Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

- syntax and semantics of LTL ⟵
- automata-based LTL model checking
- complexity of LTL model checking

Computation-Tree Logic

Equivalences and Abstraction

Linear Temporal Logic (LTL)

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$$\varphi \ ::= \ \mathit{true} \ \big| \ a \ \big| \ \varphi_1 \wedge \varphi_2 \ \big| \ \neg\varphi \ \big| \ \bigcirc \varphi \ \big| \ \varphi_1 \,\mathsf{U}\, \varphi_2$$

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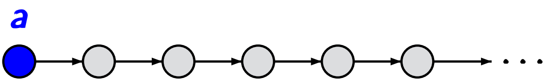

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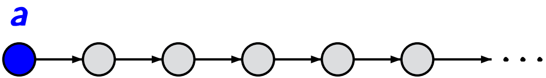
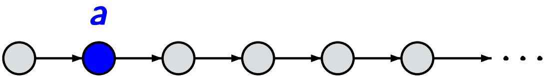


next operator $\bigcirc a$



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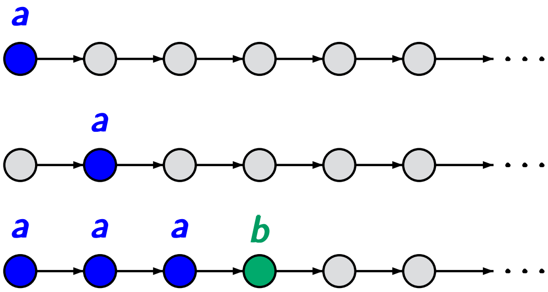

Derived operators in LTL

$$\varphi \ ::= \ \mathit{true} \mid a \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc \varphi \mid \varphi_1 \, \mathsf{U} \, \varphi_2$$

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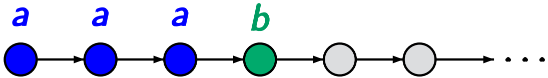
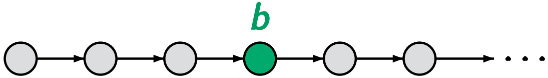


eventually
$\Diamond b$



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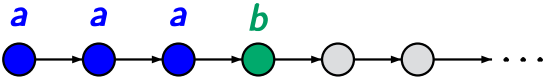

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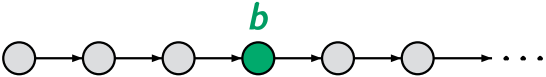
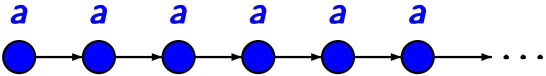
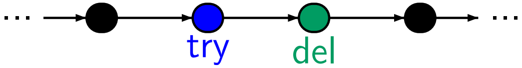


always
$\Box a$



Next $\bigcirc$, until U and eventually $\Diamond$

$\Box\,(\text{try\_to\_send} \rightarrow \bigcirc\ \text{delivered})$



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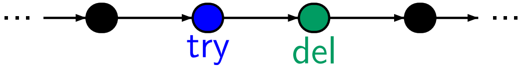

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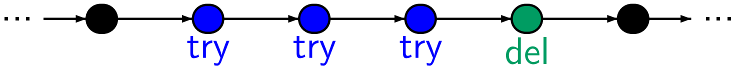

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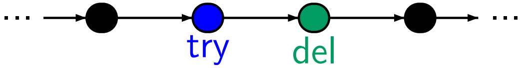

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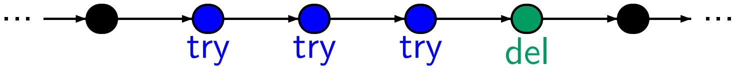

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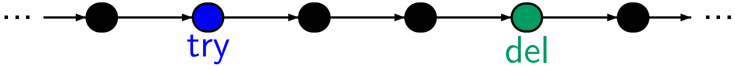

Examples for LTL formulas

$$\varphi \ ::= \ \mathit{true} \ \big| \ a \ \big| \ \varphi_1 \wedge \varphi_2 \ \big| \ \neg\varphi \ \big| \ \bigcirc \varphi \ \big| \ \varphi_1 \,\mathsf{U}\, \varphi_2$$

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traffic light: $\Box(\textit{yellow} \vee \bigcirc\neg \textit{red})$

Infinitely often and eventually forever

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e.g., unconditional fairness $\Box\Diamond \mathit{crit}_i$

strong fairness $\Box\Diamond \mathit{wait}_i \rightarrow \Box\Diamond \mathit{crit}_i$

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	weak fairness	$\Diamond\Box \mathit{wait}_i \rightarrow \Box\Diamond \mathit{crit}_i$

LTL-semantics

interpretation of **LTL** formulas over traces, i.e., infinite words over 2^{AP}

formalized by a satisfaction relation $\models$ for

- LTL formulas and
- infinite words $\sigma = A_0\,A_1\,A_2\,\ldots \in \left(2^{AP}\right)^{\omega}$

Semantics of LTL over infinite words

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$\sigma \models a$ iff $A_0 \models a$,i.e., $a \in A_0$

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$\sigma \models \varphi_1 \wedge \varphi_2$	iff	$\sigma \models \varphi_1$ and $\sigma \models \varphi_2$

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$\sigma \models \bigcirc\varphi$	iff	$\mathit{suffix}(\sigma, 1) = A_1\, A_2\, A_3\, \ldots \models \varphi$

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\sigma \models \bigcirc\varphi & \text{iff} & \mathit{suffix}(\sigma, 1) = A_1\, A_2\, A_3\, \ldots \models \varphi \\
\sigma \models \varphi_1 \,\mathsf{U}\, \varphi_2 & \text{iff} & \text{there exists } j \geq 0 \text{ such that}
\end{array}
$$

$$\mathit{suffix}(\sigma, j) = A_j\, A_{j+1}\, A_{j+2}\, \ldots \models \varphi_2 \quad \text{and}$$

$$\mathit{suffix}(\sigma, i) = A_i\, A_{i+1}\, A_{i+2}\, \ldots \models \varphi_1 \quad \text{for } 0 \leq i < j$$

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LT property of formula φ:

$$Words(\varphi) \overset{\text{def}}{=} \left\{\sigma \in \left(2^{AP}\right)^{\omega} : \sigma \models \varphi\right\}$$

LTL-semantics of derived operators $\Diamond$ and $\Box$

for $\sigma = A_0\,A_1\,A_2 \ldots \in \left(2^{AP}\right)^\omega$:

$$\vdots$$

$\sigma \models \varphi_1 \,\mathsf{U}\, \varphi_2$ iff there exists $j \geq 0$ such that

$A_j\,A_{j+1}\,A_{j+2} \ldots \models \varphi_2$ and

$A_i\,A_{i+1}\,A_{i+2} \ldots \models \varphi_1$ for $0 \leq i < j$

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\sigma \models \Diamond \varphi & \text{iff} & \text{there exists } j \geq 0 \text{ such that} \\
 & & A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi
\end{array}
$$

LTL-semantics of derived operators $\lozenge$ and $\square$

for $\sigma = A_0 A_1 A_2 \ldots \in (2^{AP})^\omega$:

$$
\begin{array}{lll}
 & \vdots & \\
\sigma \models \varphi_1 \, \mathsf{U} \, \varphi_2 & \text{iff} & \text{there exists } j \geq 0 \text{ such that} \\
 & & A_j A_{j+1} A_{j+2} \ldots \models \varphi_2 \quad \text{and} \\
 & & A_i A_{i+1} A_{i+2} \ldots \models \varphi_1 \quad \text{for } 0 \leq i < j \\
\sigma \models \lozenge \varphi & \text{iff} & \text{there exists } j \geq 0 \text{ such that} \\
 & & A_j A_{j+1} A_{j+2} \ldots \models \varphi \\
\sigma \models \square \varphi & \text{iff} & \text{for all } j \geq 0 \text{ we have:} \\
 & & A_j A_{j+1} A_{j+2} \ldots \models \varphi
\end{array}
$$

LTL semantics over TS

given a TS $\mathcal{T} = (S, Act, \to, S_0, AP, L)$

define satisfaction relation $\models$ for

- **LTL** formulas over AP
- the maximal path fragments and states of $\mathcal{T}$

assumption: $\mathcal{T}$ has no terminal states, i.e.,
all maximal path fragments in $\mathcal{T}$ are infinite

LTL semantics over paths of TS

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given: TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$
without terminal states

LTL formula φ over AP

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LTL formula φ over AP

interpretation of φ over infinite path fragments

$$\pi = s_0\, s_1\, s_2 \ldots \models \varphi \quad \text{iff} \quad trace(\pi) \models \varphi$$

LTL semantics over paths of TS

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$$\pi = s_0\, s_1\, s_2 \ldots \models \varphi \quad \text{iff} \quad trace(\pi) \models \varphi$$

$$\text{iff} \quad trace(\pi) \in Words(\varphi)$$

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LTL formula φ over AP

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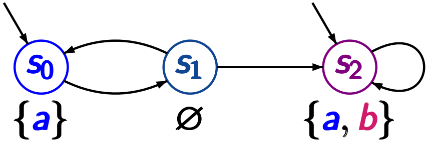
$$\text{iff} \quad trace(\pi) \in Words(\varphi)$$

remind: LT property of an LTL formula:

$$Words(\varphi) = \left\{\sigma \in \left(2^{AP}\right)^{\omega} : \sigma \models \varphi\right\}$$

Example: LTL-semantics over paths



$AP = \{a, b\}$

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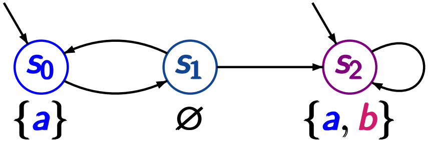

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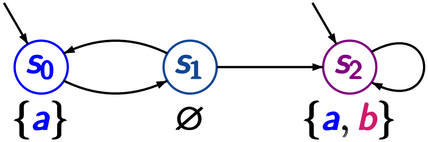

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path $\pi = s_0\, s_1\, s_2\, s_2\, s_2\, s_2 \ldots$ $\quad$ $trace(\pi) = \{a\}\, \varnothing\, \{a, b\}^\omega$

$\pi \models a$

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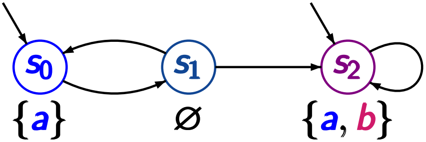

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$\pi \models a$, but $\pi \not\models b$ $\quad$ as $L(s_0) = \{a\}$

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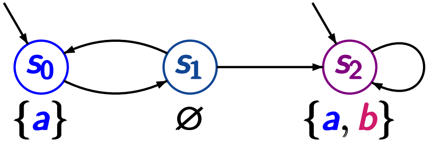

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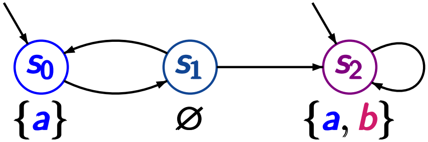
path $\pi = s_0\, s_1\, s_2\, s_2\, s_2\, s_2 \ldots$ $\quad$ $trace(\pi) = \{a\}\, \varnothing\, \{a, b\}^{\omega}$

$\pi \models a$, but $\pi \not\models b$ $\quad$ as $L(s_0) = \{a\}$

$\pi \models \bigcirc(\neg a \wedge \neg b)$

Example: LTL-semantics over paths



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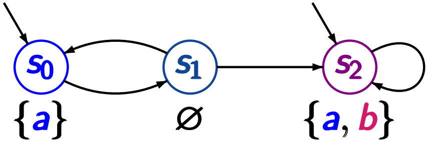

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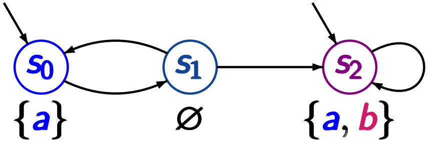

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path $\pi = s_0\, s_1\, s_2\, s_2\, s_2\, s_2 \dots$ $\qquad$ $trace(\pi) = \{a\}\, \varnothing\, \{a, b\}^\omega$

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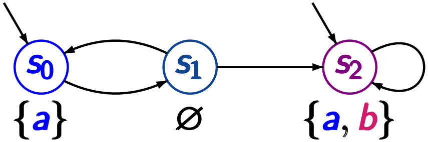

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Example: LTL-semantics over paths

$$\begin{tikzcd} \arrow[r] & s_0 \arrow[r, bend right] & s_1 \arrow[l, bend right] \arrow[r] & s_2 \arrow[loop right] & \arrow[l] \end{tikzcd}$$

s_0: $\{a\}$, s_1: $\varnothing$, s_2: $\{a, b\}$

$AP = \{a, b\}$

path $\pi = s_0\, s_1\, s_2\, s_2\, s_2\, s_2 \ldots$ $\quad$ $trace(\pi) = \{a\}\, \varnothing\, \{a, b\}^\omega$

$\pi \models a$, but $\pi \not\models b$	as $L(s_0) = \{a\}$
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$\pi \models \bigcirc\bigcirc(a \wedge b)$	as $L(s_2) = \{a, b\}$
$\pi \models (\neg b)\, \mathsf{U}\, (a \wedge b)$	as $s_0, s_1 \models \neg b$
	and $s_2 \models a \wedge b$

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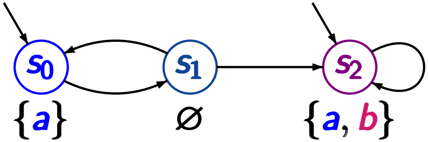

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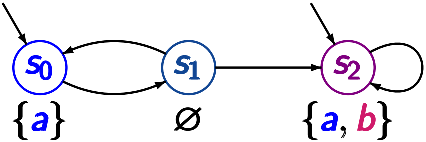
$\pi \models (\neg b) \mathsf{U} (a \wedge b)$	as $s_0, s_1 \models \neg b$
$\pi \models (\neg b) \mathsf{U} \Box(a \wedge b)$	and $s_2 \models a \wedge b$



$AP = \{a, b\}$

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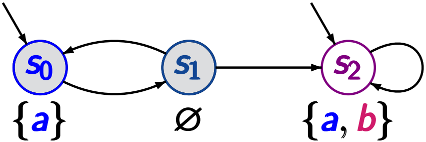
Correct or wrong ?



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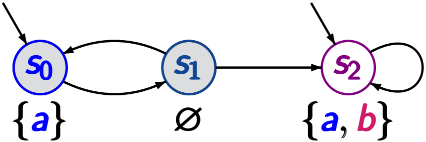

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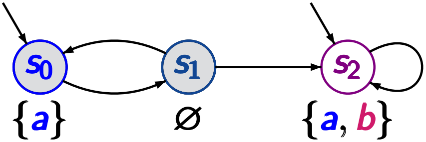

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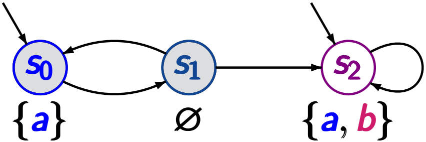

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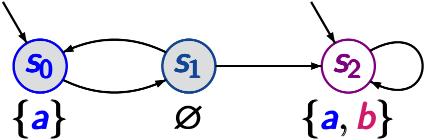
$\pi \models \Diamond b \rightarrow (a \,\mathsf{U}\, b)$?



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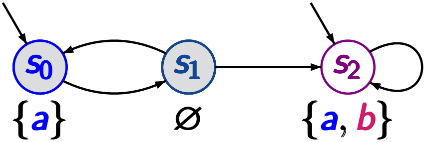

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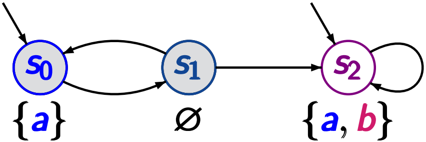

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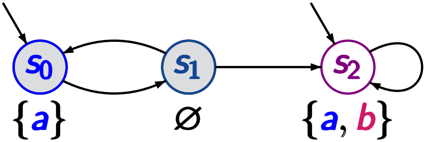

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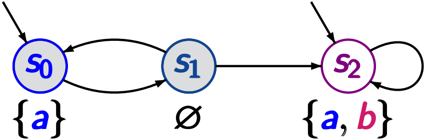
$\pi \models \Box a$?



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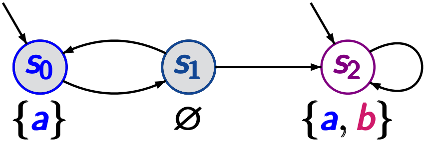

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$\pi \models \Box \Diamond a$?

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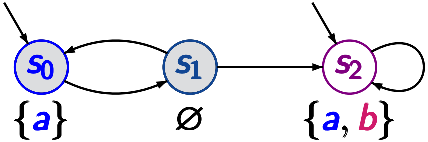

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$\pi \not\models \Box a$	as $s_1 \not\models a$
$\pi \models \Box \Diamond a$	as $\Box \Diamond \mathrel{\widehat{=}}$ infinitely often

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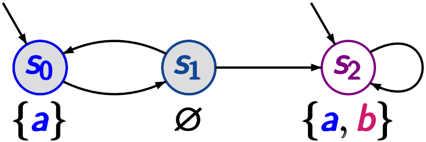

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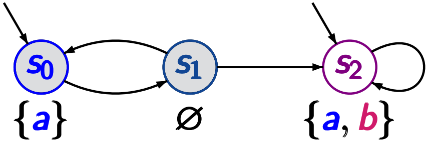

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$\pi \models \Box\Diamond a$ $\qquad$ as $\Box\Diamond \mathrel{\widehat{=}}$ infinitely often

$\pi \not\models \Diamond\Box a$ $\qquad$ as $\Diamond\Box \mathrel{\widehat{=}}$ eventually forever

LTL-semantics of derived operators

for $\sigma = A_0\, A_1\, A_2\, \ldots \in \left(2^{AP}\right)^{\omega}$:

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$\sigma \models \Box\varphi$	iff	for all $j \geq 0$ we have: $A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi$

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$\sigma \models \Box\Diamond\varphi$	iff	there are infinitely many $j \geq 0$ s.t. $A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi$
$\sigma \models \Diamond\Box\varphi$	iff	for almost all $j \geq 0$ we have: $A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi$

LTL semantics over the states of a TS

LTL semantics over the states of a TS

given: TS $\mathcal{T} = (S, Act, \to, S_0, AP, L)$
without terminal states

LTL formula φ over AP

interpretation of φ over infinite path fragments

$$\pi = s_0\, s_1\, s_2 \ldots \models \varphi \quad \text{iff} \quad trace(\pi) \models \varphi$$

interpretation of φ over states:

$$s \models \varphi \quad \text{iff} \quad trace(\pi) \models \varphi \text{ for all } \pi \in Paths(s)$$

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satisfaction relation for LT properties

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Interpretation of LTL formulas over TS

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given: TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$
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LTL formula φ over AP

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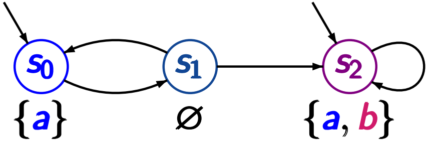
satisfaction relation for LT properties

Which formulas hold for $\mathcal{T}$?



$$AP = \{a, b\}$$

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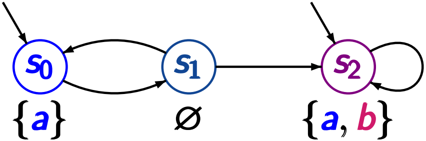
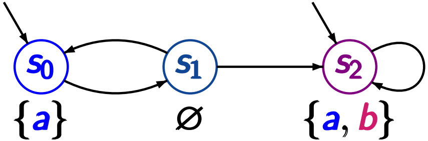


$AP = \{a, b\}$

$\mathcal{T} \models a$

Which formulas hold for $\mathcal{T}$?



$AP = \{a, b\}$

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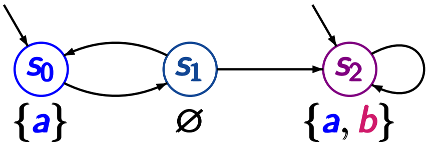

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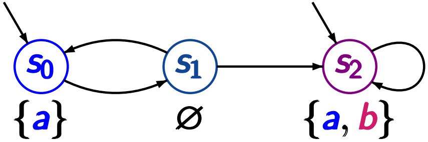

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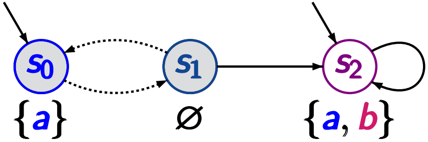

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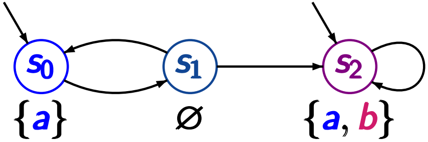

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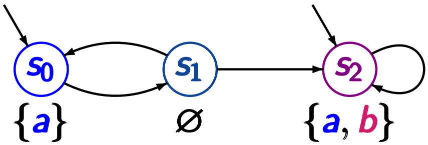

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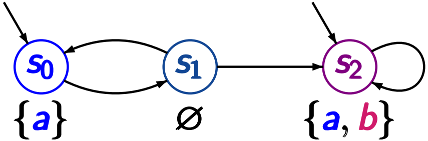

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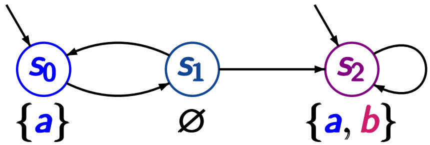

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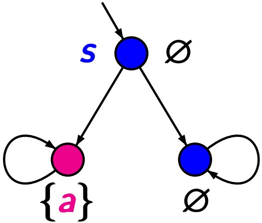

$s \not\models \Diamond a$ and $s \not\models \neg\Diamond a$

Provide an LTL formula over $AP = \{a, b\}$ for ...

- set of all words $A_0\,A_1\,A_2\,\ldots \in \left(2^{AP}\right)^\omega$ such that:

$$\forall i \geq 0.\ (\ a \in A_i \implies i \geq 1 \wedge b \in A_{i-1}\ )$$

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Examples:

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$$\vdots$$

all equivalences
from propositional logic

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Examples:

$$\begin{aligned} \varphi_1 \vee \varphi_2 &\equiv \varphi_2 \vee \varphi_1 \\ \neg\neg\varphi &\equiv \varphi \\ &\vdots \\ \neg\bigcirc\varphi &\equiv \bigcirc\neg\varphi \end{aligned}$$

all equivalences
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Self-duality of the next operator

$$\boxed{\varphi_1 \equiv \varphi_2 \quad \text{iff} \quad \mathit{Words}(\varphi_1) = \mathit{Words}(\varphi_2)}$$

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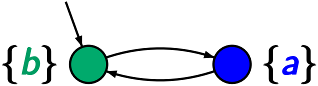

$$\models \Diamond b \wedge \Diamond a$$
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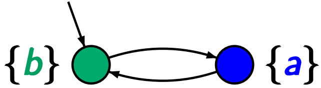

$\models \Diamond b \wedge \Diamond a$

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---

similarly: $\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$

$\Box(\varphi \vee \psi) \not\equiv \Box\varphi \vee \Box\psi$

Expansion laws

Expansion law for U

until: $\varphi \,\mathsf{U}\, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

Expansion laws for U and $\Diamond$

until: $\varphi \,\mathsf{U}\, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

eventually: $\Diamond\psi \;\equiv\; \psi \;\vee\; \bigcirc\Diamond\psi$

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note:
$$\begin{aligned} \Diamond\psi &= \mathit{true} \, \mathsf{U} \, \psi \\ &\equiv \psi \lor (\mathit{true} \land \bigcirc(\underbrace{\mathit{true} \, \mathsf{U} \, \psi}_{=\Diamond\psi})) \\ &\equiv \psi \lor \bigcirc\Diamond\psi \end{aligned}$$

Expansion laws for U, $\Diamond$ and $\Box$

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always: $\Box\psi \equiv$ **?**

Expansion laws for U, $\Diamond$ and $\Box$

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eventually:	$\Diamond\psi$	$\equiv$	ψ	$\vee$	$\bigcirc\Diamond\psi$
always:	$\Box\psi$	$\equiv$	ψ	$\wedge$	$\bigcirc\Box\psi$

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\end{aligned}
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&\equiv \psi \wedge \bigcirc\neg\Diamond\neg\psi \quad \longleftarrow \boxed{\text{self duality of } \bigcirc}
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Expansion laws are fixed point equations

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... don't yield a complete characterization, e.g.,

$$\boxed{\begin{array}{lcl} \mathit{false} & \equiv & a \wedge \bigcirc \mathit{false} \\ \Box a & \equiv & a \wedge \bigcirc \Box a \end{array}}$$

consider $\psi = a$

Expansion laws are fixed point equations

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least fixed point

eventually: $\Diamond\psi \;\equiv\; \psi \vee \bigcirc\Diamond\psi$

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greatest fixed point

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Expansion law for U

The LTL formula $\chi = \varphi \, \mathsf{U} \, \psi$ is the least solution of

$$\chi \equiv \psi \vee (\varphi \wedge \bigcirc \chi)$$

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The LTL formula $\chi = \varphi \, \mathbf{U} \, \psi$ is the least solution of

$$\chi \equiv \psi \vee (\varphi \wedge \bigcirc \chi)$$

i.e., $\mathit{Words}(\varphi \, \mathbf{U} \, \psi)$ least LT-property E s.t.

$$E = \mathit{Words}(\psi) \cup \left\{ A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi) : A_1 A_2 \ldots \in E \right\}$$

The LTL formula $\chi = \varphi \,\mathsf{U}\, \psi$ is the least solution of

$$\chi \equiv \psi \vee (\varphi \wedge \bigcirc \chi)$$

i.e., $\mathit{Words}(\varphi \,\mathsf{U}\, \psi)$ least LT-property E s.t.

$$E = \mathit{Words}(\psi) \cup \{A_0A_1A_2\ldots \in \mathit{Words}(\varphi) : A_1A_2\ldots \in E\}$$

It even holds that $\mathit{Words}(\varphi \,\mathsf{U}\, \psi)$ least LT-property E s.t.

$$(1) \quad \mathit{Words}(\psi) \subseteq E$$

$$(2) \quad \{A_0A_1A_2\ldots \in \mathit{Words}(\varphi) : A_1A_2\ldots \in E\} \subseteq E$$