3F1 Signals and Systems[1]

L1. Introduction and Recap

1.1 Euler Method for Solving ODE

- Given the gradient (rate of change) of some variable over time $\frac{dy}{dt} = f(y,t)$
- Define a fixed timestep δt. Then, ignoring initial conditions, we can derive an approximate solution for the next timestep:

$$y(t + \delta t) = y(t) + \delta t \times \frac{dy}{dt} = y(t) + \delta t \times f(y,t)$$

1.2 Pros and Cons of Discrete Time Methods

- Examples using digital signal processors (Digital circuits) over the analogue/continuous (Analog electronics) processes are: digital audio, digital communication link, image processing and speech synthesis.
- Note Discrete Time is different from digital signals, **discrete time signals have quantized time interval but the output is not quantized, digital outputs are quantized**. Below are the advantages and disadvantages of discrete time methods:

Advantages	Disadvantages
Algorithms can be implemented on any hardware	δt determines the stability which is hard to guarantee
Flexible and easy to modify	Need ADC/DAC hardware
Easy to implement complex control systems	Finite sampling rate

1.3 LTI Systems Revision

- A system is **linear** if it satisfies the Principal of Superposition:
  if $\mathcal{L}(u_1(t)) = y_1(t)$ and $\mathcal{L}(u_2(t)) = y_2(t)$, then for any scalars α_1, α_2:

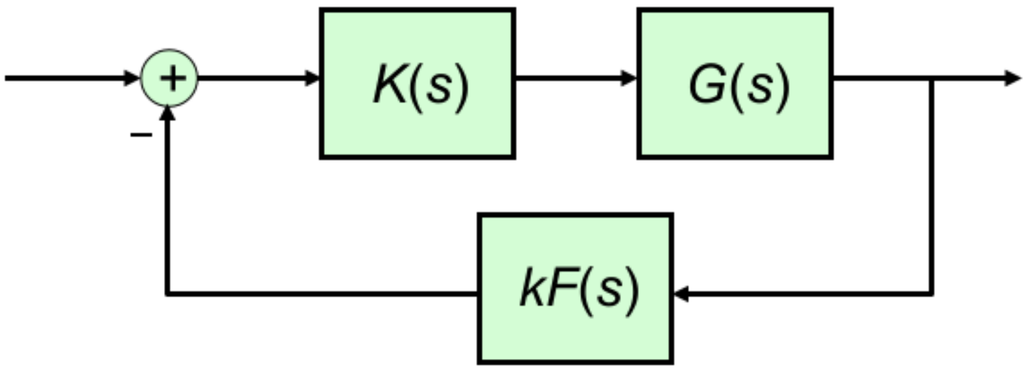
$$\mathcal{L}(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \mathcal{L}(u_1(t)) + \alpha_2 \mathcal{L}(u_2(t)) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

- A system is **time-invariant** if $\mathcal{L}(u(t)) = y(t)$ then $\mathcal{L}(u(t+T)) = y(t+t)$ for any time interval T

- Note nearly all real world systems are not time invariant, i.e. the structural integrity of a bridge changes over time.
- A system is *stable* if the transfer function $G(s)$ has no poles in the right half plane or on the imaginary axis.
- For a stable LIT system $G(s)$, the steady state response for an input $u(t) = \sin(\omega t)$ is:

$$y_{ss}(t) = |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

- The Nyquist Stability Criterion gives a test for the closed loop system to be stable. Consider the following feedback system:
  - First plot the Nyquist diagram of $F(s)G(s)K(s)$, which is the locus of $F(j\omega)G(j\omega)K(j\omega)$ as ω varies from $-\infty$ through to 0 to ∞
  - Let N be the number of anti-clockwise encirclements of the $-1/k$ point of the Nyquist diagram
  - Closed loop stability is achieved $\iff N = \#\text{ of RHP poles of }F(s)G(s)K(s)$



- MATLAB Nyquist Implementation (Non library):

  In order to plot the nyquist plot for $G(s) = \dfrac{1}{(s+2)(s+4)(s+6)}$:

```
% create symbolics and transfer functions
syms s w
G = 1/((s+2) * (s+4) * (s+6));

% substitute jw into the transfer function
G_w = subs(G, s, j*w);

% sweep across all frequencies
W = [-100 : 0.1 : 100];
Nyq = eval(subs(G_w, w, W));

% plot the real part against imaginary part
x = real(Nyq)
y = imag(Nyq)
```

- MATLAB Nyquist Implementation (Library)

```
% create transfer function
G = tf ([0 0 0 1],[1 12 44 48])
% plot with built in function
nyquist(G)
grid on
```

L2. Z-transform

2.1 Introduction

We know Laplace transform maps a signal from time domain into s domain, it operates on continuous signals $x(t)$. What about its equivalent for discrete signals?
A discrete time signal is a number sequence with discrete intervals T:

$$[x(0), x(T), x(2T), ...] \text{ or } [x_0, x_1, x_2, ...]$$

T is also known as the **sampling period**, i.e. the intervals between two signals is the period of time we wait before we take a sample of the continuous signal $x(t)$. A standard notation is:

$$\{x(kT)\}_{k\geq 0} \text{ or } \{x_k\}_{k\geq 0}$$

Analogous to Laplace transform, the z-transform for

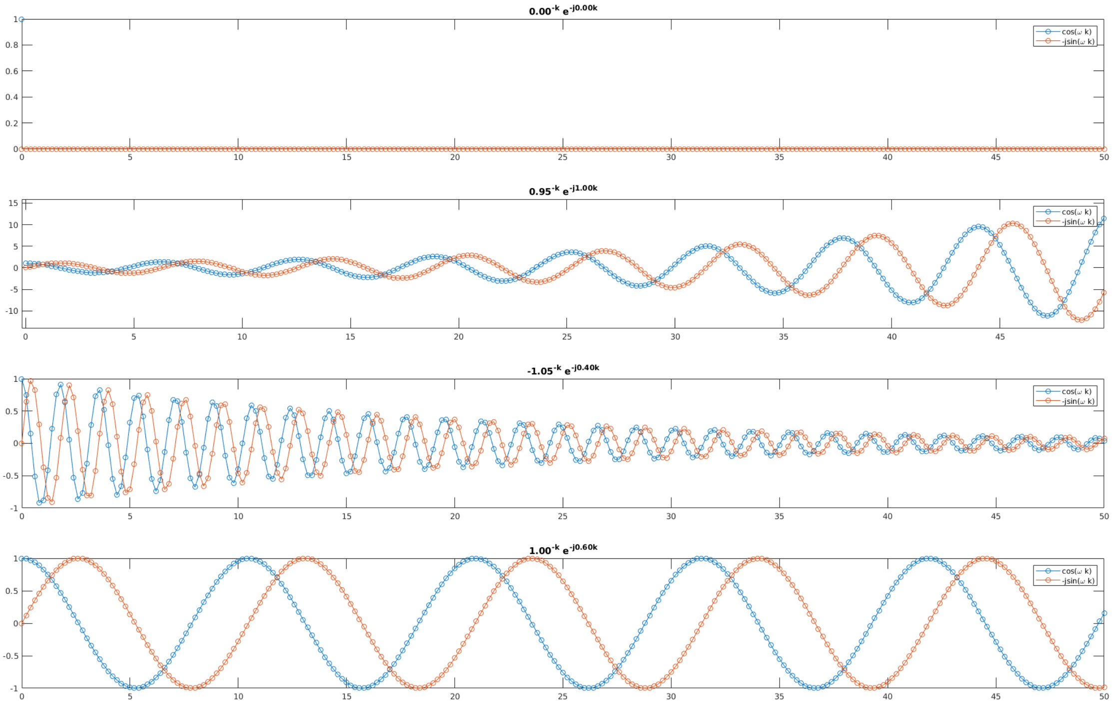
$$\{x(kT)\}_{k\geq 0}$$

is :

$$\sum_{k=1}^{\infty} x(kT)e^{-skT} = \sum_{k=1}^{\infty} x(kT)z^{-k}$$

We use $z = e^{sT}$ to make the notation easier as e^{sT} always appears as a group.

But what exactly is z-transform? We recall from IB that e^{sT} is equal to $e^{\sigma+j\omega} = re^{j\omega}$, now the important thing is to visualize possibilities for $z^{-k} = r^{-k}e^{-j\omega k}$. **k in this case is the index of our signal, we can literally interpret it as equivalent to time in a continuous signal**. Just like how we plot time varying functions, here z^{-k} is an 'index' varying function with r and ω interpreted as the amplitude and frequency of some signal sampled with time period T. Figures below show a set of possible signals generated by z^{-k}:



So z^{-k} generates all possible signals ranging from a train of ones ($r = 0, \omega = 0$) to exponentially decaying and increasing sinusoids. This is what we mean by mapping a signal from k to s domain. But what does the dot multiplication mean for each $x(kT)z^{-k}$? Without diving into details, say if we have $x(kT) = [3, -3, 3, -3, ...]$ and $z^{-k}_{|s=0} = [1, 1, 1, 1...]$, multiplying these together, we get $\sum_{k=1}^{\infty} x(kT)z^{-k}_{|s=0} = 0$, the zero here can be interpreted as a form of correlation between two functions, i.e how much component there is in one function from another, or the response of the system had we excited it with that signal. Putting all these together, we see z-transform tells us the response of our system at all possible input signals, just like Laplace transform except we are mapping from k to s.

2.2 Properties of Z-Transform

- **Linearity**

$$\mathcal{Z}(\alpha_{x_k} + \beta y_k) = \alpha \mathcal{Z}(\{x_k\}) + \beta \mathcal{Z}(\{y_k\})$$

- **Time delay**
  States a time delay equals to multiplying by z^{-1} in z domain, for a signal $\{x_k\}$:

$$\{x_k\} = (x_0, x_1, x_2, ...) \rightarrow \{x_{k-1}\} = (x_{-1}, x_0, x_1, ...)$$

  Start by writing a few terms down:

$$\mathcal{Z}(\{x_{k-1}\}) = \sum_{k=1}^{\infty} x_{k-1} z^{-k} = x_{-1} + x_0 z^{-1} + x_1 z^{-2} + x_2 z^{-3} + \ldots$$

$$\mathcal{Z}(\{x_{k-1}\}) = x_{-1} + \sum_{i=0}^{\infty} x_i z^{-i-1} = x_{-1} + z^{-1} X(z)$$

- **Time advance**

  Also starts by writing down a few terms:

$$\mathcal{Z}(\{x_{k-1}\}) = \sum_{k=1}^{\infty} x_{k+1} z^{-k} = x_1 + x_2 z^{-1} + x_3 z^{-2} + x_4 z^{-3} + \ldots$$

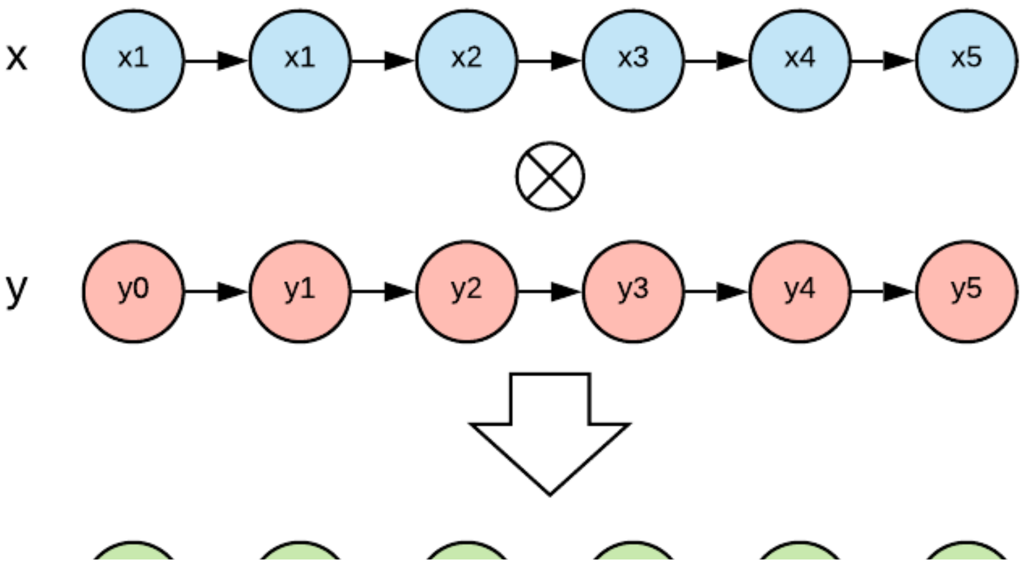
$$\mathcal{Z}(\{x_{k-1}\}) = \sum_{i=1}^{\infty} x_i z^{-i+1} = -x_0 z + z \sum_{i=0}^{\infty} x_i z^{-i} = -z x_0 + z X(z)$$

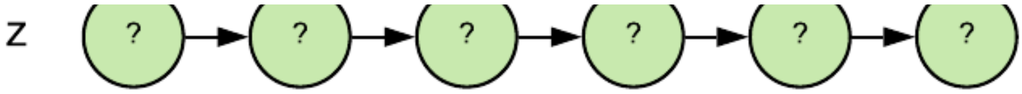
- **Initial value theorem**

$$x_0 = \lim_{z \to \infty} (x_0 + x_1/z + x_2/z^2 + \ldots) = \lim_{z \to \infty} X(z)$$

- **Convolution**

  This one actually requires a bit more thought. Let us write down two signals $\{x_k\} = [1, 2, 3, 4, 5, 6]$ and $\{y_k\} = [7, 8, 9, 10, 11, 12]$, what is $x_k \star y_k$? Let us also have some naming convention, $\{x_k\}$ and $\{y_k\}$ are *signals*, their contents are called *events* indexed by n, i.e. for signal x, at $n = 0$, there is an event $x[0] = 1$. When convolving the two signals together, we wish to know $\{z_k\}$ with each event denoted by $z[n]$. This is shown in the picture below (*typo, first x1 should be x0*):

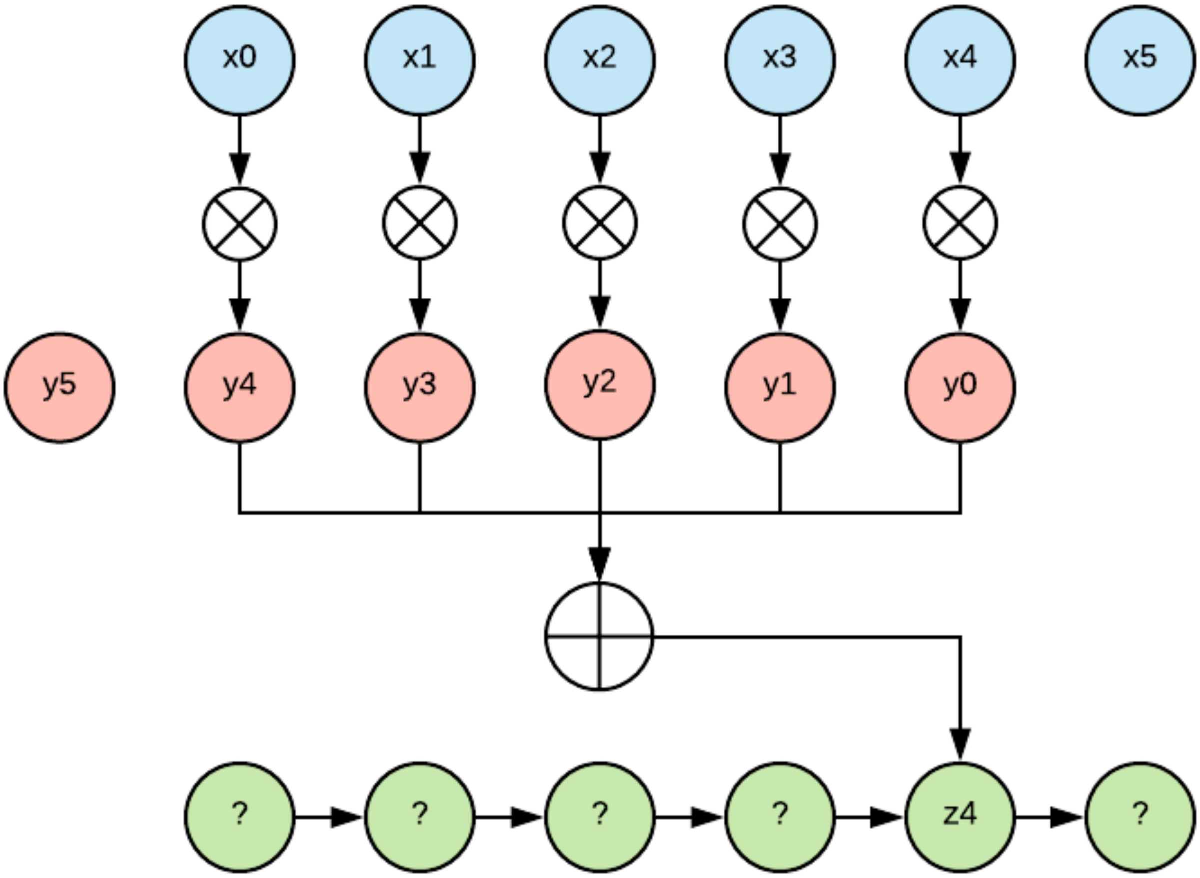




So what is each event $z[n]$ in out output z? First, define what convolution is mathematically:

$$z[n] = \sum_{i=0}^{n} x_n y_{n-i}$$

Picture below shows how $z[4]$ is calculated:



If we think about what this actually means, at the index 4, the result z_4 is due to the fact that the latest event x_4 triggered the initial response y_0, the previous event x_3 is now triggering y_1 as one time interval has passed, same goes for all previous events. The outcome z_4 is the combination of previous events' residual effects plus current event's immediate effect. Finally, we are able to write, for our train of outputs $\{z_k\}$:

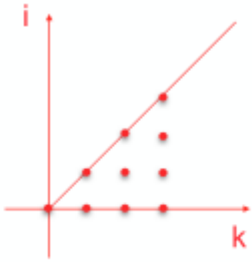
$$\{z_k\} = \{x_k\} \star \{y_k\} = \sum_{i=0}^{k} x_k y_{k-i}$$

Which is just the previous equation but for every single event up to k. Also for the picture above, the same result could be obtained by swapping x and y signals' order, hence we can also write:

$$\{z_k\} = \sum_{i=0}^{k} x_{k-i} y_i$$

The z transform of a convolution requires swapping the order of summation, the derivation and "helper" diagram are displayed below, note all we are doing is instead of summing from left to right, we index from bottom to top.

$$\begin{aligned}\mathcal{Z}\left(\{x_k\} \star \{y_k\}\right) &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} x_{k-i} y_i\right) z^{-k} \\ &= \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} x_{k-i} y_i z^{-k} \\ &= \sum_{i=0}^{\infty} y_i z^{-i} \sum_{k=i}^{\infty} x_{k-i} z^{-(k-i)} = X(z)Y(z)\end{aligned}$$



2.3 Pulse Response of LTI Systems
Just like continuous time LTI systems, any discrete-time LTI system can be represented as a convolution. For any input $\{u_k\}_{k\geq 0}$, with pulse response $\{g_k\}$:

$$y_k = \sum_{i=0}^{k} u_i g_{k-i} = \sum_{i=0}^{k} u_{k-i} g_i$$

that is,

$$\{y_k\} = \{g_k\} \star \{u_k\}$$

To show the above statement is true, we can breakdown any input the superposition of a train of impulsive inputs:

$$(u_0, u_1, u_2, ...) = u_0(1,0,0,...) + u_1(0,1,0,...) + u_2(0,0,1,...) + ...$$

For each impulsive input, through the linearity of the system G, their output at is:

$$\{y_k\} = u_0(g_0, g_1, g_2, ...) + u_1(0, g_0, g_1, ...) + u_3(0, 0, g_0, g_1, ...) + ... = \sum_{i=0}^{k} u_i g_{k-i}$$

The equation above is valid since $\{g_k\} = 0, \ \forall k < 0$

2.4 FIR, IIR and Causality

Digital filters ("Discrete time systems") whose pulse response terminates after a finite number of steps :

$$\{g_k\} = (g_0, g_1, ..., g_n, 0, 0, ...0)$$

are called **Finite Impulse Response (FIR)** systems. Otherwise the system is called **Infinite Impulse Response (IIR)**. Since the sequence is finite, a FIR system's transfer function can be fully expressed:

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + ... + g_n z^{-n} = \frac{z^n g_0 + ... + g_n}{z^n}$$

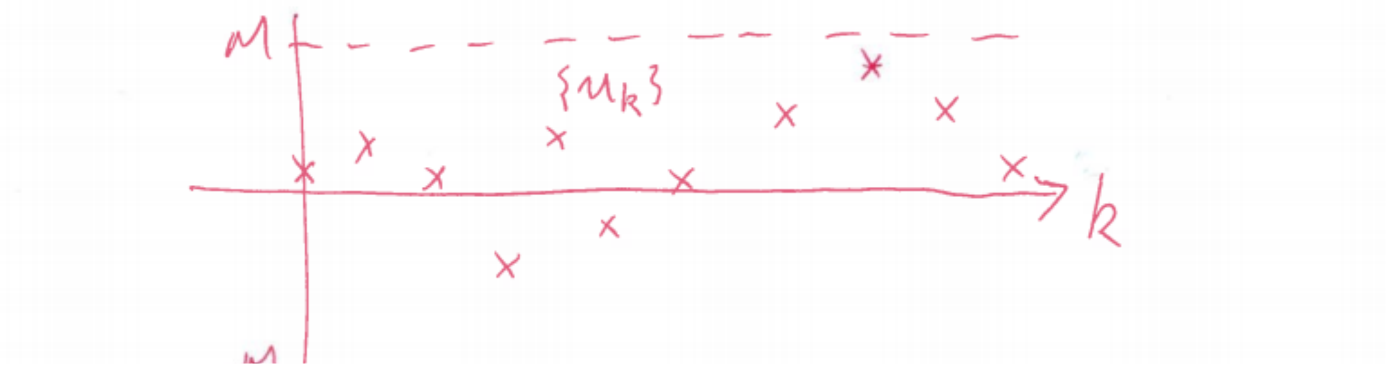
This shows that **all of the pulse of the transfer function of an FIR filter are at** $z = 0$
For causal systems, $G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + ...$ therefore $G(z)$ written in fractional form:

$$G(z) = \frac{b_0 z^m + b_1 z^{m-1} + ... + b_m}{z^n + a_1 z^{n-1} + ... + a_n}$$

for the system to be causal $m \leq n$

L4. BIBO Stability

The signal $\{u_k\}$ is bounded if there exists a positive constant M such that $|u_k| < M$ for all k.
This demonstrated in the sketch below:



A discrete time system is stable if **bounded inputs give bounded outputs** (BIBO). There are three equivalent theorems related to any LTI systems where the pulse response of G is $\{g_k\}_{k\geq 0}$:

- 1. G is stable
- 2. All of the roots, p_i of the denominator of $G(z)$ satisfies $|p_i| < 1$
- 3. \sum_{k=0}^{\infty} |g_k| is finite

  The first proof goes from (1) to (2):

(in part) Suppose $p_1, p_2, p_3, \ldots$ are distinct. Then we can decompose G using partial fractions:

$$G(z) = \frac{\alpha_1}{1 - p_1 z^{-1}} + \ldots + \frac{\alpha_n}{1 - p_n z^{-1}}$$

Then

$$g_k = \alpha_1 p_1^k + \alpha_2 p_2^k + \ldots + \alpha_n p_n^k$$

Now suppose $|p_i| > 1$ for some i. Then g_k is unbounded. Therefore a pulse input (bounded) gives an unbounded output and G is not stable.

The second proof goes from (3) to (1):

Let $\{u_k\}$ be a bounded input, i.e. $|u_k| < M$ for $k \geq 0$. Then the output, $\{y_k\}$ is given by:

$$y_k = \sum_{i=0}^{k} g_i u_{k-i}$$

Then

$$|y_k| = \left| \sum_{i=0}^{k} g_i u_{k-i} \right|$$

$|u_k| < M \ \forall k.$

$$\leq \sum_{i=0}^{k} |g_i| |u_{k-i}|$$

$$\leq \sum_{i=0}^{k} |g_i| M = M \sum_{i=0}^{k} |g_i|$$

$$\leq M \sum_{i=0}^{\infty} |g_i|$$

So $\sum_{i=0}^{\infty} |g_i|$ finite $\Rightarrow \{y_k\}$ bounded $\therefore$ system is stable $\square$

All this proof is saying is to 1. write the output as a convolution between impulse response and input, 2. since input is bounded by M then replace it and extract it out. Then using (3), theorem (1) is shown.

1. Created by: Tom Xiaoding Lu on 10/04/18 ↩