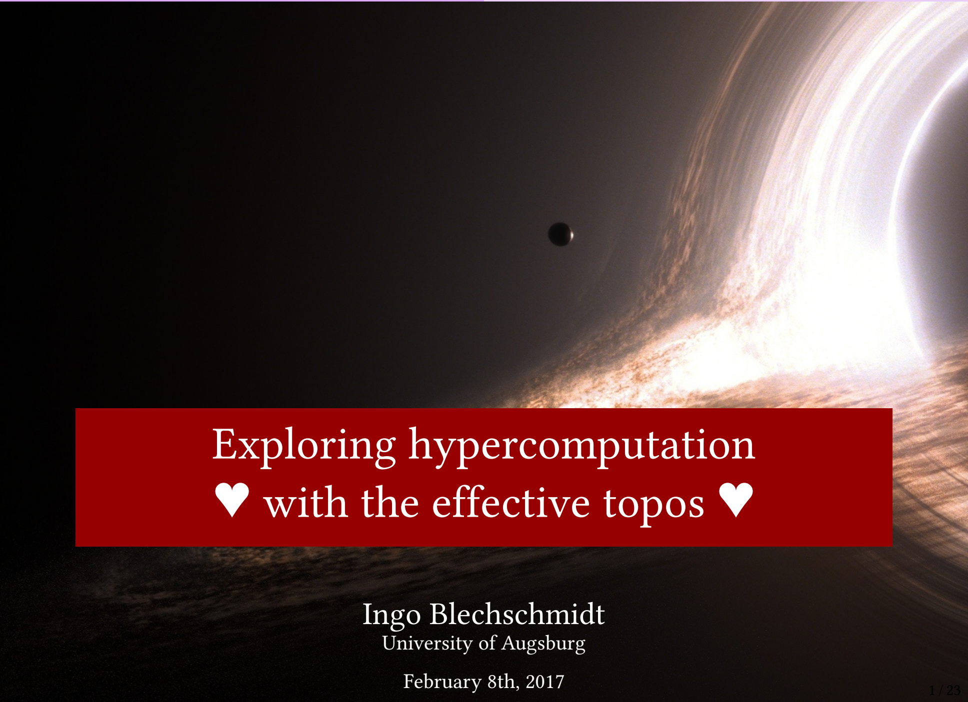


Exploring hypercomputation ♥ with the effective topos ♥

Ingo Blechschmidt
University of Augsburg

February 8th, 2017

1. Crash course on ordinal numbers

2. (Super) Turing machines
   - Bacis on Turing machines
   - Bacis on super Turing machines
   - The power of super Turing machines
   - Outlook on the bigger theory

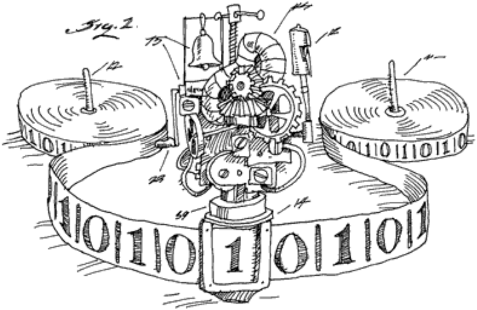
3. The effective topos
   - First steps in the effective topos
   - The wonder of constructive logic
   - Effective content of classical tautologies
   - Wrapping up

Part I

A crash course on ordinal numbers

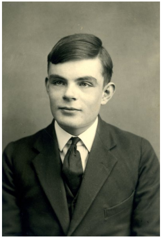
Part II

(Super) Turing machines

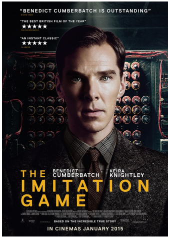


Basics on Turing machines

- Turing machines are idealised computers operating on an **infinite tape** according to a **finite list** of rules.
- The concept is astoundingly robust.
- A subset of $\mathbb{N}$ is **enumerable by a Turing machine** if and only if it's a Σ_1-set.



Alan Turing
(* 1912, † 1954)



worth seeing

Alison Bechdel
(* 1960)

Super Turing machines

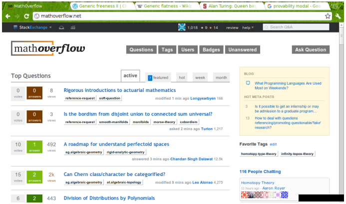
With super Turing machines, the time axis is more interesting:

- normal: $0, \; 1, \; 2, \; \ldots$
- super: $0, \; 1, \; 2, \; \ldots, \; \omega, \; \omega+1, \; \ldots, \; \omega \cdot 2, \; \omega \cdot 2 + 1, \; \ldots \ldots \ldots$

On reaching a limit ordinal time step like ω or $\omega \cdot 2$,

- the machine is put into a designated state,
- the read/write head is moved to the start of the tape, and
- the tape is set to the “lim sup” of all its previous contents.

Joel David Hamkins



MathOverflow

Andy Lewis

A question to you

What's the behaviour of this super Turing machine?

In the start state and the limit state, check whether the current cell contains a "1".

- If yes, then stop.
- If not, then flash that cell: set it to "1", then reset it to "0". Then unremittingly move the head rightwards.

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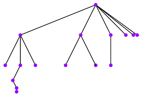
Super Turing machines can break out of (some kinds of) infinite loops.

What can super Turing machines do?

- Everything ordinary Turing machines can do.
- Verify number-theoretic statements.
- Decide whether a given ordinary Turing machine halts.
- Simulate super Turing machines.
- Decide Π^1_1- and Σ^1_1-statements:
  - "For any function $\mathbb{N} \to \mathbb{N}$ it holds that …"
  - "There is a function $\mathbb{N} \to \mathbb{N}$ such that …"



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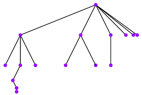
But: Super Turing machines can't calculate all functions and can't write all 0/1-sequences to the tape.



Fun facts

- Any super Turing machine either halts or gets caught in an unbreakable infinite loop after **countably many steps**.
- An ordinal number α is **clockable** iff there is a super Turing machine which halts precisely after time step α.
  - Speed-up Lemma: If $\alpha + n$ is clockable, then so is α.
  - Big Gaps Theorem
  - Many Gaps Theorem
  - Gapless Blocks Theorem
- **Lost Melody Theorem:** There are 0/1-sequences which a super Turing machine can recognise, but not write to the tape.

Part III

The effective topos

The effective topos

- "$1 + 1 = 2$."
- "Any number is either prime or not."
- "Any function $\mathbb{N} \to \mathbb{N}$ is either the zero function or not."
- "Any function $\mathbb{N} \to \mathbb{N}$ is computable by a Turing machine."
- "Any function $\mathbb{R} \to \mathbb{R}$ is continuous."

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First steps in the effective topos

$\mathrm{Eff}(\mathrm{TM}) \models$ "For any number n there is a prime $p > n$."

means:

There is a Turing machine which reads a number n as input and outputs a prime number $p > n$.

"Realisability Theory"

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$\mathrm{Eff}(\mathrm{TM}) \models$ "Any number is either prime or not prime."

means:

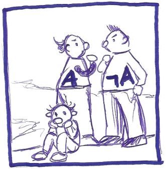
There is a Turing machine which reads a number n as input and outputs YES or NO depending on whether n is prime or not.

What's true in alternate toposes?

Metatheorem: If a statement has a **constructive proof**, then it holds in **any topos**.

Constructive logic is like classical logic, except we don't suppose the **law of excluded middle** (LEM), which says:

- "Any statement is either true or not true."
- "If a statement is *not not* true, then it's true."



Nonconstructive proofs

Theorem. There are **irrational** numbers x und y such that x^y is rational.

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Proof. Either $\sqrt{2}^{\sqrt{2}}$ is rational or not.

1. In the first case we are done.
2. In the second case we set $x := \sqrt{2}^{\sqrt{2}}$ and $y := \sqrt{2}$. Then $x^y = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$ is rational.

Appreciating constructive logic

At first sight, dropping the axiom of excluded middle looks like a sad thing to do. It's a useful axiom! However:

- The axiom is not needed as often as one would think.
- The abstinence is good for your mental hygiene.
- Constructive logic allows for finer distinctions.
- From constructive proofs one can mechanically extract programs which witness the proved statements.
- **Dropping the law of excluded middle allows to add curious unconvential axioms.**

LEM for equality of functions

$\mathrm{Eff}(\mathrm{TM}) \models$ "Any function $f : \mathbb{N} \to \mathbb{N}$ is either the zero function or not."

means:

There is a Turing machine which reads the source of a Turing machine M, which calculates a function $\mathbb{N} \to \mathbb{N}$, as input, and finds out whether M always yields zero or not.

That's false.

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The statement is true in $\mathrm{Eff}(\mathrm{STM})$, the effective topos associated to super Turing machines.

LEM for the halting problem

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"Every real number is either zero or not zero."

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For a Turing machine M consider the real number $0.000\ldots$ whose n'th decimal digit is a one iff M halts after step n.

For a real number x consider the Turing machine which searches the digits of x for a nonzero digit.

Markov's principle

$\mathrm{Eff}(\mathrm{TM}) \models$ "For any function $f : \mathbb{N} \to \mathbb{N}$ which is not the zero function, there is a number $n \in \mathbb{N}$ such that $f(n) \neq 0$."

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That's true! By unbounded search.

Searching uncountable sets

> "For any function $f : \mathbb{N} \to \mathbb{N}$ from numbers to numbers, there either exists a number n such that $f(n)$ is one or there is no such number."

This statement is false in $\mathrm{Eff}(\mathrm{TM})$.

> "For any function $P : L(\mathbb{B}) \to \mathbb{B}$ from infinite lists of booleans to booleans, there either exists a list x such that $P(x)$ is true or there is no such list."

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The Church–Turing thesis

The **Church–Turing thesis** states:

> If a function $f : \mathbb{N} \to \mathbb{N}$ is calculable in the real world, then it's also calculable by a Turing machine.

$\mathrm{Eff}(\mathrm{TM}) \models$ "Any function $f : \mathbb{N} \to \mathbb{N}$ is calculable by a Turing machine."

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> There is a Turing machine which reads the source of a Turing machine calculating a function $f : \mathbb{N} \to \mathbb{N}$ as input and outputs the source of a Turing machine which calculates f.
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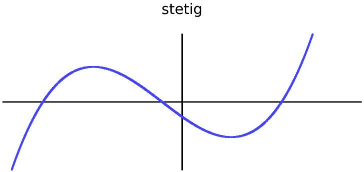
In $\mathrm{Eff}(\mathrm{STM})$ the statement is false.

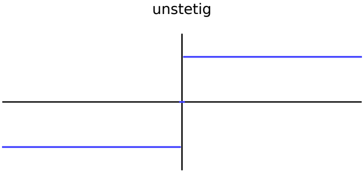
Automatic continuity

The following statement is wildly **false** in Set:

"Every function $f : \mathbb{R} \to \mathbb{R}$ is continuous."

A function f is **continuous** if and only if, for calculating $f(x)$ to finitely many digits, finitely many digits of x suffice.





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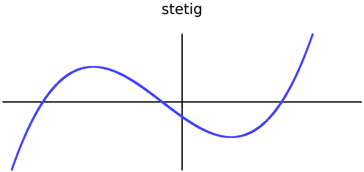

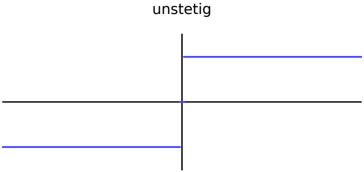

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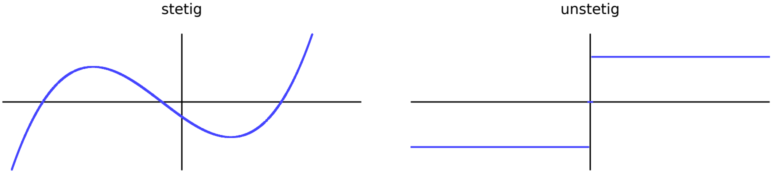

True in $\mathrm{Eff}(\mathrm{TM})$. True in $\mathrm{Eff}(\mathrm{RW})$, if private communication channels are possible and only finitely many computational steps can be executed in finite time.

Wrapping up

- Effective toposes are a good vehicle for studying the nature of computation.
- The effective toposes build links between constructive mathematics and programming.
- Toposes allow for curious dream axioms.
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There is more to mathematics than the standard topos.