3F1 Signals and Systems: Handout 16

Course summary

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Continuous-time signals and systems

Differential equation

$y + a_1 y' + a_2 y'' = u + b_1 u'$

Impulse response

$g(t) = A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t}$

Transfer function

$G(\omega)$ or $H(G) = \frac{1+b_1 s}{1+a_1 s+a_2 s^2}$

Figure: The 3 equivalent descriptions of a continuous-time LTIS

Tools:

- Bode diagrams (stationary response to sinusoidals)
- Nyquist diagrams (stability of closed-loop system)

Discrete-time signals and systems

Difference equation
$$s_k + a_1 s_{k-1} + a_2 s_{k-2} = u_k + b_1 u_{k-1}$$

Delta response
$$h_k = A_1 q_1^k + A_2 q_2^k$$

Transfer function
$$H(z) = \frac{1+b_1 z^{-1}}{1+a_1 z^{-1}+a_2 z^{-2}}$$

Figure: The 3 equivalent descriptions of a discrete-time LTIS

Tools:
- Bode diagrams (stationary response to sinusoidals)
- Nyquist diagrams (stability of closed-loop system)

Discrete Transforms

- For a signal $\{u_k\}$, the z transform is defined as

$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k}$$

- For a signal $\{u_k\}$, the DTFT is defined as

$$U(\theta) = \sum_{k=-\infty}^{\infty} u_k e^{-jk\theta}$$

- both transforms have a convolution property, shift properties, conjugate symmetry properties (the DTFT also has a reverse conjugate symmetry property)
- both transforms can be inverted by inspection but the DTFT also has an inversion expression

$$s_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\theta) e^{jk\theta} d\theta$$

Stability

Theorem (Conditions for stability of a discrete time system)

Let G be a discrete time system with a rational transfer function,

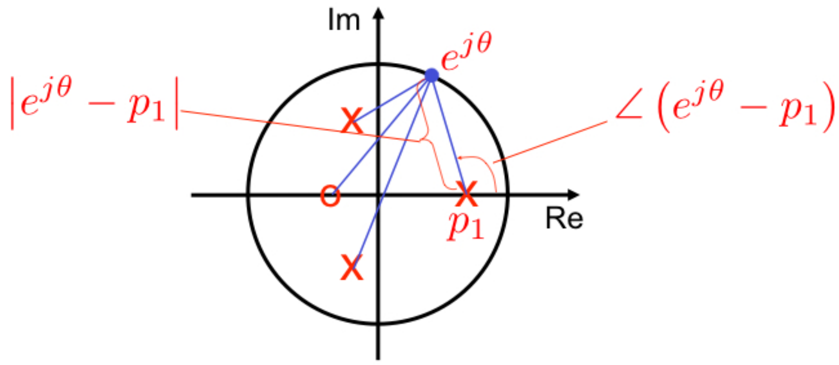
$$G(z) = \frac{b(z)}{a(z)} = \frac{b_0 + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

with no common factors between $b(z)$ and $a(z)$. Let the pulse response of G be $\{g_k\}_{k \geq 0}$. Then the following are equivalent:

1. *G is stable*
2. *All of the roots p_i of $a(z)$ (i.e. poles) satisfy $|p_i| < 1$*
3. *$\sum_{k=0}^{\infty} |g_k|$ is finite*

Bode diagram for stable systems

- $G(z) = c\frac{\prod_{k=1}^{m}(z - z_k)}{\prod_{k=1}^{n}(z - p_k)}$



- $|G(e^{j\theta})| = |c|\frac{\prod_{k=1}^{m}|e^{j\theta} - z_k|}{\prod_{k=1}^{n}|e^{j\theta} - p_k|}$

  $$|G(e^{j\theta})|_{dB} = 20\log(|G(e^{j\theta})|)$$
  $$= 20\Big(\log|c| + \sum_{k=1}^{m}\log|e^{j\theta} - z_k| - \sum_{k=1}^{n}\log|e^{j\theta} - p_k|\Big)$$

- $\angle G(e^{j\theta}) = \angle(c) + \sum_{k=1}^{m}\angle(e^{j\theta} - z_k) - \sum_{k=1}^{n}\angle(e^{j\theta} - p_k)$

The Nyquist stability criterion

The closed loop system is stable if and only if the number of encirclements of $-1/K$ by $G(e^{j\theta})$ as θ increases from $-\pi$ to π equals the number of open loop poles strictly outside the unit circle.

Algebraic transformations from Laplace to z domain

$$H(z) = H_c(s)_{s=\psi(z)} \text{ where } \psi(\cdot) \text{ is given by}$$

Euler's method or Forward difference

$$s = \frac{z-1}{T} \qquad \text{(intuition: linear approximation)}$$

Backward difference

$$s = \frac{1-z^{-1}}{T} \qquad \text{(intuition } \dot{x} \simeq \frac{x(t)-x(t-T)}{T}\text{)}$$

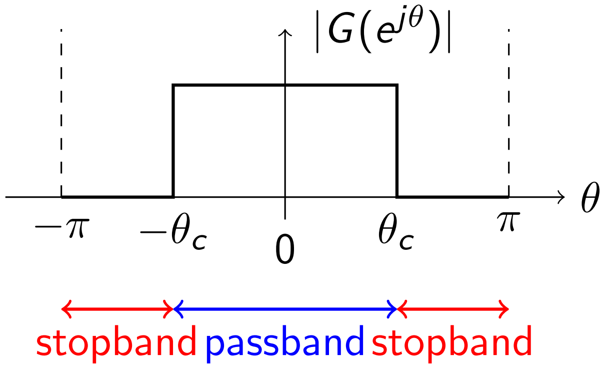
Bilinear (Tustin's) transformation

$$s = \frac{2}{T}\frac{z-1}{z+1} \qquad \text{(or simply } \frac{z-1}{z+1}\text{)}$$

Response invariant transforms of continuous time systems

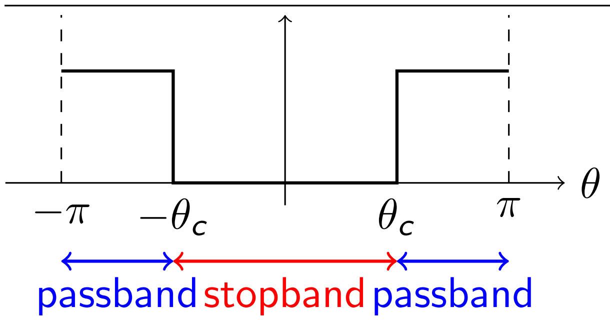
- impulse invariant
- step invariant
- ramp invariant

Filter design: desired frequency responses



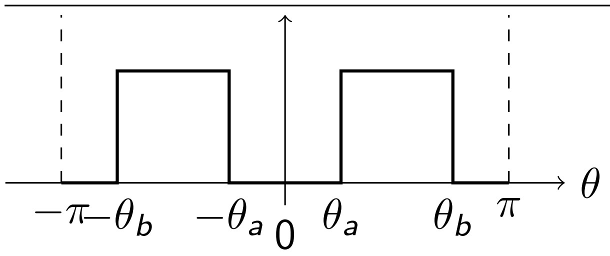
Lowpass:

$$G(e^{j\theta}) = \begin{cases} 1 & |\theta| \leq \theta_c \\ 0 & |\theta| > \theta_c \end{cases}$$



Highpass:

$$G(e^{j\theta}) = \begin{cases} 0 & |\theta| \leq \theta_c \\ 1 & |\theta| > \theta_c \end{cases}$$



Bandpass:

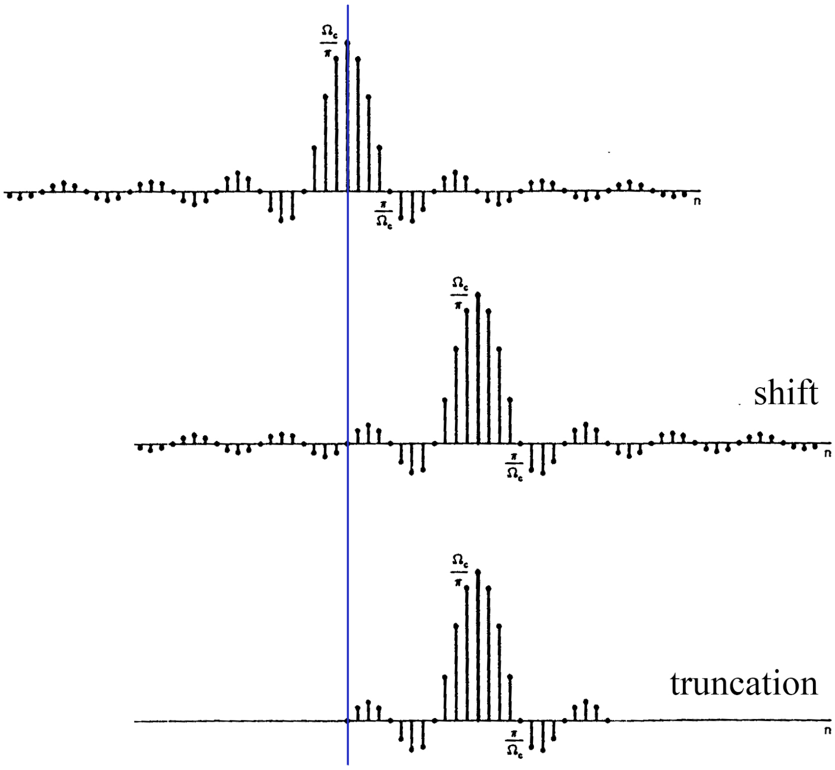
$$G(e^{j\theta}) = \begin{cases} 1 & \theta_a \leq |\theta| \leq \theta_b \\ 0 & \text{otherwise} \end{cases}$$

FIR filter design

h_k is the non-causal impulse response of the ideal filter

New (finite impulse response) filter G derived by shift and truncation at $N+1$ samples of the ideal response

Causality is recovered. Intuition: truncation of "small" samples has modest impact...



Use windowing!

IIR filter design

→ use algebraic transforms starting from continuous time designs

DFT is a matrix multiplication

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)^2} \end{bmatrix}$$

- $W_N = e^{-j\frac{2\pi}{N}}$
- it has the convolution property, time shift properties, conjugate symmetry properties
- **all properties are cyclic/circular**
- the FFT performs the matrix multiplication in $O(N \log N)$

Random processes

For stationary ergodic random processes,

		Discrete time	Continuous time
Time	Auto-correl. function	$r_{XX}[k] = E[X_\ell X_{\ell+k}]$ $r_{XY}[k] = E[X_\ell Y_{\ell+k}]$	$r_{XX}(\tau) = E[X(t)X(t+\tau)]$ $r_{XY}(\tau) = E[X(t)Y(t+\tau)]$
Frequency	PSD, CSD	$\mathcal{S}_{XX}(\theta), \mathcal{S}_{XY}(\theta)$	$\mathcal{S}_{XX}(\omega), \mathcal{S}_{XY}(\omega)$
Linear filtering		$\begin{cases} \mathcal{S}_{YY}(\theta) = \lvert H(\theta)\rvert^2 \mathcal{S}_{XX}(\theta) \\ \mathcal{S}_{XY}(\theta) = H(\theta)\mathcal{S}_{XX}(\theta) \end{cases}$	$\begin{cases} \mathcal{S}_{YY}(\omega) = \lvert H(\omega)\rvert^2 \mathcal{S}_{XX}(\omega) \\ \mathcal{S}_{XY}(\omega) = H(\omega)\mathcal{S}_{XX}(\omega) \end{cases}$

- PSD: power spectral density, CSD: cross spectral density
- most of the content of these 3 last lectures aims to motivate and explain the content of this slide