Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

- regular safety properties
- ω-regular properties $\longleftarrow$
- model checking with Büchi automata

Linear Temporal Logic

Computation-Tree Logic

Equivalences and Abstraction

Regular LT properties

idea: define regular **LT** properties to be those languages of infinite words over the alphabet 2^{AP} that have a representation by a finite automata

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  - ω-automata, i.e., acceptors for infinite words
  - ω-regular expressions

Regular expressions

remind: syntax and semantics of regular expressions over some alphabet $\Sigma = \{A, B, \ldots\}$

Regular expressions over Σ

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$\mathcal{L}(\emptyset) = \varnothing$	$\mathcal{L}(\epsilon) = \{\varepsilon\}$	$\mathcal{L}(A) = \{A\}$
$\mathcal{L}(\alpha_1 + \alpha_2) \ = \ \mathcal{L}(\alpha_1) \cup \mathcal{L}(\alpha_2)$		union
$\mathcal{L}(\alpha_1 . \alpha_2) \ = \ \mathcal{L}(\alpha_1)\mathcal{L}(\alpha_2)$		concatenation
$\mathcal{L}(\alpha^*) \ = \ \mathcal{L}(\alpha)^*$		Kleene closure

ω-regular expressions

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regular expressions + ω-operator α^ω

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for $L \subseteq \Sigma^*$:

$$L^\omega \stackrel{\text{def}}{=} \left\{ w_1 w_2 w_3 \ldots : w_i \in L \text{ for all } i \geq 1 \right\}$$

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note: $L^\omega \subseteq \Sigma^\omega$ if $\varepsilon \notin L$

Syntax and semantics of ω-regular expressions

syntax of ω-regular expressions over alphabet Σ:

$$\gamma \;=\; \alpha_1.\beta_1^{\omega} + \ldots + \alpha_n.\beta_n^{\omega} \quad \text{where}$$

α_i, β_i are regular expressions over Σ s.t. $\varepsilon \notin \mathcal{L}(\beta_i)$

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A language $L \subseteq \Sigma^\omega$ is called ω-regular iff there exists an ω-regular expression γ s.t.

$$L = \mathcal{L}_\omega(\gamma)$$

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alphabet $\Sigma = \{A, B\}$

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$$(B^*.A^+.B)^*.B^\omega + (B^*.A^+.B)^\omega$$

where $\alpha^+ \stackrel{\text{def}}{=} \alpha.\alpha^*$.

Provide an ω-regular expression for ...

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- set of all infinite words where each A is followed eventually by letter B

$$(B^*.A^+.B)^*.B^\omega + (B^*.A^+.B)^\omega \equiv (A^*.B)^\omega$$

where $\alpha^+ \stackrel{\text{def}}{=} \alpha.\alpha^*$.

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- invariant with invariant condition $a \vee \neg b$

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> Let Φ be an invariant condition and let
>
> $$\{A \subseteq AP : A \models \Phi\} = \{A_1, \dots, A_k\}$$
>
> Then: invariant "always Φ" $\widehat{=}$ $(A_1 + \dots + A_k)^\omega$

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  Indeed: each invariant is ω-regular

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  $$\big((\emptyset + \{b\})^{*}.(\{a\} + \{a, b\})\big)^{\omega}$$

ω-regular properties

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$$(2^{AP})^*.(\{a\} + \{a, b\}).(2^{AP})^\omega$$

where $2^{AP} \mathrel{\widehat{=}} \emptyset + \{a\} + \{b\} + \{a, b\}$

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Symbolic notation

symbolic notation for ω-regular properties

... using formulas instead of sums ....

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Nondeterministic Büchi automata (NBA)

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syntax as for **NFA**

$\uparrow$

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semantics: language of infinite words

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run for a word $A_0\, A_1\, A_2\, \ldots \in \Sigma^\omega$:

state sequence $\pi = q_0\, q_1\, q_2\, \ldots$ where $q_0 \in Q_0$
and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

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run π is accepting if $\overset{\infty}{\exists} \, i \in \mathbb{N}. \; q_i \in F$

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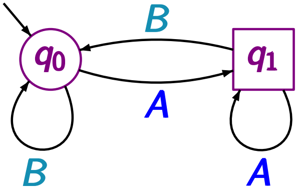
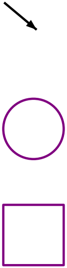
accepted language $\mathcal{L}_\omega(\mathcal{A}) \subseteq \Sigma^\omega$ is given by:

$\mathcal{L}_\omega(\mathcal{A}) \stackrel{\text{def}}{=}$ set of infinite words over Σ that have an accepting run in $\mathcal{A}$

Notations in pictures for NBA

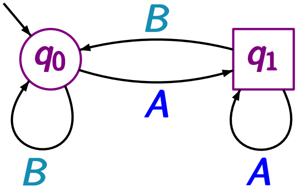


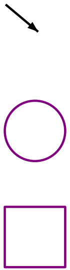


initial state

nonfinal state

final state





initial state

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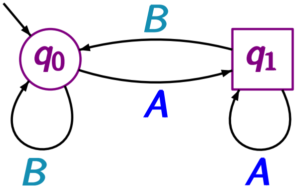
NBA with state space $\{q_0, q_1\}$

q_0 initial state

q_1 accept state

alphabet $\Sigma = \{A, B\}$

Examples for NBA over $\Sigma = \{A, B\}$



accepted language: **?**

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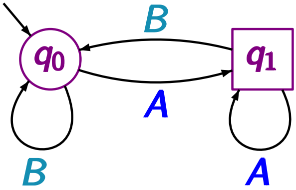

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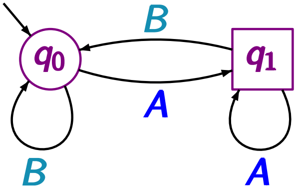

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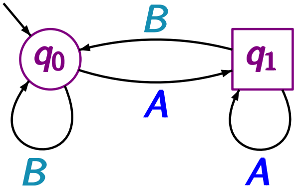
set of all infinite words that contain infinitely many A's

$$(B^*.A)^\omega$$



accepted language:
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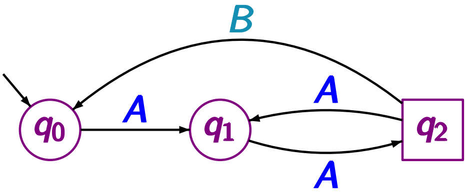
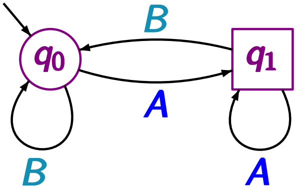


Examples for NBA over $\Sigma = \{A, B\}$



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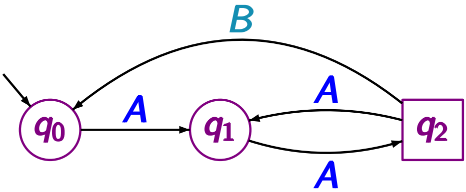

$A\,A\,B\,A\,A\,B\,A\,A\,B\ldots$
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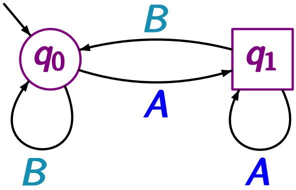

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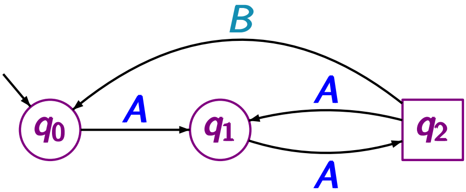
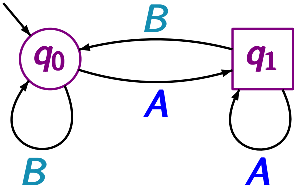


accepted language:
"every B is preceded by a positive even number of A's"

$$\left.\begin{array}{l} A\,A\,B\,A\,A\,B\,A\,A\,B\ldots \\ A\,A\,A\,A\,A\,A\,A\,A\,A\ldots \end{array}\right\}$$ accepted words

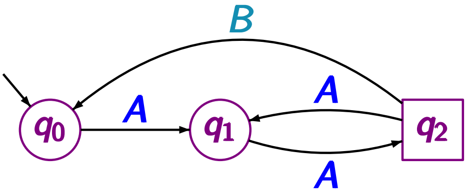
Examples for NBA over $\Sigma = \{A, B\}$



accepted language:

set of all infinite words that contain infinitely many A's

$$(B^*.A)^\omega$$



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$$((A.A)^+.B)^\omega + ((A.A)^+.B)^*.A^\omega$$

NBA for LT properties

NBA for LT properties

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \to 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

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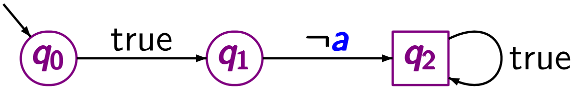
accepted language $\mathcal{L}_\omega(\mathcal{A})$ is an LT-property:

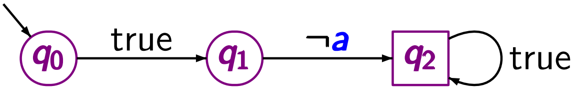
$\mathcal{L}_\omega(\mathcal{A})$ = set of infinite words over 2^{AP} that have an accepting run in $\mathcal{A}$

NBA for LT properties



$$\mathcal{L}_\omega(\mathcal{A}) = ?$$

set of atomic propositions $AP = \{a, b\}$



$$\mathcal{L}_\omega(\mathcal{A}) \;\widehat{=}\; \text{true}.\,\neg a.\,\text{true}^\omega$$

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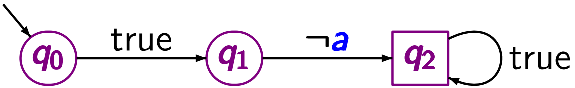
NBA for LT properties



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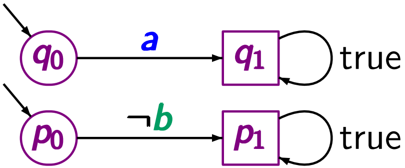

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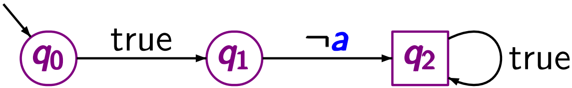

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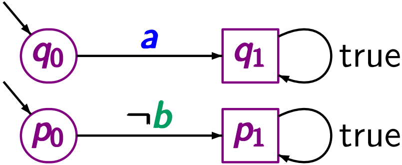
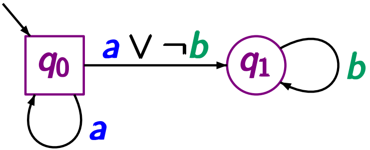


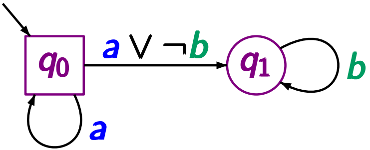
$$(a \vee \neg b).\text{true}^\omega$$

set of atomic propositions $AP = \{a, b\}$

NBA for LT properties

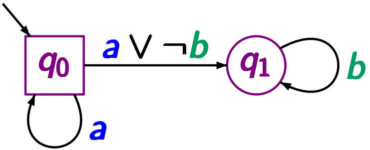


NBA for LT properties



"always a" $\widehat{=}$ a^ω

NBA for LT properties



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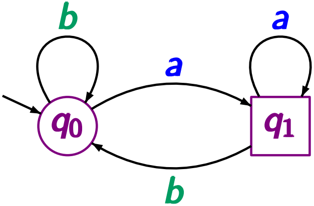
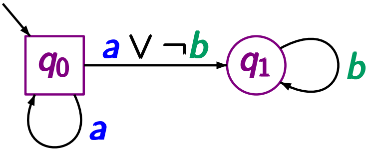


NBA for LT properties



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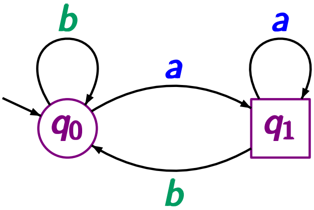
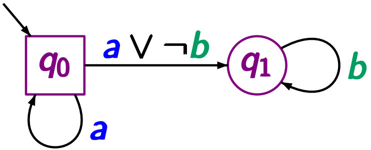


"infinitely often a and ..."

NBA for LT properties



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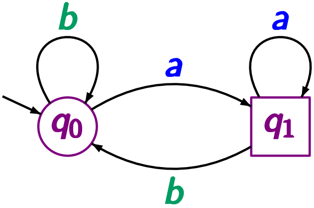
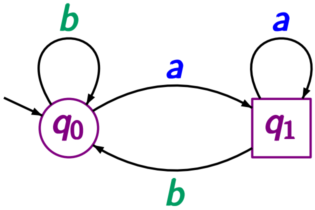


"infinitely often a and always $a \vee b$"

$$\mathrel{\widehat{=}} \big((a \vee b)^*.a \big)^\omega$$

NBA for LT properties



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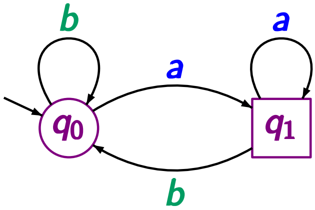
"infinitely often a"

$$\big((\neg a)^*.a\big)^\omega$$

NBA for LT properties



"infinitely often a and always $a \vee b$"

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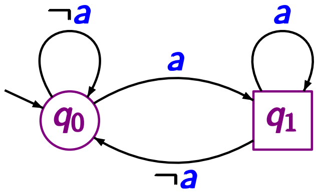

"infinitely often a"

$$\big((\neg a)^*.a\big)^\omega$$

From NBA to ω-regular expressions

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For each NBA $\mathcal{A}$ there is an ω-regular expression γ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

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Proof. Let $\mathcal{A}$ be an NBA $(Q, \Sigma, \delta, Q_0, F)$

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Proof. Let $\mathcal{A}$ be an NBA $(Q, \Sigma, \delta, Q_0, F)$ and $q, p \in Q$.

Let $\mathcal{A}_{q,p}$ be the NFA $(Q, \Sigma, \delta, q, \{p\})$.

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$$\mathcal{L}_\omega(\mathcal{A}) = \bigcup_{q \in Q_0} \bigcup_{p \in F} \mathcal{L}(\mathcal{A}_{q,p}) \left(\mathcal{L}(\mathcal{A}_{p,p}) \setminus \{\varepsilon\} \right)^\omega$$

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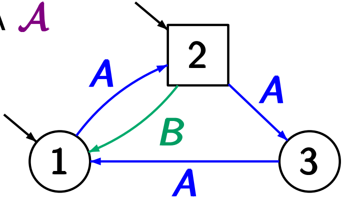
is ω-regular as $\mathcal{L}(\mathcal{A}_{q,p})$ and $\mathcal{L}(\mathcal{A}_{p,p}) \setminus \{\varepsilon\}$ are regular

Example: NBA $\rightsquigarrow$ ω-regular expression

NBA $\mathcal{A}$



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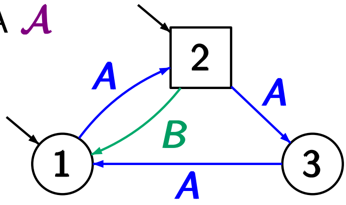

$$\mathcal{L}_\omega(\mathcal{A}) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$$

$$L_{12} = \mathcal{L}(\mathcal{A}_{12})$$
$$L_{22} = \mathcal{L}(\mathcal{A}_{22})$$
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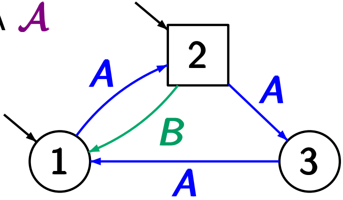

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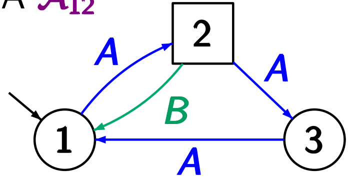
NFA $\mathcal{A}_{12}$



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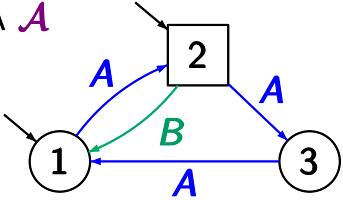

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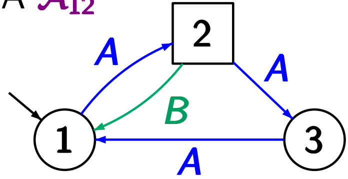

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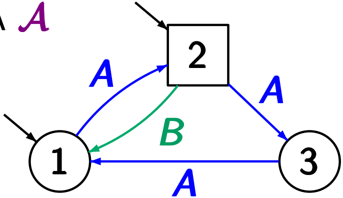

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$L_{12} \mathrel{\widehat{=}} A.(B.A + A.A.A)^*$ $\qquad$ $L'_{22} \mathrel{\widehat{=}} (B.A + A.A.A)^+$

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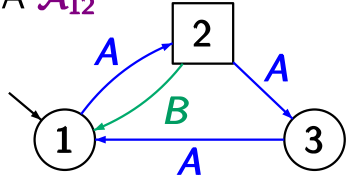

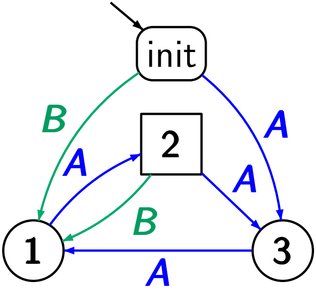

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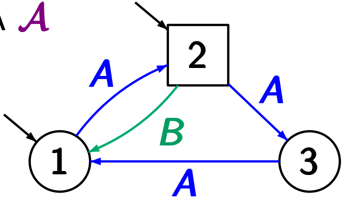

language of $\mathcal{A}$:

$$A.(B.A + A.A.A)^\omega \\ + \ (B.A + A.A.A)^\omega$$

$L_{12} \mathrel{\widehat{=}} A.(B.A + A.A.A)^*$

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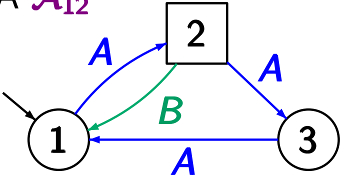

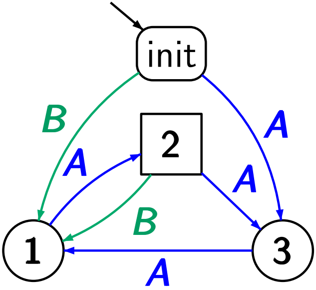

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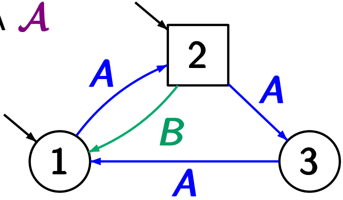
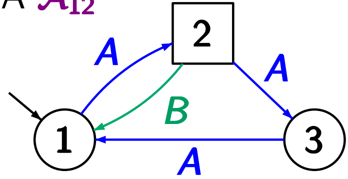


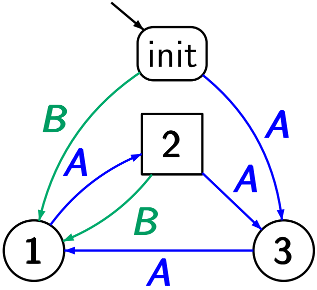
language of $\mathcal{A}$:

$$\begin{aligned} & A.(B.A + A.A.A)^{\omega} \\ + \ & (B.A + A.A.A)^{\omega} \\ \equiv \ & (A + \varepsilon).(B.A + A.A.A)^{\omega} \end{aligned}$$

$L_{12} \mathrel{\widehat{=}} A.(B.A + A.A.A)^{*}$ $\qquad$ $L'_{22} \mathrel{\widehat{=}} (B.A + A.A.A)^{+}$

NFA $\mathcal{A}_{12}$





From ω-regular expressions to NBA

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For each ω-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an **NBA** $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

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Proof.

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Proof. consider **NFA** $\mathcal{A}_i$ for α_i and $\mathcal{B}_i$ for β_i

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- construct **NBA** $\mathcal{B}_i^\omega$ for β_i^ω

From ω-regular expressions to NBA

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- construct **NBA** $\mathcal{B}_i^\omega$ for β_i^ω
- construct **NBA** $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i.\beta_i^\omega$

For each ω-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

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- construct **NBA** for $\bigcup\limits_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$

From ω-regular expressions to NBA

For each ω-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

Proof. consider NFA $\mathcal{A}_i$ for α_i and $\mathcal{B}_i$ for β_i

- construct NBA $\mathcal{B}_i^\omega$ for β_i^ω
- construct NBA $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i.\beta_i^\omega$
- construct **NBA** for $\bigcup\limits_{1 \le i \le n} \mathcal{L}_\omega(\mathcal{C}_i)$ $\longleftarrow$

NBA are closed under union

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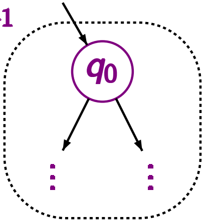
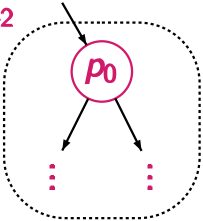
NBA $\mathcal{A}_1$

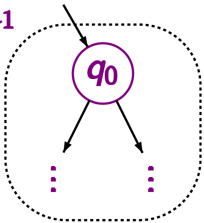


NBA $\mathcal{A}_2$



NBA are closed under union

NBA $\mathcal{A}_1$



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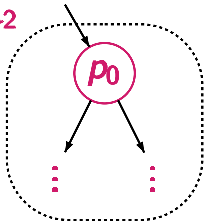
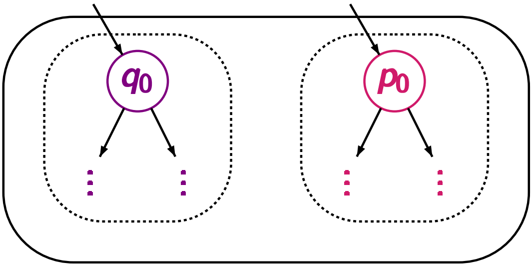


NBA for $\mathcal{L}_\omega(\mathcal{A}_1) \cup \mathcal{L}_\omega(\mathcal{A}_2)$



For each ω-regular expression

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Concatenation of an NFA and an NBA

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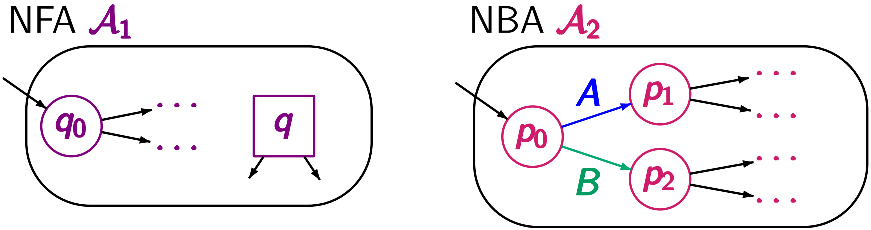

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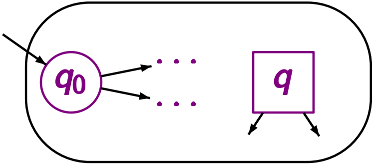
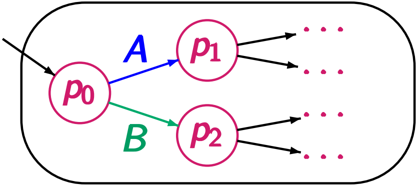
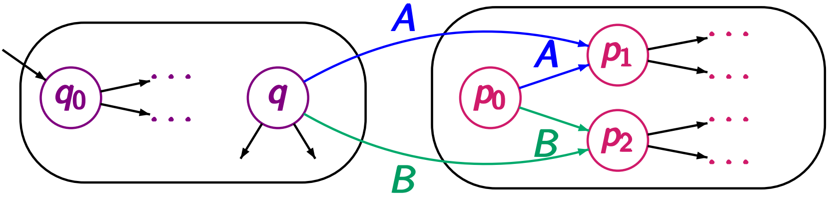


NBA $\mathcal{A}_2$



NBA for $\mathcal{L}(\mathcal{A}_1).\mathcal{L}_\omega(\mathcal{A}_2)$:



Concatenation of an NFA and an NBA

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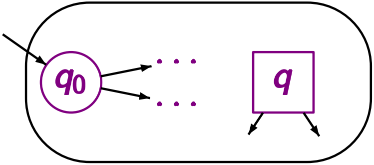

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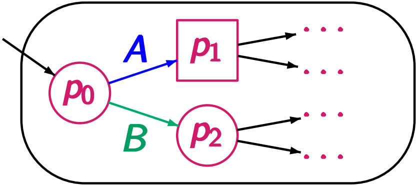

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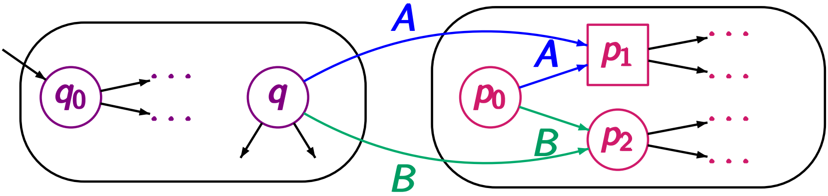

accept states as in $\mathcal{A}_2$

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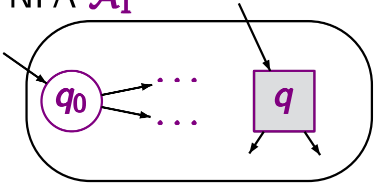

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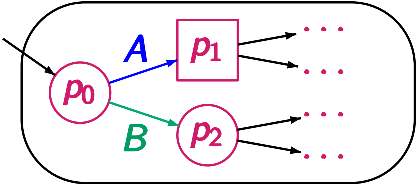

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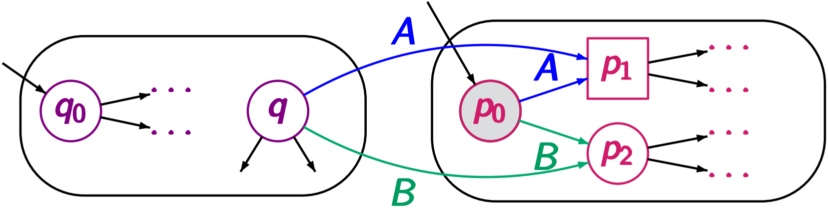

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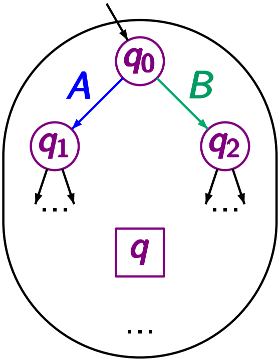
ω-operator for NFA

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$ $\rightsquigarrow$ NBA $\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$



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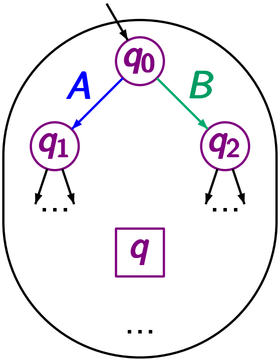

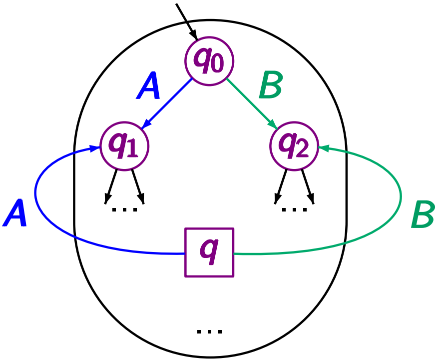

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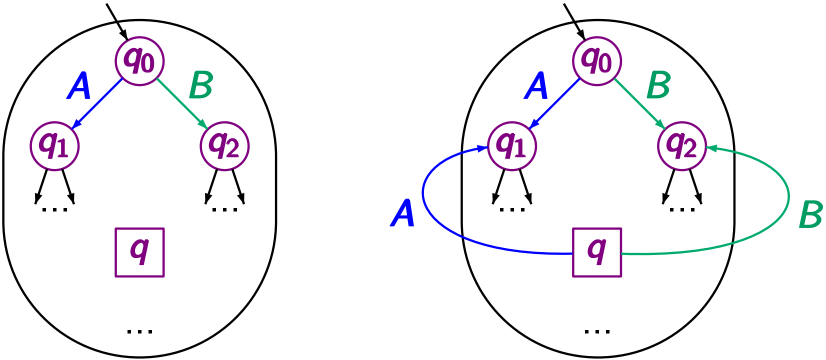

wrong !

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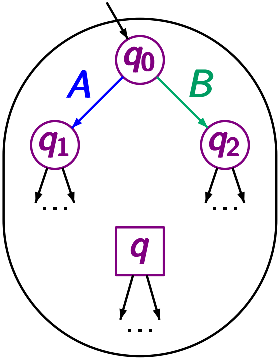

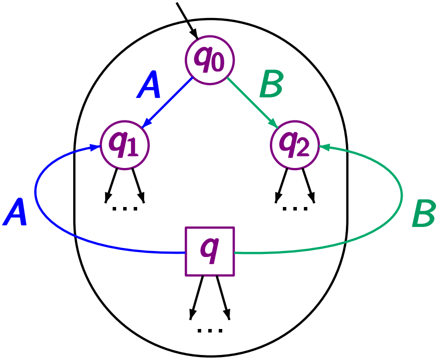

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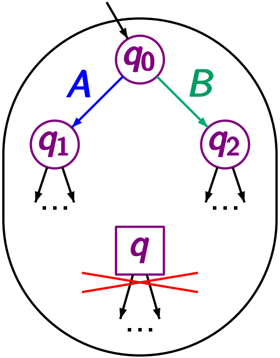
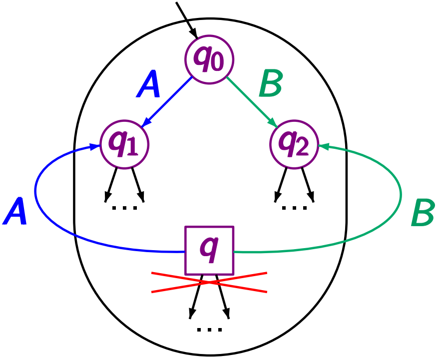
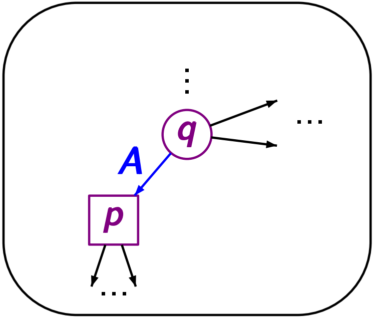




wrong !

... correct, if $\delta(q, x) = \varnothing \quad \forall q \in F \ \forall x \in \Sigma$

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$ $\Longrightarrow$ **NFA** $\mathcal{B}$ for L s.t. all final states are terminal



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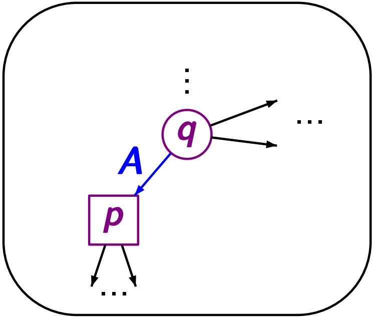

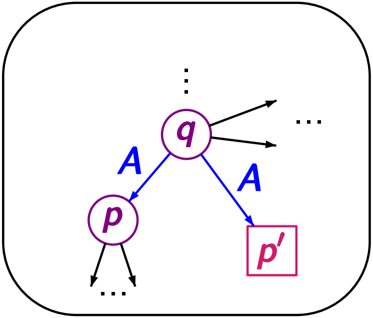

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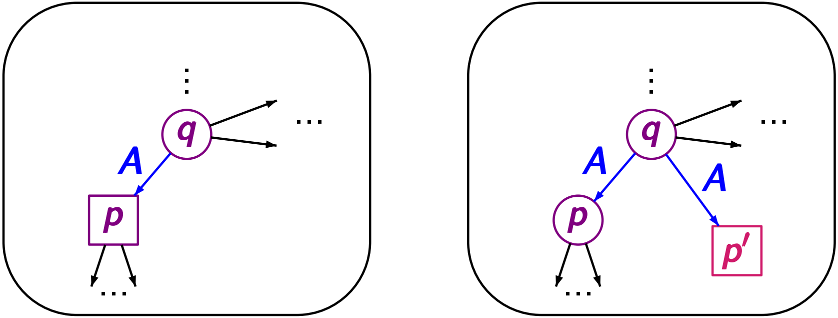

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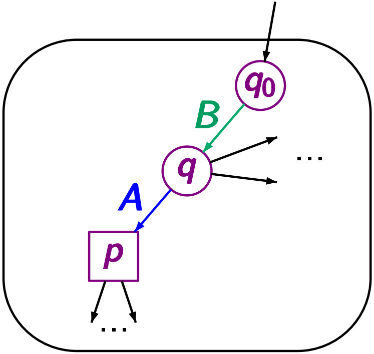

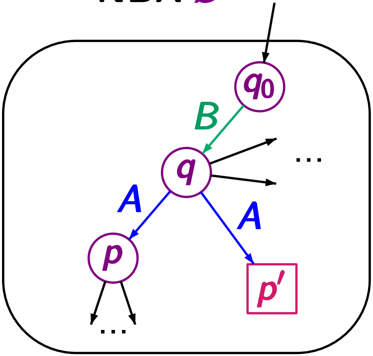

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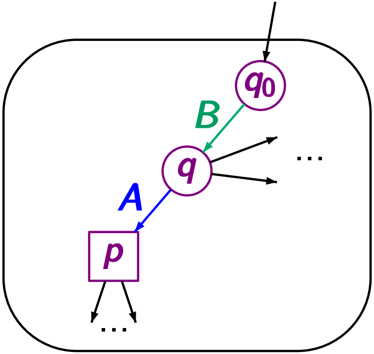

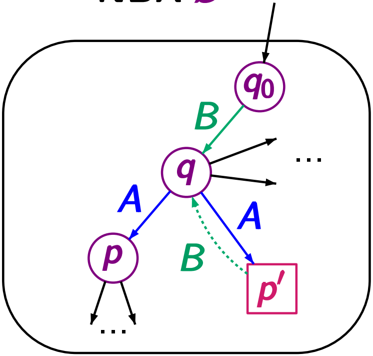

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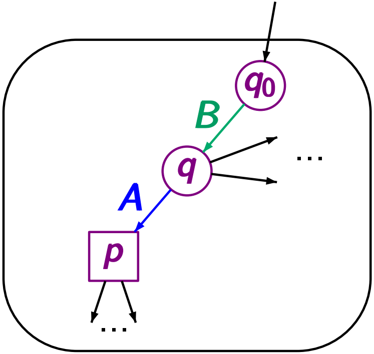
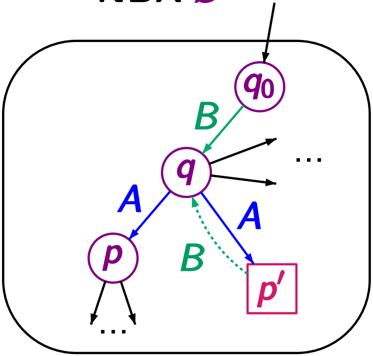
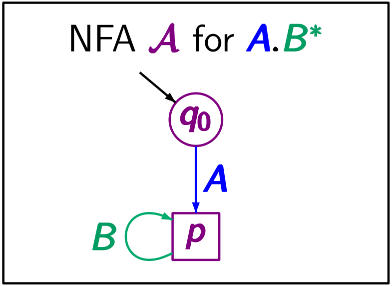




$$\mathcal{L}(\mathcal{A})^\omega = \mathcal{L}_\omega(\mathcal{B}^\omega)$$

Example: ω-operator for NFA



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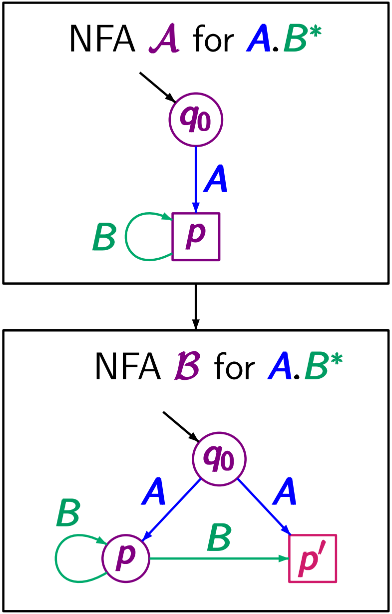

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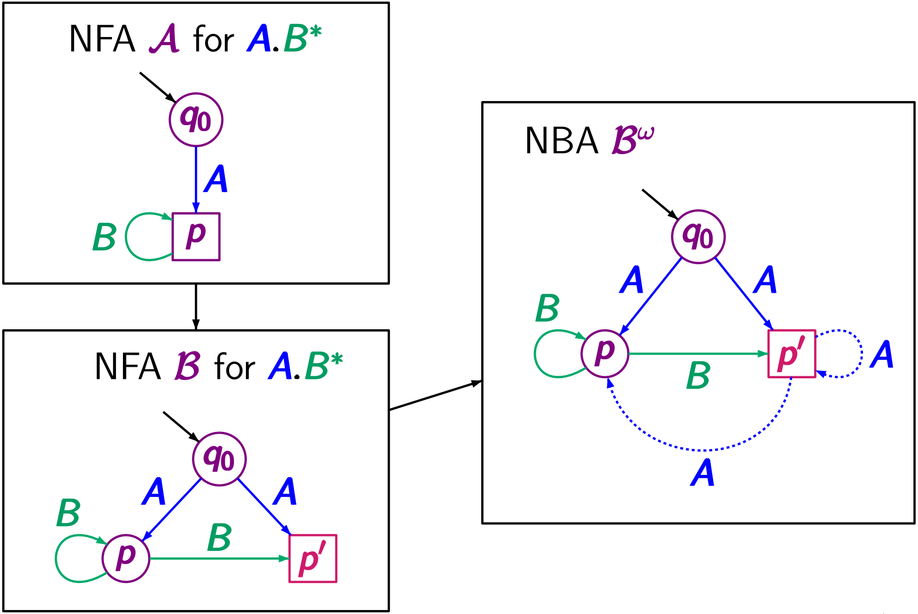

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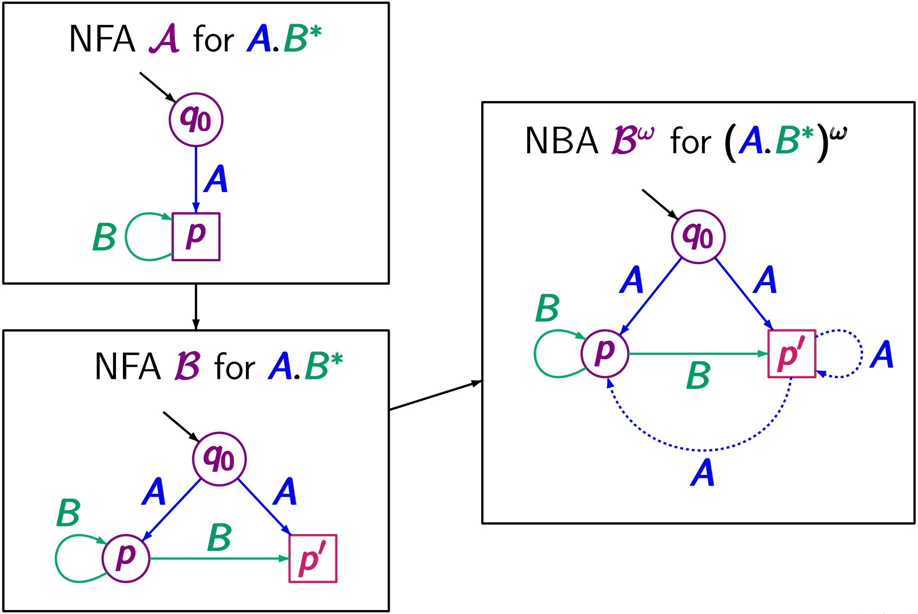

Equivalence of ω-regular expressions and NBA

(1) For each NBA $\mathcal{A}$ there exists an ω-regular expression γ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

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If E be an LT property then:

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Corollary:

If E be an LT property, i.e., $E \subseteq (2^{AP})^\omega$, then:

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Closure properties of ω-regular properties

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remind: Kleene's theorem for regular languages:

The class of regular languages is closed under

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- *complementation:*
  much more difficult than for NFA,
  via other types of ω-automata

Nonemptiness for NBA

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given: NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

question: does $\mathcal{L}_\omega(\mathcal{A}) \neq \varnothing$ hold ?

Nonemptiness for NBA

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\mathcal{L}_\omega(\mathcal{A}) \neq \varnothing \quad \text{iff} \quad \exists q_0 \in Q_0 \; \exists p \in F \; \exists x \in \Sigma^* \; \exists y \in \Sigma^+ . \; p \in \delta(q_0, x) \cap \delta(p, y)$$

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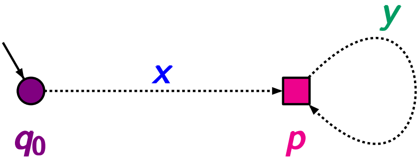
↑

there exists a reachable accept state $p \in F$
that belongs to a cycle



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&\qquad\qquad p \in \delta(q_0, x) \cap \delta(p, y) \\
&\text{iff} \quad \text{there exist finite words } x, y \in \Sigma^* \\
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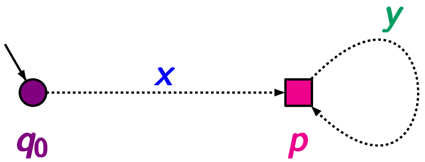
\end{aligned}$$



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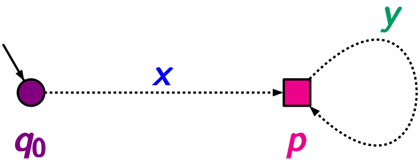
"ultimatively periodic words"



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The emptiness problem for NBA is solvable by means of graph algorithms in time $\mathcal{O}(\,poly(\mathcal{A})\,)$

Deterministic Büchi automata (DBA)

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A DBA is an NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ such that

- $\mathcal{A}$ has a unique initial state,
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notation: $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if $Q_0 = \{q_0\}$

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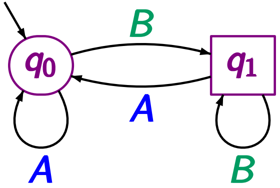

alphabet $\Sigma = \{A, B\}$

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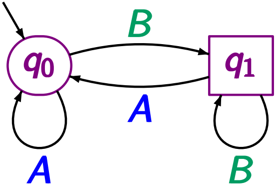
- $|\delta(q, A)| \leq 1$ for all $q \in Q$ and $A \in \Sigma$

notation: $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if $Q_0 = \{q_0\}$



DBA for "infinitely often B"

alphabet $\Sigma = \{A, B\}$

Determinization by powerset construction

well-known:

the powerset construction for the determinization (and complementation) of finite automata **(NFA)**

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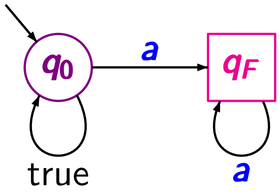
question:

does the powerset construction also work for Büchi automata **(NBA)** **?**

NBA for "eventually forever a"



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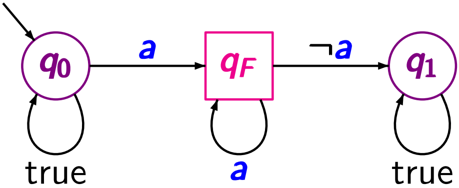
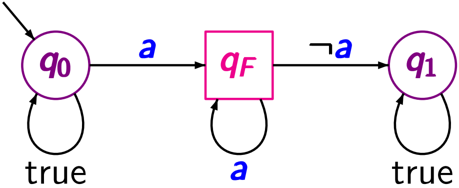
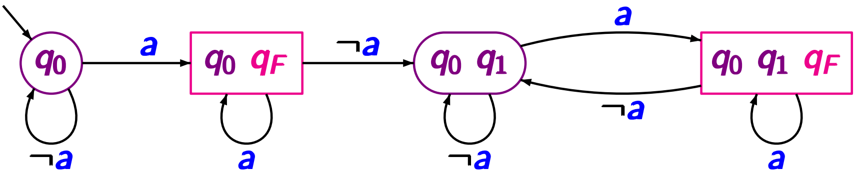


Determinization by powerset construction

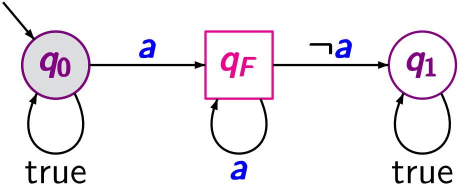
NBA for "eventually forever a"



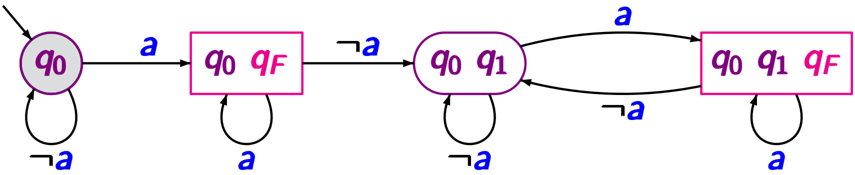
powerset construction



NBA for "eventually forever a"



powerset construction



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Determinization by powerset construction

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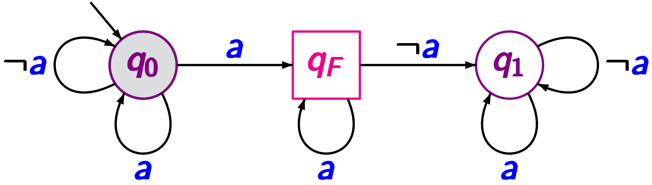
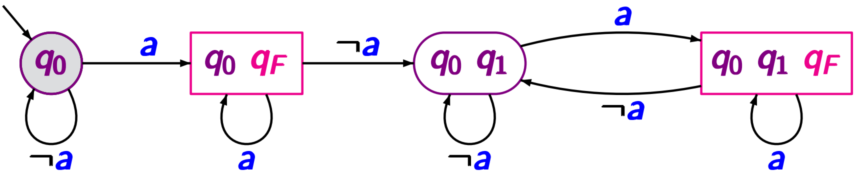


powerset construction



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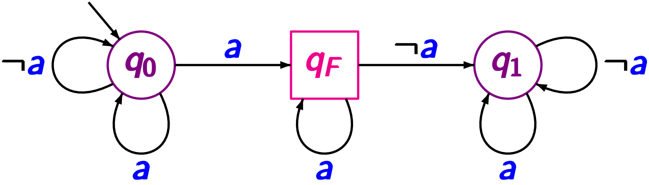
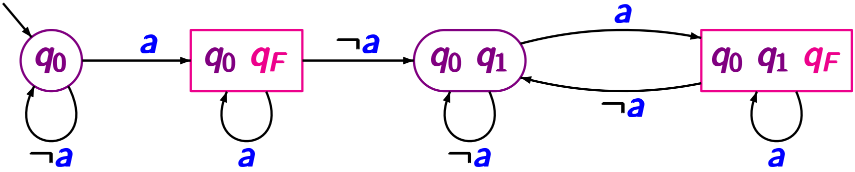
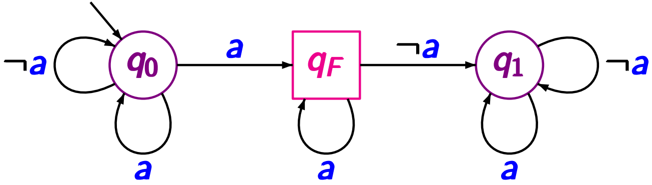


powerset construction

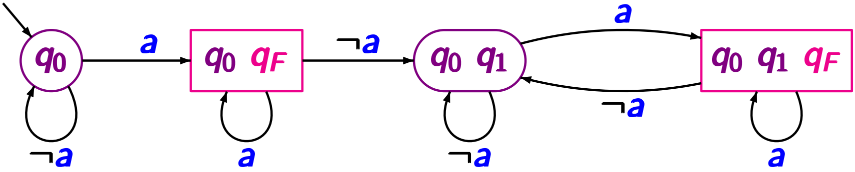


DBA for "infinitely often a"

NBA for "eventually forever a"



powerset construction



DBA for "infinitely often a"

Complementation of DBA

Complementation of DBA

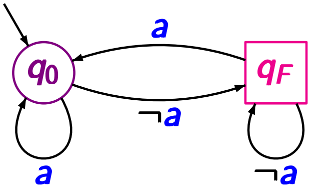
well-known:

DFA can be complemented by complementation of the acceptance set

question:

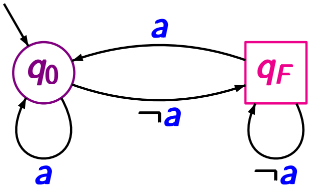
does this also work for **DBA** ?

Complementation of DBA



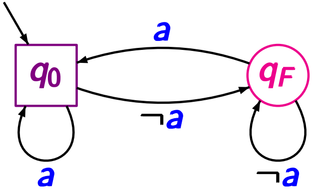
DBA for
"infinitely often $\neg a$"

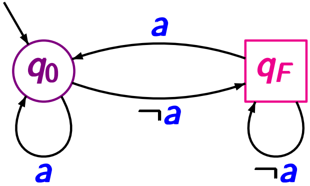
Complementation of DBA



DBA for
"infinitely often ¬a"

complement automaton





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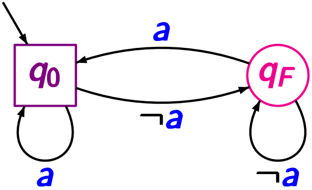
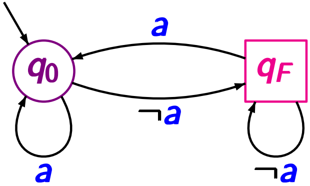


DBA for
"infinitely often a"

Complementation ⟵ $\boxed{\text{fails for DBA}}$



DBA for
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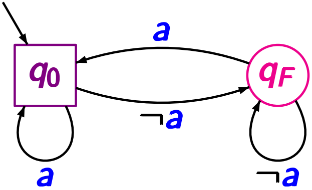

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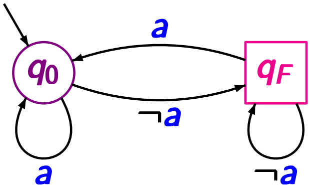

DBA for
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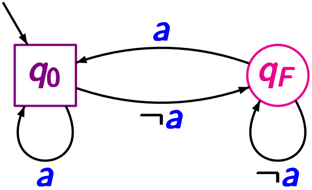

DBA for
"infinitely often a"

There is **no DBA** for the LT-property
"eventually forever a"

On the power of DBA

There is no DBA $\mathcal{A}$ over the alphabet $\Sigma = \{A, B\}$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega\big((A + B)^*.A^\omega \big)$

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Proof: apply the above theorem for $A = \{a\}$, $B = \varnothing$

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The class of **DBA**-recognizable languages is a proper subclass of the class of ω-regular languages

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The class of **DBA**-recognizable languages is a proper subclass of the class of ω-regular languages and is not closed under complementation.

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The class of **DBA**-recognizable languages is a proper subclass of the class of ω-regular languages and is not closed under complementation.

$(A^*.B)^\omega$ "infinitely many B's" DBA-recognizable

$(A+B)^*.A^\omega$ "only finitely many B's" not DBA-recognizable

Generalized NBA (GNBA)

Generalized NBA (GNBA)

A generalized nondeterministic Büchi automaton is a tuple

$$\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$$

where Q, Σ, δ, Q_0 are as in NBA, but $\mathcal{F}$ is a set of accept sets, i.e., $\mathcal{F} \subseteq 2^Q$.

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$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists}\, i \in \mathbb{N} \ \text{ s.t. } \ q_i \in F$$

GNBA $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ as NBA, but $\mathcal{F} \subseteq 2^Q$

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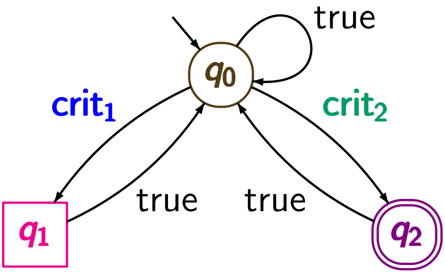
$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists}\, i \in \mathbb{N} \text{ s.t. } q_i \in F$$

accepted language:

$$\mathcal{L}_\omega(\mathcal{G}) \ \overset{\text{def}}{=} \ \{\sigma \in \Sigma^\omega : \sigma \text{ has an accepting run in } \mathcal{G}\ \}$$

Example: GNBA for liveness property

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{crit_1, crit_2\}$



$$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$$

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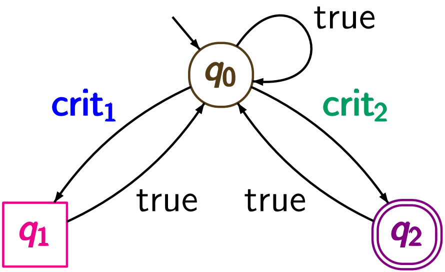

$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

specifies the LT-property

"infinitely often $crit_1$ and infinitely often $crit_2$"

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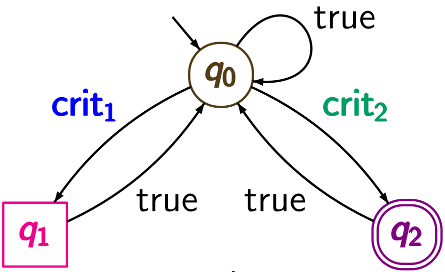

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note: $q_0 \xrightarrow{A} q_1$ implies $A \models crit_1$

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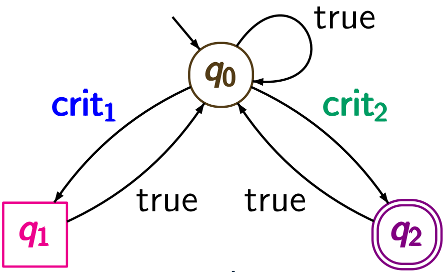

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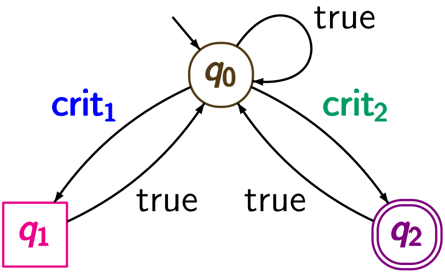
$q_0 \xrightarrow{A} q_2$ implies $A \models crit_2$

hence: if $A_0\, A_1\, A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$ then

$$\overset{\infty}{\exists}\, i \geq 0.\ crit_1 \in A_i \ \wedge\ \overset{\infty}{\exists}\, i \geq 0.\ crit_2 \in A_i$$

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{crit_1, crit_2\}$



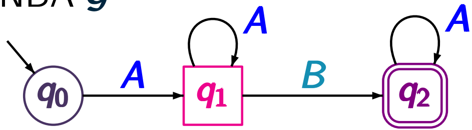
$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

all words $A_0\, A_1\, A_2 \ldots \in \Sigma^\omega$ s.t. $\overset{\infty}{\exists}\, i \geq 0.\ crit_1 \in A_i$ and $\overset{\infty}{\exists}\, i \geq 0.\ crit_2 \in A_i$ have an accepting run of the form:

$$q_0 \ldots q_0\, q_1\, q_0 \ldots q_0\, q_2\, q_0 \ldots q_0\, q_1\, q_0 \ldots q_0\, q_2\ \ldots$$

Examples: GNBA over $\Sigma = \{A, B\}$

GNBA $\mathcal{G}$



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Examples: GNBA over $\Sigma = \{A, B\}$

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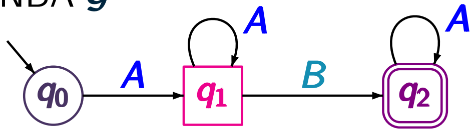
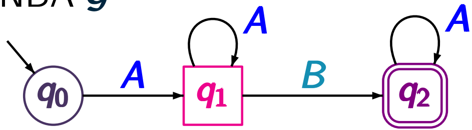


$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

$\mathcal{L}_\omega(\mathcal{G}) =$ **?**

Examples: GNBA over $\Sigma = \{A, B\}$

GNBA $\mathcal{G}$



$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

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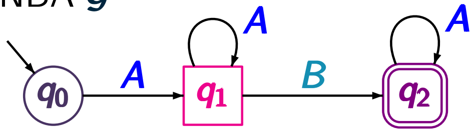
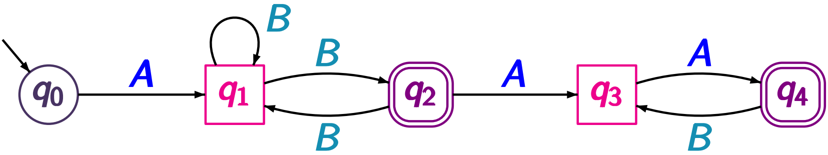
GNBA $\mathcal{G}$



$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

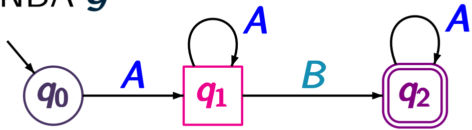
$\mathcal{L}_\omega(\mathcal{G}) = \varnothing$

GNBA $\mathcal{G}'$ with $\mathcal{F}' = \{\{q_1, q_3\}, \{q_2, q_4\}\}$



Examples: GNBA over $\Sigma = \{A, B\}$

GNBA $\mathcal{G}$



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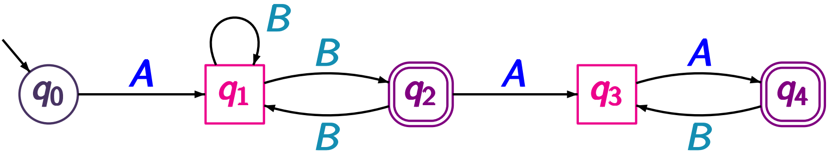
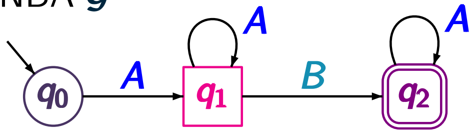


accepted language: **?**

Examples: GNBA over $\Sigma = \{A, B\}$

GNBA $\mathcal{G}$



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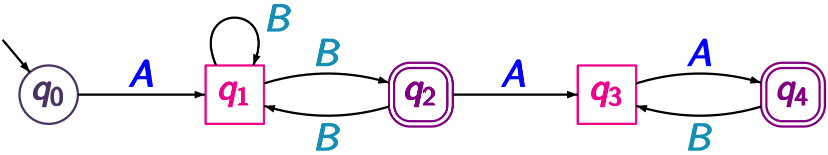
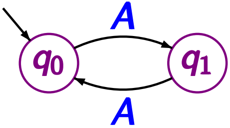


accepted language: $A.B^\omega + A.B^+.A.(A.B)^\omega$

Empty acceptance condition

Empty acceptance condition

NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:



acceptance set $F = \varnothing$

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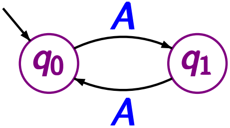

set of acceptance sets

$\mathcal{F} = \varnothing$

Empty acceptance condition

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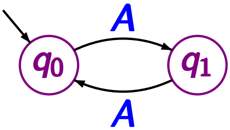

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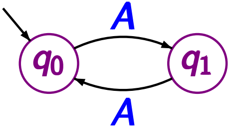

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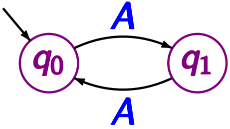

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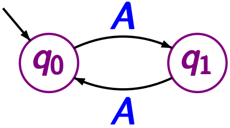
GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:



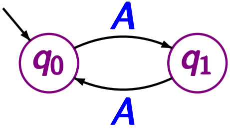
set of acceptance sets

$\mathcal{F} = \varnothing$

$\mathcal{L}_\omega(\mathcal{G}) = \textbf{?}$

Empty acceptance condition

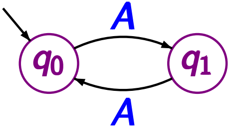
NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:



acceptance set $F = \varnothing$

$\mathcal{L}_\omega(\mathcal{A}) = \varnothing$

GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:



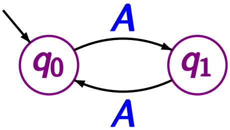
set of acceptance sets

$\mathcal{F} = \varnothing$

$\mathcal{L}_\omega(\mathcal{G}) = \{A^\omega\}$

Empty acceptance condition

NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:



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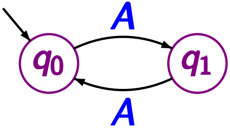
GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:



set of acceptance sets

$\mathcal{F} = \varnothing$

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$$\mathcal{L}_\omega(\mathcal{G}) = \left\{ \begin{array}{l} \text{set of all infinite words} \\ \text{that have an infinite run} \end{array} \right.$$

Correct or wrong?

For every GNBA $\mathcal{G}$ there exists a GNBA $\mathcal{G}'$ such that

- $\mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{G}')$
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correct

$$\begin{array}{rcl} \text{GNBA } \mathcal{G} & = & (Q, \Sigma, \delta, Q_0, \varnothing) \\ \Downarrow & & \\ \text{GNBA } \mathcal{G}' & = & (Q, \Sigma, \delta, Q_0, \{Q\}) \end{array}$$

From GNBA to NBA

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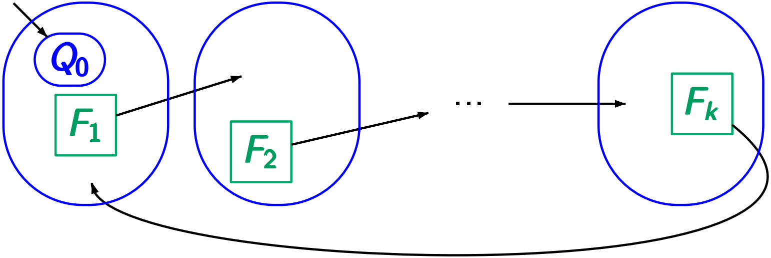

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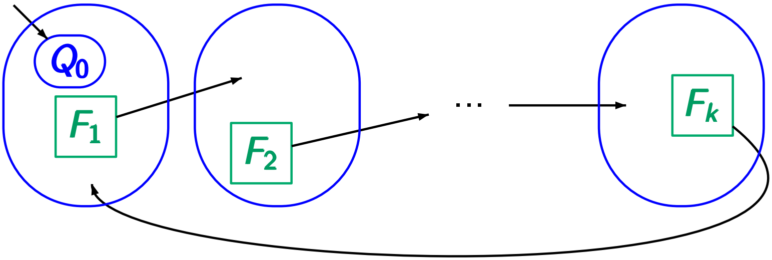
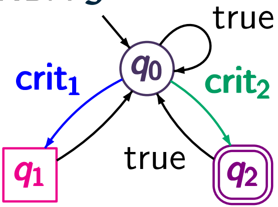


size of the NBA: $\mathit{size}(\mathcal{A}) = \mathcal{O}(\mathit{size}(\mathcal{G}) \cdot |\mathcal{F}|)$

Example: from GNBA to NBA

GNBA $\mathcal{G}$



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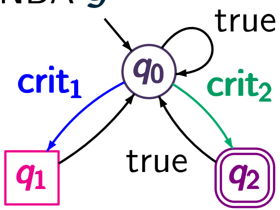

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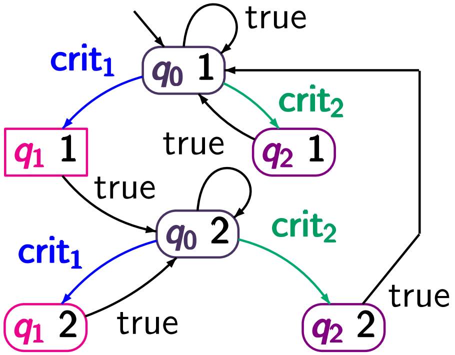

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Intersection for NBA

$$\left.\begin{aligned} \mathcal{A}_1 &= (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 &= (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{aligned}\right\} \text{ two } \textbf{NBA}$$

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recall:
intersection for finite automata **NFA** $\mathcal{A}_1$ and $\mathcal{A}_2$ is realized by a product construction that

- runs $\mathcal{A}_1$ and $\mathcal{A}_2$ in parallel (synchronously)
- checks whether both end in a final state

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$\leadsto$ product of $\mathcal{A}_1$ and $\mathcal{A}_2$ yields a **GNBA**

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