

Chapter 1

Introduction

1.1 Motivation

The topic of verification and termination analysis of software increases in importance with the development of new programs. Even though that for Touring Complete programming languages the *halting problem* is undecidable, and therefore no complete and sound method can exist, a verity of approaches to determine termination are researched and still being developed. These approaches can determine termination on programs, which match certain criteria in form of structure, composition or using only a closed set of operations for example only linear updates of variables.
Given a tool, which can provide a sound and in many scenarios applicable mechanism to prove termination, an optimized framework could analyse written code and find bugs before the actual release of the software [VDDS93]. Contemplating that automatic verification can be applied to termination proved software the estimated annual US Economy loses of $60 billion each year in costs associated software maintenance could be reduced significantly [ZC09].

1.2 *AProVE*

One promising approach is the tool AProVE (Automated Program Verification Environment) developed at the RWTH Aachen by the Lehr- und Forschungsgebiet Informatik 2. The *AProVE*-tool (further only called AProVE) for automatic termination and complexity proving works with different programming languages of major language paradigms like Java (object oriented), Haskell (functional), Prolog (logical) as well as rewrite systems.

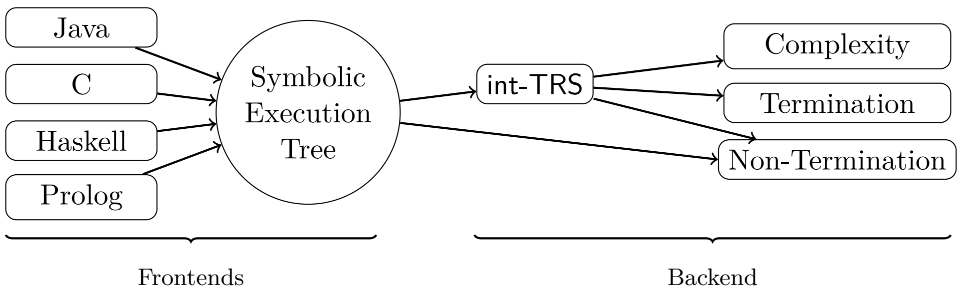


Figure 1.1: Schematic partition of the derivation process of AProVE adapted from [GAB+17]

AProVE is able to unify different languages into one structure by converting programs of specific languages like C into *Low Level Virtual Machine(llvm)*-code using the tool Clang [1]. Among others these llvm-programs can be converted into a so called *Symbolic Execution Graph*. This graph represents all possible computations of the input program. If this graph contains lasso's, which are strongly connected components (SCC) and the corresponding path from the root to the SCC, AProVE derives so called (integer) term rewrite systems (further only called int-TRS)[2]. By adding conditions to the int-TRS rules the solution space gets restricted and therefore the int-TRS under-approximates. From that it is proven that the non-termination of (at least) one int-TRS implies non-termination of the program. A more detailed description of the process is stated in [HEF+17]
The conversion of different languages into int-TRS and subsequently applying various different approaches is what makes this tool strong in meanings of proofing [GAB+17].

1.3 Overview

This paper provides the introduction to the topic of termination analysis. We focus on the very basic steps, because of the huge variety of possible approaches and related methods. Any further knowledge about termination analysis techniques and how they are applied within AProVE can be found in the related papers [GAB+17], [GSKT06], [GTSKF03].
Within Chapter 2 some preliminaries used within the paper are defined to create a well-defined base for any further argumentation and derivation. It covers the most essential definition of the this thesis, which is the nontermination, topics of basic knowledge about *Integer Term Rewrite Systems* and it's within this approach considered subset based on it's structure. Also the definition of the *geometric nontermination argument*, which builds the main constituent, and any strongly related matrices are defined. Also we define a tree-structure, which we use to handle arithmetical terms containing variables. Last we take a glimpse at the topic of SMT-solving and declare the essential parts used within the implementation of the approach.
The main chapter, which is Chapter 3, deals with the derivation of the *STEM* part, for constant or variable terms, the derivation of the *LOOP* with all it's matrices and finally the derivation of the SMT-Problem, which provides a *geometric nontermination argument* if it exists.
At the end, we want to take a look at the usability of the approach itself. Also we want to point out possible adaptations and improvements of the implementation of this approach.

[1] further information: `https://clang.llvm.org/`
[2] a mathematical definition can be found within [FGP+09]

Chapter 2

Preliminaries

In order to be able to explain the non-termination approach we have to declare, what nontermination means, which programs are considered within the Geometric Nontermination and present the *geometric nontermination argument* building the core of the approach. Furthermore we have to define a few structures we work on. Further we have to define what an *SMT-solver* is and how it is used within this implementation.

2.1 Non-Termination

The definition of non-termination is the most essential considering a technique proving it. Non-termination can be defined as a specific input to a program p, such that p runs in an infinite loop. Proving termination is much harder than proving non-termination, since we only have to determine one case, which fit's the condition of running into an infinite loop.
Non-termination is obviously still undecidable, since otherwise the halting problem would be decidable, nevertheless there are a large variety of possibilities to prove non-termination as we will see in this paper.

2.2 Integer Term Rewrite System(int-TRS)

In order to apply the upcoming procedure we have to define what structure the approach works on. The derivation of an int-TRS from source code is described in Section 1.2. Considering the following approach we will look at a int-TRS in a more superficial way. The composition of a int-TRS considered in this paper is shown in Figure 2.1.

$$
\begin{array}{lcl}
\overbrace{f_x}^{(1)} & \rightarrow & \overbrace{f_y}^{(2)}(v_1, \dots v_n) : | : cond_1 \\
f_y(\underbrace{v_1, \dots v_n}_{(3)}) & \rightarrow & f_y(\underbrace{v'_1, \dots v'_n}_{(3)}) : | : \underbrace{cond_2}_{(4)}
\end{array}
$$

Figure 2.1: The structure of a int-TRS considered in this paper.

The considered program shown in Figure 2.1 consists of a set of structure elements, whose definition is necessary:

(1) A function symbol consisting of no variables is a *start function symbol* and is the first symbol to use. Further explanation in (line 1) and Figure 2.3.

(2) A function symbol denoting a current program-state

(3) The variables of a function symbol denoting the change of the variables by applying the term rewriting rule. The value of $v'_i \in \mathbb{Z}$ is a linear update of the variables $v_j \in \mathbb{Z}$ $1 \leq j \leq n$ in standard linear integer form. [1]

(4) The conditional term of the form (in)equation$_1$ && ... && (in)equation$_m$ $m \in \mathbb{N}$, where (in)equation$_i$ contains not only v_j $1 \leq j \leq n$, but can also introduce new variables through equality. The *(in)equalities* are defined further in Section 3.2.1.

(line 1) The first line is the rewriting rule the program starts with and can be seen as a declaration of initial values of some variables. An example is shown in Figure 2.3

(line 2) Such a self-looping rule considered within this approach to define further computations. Other looping rules will be presented in Section 4.3.2.

2.3 Geometric Nontermination Argument (GNA)

Adapted from Jan Leikes and Matthias Heizmanns paper *Geometric Nontermination Arguments* [LH14] we will define the considered programs, define the *STEM* and *LOOP* and finally state the definition of *Geometric Nontermination Arguments*.

2.3.1 Considered Programs

The considered programs in the Geometric Nontermination are not bound to a special programming language. The paper works on so called Linear-Lasso Programs, which in fact are also used within AProVE to derive the so called (int-)TRS, stated in Section 1.2. Because of the also stated conversion of the language into a *Symbolic Execution Graph* and further analysis the applicability of *geometric nontermination argument*'s are not bound to any programming language.
In order to define the specific conditions under which we can use the approach, we take the language Java as an example.

2.3.2 Structure

The structure of the considered programs is quite simple. They contain an optional declaration of the used variables and a *while*-loop. Even though Java would not accept this the conversion to llvm would still be sound. An example of a fulfilling Java program is shown in Figure 2.2.

- The *STEM*:
  The initialization and optional declaration of variables used within the *while*-loop. In Figure 2.2 line 3 and 4 are considered the *STEM*. Also only b is declared with a value.

[1] The standard linear integer form has the following pattern: $a_1 * v_1 + \cdots + a_n * v_n + c$, where $a_i, v_i, c \in \mathbb{Z}$, $1 \leq i \leq n$. Also it is important that $a_i * v_i$ has this order and not $v_i * a_i$

- The guard:
  The guard of the *while*-loop is essential to restrict a as we will see in Section 3.1.2. With the restriction of $a + b \geq 4$ we can prove termination for $a < 3$ without further analysis, and also, in order to prove nontermination, assume that $a \geq 3$.

- The linear Updates:
  The updates of the variables within the *while*-loop are the most essential part for termination, since their value determine if the guard still holds. The approach works with only linear updates of the variables, so for every variable v_i where $1 \leq i \leq n$ we can have a $v_i = a_1 * v_1 + ... + a_n * v_n + c$ with $n \in \mathbb{N}$. Note since we work on int-TRS it is sufficient for a_i to be in $\mathbb{Z}$.

```
1  int main(){
2
3    int a;
4    int b=1;
5
6    while(a+b>=4){
7      a=3*a+b;
8      b=2*b-5;
9    }
10 }
```

the *STEM*

the guard

the linear update

Figure 2.2: A Java program fulfilling the conditions mentioned in Section 2.3.2 to be applicable

The guard and linear updates together form the so called *LOOP*.

Through the in Section 2.2 described procedure and given structure we receive the to Figure 2.2 corresponding int-TRS shown in Figure 2.3. As we can see the original program can be recognized quite easily. The first rule in line 1 denotes the *STEM*, while the second line equals the *LOOP*.

1 $\overbrace{f_1 \rightarrow f_2(1+3*v_1,-3) : | : v_1 > 2 \ \&\& \ 8 < 3*v_1}^{STEM}$

2 $\underbrace{\underbrace{f_2(v_1,v_2) \rightarrow f_2(3*v_1+v_2,v_3)}_{\texttt{linear update}} : | : \underbrace{v_1+v_2 > 3 \ \&\& \ v_1 > 6 \ \&\& \ 3*v_1 > 20 \ \&\& \ 5+v_3 = 2*v_2 \ \&\&}_{\texttt{guards}}}_{LOOP}$

Figure 2.3: The int-TRS corresponding to the Java program in Figure 2.2

Neglecting the conditional terms for now the declaration of v_2' in line 1 to -3, because of the one circle the *Symbolic Execution Graph* has to compute in order to find a lasso. Starting with $b = 1$ one step would be the computation of $b = 2*1-5 = -3$. The definition of v_1' is more difficult and will be shown within Section 3.1. Also the update for v_1' within line 2 is the same as in Figure 2.2 line 7. The definition of $v_2' = v_3$ is fundamental and not as simple as v_1', since v_3 is a new variable introduced within the *guards* through the equality $5 + v_3 = 2*v_2$. The handling of such variables will be explained in Section 3.2.1 and Section 3.2.2.

2.3.3 Necessary Definitions

In order to be able to define the key element of this approach, the *geometric nontermination argument*, we have to define a number of matrices and constant vectors, which are used to derive such a *geometric nontermination argument*.

Definition 2.3.1 (*STEM*)**.** *The STEM is denoted as a vector $x \in \mathbb{Z}^n$, where n is the number of variables within the rule of the start function symbols right hand side of a* int-TRS*. The values of x can be constants or defined by conditions. Examples are shown within Section 3.1.*

Definition 2.3.2 (Guard Matrix, Guard Constants)**.** *Let $n \in \mathbb{N}$ be the number of distinct variables, v_i $1 \leq i \leq n$ the i-th distinct variable names occurring on the left hand side, $m \in \mathbb{N}$ be the number of guards not containing equality, $a_{i,j} \in \mathbb{Z}$ $1 \leq i \leq n$, $1 \leq j \leq m$ the factor of v_i in g_j and $c_i \in \mathbb{Z}$ be the constant term within r_j.*

Then the Guard Matrix $G \in \mathbb{Z}^{m \times n}$ is defined as $G_{i,j} = a_{i,j}$ and Guard Constants $g \in \mathbb{Z}^m$ are defined as $g_i = c_i$.

Newly introduced variables must not be represented by a column of the Guard Matrix, but create substitutions further used in Section 3.1.2, Section 3.2.1 and Section 3.2.2.

Example 1. *The corresponding Guard Matrix to Figure 2.3 is* $G = \begin{pmatrix} -1 & -1 \\ -1 & 0 \\ -3 & 0 \\ 0 & 2 \end{pmatrix}$ *and the Guard Constants is* $g = \begin{pmatrix} -4 \\ -7 \\ -21 \\ -6 \end{pmatrix}$
The normalization of the guards r_i to the form $a_{i,1}v_1 + \dots a_{i,n}v_n \leq c$ transforms for example the guard r_1 in the following way

$$r_1 \Leftrightarrow v_1 + v_2 > 3 \Leftrightarrow -v_1 - v_2 < -3 \Leftrightarrow -v_1 - v_2 \leq -4$$

Definition 2.3.3 (Update Matrix, Update Constants)**.** *Let $n \in \mathbb{N}$ be the number of distinct variables of the left hand side, v_i $1 \leq i \leq n$ the i-th distinct variable name, $m \in \mathbb{N}$ the arity of the function symbol of the right hand side, v'_i $1 \leq i \leq m$ the i-th variable definition of the right hand sight's function symbol, $a_{i,j} \in \mathbb{Z}$ $1 \leq i \leq n$ $1 \leq j \leq m$ be the factor of variable v_i in variable definition v'_i and $c_i \in \mathbb{Z}$ $1 \leq i \leq m$ the constant term of v'_i.*

Then the Update Matrix $U \in \mathbb{Z}^{m \times n}$ is defined as $U_{i,j} = a_{i,j}$ and Update Constants $u \in \mathbb{Z}^m$ are defined as $g_i = c_i$.

Regarding the new variable v_3, we have to substitute in order to keep the desired size of the matrix. This procedure is further defined within Section 3.2.1 and Section 3.2.2.

Example 2. *The corresponding Update Matrix to Figure 2.3 is* $U = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ *and the Update Constants are* $u = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$.

The second row is not as obvious as the first row and will be derived in detail within Section 3.2.2.

Definition 2.3.4 (Iteration Matrix, Iteration Constants). *Let G be the Guard Matrix, g the Guard Constants, U the Update Matrix, u the Update Constants, $n \in \mathbb{N}$ the number of variables and $m \in \mathbb{N}$ the number of conditional terms.*
Also let $\mathbf{0}$ be a matrix of the size of G with only entry's 0 and I denote the identity matrix having the same dimension as U.

The Iteration Matrix $A \in \mathbb{Z}^{2*n+m \times 2*n}$, which defines one complete execution of the LOOP, and the Iteration Constants $b \in \mathbb{Z}^{2*n+m}$ is defined as

$$A = \begin{pmatrix} G & \mathbf{0} \\ U & -I \\ -U & I \end{pmatrix} \text{ and } b = \begin{pmatrix} g \\ -u \\ u \end{pmatrix} \text{ [LH14]}$$

Definition 2.3.5 (LOOP). *The LOOP is defined as a tuple (A, b), where A is the Iteration Matrix and b the Iteration Constants of an int-TRS.*

Now we can define the key element, which was originally defined for linear lasso programs.

Definition 2.3.6 (Geometric Non Termination Argument). *A tuple of the form:*

$$(x, y_1, \ldots, y_k, \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_{k-1})$$

is called a geometric nontermination argument off size k for a program $= (STEM, LOOP)$ with n variables iff all of the following statements hold:

$$\begin{aligned} &\textit{(domain)}\ x, y_1, \ldots, y_k \in \mathbb{R}^n,\ \lambda_1, \ldots \lambda_k, \mu_1, \ldots \mu_{k-1} \geq 0 \\ &\textit{(init)}\ x \text{ represents the start term (STEM)} \\ &\textit{(point)}\ A \begin{pmatrix} x \\ x + \sum_i y_i \end{pmatrix} \leq b \\ &\textit{(ray)}\ A \begin{pmatrix} y_i \\ \lambda_i y_i + \mu_{i-1} y_{i-1} \end{pmatrix} \leq 0 \text{ for all } 1 \leq i \leq k \end{aligned}$$

Note that $y_0 = \mu_0 = 0$ is set for the ray instead of a case distinction. [LH14]

The usage of such a *geometric nontermination argument* is justified by the following sentence:

Sentence 2.1. *If a geometric nontermination argument a for a program p exists, then p does not terminate. [LH14]*

2.4 Reverse-Polish-Notation-Tree

Within the program of deriving a *geometric nontermination argument* it happens that we get a mathematical term in the so-called *Polish Notation* or *Reverse Polish Notation in prefix notation*, which is a special form of rewriting a, in our case linear, expression to compute the solution efficiently using a stack. [Wik17a] Within our program we use this kind of notation to parse it into our own tree-structure to do further analysis.

As shown in Figure 2.4 we have an *abstract* root, subclasses for every occurring type of element within the int-TRS, a *static* parsing of a given term and an exception for parsing exceptions. An example of the *Reverse Polish Notation Tree*'s usage is shown in Figure 2.5

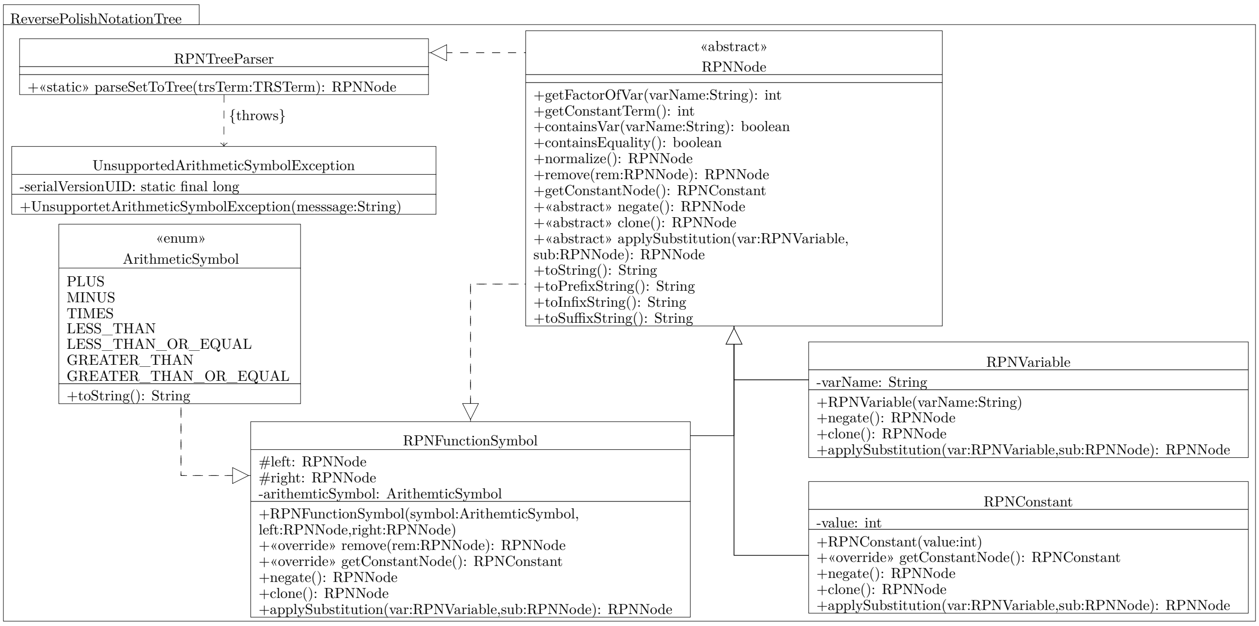


Figure 2.4: The class diagram of the *Reverse Polish Notation Tree* within the *geometric nontermination analysis*

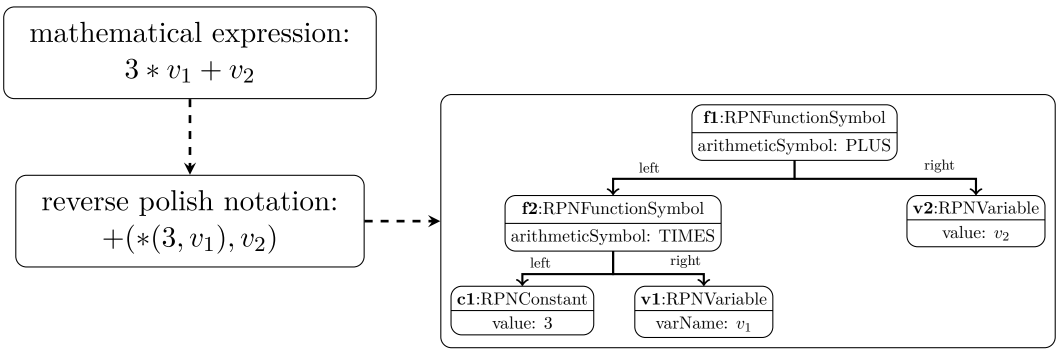


Figure 2.5: An example of the representation of the term $3*v_1+v_2$ as a graph using the *Reverse Polish Notation Tree* of Section 2.4

2.5 SMT-Problem

Also we have to consider an *Satisfiability Modulo Theorie*-Problem (SMT-Problem), we have to solve to derive a *geometric nontermination argument* fulfilling all the criterias of Definition 2.3.6. Since SMT-Problem solving is a big research topic on it's own we only consider the very basic of SMT-Solving necessary to understand how the program solves the problem.

Within this approach we use the so called *Basic Structures* defined within AProVE to add assertions to the *SMT-solver* using the *SMTFactory*. An example of the structure of the assertions can be found in Figure 2.6.

$$\underbrace{\underbrace{\underbrace{\underbrace{3}_{(1)} * \underbrace{v_1}_{(2)}}_{(3)} \underbrace{+}_{(4)} \quad \dots}_{(5)} \quad \underbrace{\leq}_{(6)} \quad \underbrace{5}_{(7)}}_{(8)}$$

Figure 2.6: An example to show the structure of an assertion used for the *SMT-solver*

Such an example assertion can be split into different parts:

(1) *PlainIntegerConstants* as coefficients

(2) *PlainIntegerVariables'* as variables the *SMT-solver* should derive values for such that all assertions are satisfied

(3) A coefficient multiplied with a variables is represented by an *PlainIntegerOperation* with *ArithmeticOperationType MUL* to denote multiplication

(4) An *ArithmeticOperationType* of type *ADD* to denote addition

(5) The left hand side is one big *PlainIntegerOperation* consisting of the addition(4) of the multiplication (3) of coefficient's (1) and variables (2).

(6) The *IntegerRelationType* defining the assertion. We only use the *EQ (equal)* or *LE (less than or equal)* relations.

(7) The right hand side is only one *PlainIntegerConstant*

(8) The whole line is a *PlainIntegerRelation*, which can be transformed into the *SMTExpressionFormat* the *SMT-solver* uses.

We use a solver within APro VE to create a bunch of assertions restricting the possible solution space. Since we operate in integer arithmetic and use linear equations we can restrict the solver to only use *quantifier free linear integer arithmetic*. In order to solve the problem given by the assertions the solver tries to derive a model satisfying all of them or derive an unsatisfiable core. [Áb16]

Example 3. *Consider the following assertions that should hold:*

$$x \leq y \quad x > 5 \quad x + y \leq 20 \quad y \neq 10$$

Then a possible model would be $m_1 = \{x = 6, y = 6\}$*. An other model would be* $m_2 = \{x = 6, y = 7\}$*. However If we change the third rule to* $x + y \leq 10$ *there is no model to the problem and we would receive the unsatisfiable core* $c = \{x \leq y,\ x > 5,\ x + y \leq 10\}$*.*

Since for Definition 2.3.6 the existence of a model is the crucial information, the model which should be derived is arbitrary among the set of possible models.

Further knowledge about SMT-Problem solving can be gathered from the lecture "Introduction to Satisfiability Checking" or the SMT-RAT toolbox for Strategic and Parallel SMT Solving by Prof. Dr. Erika Ábrahám and her team at the *RWTH Aachen University* [CKJ+15].

Chapter 3

Geometric Non-Termination

Now that all preliminaries are stated we can start to take a look at how the approach works within AProVE. To find a *geometric nontermination argument* and so prove nontermination we use AProVE to generate an int-TRS of a given program, which is defined within Section 1.2. Based on the calculated int-TRS we derive the *STEM*, the *LOOP* and then generate an SMT-Problem using Definition 2.3.6 and compute a *geometric nontermination argument*, which would be a prove of nontermination, or state that no *geometric nontermination argument* can exist, which does not infer termination nor nontermination.

3.1 Derivation of the *STEM*

The derivation of the *STEM* is the first step in order to derive a *geometric nontermination argument*. As described in Section 2.3.2 the *STEM* defines the variables before iterating through the *LOOP*. Owed to the fact, that AProVE has to find the lasso to derive an int-TRS within the generated *Symbolic Execution Graph* one iteration through the *LOOP* will be calculated. Obviously this does not falsify the result. If it does not terminate i will still not terminate after one iteration and if it terminates after n iterations and we compute one it will still terminate after $n-1$ iterations.

Within the derivation of the *STEM* we distinguish between two cases discussed in the following sections.

3.1.1 Constant *STEM*

The constant stem is the easiest case to derive the *STEM* from. It has the form:

$$f_x \to f_y(c_1, \dots c_n) : | : TRUE$$

Here the $c_1, \dots c_n \in \mathbb{Z}$ denote constant numbers.

An example of a constant *STEM* is shown in Figure 3.1. The values of x can be directly read from the right hand side and need no further calculations.

$f_1 \to f_2(10, 2) : | : TRUE$

Figure 3.1: An example of a constant int-TRS rule to derive the *STEM*. The *STEM* in this case would be $\begin{pmatrix} 10 \\ 2 \end{pmatrix}$

3.1.2 Variable *STEM*

The more complex case is given if the start function symbol has the following form:

$$f_x \to f_y(v_1, \ldots v_n) : | : cond$$

where v_i $1 \leq i \leq n$ is either a constant term like in Section 3.1.1 or a linear update defined by the *cond* term. An example for such a *STEM* is shown in Figure 3.2. In order to derive terms in $\mathbb{Z}$ an SMT-Problem needs to be solved. We can compute the *Guard Matrix*, *Guard Constants*, *Update Matrix* and *Update Constants* of the start function symbol and use the *SMTFactory*, which is explained within Section 2.5, to create the assertions leading to either an assignment of the *STEM* x to a value or to a unsatisfiable core. Such a core would state, that the *while*-Loop would not hold after any assignment. If such a constellation entails termination or if it just does not entail nontermination needs further observance not provided in this paper.
The handling of equations within the guard term is described within Section 3.2.1 and underlies the same procedure regarding the stem.

$f_1 \to f_2(1 + 3 * v, 2) : | : v > 2$ && $8 < 3 * v$

Figure 3.2: An example of a variable int-TRS rule to derive the *STEM* from. In order to derive the *STEM*, an v fulfilling the conditions need to be found using an SMT-Solver. Since $v = 3$ is the first number provided by the *SMT-solver* in $\mathbb{Z}$ that satisfies the guards the *STEM* would be $\begin{pmatrix} 1 + 3 * 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$. Note that $v = 4$ would be equally permissible.

3.2 Derivation of the *LOOP*

The derivation of the *LOOP* is pretty straight forward applying Definition 2.3.2, Definition 2.3.3 to a looping rule and then computing *Iteration Matrix* and *Iteration Constants* using Definition 2.3.4, if no guards with equalities ("=") occur. Then we have to perform previous steps to apply the mentioned definitions.
Let f_x be the starting function symbol given by the int-TRS and r_i be a rule, with

$f_x \to f_y(v_1, \ldots, v_n) : | : cond_1$

then we take the in lexicographical order first rule r_l of the form

$f_y(v_1, \ldots, v_n) \to f_y(v'_1, \ldots, v'_n) : | : cond_2$

and compute the *Iteration Matrix* and *Iteration Constants* according to r_l. If we observe, that a second rule that could possibly chosen exists we encounter non-determinism, which is not supported so far.

3.2.1 The *Guard Matrix* and *Guard Constants*

The derivation of the *Guard Matrix* and *Guard Constants* can be achieved by applying the Definition 2.3.2 to the guards of the given rule r_l. For that we create G as the coefficient matrix. The size of G is determined by the arity of the function symbol of r_l and the number of guards not containing "=". In Section 3.2.1 we show the dealing with guards containing "=" with Example 4 as an example. The first step is to define the desired form of the guards. For that we introduce the *standard guard form*, the guards are given from the *Symbolic Execution Graph*, and the desired *strict guards form*.

Definition 3.2.1 (standard guard form)**.** *A guard g is in standard guard form iff $g := \varphi \circ c$, with φ in standard linear integer form and $\circ \in \{<, >, \leq, \geq, =\}$.*
A condition to a rule cond is in standard guard form iff

$$cond = \{g | g \text{ guard}, g \text{ is in standard guard form}\}$$

The condition given be the *Symbolic Execution Graph* is one rule r , which represents a set G in standard guard form as

$$r = \&\&(g_1, (\&\&(\dots, (\&\&(g_{n-1}, g_n)) \dots))),$$

where $g_i \in G$.
The easiest way to retrieve the guards g_i is by using Algorithm 1.

Algorithm 1 Retrieving a set of guards G from a rule r of the form stated in Section 3.2.1

function COMPUTEGUARDSET(Rule r) ▷ r has to be a rule representing a *cond*-term
  Stack $stack \leftarrow r$
  Set $guards$
  while $!stack.isEmpty()$ **do**
    $item \leftarrow stack.pop$
    if item is of the form $\&\&(x_1, x_2)$ **then** ▷ break up concatenation of two guards
      add x_1 and x_2 to $stack$ ▷ and add them individually
    else
      add $item$ to $guards$ ▷ if it is no concatenation it is a single guard
    end if
  end while
  return $guards$
end function

So we get a set $G = \{g \mid g \text{ is in standard guard form}\}$. Now we want to compute the desired form, the *strict guard form*, from which we can derive the *Guard Matrix* and *Guard Constants*.

Definition 3.2.2 (strict guard form)**.** *A guard g is in strict guard form iff $g := \varphi \leq c$, with φ in standard linear integer form and $c \in \mathbb{Z}$ subtracted by a possible constant c_φ previously included by φ*

To transfer a guard from standard guard form to strict guard form we have to apply the two following steps:

1. **rewrite equations**
   If the guards contain a guard with the symbol "=" we have to rewrite the "new" variable. To define, which are new variables and substitute these, we perform the following algorithm:

Algorithm 2 The algorithm to handle equalities that introduce new variables within the guards.

function FILTEREQUALITIES(G) ▷ G is in standard guard form
  $V_{left} = \{v \mid$ the left hand side of the rule contains $v\}$
  $V_{right} = \{v \mid$ the right hand side of the rule contains $v\}$
  $V_{sub} = V_{right} - V_{left}$
  define substitution $\theta = \{\}$
  while $V_{sub} \neq \emptyset$ **do**
    select $s \in V_{sub}$
    select $g_s \in \{g \in G \mid g$ contains " = "$\}$ ▷ Should only be one guard
    remove g_s from G
    rewrite g_s to the form $s = \psi$
    $\theta = \theta\{s/\psi\}$ ▷ update substitution
    for all $g \in G$ **do** ▷ apply Substitution
      $g = \theta g$
    end for
    remove s from V_{sub}
  end while
  return G
end function

The result of that is a set G', which satisfy the condition of occurring variables.

2. **normalizing**
Given the new set G' we have to normalize the inequalities to achieve strict guard form. This is done with two steps:

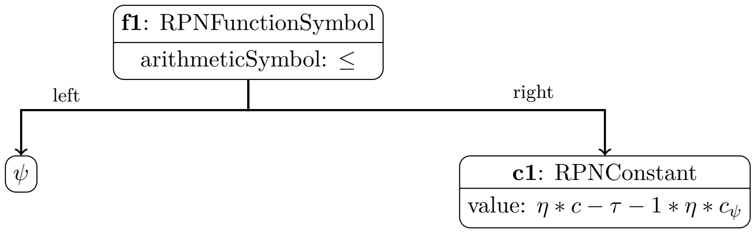
   (a) rewrite a guard g_i of the form $g_i \Leftrightarrow \psi + c_\psi \circ c$, where $\circ \in \{<, >, \leq, \geq\}$ to the form $\eta * \psi + \eta * c_\psi \leq \eta * c - \tau$ depending on $\circ$.

$\circ$	η	τ	$\eta * \psi + \eta * c_\psi \leq \eta * c - \tau$
$<$	1	1	$\psi + c_\psi \leq c - 1$
$>$	-1	1	$-\psi - c_\psi \leq -c - 1$
$\leq$	1	0	$\psi + c_\psi \leq c$
$\geq$	-1	0	$-\psi - c_\psi \leq -c$

   η is the indicator of inverting the guard to convert $\geq$ $(>)$ to $\leq$ $(<)$
   τ can be seen as the subtraction of 1 to receive the $\leq$ instead of a $<$.

   (b) transfer the c_ψ to the right side to only contain one constant term located on the right side. So the final form is $\eta * \psi \leq \eta * c - \tau - 1 * \eta * c_\psi$, where $\eta * c - \tau - 1 * \eta * c_\psi$ is constant term and ψ is in standard linear integer form without a constant term.

After that every guard is in strict guard form. So all we have to do in order to now compute the *Guard Matrix* and the *Guard Constants* is to apply Algorithm 3, to determine the coefficients stored in the *Guard Matrix*, and determine the constant terms.
Regarding the normalization, which is implemented on the *Reverse Polish Notation Tree*, we could apply Algorithm 4 to compute the constant terms, but also we can use the normalized guard to compute the constant term within linear time. The implementation of the transformation guarantees the following form for a guard in strict guard form:



So the constant term can simply be read off from the right child-node of the *RPNFunctionSymbol* "$\leq$", neglecting the left ψ-term.

Example 4. *This example is based in the* int-TRS *from Figure 2.3. Regarding the* int-TRS *we have the guard-term:*

$$v_1 + v_2 > 3 \,\&\&\, v_1 > 6 \,\&\&\, 3 * v_1 > 20 \,\&\&\, 5 + v_3 = 2 * v_2 \,\&\&\, v_3 < -10$$

which lead to the set G using Algorithm 1:

$$\{v_1 + v_2 > 3,\ v_1 > 6,\ 3 * v_1 > 20,\ 5 + v_3 = 2 * v_2,\ v_3 < -10\}$$

Starting with Algorithm 2 to handle equalities:

$$\begin{aligned}
&\textit{(line 2-4)} && \text{We compute } V_{left} = \{v_1, v_2\},\ V_{right} = \{v_1, v_2, v_3\} \text{ so } V_{sub} = \{v_3\}\\
&\textit{(line 5)} && \text{begin with } \theta = \{\}\\
&\textit{(line 7,8)} && \text{Since obviously } V_{sub} \neq \emptyset \text{ we select } s = v_3 \text{ and select } g_s \Leftrightarrow 5 + v_3 = 2 * v_2\\
&\textit{(line 9,10)} && g_s \text{ rewritten to the form } s = \psi \text{ then follows with } v_3 = 2 * v_2 - 5\\
&\textit{(line 11)} && \theta = \theta\{s/2 * v_2 - 5\} = \{s/2 * v_2 - 5\}\\
&\textit{(line 12-15)} && G = \{v_1 + v_2 > 3,\ v_1 > 6,\ 3 * v_1 > 20,\ 2 * v_2 - 5 < -10\}\\
&\textit{(end)} && \text{Since } V_{sub} = \emptyset \text{ return } G
\end{aligned}$$

Starting with the normalization:
Applying the stated rule for the different inequation-signs we receive the new guards:

$$\{-1 * v_1 + -1 * v_3 \leq -4,\ -1 * v_1 \leq -7,\ -3 * v_1 \leq -21,\ 2 * v_2 \leq -6\}$$

Note that the writing of for example $-1 * v_1$ is wanted in order to be able to neglect the case of for example $-2 * -v_1$ if -2 gets multiplied.

From that we apply Algorithm 3 and receive the Guard Matrix $G = \begin{pmatrix} -1 & -1 \\ -1 & 0 \\ -3 & 0 \\ 0 & 2 \end{pmatrix}$. *Also using the stated structure details we can derive the Guard Constants* $g = \begin{pmatrix} -4 \\ -7 \\ -21 \\ -6 \end{pmatrix}$

3.2.2 The *Update Matrix* and *Update Constants*

The *Update Matrix* and *Update Constants* can be derived quite easily. The updates within the function symbol neither contain an equation nor an inequation sign. Therefore no new variables can be initialized. Since it is still possible, that some of the, within the guard part instantiated, variables appear within the update we have to apply the final set of substitutions θ from Algorithm 2 to the linear update.

Example 5. *This example is based in the int-TRS from Figure 2.3 in combination with the Example 4 providing V_{sub} and θ.*
Since the set of substitutions $\theta = \{v_3/2 * v_2 - 5\}$ is not empty and within the update given by

$(3 * v_1 + v_2,\ v_3)$

contains $v_3 \in V_{sub}$ we have to apply the substitution and receive:

$(3 * v_1 + v_2,\ 2 * v_2 - 5)$

After restraining the updates to only mention the desired variables we can introduce Algorithm 3 as a procedure to compute the coefficient of a given variable. It performs a recursive search on the tree and uses the standard linear integer form definition that a coefficient is always the left child of the multiplication with it's corresponding variable. The procedure works like the following:

Algorithm 3 Derivation of a coefficient within an *Reverse Polish Notation Tree*

```
function getCoefficient(query)
    if node == query then                              ▷ query is a variable name
        return 1
    else if node does not contain query then           ▷ tree does not contain query
        return 0
    end if

    if node represents PLUS then                       ▷ Choose the subtree containing the query
        if left side contains query then
            return getCoefficient(query)
        else
            return getCoefficient(query)
        end if
    end if
    if node represents TIMES then                      ▷ Retrieve value
        if node.right == query then
            return node.left.value
        end if
    end if
end function
```

An example derivation of a factor using Algorithm 3 is shown in Figure 3.3.

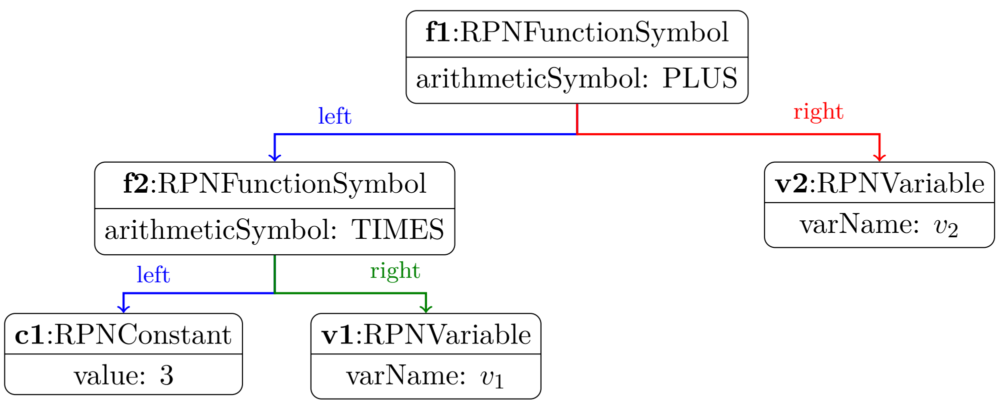


Figure 3.3: An example of deriving the coefficient of a given formula and a variable as query. This example uses the *Reverse Polish Notation Tree* of Figure 2.5 and variable v_1 as the query.

The red-arrow stands for the neglected right subtree of the root node, which can be neglected because the query is not contained. The blue-arrows show the path to the subtree further investigated. The green-arrow determines, that the right child node is the query so the left child node has to be the coefficient. Since the underlying update is in standard linear integer form the left subtree has to be a *RPNConstant*.

The *Update Constants* can be derived by an simplification of Algorithm 3, since we only have to retrieve the constant term within the tree. The corresponding derivation is given by Algorithm 4.

Algorithm 4 Derivation of a constant term within an *Reverse Polish Notation Tree*

function GETCONSTANTTERM
    if this is a constant **then**
        return this.value
    end if

    $flip \leftarrow 1$
    if this represents MINUS **then** ▷ flip result in case of prev. negation
        $flip \leftarrow -1$
    end if
    if this represents sth. $\neq$ TIMES **then**
        $left \leftarrow left.getConstantTerm()$ ▷ recursive calls
        $right \leftarrow right.getConstantTerm() * flip$
        return $left + right$
    end if
end function

An example of a constant term using Algorithm 4 can be found in Figure 3.4.

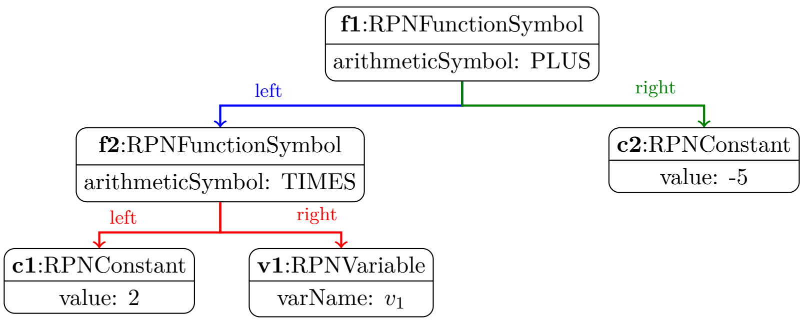


Figure 3.4: An example derivation of a constant term in the second variable update of the example in Example 5. Here red stands for neglected paths, blue stands for considered paths/recursive calls, and green stands for a found constant term.

Since a constant $c < 0$ can stored in a constellation shown in Figure 3.5 we consider a variable $flip$ to store a sign change occurring for a subtraction. Knowing that the standard linear integer form is used all occurs of a multiplication can be neglected.
Through the standard linear integer form one of the recursive calls has to be 0 since only one constant term exists.

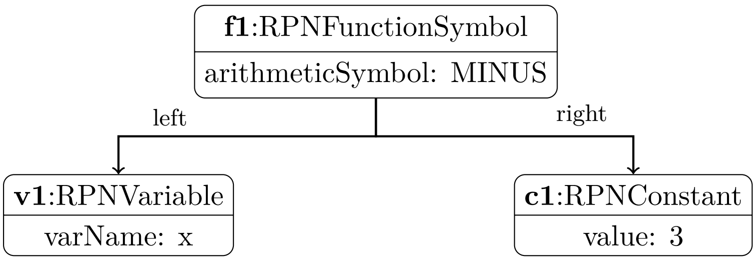


Figure 3.5: The *Reverse Polish Notation Tree* of the term $x - 3$, where the *flip* of Algorithm 4 has to be used. This constellation can not be universally neglected. The value recursively found for the constant term would be $(-1) * 3 = -3$

Using Algorithm 3 and Algorithm 4 one can derive the *Update Matrix* $U \in \mathbb{Z}^{n \times n}$ and *Update Constants* $u \in \mathbb{Z}^n$ for a rule r_j of the form

$$r_j := f_y(v_1, \dots v_n) \to f_y(v_1', \dots v_n') : | : cond$$

so that the following holds:

$$U \times \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + u = \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix}$$

3.2.3 The Iteration Matrix

The *Iteration Matrix* and *Iteration Constants* are a composition of the previously derived *Iteration-* and *Guard Matrix* respectively *Iteration-* and *Guard Constants*.
As stated in Definition 2.3.4 the *Iteration Matrix* and *Iteration Constants* can be computed as

$$A = \begin{pmatrix} G & \mathbf{0} \\ M & -I \\ -M & I \end{pmatrix} \text{ and } b = \begin{pmatrix} g \\ -u \\ u \end{pmatrix} \text{ [LH14]}$$

Given G, g, U and u computing A and b is simply inserting and creating a matrix $\mathbf{0} \in \{0\}^{m \times n}$ and identity-matrix $I \in \{0, 1\}^{n \times n}$, where n is the number of distinct not newly introduced variables and m the number of guards without equations.

3.3 Derivation of the *SMT*-Problem

The existence of a *geometric nontermination argument* is checked using an SMT solver, presented in Section 2.5, which will either give us a model satisfying the constraints or proof the non existence by giving an unsatisfiable core.
The constraints the SMT solver has to fulfil are the four criteria mentioned within Definition 2.3.6, which are non-linear. So the satisfiability of these is decidable. Since we derive the deterministic update as *Update Matrix* we can further compute it's eigenvalues and assign these to $\lambda_1, \dots \lambda_k$, receive linear constraints and thus can decide existence efficiently. [LH14].
So the next step in order to proof non termination is to compute the eigenvalues of the *Update*

Matrix. This is done by the *Apache math3* library [1] because of performance reasons. Computation of such matrices can be very costly if programmed inefficiently. After computing the eigenvalues, we have set values for the *STEM* x and $\lambda_1, \dots \lambda_k$ as constant values.
Using the *SMTFactory*, which offers methods to create the within Section 2.5 and Figure 2.6 stated structure, we are able to create assertions and add them to the *SMT-solver*, such that the following holds:

If the *SMT-solver*, with assertions $a_1^p, \dots a_n^p$ created from program p,has a model m then m defines variables $y_1, \dots, y_k$ and μ_1, μ_{k-1} within $\mathbb{N}$ such that $(x, y_1, \dots, y_k, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_{k-1})$ is a *geometric nontermination argument*.

3.3.1 The Domain Criteria

$$\text{(domain)} \quad x, y_1, \dots, y_k \in \mathbb{R}^n, \ \lambda_1, \dots \lambda_k, \mu_1, \dots \mu_{k-1} \geq 0$$

(see: Definition 2.3.6)

The *Domain Criteria* for x and $y_1, \dots y_k$ are trivial, because at no point of computation we would consider a vector $v \in \mathbb{C}$. The arity of x is set within the derivation of the *STEM* (see: Section 3.1) and sets the starting values for the n-variables. The arity of every y_i is determined within the assertions of the *Point Criteria* and the *Ray Criteria*.
Therefore this criteria adds no further assertions towards the *SMT-solver*.

3.3.2 The Initiation Criteria

(init) x represents the *start term* (*STEM*)

(see: Definition 2.3.6)

The *Initiation Criteria* is quite trivial to mention within the *SMT-solver*, since we defined the *STEM* x within Section 3.1 to be exactly the *start term*.
So this criteria also adds no further assertions towards the *SMT-solver*.

3.3.3 The Point Criteria

$$\text{(point)} \quad A \begin{pmatrix} x \\ x + \sum_i y_i \end{pmatrix} \leq b$$

(see: Definition 2.3.6)

The *Point Criteria* is the first criteria to add assertions towards the *SMT-solver*.
Since within the *Iteration Matrix* A the *Update Matrix* U is contained twice with a different sign the *Iteration Matrix* creates ,through the *Point Criteria* exactly, opposite signed rules for the last $2n$ rows. This means that, even though within the *Point Criteria* the relation is $\leq$, the last $2n$ have to fulfil the equality of the rows.

[1] the mentioned method can be found under [Apa17]

Let $s \in \mathbb{R}^n$ for $1 \leq i \leq n$ be $s_i = x_i + \sum_j (y_j)_i$, where $(y_j)_i$ denotes the i-th entry of y_j. Then the *Point Criteria* can be rewritten to:

$$A \begin{pmatrix} x \\ s \end{pmatrix} \leq b$$

$$\Leftrightarrow \begin{pmatrix} & G & & 0 & \dots & 0 \\ a_{1,1} & \dots & a_{1,n} & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} & 0 & \dots & -1 \\ -a_{1,1} & \dots & -a_{1,n} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & \dots & -a_{n,n} & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_n \end{pmatrix} \leq \begin{pmatrix} g \\ -u_1 \\ \vdots \\ -u_n \\ u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$\Rightarrow Gx \leq g$, which means that the guards have to hold. Note that s_1 to s_n get multiplied with 0 so don't need to be considered. Also the following has to hold:

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} & 0 & \dots & -1 \\ -a_{1,1} & \dots & -a_{1,n} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & \dots & -a_{n,n} & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_n \end{pmatrix} \leq \begin{pmatrix} -u_1 \\ \vdots \\ -u_n \\ u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{1,1} * x_1 & \dots & a_{1,n} * x_n & -1 * s_1 & \dots & 0 * s_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} * x_1 & \dots & a_{n,n} * x_n & 0 * s_1 & \dots & -1 * s_n \\ -a_{1,1} * x_1 & \dots & -a_{1,n} * x_n & 1 * s_1 & \dots & 0 * s_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} * x_1 & \dots & -a_{n,n} * x_n & 0 * s_1 & \dots & 1 * s_n \end{pmatrix} \leq \begin{pmatrix} -u_1 \\ \vdots \\ -u_n \\ u_1 \\ \vdots \\ u_n \end{pmatrix}$$

By looking closely one can see that for every line l_i $1 \leq i \leq n$ with

$$l_i^{\text{left}} \leq l_i^{\text{right}}$$

there is a rule l_{i+n} with

$$-l_i^{\text{left}} \leq -l_i^{\text{right}},$$

which can be rewritten as n rules of the form:

$$l_i^{\text{left}} = l_i^{\text{right}}$$

So using the *SMTFactory* we create such variables s_i and add the n assertions determined above. The guards can be neglected, since if they are constant there are no guards and if there are, we derive the *STEM* to fulfil these, so adding them would not give any advantage.
Since variable vectors are represented as a *Reverse Polish Notation Tree* we can use an implemented method to calculate the multiplication, normalize the outcome and parse the *Reverse Polish Notation Tree* into an assertion all featured by the *SMTFactory*.
The assertion ensuring that the new variables s_i are the sum of the i-th value of the y's is added

within Section 3.3.5.

3.3.4 The Ray Criteria

$$\text{(ray)} \quad A\begin{pmatrix} y_i \\ \lambda_i y_i + \mu_{i-1} y_{i-1} \end{pmatrix} \leq 0 \text{ for all } 1 \leq i \leq k$$

(see: Definition 2.3.6)

The *Ray Criteria* is the hardest criteria in terms of asserting, because of it's way of computation. The computation can be split into two parts on it's own.

$i = 1$:

For $i = 0$ the second addend $\mu_{i-1}y_{i-1}$ is equal to 0, because of Definition 2.3.6 that $y_0 = \mu_0 = 0$. So with λ_1 being the first eigenvalue of the *Update Matrix* we get that $A\begin{pmatrix} y_1 \\ \lambda_1 y_1 \end{pmatrix} \leq 0$.
Through A and the *Domain Criteria* we know, that every $y_i \in \mathbb{R}^n$ so we add n new variables $y_{1,i}$, such that $y_1 = \begin{pmatrix} y_{1,1} \\ \vdots \\ y_{1,n} \end{pmatrix}$, multiply the *Update Matrix A* with the new vector regarding the substitution and create an assertion per row using the *SMTFactory*, the *IntegerRelationType LE* (denotes: less than or equal) and as the right hand side constant a 0.

$i > 1$:

Since we don't have any concrete values for any y_i or μ_i so far the solving of the problem with the term $\mu_{i-1}y_{i-1}$ is not linear and therefore the computation has to be either performed in *quantifier free non-linear integer arithmetic* or iterated over possible entry's for the μ's.
In the implemented approach the *quantifier free non-linear integer arithmetic* is used. Even if it's generally undecidable there are implementations over finite domains semi-deciding the problems. [BDE+14] [GAB+16]
Further comment about the usage of QF_NIA can be found within Chapter 4.
With λ_i being the i-th eigenvalue of the *Update Matrix* we can compute the result of the multiplication as in the $i = 0$ case, but have to normalize the outcome using the basic distributive property in order to handle it within an *Reverse Polish Notation Tree* and correctly generate an assertion from it.

So for every step $i > 1$ we add n new variables $y_{i,n}$ such that $y_i = \begin{pmatrix} y_{i,1} \\ \vdots \\ y_{i,n} \end{pmatrix}$ and a new variable μ_{i-1} such that

$$\lambda_i y_i + \mu_{i-1} y_{i-1} \Leftrightarrow \lambda_i \begin{pmatrix} y_{i,1} \\ \vdots \\ y_{i,n} \end{pmatrix} + \mu_{i-1} \begin{pmatrix} y_{i-1,1} \\ \vdots \\ y_{i-1,n} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \lambda_i y_{i,1} + \mu_{i-1} y_{i-1,1} \\ \vdots \\ \lambda_i y_{i,n} + \mu_{i-1} y_{i-1,n} \end{pmatrix}$$

As in the other case we can simply compute the multiplication with the *Update Matrix A* , normalize the outcome and analogously create an assertion per row with the *SMTFactory*.

Note that $y_{i-1,n}$ represent the values of the previous step and therefore not only already exist, but also create a lattice of restrictions for the $y_{i,j}$ since the values depend highly on the previous values. At this point the relation of rewriting the problem as a geometric series, like it is done in the underlying paper [LH14], is quite comprehensibly.

3.3.5 Additional assertion

The final step of asserting needs to be done, because of the restriction of the variables from the *Ray Criteria* in Section 3.3.4 to the sum from the *Point Criteria* in Section 3.3.3.
The assertion, that needs to be added has the following form:

$$s_i = y_{1,i} + \cdots + y_{n,i}$$

This ensures that the values of y sum up to the values determined in the *Point Criteria*.

After the adding the *Additional assertion* from Section 3.3.5 the *SMT-solver* contains all the restrictions to compute a *geometric nontermination argument* or an unsatisfiable core for the given program.
If a *geometric nontermination argument* is found it is stored as an instance of the corresponding class and given to AProVE as a proof.

3.4 Verification of the Geometric Non-Termination Argument

An instance of a *geometric nontermination argument* can be rechecked giving the *Iteration Matrix* and *Iteration Constants* if all of the four criteria of Definition 2.3.6 by simply computing and checking if the conditions hold. In the whole chapter we work with the example used throughout the whole thesis with it's Java-code in Figure 2.2 its corresponding int-TRS in Figure 2.3. The in Chapter 3 computed *Iteration Matrix* and *Iteration Constants* based on the *Guard* and *Update Matrix/Constants* and also the Eigenvalues as the λ's.

The implemented technique provides a model m from the *SMT-solver* and it's assertions stated in Section 3.3. From the model m the technique filters the values and stores the in the *geometric nontermination argument*-class, which is given to AProVE as a witness. Such an *geometric nontermination argument*-object stores all variable declarations needed in order to revalidate itself to a given scenario. If one has an other technique of derivation it is possible to compute the matrices and set the values for an *geometric nontermination argument*-object and validate it, either proving it or mentioning, which criteria does not hold.

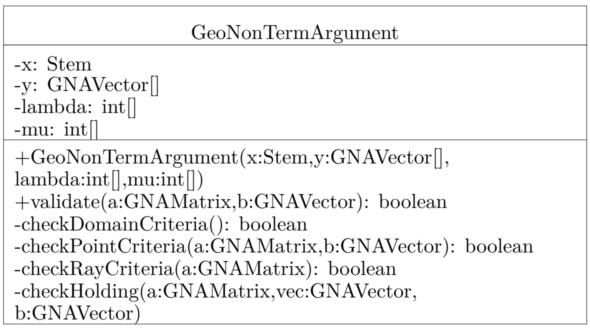


Figure 3.6: The Java-class of a *geometric nontermination argument* given to AProVE as a witness of nontermination

From Chapter 3 we get the following *Iteration Matrix A* and *Iteration Constants b*:

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -3 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} -4 \\ -7 \\ -21 \\ -6 \\ 0 \\ 5 \\ 0 \\ -5 \end{pmatrix}$$

The from the technique given witness is the *geometric nontermination argument* of size 2 with the following values:

STEM	y_1	y_2	λ_1	λ_2	μ_1
$\begin{pmatrix} 10 \\ -3 \end{pmatrix}$	$\begin{pmatrix} 9 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 8 \\ -8 \end{pmatrix}$	3	2	0

From that we can start and validate every criteria stated in Definition 2.3.6.

3.4.1 (domain-criteria) $x, y_1, \ldots, y_k \in \mathbb{R}^n$, $\lambda_1, \ldots \lambda_k, \mu_1, \ldots \mu_{k-1} \geq 0$

The validity of this criteria is given in Section 3.4 with $n = 2$. This should not be surprising, since we defined the vectors to this dimension and the λ's and μ's to be ≥ 0 in Section 3.3.

3.4.2 (init-criteria) x represents the *start term* (*STEM*)

Also the given *STEM* meets the conditions as already derived in Section 3.1.2.

3.4.3 (point-criteria) $A \begin{pmatrix} x \\ x + \sum_i y_i \end{pmatrix} \leq b$

Now we have to compute the sum of the y_i's and observe if the inequation holds. So with the values given in Section 3.4 we receive:

$$A \begin{pmatrix} x \\ x + \sum_i y_i \end{pmatrix} \leq b \Leftrightarrow A \begin{pmatrix} 10 \\ -3 \\ 10 + (9 + 8) \\ -3 + (0 + (-8)) \end{pmatrix} \leq b \Leftrightarrow A \begin{pmatrix} 10 \\ -3 \\ 27 \\ -11 \end{pmatrix} \leq b \Leftrightarrow \begin{pmatrix} -7 \\ -10 \\ -30 \\ -6 \\ 0 \\ 5 \\ 0 \\ -5 \end{pmatrix} \leq \begin{pmatrix} -4 \\ -7 \\ -21 \\ -6 \\ 0 \\ 5 \\ 0 \\ -5 \end{pmatrix}$$

Not only one can see, that every row on it's own satisfies the desired inequation, we can also observe the within Section 3.3.3 stated equality hold for the last $2n = 2 * 2 = 4$ rows. The point-criteria obviously holds.

3.4.4 (ray-criteria) $A\begin{pmatrix} y_i \\ \lambda_i y_i + \mu_{i-1} y_{i-1} \end{pmatrix} \leq 0$ for all $1 \leq i \leq k$

Because the *geometric nontermination argument* is of size 2 we have only 2 ray-criteria subformulas that need to be tested

r_1**: i=1**

$$A\begin{pmatrix} y_1 \\ \lambda_1 y_1 \end{pmatrix} \leq 0 \Leftrightarrow A\begin{pmatrix} 9 \\ 0 \\ 3*9 \\ 3*0 \end{pmatrix} \leq 0 \Leftrightarrow A\begin{pmatrix} 9 \\ 0 \\ 27 \\ 0 \end{pmatrix} \leq 0 \Leftrightarrow \begin{pmatrix} -9 \\ -9 \\ -27 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

r_2**: i=2**

$$A\begin{pmatrix} y_2 \\ \lambda_2 y_2 + \mu_1 y_1 \end{pmatrix} \leq 0 \Leftrightarrow A\begin{pmatrix} 8 \\ -8 \\ 2*8+0*9 \\ 2*(-8)+0*0 \end{pmatrix} \leq 0 \Leftrightarrow A\begin{pmatrix} 8 \\ -8 \\ 16 \\ -16 \end{pmatrix} \leq 0 \Leftrightarrow \begin{pmatrix} 0 \\ -8 \\ -24 \\ -16 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since all r_i $1 \leq i \leq 2$ are computed and checked that they hold also the ray-criteria holds.

Observing that all criteria hold the whole *geometric nontermination argument* is suitable to Definition 2.3.6 and therefore is a witness of nontermination regard the program stated in Figure 2.2.

Chapter 4

Evaluation and Benchmark

In this chapter we want to take a look at the implementation and evaluate, if the approach itself is useful in terms of applicable cases or if the approach works only on very exotic and uncommon preconditions.
Also we want to take a look at the benchmarks of the implementation, in terms of storage and computational efficiency.
Further we want to outline improvement possibilities of the implementation and problems within, where an efficient solution is not quite obvious.

4.1 Evaluation of the approach

The approach provides a sound and complete solution to specific type of programs. Given a tool like AProVE, which provides an int-TRS the further computation that has to be done can be solved quite efficient. Using an state of the art *SMT-solver* and the definition of λ_i to be the i-th eigenvalue the problem can also be resolved efficiently for given μ's. If the μ's are not given the problem is undecidable, which makes it still useful within AProVE, but not as strong as before.
Since the paper itself does not mention equalities within the guards the substitution of newly introduced variables as handled within Section 3.2.1 is necessary in order to apply the definition. Such a substitution is very costly in computation time and also highly error prone.

4.2 Benchmarks

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4.3 Possible Improvement of the Implementation

The implementation of the approach is fully functional under the circumstances mentioned, like for example the defined structure in Section 2.3.2. Nevertheless also this implementation has certain cases in, which it does not perform as efficient as it could, or where it can be improved in terms of applicability. So here we state the possible improvements of the implementation to make it universally more useful and therefore stronger or more efficient.

4.3.1 ***SMT-solver*** logic

As already stated in Section 3.3 the problematic of the μ's can lead to a shift into undecidability, since the solvating of variable multiplication on integers (*quantifier free non-linear integer arithmetic*) is undecidable. Also mentioned in Section 3.3 there are a bunch of approaches, which lead to semi-decidability and therefore to the possibility to still use the variable multiplication within the problem if the μ's can be restricted to a finite domain.
A possible improvement could be an iteration over different values of the μ's. The number of problems, that have to be solved would be blow up, but the problem itself would always be decidable.
A reliable case-study of a large set of examples could underline the necessity of the iteration, since we wouldn't be able to derive a *geometric nontermination argument* using the *quantifier free non-linear integer arithmetic*. It could also lead to the overhead of computational cost using the iterative method, which can be useful if the problem does not have any time restrictions of the deciding process, but is not suitable within the competitions AProVE participates, like the *International Competition of Termination Tools* [1] or *International Competition on Software Verification* [2]. [I217]
The *Termination Competition 2017*, which is organized by the *International Competition of Termination Tools*, for example has a time limit of 300 seconds and only allows 4 core usage, which makes an iterative method very costly. [Wik17b]

4.3.2 int-TRS program structure

As stated in Section 2.3.2 we restrict this implementation to the form

$$
\begin{array}{lll}
f_x & \to f_y(v_1, \dots v_n) & : | : cond_1 \\
f_y(v_1, \dots v_n) & \to f_y\ (v'_1, \dots v'_n) & : | : cond_2
\end{array}
$$

which obviously is a restriction, because int-TRS's of the form

$$
\begin{array}{lll}
f_x & \to f_y(v_1, \dots v_n) & : | : cond_1 \\
f_{y_1}(v_1, \dots v_n) & \to f_{y_2}\ (v'_1, \dots v'_n) & : | : cond_2 \\
\vdots & & \\
f_{y_k}(v_1, \dots v_n) & \to f_{y_1}\ (v'_1, \dots v'_n) & : | : cond_{k+1}
\end{array}
$$

could possibly considered using the method k times. Analysing such int-TRS would make the implementation much stronger in terms of proofing.
An other possible variation of the considered int-TRS could be like the following

$$
\begin{array}{lll}
f_x & \to f_y(v_1, \dots v_n) & : | : cond_1 \\
f_{y_1}(v_1, \dots v_n) & \to f_{y_2}(v'_1, \dots v'_m) & : | : cond_2 \\
f_{y_2}(v_1, \dots v_m) & \to f_{y_1}(v'_1, \dots v'_n) & : | : cond_3
\end{array}
$$

where $m \neq n$, but the values v'_i $1 \leq i \leq m$ is computed as a linear update of the values v_j $1 \leq j \leq n$.

[1] further information: `http://termination-portal.org/wiki/Termination_Competition`
[2] further information: `https://sv-comp.sosy-lab.org/2017/`

These are only two alternations of the considered structure, which would be also recommended to implement in order to create a more universal applicable method.

4.3.3 *Reverse Polish Notation Tree*

Within Section 2.4 we defined the *Reverse Polish Notation Tree*, on which we base the arithmetic computations and statements. AProVE also has a tree structure it handles such statements in. The *Reverse Polish Notation Tree* structure exists mainly because of two reasons:

1. the structure AProVE bases these statements on is much more complex, but also much more powerful, which made programming a lot more difficult. Parsing it into a tree, which can only contain elements expected to be in such expressions not only works equivalently it also prevents errors if AProVE's structure gets extended or changed. The *RPNTreeParse* handles the conversion and therefore can be seen as an adapter, which filters every int-TRS that must not occur in *Geometric Nontermination Analysis* as stated in this thesis.

2. many algorithms are difficult to encode if not programmed recursively. Since coding in the already stated classes was no option, and inheritance would not work because of accessing problems, creating my own structure was a simple work-around.

The examples worked with have been small enough to not create any problems with the conversion and possible less efficient methods, but if applied to huge problems a converting of the approach to work on the structure AProVE proposes would be the better way.

Chapter 5

Related Work

The first related source that should be mentioned is the research of Jan Leike and Matthias Heizmann. These two researchers from the Australian National University and University of Freiburg wrote the *geometric nontermination argument*-Paper this paper relies on. Their definition and proof of the *geometric nontermination argument* in combination with the derivation on lasso-programs is the base of all computation explained throughout Chapter 2 and Chapter 3. [LH14]
In that context also Ultimate LassoRanker should be mentioned. A tool for termination and nontermination arguments for linear lasso programs by Jan Leike and Matthias Heizmann also containing *geometric nontermination argument*'s. [LH17]
Further A. Tiwari considered linear loop programs not over the naturals, but over the reals. For such programs he proofed that they in fact are decidable in terms of termination if they met the condition of only strict guards. [Tiw04]
An other interesting related work is written by R. Rebiha et al., which contains the generalization of the eigenvalues not only to be natural but also can be within the reals. [RMM14] Also J. Ouaknine generalized from the basic approach by mentioning integer lasso programs, where the corresponding *Update Matrix* is diagonalizable. [OPW14]
An extension of the *geometric nontermination argument* approach can be found within the work of C. David et al. providing a technique also applicable to non deterministic programs. It uses a constraint-based synthesis of recurrence sets, which are apart from others defined by A. Tiwari in [Tiw04]. Also it can work with second order theory for bit vectors. These can be used to find nonterminating lassos, which do not have a *geometric nontermination argument*. The downside of such an extension is the problem of solving an $\exists\forall\exists$-constraint. [DKL15] [LH14]

Last but not least the often mentioned tool AProVE is to be mentioned. Using a verity of techniques including lasso's within a range of proofing attempts this tool provides a series of promising techniques implemented.

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