Invariant

Let E be an LT property over AP.

E is called an invariant if there exists a propositional formula Φ over AP such that

$$E = \left\{ A_0 A_1 A_2 \ldots \in \left(2^{AP}\right)^\omega : \forall i \geq 0.\, A_i \models \Phi \right\}$$

Φ is called the invariant condition of E.

Safety properties

state that "nothing bad will happen"

- mutual exclusion: *never* $crit_1 \wedge crit_2$
- deadlock freedom: e.g., for dining philosophers

  never $\bigwedge_{0 \leq i < n} wait_i$
- German traffic lights:

  every red phase is preceded by a yellow phase
- beverage machine:

  no drink must be released if the user did not enter a coin before

  the total number of entered coins is never less than the total number of released drinks

Safety properties

state that "nothing bad will happen"

invariants: ⟵ "no **bad state** will be reached"

- mutual exclusion: *never* $crit_1 \wedge crit_2$
- deadlock freedom: *never* $\bigwedge_{0 \le i < n} wait_i$

other safety properties: ⟵ "no **bad prefix**"

- German traffic lights:
  every red phase is preceded by a yellow phase
- beverage machine:
  the total number of entered coins is never less than the total number of released drinks

Bad prefixes of safety properties

- traffic lights:

every red phase is preceded by a yellow phase

↑

bad prefix: finite trace fragment where a red phase appears without being preceded by a yellow phase
e.g., $\ldots \{●\}\{●\}$

- beverage machine:

the total number of entered coins is never less than the total number of released drinks

↑

bad prefix, e.g., $\{pay\}\{drink\}\{drink\}$

Definition of safety properties

Let E be a LT property over AP, i.e., $E \subseteq (2^{AP})^\omega$.

E is called a safety property if for all words

$$\sigma \;=\; A_0\,A_1\,A_2\ldots \in \left(2^{AP}\right)^\omega \setminus E$$

there exists a finite prefix $A_0\,A_1\ldots A_n$ of σ such that *none* of the words $A_0\,A_1\ldots A_n\,B_{n+1}\,B_{n+2}\,B_{n+3}\ldots$ belongs to E, i.e.,

$$E \cap \left\{\sigma' \in (2^{AP})^\omega : A_0\ldots A_n \text{ is a prefix of } \sigma'\right\} = \varnothing$$

Such words $A_0\,A_1\ldots A_n$ are called bad prefixes for E.

$E =$ set of all infinite words that do *not* have a bad prefix

Definition of safety properties

Let E be a LT property over AP, i.e., $E \subseteq (2^{AP})^\omega$.

E is called a safety property if for all words

$$\sigma \;=\; A_0\,A_1\,A_2\ldots \in \left(2^{AP}\right)^\omega \setminus E$$

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Such words $A_0\,A_1\ldots A_n$ are called bad prefixes for E.

$$\underset{\uparrow}{BadPref_E} \overset{\text{def}}{=} \text{set of bad prefixes for } E \;\subseteq \left(2^{AP}\right)^+$$

briefly: $BadPref$

Definition of safety properties

Let E be a LT property over AP, i.e., $E \subseteq (2^{AP})^{\omega}$.

E is called a safety property if for all words

$$\sigma \;=\; A_0\, A_1\, A_2 \ldots \in \left(2^{AP}\right)^{\omega} \setminus E$$

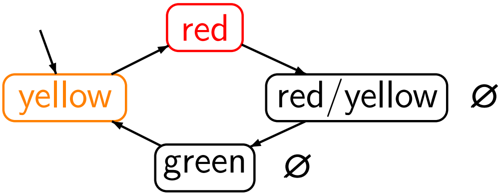
there exists a finite prefix $A_0\, A_1 \ldots A_n$ of σ such that *none* of the words $A_0\, A_1 \ldots A_n\, B_{n+1}\, B_{n+2}\, B_{n+3} \ldots$ belongs to E, i.e.,

$$E \cap \left\{ \sigma' \in (2^{AP})^{\omega} : A_0 \ldots A_n \text{ is a prefix of } \sigma' \right\} = \varnothing$$

Such words $A_0\, A_1 \ldots A_n$ are called bad prefixes for E.

minimal bad prefixes: any word $A_0 \ldots A_i \ldots A_n \in \mathit{BadPref}$ s.t. no proper prefix $A_0 \ldots A_i$ is a bad prefix for E

Safety property for a traffic light

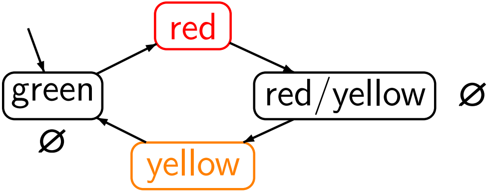


"every red phase is preceded by a yellow phase"

hence: $\mathcal{T} \models E$

E = set of all infinite words $A_0 \, A_1 \, A_2 \ldots$ over 2^{AP} such that for all $i \in \mathbb{N}$:

$$red \in A_i \implies i \geq 1 \text{ and } yellow \in A_{i-1}$$

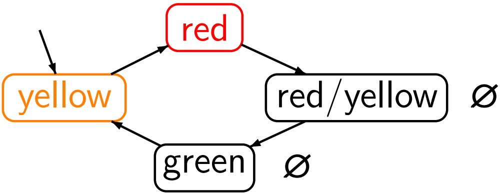


$\mathcal{T} \not\models E$

minimal bad prefix:

$\varnothing \, \{red\}$

Safety property for a traffic light



"every red phase is preceded by a yellow phase"

hence: $\mathcal{T} \models E$

$$
\boxed{
\begin{array}{rcl}
E & = & \text{set of all infinite words } A_0\,A_1\,A_2\ldots \\
 & & \text{over } 2^{AP} \text{ such that for all } i \in \mathbb{N}\text{:} \\
 & & \quad red \in A_i \implies i \geq 1 \text{ and } yellow \in A_{i-1}
\end{array}
}
$$

is a safety property over $AP = \{red, yellow\}$ with

$$
\begin{array}{rcl}
BadPref & = & \text{set of all finite words } A_0\,A_1\,\ldots\,A_n \\
 & & \text{over } 2^{AP} \text{ s.t. for some } i \in \{0, \ldots, n\}\text{:} \\
 & & \quad red \in A_i \wedge (i{=}0 \vee yellow \notin A_{i-1})
\end{array}
$$

Satisfaction of safety properties

Let $E \subseteq (2^{AP})^\omega$ be a safety property, $\mathcal{T}$ a TS over AP.

$$\begin{aligned} \mathcal{T} \models E \quad & \text{iff} \quad \mathit{Traces}(\mathcal{T}) \subseteq E \\ & \text{iff} \quad \mathit{Traces}_{fin}(\mathcal{T}) \cap \mathit{BadPref} = \varnothing \\ & \text{iff} \quad \mathit{Traces}_{fin}(\mathcal{T}) \cap \mathit{MinBadPref} = \varnothing \end{aligned}$$

$\mathit{BadPref}$ = set of all bad prefixes of E

$\mathit{MinBadPref}$ = set of all minimal bad prefixes of E

$\mathit{Traces}(\mathcal{T})$ = set of traces of $\mathcal{T}$

$\mathit{Traces}_{fin}(\mathcal{T})$ = set of finite traces of $\mathcal{T}$

$= \{ \mathit{trace}(\widehat{\pi}) : \widehat{\pi}$ is an initial, finite path fragment of $\mathcal{T} \}$

Every invariant is a safety property.

correct.

Let E be an invariant with invariant condition Φ.

- bad prefixes for E: finite words $A_0 \ldots A_i \ldots A_n$ s.t.

$$A_i \not\models \Phi \text{ for some } i \in \{0, 1, \ldots, n\}$$

- minimal bad prefixes for E:
  finite words $A_0\, A_1 \ldots A_{n-1}\, A_n$ such that

$$A_i \models \Phi \text{ for } i = 0, 1, \ldots, n{-}1, \text{ and } A_n \not\models \Phi$$

$\varnothing$ is a safety property

correct

- all finite words $A_0 \dots A_n \in (2^{AP})^+$ are bad prefixes
- $\varnothing$ is even an invariant (invariant condition ***false***)

$(2^{AP})^\omega$ is a safety property

correct

$$\text{“For all words} \in \underbrace{(2^{AP})^\omega \setminus (2^{AP})^\omega}_{=\varnothing} \dots\text{”}$$

For a given infinite word $\sigma = A_0\, A_1\, A_2\, \ldots$, let

$$pref(\sigma) \stackrel{\text{def}}{=} \text{set of all nonempty, finite prefixes of } \sigma$$
$$= \{ A_0\, A_1 \ldots A_n \,:\, n \geq 0 \}$$

For $E \subseteq \left(2^{AP}\right)^{\omega}$, let $pref(E) \stackrel{\text{def}}{=} \bigcup_{\sigma \in E} pref(\sigma)$

Given an LT property E, the prefix closure of E is:

$$cl(E) \stackrel{\text{def}}{=} \{\sigma \in (2^{AP})^{\omega} : pref(\sigma) \subseteq pref(E)\}$$

For any infinite word $\sigma \in \left(2^{AP}\right)^{\omega}$, let

$$pref(\sigma) \;=\; \text{set of all nonempty, finite prefixes of } \sigma$$

For any LT property $E \subseteq \left(2^{AP}\right)^{\omega}$, let

$$pref(E) \;=\; \bigcup_{\sigma \in E} pref(\sigma) \text{ and}$$

$$cl(E) \;=\; \left\{\sigma \in (2^{AP})^{\omega} : pref(\sigma) \subseteq pref(E)\right\}$$

Theorem:

E is a safety property iff $cl(E) = E$

Safety and finite trace inclusion

remind: LT properties and trace inclusion:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces(\mathcal{T}_1) \subseteq Traces(\mathcal{T}_2)$$

iff for all LT properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

safety properties and finite trace inclusion:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Safety and finite trace inclusion

$$\mathit{Traces}_{\mathit{fin}}(\mathcal{T}_1) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longrightarrow$": obvious, as for safety property E:

$$\mathcal{T} \models E \quad \text{iff} \quad \mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \cap \mathit{BadPref} = \varnothing$$

Hence:

If $\mathcal{T}_2 \models E$ and $\mathit{Traces}_{\mathit{fin}}(\mathcal{T}_1) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}_2)$ then:

$$\begin{aligned} & \mathit{Traces}_{\mathit{fin}}(\mathcal{T}_1) \cap \mathit{BadPref} \\ \subseteq \ & \mathit{Traces}_{\mathit{fin}}(\mathcal{T}_2) \cap \mathit{BadPref} = \varnothing \end{aligned}$$

and therefore $\mathcal{T}_1 \models E$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

for each transition system $\mathcal{T}$:

$$pref(Traces(\mathcal{T})) = Traces_{fin}(\mathcal{T})$$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property

as $cl(E) = E$

set of bad prefixes: $\left(2^{AP}\right)^+ \setminus Traces_{fin}(\mathcal{T}_2)$

Safety and finite trace inclusion

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

Proof "$\Longleftarrow$": consider the LT property

$$E = cl(Traces(\mathcal{T}_2)) = \{\sigma : pref(\sigma) \subseteq Traces_{fin}(\mathcal{T}_2)\}$$

Then, E is a safety property and $\mathcal{T}_2 \models E$.

By assumption: $\mathcal{T}_1 \models E$ and therefore $Traces(\mathcal{T}_1) \subseteq E$.

Hence:
$$\begin{aligned} Traces_{fin}(\mathcal{T}_1) &= pref(Traces(\mathcal{T}_1)) \\ &\subseteq pref(E) = pref(cl(Traces(\mathcal{T}_2))) \\ &= Traces_{fin}(\mathcal{T}_2) \end{aligned}$$

Safety and finite trace equivalence

safety properties and finite trace inclusion:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces_{fin}(\mathcal{T}_1) \subseteq Traces_{fin}(\mathcal{T}_2)$$

iff for all safety properties E: $\mathcal{T}_2 \models E \implies \mathcal{T}_1 \models E$

safety properties and finite trace equivalence:

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are TS over AP then:

$$Traces_{fin}(\mathcal{T}_1) = Traces_{fin}(\mathcal{T}_2)$$

iff $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the same safety properties

Summary: trace relations and properties

trace inclusion

$Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$ iff

for all LT properties E: $\quad \mathcal{T}' \models E \Longrightarrow \mathcal{T} \models E$

finite trace inclusion

$Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$ iff

for all safety properties E: $\mathcal{T}' \models E \Longrightarrow \mathcal{T} \models E$

Summary: trace relations and properties

trace equivalence

$\mathit{Traces}(\mathcal{T}) = \mathit{Traces}(\mathcal{T}')$ iff

$\mathcal{T}$ and $\mathcal{T}'$ satisfy the same LT properties

finite trace equivalence

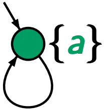
$\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) = \mathit{Traces}_{\mathit{fin}}(\mathcal{T}')$ iff

$\mathcal{T}$ and $\mathcal{T}'$ satisfy the same safety properties

If $Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$
then $Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$.

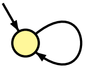
correct, since

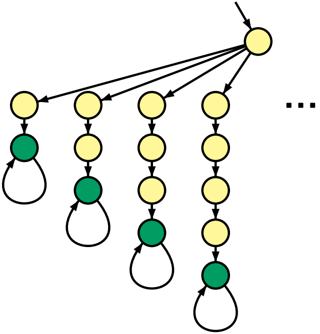
$$\begin{aligned} Traces_{fin}(\mathcal{T}) &= \text{set of all finite nonempty prefixes} \\ &\ \text{of words in } Traces(\mathcal{T}) \\ &= pref(Traces(\mathcal{T})) \end{aligned}$$



$$\begin{aligned} Traces(\mathcal{T}) &= \{ \{a\}^\omega \} \\ Traces_{fin}(\mathcal{T}) &= \{ \{a\}^n : n \geq 1 \} \end{aligned}$$

Finite trace relations versus trace relations

$\mathcal{T}$ 

$\mathcal{T}'$ 

$$Traces(\mathcal{T}) = \{\varnothing^\omega\}$$

$$Traces_{fin}(\mathcal{T}) = \{\varnothing^n : n \geq 0\}$$

$$Traces(\mathcal{T}') = \{\varnothing^n\{b\}^\omega : n \geq 2\}$$

$$Traces_{fin}(\mathcal{T}') = \{\varnothing^n : n \geq 0\} \cup \{\varnothing^n\{b\}^m : n \geq 2 \wedge m \geq 1\}$$

$Traces(\mathcal{T}) \not\subseteq Traces(\mathcal{T}')$, but
$Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$

LT property
$E \mathrel{\widehat{=}}$ "eventually b"

$\mathcal{T} \not\models E, \quad \mathcal{T}' \models E$

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states,
i.e., all paths of $\mathcal{T}$ are infinite

(2) $\mathcal{T}'$ is finite.

Then:

$$\begin{array}{rl} & \mathit{Traces}(\mathcal{T}) \subseteq \mathit{Traces}(\mathcal{T}') \\ \text{iff} & \mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}') \end{array}$$

"$\Longrightarrow$": holds for all transition systems

"$\Longleftarrow$": suppose that (1) and (2) hold and that

(3) $\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) \subseteq \mathit{Traces}_{\mathit{fin}}(\mathcal{T}')$

Show that $\mathit{Traces}(\mathcal{T}) \subseteq \mathit{Traces}(\mathcal{T}')$

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states

(2) $\mathcal{T}'$ is finite

(3) $Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$

Then $Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$

Proof: Pick some path $\pi = s_0\, s_1\, s_2 \ldots$ in $\mathcal{T}$ and show that there exists a path

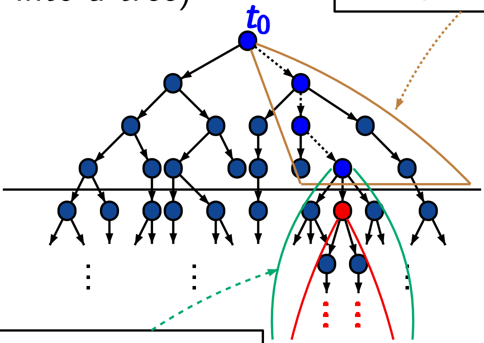
$$\pi' = t_0\, t_1\, t_2 \ldots \text{ in } \mathcal{T}'$$

such that $trace(\pi) = trace(\pi')$

finite TS $\mathcal{T}'$

paths from state t_0
(unfolded into a tree)

contains all path fragments with trace $A_0\,A_1 \ldots A_n$
in particular: $t_0\,t_1 \ldots t_n$



finite until depth $\leq n$

contains infinitely many path fragments $t_n\,s^m_{n+1} \ldots s^m_m$

there exists $t_{n+1} \in Post(t_n)$
s.t. $t_{n+1} = s^m_{n+1}$ for infinitely many m

Finite trace and trace inclusion

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are TS over AP such that

(1) $\mathcal{T}$ has no terminal states

(2) $\mathcal{T}'$ is finite $\longleftarrow$ image-finiteness is sufficient

(3) $Traces_{fin}(\mathcal{T}) \subseteq Traces_{fin}(\mathcal{T}')$

Then $Traces(\mathcal{T}) \subseteq Traces(\mathcal{T}')$

image-finiteness of $\mathcal{T}' = (S', Act, \rightarrow, S'_0, AP, L')$:

- for each $A \in 2^{AP}$ and state $s \in S'$:

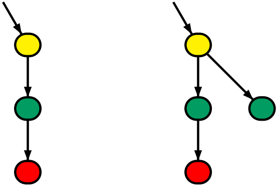
  $\{t \in Post(s) : L'(t) = A\}$ is finite

- for each $A \in 2^{AP}$: $\{s_0 \in S'_0 : L'(s_0) = A\}$ is finite

Trace equivalence vs. finite trace equivalence

Whenever $\mathit{Traces}(\mathcal{T}) = \mathit{Traces}(\mathcal{T}')$ then
$\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) = \mathit{Traces}_{\mathit{fin}}(\mathcal{T}')$

while the reverse direction does not hold in general
(even not for finite transition systems)



finite trace equivalent,
but *not* trace equivalent

Trace equivalence vs. finite trace equivalence

Whenever $\mathit{Traces}(\mathcal{T}) = \mathit{Traces}(\mathcal{T}')$ then
$\mathit{Traces}_{\mathit{fin}}(\mathcal{T}) = \mathit{Traces}_{\mathit{fin}}(\mathcal{T}')$

The reverse implication holds under additional assumptions, e.g.,

- if $\mathcal{T}$ and $\mathcal{T}'$ are finite and have no terminal states
- or, if $\mathcal{T}$ and $\mathcal{T}'$ are AP-deterministic

Liveness

"liveness: something good will happen."

"event *a* will occur eventually"

e.g., termination for sequential programs

"event *a* will occur infinitely many times"

e.g., starvation freedom for dining philosophers

"whenever event *b* occurs then event *a* will occur sometimes in the future"

e.g., every waiting process enters eventually its critical section

which property type? LF2.6-2

- Each philosopher thinks infinitely often.

  liveness

- Two philosophers next to each other never eat at the same time.

  invariant

- Whenever a philosopher eats then he has been thinking at some time before.

  safety

- Whenever a philosopher eats then he will think some time afterwards.

  liveness

- Between two eating phases of philosopher $\boldsymbol{i}$ lies at least one eating phase of philosopher $\boldsymbol{i+1}$.

  safety

Liveness

many different formal definitions of liveness
have been suggested in the literature

here: one just example for a formal definition
of liveness

Definition of liveness properties

Let E be an LT property over AP, i.e., $E \subseteq \left(2^{AP}\right)^{\omega}$.

E is called a liveness property if each finite word over AP can be extended to an infinite word in E, i.e., if

$$pref(E) \;=\; \left(2^{AP}\right)^{+}$$

Examples:

- each process will eventually enter its critical section
- each process will enter its critical section infinitely often
- whenever a process has requested its critical section then it will eventually enter its critical section

Examples for liveness properties

An LT property E over AP is called a liveness property if $\mathit{pref}(E) = (2^{AP})^+$

Examples for $AP = \{\mathit{crit}_i : i = 1, \ldots, n\}$:

- each process will eventually enter its critical section

E = set of all infinite words $A_0\, A_1\, A_2 \ldots$ s.t.

$$\forall i \in \{1, \ldots, n\}\ \exists k \geq 0.\ \mathit{crit}_i \in A_k$$

Examples for liveness properties

An LT property E over AP is called a liveness property if $pref(E) = (2^{AP})^+$

Examples for $AP = \{crit_i : i = 1, \ldots, n\}$:

- each process will eventually enter its critical section
- each process will enter its critical section infinitely often

E = set of all infinite words $A_0\, A_1\, A_2 \ldots$ s.t.

$$\forall i \in \{1, \ldots, n\} \ \stackrel{\infty}{\exists}\, k \geq 0.\ crit_i \in A_k$$

Examples for liveness properties

An LT property E over AP is called a liveness property if $pref(E) = (2^{AP})^+$

Examples for $AP = \{wait_i, crit_i : i = 1, \ldots, n\}$:

- each process will eventually enter its critical section
- each process will enter its crit. section inf. often
- whenever a process is waiting then it will eventually enter its critical section

E = set of all infinite words $A_0\, A_1\, A_2 \ldots$ s.t.

$$\forall i \in \{1, \ldots, n\}\ \forall j \geq 0.\ wait_i \in A_j \longrightarrow \exists k > j.\ crit_i \in A_k$$

Decomposition theorem

> For each LT-property E, there exists a safety property $SAFE$ and a liveness property $LIVE$ s.t.
>
> $$E \;=\; SAFE \cap LIVE$$

Proof: Let $SAFE \stackrel{\text{def}}{=} cl(E)$

$$LIVE \stackrel{\text{def}}{=} E \cup \left((2^{AP})^{\omega} \setminus cl(E) \right)$$

Show that:

- $E = SAFE \cap LIVE$ $\quad\surd$
- $SAFE$ is a safety property as $cl(SAFE) = SAFE$
- $LIVE$ is a liveness property, i.e., $pref(LIVE) = (2^{AP})^{+}$

Being safe and live

Which LT properties are both a safety and a liveness property?

answer: The set $(2^{AP})^\omega$ is the only LT property which is a safety property and a liveness property

- $(2^{AP})^\omega$ is a safety and a liveness property: $\surd$
- If E is a liveness property then

$$\begin{aligned} & pref(E) = (2^{AP})^+ \\ \Longrightarrow \quad & cl(E) = (2^{AP})^\omega \end{aligned}$$

If E is a safety property too, then $cl(E) = E$.
Hence $E = cl(E) = (2^{AP})^\omega$.