Most ways of deriving the volume, mass, or moment of inertia of a solid are either expensive or yield an approximate result. Here is a technique that is both efficient and exact.

A Symbolic Method for Calculating the Integral Properties of Arbitrary Nonconvex Polyhedra

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Volume, center of mass, moments of inertia, and similar properties of solids are defined by triple integrals over subsets of three-dimensional Euclidean space. Such quantities figure prominently in static and dynamic simulation equations, where the mass of an object or the effects during rotation must be calculated prior to manufacture, for instance; and therefore, the ability to compute the integral properties for geometrically complex solids is an important goal in CAD/CAM, robotics, and other fields.

Integral properties of a solid Q are defined as the volume integral of a function f over the solid

$$I = \int_Q f(x,y,z)\,dv$$

Most computational studies of multiple integrals deal with problems in which the domain Q is geometrically simple but the integrand f is complicated. However, in the calculation of mass, moment of inertia, etc., we confront the converse problem: the function f is usually simple but the domain Q may be very complicated.

Lee and Requicha[1,2] have reviewed several representation-oriented algorithms for evaluating the triple integral described above. The known methods for representing solids include primitive instancing, disjoint decomposition, constructive solid geometry, simple sweeping, and boundary representation. Each of these methods poses particular difficulties in the transfer of theory into practical application.

Objects described by primitive instancing belong to a finite number of primitive solid families, each of which is characterized by a finite number of parameters. Algorithms for computing integral properties of objects represented by primitive instancing are primitive specific: that is, a special formula or method is developed for each primitive. As the number and complexity of primitives in a representation scheme increase, therefore, so do the programming effort and the size of the software library.

Decomposition methods suffer from similar drawbacks. Disjoint decomposition partitions a solid into smaller solids. Three-dimensional triangulation decomposes a solid into a union of tetrahedra. Octree decompositions[3] generate displays of cubical solids whose linear dimensions are power-of-two multiples of some minimal size. An integral over a solid is then the sum of integrals over each small solid. However, generating an appropriate decomposition for a solid is usually expensive and requires considerable human labor and computation time.

Likewise, constructive solid geometry representation suffers from some relative inefficiencies. In this scheme, objects are described by the union, intersection, and subtraction of primitive solids. A CSG representation is a tree with branching nodes representing operators and leaves representing primitive solids. Lee and Requicha[2] exploit a divide-and-conquer method for computing the integral properties of solids represented by CSG by recursively applying the formulas

$$\int_{A \cup^* B} f\,dv = \int_A f\,dv + \int_B f\,dv - \int_{A \cap^* B} f\,dv$$

$$\int_{A -^* B} f\,dv = \int_A f\,dv - \int_{A \cap^* B} f\,dv$$

However, one must first solve the basic problem of evaluating an integral over the intersection of a number of primitive solids, and this involves extensive computation time.

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Other methods, though theoretically exact, may stretch programming skills to their limit. Sweeping representations describe a volume by an object moving along a trajectory, generally with translational and/or rotational motions. The integral properties of solids represented by translational and rotational sweeping may be computed by exploiting dimensional separability to convert a triple integral into a double integral. It remains a difficult challenge to devise a convenient algorithm for this technique, however.

Finally, some methods trade exactness for efficiency. Integral properties of solids represented by boundary representations can be evaluated by generating a collection of quasi-disjoint cells whose union approximates the solid, and computing the integral properties of the solid by adding the contributions of each individual cell. Cohen and Hickey[4] also introduced two algorithms for computing volumes of convex polyhedra. The first algorithm consists of triangulating a given solid into simplices and adding their individual volumes together. The idea is similar to that contained in this article. However, our algorithm is also applicable to the calculation of arbitrary polynomial functions. The second algorithm is of the Monte Carlo genre but is specialized to take advantage of the convexity of polyhedra. The result is yet approximate however, and accuracy can be increased only at the expense of additional computer time.

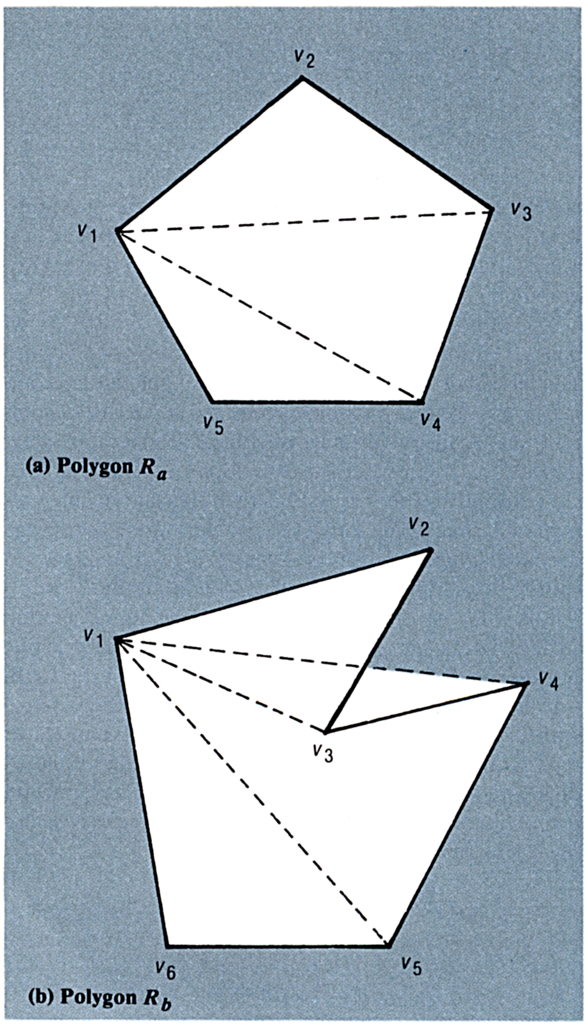

Figure 1. A convex polygon composed of five vertices and dissected into three triangles (a); a concave polygon composed of six vertices and dissected into four triangles (b).

In each of the above methods, objects with complicated boundaries thwart the goals of exactness and ease of execution. By contrast, we present here a simple method for evaluating the integral of an arbitrary polynominal function over an arbitrary, possibly nonconvex polyhedron represented by boundary representation.

A direct integral over a polyhedron can be evaluated by taking a central projection and adding the appropriate contributions of the cones defined by the faces of the object with respect to the center of projection.[5] The divergence theorem provides an alternative method for evaluating the integral properties of solids by simply integrating over their boundaries:

$$\int_Q f(x,y,z)\,dv = \int_Q div(\mathbf{g})\,dv = \int_{\partial Q} \mathbf{g}\cdot\mathbf{n}\,ds$$

where $\mathbf{g}$ is a vector function satisfying that the divergence of $\mathbf{g}$ equals the function f, ∂Q is the boundary of Q, $\mathbf{n}$ is the unit outward normal vector of the boundary, and ds is the surface differential.

We first present a general formula for direct evaluation of the integral of a polynominal over a 3-D simplex. An integral over a polyhedron can then be easily calculated by using the central projection method and decomposing a polyhedron systematically into a set of simplices and accumulating the results from each simplex based on this formula. This method adopts a systematic and automatic decomposition. It is analytically exact but the practical accuracy of the result is within the accuracy of floating-point arithmetic. Furthermore, the time complexity of this method is linearly proportional to the number of vertices of a polyhedron.

Symbolic evaluation

The area of triangle T with vertices (v_0, v_1, v_2) equals the outer product of the two vectors $\mathbf{r}_1 = (v_0, v_1)$ and $\mathbf{r}_2 = (v_0, v_2)$:

$$\text{Area }(T) = \tfrac{1}{2}(\mathbf{r}_1 \times \mathbf{r}_2) = \tfrac{1}{2}(y_1 z_2 - y_2 z_1, z_1 x_2 - x_1 z_2, x_1 y_2 - x_2 y_1), \text{ where} \quad \mathbf{r}_1 = (x_1, y_1, z_1) \text{ and } \mathbf{r}_2 = (x_2, y_2, z_2) \tag{1}$$

Here the symbol $\times$ denotes the cross products of two vectors. The area is defined as a vector that not only has a quantity but also an associated orientation. The area of an arbitrary closed planar region R is

$$\text{Area }(R) = \tfrac{1}{2}\oint_{\partial R} \mathbf{r} \times \mathbf{dl} \tag{2}$$

where ∂R is the boundary of the region R and $\mathbf{dl}$ is the differential tangent vector of the boundary. The area of a planar polygon $R = (v_1, v_2, \ldots, v_n)$ with n vertices, $\{v_i = (x_i, y_i, z_i)\}$, is

$$\text{Area}(R) = \tfrac{1}{2} \oint_{\partial R} \mathbf{r} \times \mathbf{dl}$$
$$= \tfrac{1}{2} \sum_{i=1}^{n-2} (v_1, v_{i+1}) \times (v_1, v_{i+2}) \quad \textbf{(3)}$$

Using v_1 as a projection origin, we dissect the planar polygon sequentially into a series of triangles formed by v_1 and each directed edge of the polygon in sequence. The area of the polygon is the vector sum of the areas of all triangles as defined in Equation 1. The dissected triangles are not necessarily mutually disjoint, however, if a polygon is concave. Equation 3 nevertheless holds for either convex or nonconvex polygons.

For example, in Figure 1a, polygon R_a is a convex polygon composed of five vertices. Using v_1 as a projection origin, R_a is dissected sequentially into three triangles $\{T_i = (v_1, v_{i+1}, v_{i+2})\}$ for $i=1,2,3$. The area of R_a equals the sum of the areas of all triangles. In Figure 1(b), polygon R_b is concave and composed of six vertices. R_b can be dissected into four triangles $\{T_i = (v_1, v_{i+1}, v_{i+2})\}$ for $i=1,...,4$, where the triangles $T_1 = (v_1, v_2, v_3)$ and $T_2 = (v_1, v_3, v_4)$ are not disjoint. The area of R_b equals the sum of the appropriately signed areas of all triangles, i.e., $R_b = T_1 - T_2 + T_3 + T_4$.

The integral properties of a polyhedron Q can be described by

$$I = \int_Q f(x,y,z)\, dv \quad \textbf{(4)}$$

where function f is a polynomial. With a linear transformation defined as

$$\begin{cases} x = g_x(u,v,w) \\ y = g_y(u,v,w) \\ z = g_z(u,v,w) \end{cases}$$

Equation 4 becomes

$$I = \iiint_Q f(g_x, g_y, g_z)\, |J|\, dudvdw \quad \textbf{(5)}$$

where the Jacobian J

$$J = \begin{vmatrix} \frac{\partial g_x}{\partial u} & \frac{\partial g_x}{\partial v} & \frac{\partial g_x}{\partial w} \\ \frac{\partial g_y}{\partial u} & \frac{\partial g_y}{\partial v} & \frac{\partial g_y}{\partial w} \\ \frac{\partial g_z}{\partial u} & \frac{\partial g_z}{\partial v} & \frac{\partial g_z}{\partial w} \end{vmatrix} \quad \textbf{(6)}$$

The integrand f in Equation 4 is a polynomial that can be generally represented as

$$f(x,y,z) = \sum_{n_1,n_2,n_3} x^{n_1} y^{n_2} z^{n_3},$$

where n_1, n_2, and n_3 are integers.

To compute the integral in Equation 4, we can focus our attention only on one term:

$$I = \iiint_Q x^{n_1} y^{n_2} z^{n_3}\, dxdydz \quad \textbf{(7)}$$

First let's look at a simple case where Q is a 3-D simplex, i.e., a tetrahedron, with four vertices (v_0, v_1, v_2, v_3), the vertex v_0 located at the origin. The coordinates of the vertices are

$$\begin{cases} v_0 = (0,0,0) \\ v_1 = (x_1, y_1, z_1) \\ v_2 = (x_2, y_2, z_2) \\ v_3 = (x_3, y_3, z_3) \end{cases} \quad \textbf{(8)}$$

We define a linear transformation $\mathbf{T}$ as

$$\mathbf{T} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \quad \textbf{(9)}$$

which relates the old coordinate system (x,y,z) with the new system (X,Y,Z) by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad \textbf{(10)}$$

Under this transformation, the tetrahedron $Q = (v_0, v_1, v_2, v_3)$ in Equation 8 is transformed into an orthogonal unit tetrahedron $W = (v_0', v_1', v_2', v_3')$ with coordinates

$$\begin{aligned} v_0' &= (0,0,0) \\ v_1' &= (1,0,0) \\ v_2' &= (0,1,0) \\ v_3' &= (0,0,1) \text{ respectively.} \end{aligned} \quad \textbf{(11)}$$

Based on the transformation in Equation 5, the integral in Equation 7 becomes

$$\begin{aligned} I &= \iiint_Q x^{n_1} y^{n_2} z^{n_3}\, dxdydz \\ &= ||T|| \iiint_W (x_1 X + x_2 Y + x_3 Z)^{n_1} \\ &\qquad (y_1 X + y_2 Y + y_3 Z)^{n_2} \\ &\qquad (z_1 X + z_2 Y + z_3 Z)^{n_3}\, dXdYdZ \end{aligned} \quad \textbf{(12)}$$

where the Jacobian $||T||$ equals the absolute value of the determinant of the matrix $\mathbf{T}$.

Next we present a formula for evaluating the integral of a polynomial $x^{n_1} y^{n_2} z^{n_3}$ over an orthogonal unit tetrahedron W described in Equation 11.

$$\begin{aligned} &\int_W x^{n_1} y^{n_2} z^{n_3}\, dv \\ &= \int_0^1 \int_0^{1-z} \int_0^{1-z-y} x^{n_1} y^{n_2} z^{n_3}\, dxdydz \\ &= \frac{n_1!\, n_2!\, n_3!}{(n_1 + n_2 + n_3 + 3)!} \end{aligned} \quad \textbf{(13)}$$

The details are shown in the Appendix. We can see that an arbitrary tetrahedron can always be transformed to an orthogonal unit tetrahedron by means of a simplex-dependent transformation matrix $\mathbf{T}$ as in Equation 9. Therefore, an integral of a polynomial over a tetrahedron can be evaluated symbolically by Equations 12 and 13.

To calculate the integral in Equation 12, first of all we decompose the integrand

$$\begin{aligned} I = ||T|| \int_W &(x_1 X + x_2 Y + x_3 Z)^{n_1} \\ &(y_1 X + y_2 Y + y_3 Z)^{n_2} \\ &(z_1 X + z_2 Y + z_3 Z)^{n_3}\, dV \end{aligned}$$

$$= \|T\| \sum_i \sum_j \sum_k c(i,j,k) \int_W X^i Y^j Z^k dV \quad \textbf{(14)}$$

$$= \|T\| \sum_i \sum_j \sum_k c(i,j,k) \frac{i!\ j!\ k!}{(i+j+k+3)!}$$

Here the function $c(i,j,k)$ represents the coefficient of a term $X^i Y^j Z^k$ in the expansion of the integrand, which can be loosely described by

$$(x_1x+x_2y+x_3z)^{n_1}$$
$$(y_1x+y_2y+y_3z)^{n_2}$$
$$(z_1x+z_2y+z_3z)^{n_3}$$

$$= \sum_{i+j+k=n_1+n_2+n_3} c(i,j,k) x^i y^j z^k$$

The following examples show how to calculate the volume, center of mass, and moments of inertia of a 3-D simplex. The volume of a tetrahedron Q as described in Equation 8 is

$$V = \int_Q dv = \|T\| \int_W dV = \|T\| \frac{0!}{3!} = \frac{\|T\|}{6} \quad \textbf{(15)}$$

The barycenter (x_o, y_o, z_o) of the tetrahedron Q is

$$x_o = \int_Q x dv / V$$
$$= \|T\| \iiint_W (x_1X+x_2Y+x_3Z) dX dY dZ / V$$
$$= \tfrac{1}{4}(x_1+x_2+x_3) \quad \textbf{(16)}$$
$$y_o = \tfrac{1}{4}(y_1+y_2+y_3)$$
$$z_o = \tfrac{1}{4}(z_1+z_2+z_3)$$

The moments of inertia of the tetrahedron Q are

$$I_{xx} = \int_Q x^2 dv$$
$$= \frac{V}{10}(x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3)$$

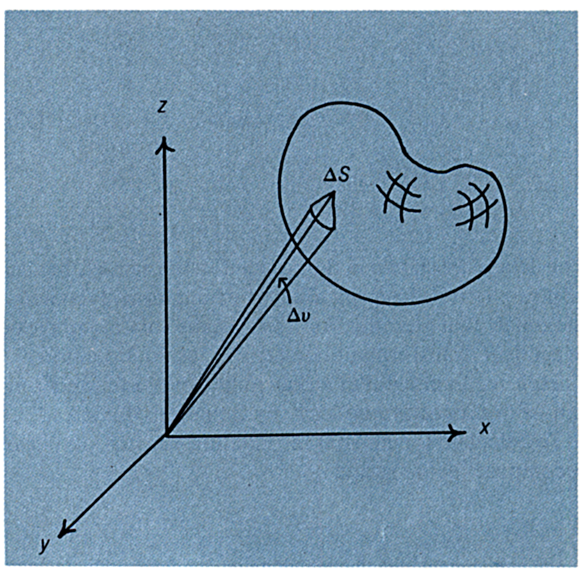

Figure 2. A cone is expanded by a small portion of the face of a polyhedron with respect to the origin.

$$I_{yy} = \int_Q y^2 dv$$
$$= \frac{V}{10}(y_1^2+y_2^2+y_3^2+y_1y_2+y_1y_3+y_2y_3)$$

$$I_{zz} = \int_Q z^2 dv$$
$$= \frac{V}{10}(z_1^2+z_2^2+z_3^2+z_1z_2+z_1z_3+z_2z_3)$$

$$I_{xy} = \int_Q xy dv \quad \textbf{(17)}$$
$$= \frac{V}{20}[2(x_1y_1+x_2y_2+x_3y_3)$$
$$+ (x_1y_2+x_2y_1+x_1y_3+x_3y_1+x_2y_3+x_3y_2)]$$

$$I_{yz} = \frac{V}{20}[2(z_1y_1+z_2y_2+z_3y_3)$$
$$+ (z_1y_2+z_2y_1+z_1y_3+z_3y_1+z_2y_3+z_3y_2)]$$

$$I_{zz} = \frac{V}{20}[2(x_1z_1+x_2z_2+x_3z_3)$$
$$+ (x_1z_2+x_2z_1+x_1z_3+x_3z_1+x_2z_3+x_3z_2)]$$

Higher order moments of the tetrahedron Q can also be calculated in a similar way.

Outline of the method

The integral properties of a solid Q defined in a polar coordinate system can be described as

$$I = \int_Q f(r,\theta,\phi) dv \quad \textbf{(18)}$$

Let $\mathbf{G}(r, \theta, \phi)$ be a vector function and $g(r, \theta, \phi)$ be a scalar function which is equal to the divergence of $\mathbf{G}$

$$\mathbf{G}(r, \theta, \phi) = g(r, \theta, \phi)\ \mathbf{r} \text{ and}$$
$$\nabla \cdot \mathbf{G} = f$$

where $\mathbf{r}$ is the unit radial vector. Functions $g(r, \theta, \phi)$ and f are therefore related by

$$f = \frac{\partial g(r, \theta, \phi)}{\partial r} + 2\frac{g(r, \theta, \phi)}{r}$$

$$g = \frac{1}{r^2} \int_0^r r'^2 f(r', \theta, \phi) dr'$$

Through the divergence theorem, the integral in Equation 18 becomes

$$I = \int_Q f(r, \theta, \phi) dv$$
$$= \int_Q \nabla \cdot \mathbf{G} dv$$
$$= \int_{\partial Q} \mathbf{G} \cdot \mathbf{ds} \quad \textbf{(19)}$$
$$= \int_{\partial Q} g(r, \theta, \phi) \mathbf{r} \cdot \mathbf{n} ds$$

Therefore, the integral can be represented as a surface integral over the boundary of the polyhedron. For example, in Figure 2, where the integral is taken over a small volume Δv, a cone that is expanded by a small face Δs of the polyhedron with respect to the origin, the result is

$$\int_{\Delta v} f(r,\theta,\phi)dv = \int_{\partial\Delta v} g(r,\theta,\phi)\mathbf{r}\cdot\mathbf{n}ds \tag{20}$$

Since the outward normal vector $\mathbf{n}$ of the wall of the cone is orthogonal to the radial vector $\mathbf{r}$, i.e., $\mathbf{r}\cdot\mathbf{n}=0$, the integral in Equation 20 is valuable only on the face Δs:

$$\int_{\Delta v} f(r,\theta,\phi)dv = \int_{\Delta s} g(r,\theta,\phi)\mathbf{r}\cdot\mathbf{n}ds \tag{21}$$

Let's call the face that expands a cone the *base* of the cone. A volume integral over a cone can therefore be represented in terms of a surface integral over the base of the cone. The integral in Equation 19 thus equals a sum of appropriately signed volume integrals over cones expanded by faces of Q, such that

$$\begin{aligned} I &= \int_Q f(r,\theta,\phi)\,dv \\ &= \int_{\partial Q} g(r,\theta,\phi)\mathbf{r}\cdot\mathbf{n}ds \\ &= \sum_{\Delta v_i} S(\Delta v_i)\int_{\Delta v_i} f(r,\theta,\phi)\,dv \end{aligned} \tag{22}$$

We introduced a sign function $S(\Delta v_i)$ in the above equation. The sign function is defined on a cone Δv_i as

$$\begin{aligned} S(\Delta v_i) &= +1 \text{ if } \Delta v_i \text{ is coherent with } Q \\ &= -1 \text{ otherwise.} \end{aligned}$$

Significantly, the $\mathbf{n}$ in Equation 19 is the outward normal vector of the boundary of the polyhedron Q. However, the $\mathbf{n}$ in Equation 21 is the outward normal vector of the boundary of the cone Δv_i. They may point in opposite directions. They point in the same direction only when the interiors of Q and Δv_i occupy the same half-space divided by the base Δs_i of the cone. In this case, we say that Δv_i and Q are coherent. The sign function $S(\Delta v_i)$ determines whether the integral over Δv_i has a positive or negative contribution to the integral over Q. If Δv_i is coherent with Q, the integral over Δv_i has a positive contribution to Q; otherwise it has a negative contribution.

In Figure 3, a tetrahedron $Q=(v_1, v_2, v_3, v_4)$ has four faces, $F_1=(v_2, v_4, v_3)$, $F_2=(v_1, v_3, v_4)$, $F_3=(v_1, v_4, v_2)$, and $F_4=(v_1, v_2, v_3)$. The order of the vertices in each face is specified clockwise so that the normal vector of the face always points away from the tetradron. A face F_i of Q with the origin O forms a new tetrahedron $Q_i=(O, v_m, v_j, v_k)$, where (v_m, v_j, v_k) are vertices of F_i. Four new tetrahedra, $Q_1=(O, v_2, v_3, v_4)$, $Q_2=(O, v_1, v_3, v_4)$, $Q_3=(O, v_1, v_2, v_4)$, and $Q_4=(O, v_1, v_2, v_3)$ are thus formed. As we can tell, tetrahedra Q_1, Q_2, and Q_3 are coherent, but Q_4 is incoherent with Q. Since each tetrahedron has a vertex located at the origin, an integral over these tetrahedra can be evaluated symbolically with the method illustrated in the last section.

An integral over the tetrahedron Q thus equals the sum of the appropriately signed integral over each newly formed tetrahedron Q_i:

$$\begin{aligned} I &= \int_Q f(x,y,z)\,dv \\ &= \sum_{Q_1,Q_2,Q_3,Q_4} S(Q_i)\int_{Q_i} f(x,y,z)\,dv \end{aligned} \tag{23}$$

As we know, the sign functions are $S(Q_1)=S(Q_2)=S(Q_3)=1$, and $S(Q_4)=-1$. Let the coordinates of the vertices of a face $F_i=(v_m, v_j, v_k)$ be

$$\begin{cases} v_m &= (x_m, y_m, z_m) \\ v_j &= (x_j, y_j, z_j) \\ v_k &= (x_k, y_k, z_k) \end{cases}$$

A linear transformation $\mathbf{T}_i$ can be defined as

$$\mathbf{T}_i = \begin{pmatrix} x_m & x_j & x_k \\ y_m & y_j & y_k \\ z_m & z_j & z_k \end{pmatrix} \tag{24}$$

As we know, the determinant of the transformation matrix is positive if Q_i is coherent with the base F_i, i.e., $|T_i|>0$; if Q_i is incoherent with F_i, the determinant is negative. The determinant $|T_i|$ implicitly represents the effects of the sign function $S(Q_i)$ and the Jacobian $||T_i||$. Equation 23 thus becomes

$$\begin{aligned} I &= \int_Q f(x,y,z)\,dv \\ &= \sum_{Q_i} S(Q_i)\int_{Q_i} f(x,y,z)\,dv \\ &= \sum_{Q_i} S(Q_i)\,||T_i||\int_{W_i} f(g_x,g_y,g_z)\,dV \\ &= \sum_{Q_i} |T_i|\int_{W_i} f(g_x,g_y,g_z)\,dV \end{aligned} \tag{25}$$

where functions g_x, g_y, and g_z are defined as

$$\begin{cases} x &= g_x(X,Y,Z) = x_mX + x_jY + x_kZ \\ y &= g_y(X,Y,Z) = y_mX + y_jY + y_kZ \\ z &= g_z(X,Y,Z) = z_mX + z_jY + z_kZ \end{cases}$$

A polyhedron Q composed of f faces, e edges, and v vertices can be represented by $Q=(F,E,V)$, where $F=(F_1, \ldots, F_f)$ is a set of faces, $E=(E_1, \ldots, E_e)$ is a set of edges, and $V=(v_1, \ldots, v_v)$ is a set of ver-

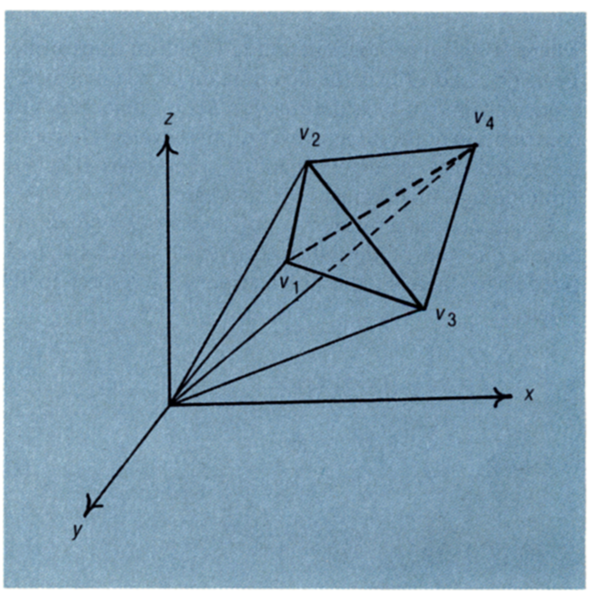

Figure 3. A tetrahedron with four faces expands into four new tetrahedra with respect to the origin.

tices. A face F_i is composed of f_i vertices, $F_i = (v_1^i, \ldots, v_{f_i}^i)$, where a vertex $v_j^i \in V$ and the vertices are specified in an order such that the normal vector of F_i points away from the polyhedron.

A face can also be a nonconvex polygon. As described in the second section, a face F composed of f vertices can be dissected sequentially into $f-2$ triangles. A face $F_i = \{v_1^i, v_2^i, \ldots, v_{f_i}^i\}$ can be dissected into $f_i - 2$ triangles, $\{T_j^i = (v_1^i, v_{j+1}^i, v_{j+2}^i)\}$, by using the vertex v_1^i as the projection origin. A triangle T_j^i with the origin O forms a tetrahedron $Q_j^i = (O, v_1^i, v_{j+1}^i, v_{j+2}^i)$. A face F_i with respect to the origin therefore expands a set of tetrahedra $\{Q_j^i\}$. An integral over a cone Y_i that is expanded by a face F_i with respect to the origin thus equals the sum of appropriately signed integrals over all tetrahedra $\{Q_j^i\}$. Let the coordinates of the vertices of $Q_j^i = (O, v_1^i, v_{j+1}^i, v_{j+2}^i)$ be

$$\begin{cases} O &= (0, 0, 0) \\ v_1^i &= (x_1^i, y_1^i, z_1^i) \\ v_{j+1}^i &= (x_{j+1}^i, y_{j+1}^i, z_{j+1}^i) \\ v_{j+2}^i &= (x_{j+2}^i, y_{j+2}^i, z_{j+2}^i) \end{cases}$$

As described before, a linear transformation $\mathbf{T}_j^i$ can then be defined as

$$\mathbf{T}_j^i = \begin{pmatrix} x_j^i & x_{j+1}^i & x_{j+2}^i \\ y_1^i & y_{j+1}^i & y_{j+2}^i \\ z_1^i & z_{j+1}^i & z_{j+2}^i \end{pmatrix} \tag{26}$$

An integral over a cone Y_i expanded by a face F_i equals the sum of integrals over all tetrahedra $\{Q_j^i\}$

$$\begin{aligned} I &= \int_{Y_i} f(x, y, z)\,dv \\ &= \sum_{Q_j^i} S(Q_j^i) \int_{Q_j^i} f(x,y,z)\,dv \\ &= \sum_{W_j^i} |T_j^i| \int_{W_j^i} f(g_x, g_y, g_z)\,dV \end{aligned} \tag{27}$$

where W_j^i is an orthogonal unit tetrahedron transformed from Q_j^i, and $|T_j^i|$ is the determinant of the matrix $\mathbf{T}_j^i$. Notice again that the sign function $S(Q_j^i)$ determines the contribution of the integral over Q_j^i to the integral over the cone, and that the determinant $|T_j^i|$ combines effects of both the sign function and the Jacobian.

Finally, an integral over a polyhedron $Q = (F, E, V)$ equals the sum of the integrals over all cones $\{Y_i\}$ expanded by faces of the polyhedron with respect to the origin O:

$$\begin{aligned} I &= \int_Q f(x, y, z)\,dv \\ &= \sum_{F_i} \int_{Y_i} f(x,y,z)\,dv \\ &= \sum_{F_i} \sum_{Q_j^i} \int_{Q_j^i} f(x,y,z)\,dv \\ &= \sum_{F_i} \sum_{W_j^i} |T_j^i| \int_{W_j^i} f(g_x, g_y, g_z)\,dV \end{aligned} \tag{28}$$

The functions g_x, g_y, and g_z are dependent on the simplex Q_j^i and are defined as

$$\begin{cases} x = g_x(X, Y, Z) = x_1^i X + x_{j+1}^i Y + x_{j+2}^i Z \\ y = g_y(X, Y, Z) = y_1^i X + y_{j+1}^i Y + y_{j+2}^i Z \\ z = g_z(X, Y, Z) = z_1^i X + z_{j+1}^i Y + z_{j+2}^i Z \end{cases}$$

We have described a simple and systematic method for calculating the integral of a polynomial function over an arbitrary nonconvex polyhedron. Computation is divided into integrals over a set of cones formed by faces of the polyhedron with respect to the origin. An integral over a cone is then divided into integrals over a set of tetrahedra formed by the triangles dissected from the face with respect to the origin.

Although Equations 12, 13, 27, and 28 look complicated, the basic idea is very simple. An integral over a polyhedron $Q = (F,E,V)$ equals the sum of the appropriately signed integrals over cones formed by faces of Q with respect to the origin, as Equation 28 illustrates. An integral over a cone equals the sum of the appropriately signed integrals over a set of dissected tetrahedra, shown in Equation 27. Equation 13 supplies a direct solution to the evaluation of the integral of a polynomial function $f(x, y, z) = x^{n_1} y^{n_2} z^{n_3}$ over an orthogonal unit tetrahedron. Equation 12 then shows that an arbitrary tetrahedron can be transformed to an orthogonal unit tetrahedron; therefore, an integral over the tetrahedron can be computed on the basis of the result from Equation 13.

The implementation of the method is also very simple. The computation involves sequential scanning over the faces of the polyhedron and sequential dissecting of the faces into triangles. Since a face F_i with e_i edges is dissected sequentially into $(e_i - 2)$ triangles, it requires $\Sigma(e_i - 2)$ computations. Since an edge is counted twice during the whole scanning procedure, we arrive at

$$\sum_{F_i} (e_i - 2) = 2E - 2F$$

where E is the total number of edges and F is the total number of faces of the polyhedron. According to the Euler equation, in which $V - E + F = 2$, the total number of computations equals $2(V-2)$, i.e.,

$$\sum_{F_i} (e_i - 2) = 2(V-2)$$

Therefore, the time complexity of the method is linearly proportional to V, the number of vertices of the polyhedron.

Note that the numerical accuracy of this method can be improved by shifting the origin of the coordinate system to the barycenter of the object, especially to prevent the occurrence of long, thin tetrahedra when an object is far removed from the origin. By using a more sophisticated 3-D triangulation, one can improve the efficiency of the algorithm. Also, this method can be generalized to solids in higher dimensions. Also, its computation time does not increase linearly with the number of integral quantities computed, since only one matrix determinant $|T|$ needs to be computed when several integral quantities are calculated at the same time. ■

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Appendix

The beta and gamma functions are defined as

$$\beta(n_1+1, n_2+1) = \int_0^1 x^{n_1}(1-x)^{n_2}dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^n dx = n!$$

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

The derivation of Equation 13 is

$$I = \int_W x^{n_1}y^{n_2}z^{n_3}dxdydz$$

$$= \int_0^1 \int_0^{1-z} \int_0^{1-z-y} x^{n_1}y^{n_2}z^{n_3}dxdydz$$

$$= \frac{1}{n_1+1}\int_0^1 \int_0^{1-z} (1-z-y)^{n_1+1}$$

$$y^{n_2}z^{n_3}dydz$$

$$= \frac{1}{n_1+1}\int_0^1 \int_0^{1-z} (1-z)^{n_1+1}$$

$$\left(1-\frac{y}{1-z}\right)^{n_1+1} y^{n_2}z^{n_3}dydz$$

Let $Y = \dfrac{y}{1-z}$, $y = (1-z)Y$, then

$$dy = (1-z)dY$$

$$I = \frac{1}{n_1+1}\int_0^1 \int_0^1 (1-z)^{n_1+1}$$

$$(1-Y)^{n_1+1}(1-z)^{n_2}Y^{n_2}z^{n_3}(1-z)dYdz$$

$$= \frac{1}{n_1+1}\int_0^1 \int_0^1 (1-Y)^{(n_1+1)}Y^{n_2}$$

$$(1-z)^{(n_1+n_2+2)}z^{n_3}dYdz$$

$$= \frac{1}{n_1+1}\int_0^1 \beta(n_2+1, n_1+2)$$

$$(1-z)^{(n_1+n_2+2)}z^{n_3}dz$$

$$= \frac{\beta(n_2+1, n_1+2)}{n_1+1}\int_0^1 (1-z)^{(n_1+n_2+2)}z^{n_3}dz$$

$$= \frac{\beta(n_2+1, n_1+2)}{n_1+1}\beta(n_3+1, n_1+n_2+3)$$

$$= \frac{1}{n_1+1}\,\frac{n_2!(n_1+1)!}{(n_1+n_2+2)!}\,\frac{n_3!(n_1+n_2+2)!}{(n_1+n_2+n_3+3)!}$$

$$= \frac{n_1!\,n_2!\,n_3!}{(n_1+n_2+n_3+3)!}$$

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