Prof. Dr. Ir. Dr. h. c. Joost-Pieter Katoen
Christian Hensel, Matthias Volk

# Introduction to Model Checking (Summer Term 2018)

# — Solution 9 (due 2nd July) —

## General Remarks

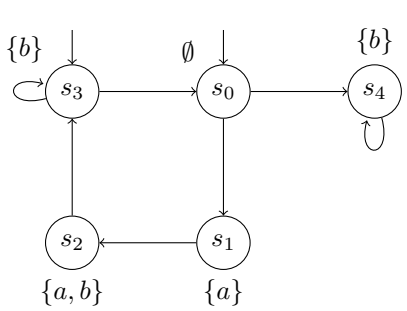
- The exercises are to be solved in groups of *three* students.
- You may hand in your solutions for the exercises just before the exercise class starts at 12:15 or by dropping them into the "Introduction to Model Checking" box at our chair *before 12:00*. Do *not* hand in your solutions via L2P or via e-mail.
- If a task asks you to justify your answer, an explanation of your reasoning is sufficient. If you are required to prove a statement, you need to give a *formal* proof.

## Exercise 1 (4 Points)

Consider the following CTL formulae and the transition system TS outlined on the right:

$\Phi_1 = \forall(a \,\mathsf{U}\, b) \vee \exists \bigcirc \forall\Box\, b$
$\Phi_2 = \forall\Box \forall(a \,\mathsf{U}\, b)$
$\Phi_3 = (a \wedge b) \rightarrow \exists\Box \exists \bigcirc \forall(b \,\mathsf{W}\, a)$
$\Phi_4 = \forall\Box \exists\Diamond \neg(a \vee b)$

TS :




Give the satisfaction sets $Sat(\Phi_i)$ for each CTL formula $\Phi_i$, $1 \leq i \leq 4$.
Does TS $\models \Phi_i$ hold?

### Solution:

For each of the CTL state formulae $\Phi_i$ (and each state subformulae), we have to compute

$$Sat(\Phi_i) = \{s \in S \mid s \models \Phi_i\}$$

From this, we can decide TS $\models \Phi_i$ by checking $S_0 \subseteq Sat(\Phi_i)$.

- $\Phi_1 = \forall(a \,\mathsf{U}\, b) \vee \exists \bigcirc \forall\Box\, b$:
  We compute the satisfaction sets:

$$\begin{aligned} Sat\big(\forall(a \,\mathsf{U}\, b)\big) &= \{s_1, s_2, s_3, s_4\} \\ Sat\big(\forall\Box\, b\big) &= \{s_4\} \\ Sat\big(\exists \bigcirc \forall\Box\, b\big) &= \{s_0, s_4\} \\ Sat\big(\Phi_1\big) &= \{s_0, s_1, s_2, s_3, s_4\} \end{aligned}$$

  Thus, $S_0 \subseteq Sat(\Phi_1) \implies$ TS $\models \Phi_1$.

- $\Phi_2 = \forall\Box\forall(a \,\mathsf{U}\, b)$:
  We compute the satisfaction sets:

$$\begin{aligned} Sat\big(\forall(a \,\mathsf{U}\, b)\big) &= \{s_1, s_2, s_3, s_4\} \\ Sat\big(\Phi_2\big) &= \{s_4\} \end{aligned}$$

  Thus, $S_0 \not\subseteq Sat(\Phi_2) \implies \text{TS} \not\models \Phi_2$.

- $\Phi_3 = (a \wedge b) \rightarrow \exists\Box\exists\bigcirc\forall(b \,\mathsf{W}\, a)$:
  We compute the satisfaction sets:

$$\begin{aligned} Sat\big(a\big) &= \{s_1, s_2\} \\ Sat\big(b\big) &= \{s_3, s_4\} \\ Sat\big(a \wedge b\big) &= \{s_2\} \\ Sat\big(\forall(b \,\mathsf{W}\, a)\big) &= \{s_1, s_2, s_4\} \\ Sat\big(\exists\bigcirc\forall(b \,\mathsf{W}\, a)\big) &= \{s_0, s_1, s_4\} \\ Sat\big(\exists\Box\exists\bigcirc\forall(b \,\mathsf{W}\, a)\big) &= \{s_0, s_4\} \\ Sat\big(\Phi_3\big) &= \{s_0, s_1, s_3, s_4\} \end{aligned}$$

  Thus, $S_0 \subseteq Sat(\Phi_3) \implies \text{TS} \models \Phi_3$.

- $\Phi_4 = \forall\Box\exists\Diamond\neg(a \vee b)$:
  We compute the satisfaction sets:

$$\begin{aligned} Sat\big(a\big) &= \{s_1, s_2\} \\ Sat\big(b\big) &= \{s_3, s_4\} \\ Sat\big(a \vee b\big) &= \{s_1, s_2, s_3, s_4\} \\ Sat\big(\neg(a \vee b)\big) &= \{s_0\} \\ Sat\big(\exists\Diamond\neg(a \vee b)\big) &= \{s_0, s_1, s_2, s_3\} \\ Sat\big(\Phi_4\big) &= \emptyset \end{aligned}$$

  Thus, $S_0 \not\subseteq Sat(\Phi_4) \implies \text{TS} \not\models \Phi_4$.

## Exercise 2 (5 Points)

Prove that $Sat(\exists(\Phi \,\mathsf{W}\, \Psi))$ is the largest set $T$ such that

$$T \subseteq Sat(\Psi) \;\cup\; \{s \in Sat(\Phi) \mid \text{Post}(s) \cap T \neq \emptyset\}. \tag{9.1}$$

### Solution:

We first prove that $T = Sat(\exists(\Phi \,\mathsf{W}\, \Psi))$ satisfies (9.1). For $s \in T$ we obtain by the expansion law

$$\exists(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{W}\, \Psi))$$

that either $s \in Sat(\Psi)$ or $s \in Sat(\Phi)$ *and* there exists $s' \in \text{Post}(s)$ with $s' \in T$. Hence, $s \in Sat(\Psi) \cup \{s \in Sat(\Phi) \mid \text{Post}(s) \cap T \neq \emptyset\}$ and $T$ satisfies (9.1).

It remains to show that $Sat(\exists(\Phi \,\mathsf{W}\, \Psi))$ is the *largest* set that satisfies (9.1). Let $T$ be a set of states such that $T$ satisfies (9.1). We prove that $T \subseteq Sat(\exists(\Phi \,\mathsf{W}\, \Psi))$. For this, let $s_1 \in T$.

- If $s_1 \in Sat(\Psi)$ then $s_1 \in Sat(\exists(\Phi \,\mathsf{W}\, \Psi))$.
- Otherwise $s_1 \in Sat(\Phi)$ and there exists $s_2 \in \text{Post}(s_1)$ with $s_2 \in T$.

– If $s_2 \in Sat(\Psi)$ then there is a path $\pi \in Paths(s_1)$ with the prefix $\widehat{\pi} = s_1 s_2$. $\pi$ satisfies $\Phi \mathsf{W} \Psi$ and therefore $s_1 \in Sat(\exists(\Phi \mathsf{W} \Psi))$.

– Otherwise $s_2 \notin Sat(\Psi)$ and since $s_2 \in T$ it is $s_2 \in Sat(\Phi)$ and there exists $s_3 \in \mathrm{Post}(s_2)$ with $s_3 \in T$. Continuing this inductive reasoning, we either obtain an infinite path $\pi = s_1 s_2 \ldots$ with $\pi \models \Box \Phi$ *or* an initial path fragment $\widehat{\pi} = s_1 s_2 s_3 \ldots s_n$ where for all $0 \leq i < n$ we have $s_n \models \Phi$ and $s_n \models \Psi$. In both cases, there is a path $\pi \in Paths(s)$ with $\pi \models \Phi \mathsf{W} \Psi$ and therefore $s \in Sat(\exists(\Phi \mathsf{W} \Psi))$.

## Exercise 3 (2 + 2 Points)

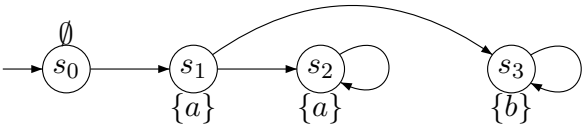
(a) Using an appropriate theorem from the lecture, prove that there does not exist an equivalent LTL-formula for the CTL-formula $\Phi_1 = \forall\Diamond(a \wedge \exists \bigcirc a)$.

(b) Now prove directly (i.e. without the theorem from the lecture) that there does not exist an equivalent LTL-formula for the CTL-formula $\Phi_2 = \forall\Diamond \exists \bigcirc \forall\Diamond \neg a$.
*Hint: Argue by contraposition. In particular, think about trace inclusion versus CTL-equivalence.*

### Solution:

(a) We prove, that there is no equivalent LTL-formula for the CTL-formula $\Phi_1 = \forall\Diamond(a \wedge \exists \bigcirc a)$. By applying the theorem from the lecture, we know that if there is an equivalent LTL formula, it can be obtained by eliminating the path quantifiers from $\Phi_1$. For $\Phi_1$ we obtain the LTL-formula $\varphi_1 = \Diamond(a \wedge \bigcirc a)$ and now prove that $\Phi_1 \not\equiv \varphi_1$.

For this, consider the following transition system TS below.

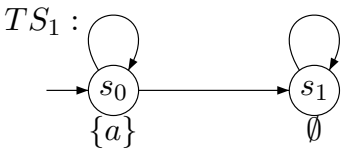



We have $TS \not\models_{\mathrm{LTL}} \varphi_1$ because of the path $\pi = s_0 s_1 s_3^\omega \in Paths(s_0)$ whose trace $\emptyset\,\{a\}\,\{b\}^\omega \not\models \varphi_1$. On the other hand, we have $TS \models_{\mathrm{CTL}} \Phi_1$, because $s_1 \models a \wedge \exists \bigcirc a$ and all paths in $Paths(s_0)$ eventually visit $s_1$.

(b) Now, we prove that there exists no equivalent LTL-formula for the CTL-formula $\Phi_2 = \forall\Diamond \exists \bigcirc \forall\Diamond \neg a$ without the theorem from the lecture.

We do so by contradiction. Assume there exists an LTL formula $\varphi_2$ such that $\Phi_2 \equiv \varphi_2$.

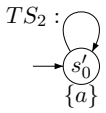
For the transition system $\mathrm{TS}_1$




we have $Sat(\forall\Diamond \neg a) = \{s_1\}$ and therefore $Sat(\exists \bigcirc \forall\Diamond \neg a) = \{s_0, s_1\}$ and trivially $Sat(\Phi_2) = \{s_0, s_1\}$. It follows $\mathrm{TS}_1 \models_{\mathrm{CTL}} \Phi_2$. Because of the assumption $\Phi_2 \equiv \varphi_2$, also $\mathrm{TS} \models_{\mathrm{LTL}} \varphi_2$ holds. We have

$$\mathrm{TS} \models_{\mathrm{LTL}} \varphi_2 \iff Traces(\mathrm{TS}) \subseteq Words(\varphi_2). \tag{9.2}$$

Now consider the trace $\sigma = \{a\}^\omega \in Traces(\mathrm{TS}_1)$, which by (9.2) is also in $Words(\varphi_2)$. For the transition system $\mathrm{TS}_2$




we have $\mathit{Traces}(\mathrm{TS}_2) = \{\{a\}^\omega\} \subseteq \mathit{Words}(\varphi_2)$, which implies $\mathrm{TS}_2 \models_{\mathrm{LTL}} \varphi_2$. On the other hand, we obviously have $\mathrm{TS}_2 \not\models_{\mathrm{CTL}} \Phi_2$ because $\neg a$ is not fulfilled by any state in $\mathrm{TS}_2$. This contradicts $\Phi_2 \equiv \varphi_2$.

## Exercise 4★ (4 + 7* Points)

(a) For an arbitrary transition system $\mathrm{TS} = (S, \mathrm{Act}, \to, S_0, \mathrm{AP}, L)$ without terminal states, let $\mathrm{Reach}_{\mathrm{TS}}(s)$ denote the states reachable from $s \in S$ in TS. In other words $s' \in \mathrm{Reach}_{\mathrm{TS}}(s)$ if and only if there exists a path $\pi = s_0 s_1 \ldots \in \mathit{Paths}(s)$ such that there exists $i \geq 0$ with $s_i = s'$. Show that for all $s \in S$ the following is equivalent:

1. $s \models_{\mathrm{LTL}} \Box\, a$,
2. $s \models_{\mathrm{CTL}} \forall\Box\, a$,
3. $\forall s' \in \mathrm{Reach}_{\mathrm{TS}}(s)\ .\ s' \models a$, and
4. $\forall s' \in \mathrm{Reach}_{\mathrm{TS}}(s)\ .\ s' \models_{\mathrm{CTL}} \forall\Box\, a$.

*Hint:* You may use the fact that for all $s, s', s'' \in S$ we have $s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \wedge s'' \in \mathrm{Reach}_{\mathrm{TS}}(s') \implies s'' \in \mathrm{Reach}_{\mathrm{TS}}(s)$ or, equivalently, $s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \implies \mathrm{Reach}_{\mathrm{TS}}(s') \subseteq \mathrm{Reach}_{\mathrm{TS}}(s)$.

(b) ***This subexercise does not count towards the total number of points that you can achieve. Not solving it does not decrease the percentage of points you achieved while solving it may increase it.**

Prove

$$\forall\Big(a \,\mathsf{U}\, (b \wedge \forall\Box\, a)\Big) \equiv \Box\, a \wedge \Diamond\, b.$$

### Solution:

(a) 1 $\iff$ 2:

$$\begin{aligned} & s \models_{\mathrm{LTL}} \Box\, a \\ \iff\ & \forall \pi \in \mathit{Paths}(s).\pi \models \Box\, a \\ \iff\ & s \models_{\mathrm{CTL}} \forall\Box\, a. \end{aligned}$$

3 $\iff$ 2 $\iff$ $\neg$2 $\iff$ $\neg$3:

$$\begin{aligned} & s \not\models_{\mathrm{CTL}} \forall\Box\, a \\ \iff\ & s \models_{\mathrm{CTL}} \neg\forall\Box\, a \\ \iff\ & s \models_{\mathrm{CTL}} \exists\Diamond\, \neg a \\ \iff\ & \exists \pi = s_0 s_1 \ldots \in \mathit{Paths}(s)\ .\ \exists i \geq 0\ .\ s_i \models \neg a \\ \iff\ & \exists \pi = s_0 s_1 \ldots \in \mathit{Paths}(s)\ .\ \exists i \geq 0\ .\ s_i \not\models a \\ \iff\ & \exists s_i \in \mathrm{Reach}_{\mathrm{TS}}(s)\ .\ s_i \not\models a \end{aligned}$$

$3 \implies 4 \quad \iff \quad \neg 4 \implies \neg 3$:

$$
\begin{aligned}
&\neg \forall s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, s' \models \forall \Box\, a \\
&\iff \exists s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, s' \not\models_{\mathrm{CTL}} \forall \Box\, a \\
&\iff \exists s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, s' \models \neg \forall \Box\, a \\
&\iff \exists s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, s' \models \exists \Diamond\, \neg a \\
&\iff \exists s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, \exists \pi = s_0 s_1 \ldots \in \mathit{Paths}(s') \,.\, \exists i \geq 0 \,.\, s_i \models \neg a \\
&\iff \exists s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, \exists s'' \in \mathrm{Reach}_{\mathrm{TS}}(s') \,.\, s'' \not\models a \\
&\overset{\text{hint}}{\implies} \exists s'' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, s'' \not\models a
\end{aligned}
$$

$4 \implies 3 \quad \iff \quad \neg 3 \implies \neg 4$:

$$
\begin{aligned}
&\neg \forall s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, s' \models a \\
&\iff \exists s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, s' \not\models a \\
&\iff \exists \pi = s_0 s_1 \ldots \in \mathit{Paths}(s) \,.\, \exists i \geq 0 \,.\, s_i \not\models a \\
&\iff s \models \exists \Diamond\, \neg a \\
&\iff s \not\models \neg \exists \Diamond\, \neg a \\
&\iff s \not\models \forall \Box\, a \\
&\implies \exists s' \in \mathrm{Reach}_{\mathrm{TS}}(s) \,.\, s' \not\models \forall \Box\, a
\end{aligned}
$$

(b) Let $\mathrm{TS} = (S, \mathrm{Act}, \to, S_0, \mathrm{AP}, L)$ without terminal states and $s \in S$. We start by showing

$$\underbrace{\Box\, a \wedge \Diamond\, b}_{\varphi} \equiv \underbrace{a \,\mathsf{U}\, (b \wedge \Box\, a)}_{\varphi'}$$

$\implies$:

$$
\begin{aligned}
&A_0 A_1 \ldots \models \Box\, a \wedge \Diamond\, b \\
&\implies \forall i \geq 0 . A_i \ldots \models a \wedge \exists j \geq 0 . A_j \ldots \models b \\
&\implies \forall i \geq 0 . A_i \ldots \models a \wedge \exists j \geq 0 . A_j \ldots \models b \wedge \forall k \geq j . A_k \ldots \models a \\
&\implies \forall i \geq 0 . A_i \ldots \models a \wedge \exists j \geq 0 . A_j \ldots \models b \wedge \forall k \geq j . A_k \ldots \models a \wedge \forall 0 \leq k < j . A_k \ldots \models a \\
&\implies \exists j \geq 0 . A_j \ldots \models b \wedge \forall k \geq j . A_k \ldots \models a \wedge \forall 0 \leq k < j . A_k \ldots \models a \\
&\implies \exists j \geq 0 . A_j \ldots \models b \wedge \Box\, a \wedge \forall 0 \leq k < j . A_k \ldots \models a \\
&\implies A_0 A_1 \ldots \models a \,\mathsf{U}\, b \wedge \Box\, a
\end{aligned}
$$

$\impliedby$:

$$
\begin{aligned}
&A_0 A_1 \ldots \models a \,\mathsf{U}\, b \wedge \Box\, a \\
&\implies \exists j \geq 0 . A_j \ldots \models b \wedge \Box\, a \wedge \forall 0 \leq k < j . A_k \ldots \models a \\
&\implies \exists j \geq 0 . A_j \ldots \models b \wedge \forall k \geq j . A_k \ldots \models a \wedge \forall 0 \leq k < j . A_k \ldots \models a \\
&\implies \exists j \geq 0 . A_j \ldots \models b \wedge \forall k \geq j . A_k \ldots \models a \wedge \forall 0 \leq k < j . A_k \ldots \models a \wedge \forall \ell \geq 0 . A_\ell \ldots \models a \\
&\implies \exists j \geq 0 . A_j \ldots \models b \wedge \forall \ell \geq 0 . A_\ell \ldots \models a \\
&\implies A_0 A_1 \ldots \models \Diamond\, b \wedge \Box\, a
\end{aligned}
$$

That is, the equivalence $\forall\Big(a \;\mathsf{U}\; (b \wedge \forall \Box\, a)\Big) \equiv \Box\, a \wedge \Diamond\, b$ *might* hold, because — according to the theorem from the lecture — $\Box\, a \wedge \Diamond\, b$ is equivalent to the only possible equivalent candidate $a \,\mathsf{U}\, (b \wedge \Box\, a)$. However, it might still be the case that there is no equivalent LTL formula for $\forall\Big(a \;\mathsf{U}\; (b \wedge \forall \Box\, a)\Big)$ and, in particular, $\forall\Big(a \;\mathsf{U}\; (b \wedge \forall \Box\, a)\Big) \not\equiv a \,\mathsf{U}\, (b \wedge \Box\, a)$.

We now prove $\forall\Big(a \,\mathsf{U}\, (b \wedge \forall\Box\, a)\Big) \equiv a \,\mathsf{U}\, (b \wedge \Box\, a)$.

$\Longrightarrow$: Let $s \models \forall\Big(a \,\mathsf{U}\, (b \wedge \forall\Box\, a)\Big)$. Using the semantics, we have

$$\begin{aligned} & s \models \forall\Big(a \,\mathsf{U}\, (b \wedge \forall\Box\, a)\Big) \\ \iff & \forall\pi \in \mathit{Paths}(s).\exists i \geq 0.\Big(\pi[i] \models b \wedge \forall\Box\, a\Big) \wedge \forall 0 \leq j < i.\pi[j] \models a \end{aligned}$$

We fix an arbitrary path $\pi = s_0 s_1 \ldots \in \mathit{Paths}(s)$ and have that

$$\begin{aligned} & \exists i \geq 0.\Big(s_i \models b \wedge \forall\Box\, a\Big) \wedge \forall 0 \leq j < i.s_j \models a \\ \Longrightarrow\ & \exists i \geq 0.s_i \models b \wedge s_i \models \forall\Box\, a \wedge \forall 0 \leq j < i.s_j \models a \end{aligned}$$

In particular, $s_i \models \forall\Box\, a \iff \forall\pi' \in \mathit{Paths}(s_i).\pi' \models \Box\, a$. As $s_i s_{i+1} \ldots \in \mathit{Paths}(s_i)$, we have that $\forall j \geq i.s_j \models a$. Therefore

$$\begin{aligned} & \exists i \geq 0.s_i \models b \wedge s_i \models \forall\Box\, a \wedge \forall 0 \leq j < i.s_j \models a \\ \Longrightarrow\ & \exists i \geq 0.s_i \models b \wedge \forall j \geq i.s_j \models a \wedge \forall 0 \leq j < i.s_j \models a \\ \Longrightarrow\ & \exists i \geq 0.s_i \models b \wedge \forall j \geq 0.s_j \models a \\ \Longrightarrow\ & s_0 s_1 \ldots \models \Diamond\, b \wedge \Box\, a. \end{aligned}$$

$\Longleftarrow$: Let $s \models \Box\, a \wedge \Diamond\, b$. In particular, for all $\pi \in \mathit{Paths}(s).\pi \models \Box\, a$ and therefore $s \models \Box\, a$. From previous tasks, we have

$$\forall s' \in \mathrm{Reach}_{\mathrm{TS}}(s)\ .\ s' \models a \quad \text{and} \tag{9.3}$$
$$\forall s' \in \mathrm{Reach}_{\mathrm{TS}}(s)\ .\ s' \models_{\mathrm{CTL}} \forall\Box\, a \tag{9.4}$$

We have

$$\begin{aligned} & s \models \Box\, a \wedge \Diamond\, b \\ \Longrightarrow\ & s \models \Box\, a \wedge s \models \Diamond\, b \\ \Longrightarrow\ & s \models \Box\, a \wedge \forall\pi = s_0 s_1 \in \mathit{Paths}(s) \exists i \geq 0.s_i \ldots \models_{LTL} b \\ \Longrightarrow\ & s \models \Box\, a \wedge \forall\pi = s_0 s_1 \in \mathit{Paths}(s) \exists i \geq 0.s_i \models_{CTL} b \\ \overset{(9.3)}{\Longrightarrow}\ & s \models \Box\, a \wedge \forall\pi = s_0 s_1 \in \mathit{Paths}(s) \exists i \geq 0.s_i \models_{CTL} b \wedge \forall\Box\, a \\ \overset{(9.4)}{\Longrightarrow}\ & s \models \Box\, a \wedge \forall\pi = s_0 s_1 \in \mathit{Paths}(s) \exists i \geq 0.s_i \models_{CTL} (b \wedge \forall\Box\, a) \wedge \forall 0 \leq j < i.s_j \models_{CTL} a \\ \Longrightarrow\ & \forall\pi = s_0 s_1 \in \mathit{Paths}(s) \exists i \geq 0.s_i \models_{CTL} (b \wedge \forall\Box\, a) \wedge \forall 0 \leq j < i.s_j \models_{CTL} a \\ \Longrightarrow\ & s \models \forall\Big(a \,\mathsf{U}\, (b \wedge \forall\Box\, a)\Big) \end{aligned}$$

## Exercise 5$^\star$ (1+1+1 Points)

Consider the following CTL formulae:

$$\Phi_1 = \forall \bigcirc \Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \exists\Box \forall \bigcirc\, a\Big)$$
$$\Phi_2 = \forall\big(\neg a \,\mathsf{W}\, (b \rightarrow \forall \bigcirc\, c)\big)$$

(a) Transform $\Phi_1$ into PNF.

(b) Transform $\Phi_1$ into ENF.

(c) Transform $\Phi_2$ into ENF.

**Solution:**

(a)

$$\begin{aligned}
\Phi_1 &= \forall \bigcirc \Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \exists\Box \forall \bigcirc\, a\Big) \\
&\equiv \forall \bigcirc \Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \neg\forall\Diamond \neg\forall \bigcirc\, a\Big) \\
&\equiv \forall \bigcirc \Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \neg\forall(\mathit{true} \,\mathsf{U}\, \neg\forall \bigcirc\, a)\Big) \\
&\equiv \forall \bigcirc \Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \exists\big((\mathit{true} \wedge \neg(\neg\forall \bigcirc\, a)) \,\mathsf{W}\, (\neg \mathit{true} \wedge \neg(\neg\forall \bigcirc\, a))\big)\Big) \\
&\equiv \forall \bigcirc \Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \exists\big((\mathit{true} \wedge \forall \bigcirc\, a) \,\mathsf{W}\, (\neg \mathit{true} \wedge \forall \bigcirc\, a)\big)\Big) \\
&\equiv \forall \bigcirc \Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \exists\big((\forall \bigcirc\, a) \,\mathsf{W}\, \mathit{false}\big)\Big)
\end{aligned}$$

(b)

$$\begin{aligned}
\Phi_1 &= \forall \bigcirc \Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \exists\Box \forall \bigcirc\, a\Big) \\
&\equiv \neg\exists \bigcirc \neg\Big(\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \vee \exists\Box \forall \bigcirc\, a\Big) \\
&\equiv \neg\exists \bigcirc \Big(\neg\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \wedge \neg\exists\Box \forall \bigcirc\, a\Big) \\
&\equiv \neg\exists \bigcirc \Big(\neg\exists\big(\neg a \,\mathsf{U}\, (b \wedge \neg c)\big) \wedge \neg\exists\Box \neg\exists \bigcirc\, \neg a\Big)
\end{aligned}$$

(c)

$$\begin{aligned}
\Phi_2 &= \forall\big(\neg a \,\mathsf{W}\, (b \rightarrow \forall \bigcirc\, c)\big) \\
&\equiv \neg\exists\Big(\big(\neg a \wedge \neg(b \rightarrow \forall \bigcirc\, c)\big) \,\mathsf{U}\, \big(\neg(\neg a) \wedge \neg(b \rightarrow \forall \bigcirc\, c)\big)\Big) \\
&\equiv \neg\exists\Big(\big(\neg a \wedge \neg(\neg b \vee \forall \bigcirc\, c)\big) \,\mathsf{U}\, \big(a \wedge \neg(\neg b \vee \forall \bigcirc\, c)\big)\Big) \\
&\equiv \neg\exists\Big(\big(\neg a \wedge b \wedge \neg\forall \bigcirc\, c\big) \,\mathsf{U}\, \big(a \wedge b \wedge \neg\forall \bigcirc\, c\big)\Big) \\
&\equiv \neg\exists\Big(\big(\neg a \wedge b \wedge \exists \bigcirc\, \neg c\big) \,\mathsf{U}\, \big(a \wedge b \wedge \exists \bigcirc\, \neg c\big)\Big)
\end{aligned}$$