Answers to questions in Lab 1: Filtering operations

Name: Timotheos Souroulla Program: Systems, Control and Robotics (TSCRM)

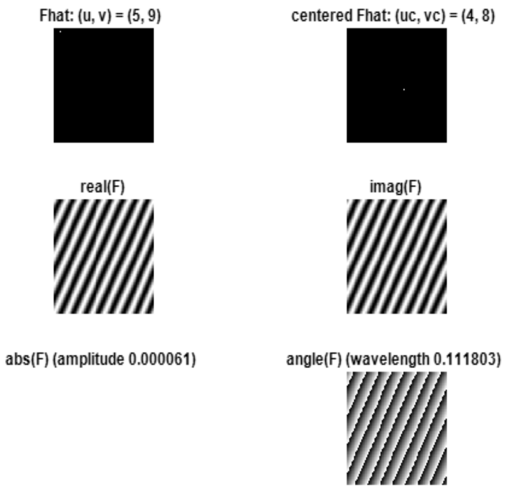
Instructions: Complete the lab according to the instructions in the notes and respond to the questions stated below. Keep the answers short and focus on what is essential. Illustrate with figures only when explicitly requested.

Good luck!

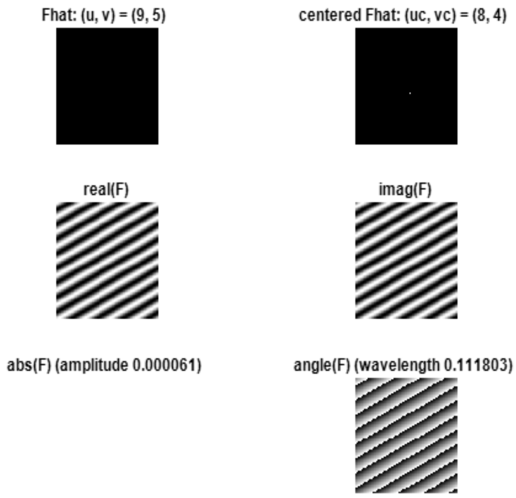
Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9), (17, 121), (5, 1) and (125, 1) respectively. What do you observe?

Answers:

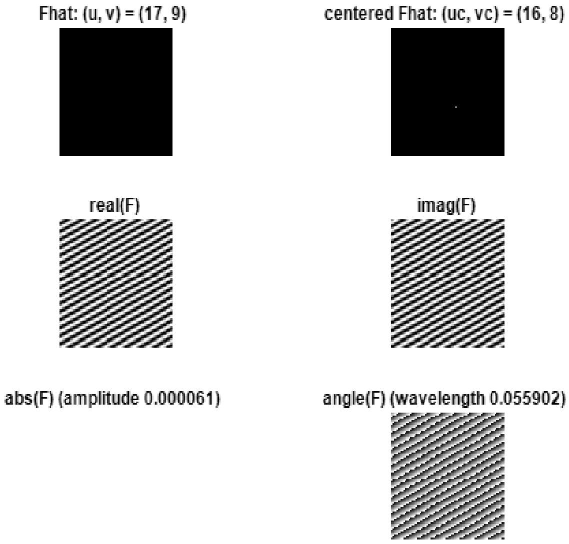
(5, 9):



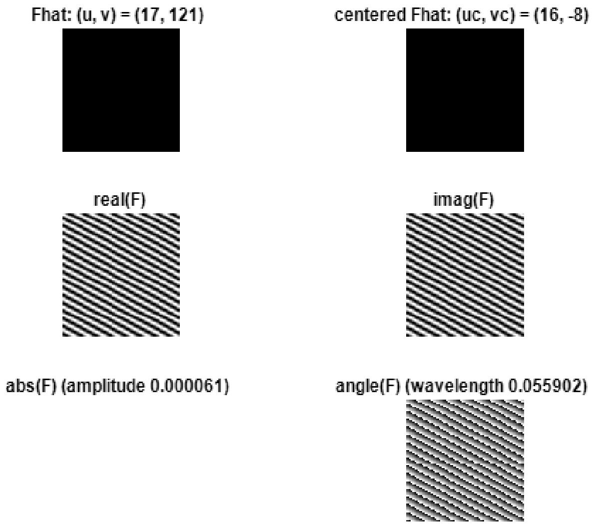
(9,5):



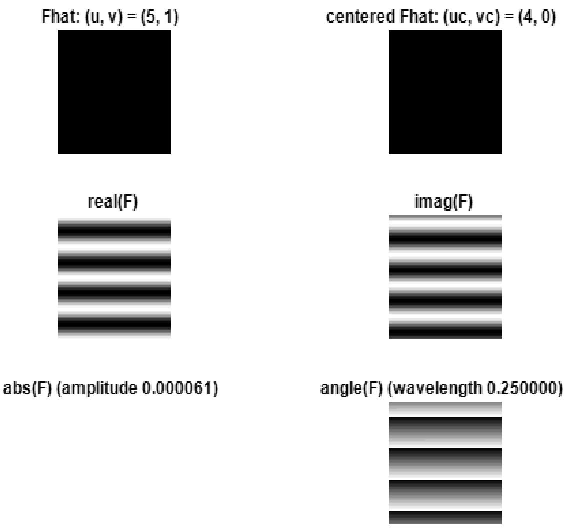
(17,9):



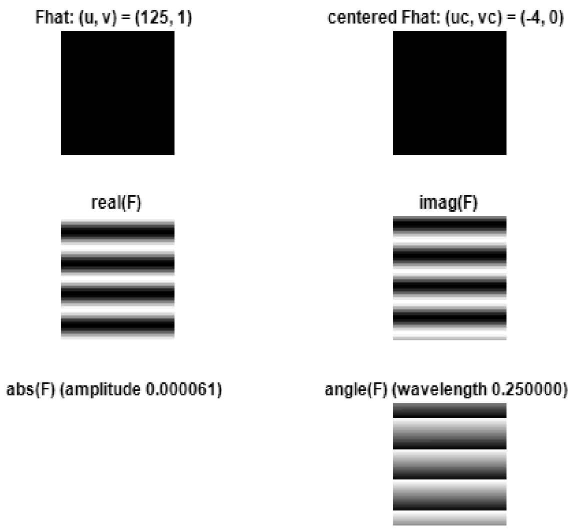
(17,121):



(5,1):



(125,1):



Question 2: Explain how a position (p, q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a MATLAB figure.

Answers:

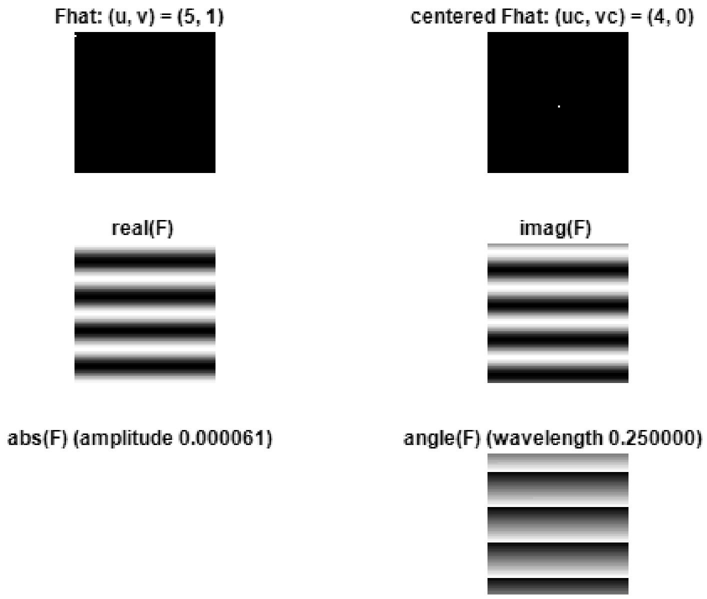
Consider a case e.g. (p,q) = (4,0) is equal to 1 and all remaining points are equal to 0.
So, this equation tells us that:

1. In the frequency domain we have a point (4,0) at which the power is highest. The frequency is 0 in x direction and 4 in y direction. So, it is like having a 2D Dirac function at the point (4,0).
2. If we use inverse Fourier transform on a 2D Dirac function, we will get an exponential delay, which is equivalent as having a sine wave (by using Euler's Formula).

According to the equation:

$$f(m,n) = \frac{1}{\sqrt{MN}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{f}(u,v) e^{+2\pi i(\frac{mu}{M} + \frac{nv}{N})}$$

for (u, v) = (p, q) = (4, 0):



Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in the notes. Complement the code (variable amplitude) accordingly.

Answers:
From equation (4) we can observe that each pixel of the image has a real and an imaginary part. So, the amplitude of each pixel is the magnitude of its complex value. In our case, we only have one non-zero point (p,q), so the total amplitude of our image is just the magnitude of F(p,q).

$$\text{Fourier spectrum: } |\hat{f}(\omega)| = \sqrt{Re^2(\omega) + Im^2(\omega)}$$

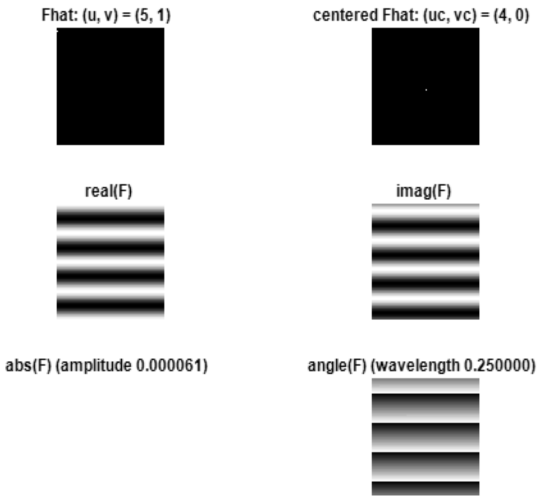
Where ω is the non-zero points in the Fourier domain.

Question 4: How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

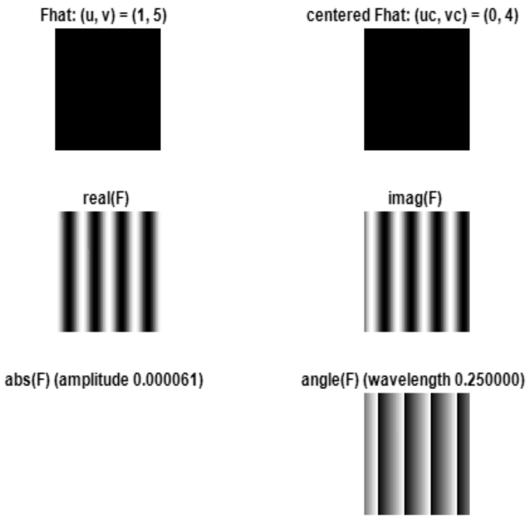
Answers:

$$f(m,n) = \frac{1}{\sqrt{MN}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{f}(u,v) e^{+2\pi i(\frac{mu}{M} + \frac{nv}{N})} \qquad \lambda = \frac{1}{\sqrt{u^2+v^2}}$$

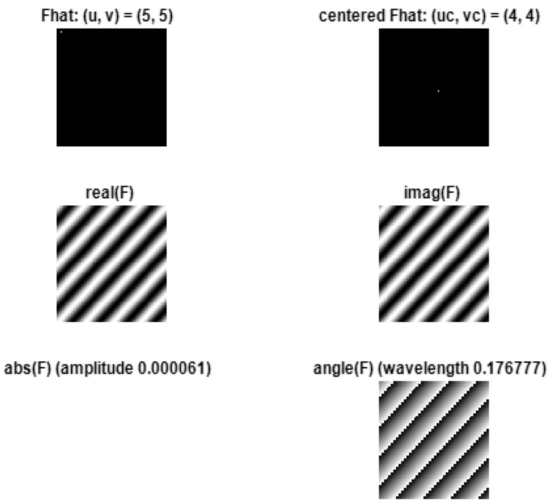
$p \neq 0, q = 0$:



$p = 0, q \neq 0$:



p = q = constant:

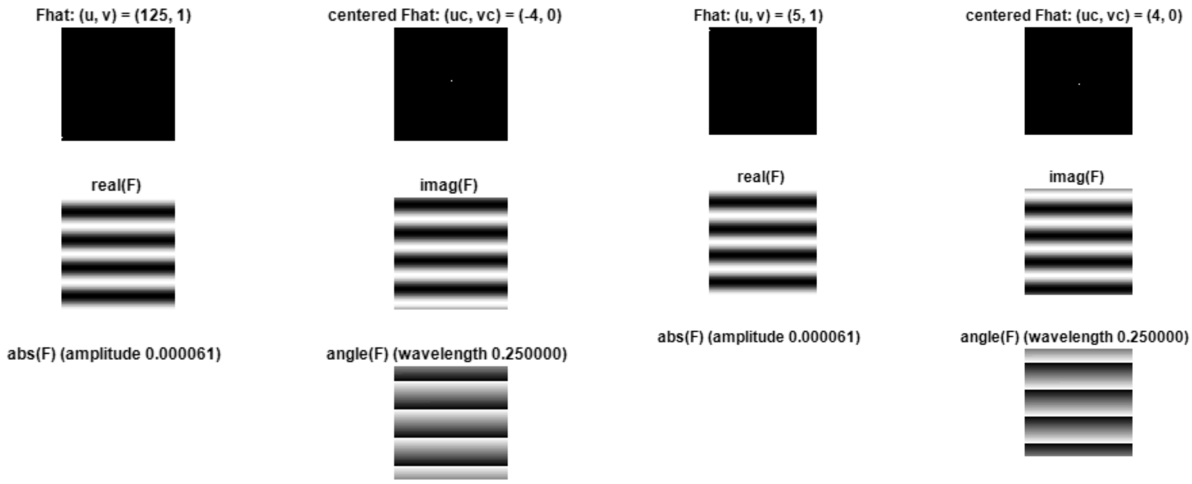


If we have a high value for p, then the frequency on the y-axis will be bigger. The same thing happens with q, but in frequency on the x-axis. So, as far as we are away from the center, then the wavelength of the sine waves will be lower, which leads to higher frequency of the image, and consequently, the image will consist many changes and sharp edges.

Question 5: What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically with MATLAB!

Answers:
If we pass the center of the image, or half the image size, then the imaginary part of the image will become negative (or positive if it was already negative), and so we will have its complex conjugate. The angle part of the image will be flipped in the direction of the point that passed the center. For example, if a point passes the center in the x direction, then the image will be flipped in the x-axis. If both points pass the center, then the image will be flipped in both axes.

e.g.:



In this example, our non-zero point in Fourier domain is (-4,0) and the new point is (4,0). The real part of the image does not change, but the imaginary part is flipped upside down. This leads to the change of the angle of the image, which is flipped in the same direction as the imaginary part. We can see from the imaginary part of the image that it flipped around the x-axis (The bottom of the image is white and then it becomes black).

Question 6: What is the purpose of the instructions following the question *What is done by these instructions?* in the code?

Answers:

The purpose of these instructions is to use the center of the image as the reference point instead of the top left point, which in MATLAB is the point (1,1). This is used only when plotting the Fourier spectrum of the image so it can be seen easier visually.

Question 7: Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Answers:

According to the equation:

$$\hat{f}(u,v) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) e^{-2\pi i\left(\frac{mu}{M} + \frac{nv}{N}\right)}$$

If in spatial domain there are not so many alterations, then this will lead to sine waves with low frequencies in the Fourier domain. These low frequencies have small value for u and v. In addition, if we take into consideration that the top left corner of the image in MATLAB is noted as the point (1,1). we can conclude that low values of u and v will be concentrated at the borders of the image. Low values of u will be represented on the left side of the image, while low values of v will be represented in the top side of the image.

Question 8: Why is the logarithm function applied?

Answers:

In a grayscale image we can represent values between [0 255], where 0 is a black pixel and 255 is a white pixel. In the Fourier domain we have complex numbers and for this reason we are using the abs() function to get their magnitude. But we are getting values much greater than 255, so we are using the logarithm function to ensure that our values of the Fourier domain are inside the boundaries [0 255], so that they can be represented as a grayscale image.

Question 9: What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answers:

From our results, we can conclude that the Fourier domain retains the property of linearity. When two functions are added in the spatial domain, then their Fourier domain will be the sum of their individual Fourier transforms. Spatial domain also preserves the property of linearity.

$$\mathcal{F}\left[a\,\mathrm{f}_1(m,n) + b\,\mathrm{f}_2(m,n))\right] = a\,\hat{f}_1(u,v) + b\,\hat{f}_2(u,v)$$

$$a\,\mathrm{f}_1(m,n) + b\,\mathrm{f}_2(m,n)) = \mathcal{F}^{-1}\left[a\,\hat{f}_1(u,v) + b\,\hat{f}_2(u,v)\right]$$

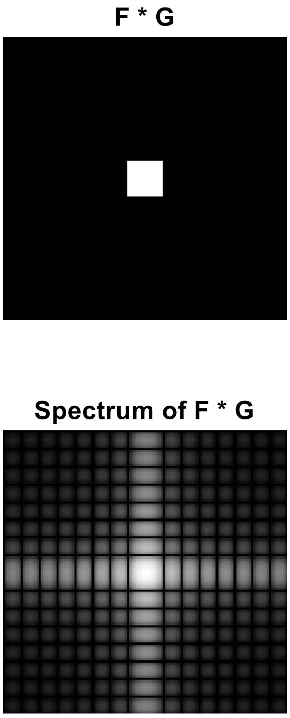
Question 10: Are there any other ways to compute the last image? Remember what multiplication in Fourier domain equals to in the spatial domain! Perform these alternative computations in practice.

Answers:

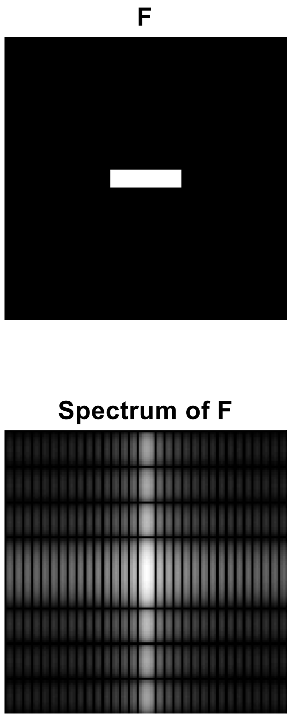
To construct the last image, we multiplied the two functions elementwise in the spatial domain. Another way of getting the same image is to transform F and G in the Fourier domain, and then convolve them. Finally, we need to use inverse Fourier transform to get the same image.

Multiplication in spatial domain is convolution in Fourier domain, and vice versa.

Q10)



Q11)



Question 11: What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

Answers:

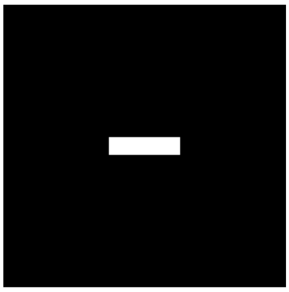
Using the two results we got from this exercise and the previous one we can conclude that if the image is narrow in spatial domain then it will become wide in Fourier domain, and vice versa. For these examples we had a square shaped white box and rectangle one in spatial domain. The Fourier transform of the square wave is a Sinc function equally distributed in both p and q directions, whereas in the case of the rectangle, the Fourier transform is a Sinc function with more frequency components in the direction where the rectangle is wider.

Question 12: What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.

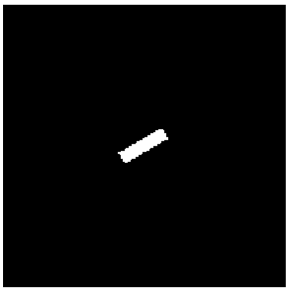
Answers:

We can clearly see from the figures obtained that rotation in spatial domain leads to rotation in Fourier domain.
The restored image, image after rotating back the image to its initial position, is the same as the original image, but with some added artifacts in high frequencies. These artifacts are showed up due to noise occurred in the Fourier transform step of the image. In addition, we lose information in the corners of the Fourier domain, since we rotated the Fourier domain instead of the spatial domain. We should rotate the image back to its initial position in the spatial domain instead of rotating it back in the Fourier domain.

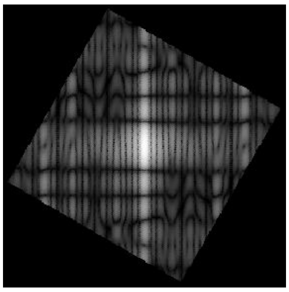
F



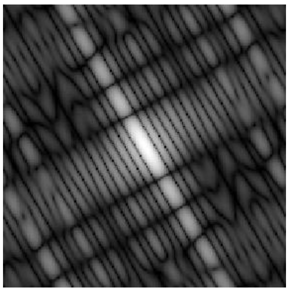
G, rotated by 30



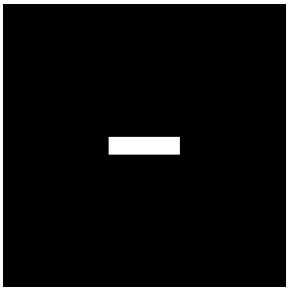
Spectrum of F



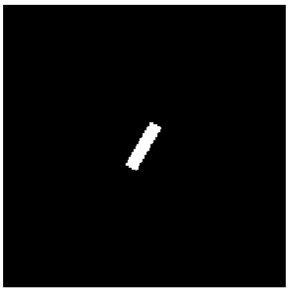
Spectrum of G



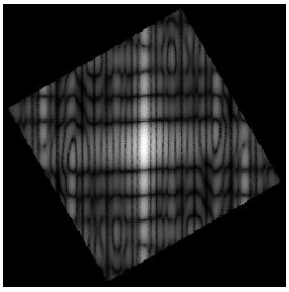
F



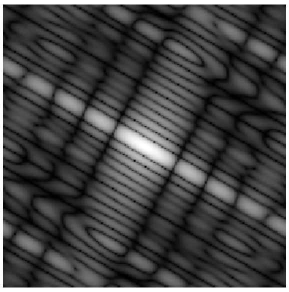
G, rotated by 60



Spectrum of F



Spectrum of G



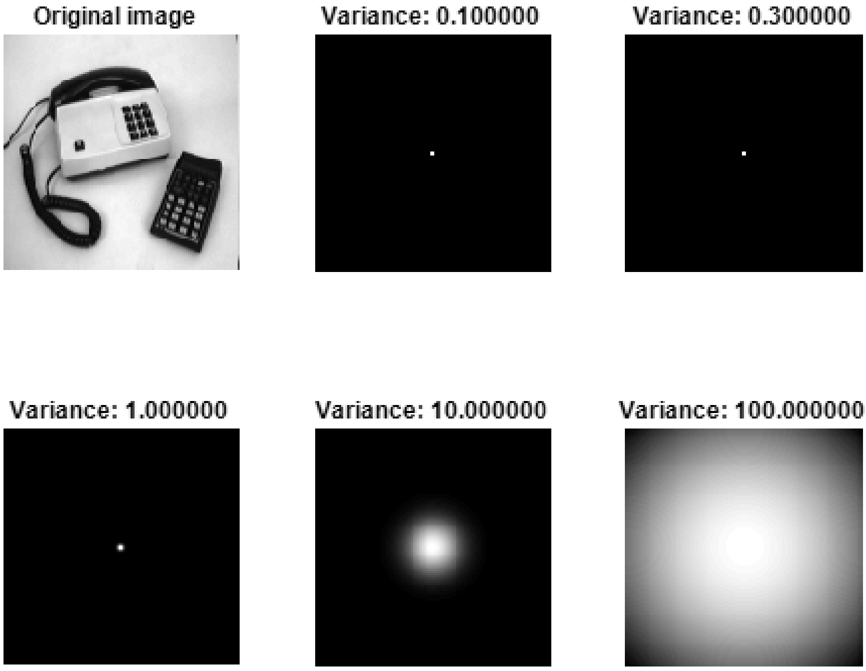
Question 13: What information is contained in the phase and the magnitude of the Fourier transform?

Answers:

Phase defines how waveforms are shifted along its direction and where edges will end up in the image (most important).
Magnitude defines how large the waveforms are, and at the same time it defines what grey-levels are on either side of the edge (less important).

Question 14: Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and 100.0?

Answers:



<u>t = 0.1:</u>

variance =

$$\begin{matrix} 0.2500 & -0.0000 \\ -0.0000 & 0.2500 \end{matrix}$$

<u>t = 0.3</u>

variance =

$$\begin{matrix} 0.2500 & -0.0000 \\ -0.0000 & 0.2500 \end{matrix}$$

<u>t = 1.0</u>

variance =

$$\begin{matrix} 1.0000 & -0.0000 \\ -0.0000 & 1.0000 \end{matrix}$$

<u>t = 10.0</u>

variance =

100.0000   0.0000
  0.0000   100.0000

<u>t = 100.0</u>

variance =

1.0e+03 *

1.2922   0.0000
0.0000   1.2922

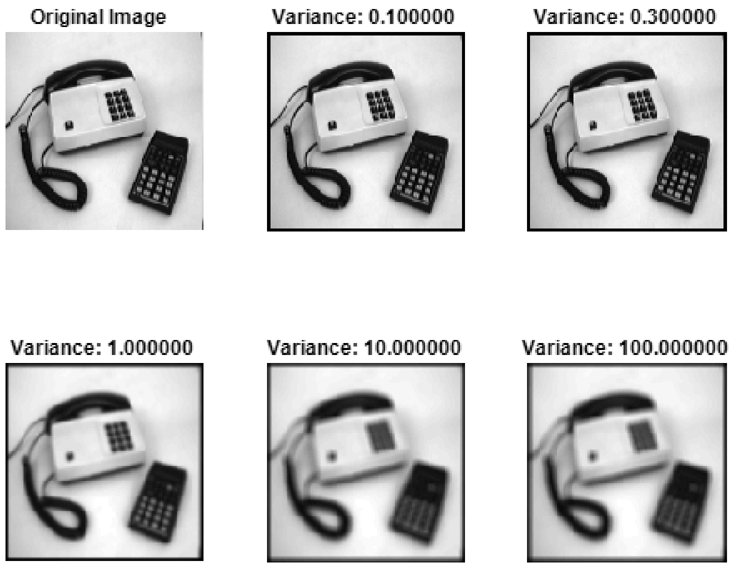
Question 15: Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.

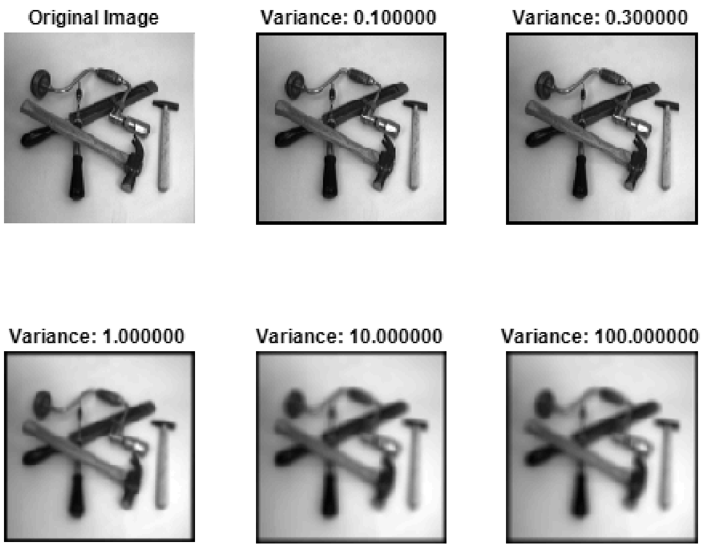
Answers:

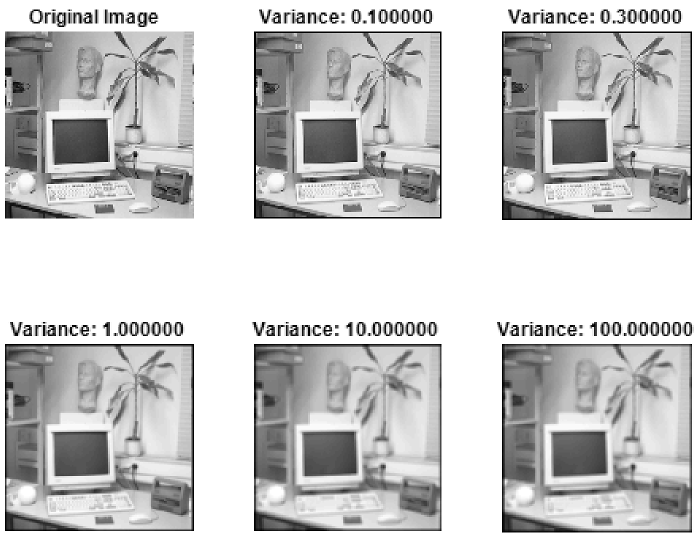
The results are different from the estimated variance, but their values are very similar. The result is multiplied by the Identity matrix, but with a different constant, instead of the value of t. So, the results correspond to the ideal continuous case because of the multiplication of a constant with the identity matrix.

Question 16: Convolve a couple of images with Gaussian functions of different variances (like t = 1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

Answers:







The higher the value of the variance of a Gaussian filter, the blurrier the image will become. This is happening due to the effect of neighboring pixels have on the middle pixel.

Question 17: What are the positive and negative effects for each type of filter? Describe what you observe and name the effects that you recognize. How do the results depend on the filter parameters? Illustrate with MATLAB figure(s).

Answers:

Gaussian Filter:

(+) Removes Gaussian noise
(+) Rotationally symmetric
(+) The degree of smoothing is controlled by σ => controlled by the user
(-) Image becomes blurry
(-) Need to apply a mean filter or a sharpening mask to completely restore the image

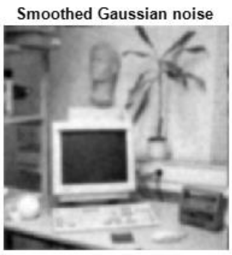
Median Filter:

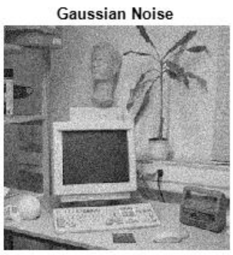
(+) Eliminates Salt and Pepper Noise
(+) Preserves monotonic structures (shading) and position of step edges
(-) Creates painting-like images
(-) Need to apply a sharpness filter to completely restore the image

Low-Pass Filter:

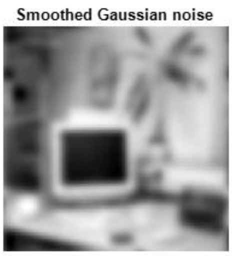
(+) Removes high frequencies, which are usually noise
(-) Removes high frequency components like sharp edges and sharp details
(-) Need to apply a smoothing filter to completely restore the image

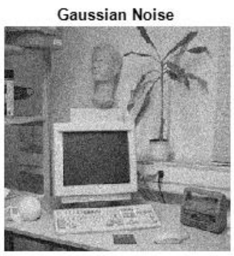
Gaussian Filter:





high value: Blurred image





low value: Does not remove the Gaussian noise

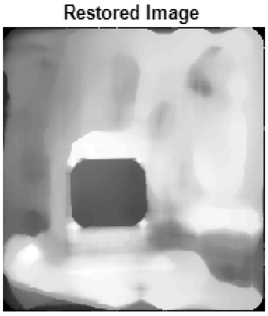
Median filter:

Restored Image

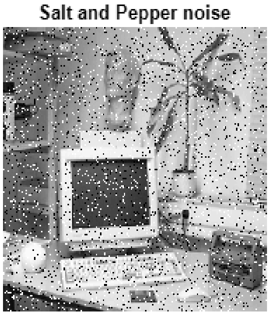
Salt and Pepper noise

High Value: The image is painting-like and blurry

Restored Image



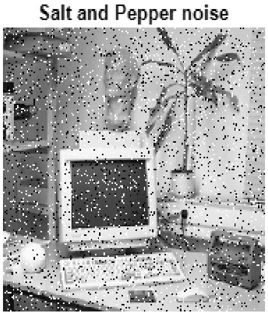
Salt and Pepper noise



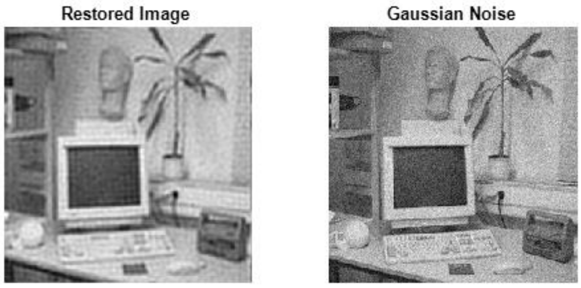
Low Value: Salt and Pepper noise is not eliminated

Restored Image

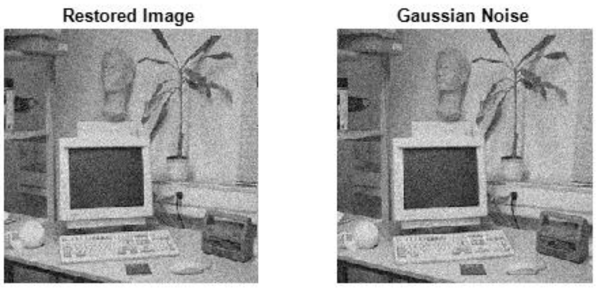
Salt and Pepper noise



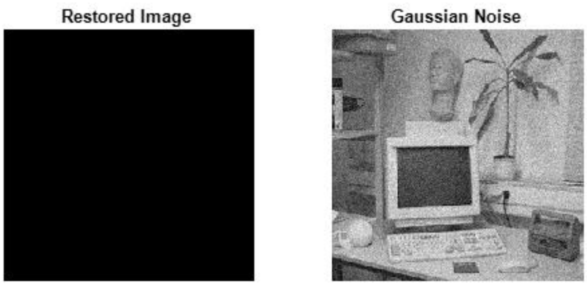
Low-Pass filter:



High Value: High frequencies are not eliminated, so noise is not cancelled out



Low Value: No frequencies pass through the filter, so we get a black image



Question 18: What conclusions can you draw from comparing the results of the respective methods?

Answers:

We conclude that each filter has its benefits and its costs, for example the median filter eliminates salt and pepper noise, but the resulted image is painting-like. Another conclusion that we had, is that for different type of noise we should use a different filter, and in order to completely restore an image we could combine two or more filters.
So, while trying to eliminate noise, we should first identify the type of noise (by using the histogram of a part of the image that should have the same intensity all over it) at the image and then decide which filter to use.

Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

Answers:

We observe that the image preserves its characteristics, but its quality is lower, which leads to the blocking of the image. This is happening because we subsample the original image and we set the value of not sampled pixels with the value of their neighbors. In addition, when subsampling is applied, then the size of the image will be reduced.
When we filter the image with low quality, we do not preserve the main characteristics of the original image. Both filters seem to have the same results. As seen from the figures, the resulted image for iteration i =4 is not even close to the original one.

Example: When we subsample with 2:

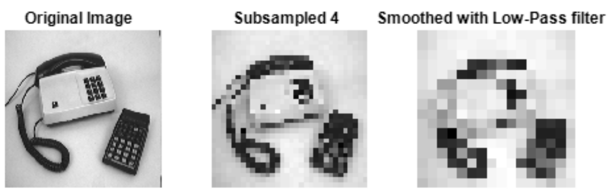
Intensity values of a part of an image:

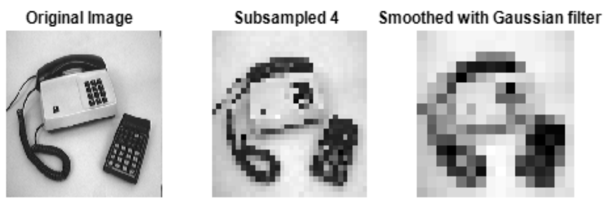
125	100	125	90	125	85

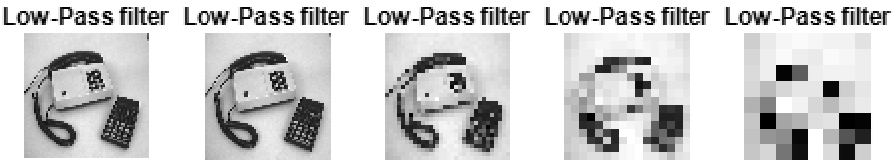
Intensities of the same image after the subsampling:

125	125	125	125	125	125

It can be seen the details in this part of the image are gone, so quality of the image is not preserved.







Question 20: What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answers:

When we subsample, we lose frequencies from the original image and side effects are created. We may lose edges and sharp details, which correspond to high frequencies, or we may lose monotonic structures (shading), which correspond to low frequencies.
When we apply a low-pass filter, then high frequencies are eliminated, and low frequencies are preserved. Side effects may be generated due to aliasing or folding effect, so artifacts are introduced to the resulted image.
If subsampling rate is wrongly selected, then aliasing (or folding effect) will be created. This is the main reason why Gaussian and low-pass filters become weak, and as a result, the filtered image becomes worse than the prefiltered image. Main characteristics of the original image are lost, because of the frequency elimination happened during the subsampling stage.
