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Introduction to Model Checking (Summer Term 2018)

— Solution 5 (due 4th June) —

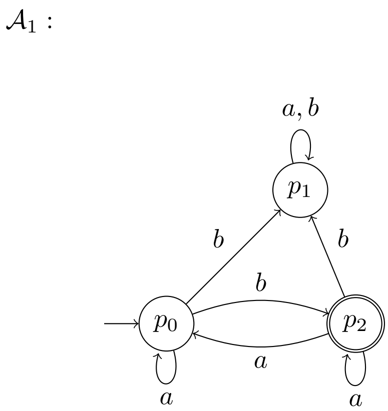
General Remarks

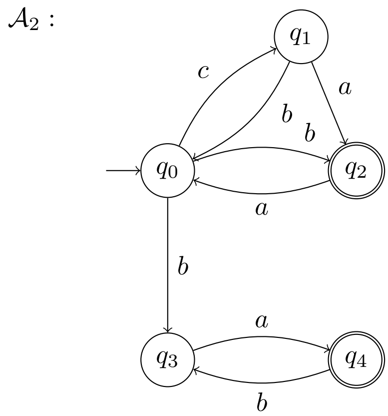
- The exercises are to be solved in groups of *three* students.
- You may hand in your solutions for the exercises just before the exercise class starts at 12:15 or by dropping them into the "Introduction to Model Checking" box at our chair *before 12:00*. Do *not* hand in your solutions via L2P or via e-mail.

Exercise 1★ (2+2+3 Points)

In the following we have $\Sigma = \{a, b, c\}$.

(a) Consider the following NBA $\mathcal{A}_1, \mathcal{A}_2$.





For each NBA $\mathcal{A}_i$ give an ω-regular expression α_i which characterizes the language accepted by the NBA, i.e., $\mathcal{L}_\omega(\alpha_i) = \mathcal{L}_\omega(\mathcal{A}_i)$.

(b) Consider the following descriptions of ω-regular languages $\mathcal{L}^i_\omega$.

(i) $\mathcal{L}^1_\omega$: a occurs infinitely many times. In between two successive a either

- an odd number of b and no c, or
- an even number of c and no b

has to occur.

(ii) $\mathcal{L}^2_\omega$:

- If c occurs only finitely many times then a and b occur infinitely many times.
- If c occurs infinitely many times then a and b occur only finitely many times.

For each language $\mathcal{L}^i_\omega$ give an NBA $\mathcal{B}_i$ which accepts the language.

(c) Consider again the languages from (b). For each language $\mathcal{L}^i_\omega$ give a DBA $\mathcal{D}_i$ which accepts the language. If you can not find a DBA, justify why there exist no DBA accepting the language.

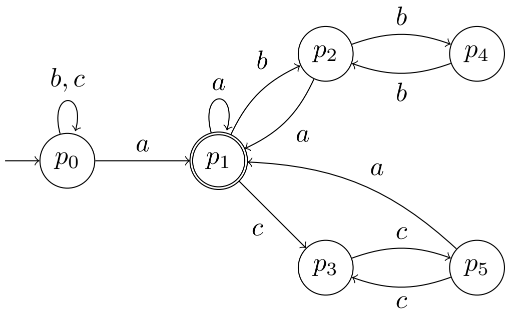
Solution:

(a) The ω-regular expressions are as follows:

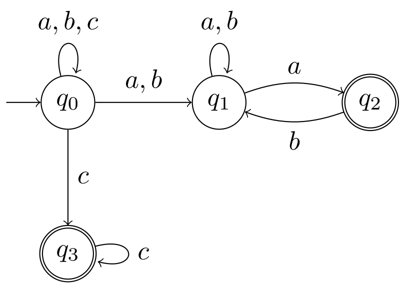
- For $\mathcal{A}_1$: $\alpha_1 = (a^*ba)^\omega + (a^*ba)^*ba^\omega$.
- For $\mathcal{A}_2$: $\alpha_2 = \big((cb)^*(caa + ba)\big)^\omega + \big((cb)^*(caa + ba)\big)^* b(ab)^\omega$.
  The second part of the ω-regular expression is actually already incorporated in the first part. Therefore, the following is also a valid solution: $\alpha_2' = \big((cb)^*(caa + ba)\big)^\omega$.

(b) Note that 0 is also an even number.

$\mathcal{B}_1$:

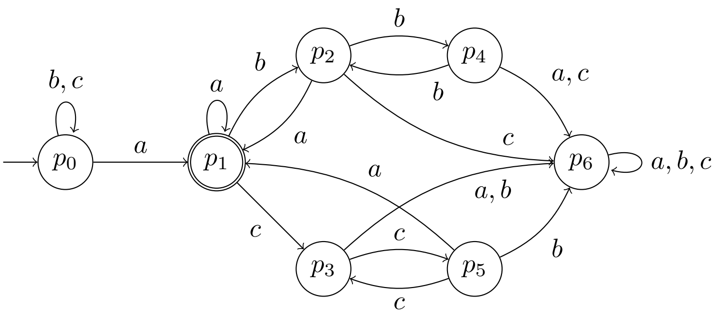


$\mathcal{B}_2$:



(c) (i) The NBA $\mathcal{B}_1$ from (b) is already a DBA. We can further transform this into a non-blocking DBA $\mathcal{D}_1$ by adding a trap state.

$\mathcal{D}_1$:



(ii) The language $\mathcal{L}_\omega^2$ cannot be accepted by a DBA. Intuitively, the decision whether c will be seen finitely or infinitely many times can be delayed for arbitrary many steps. Thus, this decision can not be resolved deterministically.

In the following we show in greater detail that there is no DBA accepting the language $\mathcal{L}_\omega^2$. Assume there exists a DBA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega^2$.

First, we define the transitive closure δ^* of δ as $\delta^*(s, wa) = \delta(\delta^*(s, w), a)$ and $\delta^*(s, \varepsilon) = s$ for $s \in Q$, $a \in \Sigma$, $w \in \Sigma^*$ and ε the empty word.

Consider the word $w_1 = c^\omega$. Since $w_1 \in \mathcal{L}_\omega^2$, there exists a (uniquely determined) accepting state $q_1 \in F$ and $n_1 \geq 1$ such that $\delta^*(q_0, c^{n_1}) = q_1 \in F$. Since $w_2 = c^{n_1}ac^\omega \in \mathcal{L}_\omega^2$, there also exists an accepting state $q_2 \in F$ and $n_2 \geq 1$ such that $\delta^*(q_0, c^{n_1}ac^{n_2}) = q_2 \in F$. We continue this process and get a sequence $n_1, n_2, n_3, \ldots$ of natural numbers and a sequence $q_1, q_2, q_3, \ldots$ of accepting states such that

$$\delta^*(q_0, c^{n_1}ac^{n_2}ac^{n_3}a \ldots c^{n_{i-1}}ac^{n_i}) = q_i \in F$$

Since there are only finitely many states, there exists $i < j$ such that

$$\delta^*(q_0, c^{n_1}ac^{n_2}a \ldots c^{n_i}) = \delta^*(q_0, c^{n_1}ac^{n_2}a \ldots c^{n_i}a \ldots c^{n_j})$$

Therefore, $\mathcal{A}$ has an accepting run on

$$c^{n_1}ac^{n_2}a \ldots c^{n_i}(ac^{n_{i+1}}a \ldots ac^{n_j})^\omega$$

However, this word has infinitely occurences of a and c and thus does not belong to $\mathcal{L}_\omega^2$. $\lightning$

Exercise 2 (3 Points)

Provide an example for a liveness property P_{live} that is *not* ω-regular. Show that P_{live} is indeed a liveness property and prove that P_{live} is not ω-regular.

Hint: Think about words of the form $\{a\}\{b\}\{a\}\{a\}\{b\}\{a\}\{a\}\{a\}\{b\}\ldots$.

Solution:

Let $\Sigma = \{a, b\}$ and $\sigma^k = \{a\}^k \{b\}$. Consider the following property over Σ:

$$P_{live} = \{A_0 A_1 A_2 \cdots \in (2^{\mathrm{AP}})^\omega \mid \exists i \geq 0 : A_i A_{i+1} A_{i+2} A_{i+3} A_{i+4} \cdots = \sigma^1 \sigma^2 \sigma^3 \ldots\}$$

We show that P_{live} is indeed a liveness property. Take any finite word $\rho \in (2^{\mathrm{AP}})^+$. We can extend ρ to $\rho\sigma^1\sigma^2\sigma^3 \cdots \in P_{live}$. Thus, $\mathit{pref}(P_{live}) = (2^{\mathrm{AP}})^+$ and P_{live} is indeed a liveness property.

We show that P_{live} is not ω-regular. Assume P_{live} is a ω-regular language. Then there exists an NBA $\mathcal{A}$ accepting the language, i.e., $\mathcal{L}_\omega(\mathcal{A}) = P_{live}$. From the lecture we know:

$$\mathcal{L}_\omega(\mathcal{A}) \neq \emptyset \text{ iff there exist finite words } x, y \in \Sigma^* \text{ s.t. } y \neq \varepsilon \text{ and } xy^\omega \in \mathcal{L}_\omega(\mathcal{A})$$

Thus, the word xy^ω is ultimately periodic. However, no word in P_{live} is ultimately periodic. $\lightning$

In the following we show that P_{live} is not ω-regular in greater detail. We use the transitive closure δ^* as defined in the solution to exercise 1(c)(ii).

Now assume P_{live} is ω-regular. Then there exists an NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ which accepts the language, i.e., $\mathcal{L}_\omega(\mathcal{A}) = P_{live}$. As $\mathcal{A}$ has a finite state space, for each word $w \in \mathcal{L}_\omega(\mathcal{A})$ there exists an accepting state $s_w \in F$ which is visited infinitely often. That means each word w has finite (non-empty) fragments u_w and v_w with $w = u_w v_w \ldots$, such that $s_w \in \delta^*(s_0, u_w)$ and $s_w \in \delta^*(s_w, v_w)$ for $s_0 \in Q_0$. Consider the word $w' = u_w(v_w)^\omega$. As the accepting state s_w is visited infinitely often for w' this word is also accepted by $\mathcal{A}$, i.e., $w' \in \mathcal{L}_\omega(\mathcal{A})$. However, the word w' is periodic and thus $w' \notin P_{live}$.

More explicitly we have to distinguish two cases for w':

1. $\{b\}$ is not contained in v_w. In that case $w' = u_w \{a\}^\omega$ and thus $w' \notin P_{live}$.
2. $\{b\}$ is contained in v_w. Let $n = |v_w|$ be the length of v_w. As $\{b\}$ is in v_w we have at most $n-1$ consecutive $\{a\}$ before the next $\{b\}$. But then A_n will never be contained in w' and thus $w' \notin P_{live}$.

Therefore $\mathcal{A}$ accepts words which are not in P_{live}, i.e., $\mathcal{L}_\omega(\mathcal{A}) \neq P_{live}$. $\lightning$

Exercise 3 (2+2+2+1 Points)

(a) Provide NBA $\mathcal{A}_1$ and $\mathcal{A}_2$ for the languages given by the ω-regular expressions $\alpha_1 = (AC + B)^* B^\omega$ and $\alpha_2 = (B^* AC)^\omega$.

(b) Apply the product construction to obtain a GNBA $\mathcal{G}$ with $\mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$.

(c) Transform the GNBA $\mathcal{G}$ into an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$.

(d) Justify, why $\mathcal{L}_\omega(\mathcal{G}) = \emptyset$ on the level of the GNBA $\mathcal{G}$.

Hint: For a GNBA $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ with at least one element in $\mathcal{F} = \{F_1, \ldots, F_k\}$. Let $\mathcal{A} = (Q', \Sigma, \delta', Q'_0, F')$ be an NBA with

- $Q' = Q \times \{1, \ldots, k\}$,
- $Q'_0 = Q_0 \times \{1\}$,
- $F' = F_1 \times \{1\}$, and

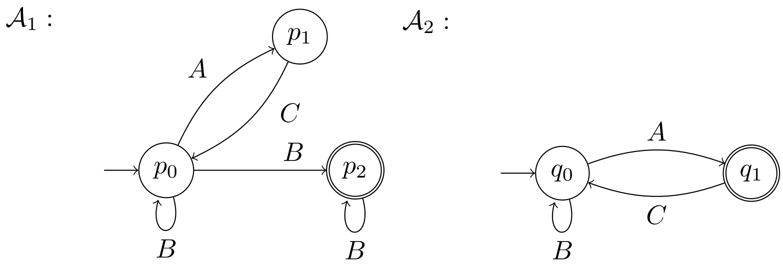
for all $A \in \Sigma$ it is

$$\delta'(\langle q, i\rangle, A) = \begin{cases} \{\langle q', i\rangle \mid q' \in \delta(q, A)\} & \text{if } q \notin F_i \\ \{\langle q', (i \mod k) + 1\rangle \mid q' \in \delta(q, A)\} & \text{if } q \in F_i. \end{cases}$$

Then $\mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{A})$.

Solution:

(a) Let $\Sigma = \{A, B, C\}$. The NBA $\mathcal{A}_1 = \left(Q_1, \Sigma, \delta_1, Q_{0,1}, \underbrace{F_1}_{\{p_2\}}\right)$ and $\mathcal{A}_2 = \left(Q_2, \Sigma, \delta_2, Q_{0,2}, \underbrace{F_2}_{\{q_1\}}\right)$ for the languages are depicted below:

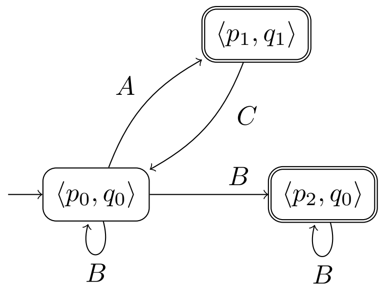


(b) Applying the product construction yields the following GNBA:

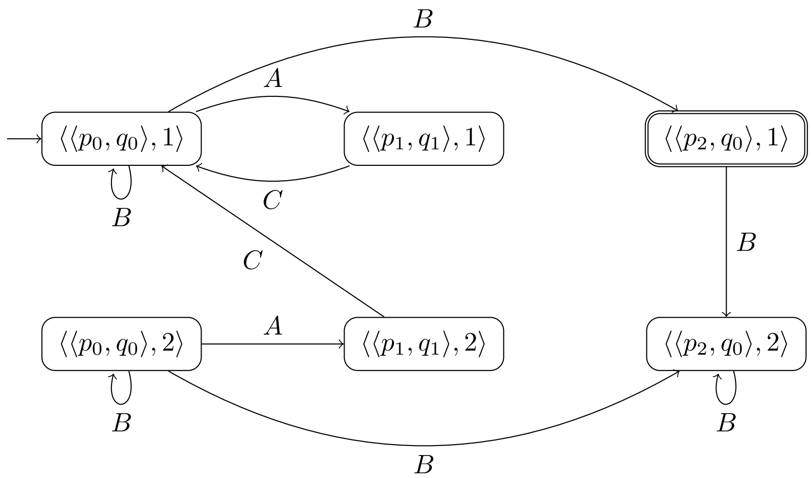
$$\mathcal{G} = (Q_1 \times Q_2, \Sigma, \delta, Q_{0,1} \times Q_{0,2}, \mathcal{F}) \text{ where}$$

- $\mathcal{F} = \{F_1 \times Q_2\} \cup \{Q_1 \times F_2\} = \{\{\langle p_2, q_0\rangle, \langle p_2, q_1\rangle\}, \{\langle p_0, q_1\rangle, (\langle p_1, q_1\rangle, \langle p_2, q_1\rangle\}\}$

The automaton $\mathcal{G}$ is as follows (only reachable states are depicted, all unreachable states have no outgoing transitions):



(c) Given $\mathcal{G}$, we construct an equivalent NBA $\mathcal{A}$ as follows:



The two states $\langle\langle p_0, q_0\rangle, 2\rangle$ and $\langle\langle p_1, q_1\rangle, 2\rangle$ are not reachable. They do therefore not influence the accepted languages and may be omitted.

(d) According to the acceptance condition of generalized Büchi automata, $\mathcal{G}$ accepts an input word α if and only if for each $F \in \mathcal{F}$ some state $s \in F$ is visited infinitely often. But as soon as (p_2, q_0) is visited, $(p_1, q_1) \in F_1$ is not reachable any longer.
Therefore $\mathcal{G}$ only accepts the empty language, i.e., $\mathcal{L}_\omega(\mathcal{G}) = \emptyset$.

The same can be seen on the NBA $\mathcal{A}$. From the lecture, we know that the language of an NBA is non-empty if and only if an accepting state p is reachable from the initial state that can reach itself via some non-empty word. The only accepting state in $\mathcal{A}$ is $\langle\langle p_2, q_0, 1\rangle\rangle$, which is reachable. However, once in this state, there is no way to return to it and, by the theorem, the language of $\mathcal{A}$ is empty.

Exercise 4 (1+2 Points)

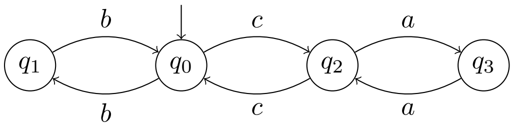
A nondeterministic Muller automaton is a quintuple $\mathcal{A} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ where Q, Σ, δ and Q_0 are as for NBA and $\mathcal{F} \subseteq 2^Q$. For an infinite run $\rho = q_0 q_1 q_2 \ldots$ of $\mathcal{A}$, let

$$\mathit{inf}(\rho) := \{q \in Q \mid \exists^\infty i \geq 0.\ q_i = q\}\,.$$

Let $\alpha \in \Sigma^\omega$.

$$\mathcal{A} \text{ accepts } \alpha \iff \text{exists infinite run } \rho \text{ of } \mathcal{A} \text{ on } \alpha \text{ s.t. } \mathit{inf}(\rho) \in \mathcal{F}.$$

(a) Consider the following Muller automaton $\mathcal{A}$ with $\mathcal{F} = \{\{q_2, q_3\}, \{q_1, q_3\}, \{q_0, q_2\}\}$:



Give the language accepted by $\mathcal{A}$ by means of an ω-regular expression.

(b) Show that every GNBA $\mathcal{G}$ can be transformed into a nondeterministic Muller automaton $\mathcal{A}$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{G})$ by defining the corresponding transformation.

Solution:

(a) First, we observe that there is no run in $\mathcal{A}$ that visits $\{q_1, q_3\}$ infinitely often, but visits the states $\{q_0, q_2\}$ only finitely often. Hence, there is no run that is accepting because of $\{q_1, q_3\}$.

Since both accepting sets $\{q_0, q_2\}$ and $\{q_2, q_3\}$ ensure that from some point on, only a or c, respectively, are seen, the language of the automaton can be represented as

$$\mathcal{L}_\omega((bb + c(aa)^*c)^*c.a^\omega + (bb + c(aa)^*c)^*.c^\omega)$$

(b) Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ be a GNBA with accepting sets $\mathcal{F} \subseteq 2^Q$. We can transform $\mathcal{A}$ to an NBA $\mathcal{A}' = (Q', \Sigma, \delta', Q'_0, F')$ $(F' \subseteq Q)$ as seen in the lecture. Let $\mathcal{A}'' = (Q', \Sigma, \delta', Q'_0, \mathcal{F}'')$ with $\mathcal{F}'' = \left\{F'' \in 2^Q \mid F'' \cap F \neq \emptyset\right\}$ be a nondeterministic Muller automaton. Then $\mathcal{A}''$ accepts the same language as $\mathcal{A}'$ (and therefore also $\mathcal{A}$) since we define all sets of states to be accepting as long as they contain at least one accepting state of the NBA, which effectively mimics the Büchi accepting condition.