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Harmonic Motion Analysis

Abstract: In this lab we experimented with harmonic motion analysis. We studied the influence of damping on both driven and un-driven harmonic oscillators using a Pasco Driven Harmonic Motion Analyzer. With this apparatus we were able to determine the resonant period T_d, resonant frequency ω_d, un-driven damping constant $\gamma_{nodrive}$, and driven damping constant γ_{drive}. After analyzing the data we found a definitive relationship between the distance between the magnets (which also affected their proximity to the damping rod) and the value of the damping constant. The closer the magnets were to each other, the greater the damping constant. Although the values for the damping constant were different in the driven and un-driven cases, they were within uncertainty of each other.

Introduction:

In this lab we are studying harmonic motion analysis by examining the influence of damping and studying resonant conditions on a harmonic oscillator.

Objective: First we will take advantage of resonant conditions to acquire data for the resonant period (T) and resonant frequency (f) using an apparatus. Next, we will calculate the damping constant (**γ**) in both driven and un-driven oscillators and the affect of magnetic damping on the oscillators. Finally, we will compare the results for (**γ**) in both driven and un-driven oscillators to each other and draw a conclusion about the affect of magnetic damping on the oscillators.

In order to analyze the harmonic oscillator we will need to use several equations. The derivations below are based on information from *Analytical Mechanics* (Fowles & Cassiday 4th Edition). The derivations are as follows:

F= force, m = mass, a = acceleration, t = time, A = amplitude,

F = ma (equation 1)

where a = $\frac{dv}{dt} = \frac{d^2x}{dt^2}$

Thus:

F = $m\frac{d^2x}{dt^2}$ (equation 2)

The potential energy function for the spring and mass at the equilibrium position have similar behavior with the restoring force given by Hooke's Law.

F = $m\frac{d^2x}{dt^2}$ = -kx

Thus:

$\frac{dv}{dt} = \frac{d^2x}{dt^2}$ + kx = 0 (Equation 3)

where k = spring constant, x = position as a function of time.

The solution equation is:

$x = A_1 \cos(\omega_0 t + \theta_0)$ (Equation 4) where A_1 = amplitude of oscillation, ω_0 = natural frequency, θ_0 = a phase factor obtained from initial conditions.

and

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \text{(Equation 5)}$$

When there is linear damping there is another force in the differential equation giving:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \qquad \text{(Equation 6)}$$

where c = damping constant

When under-damped (which is what we are looking at),

$$x = A_1 e^{-\gamma t}\cos(\omega_d t + \theta_0) \qquad \text{(Equation 7)}$$

where γ = damping factor = $\frac{c}{2m}$, ω_d = drive frequency $= \sqrt{\omega_0{}^2 - \gamma^2}$

Measuring the amplitude of a decaying oscillation on two oscillations separated by n cycles (from Equation 7) yields:

$$\frac{A_0}{A_n} = e^{\gamma n T_d} \qquad \text{(Equation 8)}$$

where T_d = drive frequency = $\frac{2\pi}{\omega_d}$, A_0 = initial amplitude, A_n = amplitude after n cycles.

If we look at a periodic driving force applied to the oscillator this results in the equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = kA_{drive}\cos(\omega t) \qquad \text{(Equation 9)}$$

A_{drive} = drive amplitude

There is a decaying portion and a steady-state portion. The decaying portion goes away leaving us with the steady-state portion as:

$$x = A\cos(\omega t + \phi) \qquad \text{(Equation 10)}$$

With:

$$A = \frac{kA_{drive}}{\sqrt{(k - m\omega^2)^2 + c^2\omega^s}} \qquad \text{(Equation 11)}$$

$$\tan(\phi) = \frac{c\omega}{k - m\omega^2} \qquad \text{(Equation 12)}$$

When the drive frequency is adjusted to the point where the amplitude is maximum, resonance is achieved. Resonant frequency is:

$$\omega_r = \sqrt{\omega_0^{\ 2} - 2\gamma^2} \qquad \text{(Equation 13)}$$

When $\omega = \omega_0, \phi = 90^0$ as can be seen here:

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)}, \text{ as } \cos(\phi) \to 0, \phi = 90^0$$

$$\tan(\phi) = \frac{\frac{c\omega}{m}}{\frac{k}{m} - \omega^2} = \frac{2\gamma\omega}{\omega_0^{\ 2} - \omega^2} = \frac{2\gamma\omega_0}{\omega_0^{\ 2} - \omega_0^{\ 2}} = \frac{2\gamma\omega_0}{0} = \infty$$

$$\tan(\phi) = \infty \Rightarrow \phi = \frac{\pi}{2} = 90^0$$

And in a weakly damped oscillator:

$$A_{res} \approx \frac{\omega A_{drive}}{c/m} = \frac{\omega A_{drive}}{2\gamma} \qquad \text{(Equation 14)}$$

A_{res} = resonant amplitude

For this lab we will need the following equations (Equations 16 and 17 have been rearranged from equations above for our intended use in this lab):

$$\omega_{drive} = 2\pi f_{res} \qquad \text{(Equation 15)}$$

$$\gamma_{nodrive} = \frac{\ln\left(\frac{A_0}{A_n}\right)}{nT_d} \qquad \text{(Equation 16)}$$

$$\gamma_{drive} = \frac{\omega_d A_d}{2(A_{res})} \qquad \text{(Equation 17)}$$

Note:

ω = frequency

ω_0 = natural frequency

ω_r = resonant frequency

ω_d = drive frequency

γ = γ = damping constant

γ_{drive} = driven damping constant

A_n = amplitude after n cycles

A_0 = initial amplitude

A_{res} = resonant amplitude

A_d = drive amplitude

$\gamma_{nodrive}$ = un-driven damping constant

T_d = driven period

Procedure:

The equipment used for this lab is a Pasco Driven Harmonic Motion Analyzer.

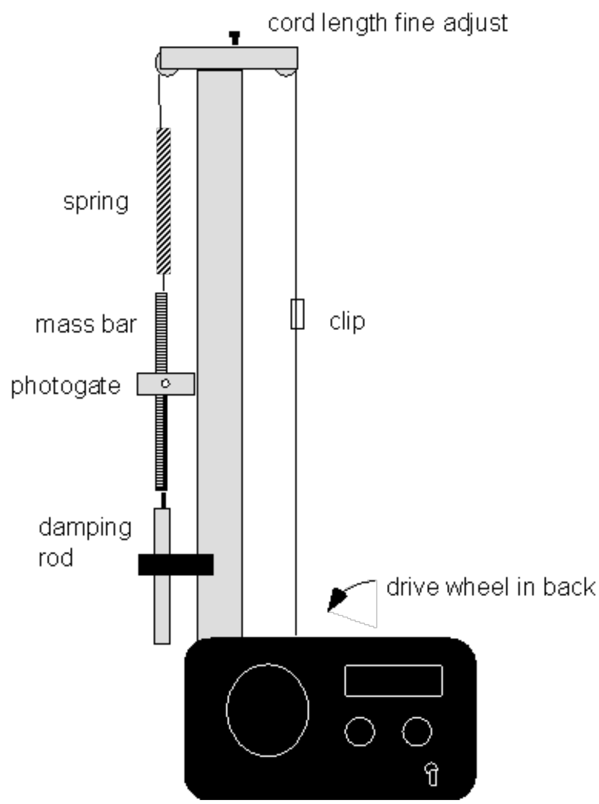
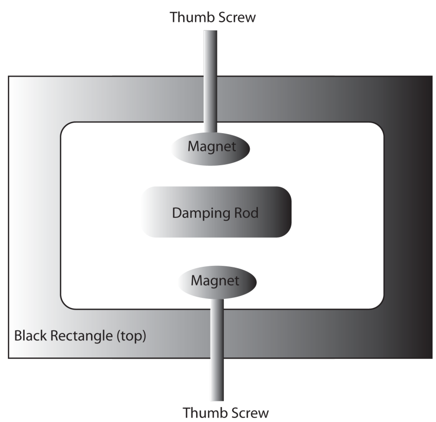


*Diagram from www.oberlin.edu

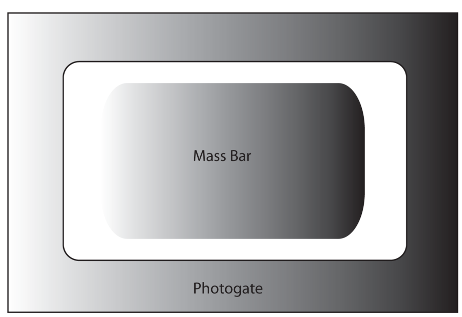
The controls on the front of the apparatus consist of: a switch to turn the drive motor on/off; a digital readout display; a dial to change the drive frequency; a phase indicator; and a function dial to switch the digital readout display to period, amplitude, or frequency.

We started with the apparatus setup as shown in the apparatus diagram. The black metal rectangle (shown with the damping rod through it) has a hole through it that the damping rod glides through freely. There are two magnets inside the black metal rectangle as shown here (top down view).



We first rotated the thumb screws outward to get the magnets as far away as possible from the metal damping rod.

We then had to align the mass bar as shown below so that it went through the photogate without touching any of the sides to avoid friction (top down view).



Resonant Frequency Measurement:

We next determined the resonant frequency in three different ways.

The first way we determined the resonant frequency was to leave the drive motor off. We then set the function dial to period. We pulled the damping rod down approximately two centimeters and released it. We then recorded the value of the period shown on the display.

The second way that we determined the resonant frequency was we turned on the drive motor, set the function dial to amplitude, and varied the drive frequency dial until we observed the maximum amplitude. While the apparatus was at maximum amplitude we switched the function dial back to frequency and recorded the resonant frequency.

The third way that we determined the resonant frequency was to adjust the drive frequency dial until the phase indicator showed a lag of 90 degrees. Making sure the function dial was set to frequency we recorded the resonant frequency at a 90 degree lag.

<u>Damping Factor as a Function of Magnet-Rod Distance Procedure:</u>

Next we measured the damping factor γ as a function of magnet-rod separation. We double-checked the alignment of the mass bar and that the magnets were as far from the damping rod as possible.

We pulled the damping rod down as far as we practically could and released. Keeping the peak-to-peak amplitude less than 140mm we recorded the first reliable amplitude measurement (always making sure there was tension in the spring). Then we recorded the amplitude exactly three full cycles later.

We repeated this procedure two more times, but altered the number of cycles for the second step. One time we waited five full cycles for the second amplitude measurement. One time we waited ten full cycles for the second amplitude measurement.

<u>Damping Factor as a Function of Magnet-Rod Distance (Alternate Distances):</u>

Next we measured and recorded the distance between the two magnets using calipers.

Then we moved the magnets closer together by turning each thumb screw a full turn in. We measured and recorded this distance with calipers. With the magnets in their new position we repeated the *Damping Factor as a Function of Magnet-Rod Distance Procedure*.

We repeated this for a total of five different magnet separations moving them in a full turn for each new measurement.

Distance Dependence of Damping Constant in Driven Oscillations:

Lastly we examined the damping constant in driven oscillations.

Using the thumb screws we took the magnets back out to the position where they were the furthest distance apart. We turned the drive motor on. The apparatus was still at resonance from the previous setting. Once the oscillations reached steady-state, we switched the function dial to the amplitude setting and recorded the peak-to-peak amplitude.

Then using the same magnet distances as we did in the previous procedure, we repeated this procedure at the five different magnet distances and recorded the peak-to-peak amplitude at each distance.

Results:

Initial Resonant Frequency Measurement:

Our initial measurements for resonant frequency obtained from experimentation can be found in table 1 columns two, three, and four as follows:

Resonant Frequency **Table 1**

Method of Measurement	T (s)	A (mm)	Resonant f (Hz)	ω_d (rad/s)	T_d (s)
1 - No drive, Pull rod 2cm & release	0.450		2.22		
2 - Drive on, find max A, record f	0.459	98.0	2.18	13.70	0.459
3 -Drive on, let lag 90 degrees, record f	0.459		2.18		

For further calculations we needed to calculate ω_d (note: $T = T_d$). The calculation for ω_d using equation (15) is:

$$\omega_d = 2\pi f_{res} = 2\pi(2.18) \approx 13.70 rad / s$$

Damping Factor (γ) as a Function of Magnet-Rod Separation:

First we looked at the damping factor (γ) as a function of magnet-rod separation in un-driven oscillations. The measured and calculated values for **γ** as a function of magnet-rod separation can be found in table 2:

γ & Magnet Dist No Drive Table 2

Trial	Mag Dist (mm)	A (mm) initial	A (mm) 3 cycles	A (mm) 5 cycles	A (mm) 10 cycles	Dist uncert (mm)	γ (1/s)	γ_{ave} (1/s)	γ_{uncert} (1/s)
1 - Magnets all the way out	21.300	79.0	61.0			0.005	0.188		
	21.300	81.0		57.0		0.005	0.153	0.169	0.019
	21.300	79.0			37.0	0.005	0.165		
2 - Magnets one turn in	18.840	81.0	57.0			0.005	0.255		
	18.840	80.0		51.0		0.005	0.196	0.216	0.039
	18.840	89.0			36.0	0.005	0.197		
3- Magnets two turns in	16.320	84.0	51.0			0.005	0.362		
	16.320	79.0		45.0		0.005	0.245	0.287	0.076
	16.320	86.0			27.0	0.005	0.252		
4 - Magnets three turns in	13.800	79.0	46.0			0.005	0.393		
	13.800	88.0		37.0		0.005	0.378	0.388	0.010
	13.800	85.0			14.0	0.005	0.393		
5 - Magnets four turns in	11.310	87.0	37.0			0.005	0.621		
	11.310	83.0		22.0		0.005	0.579	0.590	0.031
	11.310	82.0			6.0	0.005	0.570		
6 - Magnets five turns in	8.850	86.0	27.0			0.005	0.841		
	8.850	88.0		12.0		0.005	0.868	0.889	0.068
	8.850	81.0			1.0	0.005	0.957		

Since the uncertainty between the three values of $\boldsymbol{\gamma}_{ave}$ was much more significant than any uncertainty introduced by the amplitude we were able to determine $\boldsymbol{\gamma}_{best}$ by simply taking $\boldsymbol{\gamma}_{ave}$ and using the values for $\boldsymbol{\gamma}_{ave}$ to determine uncertainty in **γ.** An example calculation for trial 1 using equation (16) is as follows:

$$\gamma_{nodrive(1a)} = \frac{\ln\left(\frac{A_0}{A_n}\right)}{nT_d} = \frac{\ln\left(\frac{79}{61}\right)}{3(.459)} \approx 0.188(1/s)$$

$$\gamma_{nodrive(1b)} = \frac{\ln\left(\frac{A_0}{A_n}\right)}{nT_d} = \frac{\ln\left(\frac{81}{57}\right)}{5(.459)} \approx 0.153(1/s)$$

$$\gamma_{nodrive(1c)} = \frac{\ln\left(\frac{A_0}{A_n}\right)}{nT_d} = \frac{\ln\left(\frac{79}{7}\right)}{10(.459)} \approx 0.165(1/s)$$

$$\gamma_{nodrive(ave)} = \frac{(0.188+0.153+0.165)}{3} \approx 0.169(1/s)$$

To find **γ**, we must take $\boldsymbol{\gamma}_{ave}$ and compare it against the three $\gamma_{nodrive(1)}$ values. We must find the greatest discrepancy in value between $\boldsymbol{\gamma}_{ave}$ and $\gamma_{nodrive(1)}$ to determine the uncertainty. In this case:

$$uncert = 0.188 - 0.169 = 0.019(1/s)$$

Thus:

γ = $0.169 \pm 0.019\,(1/\mathrm{s})$

All further calculations can be found in table 2.

Damping Factor (γ) as a Function of Magnet-Rod Separation in Driven Oscillations:

Next we looked at the damping factor (γ) as a function of magnet-rod separation in driven oscillations. The measured and calculated values for **γ** as a function of magnet-rod separation in driven oscillations can be found in table 3:

γ & Magnet Dist Driven Table 3

Trial	Mag Dist (mm)	A (mm) pk-to-pk	A (mm) res	A_{dr} (mm)	A_{dr} (mm) uncert	A uncert (mm)	γ_{best} (1/s)	γ_+ (1/s)	γ_- (1/s)	γ_{uncert} (1/s)
1 - Magnets all the way out	21.300	96.0	48.0	1.1	0.1	0.500	0.157	0.173	0.141	0.016
2 - Magnets one turn in	18.840	84.0	42.0	1.1	0.1	0.500	0.179	0.198	0.161	0.019
3 - Magnets two turns in	16.320	63.0	31.5	1.1	0.1	0.500	0.239	0.265	0.214	0.026
4 - Magnets three turns in	13.800	44.0	22.0	1.1	0.1	0.500	0.343	0.382	0.304	0.040
5 - Magnets four turns in	11.310	29.0	14.5	1.1	0.1	0.500	0.520	0.587	0.457	0.067
6 - Magnets five turns in	8.850	18.0	9.0	1.1	0.1	0.500	0.837	0.967	0.721	0.130

Since in this case the values for **γ** were significantly affected by any uncertainty in amplitude, we needed to take that uncertainty into consideration when solving for **γ** using equation (17).

An example calculation for trial 1 is as follows:

$$\gamma_{best} = \frac{\omega_d A_d}{2(A_{res})} = \frac{(13.69)(1.1)}{2(48)} \approx 0.157(1/s)$$

$$\gamma_+ = \frac{\omega_d A_d}{2(A_{res})} = \frac{(13.69)(1.1+0.1)}{2(48-0.5)} \approx 0.173(1/s)$$

$$\gamma - = \frac{\omega_d A_d}{2(A_{res})} = \frac{(13.69)(1.1-0.1)}{2(48+0.5)} \approx 0.141(1/s)$$

Thus:

γ = 0.157 ± 0.016 (1/s)

All further calculations can be found in table 3.

Conclusion:

We were able to determine the resonant period T_d, resonant frequency ω_d, un-driven damping constant $\gamma_{nodrive}$, and driven damping constant γ_{drive}. In both driven and un-driven oscillations we were able to determine that there was a definite relationship between the damping constant and the distance of the magnets from the damping rod. The closer the magnets were to each other (thus both closer to the rod) the higher the damping constant. This makes sense qualitatively. Since the rod is made of steel, the closer the magnets are to it, the more they will dampen its movement. When comparing the two values of the damping constant there was a small difference, however for each magnet distance the values for both the driven and un-driven damping constant were within uncertainty of each other. The difference is likely due to sources of error when experimenting with the un-driven damping constant. In that case, there was some unavoidable friction between the mass bar and the photogate and the results were dependent upon the limitations of human eyes and hands. In experimentation with the driven oscillator, the machine provided the data