

# Alternate mathematical universes

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## On truth

“There are infinitely many prime numbers.”

“There are exactly five platonic solids.”

“There are more real numbers than integers.”

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- Mathematical statements are thought to be absolutely objective and independent of human sentiments.
- But in fact, their truth depends on the **choice of mathematical universe** (“topos”).
- The usual axioms of logic hold in the **standard topos** Set.
- In other topoi, slightly different axioms are true.

# Examples

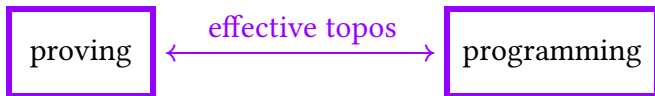
There are topoi in which ...

- there is an intermediate infinity between  $\mathbb{N}$  and  $\mathbb{R}$ ,
- there is no such intermediate infinity,
- there are **infinitesimal real numbers**  $\varepsilon$  such that  $\varepsilon \neq 0$  but  $\varepsilon^2 = 0$ , or
- any function  $\mathbb{R} \rightarrow \mathbb{R}$  is a polynomial.

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# The effective topos

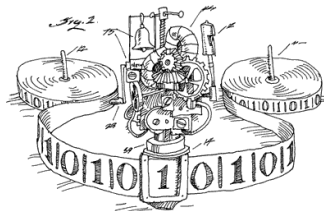
- “ $1 + 1 = 2$ .”  
True in Set, true in  $\text{Eff}(\text{TM})$ .
- “Any number is either prime or not.”  
Trivially true in Set, nontrivially true in  $\text{Eff}(\text{TM})$ .
- “Any function  $\mathbb{N} \rightarrow \mathbb{N}$  is either the zero function or not.”  
Trivially true in Set, false in  $\text{Eff}(\text{TM})$ .
- “Any function  $\mathbb{N} \rightarrow \mathbb{N}$  is computable by a Turing machine.”  
False in Set, trivially true in  $\text{Eff}(\text{TM})$ .
- “Any function  $\mathbb{R} \rightarrow \mathbb{R}$  is continuous.”  
False in Set, true in  $\text{Eff}(\text{TM})$ .

# The effective topos

There are many **models of computation**:

- Turing machines
- Lambda calculus
- Perl programs (running on idealized computers)
- Computers in the real physical world

For any such model  $\mathcal{M}$ , there is a special topos  $\text{Eff}(\mathcal{M})$ , the associated **effective topos**.





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$\text{Eff}(\mathcal{M}) \models$  “For any number  $n$  there is a prime  $p > n$ .”

means:

There is a program which reads a number  $n$  as input and outputs a prime number  $p > n$ .

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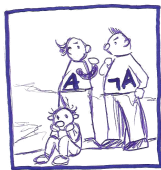
There is a program which reads a number  $n$  as input and outputs YES or NO depending on whether  $n$  is prime or not.

# What's true in alternate topoi?

**Metatheorem:** If a statement has a **constructive proof**, then it holds in **any topoi**.

Constructive logic is like classical logic, except we don't suppose the **law of excluded middle** (LEM), which says:

- “Any statement is either true or not true.”
- “If a statement is *not not* true, then it's true.”



# Nonconstructive proofs

A number is **rational** if and only if its decimal digits eventually repeat.

- $\frac{21}{13}$  and 37 are rational.
- $\sqrt{2}$  and  $\pi$  are irrational.



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- $\sqrt{2}$  and  $\pi$  are irrational.

**Theorem.** There are **irrational** numbers  $x$  and  $y$  such that  $x^y$  is rational.

**Proof.** Either  $\sqrt{2}^{\sqrt{2}}$  is rational or not.

- 1 In the first case we are done.
- 2 In the second case we set  $x := \sqrt{2}^{\sqrt{2}}$  and  $y := \sqrt{2}$ .  
Then  $x^y = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$  is rational.



# Appreciating constructive logic

At first sight, dropping the axiom of excluded middle looks like a sad thing to do. It's a useful axiom! However:

- The axiom is not needed as often as one would think.
- The abstinence is good for your mental hygiene.
- Constructive logic allows for finer distinctions.
- From constructive proofs one can mechanically extract programs which witness the proved statements.
- **Dropping the law of excluded middle allows to add curious unconventional axioms.**

# LEM for equality of functions

$\text{Eff}(\text{TM}) \models$  “Any function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is either the zero function or not.”

means:

There is a program which reads the source of a program  $M$ , which calculates a function  $\mathbb{N} \rightarrow \mathbb{N}$ , as input, and finds out whether  $M$  always yields zero or not.

That's false.

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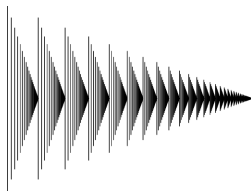
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That’s false.

The statement is true in  $\text{Eff}(\text{STM})$ , the effective topos associated to **super Turing machines**.



# LEM for the halting problem

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For a Turing machine  $M$  consider the real number  $0.000\dots$ , whose  $n$ ’th decimal digit is a one if and only if  $M$  halts after step  $n$ .

For a real number  $x$  consider the Turing machine which searches the digits of  $x$  for a nonzero digit.

# Markov's principle

$\text{Eff}(\text{TM}) \models$  “For any function  $f : \mathbb{N} \rightarrow \mathbb{N}$  which is not the zero function, there is a number  $n \in \mathbb{N}$  such that  $f(n) \neq 0$ .”

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There is a program which reads the source of a program  $M$ , which calculates a function  $\mathbb{N} \rightarrow \mathbb{N}$  which is not the zero function, as input and outputs a number  $n$  such that  $M(n)$  is not zero.



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That's true! By unbounded search.

## Searching uncountable sets

“For any function  $f : \mathbb{N} \rightarrow \mathbb{N}$  from numbers to numbers, there either exists a number  $n$  such that  $f(n)$  is one or there is no such number.”

This statement is false in  $\text{Eff}(\text{TM})$ .

“For any function  $P : L(\mathbb{B}) \rightarrow \mathbb{B}$  from infinite lists of booleans to booleans, there either exists a list  $x$  such that  $P(x)$  is true or there is no such list.”

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# The Church–Turing thesis

The **Church–Turing thesis** states: If a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is calculable in the real world, then it's also calculable by a program (running on a Turing machine).

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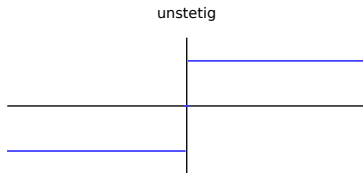
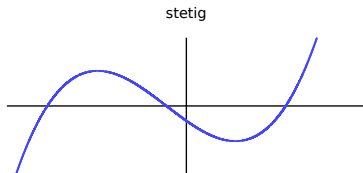
In  $\text{Eff}(\text{STM})$  and  $\text{Eff}(\lambda C)$  the statement is false.

# Automatic continuity

The following statement is wildly **false** in Set:

“Every function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous.”

A function  $f$  is **continuous** if and only if, for calculating  $f(x)$  to finitely many digits, finitely many digits of  $x$  suffice.

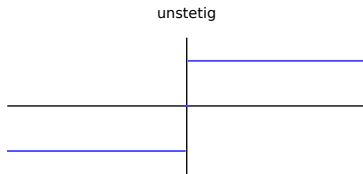
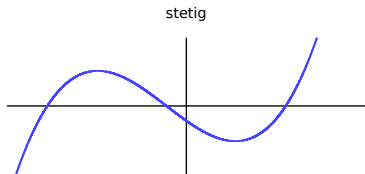


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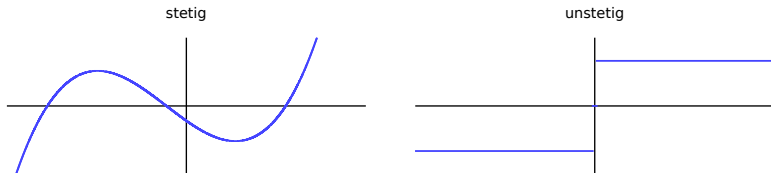
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**True** in  $\text{Eff}(\text{TM})$ . True in  $\text{Eff}(\text{RW})$ , if private communication channels are possible and only finitely many computational steps can be executed in finite time.



# Wrapping up

- Effective topoi are a good vehicle for studying the nature of computation.
- The effective topoi build links between constructive mathematics and programming.
- Topoi allow for curious dream axioms.
- Topoi have a geometric flavour: points, subtopoi, maps between topoi.

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**There is more to mathematics  
than the standard topos.**