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Exercise 1

a)

- "At next":
  $\sigma \models \varphi \text{ AX } \psi$ iff there exists a $i \geq 0$ s.t.
  $A_i A_{i+1} A_{i+2} \ldots \models \psi$ and
  $A_i A_{i+1} A_{i+2} \ldots \models \varphi$
- "While":
  $\sigma \models \varphi \text{ WH } \psi$ iff there exists k, n, s.t. for every $k \leq i \leq n$
  $A_k A_{k+1} A_{k+2} \ldots A_n \models \psi \rightarrow A_k A_{k+1} A_{k+2} \ldots A_{n-1} \models \varphi$ and
  $A_n \not\models \psi$
- "Before":
  $\sigma \models \varphi \text{ B } \psi$ iff for every $i > 0$
  $A_i \models \psi \rightarrow A_{i-1} \models \varphi$

b)

- $\varphi \text{ AX } \psi \overset{def.}{=} \mathit{true} \text{ U } (\psi \rightarrow \varphi)$
- $\varphi \text{ WH } \psi \overset{def.}{=} (\psi \wedge \varphi) \text{ U } (\neg\psi)$
- $\varphi \text{ B } \psi \overset{def.}{=} \Box(\bigcirc\psi \rightarrow \varphi)$

Exercise 2

a)

$$
\begin{aligned}
\varphi &= (a \rightarrow \bigcirc\neg b) \text{ W } (a \wedge b) \\
\neg\varphi &= \neg\big[(a \rightarrow \bigcirc\neg b) \text{ W } (a \wedge b)\big] \\
&\Leftrightarrow (\neg(a \wedge b)) \text{ U } (\neg(a \rightarrow \bigcirc\neg b) \wedge \neg(a \wedge b)) \\
&\Leftrightarrow (\neg a \vee \neg b) \text{ U } (\neg(a \rightarrow \bigcirc\neg b) \wedge \neg(a \wedge b)) \\
&\Leftrightarrow (\neg a \vee \neg b) \text{ U } (\neg(\neg a \vee \bigcirc\neg b) \wedge \neg(a \wedge b)) \\
&\Leftrightarrow (\neg a \vee \neg b) \text{ U } ((a \wedge \bigcirc b) \wedge \neg(a \wedge b)) \\
&\Leftrightarrow (\neg a \vee \neg b) \text{ U } ((a \wedge \bigcirc b) \wedge (\neg a \vee \neg b)) \\
&\Leftrightarrow (\neg a \vee \neg b) \text{ U } ((a \wedge \bigcirc b) \wedge (\neg a \vee \neg b)) \\
&\Leftrightarrow (\neg a \vee \neg b) \text{ U } (a \wedge \bigcirc b \wedge \neg b)
\end{aligned}
$$

b)

$P = Words(\varphi)$:

Is a safety property.
Every trace σ, with $\sigma \not\models \varphi$ has the prefix $\{a\}^*\{b\}$. For that we can not find any addition, s.t. φ is again fulfilled.

$P' = Words(\neg\varphi)$:

Is not a safety property.
In order to be a safety property there has to exists a finite $BadPrefix$ for every word, which is not in the language. Considering the word $\sigma = (\emptyset \vee \{a\} \vee \{b\})^\omega$, which is obviously not in $Words(\neg\varphi)$ we can not find a finite $BadPrefix$ to rule it out. So it can not be a safety property.

Exercise 3

a)

For every σ, with $\sigma \models \Diamond\Box\varphi$, also $\sigma \models \Box\Diamond\varphi$ holds, since having from some point on consecutively φ fulfilled also means to encounter infinitely many.
But the other way around does not hold, since having $\sigma = A_0A_1\ldots$, s.t. $A_i \models \varphi$ if $i \mod 2 = 0$ and $A_i \not\models \varphi$ if $i \mod 2 = 1$ (alternating fulfilling φ), then $\sigma \not\models \Diamond\Box\varphi$.
Therefore $\Diamond\Box\varphi \quad \subset \quad \Box\Diamond\varphi$

b)

$\Diamond\Box\varphi \wedge \Diamond\Box\psi \quad \equiv \quad \Diamond(\Box\varphi \wedge \Box\psi)$, since if for some point $i \geq 0$ it holds that $\Box\varphi$ and for some point $j \geq 0, j > i$ it holds that $\Box\psi$, then from J on both φ and ψ both hold continuously. So there ex. a point (j) from where $\Box\varphi \wedge \Box\psi$ holds.
Analogously if $j < i$.
If $i = j$ obviously also the first formula holds.

c)

$\varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi) \quad \subset \quad \Box\Diamond\varphi$

- $\subset$: Let $\sigma \models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$. Then φ holds at the very first element of σ. Also this element fulfills the premise of the implication. So there is an other φ in σ, since $\sigma \models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$. This new φ again fulfills the premise demand and so on. Therefore we will encounter infinitely many φ's.

- $\not\supset$: Let $\sigma = A_0A_1\ldots \models \Box\Diamond\varphi$, but the very first element $A_0 \not\models \varphi$. Then $\sigma \not\models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$, since $A_0 \models \varphi$ would have to hold.

d)

$\not\subseteq$: Let $\sigma = A_0 A_1 \ldots$, with

- $A_i \models \varphi$ for $i \in \{1,2,3,5,6,7\}$
- $A_i \models \psi$ for $i \in \{4,8\}$
- $A_9 \models \pi$

Then $\sigma \models (\varphi \text{ U } \psi) \text{ U } \pi$, but $\sigma \not\models \varphi \text{ U } (\psi \text{ U } \pi)$

$\not\supseteq$: Let $\sigma = A_0 A_1 \ldots$, with

- $A_i \models \varphi$ for $i \in \{1,2,3,4\}$
- $A_i \models \psi$ for $i \in \{5,6,7,8\}$
- $A_9 \models \pi$

Then $\sigma \not\models (\varphi \text{ U } \psi) \text{ U } \pi$, but $\sigma \models \varphi \text{ U } (\psi \text{ U } \pi)$

So $(\varphi \text{ U } \psi) \text{ U } \pi \quad \not\equiv \quad \varphi \text{ U } (\psi \text{ U } \pi)$

Exercise 4

a)

$$
\begin{aligned}
\varphi \text{ R } \psi &= \neg(\neg\varphi \text{ U } \neg\psi) \\
&= \neg(\varphi' \text{ U } \psi'), \text{ where } \varphi' = \neg\varphi \text{ and } \psi' = \neg\psi \\
&\overset{\text{exp. law of U}}{=} \neg\big(\psi' \vee (\varphi' \wedge \bigcirc(\varphi' \text{ U } \psi'))\big) \\
&= \neg\psi' \wedge \neg(\varphi' \wedge \bigcirc(\varphi' \text{ U } \psi')) \\
&= \neg\psi' \wedge (\neg\varphi' \vee \neg \bigcirc (\varphi' \text{ U } \psi')) \\
&= \neg\psi' \wedge (\neg\varphi' \vee \bigcirc\neg(\varphi' \text{ U } \psi')) \\
&\overset{\text{def of } \varphi', \psi'}{=} \neg\neg\psi \wedge (\neg\neg\varphi \vee \bigcirc\neg(\neg\varphi \text{ U } \neg\psi)) \\
&= \psi \wedge (\varphi \vee \bigcirc\neg(\neg\varphi \text{ U } \neg\psi)) \\
&\overset{\text{def. of R}}{=} \psi \wedge (\varphi \vee \bigcirc(\varphi \text{ R } \psi))
\end{aligned}
$$

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b)

$$\begin{aligned}
\varphi \text{ R } \psi &= \neg(\neg\varphi \text{ U } \neg\psi) \\
&= \neg(\varphi' \text{ U } \psi'), \text{ where } \varphi' = \neg\varphi \text{ and } \psi' = \neg\psi \\
&\overset{\text{book page 271}}{=} (\varphi' \wedge \neg\psi') \text{ W } (\neg\varphi' \wedge \neg\psi') \\
&\overset{\text{def of } \varphi',\psi'}{=} (\neg\varphi \wedge \neg\neg\psi) \text{ W } (\neg\neg\varphi \wedge \neg\neg\psi) \\
&= (\neg\varphi \wedge \psi) \text{ W } (\varphi \wedge \psi)
\end{aligned}$$