MATH 137 Fall 2020: Practice Assignment 6

Q01. For $f(x) = \dfrac{x+1}{x-1}$, find $f'(x)$ using the limit definition.

Solution. Apply the Newton quotient:

$$\begin{aligned}
f'(x) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\frac{x+1}{x-1} - \frac{a+1}{a-1}}{x-a} \\
&= \lim_{x\to a} \frac{(x+1)(a-1)-(a+1)(x-1)}{(x-a)(x-1)(a-1)} \\
&= \lim_{x\to a} \frac{(xa+a-x-1)-(xa-a+x-1)}{(x-a)(x-1)(a-1)} \\
&= \lim_{x\to a} \frac{-2(x-a)}{(x-a)(x-1)(a-1)} \\
&= \lim_{x\to a} \frac{-2}{(x-1)(a-1)} \\
&= -\frac{2}{(x-1)^2}
\end{aligned}$$

□

Q02. Let $f(x) = \dfrac{ax+b}{ax-b}$ where $a \neq 0$, $b \neq 0$.

(a) Find $f'(x)$ using any method.

Solution. First, notice that $f(x)$ is undefined at $x = \frac{b}{a}$, so we differentiate along all $x \neq \frac{b}{a}$. Apply the quotient and linear function rules:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{ax+b}{ax-b}\right) &= \frac{(ax-b)\frac{\mathrm{d}}{\mathrm{d}x}(ax+b)-(ax+b)\frac{\mathrm{d}}{\mathrm{d}x}(ax-b)}{(ax-b)^2} \\
&= \frac{(ax-b)a-(ax+b)a}{(ax-b)^2} \\
&= \frac{a(-2b)}{(ax-b)^2} \\
&= -\frac{2ab}{(ax-b)^2}
\end{aligned}$$

□

(b) Show that for $x \neq \frac{b}{a}$, $abf'(x) < 0$.

Proof. Let a and b be non-zero reals, and let $x \neq \frac{b}{a}$. Then,

$$\begin{aligned}
abf'(x) &= ab\left(\frac{2ab}{(ax-b)^2}\right) \\
&= -\frac{2a^2b^2}{(ax-b)^2}
\end{aligned}$$

Recall that the square of any non-zero number is positive. Then, we have that $a^2 > 0$, $b^2 > 0$, and $(ax-b)^2 > 0$. The last one also implies $\frac{1}{(ax-b)^2} > 0$. Multiplying,

$$\begin{aligned} \frac{a^2b^2}{(ax-b)} &> 0 \\ -2\frac{a^2b^2}{(ax-b)} &< 0 \\ abf'(x) &< 0 \end{aligned}$$

□

Q03. In each case, find $f'(x)$ using any method.

(a) $f(x) = 5^x \sin x + (x^3 + x^2)\cos x$.

Solution. Apply arithmetic rules and recall that $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(5^x \sin x) + \frac{\mathrm{d}}{\mathrm{d}x}\big((x^3 + x^2)\cos x\big) \\ &= (5^x \frac{\mathrm{d}}{\mathrm{d}x} \cos x + \cos x \frac{\mathrm{d}}{\mathrm{d}x} 5^x) + (\cos x \frac{\mathrm{d}}{\mathrm{d}x}(x^3 + x^2) + (x^3 + x^2)\frac{\mathrm{d}}{\mathrm{d}x}\cos x) \\ &= 5^x \sin x + \ln 5 \cos x 5^x + \cos x(3x^2 + 2x) + (x^3 + x^2)\sin x \\ &= \sin x(x^3 + x^2 + 5^x) + \cos x(5^x \ln 5 + 3x^2 + 2x) \end{aligned}$$

□

(b) $f(x) = \dfrac{x^2 + x - 2}{x^3 + 6}$.

Solution. Apply the quotient rule, excepting $x = \sqrt[3]{-6}$ from the domain:

$$\begin{aligned} f'(x) &= \frac{(x^3 + 6)\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + x - 2) - (x^2 + x - 2)\frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 6)}{(x^3 + 6)^2} \\ &= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)3x^2}{(x^3 + 6)^2} \\ &= -\frac{x^4 + 2x^3 - 6x^2 - 12x - 6}{(x^3 + 6)^2} \end{aligned}$$

□

(c) $f(x) = \sqrt{2\tan^2 x + 3}$.

Solution. Apply the chain rule, recalling that $\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x$.

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}\sqrt{2\tan^2 x + 3}}{\mathrm{d}(2\tan^2 x + 3)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\big(2\tan^2 x + 3\big) \\ &= \frac{1}{2\sqrt{2\tan^2 x + 3}} \cdot 2\frac{\mathrm{d}\tan^2 x}{\mathrm{d}\tan x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\tan x \\ &= \frac{1}{2\sqrt{2\tan^2 x + 3}} \cdot \left(2\frac{\mathrm{d}\tan^2 x}{\mathrm{d}\tan x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\tan x\right) \\ &= \frac{1}{2\sqrt{2\tan^2 x + 3}} \cdot 4\tan x \sec^2 x \\ &= \frac{2\tan x \sec^2 x}{\sqrt{2\tan^2 x + 3}} \end{aligned}$$

□

(d) $f(x) = 2^{\sin(\sec x)}$.

Solution. Again, simply apply the chain rule repeatedly.

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}(2^{\sin(\sec x)})}{\mathrm{d}(\sin(\sec x))} \cdot \frac{\mathrm{d}\sin(\sec x)}{\mathrm{d}(\sec x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sec x \\ &= 2^{\sin(\sec x)} \ln(2)\cos(\sec x)\sec(x)\tan(x) \end{aligned}$$ □

Q04. In each case, determine the equation of the tangent to $y = f(x)$ at the point where $x = a$.

(a) $f(x) = x^2$, $a = 3$.

Solution. □

(b) $f(x) = \cos x$, $a = -\dfrac{3\pi}{4}$.

Solution. □

(c) $f(x) = e^x$, $a = \ln \pi$.

Solution. □

(d) $f(x) = 4^x$, $a = -3$.

Solution. □

Q05. Compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in each case.

(a) $y = \cos x^2$.

Solution. □

(b) $y = \cos^2 x$.

Solution. □

Q06.

(a) Use the Chain Rule to prove that the derivative of an even function is odd.

Proof. □

(b) Using ONLY the Chain Rule and the Product Rule (and not the Reciprocal/Quotient rules), give an alternative proof of the Quotient Rule. [Hint: $\dfrac{f(x)}{g(x)} = f(x)(g(x))^{-1}$].

Proof. □

Q07. If $y = f(u)$ and $u = g(x)$ where f and g are twice differentiable functions, prove that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}.$$

Proof. $\square$