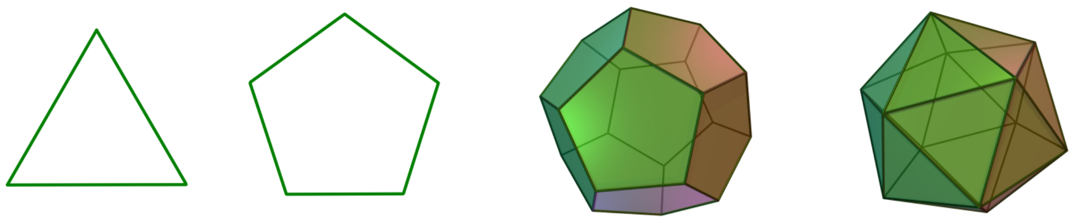
1 Polyhedral Geometry

1.a Real Vector Spaces

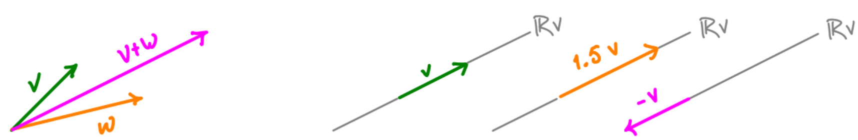
You've all seen polygons and polytopes before, maybe one of these is your favorite:



(Images from Wikipedia)

These are examples of polyhedra. To describe polyhedra carefully, we first have to say where they live – namely, we have to described "real vector spaces". We'll give a rough idea first, and then a more careful definition.

Idea: A real vector space is a space where we can add points together and rescale points by any real number.



Careful version: A *real vector space* is a set V with an addition operation

$$\begin{aligned} + : V \times V &\to V \\ (v, w) &\mapsto v + w \end{aligned}$$

and a real scalar multiplication

$$\begin{aligned} \cdot : \mathbb{R} \times V &\to V \\ (r, v) &\mapsto r \cdot v \end{aligned}$$

satisfying the following conditions. For all $u, v, w \in V$ and $r, s \in \mathbb{R}$:

- $u + (v + w) = (u + v) + w$
  (Associativity of Vector Addition)
- $v + w = v + w$
  (Commutativity of Vector Addition)

- There is an element $\mathbf{0} \in V$ with $\mathbf{0} + v = v$.
  (Identity Element of Vector Addition)

- There is an element $-v \in V$ with $v + (-v) = \mathbf{0}$.
  (Inverse Elements of Vector Addition)

- $r \cdot (s \cdot v) = (rs) \cdot v$
  (Compatibility of Scalar Multiplication with Multiplication in $\mathbb{R}$)

- For the element $1 \in \mathbb{R}$, we have $1 \cdot v = v$.
  (Identity Element of Scalar Multiplication)

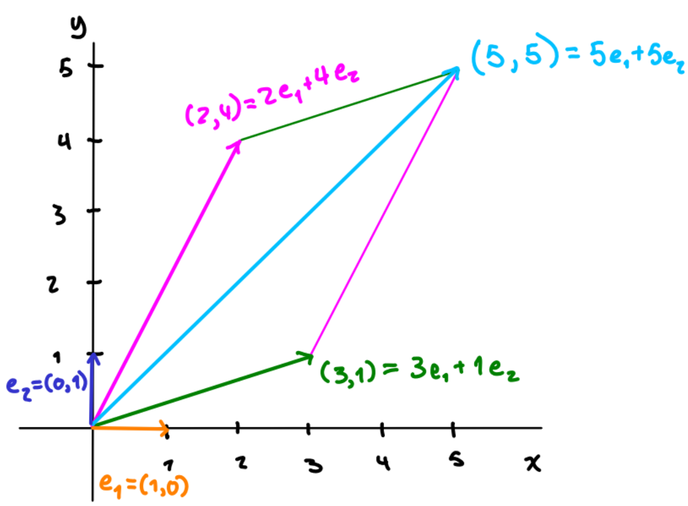
- $r \cdot (v + w) = r \cdot v + r \cdot w$
  (Distributivity of Scalar Multiplication with Respect to Vector Addition)

- $(r + s) \cdot v = r \cdot v + s \cdot v$
  (Distributivity of Scalar Multiplication with Respect to Addition in $\mathbb{R}$)

Notation: Usually, we will just write rv for $r \cdot v$.

Question: Does $\mathbb{R}$ itself have the structure of a vector space?

Question: What about $\mathbb{R}^2$ with head-to-tail/ componentwise addition? What would change if we replaced $\mathbb{R}^2$ with $\mathbb{R}^n$?



When we talk about "componentwise" addition, we are using a *basis* for our vector space.

Idea: A *basis* of V is a subset $B \subset V$ such that every element $v \in V$ can be written as a finite $\mathbb{R}$-linear combination of elements of B in exactly one way.

Careful version: A *basis* B of a real vector space V is a subset of V satisfying:

- for every finite subset $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ of B, we can only have an equality $c_1\mathbf{e}_1 + \cdots + c_m\mathbf{e}_m = \mathbf{0}$ if $c_1 = \cdots = c_m = 0$.
  (Linear Independence)
- for every $v \in V$, there is a finite subset $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ of B and associated scalars $c_1, \ldots, c_m \in \mathbb{R}$ such that $v = c_1\mathbf{e}_1 + \cdots + c_m\mathbf{e}_m$.
  (Spanning Property)

Next, given a pair of real vector spaces V and W, a map

$$f : V \to W$$

is *linear* if for every $u, v \in V$ and $r \in \mathbb{R}$:

- $f(u + v) = f(u) + f(v)$
- $f(r \cdot v) = r \cdot f(v)$

(Note that the operations on the left hand side occur in V, while those on the right hand side occur in W.)

On Homework 1, you will investigate linear maps from $\mathbb{R}^2$ to $\mathbb{R}$ and show that this collection of maps is itself a real vector space. Once again, we could replace $\mathbb{R}^2$ with $\mathbb{R}^n$ without meaningfully changing any arguments.

The space of linear maps from a real vector space V to $\mathbb{R}$ is called the *dual vector space*. It's denoted V^*.

Fact: In finite dimensions, there is a natural identification between $(V^*)^*$ and V.

This lets us treat V and V^* on equal footing – we can equally well start with V^* and think of V as the dual vector space.

We have a canonical pairing, the "evaluation pairing", between these two vector spaces:

$$\begin{aligned} \langle \cdot, \cdot \rangle : V \times V^* &\to \mathbb{R} \\ (v, \omega) &\mapsto \omega(v) = v(\omega). \end{aligned}$$

Here we view v as an element of V.

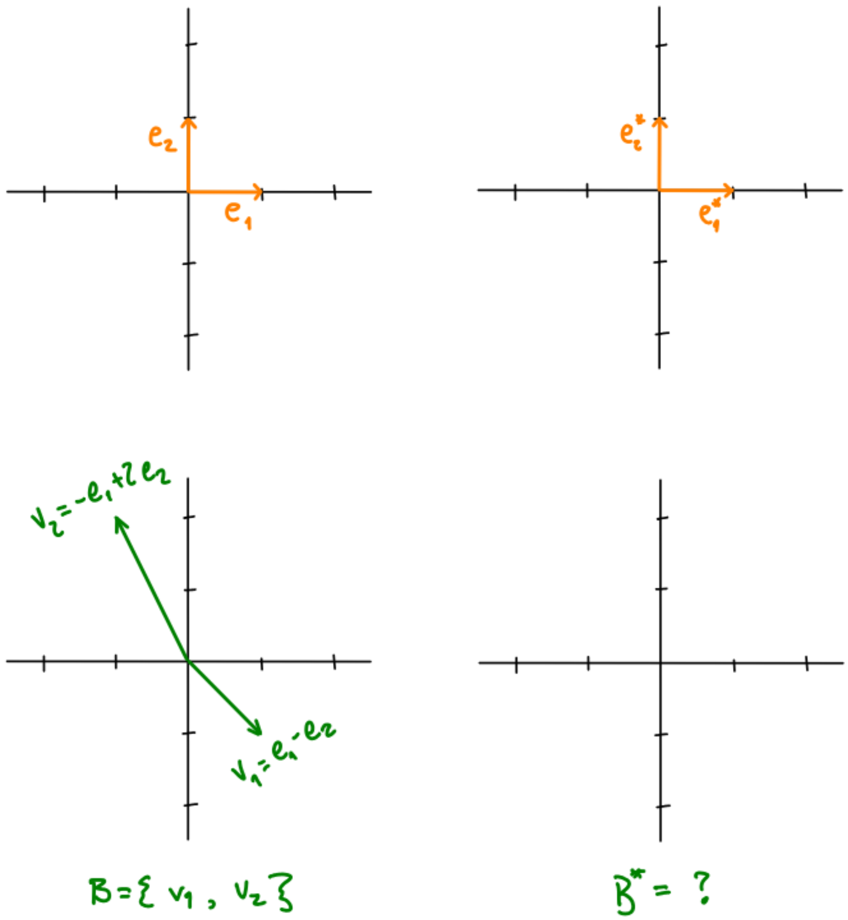
Here we view v as an element of (V*)*.

If we have already fixed a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots\}$ of V, we can use this pairing to give V^* a basis $B^* = \{\mathbf{v}_1^*, \mathbf{v}_2^*, \dots\}$ as well. We define $\mathbf{v}_i^*$ to be the unique element of V^* satisfying:

- $\langle \mathbf{v}_i, \mathbf{v}_i^* \rangle = 1$
- $\langle \mathbf{v}_j, \mathbf{v}_i^* \rangle = 0$ if $i \neq j$.

We generally abbreviate this as $\langle \mathbf{v}_j, \mathbf{v}_i^* \rangle = \delta_{ij}$. The function "$\delta_{ij}$" is called the *Kronecker delta*. It is 1 if $i = j$ and 0 if $i \neq j$.

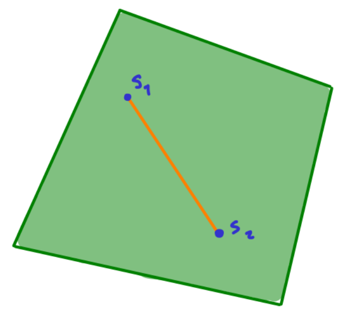
Example:



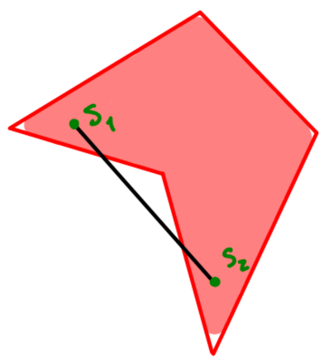
1.b Convex Polyhedra

Definition: Let S be a subset of a real vector space V. We say S is *convex* if for every pair of points s_1, s_2 in S, the line segment $\overline{s_1 s_2}$ is contained in S.

Example:

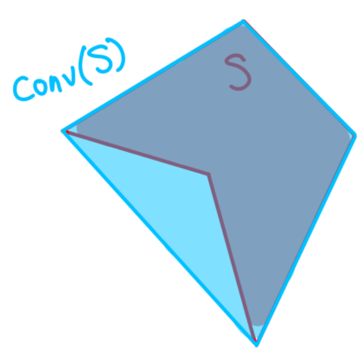


Non-example:



Definition: Let S be a subset of a real vector space V. The *convex hull* of S, denoted $\operatorname{conv}(S)$, is the smallest convex set containing S. In other words, $\operatorname{conv}(S)$ is the intersection of all convex sets containing S.

Example:



Question: Is a union of two convex sets necessarily convex?

Question: Is an intersection of two convex sets necessarily convex?

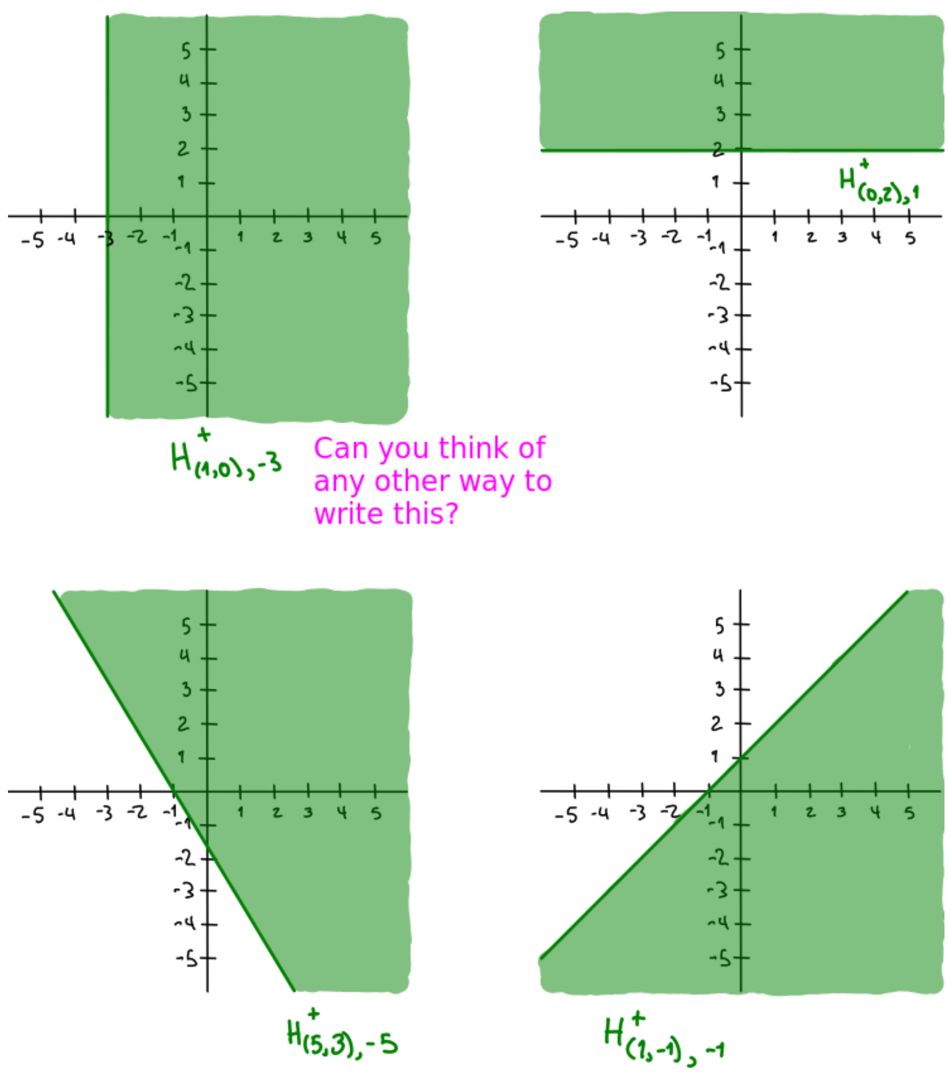
Definition: Let $w \in V^*$, with $w \neq \mathbf{0}$, and let $r \in \mathbb{R}$. The *half-space* $H^+_{w,r}$ is the subset of V defined by

$$H^+_{w,r} := \{v \in V \mid \langle v, w \rangle \geq r\}.$$

Its boundary is the *hyperplane*

$$H_{w,r} := \{v \in V \mid \langle v, w \rangle = r\}.$$

Examples:



Question: Why have I insisted that $w \neq \mathbf{0}$? What would $H^+_{\mathbf{0},1}$ and $H^+_{\mathbf{0},-1}$ look like if we dropped this non-zero condition?

Question: Is a half-space convex?

Question: Is a hyperplane convex?

This next question isn't essential for the discussion that follows, but it's a good check that you understand the material we've covered so far.

Question: Is $H_{w,r}^{+}$ or $H_{w,r}$ a vector subspace of V? Can you add some condition that will ensure one of them becomes a vector subspace?

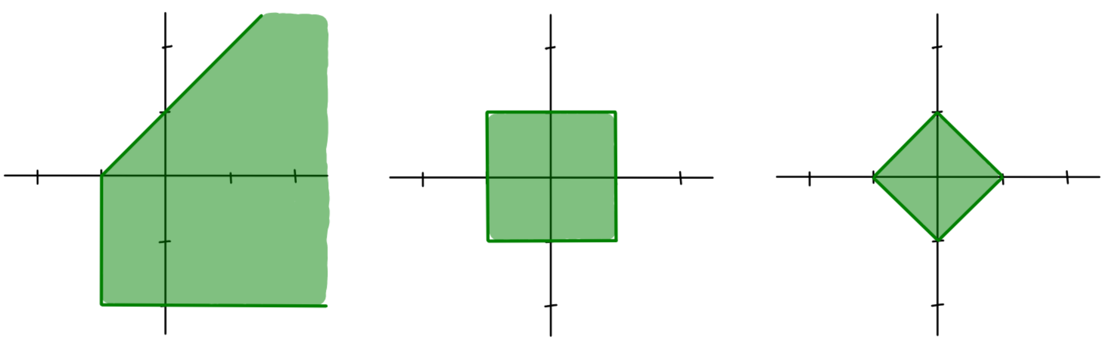
Now we are ready to introduce polyhedra and polytopes.

Definition: A *polyhedron* P in V is a subset of the form

$$P = \bigcap_{i \in I} H_{w_i, r_i}^{+},$$

where I is some finite indexing set, each w_i is a non-zero element of V^* and each r_i is in $\mathbb{R}$.

Question: Can you describe the following polyhedra as intersections of half-spaces?



Definition: A subset P of V is called a *polytope* if it is a bounded polyhedron. Equivalently, a subset P of V is called a *polytope* if

$$P = \operatorname{conv}(v_j \mid j \in J),$$

where J is some finite indexing set and each v_j is an element of V.

You should try to think about these two equivalent definitions and convince yourself that they really are describing the same subsets of V. Once you've internalized the definitions, this equivalence should feel "obviously true". I'd argue that it genuinely *is* obviously true, but this is one of those obvious statements that take a surprising amount of work to prove. The usual proof makes use of a concept called *duality* or *polarity* which we haven't introduced yet.

So, we can describe a polytope $P \subset V$ as either

$$P = \bigcap_{i \in I} H_{w_i, r_i}$$

or as

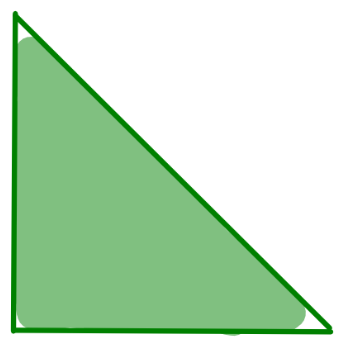
$$P = \operatorname{conv}(v_j \mid j \in J).$$

The former is called the *half-space representation* of P and the latter is called the *vertex representation* of P. On the next homework, you'll describe polytopes in terms of both.

1.c Faces of Polyhedra

Definition: Let P be a polyhedron in V. We say that a subset F of P is a *face* if there is a half-space $H^+_{w,r}$ in V with $P \subset H^+_{w,r}$ such that $F = P \cap H_{w,r}$. That is, F is a face of P if it is the intersection of P with the hyperplane at the boundary of some half-space containing P.

Example: Describe each face of the following polygon in $\mathbb{R}^2$.



There are a couple of tricky points to consider in the above example. First, I specifically stated that the polygon was in $\mathbb{R}^2$. Why? Would your answer change if we instead took it to lie in $\mathbb{R}^3$, for instance if it were lying in the xy-plane ($z = 0$) of $\mathbb{R}^3$?

Next, is the empty set $\varnothing$ a face of the polygon? Why?

Definition: We say a face F of P is a *proper face* if $F \notin \{\varnothing, P\}$.

Definition: We call the 0-dimensional faces of P *vertices*, the 1-dimensional faces of P *edges*, and the codimension 1 faces of P *facets*.

Question: If F is a face of a polyhedron P, is F itself a polyhedron?

Observation: Let $\mathcal{F}(P)$ be the set of faces of a polyhedron P. Then "$\subseteq$" is a partial order on $\mathcal{F}(P)$ and $(\mathcal{F}(P), \subseteq)$ is a poset. In fact, this poset has additional structure as well, which we will discuss soon. First, a few questions about the poset structure:

Question: Can you represent diagrammatically the poset $(\mathcal{F}(T), \subseteq)$ for a tetrahedron T?

Question: Given an arbitrary polyhedron P, does $(\mathcal{F}(P), \subseteq)$ have a least element?

Question: Does $(\mathcal{F}(P), \subseteq)$ have a greatest element in general? When does it have a greatest element?

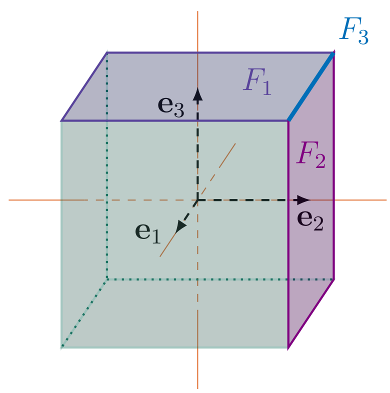
Question: If P is full dimensional, what are the maximal elements of $(\mathcal{F}(P), \subseteq)$?

Now on to the additional structure $\mathcal{F}(P)$ enjoys – it satisfies:

- If $F_1 \in \mathcal{F}(P)$ and $F_2 \in \mathcal{F}(F_1)$, then $F_2 \in \mathcal{F}(P)$. *(A face of a face is a face.)*
- If $F_1, F_2 \in \mathcal{F}(P)$, then $(F_1 \cap F_2) \in (\mathcal{F}(F_1) \cap \mathcal{F}(F_2))$. *(The intersection of two faces is a face of both.)*

That is to say, $\mathcal{F}(P)$ forms a *polyhedral complex*. We call it the *face complex* of P.

Example: Let P be the cube shown below, and F_1, F_2, and F_3 the indicated polytopes at the boundary of P. Note that F_1 and F_2 are *facets* of P. Meanwhile, $F_3 = (F_1 \cap F_2)$ is a facet of both F_1 and F_2, and a face of P.

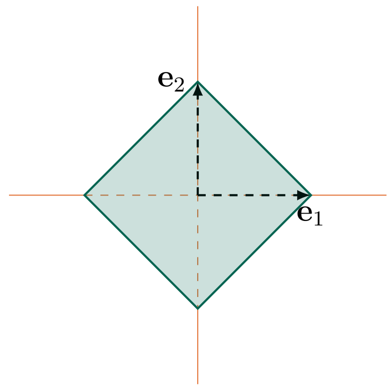
$P =$ 

1.d Duality for Polytopes and Faces

Definition: Given a polytope P in a real vector space V, the *dual* of P is

$$P^\circ := \{ w \in V^* \mid \langle v, w \rangle \geq -1 \text{ for all } v \in P \} .$$

Question: Let P be the square shown below. Can you draw P°?



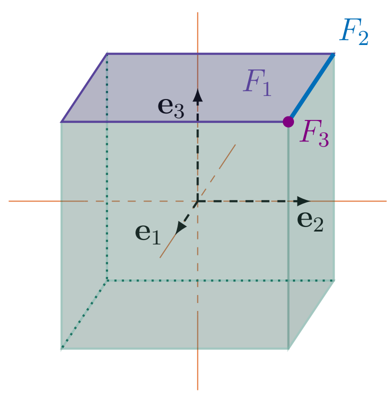
We also have a notion of duality for faces. Note that faces are themselves polytopes, and the definitions differ, so we should always try to be clear about which version of *dual* we are considering.

Definition: The *dual of a face* F of a polytope P in V is

$$\check{F} := \{w \in P^\circ \mid \langle v, w \rangle = -1\}.$$

Fact: If F is a face of P, then $\check{F}$ is a face of P°.

Question: Let P be the cube shown below, and F_1, F_2, and F_3 its indicated faces. What are P°, $\check{F}_1$, $\check{F}_2$, and $\check{F}_3$?

$P =$ 

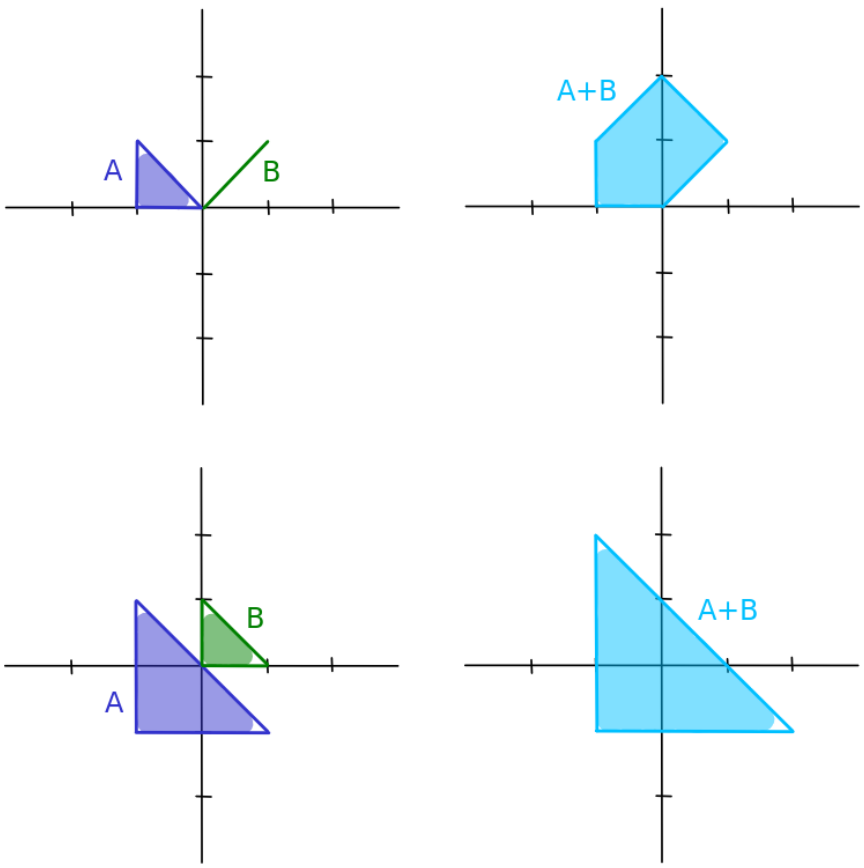
This question will be the start of your polyhedral geometry project, which will be an exploration of dual polytopes and their face complexes.

1.e Minkowski Sum

Definition: Let A and B be a pair of subsets of a vector space V. The *Minkowski sum* of A and B is

$$A + B := \{a + b \mid a \in A \text{ and } b \in B\}.$$

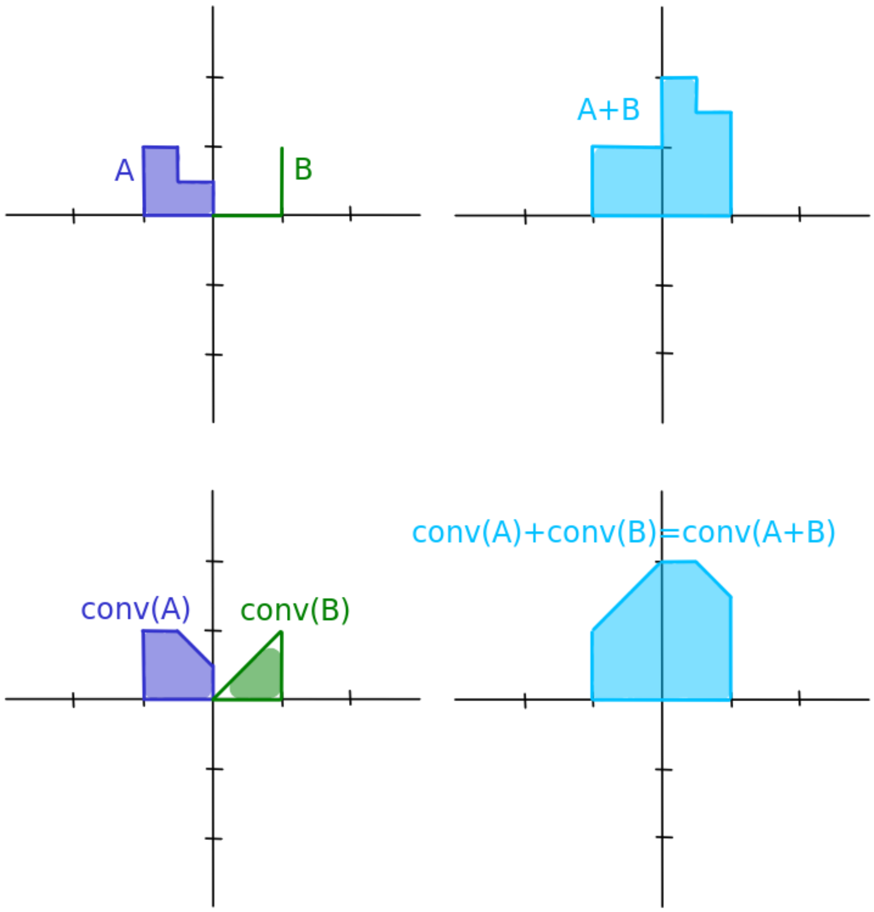
Examples:



Theorem:

$$\operatorname{conv}(A + B) = \operatorname{conv}(A) + \operatorname{conv}(B)$$

Example:



1.f Convex Functions

Definition: A function $\varphi : V \to \mathbb{R}$ is *convex* if for all $u, v \in V$ and $t \in [0,1]$, we have

$$\varphi\left((1-t)u + t\,v\right) \geq (1-t)\varphi(u) + t\,\varphi(v).$$

Warning: I'm making a non-standard choice of convention here – it's more common to have the inequality in the opposite direction. The inequality convention I've chosen is more convenient in my research area, and for a topic we'll discuss soon.

Question: What sort of function would we have if we replaced the inequality with an equality?

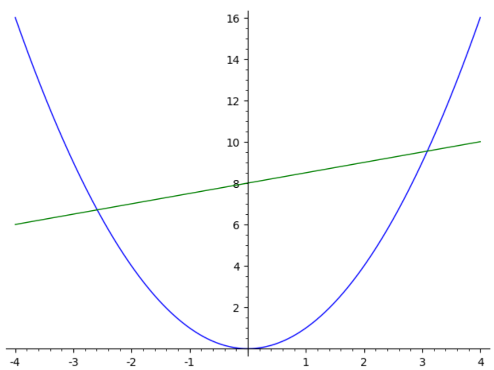
Here's an equivalent characterization of convex functions – since they are equivalent, you can use whichever version you prefer.

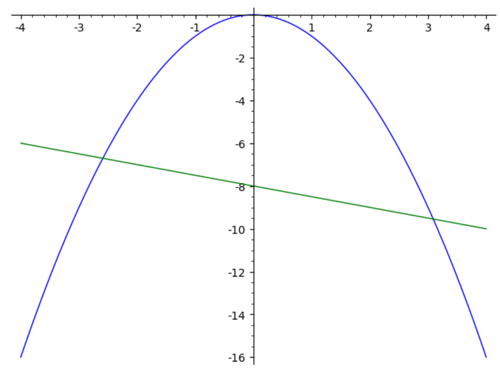
Theorem: *(Jensen's Inequality)* A function $\varphi : V \to \mathbb{R}$ is convex if and only if for all $v_1, v_2, \ldots, v_n \in V$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}_{\geq 0}$, not all 0, we have

$$\varphi\left(\frac{\sum_{i=1}^n a_i v_i}{\sum_{i=1}^n a_i}\right) \geq \frac{\sum_{i=1}^n a_i \varphi(v_i)}{\sum_{i=1}^n a_i}.$$

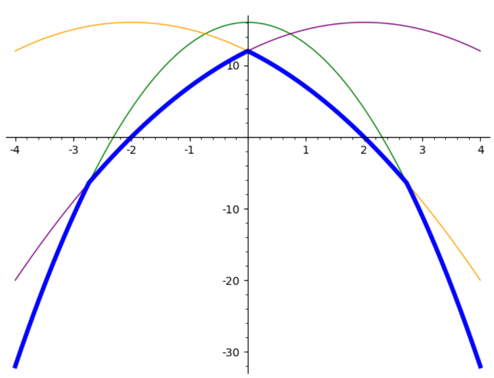
Note that left hand side of Jensen's Inequality is φ applied to the weighted average of the v_i's, where the a_i's are the weights, while the right hand side is the weighted average of the $\varphi(v_i)$'s.

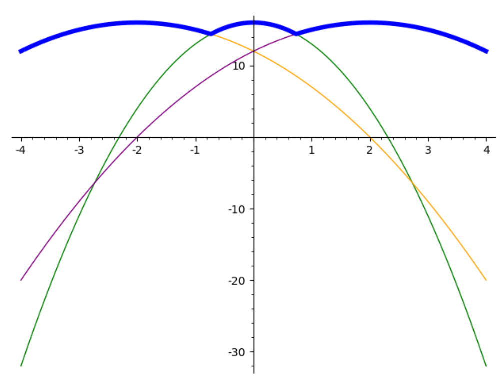
Question: Let $V = \mathbb{R}$, and let $f(x) = x^2$ and $g(x) = -x^2$. Which function f or g is convex?





Question: Let V be any real vector space, and let f and g be convex functions on V. Which function $\min(f, g)$ or $\max(f, g)$ is convex?





We can compare the equivalent characterization of convex functions described above to an equivalent characterization of the convex hull of a set and in turn of convex sets:

Theorem: Let S be a subset of a real vector space V. Then $\operatorname{conv}(S)$ is the set of weighted averages of points in S. A set is convex if and only if it is equal to its convex hull.

This leads to a nice little relationship between convex functions and convex sets.

Theorem: Let $\varphi : V \to \mathbb{R}$ be a convex function. Then

$$\Xi_{\varphi,c} := \{v \in V \mid \varphi(v) \geq c\}\,.$$

is convex.

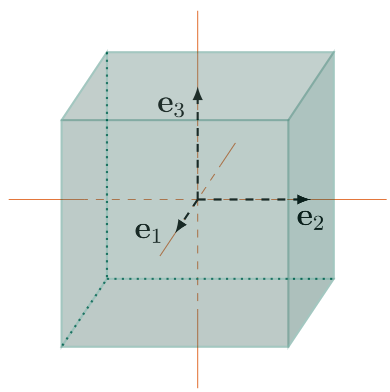
Question: Let $\varphi : V \to \mathbb{R}$ be the linear function $\langle\, \boldsymbol{\cdot}\,, w\rangle$, where $w \in V^*$. Then $\Xi_{\varphi,c}$ has another name – it's a set we've described previously. What is it?

We can use this to describe the dual P° of a polytope P as well. Denote the set of vertices of P by $\mathrm{Vert}(P)$ and let

$$\varphi = \min_{v \in \mathrm{Vert}(P)} \langle v, \,\boldsymbol{\cdot}\,\rangle.$$

Then $P^\circ = \Xi_{\varphi,-1}$.

Question: Can you describe the cube P shown below as $\Xi_{\varphi,-1}$ for some φ?

$P =$ 

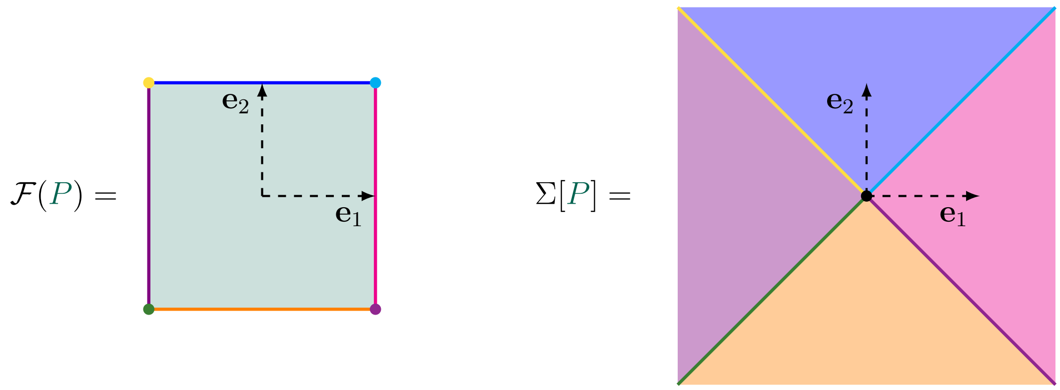
Let's now consider a full-dimensional polytope $P \subset V$ containing the origin in its interior.

Definition: Given a face F of P, the *cone of* F is $\sigma_F := \mathbb{R}_{\geq 0} \cdot F$. The *face-fan* of P, denoted $\Sigma[P]$, is

$$\Sigma[P] := \{\sigma_F \mid F \text{ is a face of } P\} \cup \{0\}\,.$$

We can think of σ_F as the set we get by taking rays from the origin through the points of F.

Example:



Definition: A function $\varphi : V \to \mathbb{R}$ is $\Sigma[P]$-*piecewise linear* if $\varphi|_\sigma$ is linear for each cone $\sigma \in \Sigma[P]$.

This means that $\varphi|_\sigma = \langle \,\cdot\, , w_\sigma \rangle$ for some $w_\sigma \in V^*$. If we can take each of these w_σ's to be lattice point of V^*, we say φ is an *integral* $\Sigma[P]$-piecewise linear function.

Question: Can you come up with any convex integral $\Sigma[P]$-piecewise linear functions for $\Sigma[P]$ the face-fan shown in the previous example?

1.g Nef Partitions

The final topic we will discuss is a fascinating duality discovered by Lev Borisov in 1993. This section is based on his paper *Towards the Mirror Symmetry for Calabi-Yau Complete Intersections in Gorenstein Toric Fano Varieties.*

Definition: Let P be a reflexive polytope in V. A decomposition of the vertices of P as a disjoint union

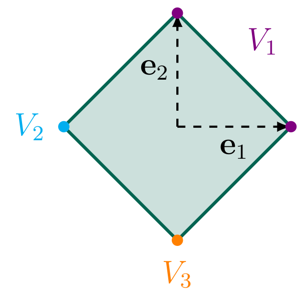
$$\mathrm{Vert}(P) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$$

is a *nef partition* if there is a collection $\varphi_1, \varphi_2, \ldots, \varphi_k$ of convex integral $\Sigma[P]$-piecewise linear functions with

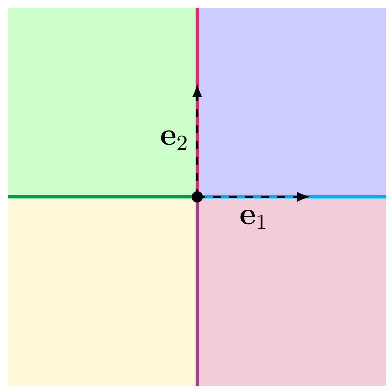
$$\varphi_i(v) = \begin{cases} -1 & \text{if } v \in V_i \\ 0 & \text{if } v \in V_j \text{ for } i \neq j. \end{cases}$$

This may look like a strange thing to define, and a weird name for it. Both the motivation and the name come from algebraic geometry – we're just exploring the polyhedral geometry part of a much deeper story with connections between several different fields of geometry.

Example: Consider the polytope P and decomposition of its vertices $\mathrm{Vert}(P) = V_1 \sqcup V_2 \sqcup V_3$ shown below.



The face-fan of P is:



Now consider the following functions:

- $\varphi_1 = \min\left(\langle \,\cdot\, , -\mathbf{e}_1^* - \mathbf{e}_2^*\rangle, \langle \,\cdot\, , -\mathbf{e}_1^*\rangle, \langle \,\cdot\, , -\mathbf{e}_2^*\rangle, \langle \,\cdot\, , \mathbf{0}\rangle\right)$
- $\varphi_2 = \min\left(\langle \,\cdot\, , \mathbf{e}_1^*\rangle, \langle \,\cdot\, , \mathbf{0}\rangle\right)$
- $\varphi_3 = \min\left(\langle \,\cdot\, , \mathbf{e}_2^*\rangle, \langle \,\cdot\, , \mathbf{0}\rangle\right)$

Let's first check that

$$\varphi_i(v) = \begin{cases} -1 & \text{if } v \in V_i \\ 0 & \text{if } v \in V_j \text{ for } i \neq j. \end{cases}$$

We want $\varphi_1(\mathbf{e}_1) = \varphi_1(\mathbf{e}_2) = -1$ and $\varphi_1(-\mathbf{e}_1) = \varphi_1(-\mathbf{e}_2) = 0$.

$$\begin{aligned} \varphi_1(\mathbf{e}_1) &= \min\left(\langle \mathbf{e}_1, -\mathbf{e}_1^* - \mathbf{e}_2^*\rangle, \langle \mathbf{e}_1, -\mathbf{e}_1^*\rangle, \langle \mathbf{e}_1, -\mathbf{e}_2^*\rangle, \langle \mathbf{e}_1, \mathbf{0}\rangle\right) \\ &= \min(-1, -1, 0, 0) \\ &= -1 \end{aligned}$$

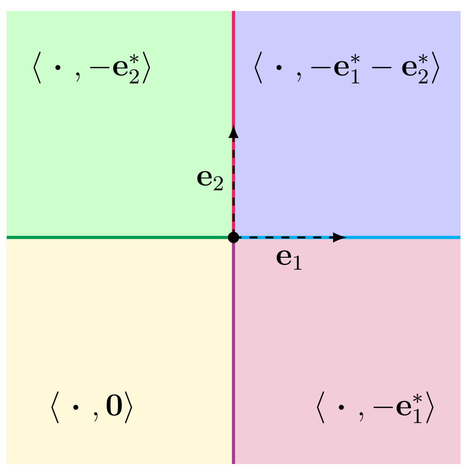
$$
\begin{aligned}
\varphi_1(\mathbf{e}_2) &= \min\left(\langle \mathbf{e}_2, -\mathbf{e}_1^* - \mathbf{e}_2^*\rangle, \langle \mathbf{e}_2, -\mathbf{e}_1^*\rangle, \langle \mathbf{e}_2, -\mathbf{e}_2^*\rangle, \langle \mathbf{e}_2, \mathbf{0}\rangle\right) \\
&= \min(-1, 0, -1, 0) \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
\varphi_1(-\mathbf{e}_1) &= \min\left(\langle -\mathbf{e}_1, -\mathbf{e}_1^* - \mathbf{e}_2^*\rangle, \langle -\mathbf{e}_1, -\mathbf{e}_1^*\rangle, \langle -\mathbf{e}_1, -\mathbf{e}_2^*\rangle, \langle -\mathbf{e}_1, \mathbf{0}\rangle\right) \\
&= \min(1, 1, 0, 0) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\varphi_1(-\mathbf{e}_2) &= \min\left(\langle -\mathbf{e}_2, -\mathbf{e}_1^* - \mathbf{e}_2^*\rangle, \langle -\mathbf{e}_2, -\mathbf{e}_1^*\rangle, \langle -\mathbf{e}_2, -\mathbf{e}_2^*\rangle, \langle -\mathbf{e}_2, \mathbf{0}\rangle\right) \\
&= \min(1, 0, 1, 0) \\
&= 0
\end{aligned}
$$

You can check that φ_2 and φ_3 have the desired values on the vertices of P as well.

Next, since each φ_i is a minimum of convex functions, each φ_i is convex. Moreover, the convex functions appearing in the minimum are all integral linear functions, so φ_i is an integral piecewise linear function. We just have to check that it is $\Sigma[P]$-*piecewise linear* – that is, we want the domains of linearity of φ_i to be unions of cones in $\Sigma[P]$. Morally, we don't want half of one cone to line in one domain of linearity and the other half in another. The restriction of φ_1 to each maximal cone of $\Sigma[P]$ is shown below.



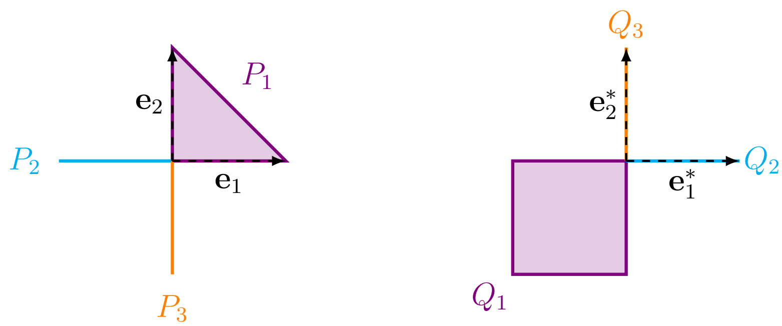
You can check that φ_2 and φ_3 are also $\Sigma[P]$-piecewise linear. We do indeed have a nef partition.

Now define the following polytopes:

$$P_i := \operatorname{conv}(\{0\} \cup V_i)$$

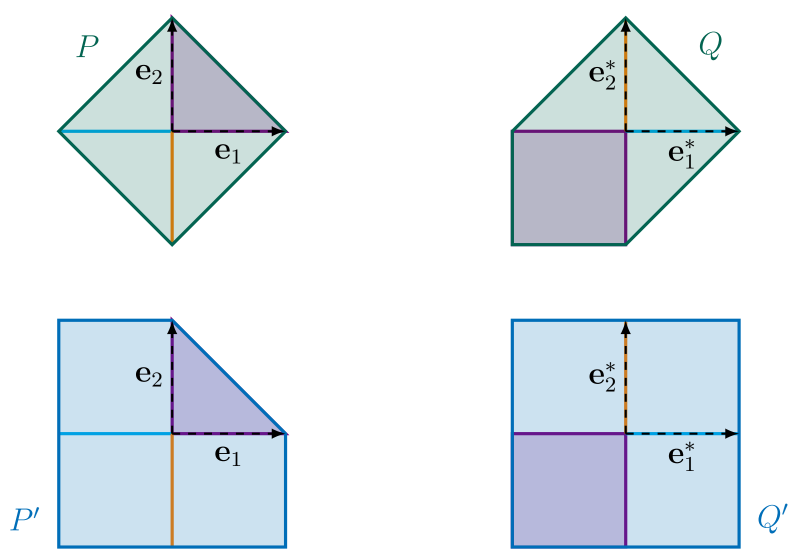
$$
\begin{aligned}
Q_i &:= \{\, w \in V^* \mid \langle u, w\rangle \geq \varphi_i(u) \text{ for all } u \in V \,\} \\
&= \{\, w \in V^* \mid \langle v, w\rangle \geq -1 \text{ if } v \in V_i, \text{ and } \langle v, w\rangle \geq 0 \text{ if } v \in V_{j\neq i} \,\}
\end{aligned}
$$

Example: Using the nef partition of the previous example, we have



Theorem: The four polytopes $P = \mathrm{conv}(P_1 \cup P_2 \cup \cdots \cup P_k)$, $Q = \mathrm{conv}(Q_1 \cup Q_2 \cup \cdots \cup Q_k)$, $P' = P_1 + P_2 + \cdots + P_k$, and $Q' = Q_1 + Q_2 + \cdots + Q_k$ are all reflexive. Moreover, the Q_i's define a nef partition of $\mathrm{Vert}(Q)$ – called the *dual nef partition* – and we have the polytope dualities $P^\circ = Q'$ and $Q^\circ = P'$. The double-dual nef partition is the original nef partition, so roles may be swapped in this picture.

Example: Continuing with our running example, we have:



Note that each of these four polytopes are reflexive – P and Q' are dual lattice polytopes as are P' and Q. What is the dual nef partition? What about the associated convex integral piecewise linear functions?