Regular LT properties

idea: define regular **LT** properties to be those languages of infinite words over the alphabet 2^{AP} that have a representation by a finite automata

- regular safety properties:
  NFA-representation for the bad prefixes

- representation other regular LT properties by
  - ω-automata, i.e., acceptors for infinite words
  - ω-regular expressions

Regular expressions over Σ

$$\boxed{\alpha \;::=\; \emptyset \mid \epsilon \mid A \mid \alpha_1 + \alpha_2 \mid \alpha_1 . \alpha_2 \mid \alpha^*}$$

where $A \in \Sigma$

semantics: $\alpha \mapsto \mathcal{L}(\alpha) \subseteq \Sigma^*$ language of finite words

$\mathcal{L}(\emptyset) = \varnothing$	$\mathcal{L}(\epsilon) = \{\varepsilon\}$		$\mathcal{L}(A) = \{A\}$
$\mathcal{L}(\alpha_1 + \alpha_2)$	$=$	$\mathcal{L}(\alpha_1) \cup \mathcal{L}(\alpha_2)$	union
$\mathcal{L}(\alpha_1 . \alpha_2)$	$=$	$\mathcal{L}(\alpha_1)\mathcal{L}(\alpha_2)$	concatenation
$\mathcal{L}(\alpha^*)$	$=$	$\mathcal{L}(\alpha)^*$	Kleene closure

regular expressions:

$$\alpha ::= \emptyset \mid \epsilon \mid A \mid \alpha_1 + \alpha_2 \mid \alpha_1 . \alpha_2 \mid \alpha^*$$

ω-regular expressions:

regular expressions $+$ ω-operator α^ω

Kleene star: "finite repetition"

ω-operator: "infinite repetition"

for $L \subseteq \Sigma^*$:

$$L^\omega \stackrel{\text{def}}{=} \{ w_1 w_2 w_3 \ldots : w_i \in L \text{ for all } i \geq 1 \}$$

note: $L^\omega \subseteq \Sigma^\omega$ if $\varepsilon \notin L$

Syntax and semantics of ω-regular expressions

syntax of ω-regular expressions over alphabet Σ:

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega \quad \text{where}$$

α_i, β_i are regular expressions over Σ s.t. $\varepsilon \notin \mathcal{L}(\beta_i)$

semantics: the language generated by γ is:

$$\mathcal{L}_\omega(\gamma) \stackrel{\text{def}}{=} \bigcup_{1 \leq i \leq n} \mathcal{L}(\alpha_i)\mathcal{L}(\beta_i)^\omega \subseteq \Sigma^\omega$$

- language of $(A^*.B)^\omega$ = set of all infinite words over $\Sigma = \{A, B\}$ containing infinitely many B's
- language of $(A^*.B)^\omega + (B^*.A)^\omega$ = set of all infinite words over Σ with infinitely many A's *or* B's = Σ^ω

ω-regular languages

syntax of ω-regular expressions over alphabet Σ:

$$\gamma = \alpha_1 . \beta_1^\omega + \ldots + \alpha_n . \beta_n^\omega \quad \text{where}$$

α_i, β_i are regular expressions over Σ s.t. $\varepsilon \notin \mathcal{L}(\beta_i)$

semantics: the language generated by γ is:

$$\mathcal{L}_\omega(\gamma) \stackrel{\text{def}}{=} \bigcup_{1 \leq i \leq n} \mathcal{L}(\alpha_i)\mathcal{L}(\beta_i)^\omega \subseteq \Sigma^\omega$$

A language $L \subseteq \Sigma^\omega$ is called ω-regular iff there exists an ω-regular expression γ s.t.

$$L = \mathcal{L}_\omega(\gamma)$$

Provide an ω-regular expression for ...

alphabet $\Sigma = \{A, B\}$

- set of all infinite words over Σ containing only finitely many A's

$$(A + B)^*.B^\omega$$

- set of all infinite words where each A is followed immediately by letter B

$$(B^*.A.B)^*.B^\omega + (B^*.A.B)^\omega$$

- set of all infinite words where each A is followed eventually by letter B

$$(B^*.A^+.B)^*.B^\omega + (B^*.A^+.B)^\omega \equiv (A^*.B)^\omega$$

where $\alpha^+ \stackrel{\text{def}}{=} \alpha.\alpha^*$.

Let E be an LT-property over AP, i.e., $E \subseteq \left(2^{AP}\right)^\omega$

> E is called an ω-regular property iff there exists an ω-regular expression γ over 2^{AP} s.t. $E = \mathcal{L}_\omega(\gamma)$

Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \vee \neg b$

  $(\emptyset + \{a\} + \{a, b\})^\omega$

> Each invariant is ω-regular

> Let Φ be an invariant condition and let
>
> $$\{A \subseteq AP : A \models \Phi\} = \{A_1, \ldots, A_k\}$$
>
> Then: invariant "always Φ" $\widehat{=} \ (A_1 + \ldots + A_k)^\omega$

ω-regular properties

Let E be an LT-property over AP, i.e., $E \subseteq \left(2^{AP}\right)^{\omega}$

E is called an ω-regular property iff there exists an ω-regular expression γ over 2^{AP} s.t. $E = \mathcal{L}_{\omega}(\gamma)$

Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \vee \neg b$

  $$(\emptyset + \{a\} + \{a, b\})^{\omega}$$

  Indeed: each invariant is ω-regular

- "infinitely often a"

  $$\big((\emptyset + \{b\})^{*}.(\{a\} + \{a, b\})\big)^{\omega}$$

ω-regular properties

Let E be an LT-property over AP, i.e., $E \subseteq 2^{AP}$.

E is called an ω-regular property iff there exists an ω-regular expression γ over 2^{AP} s.t. $E = \mathcal{L}_\omega(\gamma)$

Examples for $AP = \{a, b\}$:

- "always a" (or any other invariant)
- "infinitely often a"
- "eventually a"

$$(2^{AP})^*.(\{a\} + \{a, b\}).(2^{AP})^\omega$$

- "from some moment on a"

$$(2^{AP})^*.(\{a\} + \{a, b\})^\omega$$

Symbolic notation

Examples for $AP = \{a, b\}$

- invariant with invariant condition $a \vee \neg b$

$$(a \vee \neg b)^\omega \quad \widehat{=} \quad (\emptyset + \{a\} + \{a, b\})^\omega$$

- "infinitely often a"

$$((\neg a)^*.a)^\omega \quad \widehat{=} \quad ((\emptyset + \{b\})^*.(\{a\} + \{a, b\}))^\omega$$

- "from some moment on a":

$$true^*.a^\omega$$

- "whenever a then b will hold somewhen later"

$$((\neg a)^*.a.true^*.b)^*.(\neg a)^\omega + ((\neg a)^*.a.true^*.b)^\omega$$

Nondeterministic Büchi automata (NBA)

NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \to 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

run for a word $A_0\, A_1\, A_2\, \ldots \in \Sigma^\omega$:

state sequence $\pi = q_0\, q_1\, q_2\, \ldots$ where $q_0 \in Q_0$
and $q_{i+1} \in \delta(q_i, A_i)$ for $i \geq 0$

run π is accepting if $\overset{\infty}{\exists}\, i \in \mathbb{N}.\ q_i \in F$

Nondeterministic Büchi automata (NBA)

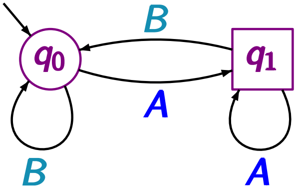
NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet
- $\delta : Q \times \Sigma \to 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

accepted language $\mathcal{L}_\omega(\mathcal{A}) \subseteq \Sigma^\omega$ is given by:

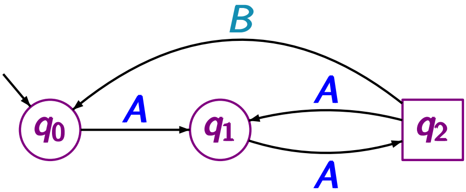
$\mathcal{L}_\omega(\mathcal{A}) \overset{\text{def}}{=}$ set of infinite words over Σ that have an accepting run in $\mathcal{A}$

Examples for NBA over $\Sigma = \{A, B\}$



accepted language:
set of all infinite words that contain infinitely many A's

$$(B^*.A)^\omega$$



accepted language:
"every B is preceded by a positive even number of A's"

$$((A.A)^+.B)^\omega + ((A.A)^+.B)^*.A^\omega$$

NBA for LT properties

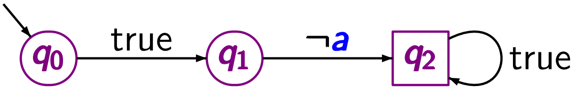
NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

- Q finite set of states
- Σ alphabet $\longleftarrow$ $\boxed{\text{here: } \Sigma = 2^{AP}}$
- $\delta : Q \times \Sigma \rightarrow 2^Q$ transition relation
- $Q_0 \subseteq Q$ set of initial states
- $F \subseteq Q$ set of final states, also called accept states

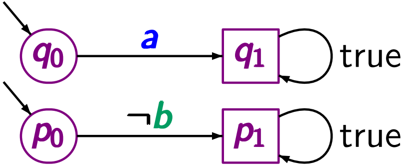
accepted language $\mathcal{L}_\omega(\mathcal{A})$ is an LT-property:

$\mathcal{L}_\omega(\mathcal{A})$ = set of infinite words over 2^{AP} that have an accepting run in $\mathcal{A}$

NBA for LT properties



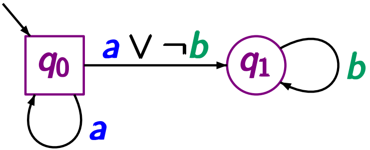
$$\mathcal{L}_\omega(\mathcal{A}) \;\widehat{=}\; \text{true}.\,\neg a.\,\text{true}^\omega$$



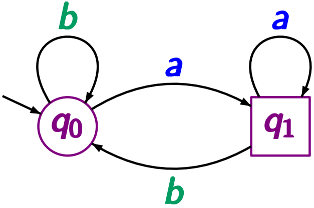
$(a \vee \neg b).\text{true}^\omega$

set of atomic propositions $AP = \{a, b\}$

NBA for LT properties



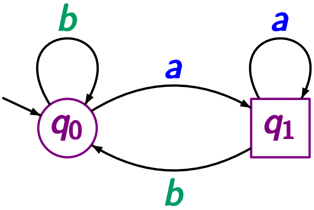
"always a" $\widehat{=}$ a^ω



"infinitely often a and always $a \vee b$"

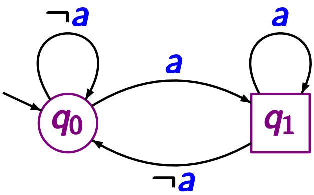
$$\widehat{=} \left((a \vee b)^*.a\right)^\omega$$

NBA for LT properties



"infinitely often a and always $a \vee b$"

$$\big((a \vee b)^*.a\big)^\omega$$



"infinitely often a"

$$\big((\neg a)^*.a\big)^\omega$$

From NBA to ω-regular expressions

For each NBA $\mathcal{A}$ there is an ω-regular expression γ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

Proof. Let $\mathcal{A}$ be an NBA $(Q, \Sigma, \delta, Q_0, F)$ and $q, p \in Q$.

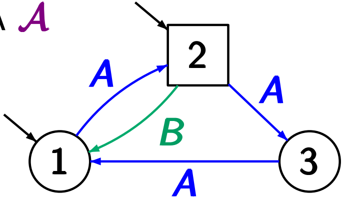
Let $\mathcal{A}_{q,p}$ be the NFA $(Q, \Sigma, \delta, q, \{p\})$. Then:

$$\mathcal{L}_\omega(\mathcal{A}) = \bigcup_{q \in Q_0} \bigcup_{p \in F} \mathcal{L}(\mathcal{A}_{q,p}) \left(\mathcal{L}(\mathcal{A}_{p,p}) \setminus \{\varepsilon\} \right)^\omega$$

is ω-regular as $\mathcal{L}(\mathcal{A}_{q,p})$ and $\mathcal{L}(\mathcal{A}_{p,p}) \setminus \{\varepsilon\}$ are regular

Example: NBA $\rightsquigarrow$ ω-regular expression

NBA $\mathcal{A}$



$$\mathcal{L}_\omega(\mathcal{A}) = L_{12}(L'_{22})^\omega \cup L_{22}(L'_{22})^\omega$$

$$L_{12} = \mathcal{L}(\mathcal{A}_{12})$$

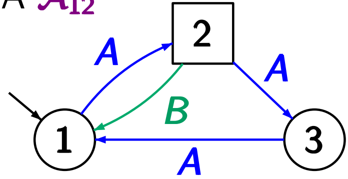
$$L_{22} = \mathcal{L}(\mathcal{A}_{22})$$

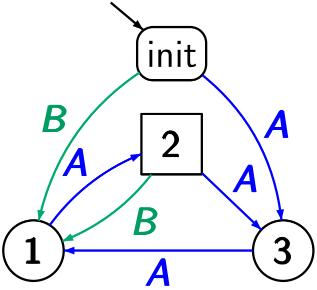
$$L'_{22} = L_{22} \setminus \{\varepsilon\}$$

$L_{12} \mathrel{\widehat{=}} A.(B.A + A.A.A)^*$

$L'_{22} \mathrel{\widehat{=}} (B.A + A.A.A)^+$

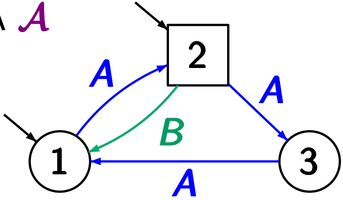
NFA $\mathcal{A}_{12}$





Example: NBA $\rightsquigarrow$ ω-regular expression

NBA $\mathcal{A}$



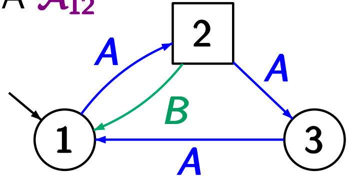
language of $\mathcal{A}$:

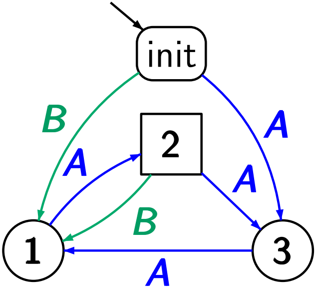
$$
\begin{aligned}
& A.(B.A + A.A.A)^\omega \\
+ \;& (B.A + A.A.A)^\omega \\
\equiv \;& (A + \varepsilon).(B.A + A.A.A)^\omega
\end{aligned}
$$

$L_{12} \mathrel{\widehat{=}} A.(B.A + A.A.A)^*$

$L'_{22} \mathrel{\widehat{=}} (B.A + A.A.A)^+$

NFA $\mathcal{A}_{12}$





From ω-regular expressions to NBA

For each ω-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

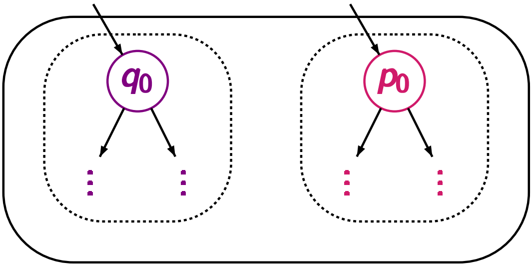
there exists an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

Proof. consider NFA $\mathcal{A}_i$ for α_i and $\mathcal{B}_i$ for β_i

- construct NBA $\mathcal{B}_i^\omega$ for β_i^ω
- construct NBA $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i.\beta_i^\omega$
- construct **NBA** for $\bigcup_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$ $\longleftarrow$

NBA are closed under union

NBA for $\mathcal{L}_\omega(\mathcal{A}_1) \cup \mathcal{L}_\omega(\mathcal{A}_2)$



From ω-regular expressions to NBA

For each ω-regular expression

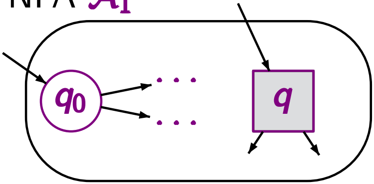
$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

there exists an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

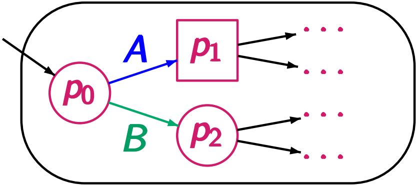
Proof. consider NFA $\mathcal{A}_i$ for α_i and $\mathcal{B}_i$ for β_i

- construct NBA $\mathcal{B}_i^\omega$ for β_i^ω
- construct **NBA** $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i.\beta_i^\omega$ $\longleftarrow$
- construct NBA for $\bigcup\limits_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$

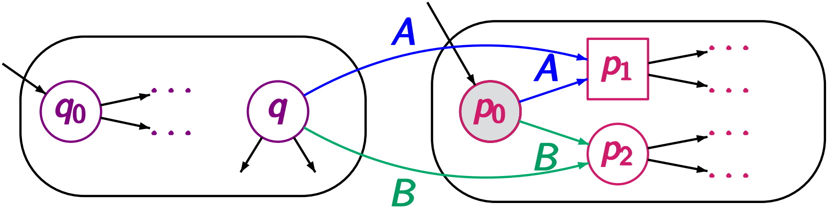
NFA $\mathcal{A}_1$



NBA $\mathcal{A}_2$



NBA for $\mathcal{L}(\mathcal{A}_1).\mathcal{L}_\omega(\mathcal{A}_2)$:



accept states as in $\mathcal{A}_2$

For each ω-regular expression

$$\gamma = \alpha_1.\beta_1^\omega + \ldots + \alpha_n.\beta_n^\omega$$

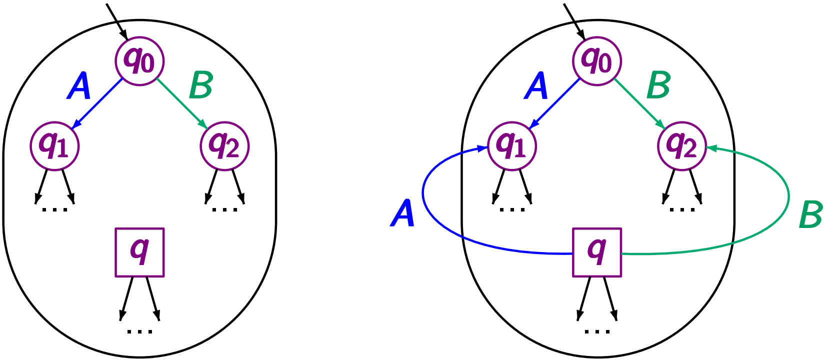
there exists an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$.

Proof. consider NFA $\mathcal{A}_i$ for α_i and $\mathcal{B}_i$ for β_i

- construct **NBA** $\mathcal{B}_i^\omega$ for β_i^ω ⟵
- construct NBA $\mathcal{C}_i = \mathcal{A}_i \mathcal{B}_i^\omega$ for $\alpha_i.\beta_i^\omega$
- construct NBA for $\bigcup\limits_{1 \leq i \leq n} \mathcal{L}_\omega(\mathcal{C}_i)$

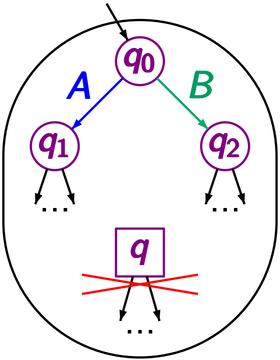
ω-operator for NFA

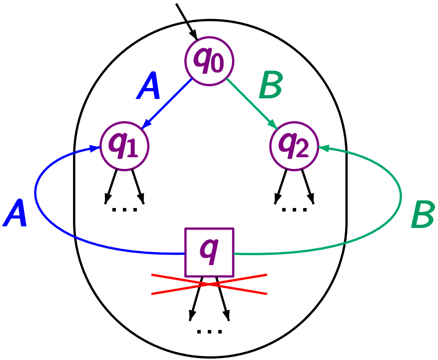
NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$ $\rightsquigarrow$ NBA $\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$



wrong !

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$ $\rightsquigarrow$ NBA $\mathcal{A}^\omega$ for language $L^\omega \subseteq \Sigma^\omega$





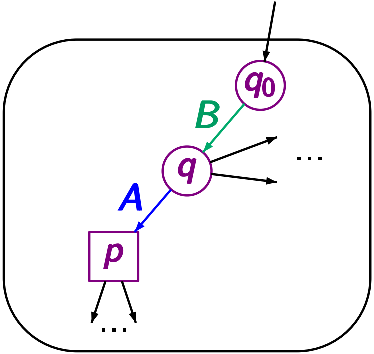
wrong !

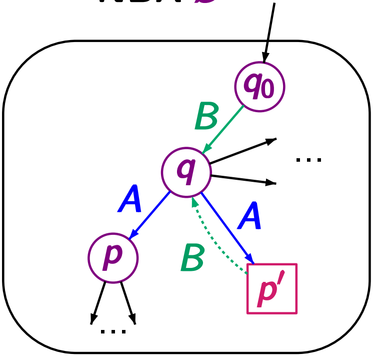
... correct, if $\delta(q, x) = \varnothing \quad \forall q \in F \; \forall x \in \Sigma$

NFA $\mathcal{A}$ for language $L \subseteq \Sigma^+$ $\Longrightarrow$ **NFA** $\mathcal{B}$ for L s.t. all final states are terminal

$\Downarrow$

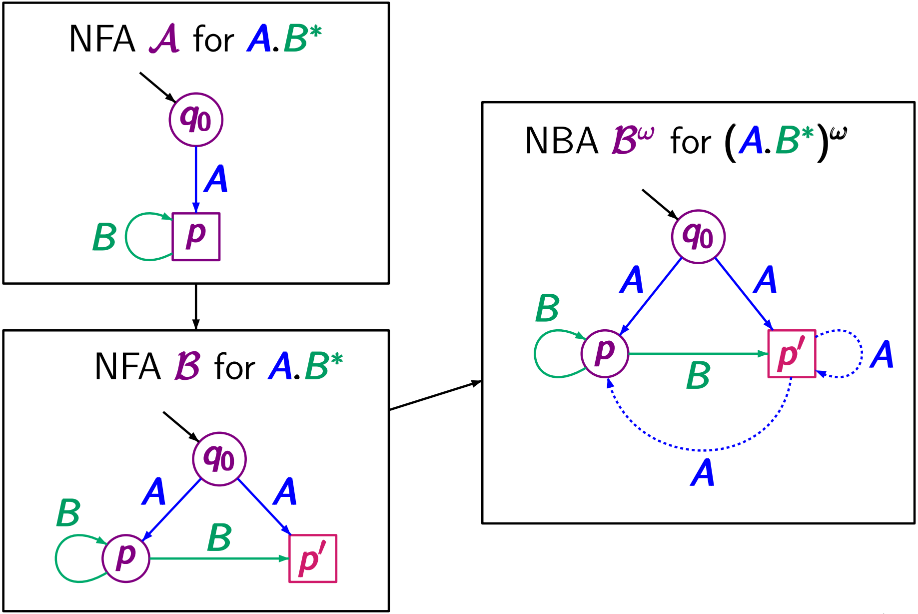
NBA $\mathcal{B}^\omega$





$$\mathcal{L}(\mathcal{A})^\omega = \mathcal{L}_\omega(\mathcal{B}^\omega)$$

Example: ω-operator for NFA



Equivalence of ω-regular expressions and NBA

(1) For each NBA $\mathcal{A}$ there exists an ω-regular expression γ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

(2) For each ω-regular expression γ there exists an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\gamma)$

Corollary:

If E be an LT property, i.e., $E \subseteq (2^{AP})^\omega$, then:

E is ω-regular iff $E = \mathcal{L}_\omega(\mathcal{A})$ for some **NBA** $\mathcal{A}$ over the alphabet 2^{AP}

remind: Kleene's theorem for regular languages:

The class of regular languages is closed under

- union, intersection, complementation
- concatenation and Kleene star

The class of ω-regular languages is closed under union, intersection and complementation.

Closure properties of ω-regular properties

The class of ω-regular languages is closed under union, intersection and complementation.

- *union:*
  obvious from definition of ω-regular expressions
- *intersection:*
  will be discussed later
  relies on a certain product construction for NBA
- *complementation:*
  much more difficult than for NFA,
  via other types of ω-automata

Nonemptiness for NBA

given: NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

question: does $\mathcal{L}_\omega(\mathcal{A}) \neq \varnothing$ hold **?**

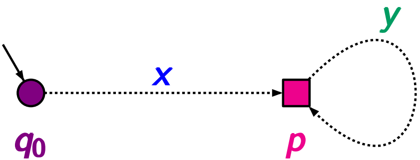
Nonemptiness for NBA

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\mathcal{L}_\omega(\mathcal{A}) \neq \varnothing \quad \text{iff} \quad \exists q_0 \in Q_0 \; \exists p \in F \; \exists x \in \Sigma^* \; \exists y \in \Sigma^+ . \\ p \in \delta(q_0, x) \cap \delta(p, y)$$

↑

there exists a reachable accept state $p \in F$ that belongs to a cycle



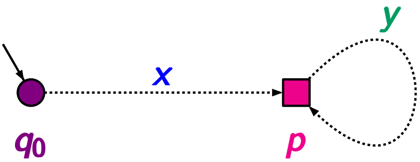
Nonemptiness for NBA

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\mathcal{L}_\omega(\mathcal{A}) \neq \varnothing \quad \text{iff} \quad \exists q_0 \in Q_0 \, \exists p \in F \, \exists x \in \Sigma^* \, \exists y \in \Sigma^+. \; p \in \delta(q_0, x) \cap \delta(p, y)$$

$$\text{iff} \quad \text{there exist finite words } x, y \in \Sigma^* \text{ s.t. } y \neq \varepsilon \text{ and } xy^\omega \in \mathcal{L}_\omega(\mathcal{A})$$

"ultimatively periodic words"



Nonemptiness for NBA

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NBA. Then:

$$\begin{array}{lll}
\mathcal{L}_\omega(\mathcal{A}) \neq \varnothing & \text{iff} & \exists q_0 \in Q_0 \ \exists p \in F \ \exists x \in \Sigma^* \ \exists y \in \Sigma^+. \\
 & & \qquad p \in \delta(q_0, x) \cap \delta(p, y) \\
 & \text{iff} & \text{there exist finite words } x, y \in \Sigma^* \\
 & & \qquad \text{s.t. } y \neq \varepsilon \text{ and } xy^\omega \in \mathcal{L}_\omega(\mathcal{A})
\end{array}$$

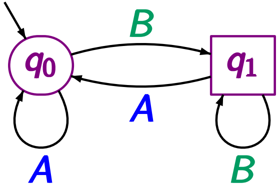
The emptiness problem for NBA is solvable
by means of graph algorithms in time $\mathcal{O}(\, poly(\mathcal{A})\,)$

Deterministic Büchi automata (DBA)

A DBA is an NBA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ such that

- $\mathcal{A}$ has a unique initial state, i.e., Q_0 is a singleton
- $\big|\delta(q, A)\big| \leq 1$ for all $q \in Q$ and $A \in \Sigma$

notation: $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ if $Q_0 = \{q_0\}$



DBA for "infinitely often B"

alphabet $\Sigma = \{A, B\}$

Determinization by powerset construction

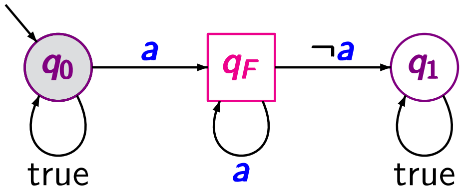
well-known:

the powerset construction for the determinization (and complementation) of finite automata **(NFA)**

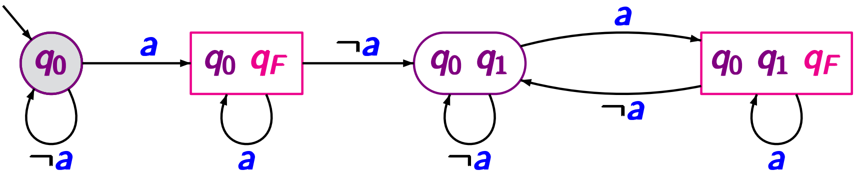
question:

does the powerset construction also work for Büchi automata **(NBA)** **?**

NBA for "eventually forever a"



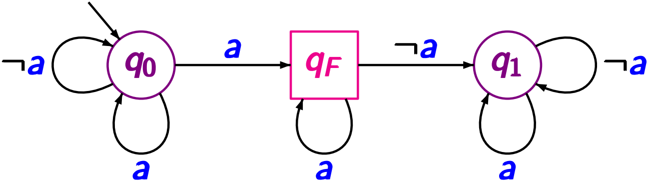
powerset construction



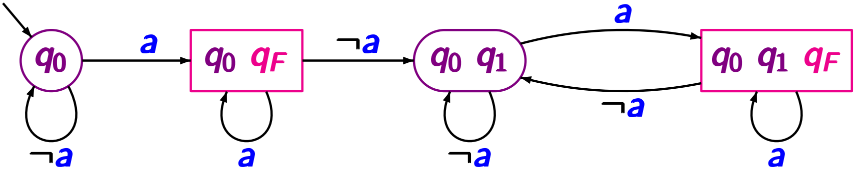
e.g., $\delta(q_0, a) = \{q_0, q_F\}$ and $\delta(q_0, \neg a) = \{q_0\}$

Powerset construction ← $\boxed{\text{fails for NBA}}$

NBA for "eventually forever a"



powerset construction



DBA for "infinitely often a"

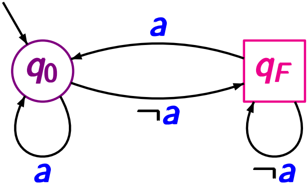
Complementation of DBA

well-known:

DFA can be complemented by complementation of the acceptance set

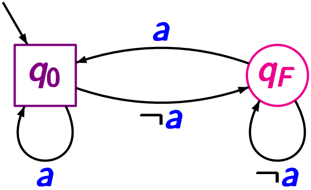
question:

does this also work for **DBA** ?



DBA for
"infinitely often $\neg a$"

complement automaton



DBA for
"infinitely often a"

There is **no DBA** for the LT-property
"eventually forever a"

There is no DBA $\mathcal{A}$ over the alphabet $\Sigma = \{A, B\}$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega\big((A + B)^*.A^\omega \big)$

Hence: there is no DBA for the LT-property "eventually forever a"

Proof: apply the above theorem for $A = \{a\}$, $B = \varnothing$

The class of **DBA**-recognizable languages is a proper subclass of the class of ω-regular languages and is not closed under complementation.

On the power of DBA

There is no DBA $\mathcal{A}$ over the alphabet $\Sigma = \{A, B\}$ such that $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega\big((A + B)^*.A^\omega \big)$

The class of **DBA**-recognizable languages is a proper subclass of the class of ω-regular languages and is not closed under complementation.

$(A^*.B)^\omega$	"infinitely many B's"	DBA-recognizable
$(A+B)^*.A^\omega$	"only finitely many B's"	not DBA-recognizable

Generalized NBA (GNBA)

A generalized nondeterministic Büchi automaton is a tuple

$$\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$$

where Q, Σ, δ, Q_0 are as in NBA, but $\mathcal{F}$ is a set of accept sets, i.e., $\mathcal{F} \subseteq 2^Q$.

A run $q_0\, q_1\, q_2 \ldots$ for some infinite word $\sigma \in \Sigma^\omega$ is called accepting if each accept set is visited infinitely often, i.e.,

$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists} i \in \mathbb{N} \ \text{ s.t. } \ q_i \in F$$

Accepted language of a GNBA

GNBA $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ as NBA, but $\mathcal{F} \subseteq 2^Q$

A run $q_0\, q_1\, q_2\, \ldots$ for some infinite word $\sigma \in \Sigma^\omega$ is accepting if

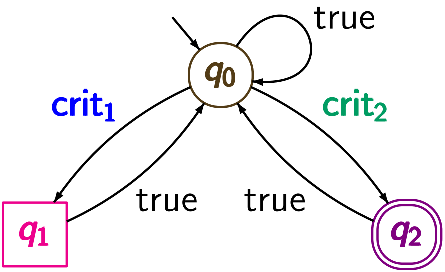
$$\forall F \in \mathcal{F} \ \overset{\infty}{\exists}\, i \in \mathbb{N} \text{ s.t. } q_i \in F$$

accepted language:

$$\mathcal{L}_\omega(\mathcal{G}) \overset{\text{def}}{=} \{\sigma \in \Sigma^\omega : \sigma \text{ has an accepting run in } \mathcal{G}\}$$

Example: GNBA for liveness property

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{crit_1, crit_2\}$



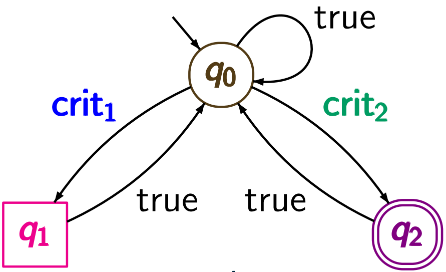
$$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$$

specifies the LT-property

"infinitely often $crit_1$ and infinitely often $crit_2$"

Example: GNBA for liveness property

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{crit_1, crit_2\}$



$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

note: $q_0 \xrightarrow{A} q_1$ implies $A \models crit_1$

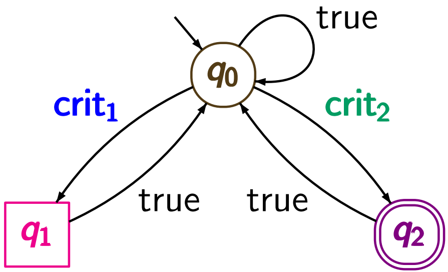
$q_0 \xrightarrow{A} q_2$ implies $A \models crit_2$

hence: if $A_0 A_1 A_2 \ldots \in \mathcal{L}_\omega(\mathcal{G})$ then

$\overset{\infty}{\exists} i \geq 0.\ crit_1 \in A_i \ \wedge\ \overset{\infty}{\exists} i \geq 0.\ crit_2 \in A_i$

Example: GNBA for liveness property

GNBA $\mathcal{G}$ over $\Sigma = 2^{AP}$ where $AP = \{crit_1, crit_2\}$



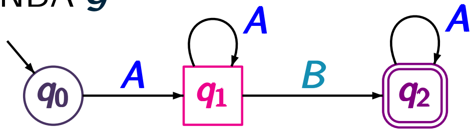
$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

all words $A_0\, A_1\, A_2 \ldots \in \Sigma^\omega$ s.t. $\overset{\infty}{\exists}\, i \geq 0.\ crit_1 \in A_i$ and $\overset{\infty}{\exists}\, i \geq 0.\ crit_2 \in A_i$ have an accepting run of the form:

$$q_0 \ldots q_0\, q_1\, q_0 \ldots q_0\, q_2\, q_0 \ldots q_0\, q_1\, q_0 \ldots q_0\, q_2\, \ldots$$

Examples: GNBA over $\Sigma = \{A, B\}$

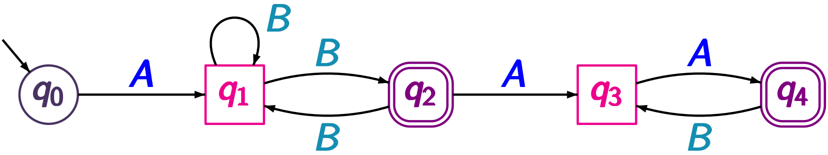
GNBA $\mathcal{G}$



$\mathcal{F} = \{\{q_1\}, \{q_2\}\}$

$\mathcal{L}_\omega(\mathcal{G}) = \varnothing$

GNBA $\mathcal{G}'$ with $\mathcal{F}' = \{\{q_1, q_3\}, \{q_2, q_4\}\}$

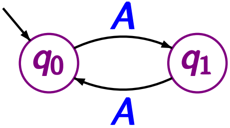


accepted language: $A.B^\omega + A.B^+.A.(A.B)^\omega$

Empty acceptance condition

Empty acceptance condition

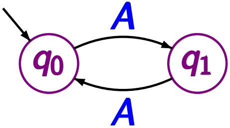
NBA $\mathcal{A}$ over $\Sigma = \{A, B\}$:



acceptance set $F = \varnothing$

$\mathcal{L}_\omega(\mathcal{A}) = \varnothing$

GNBA $\mathcal{G}$ over $\Sigma = \{A, B\}$:



set of acceptance sets

$\mathcal{F} = \varnothing$

$\mathcal{L}_\omega(\mathcal{G}) = \{A^\omega\}$

$$\mathcal{L}_\omega(\mathcal{G}) = \left\{ \begin{array}{l} \text{set of all infinite words} \\ \text{that have an infinite run} \end{array} \right.$$

For every GNBA $\mathcal{G}$ there exists a GNBA $\mathcal{G}'$ such that

- $\mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{G}')$
- the set of acceptance sets of $\mathcal{G}'$ is nonempty

correct

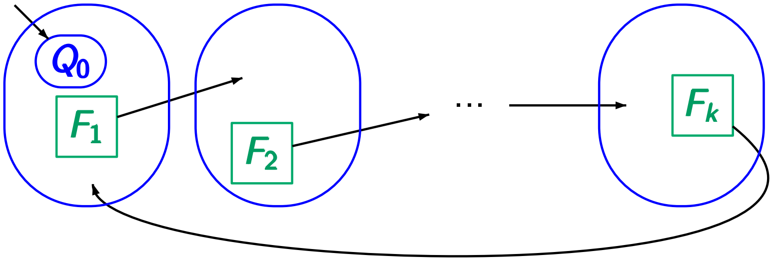
$$
\begin{array}{rcl}
\text{GNBA } \mathcal{G} & = & (Q, \Sigma, \delta, Q_0, \varnothing) \\
\Downarrow & & \\
\text{GNBA } \mathcal{G}' & = & (Q, \Sigma, \delta, Q_0, \{Q\})
\end{array}
$$

From GNBA to NBA

For each **GNBA** $\mathcal{G}$ there exists an **NBA** $\mathcal{A}$ with

$$\mathcal{L}_\omega(\mathcal{G}) = \mathcal{L}_\omega(\mathcal{A})$$

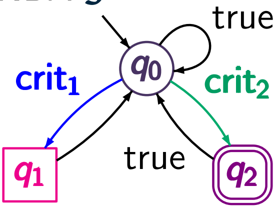
Proof. Let $\mathcal{G} = (Q, \Sigma, \delta, Q_0, \mathcal{F})$ with $\mathcal{F} = \{F_1, \ldots, F_k\}$ and $k \geq 2$. NBA $\mathcal{A}$ results from k copies of $\mathcal{G}$:



size of the NBA: $size(\mathcal{A}) = \mathcal{O}(size(\mathcal{G}) \cdot |\mathcal{F}|)$

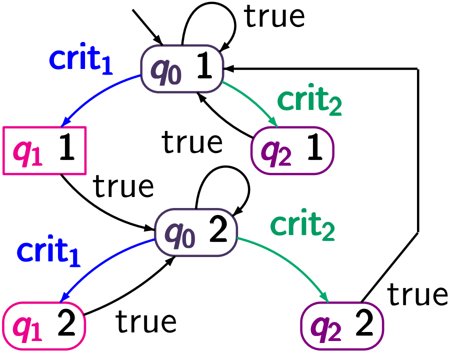
Example: from GNBA to NBA

GNBA $\mathcal{G}$



alphabet $\Sigma = 2^{AP}$ where $AP = \{\text{crit}_1, \text{crit}_2\}$

infinitely often crit$_1$ and infinitely often crit$_2$



NBA $\mathcal{A}$

Closure properties of ω-regular properties

The class of ω-regular languages is closed under union, intersection and complementation.

- *union:*
  obvious from definition of ω-regular expressions
- *intersection:* $\longleftarrow$ using **GNBA**
  via some product construction
- *complementation:*
  via other types of ω-automata
  (not discussed here)

Intersection for NBA

$$\left.\begin{aligned} \mathcal{A}_1 &= (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 &= (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{aligned}\right\} \text{ two } \textbf{NBA}$$

goal: define an **NBA** $\mathcal{A}$ s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$

idea: define $\mathcal{A}_1 \otimes \mathcal{A}_2$ as for **NFA**, i.e.,

- $\mathcal{A}_1$ and $\mathcal{A}_2$ run in parallel (synchronously)
- and check whether both are accepting

↑

i.e., both F_1 and F_2 are visited infinitely often

$\rightsquigarrow$ product of $\mathcal{A}_1$ and $\mathcal{A}_2$ yields a **GNBA**

Intersection for NBA

$$\left.\begin{array}{l} \mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1) \\ \mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2) \end{array}\right\} \text{two NBA}$$

goal: define an NBA $\mathcal{A}$ s.t. $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$

GNBA $\mathcal{G} = \mathcal{A}_1 \otimes \mathcal{A}_2$ $\rightsquigarrow$ $\boxed{\text{equivalent NBA } \mathcal{A}}$

- state space $Q = Q_1 \times Q_2$
- alphabet Σ
- set of initial states: $Q_0 = Q_{0,1} \times Q_{0,2}$
- acceptance condition: $\mathcal{F} = \{F_1 \times Q_2, Q_1 \times F_2\}$
- transition relation:

$$\delta(\langle q_1, q_2\rangle, A) = \{\langle p_1, p_2\rangle : p_1 \in \delta_1(q_1, A), p_2 \in \delta_2(q_2, A)\}$$

Summary: ω-regular languages

The class of ω-regular languages agrees with

- the class of languages given by ω-regular expressions
- the class of **NBA**-recognizable languages
- the class of **GNBA**-recognizable languages

but **DBA** are strictly less expressive

The class of ω-regular languages is closed under union, intersection and complementation.