Fundamentals of broken line convex geometry and Batyrev-Borisov duality

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1 Introduction

In this paper we study *broken line convex geometry*– a generalization of convex geometry in which the ambient space is the tropicalization of a cluster variety rather than simply a vector space, and in which broken line segments play the role ordinarily filled by line segments. We show that many classical convex geometry results remain true in this setting. For instance, versions of the following classical results remain true in broken line convex geometry:

1. A set S is convex if and only if $tS + (1-t)S = S$ for all $t \in [0, 1]$.

2. $\operatorname{conv}(S + T) = \operatorname{conv}(S) + \operatorname{conv}(T)$.

3. If φ is a convex function, then the locus where φ is at least some constant r is a convex set.

4. A bounded polyhedron is the convex hull of its vertices.

5. The dual of a convex set S is full dimensional if and only if S is strongly convex.Tim: [Or:] The dual of a convex set S is strongly convex if and only if S is full dimensional.

6. If P and P° are dual polytopes, there is a bijective, containment-reversing correspondence between the faces of P and P°.

Other aspects of the theory need a bit of modification, but remain quite pleasant. In broken line convex geometry, the faces of a polyhedral set are generally not broken line convex. However, they satisfy a rather natural weaker convexity notion– which we call *weak convexity*. They may also fail to form a complex. Nevertheless, they do have a structure reminiscent of a polyhedral complex, forming what we call a *pseudo-complex*.

As is already evident in the brief list above, one operation central to the theory of convex geometry is the Minkowski sum. As such, a key element of this story is our notion of Minkowski addition in a tropical space. It is morally the same as usual Minkowski addition, but the lack of linear structure in tropical spaces makes this addition multi-valued. See Definition 11 for the precise definition. §3 treats the interplay of this tropical Minkowski sum and the *broken line convex hull* of [CMN21]. We find that these concepts relate to each other in much the same way as the usual Minkowski sum and convex hull do. In particular, Proposition 17 is the broken line convex geometry version of Item 1 and Theorem 28 is the broken line convex geometry version of Item 2.

Next, we turn our attention to the meaning of convexity of functions in broken line convex geometry. Here again we adapt the linear definition to the tropical setting by replacing *the line segment* between a pair of points with *all broken line segments* between a pair of points:

Definition 1 (Definition 30)**.** Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be a broken line convex set. A function $\varphi : S \to \mathbb{Q}$ is *convex with respect to broken lines* if for any broken line segment $\gamma : [t_1, t_2] \to S$, we have that

$$\varphi(\gamma(t)) \geq \left(\frac{t_2 - t}{t_2 - t_1}\right) \varphi(\gamma(t_1)) + \left(\frac{t - t_1}{t_2 - t_1}\right) \varphi(\gamma(t_2))$$

for all $t \in [t_1, t_2]$.

We then give an equivalent characterization these functions in terms of structure constants of ϑ-function multiplication:[1]

Theorem 2 (Proposition 31, Remark 32)**.** *Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be broken line convex. Then $\varphi : S \to \mathbb{Q}$ is convex with respect to broken lines if and only if for all $s_1, \ldots, s_d, s \in S$, $a_1, \ldots, a_d \in \mathbb{Q}_{\geq 0}$ with $a_1 s_1, \ldots, a_d s_d$, and $(a_1 + \cdots + a_d)s$ all integral, and $\alpha^{(a_1+\cdots+a_d)s}_{a_1 s_1,\ldots,a_d s_d} \neq 0$, we have*

$$\varphi(s) \geq \sum_{i=1}^{d} \frac{a_i}{a_1 + \cdots + a_d} \varphi(s_i).$$

We use these equivalent characterizations to prove the equivalence of Gross-Hacking-Keel-Kontsevich's *min-convex* and *decreasing* definitions. See Corollary 35. We then describe other properties of functions which are convex with respect to broken lines. In particular, Proposition 37 is the broken line convex geometry version of Item 3.

After this discussion of convexity for functions, we treat *polyhedral* broken line convex geometry. The canonical pairing between tropicalizations of mirror cluster varieties affords us a natural notion of a half-space in this context.

Definition 3 (Definition 42)**.** For $y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ and $r \in \mathbb{Q}$, we call the set

$$K(y, r) := \left\{ x \in U^{\mathrm{trop}}(\mathbb{Q}) : \langle x, y \rangle \geq -r \right\}$$

a *tropical half-space.*

This in turn provides a natural analogue of a polyhedron– we say a subset $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ is *polyhedral* if it is the intersection of finitely many tropical half-spaces. (See Definition 45.) *Faces* of S are defined much like in usual convex geometry– we take a tropical half-space containing S and intersect its boundary with S. See Definition 46. As mentioned above, these faces satisfy only a weaker notion of convexity. If we choose a pair of points x_1, x_2 in a face F, we cannot say that F contains *all* broken line segments connecting x_1 and x_2. We can only say that F contains *some* broken line segment connecting x_1 and x_2. (See Corollary 51.) The failure of these faces to be broken line convex hinders another familiar property from usual convex geometry– the intersection of two faces need not be a face. As such, faces may not form a complex. They do however have a structure very reminiscent of a complex, which we refer to as a *pseudo-complex.* See Definition 52 and Proposition 53. Moreover, we show in Proposition 72 that the face pseudo-complexes of polar polytopal sets are related in precisely the same way as the linear case described in Item 6.

Finally, §6 treats a major motivation we had in writing this paper, and provides in our view particularly compelling evidence that the theory we develop here is worth studying. We state and prove a broken line convex geometry version of the duality for nef-partitions due to Borisov. See [Bor93] for the original version and 🚲Tim: [Coming soon...] for our new version. In the linear case, Borisov conjectured ([Bor93, Conjecture 3.6]):

The duality between nef-partitions of reflexive polyhedra Δ and ∇ gives rise to pairs of mirror symmetric families of Calabi-Yau complete intersections in Gorenstein toric Fano varieties P_{Δ° and P_{∇°.

His convex geometry duality and associated mirror symmetry conjecture had a profound impact on the study of mirror symmetry. Our hope is that, once we have established a

$$\{\textit{polyhedral broken line convex geometry}\} \leftrightarrow \{\textit{algebraic geometry of minimal models for cluster varieties}\}$$

dictionary, we will be able to make an analogous statement for Calabi-Yau complete intersections in Gorenstein Fano minimal models for cluster varieties.

In fact, we view this paper as part of a research program we undertook with our close collaborators L. Bossinger, M.-W. Cheung, and A. Nájera Chávez, with the goal of generalizing Batyrev and Batyrev-Borisov mirror symmetry constructions from the setting of Gorenstein Fano toric varieties to the setting of Gorenstein Fano minimal models for cluster varieties. It is our hope that the broken line convex geometry results of this paper will be an important step toward that common ultimate goal.

[1] We will discuss these ϑ-functions and structure constants in greater detail in §2. For now, a non-zero structure constant $\alpha^q_{p_1,\ldots,p_d}$ means that q is a value of the multi-valued sum $p_1 +_\vartheta \ldots +_\vartheta p_d$.

2 Background

The notion of broken line convexity used in this paper comes from [CMN21], where the main result is the equivalence of this convexity notion with the algebraic notion of *positivity* from [GHKK18]. That said, we will employ subtly different conventions and definitions here. First, as we are only ever interested in the rational points of our tropical spaces, we will always work over $\mathbb{Q}$ instead of $\mathbb{R}$.[2] Next (and relatedly), the definition of positivity in [GHKK18], and in turn that of broken line convexity in [CMN21], makes reference to *closed* sets. However, these definitions may equally well be made without requiring closure. Moreover, the proof of the equivalence of broken line convexity and positivity in [CMN21] does not rely on closure– the result still holds if closure is dropped from both definitions. We do precisely this.

Definition 4 ([CMN21])**.** A subset S of $U^{\mathrm{trop}}(\mathbb{Q})$ is *broken line convex* if for every pair of points s_1, s_2 in S, every broken line segment with endpoints s_1 and s_2 has support entirely contained in S.

This is the natural generalization of usual convexity to $U^{\mathrm{trop}}(\mathbb{Q})$, where broken line segments fill the role occupied by line segments in usual convex geometry. The aforementioned *positivity* which it is equivalent to is defined as follows:

Definition 5 ([GHKK18])**.** A subset S of $U^{\mathrm{trop}}(\mathbb{Q})$ is *positive* if for any non-negative integers a and b, and any integral tropical points $p \in a\,S(\mathbb{Z})$, $q \in b\,S(\mathbb{Z})$, and $r \in U^{\mathrm{trop}}(\mathbb{Z})$ with $\alpha^r_{p,q} \neq 0$, we have $r \in (a+b)S(\mathbb{Z})$.

In usual convex geometry, there is a canonical way to take a possibly non-convex set and replace it with a convex set which contains it– namely the convex hull. There is a completely analogous procedure here:

Definition 6 ([CMN21])**.** Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$. We define the *broken line convex hull of* S, denoted $\mathrm{conv}_{\mathrm{BL}}(S)$ to be the intersection of all broken line convex sets containing S.

Notation 7. As we will always work over $\mathbb{Q}$, when we write an interval $[t_1, t_2]$ we mean an interval in $\mathbb{Q}$ not $\mathbb{R}$.

2.1 Results from [CMMM]

There are two key results from the forthcoming paper [CMMM] that we will need throughout the course of this work. We will state them here in simplified form– the setting of [CMMM] is more general than that the current work. The reader who is – quite reasonably – hesitant to accept results whose proofs are not yet publicly available may for the time being take these two results to be conjectures upon which many aspects of the current work rely.

Let U and $U^\vee$ be mirror cluster varieties for which the full Fock-Goncharov conjecture holds. Then:

Theorem 8 ([CMMM],"Theta Recipocity")**.** *Let $x \in U^{\mathrm{trop}}(\mathbb{Z})$ and $y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Z})$. Then $x(\vartheta_y) = y(\vartheta_x)$.*

This means we have a truly canonical pairing between $U^{\mathrm{trop}}(\mathbb{Z})$ and $(U^\vee)^{\mathrm{trop}}(\mathbb{Z})$, rather than two different evaluation pairings. This pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ extends uniquely to a pairing between $U^{\mathrm{trop}}(\mathbb{Q})$ and $(U^\vee)^{\mathrm{trop}}(\mathbb{Q})$.

The other key result of [CMMM] we need is the *valuative independence theorem.*

Theorem 9 ([CMMM],"Valuative Independence")**.** *Let*

$$f = \sum_{y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Z})} c_y \vartheta_y$$

be any regular function on U and $x \in U^{\mathrm{trop}}(\mathbb{Z})$ any integral tropical point. Then

$$x(f) = \min_{c_y \neq 0} \{x(\vartheta_y)\}\,.$$

[2] The prescient reader may raise concern about placement of basepoints for broken lines in regions of dense walls. We will address this concern in §2.2 with a discussion of *broken lines* vs. *generic broken lines*, as in [CMN21], with one modification. The sequence of generic broken lines we use to define our broken lines here will live in the finite order scattering diagrams whose colimit produces the cluster scattering diagram.

Recall that integral tropical points are discrete valuations, and as such the inequality

$$x(f) \geq \min_{c_y \neq 0} \{x(\vartheta_y)\}$$

holds by definition. The valuative independence theorem replaces the inequality with an equality, essentially by eliminating the possibility of pole cancellations.

2.2 Scattering diagrams and broken lines

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To write down a scattering diagram explicitly, we choose a seed of the cluster structure. This selects a cluster torus in both U and $U^\vee$, and piecewise linearly identifies the integral tropical points of U and $U^\vee$ with the integral tropical points of the selected embedded tori, *i.e* with the cocharacter lattices of these tori. In turn, it identifies $U^{\text{trop}}(\mathbb{Q})$ and $(U^\vee)^{\text{trop}}(\mathbb{Q})$ with a pair of dual $\mathbb{Q}$-vector spaces V and V^*.

Notation 10. For each seed $\mathbf{s}$, write $\mathfrak{r}_{\mathbf{s}} : U^{\text{trop}}(\mathbb{Q}) \to V$ and $\mathfrak{r}^\vee_{\mathbf{s}} : (U^\vee)^{\text{trop}}(\mathbb{Q}) \to V^*$ for the piecewise linear identifications described above.

3 Tropical Minkowski sum

In order to generalize many convex polyhedral geometry constructions of the toric world to the setting of cluster varieties, we will need a convex tropical geometry version of the Minkowski sum. In this section we provide such a notion and illustrate some of its key properties, particularly Theorem 28 which illustrates the compatibility of this *tropical Minkowski sum* with the *broken line convex hull.* In essence, the tropical Minkowski sum of two subsets S and T of a tropical space $U^{\text{trop}}(\mathbb{Q})$ works the same way as the usual Minkowski sum of subsets of a Euclidean space– we "add" pairs of elements (s,t) with $s \in S$, $t \in T$. However, in this setting where we have only a *piecewise* linear structure, our "addition" is multivalued. The values that arise correspond to non-zero summands of products of ϑ-functions. Namely, if for some $a \in \mathbb{Z}_{>0}$, the function $\vartheta_{a\,x}$ is a non-zero summand of $\vartheta_{a\,s}\vartheta_{a\,t}$, then x is a value of the "sum" of s and t.

Definition 11. Let S and T be subsets of $U^{\text{trop}}(\mathbb{Q})$. We define the *tropical Minkowski sum* of S and T as follows:

$$\begin{aligned} S +_\vartheta T &:= \left\{x \in U^{\text{trop}}(\mathbb{Q}) : \exists s \in S, t \in T, a \in \mathbb{Z}_{>0} \text{ with } as, at, ax \in U^{\text{trop}}(\mathbb{Z}) \text{ such that } \alpha^{ax}_{as,at} \neq 0\right\} \\ &= \left\{x \in U^{\text{trop}}(\mathbb{Q}) : \exists s \in S, t \in T, \gamma : [0,\tau] \to U^{\text{trop}}(\mathbb{Q}) \text{ with } \gamma(0) = s, \gamma(\tau) = t, \gamma(\tau/2) = x/2\right\} \end{aligned}$$

where γ is a broken line segment.

The equivalence of the two descriptions in Definition 11 follows immediately from the proof of [CMN21, Theorem 6.1].

Remark 12. Consider a function $f \in \mathcal{O}(U^\vee)$ given as linear combination of products of theta functions $f = \sum_{s,t \in U^{\text{trop}}(\mathbb{Z})} c_{s,t} \vartheta_s \cdot \vartheta_t$. Since $f \in \mathcal{O}(U^\vee)$, we may also expand it as $f = \sum_{x \in U^{\text{trop}}(\mathbb{Z})} f_x \vartheta_x$. Let ϑ_{x_0} be one such non-zero summand of f. Then, there exist $s_0, t_0 \in U^{\text{trop}}(\mathbb{Z})$ with $c_{s_0,t_0} \neq 0$ such that ϑ_{x_0} is a non-zero summand of $\vartheta_{s_0} \cdot \vartheta_{t_0}$. To see this, note that we have

$$\begin{aligned} f &= \sum_{s,t \in U^{\text{trop}}(\mathbb{Z})} c_{s,t} \vartheta_s \cdot \vartheta_t \\ &= \sum_{s,t \in U^{\text{trop}}(\mathbb{Z})} \sum_{x \in U^{\text{trop}}(\mathbb{Z})} c_{s,t} \alpha^x_{s,t} \vartheta_x \\ &= \sum_{x \in U^{\text{trop}}(\mathbb{Z})} f_x \vartheta_x. \end{aligned}$$

By linear independence of theta functions, we must have that $f_x = \sum_{s,t \in U^{\text{trop}}(\mathbb{Z})} c_{s,t} \alpha^x_{s,t}$ for each $x \in U^{\text{trop}}(\mathbb{Z})$. So f_{x_0} may only be non-zero if we have some s_0, t_0 with c_{s_0,t_0} and $\alpha^{x_0}_{s_0,t_0}$ both non-zero. Note that this argument also applies if we replace the products of pairs of theta functions with products of arbitrary finite numbers of theta functions.

Remark 13. The non-negativity of scattering functions for cluster scattering diagrams implies that all structure constants $\alpha^r_{p,q}$ (or more generally $\alpha^r_{p_1,\ldots,p_d}$) are non-negative. This result is sometimes referred to as *strong positivity.* Versions of this result are due to [GHKK18, Theorem 7.5], [Man21, Proposition 2.15], and [DM21, Theorem 1.1].

Lemma 14. *Let $p, q, r \in U^{\mathrm{trop}}(\mathbb{Z})$ be such that $\alpha^r_{p,q} \neq 0$, and let $a \in \mathbb{Z}_{>0}$. Then $\alpha^{ar}_{ap,aq} \neq 0$.*

Proof. If (γ_1, γ_2) is a pair of broken lines contributing to $\alpha^r_{p,q}$, we may rescale the exponent vectors of decoration monomials as well as the supports of γ_1 and γ_2 by a factor of a to obtain a new pair broken lines (of higher multiplicity) $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ contributing to $\alpha^{ar}_{ap,aq}$. Then positivity of scattering functions implies no cancellations may occur and $\alpha^{ar}_{ap,aq} \neq 0$. □

Lemma 15. *Let $x \in U^{\mathrm{trop}}(\mathbb{Q})$, and let $a_1, \ldots, a_d$ be non-negative integers such that each $a_i x$ is integral. Then $\alpha^{(a_1+\cdots+a_d)x}_{a_1x,\ldots,a_dx} \neq 0$.*

Proof. Choose a seed $\mathbf{s}$ to identify $U^{\mathrm{trop}}(\mathbb{Q})$ with a $\mathbb{Q}$-vector space V by a map $\mathfrak{r}_{\mathbf{s}}$ as in Notation 10. Take $(\gamma_1, \cdots, \gamma_d)$ to be the collection of straight broken lines in V where the initial decoration monomial of γ_i is $z^{a_i \mathfrak{r}_{\mathbf{s}}(x)}$ and the endpoint of each γ_i is $(a_1 + \cdots + a_d)\mathfrak{r}_{\mathbf{s}}(x)$. This contributes 1 to $\alpha^{(a_1+\cdots+a_d)x}_{a_1x,\ldots,a_dx}$. (It is in fact the only contribution.) □

Lemma 16. *Let S, T and R be subsets of $U^{\mathrm{trop}}(\mathbb{Q})$. Then*

$$(S +_\vartheta T) +_\vartheta R = S +_\vartheta (T +_\vartheta R).$$

Proof. Let $x \in (S +_\vartheta T) +_\vartheta R$. Then there exists $y \in S +_\vartheta T$, $r \in R$ and $a \in \mathbb{Z}_{>0}$ with $\alpha^{ax}_{ay,ar} \neq 0$, meaning ϑ_{ax} is a non-zero summand of $\vartheta_{ay} \cdot \vartheta_{ar}$. Similarly, since $y \in S +_\vartheta T$ we have $\alpha^{by}_{bs,bt} \neq 0$ for some $s \in S$, $t \in T$ and $b \in \mathbb{Z}_{>0}$, meaning ϑ_{by} is a non-zero summand of $\vartheta_{bs} \cdot \vartheta_{bt}$. Then, since

$$\vartheta_{abx} \text{ is a non-zero summand of } \vartheta_{aby} \cdot \vartheta_{abr}$$

and

$$\vartheta_{aby} \text{ is a non-zero summand of } \vartheta_{abs} \cdot \vartheta_{abt},$$

we obtain that

$$\vartheta_{abx} \text{ is a non-zero summand of } \vartheta_{abs} \cdot \vartheta_{abt} \cdot \vartheta_{abr}.$$

Consider the expression $\vartheta_{abt} \cdot \vartheta_{abr} = \sum_{abz \in U^{\mathrm{trop}}(\mathbb{Z})} \alpha^{abz}_{abt,abr} \vartheta_{abz}$. By construction, if $\alpha^{abz}_{abt,abr} \neq 0$, then $z \in T +_\vartheta R$. We have now that

$$\vartheta_{abx} \text{ is a non-zero summand of } \sum_{abz \in U^{\mathrm{trop}}(\mathbb{Z})} \alpha^{abz}_{abt,abr} \vartheta_{abs} \cdot \vartheta_{abz}.$$

Then by Remark 12, we find that ϑ_{abx} is a non-zero summand of $\vartheta_{abs} \cdot \vartheta_{abz}$ for some $z \in T +_\vartheta R$. Consequently, $x \in S +_\vartheta (T +_\vartheta R)$. □

Proposition 17. *A subset S of $U^{\mathrm{trop}}(\mathbb{Q})$ is broken line convex if and only if for all $t \in [0, 1]$, we have*

$$tS +_\vartheta (1 - t)S = S.$$

Proof. Let S be broken line convex. Then for all a, b in $\mathbb{Z}_{\geq 0}$, $p \in aS(\mathbb{Z})$, $q \in bS(\mathbb{Z})$ and $r \in U^{\mathrm{trop}}(\mathbb{Z})$ with $\alpha^r_{p,q} \neq 0$, we have that $r \in (a + b)S$. If $z \in tS +_\vartheta (1 - t)S$, then there is some $x \in tS$, $y \in (1 - t)S$, and $c \in \mathbb{Z}_{>0}$ such that $c\,x$, $c\,y$, and $c\,z$ are in $U^{\mathrm{trop}}(\mathbb{Z})$ and $\alpha^{c\,z}_{c\,x,c\,y} \neq 0$. We can find non-negative integers a and b such that $t = \frac{a}{a+b}$ and $c = a + b$. Then $p := c\,x \in aS(\mathbb{Z})$, $q := c\,y \in bS(\mathbb{Z})$, and $r := c\,z$ must be in $(a + b)S$. It follows that $z \in S$, and $tS +_\vartheta (1 - t)S \subset S$.

On the other hand for all $z \in U^{\mathrm{trop}}(\mathbb{Q})$, we can draw a straight line segment from tz to $(1-t)z$. As such, $z \in t\{z\} +_\vartheta (1-t)\{z\}$. So if $z \in S$, then $z \in t\{z\} +_\vartheta (1-t)\{z\} \subset tS +_\vartheta (1-t)S$, and $S \subset tS +_\vartheta (1-t)S$.

Now suppose $tS +_\vartheta (1-t)S = S$ for all $t \in [0,1]$. We want to show that for all a, b in $\mathbb{Z}_{\geq 0}$, $p \in aS(\mathbb{Z})$, $q \in bS(\mathbb{Z})$ and $r \in U^{\mathrm{trop}}(\mathbb{Z})$ with $\alpha^r_{p,q} \neq 0$, we have $r \in (a+b)S$. First we address the trivial case: if $a = b = 0$ and $\alpha^r_{p,q} \neq 0$, then necessarily $p = q = r = 0 \in 0 \cdot S$. Next, assume $a > 0$ or $b > 0$, and let $t = \frac{a}{a+b}$. Write $p' := \frac{p}{a+b}$, $q' := \frac{q}{a+b}$, and $r' := \frac{r}{a+b}$, so $p' \in tS$, $q' \in (1-t)S$, and $\alpha^{(a+b)r'}_{(a+b)p',(a+b)q'} \neq 0$. This implies $r' \in tS +_\vartheta (1-t)S = S$, so $r \in (a+b)S$ as desired. □

Proposition 18. *If the subsets S and T of $U^{\mathrm{trop}}(\mathbb{Q})$ are broken line convex, then $S +_\vartheta T$ is broken line convex.*

Proof. Let $\tau \in [0,1]$.If we prove the equality

$$\tau(S +_\vartheta T) +_\vartheta (1-\tau)(S +_\vartheta T) = S +_\vartheta T,$$

then by Proposition 17 we conclude the result.

Assume that $x \in \tau(S +_\vartheta T) +_\vartheta (1-\tau)(S +_\vartheta T)$. So, there exist $y \in \tau \cdot (S +_\vartheta T)$, $z \in (1-\tau) \cdot (S +_\vartheta T)$ and $a \in \mathbb{Z}_{>0}$ such that $\alpha^{a\,x}_{a\,y,a\,z} \neq 0$, meaning ϑ_{ax} is a non-zero summand of $\vartheta_{ay} \cdot \vartheta_{az}$. Now, since $y \in \tau \cdot (S +_\vartheta T)$, there exist $s_1 \in S$, $t_1 \in T$ and $b \in \mathbb{Z}_{>0}$ such that $\alpha^{b\,y}_{b\,\tau\,s_1,b\,\tau\,t_1} \neq 0$, meaning ϑ_{by} is a non-zero summand of $\vartheta_{b\tau s_1} \cdot \vartheta_{b\tau t_1}$. Similarly, $z \in (1-\tau) \cdot (S +_\vartheta T)$ implies the existence of $s_2 \in S$, $t_2 \in T$ and $c \in \mathbb{Z}_{>0}$ such that $\alpha^{c\,z}_{c\,(1-\tau)\,s_2,c\,(1-\tau)\,t_2} \neq 0$, meaning ϑ_{cz} is a non-zero summand of $\vartheta_{c(1-\tau)s_2} \cdot \vartheta_{c(1-\tau)t_2}$. Then, it follows that

$$\begin{aligned}
&\vartheta_{abcx} \text{ is a non-zero summand of } \vartheta_{abcy} \cdot \vartheta_{abcz},\\
&\vartheta_{abcy} \text{ is a non-zero summand of } \vartheta_{abc\tau s_1} \cdot \vartheta_{abc\tau t_1},\\
&\vartheta_{abcz} \text{ is a non-zero summand of } \vartheta_{abc(1-\tau)s_2} \cdot \vartheta_{abc(1-\tau)t_2}.
\end{aligned}$$

Then, we have that

$$\vartheta_{abcx} \text{ is a non-zero summand of } \vartheta_{abc\tau s_1} \cdot \vartheta_{abc(1-\tau)s_2} \cdot \vartheta_{abc\tau t_1} \cdot \vartheta_{abc(1-\tau)t_2}. \tag{1}$$

Now, consider the expressions

$$\begin{aligned}
\vartheta_{abc\tau s_1} \cdot \vartheta_{abc(1-\tau)s_2} &= \sum_{abcs \in U^{\mathrm{trop}}(\mathbb{Z})} \alpha^{abcs}_{abc\tau s_1, abc(1-\tau)s_2} \vartheta_{abcs}, \quad \text{and}\\
\vartheta_{abc\tau t_1} \cdot \vartheta_{abc(1-\tau)t_2} &= \sum_{abct \in U^{\mathrm{trop}}(\mathbb{Z})} \alpha^{abcs}_{abc\tau t_1, abc(1-\tau)t_2} \vartheta_{abct}.
\end{aligned}$$

By Equation (1) and Remark 12 we have that ϑ_{abcx} is a non-zero summand of $\vartheta_{abcs} \cdot \vartheta_{abct}$ for some $s \in S$ and $t \in T$. Therefore, we have that $x \in S +_\vartheta T$ and we conclude that $\tau(S +_\vartheta T) +_\vartheta (1-\tau)(S +_\vartheta T) \subseteq S +_\vartheta T$.

For the other containment, consider $x \in S +_\vartheta T$. If we consider the line segment $\tau x + (1-\tau)x$, then we have that $x \in \tau\{x\} +_\vartheta (1-\tau)\{x\}$. So, since $\tau\{x\} +_\vartheta (1-\tau)\{x\} \subset \tau(S +_\vartheta T) +_\vartheta (1-\tau)(S +_\vartheta T)$ we conclude that $S +_\vartheta T \subseteq \tau(S +_\vartheta T) +_\vartheta (1-\tau)(S +_\vartheta T)$. □

Corollary 19. *Let S and T be subsets of $U^{\mathrm{trop}}(\mathbb{Q})$. Then*

$$\mathrm{conv}_{\mathrm{BL}}(S +_\vartheta T) \subseteq \mathrm{conv}_{\mathrm{BL}}(S) +_\vartheta \mathrm{conv}_{\mathrm{BL}}(T).$$

Proof. Since $S +_\vartheta T \subseteq \mathrm{conv}_{\mathrm{BL}}(S) +_\vartheta \mathrm{conv}_{\mathrm{BL}}(T)$, we have that

$$\mathrm{conv}_{\mathrm{BL}}(S +_\vartheta T) \subseteq \mathrm{conv}_{\mathrm{BL}}(\mathrm{conv}_{\mathrm{BL}}(S) +_\vartheta \mathrm{conv}_{\mathrm{BL}}(T)).$$

Note that $\mathrm{conv}_{\mathrm{BL}}(S)$ and $\mathrm{conv}_{\mathrm{BL}}(T)$ are broken line convex sets, then Proposition 18 implies that $\mathrm{conv}_{\mathrm{BL}}(S) +_\vartheta \mathrm{conv}_{\mathrm{BL}}(T)$ is a broken line convex set and consequently we obtain that $\mathrm{conv}_{\mathrm{BL}}(\mathrm{conv}_{\mathrm{BL}}(S) +_\vartheta \mathrm{conv}_{\mathrm{BL}}(T)) = \mathrm{conv}_{\mathrm{BL}}(S) +_\vartheta \mathrm{conv}_{\mathrm{BL}}(T)$. The claim follows. □

Definition 20. Let $\alpha^r_{p_1,\dots,p_d}$ denote the coefficient of ϑ_r in the expansion of $\vartheta_{p_1}\cdots\vartheta_{p_d}$. For $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ define

$$S_d := \left\{ u \in U^{\mathrm{trop}}(\mathbb{Q}) : \alpha^{(a_1+\cdots+a_d)u}_{a_1 s_1,\dots,a_d s_d} \neq 0 \text{ for some } s_1,\dots,s_d \in S \text{ and } a_1,\dots,a_d \in \mathbb{Z}_{\geq 0}, \text{ with } \sum_{i=1}^d a_i \neq 0 \right\}.$$

Lemma 21. *We have a filtration $S = S_1 \subset S_2 \subset \cdots$.*

Proof. The first equality is immediate from the definition of S_1. For the remaining containments, set $a_{d+1} = 0$ to find $S_d \subset S_{d+1}$. □

Lemma 22. *If $x \in S_{d_1}$, $y \in S_{d_2}$, and $\alpha^{(n+m)z}_{n\,x,m\,y} \neq 0$, then $z \in S_{d_1+d_2}$.*

Proof. First, since $x \in S_{d_1}$, we have $\alpha^{\mathbf{a}\,x}_{a_1 s_1,\dots,a_{d_1} s_{d_1}} \neq 0$ for some $s_1,\dots,s_{d_1} \in S$ and $a_1,\dots,a_{d_1} \in \mathbb{Z}_{\geq 0}$ with $\mathbf{a} := \sum_{i=1}^{d_1} a_i \neq 0$, and

$$\vartheta_{\mathbf{a}\,x} \text{ is a non-zero summand of } \vartheta_{a_1 s_1}\cdots\vartheta_{a_{d_1} s_{d_1}}. \tag{2}$$

Similarly, since $y \in S_{d_2}$, we have $\alpha^{\mathbf{b}\,y}_{b_1 r_1,\dots,b_{d_2} r_{d_2}} \neq 0$ for some $r_1,\dots,r_{d_2} \in S$ and $b_1,\dots,b_{d_2} \in \mathbb{Z}_{\geq 0}$ with $\mathbf{b} := \sum_{i=1}^{d_2} b_i \neq 0$, and

$$\vartheta_{\mathbf{b}\,y} \text{ is a non-zero summand of } \vartheta_{b_1 r_1}\cdots\vartheta_{b_{d_2} r_{d_2}}. \tag{3}$$

Next, since $\alpha^{(n+m)z}_{n\,x,m\,y} \neq 0$,

$$\vartheta_{(n+m)z} \text{ is a non-zero summand of } \vartheta_{n\,x}\cdot\vartheta_{m\,y}. \tag{4}$$

We claim that $\vartheta_{\mathbf{a}\,\mathbf{b}\,(n+m)z}$ is a non-zero summand of

$$\vartheta_{n\,\mathbf{b}\,a_1 s_1}\cdots\vartheta_{n\,\mathbf{b}\,a_{d_1} s_{d_1}}\cdot\vartheta_{m\,\mathbf{a}\,b_1 r_1}\cdots\vartheta_{m\,\mathbf{a}\,b_{d_2} r_{d_2}}.$$

First, using Lemma 14, we can conclude from (2) that

$$\vartheta_{n\,\mathbf{a}\,\mathbf{b}\,x} \text{ is a non-zero summand of } \vartheta_{n\,\mathbf{b}\,a_1 s_1}\cdots\vartheta_{n\,\mathbf{b}\,a_{d_1} s_{d_1}}. \tag{5}$$

The same argument applied to (3) shows

$$\vartheta_{m\,\mathbf{a}\,\mathbf{b}\,y} \text{ is a non-zero summand of } \vartheta_{m\,\mathbf{a}\,b_1 r_1}\cdots\vartheta_{m\,\mathbf{a}\,b_{d_2} r_{d_2}}. \tag{6}$$

Next, since the structure constants are non-negative, by Remark 13, (5), and (6),

$$\begin{aligned}&\text{non-zero summands of } \vartheta_{n\,\mathbf{a}\,\mathbf{b}\,x}\cdot\vartheta_{m\,\mathbf{a}\,\mathbf{b}\,y} \text{ must also be}\\ &\text{non-zero summands of } \vartheta_{n\,\mathbf{b}\,a_1 s_1}\cdots\vartheta_{n\,\mathbf{b}\,a_{d_1} s_{d_1}}\cdot\vartheta_{m\,\mathbf{a}\,b_1 r_1}\cdots\vartheta_{m\,\mathbf{a}\,b_{d_2} r_{d_2}}.\end{aligned} \tag{7}$$

Finally, we can conclude from Lemma 14 and (4) that

$$\vartheta_{(n+m)\mathbf{a}\,\mathbf{b}\,z} \text{ is a non-zero summand of } \vartheta_{n\,\mathbf{a}\,\mathbf{b}\,x}\cdot\vartheta_{m\,\mathbf{a}\,\mathbf{b}\,y}. \tag{8}$$

Combining (7) and (8) finishes the proof. □

Lemma 23. *S is positive if and only if for any $n > 0$, $a_1,\dots,a_n \in \mathbb{Z}_{\geq 0}$, $s_i \in a_i S(\mathbb{Z})$, and $r \in U^{\mathrm{trop}}(\mathbb{Z})$ with $\alpha^r_{s_1,\dots,s_n} \neq 0$, we have $r \in (a_1+\cdots+a_n)S$.*

Proof. For $n = 2$, this is the definition of positivity, so the if part holds. Next, if S is positive, we use associativity of theta function multiplication to conclude the only if part. □

Lemma 24. *For all $d \in \mathbb{Z}_{>0}$, we have $S_d \subset \mathrm{conv}_{\mathrm{BL}}(S)$.*

Proof. If $u \in S_d$, we have $\alpha^{(a_1+\cdots a_d)u}_{a_1 s_1,\ldots,a_d s_d} \neq 0$ for some $s_1,\ldots,s_d \in S$ and $a_1,\ldots,a_d \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^d a_i \neq 0$. As $S \subset \mathrm{conv}_{\mathrm{BL}}(S)$, and positivity is equivalent to broken line convexity, $(a_1+\cdots+a_d)u$ must be in $(a_1+\cdots+a_d)\,\mathrm{conv}_{\mathrm{BL}}(S)$. By Lemma 23, failure of this would contradict positivity of $\mathrm{conv}_{\mathrm{BL}}(S)$. So $u \subset \mathrm{conv}_{\mathrm{BL}}(S)$, and $S_d \subset \mathrm{conv}_{\mathrm{BL}}(S)$. □

Corollary 25. *Let S be any subset of $U^{\mathrm{trop}}(\mathbb{Q})$. Then*

$$\mathrm{conv}_{\mathrm{BL}}(S) = \bigcup_{d\geq 1} S_d.$$

Proof. By Lemma 22, the infinite union $\bigcup_{d\geq 1} S_d$ is positive, and hence broken line convex. As it is broken line convex and contains S (see Lemma 21), we find that

$$\mathrm{conv}_{\mathrm{BL}}(S) \subset \bigcup_{d\geq 1} S_d.$$

By Lemma 24, we observe the opposite inclusion:

$$\mathrm{conv}_{\mathrm{BL}}(S) \supset \bigcup_{d\geq 1} S_d.$$

□

Lemma 26. *Let S and T be subsets of $U^{\mathrm{trop}}(\mathbb{Q})$. For all $d,e \in \mathbb{Z}_{>0}$,*

$$S_d +_\vartheta T_e \subset (S +_\vartheta T)_{de}.$$

Proof. Let x be in the sum $S_d +_\vartheta T_e$. Then there is some $s \in S_d$, $t \in T_e$, and $a \in \mathbb{Z}_{>0}$ such that as, at, and ax are all integral and $\alpha^{ax}_{as,at} \neq 0$. That is,

$$\vartheta_{ax} \text{ is a non-zero summand of } \vartheta_{as}\cdot\vartheta_{at}. \tag{9}$$

Now, since $s \in S_d$, there exist $s_1,\ldots,s_d \in S$ and $b_1,\ldots,b_d \in \mathbb{Z}_{\geq 0}$ such that $\alpha^{\mathbf{b}\,s}_{b_1 s_1,\ldots,b_d s_d} \neq 0$, where $\mathbf{b} := b_1+\cdots+b_d$. That is,

$$\vartheta_{\mathbf{b}\,s} \text{ is a non-zero summand of } \vartheta_{b_1 s_1}\cdots\vartheta_{b_d s_d}. \tag{10}$$

If any of these integers b_i is 0, we may simply replace d by a smaller d' using Lemma 21. So, we may assume $b_i > 0$ for all $i \in \{1,\ldots,d\}$. Similarly, since $t \in T_e$, there exist $t_1,\ldots,t_e \in T$ and $c_1,\ldots,c_e \in \mathbb{Z}_{\geq 0}$ such that $\alpha^{\mathbf{c}\,t}_{c_1 t_1,\ldots,c_e t_e} \neq 0$, where $\mathbf{c} := c_1+\cdots+c_e$. That is,

$$\vartheta_{\mathbf{c}\,t} \text{ is a non-zero summand of } \vartheta_{c_1 t_1}\cdots\vartheta_{c_e t_e}. \tag{11}$$

As before, we may assume $c_j > 0$ for all $j \in \{1,\ldots,e\}$.

Rescaling coefficients using Lemma 14, the facts (9), (10), and (11) imply

$$\vartheta_{a\,\mathbf{b}\,\mathbf{c}\,x} \text{ is a non-zero summand of } \vartheta_{a\,\mathbf{b}\,\mathbf{c}\,s}\cdot\vartheta_{a\,\mathbf{b}\,\mathbf{c}\,t}, \tag{12}$$

$$\vartheta_{a\,\mathbf{b}\,\mathbf{c}\,s} \text{ is a non-zero summand of } \vartheta_{a\,b_1\,\mathbf{c}\,s_1}\cdots\vartheta_{a\,b_d\,\mathbf{c}\,s_d}, \tag{13}$$

and

$$\vartheta_{a\,\mathbf{b}\,\mathbf{c}\,t} \text{ is a non-zero summand of } \vartheta_{a\,\mathbf{b}\,c_1\,t_1}\cdots\vartheta_{a\,\mathbf{b}\,c_e\,t_e} \tag{14}$$

respectively. Moreover, using Lemma 15, we have that

$$\vartheta_{a\,b_i\,\mathbf{c}\,s_i} \text{ is a non-zero summand of } \vartheta_{a\,b_i\,c_1\,s_i}\cdots\vartheta_{a\,b_i\,c_e\,s_i} \tag{15}$$

and

$$\vartheta_{a\,\mathbf{b}\,c_j\,t_j} \text{ is a non-zero summand of } \vartheta_{a\,b_1\,c_j\,t_j} \cdots \vartheta_{a\,b_d\,c_j\,t_j}. \tag{16}$$

Next, using Remarks 12 and 13, the facts (12), (13), (14), (15), and (16) together imply

$$\vartheta_{a\,\mathbf{b}\,\mathbf{c}\,x} \text{ is a non-zero summand of } \prod_{\substack{i\in\{1,\dots,d\}\\ j\in\{1,\dots,e\}}} \vartheta_{ab_ic_js_i} \cdot \vartheta_{ab_ic_jt_j}. \tag{17}$$

Expanding each product $\vartheta_{ab_ic_js_i} \cdot \vartheta_{ab_ic_jt_j}$ and using Remark 12 once more, we find that

$$\vartheta_{a\,\mathbf{b}\,\mathbf{c}\,x} \text{ is a non-zero summand of } \prod_{\substack{i\in\{1,\dots,d\}\\ j\in\{1,\dots,e\}}} \vartheta_{ab_ic_jr_{ij}} \tag{18}$$

for some collections of elements $\{r_{ij} \in S +_\vartheta T : i \in \{1,\dots,d\}\,,\, j \in \{1,\dots,e\}\}$. Finally, observe that

$$\sum_{\substack{i\in\{1,\dots,d\}\\ j\in\{1,\dots,e\}}} a\,b_i\,c_j = a\,\mathbf{b}\,\mathbf{c}. \tag{19}$$

Thus, (18) and (19) imply $x \in (S +_\vartheta T)_{de}$, as claimed. □

Corollary 27. *Let S and T be subsets of $U^{\mathrm{trop}}(\mathbb{Q})$. Then*

$$\mathrm{conv}_{\mathrm{BL}}(S) +_\vartheta \mathrm{conv}_{\mathrm{BL}}(T) \subset \mathrm{conv}_{\mathrm{BL}}(S +_\vartheta T).$$

Proof. This is an immediate consequence of Corollary 25 and Lemma 26. □

Combining Corollary 19 and Corollary 27, we obtain that the tropical Minkowski sum and broken line convex hull are compatible in the following sense:

Theorem 28. *Let S and T be subsets of $U^{\mathrm{trop}}(\mathbb{Q})$. Then*

$$\mathrm{conv}_{\mathrm{BL}}(S +_\vartheta T) = \mathrm{conv}_{\mathrm{BL}}(S) +_\vartheta \mathrm{conv}_{\mathrm{BL}}(T).$$

To conclude this section, we provide another result relating the tropical Minkowski sum and broken line convex hull. It will come in handy in later sections.

Proposition 29. *Let $S = \bigcup_{i\in I} S^i$, where each $S^i \subset U^{\mathrm{trop}}(\mathbb{Q})$ is broken line convex. Then*

$$\mathrm{conv}_{\mathrm{BL}}(S) = \bigcup_{\substack{(a_i:i\in I)\in(\mathbb{Q}_{\geq 0})^I\\ \sum_{i\in I} a_i = 1}} \left({\sum_{i\in I}}_\vartheta\, a_i S^i \right). \tag{20}$$

Proof. First, let $(a_i : i \in I) \in (\mathbb{Q}_{\geq 0})^I$ with $\sum_{i\in I} a_i = 1$, and let $s \in {\sum_{i\in I}}_\vartheta a_i S^i$. Each a_iS^i is broken line convex since each S^i is. If $|I| = 1$, there is nothing to show. Next let $I = \{1, 2\}$. Then there is some $x_1 \in a_1S^1$, $x_2 \in a_2S^2$, and $c \in \mathbb{Z}_{>0}$ such that $c\,x_1$, $c\,x_2$, and $c\,s$ are all integral and $\alpha^{c\,s}_{c\,x_1,c\,x_2} \neq 0$. The case in which either a_i is zero reduces to the $|I| = 1$ case, so we may assume each a_i is non-zero. Write $a_i = \frac{n_i}{d_i}$, with $n_i, d_i \in \mathbb{Z}_{>0}$. Then $\alpha^{cd_1d_2s}_{cd_1d_2x_1,cd_1d_2x_2} \neq 0$ as well by Lemma 14. But $cd_1d_2x_1 \in cd_2n_1S^1$ and $cd_1d_2x_2 \in cd_1n_2S^2$. So [CMN21, Proposition 4.10, Theorem 6.1] implies there is a broken line segment from $\frac{d_1}{n_1}x_1 = {a_1}^{-1}x_1$ to $\frac{d_2}{n_2}x_2 = {a_2}^{-1}x_2$ passing through $\frac{d_1d_2}{d_2n_1+d_1n_2}s = (a_1 + a_2)^{-1}s = s$. Since ${a_i}^{-1}x_i \in S^i \subset S$, this implies $s \in \mathrm{conv}_{\mathrm{BL}}(S)$. Now suppose the right side of (20) is contained in the left whenever $|I| < r$, and consider the case $I=\{1,\dots,r\}$. If any $a_i=0$, we return to the $|I|<r$ case. So assume each a_i is non-zero. Let $\mathbf{a}=a_1+\cdots+a_{r-1}$, and let $a_i'=\frac{a_i}{\mathbf{a}}$ for $i\in I\setminus\{r\}=:I'$. By the induction hypothesis, we know that

$$\sum_{i\in I'}{}_{\vartheta}\, a_i' S^i \subset \mathrm{conv}_{\mathrm{BL}}\left(\bigcup_{i\in I'} S^i\right) =: S'.$$

So, $s\in \sum_{i\in I}{}_{\vartheta}\, a_i S^i \subset \mathbf{a}S' +_{\vartheta} a_r S^r$. But by the induction hypothesis, $\mathbf{a}S' +_{\vartheta} a_r S^r \subset \mathrm{conv}_{\mathrm{BL}}(S'\cup S^r) = \mathrm{conv}_{\mathrm{BL}}(S)$. So

$$\bigcup_{\substack{(a_i:i\in I)\in(\mathbb{Q}_{\geq 0})^I\\ \sum_{i\in I} a_i=1}} \left(\sum_{i\in I}{}_{\vartheta}\, a_i S^i\right) \subset \mathrm{conv}_{\mathrm{BL}}(S).$$

Now suppose $s\in\mathrm{conv}_{\mathrm{BL}}(S)$. By Corollary 25, $s\in S_d$ (from Definition 20) for some $d\in\mathbb{Z}_{>0}$. So, we can find some $s_1,\dots,s_d\in S$ and $a_1,\dots,a_d\in\mathbb{Z}_{\geq 0}$ with $a_1+\cdots+a_d\neq 0$, the tropical points $a_1\, s_1,\dots,a_d\, s_d$, and $(a_1+\cdots+a_d)s$ all integral, and the structure constant $\alpha^{(a_1+\cdots+a_d)s}_{a_1\, s_1,\dots,a_d\, s_d}\neq 0$. Each s_j is in some S^i. Let $\bigcup_{i\in I} J_i$ be a decomposition of $\{1,\dots,d\}$ as a disjoint union such that $j\in J_i$ only if $s_j\in S^i$.[3] Now we have that

$$\vartheta_{(a_1+\cdots+a_d)s} \text{ is a non-zero summand of } \prod_{i\in I}\left(\prod_{j\in J_i}\vartheta_{a_j s_j}\right) = \prod_{i\in I}\left(\sum_{x\in U^{\mathrm{trop}}(\mathbb{Z})} \alpha^{x}_{\{a_j s_j : j\in J_i\}}\vartheta_x\right).$$

By Remark 12, we can find a collection $\left\{x_i\in U^{\mathrm{trop}}(\mathbb{Z}) : i\in I, \alpha^{x_i}_{\{a_j s_j : j\in J_i\}}\neq 0\right\}$ such that

$$\vartheta_{(a_1+\cdots+a_d)s} \text{ is a non-zero summand of } \prod_{i\in I}\vartheta_{x_i}.$$

Since S^i is broken line convex, $x_i\in(\sum_{j\in J_i} a_j)S^i$. Then

$$(a_1+\cdots+a_d)s \in \sum_{i\in I}{}_{\vartheta}\, \mathbf{a}_{J_i} S^i,$$

where $\mathbf{a}_{J_i} := \sum_{j\in J_i} a_j$, and

$$s\in \sum_{i\in I}{}_{\vartheta}\, \frac{\mathbf{a}_{J_i}}{(a_1+\cdots+a_d)} S^i.$$

We conclude that

$$\mathrm{conv}_{\mathrm{BL}}(S) \subset \bigcup_{\substack{(a_i:i\in I)\in(\mathbb{Q}_{\geq 0})^I\\ \sum_{i\in I} a_i=1}} \left(\sum_{i\in I}{}_{\vartheta}\, a_i S^i\right)$$

as well. □

4 Convexity for functions on $U^{\mathrm{trop}}(\mathbb{Q})$

In this section we describe what it means for a function on $U^{\mathrm{trop}}(\mathbb{Q})$ to be convex, and we prove some key results about these convex functions.

[3] The point here is that s_j may be contained in multiple S^i's. We simply choose one such i.

Definition 30. Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be a broken line convex set. A function $\varphi : S \to \mathbb{Q}$ is *convex with respect to broken lines* if for any broken line segment $\gamma : [t_1, t_2] \to S$, we have that

$$\varphi(\gamma(t)) \geq \left(\frac{t_2 - t}{t_2 - t_1}\right) \varphi(\gamma(t_1)) + \left(\frac{t - t_1}{t_2 - t_1}\right) \varphi(\gamma(t_2)) \tag{21}$$

for all $t \in [t_1, t_2]$.

Proposition 31. *Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be broken line convex, and let $\varphi : S \to \mathbb{Q}$ be convex with respect to broken lines. If $s_1, \ldots, s_d, s \in S$, $a_1, \ldots, a_d \in \mathbb{Q}_{\geq 0}$ with $a_1 s_1, \ldots, a_d s_d$, and $(a_1 + \cdots + a_d)s$ all integral, and $\alpha^{(a_1+\cdots+a_d)s}_{a_1 s_1, \ldots, a_d s_d} \neq 0$, then*

$$\varphi(s) \geq \sum_{i=1}^{d} \frac{a_i}{a_1 + \cdots + a_d} \varphi(s_i). \tag{22}$$

Proof. Note first that if $d = 1$, the inequality trivially becomes an equality. For $d = 2$, suppose we have s_1, s_2, s, a_1, and a_2 as in the proposition statement. Assume for now that a_1 and a_2 are integral. Then by [CMN21, Proposition 4.10, Theorem 6.1] there exists some broken line segment $\gamma : [0, \tau] \to U^{\mathrm{trop}}(\mathbb{Q})$ with $\gamma(0) = s_1$, $\gamma(\tau) = s_2$, and $\gamma\left(\frac{a_2}{a_1+a_2}\tau\right) = s$. Next, if a_1 and a_2 are only rational, we can clear denominators, writing $a_1' = \lambda a_1$ and $a_2' = \lambda a_2$. By Lemma 14, $\alpha^{(a_1'+a_2')s}_{a_1' s_1, a_2' s_2} \neq 0$ as well. Thus we obtain a broken line segment $\gamma : [0, \tau] \to U^{\mathrm{trop}}(\mathbb{Q})$ with $\gamma(0) = s_1$, $\gamma(\tau) = s_2$, and $\gamma\left(\frac{a_2'}{a_1'+a_2'}\tau\right) = s$. Note however that $\frac{a_2'}{a_1'+a_2'} = \frac{a_2}{a_1+a_2}$, so we have precisely the same outcome as the case of integral coefficients.

Since φ is convex with respect to broken lines, we have

$$\varphi\left(\gamma\left(\frac{a_2}{a_1 + a_2}\tau\right)\right) \geq \left(1 - \frac{a_2}{a_1 + a_2}\right) \varphi(\gamma(0)) + \frac{a_2}{a_1 + a_2} \varphi(\gamma(\tau)),$$

so

$$\varphi\left(s\right) \geq \frac{a_1}{a_1 + a_2} \varphi(s_1) + \frac{a_2}{a_1 + a_2} \varphi(s_2).$$

This establishes the claim for $d = 2$. Next, suppose the claim holds for $d = k$. If $s_1, \ldots, s_{k+1}, s$, $a_1, \ldots, a_{k+1}$ are as in the proposition statement, then $\alpha^{(a_1+\cdots+a_{k+1})s}_{a_1 s_1, \ldots, a_{k+1} s_{k+1}} \neq 0$. That is,

$$\vartheta_{(a_1+\cdots+a_{k+1})s} \text{ is a non-zero summand of and } \prod_{i=1}^{k+1} \vartheta_{a_i s_i}.$$

Expanding the first k terms of the product and using linear independence of theta functions, we see that

$$\vartheta_{(a_1+\cdots+a_{k+1})s} \text{ must be a non-zero summand of } \vartheta_{(a_1+\cdots+a_k)s'} \vartheta_{a_{k+1} s_{k+1}}$$

for some $(a_1 + \cdots + a_k)\, s'$ with $\alpha^{(a_1+\cdots+a_k)s'}_{a_1 s_1, \ldots, a_k s_k} \neq 0$. So, by the induction hypothesis we have

$$\begin{aligned}
\varphi(s) &\geq \frac{a_1 + \cdots + a_k}{a_1 + \cdots + a_{k+1}} \varphi(s') + \frac{a_{k+1}}{a_1 + \cdots + a_{k+1}} \varphi(s_{k+1}) \\
&\geq \frac{a_1 + \cdots + a_k}{a_1 + \cdots + a_{k+1}} \left(\sum_{i=1}^{k} \frac{a_i}{a_1 + \cdots + a_k} \varphi(s_i)\right) + \frac{a_{k+1}}{a_1 + \cdots + a_{k+1}} \varphi(s_{k+1}) \\
&= \sum_{i=1}^{k+1} \frac{a_i}{a_1 + \cdots + a_{k+1}} \varphi(s_i)
\end{aligned}$$

proving the claim. $\square$

Remark 32. In fact, Proposition 31 provides an equivalent characterization of functions $\varphi : S \to \mathbb{Q}$ which are convex with respect to broken lines. That is, we also have the opposite implication. Suppose for any $s_1, \ldots, s_d, s \in S$, $a_1, \ldots, a_d \in \mathbb{Q}_{\geq 0}$ with $a_1 s_1, \ldots, a_d s_d$, and $(a_1 + \cdots + a_d)s$ all integral, and $\alpha^{(a_1+\cdots+a_d)s}_{a_1 s_1, \ldots, a_d s_d} \neq 0$, we have

$$\varphi(s) \geq \sum_{i=1}^{d} \frac{a_i}{a_1 + \cdots + a_d} \varphi(s_i).$$

Then we claim φ is convex with respect to broken lines. To see this, consider a broken line segment $\gamma : [t_1, t_2] \to S$, and let $\overline{\gamma}$ be the reparametrized broken line segment $\overline{\gamma} : [0, \tau = t_2 - t_1] \to S$ defined by $\overline{\gamma}(t) = \gamma(t - t_1)$. Clearly,

$$\varphi(\gamma(t)) \geq \left(\frac{t_2 - t}{t_2 - t_1}\right) \varphi(\gamma(t_1)) + \left(\frac{t - t_1}{t_2 - t_1}\right) \varphi(\gamma(t_2))$$

for all $t \in [t_1, t_2]$ if and only if

$$\varphi(\overline{\gamma}(t)) \geq \frac{\tau - t}{\tau} \varphi(\overline{\gamma}(0)) + \frac{t}{\tau} \varphi(\overline{\gamma}(\tau))$$

for all $t \in [0, \tau]$. We may always write $t = \frac{b}{a+b}\tau$. By [CMN21, Proposition 5.4, Theorem 6.1], we may choose a and b such that $a\,\overline{\gamma}(0)$, $b\,\overline{\gamma}(\tau)$, and $(a+b)\,\overline{\gamma}(\tau)$ are all integral and $\alpha^{(a+b)\,\overline{\gamma}(\tau)}_{a\,\overline{\gamma}(0), b\,\overline{\gamma}(\tau)} \neq 0$. Then

$$\begin{aligned}
\varphi\left(\overline{\gamma}\left(\frac{b}{a+b}\tau\right)\right) &\geq \frac{a}{a+b} \varphi\left(\overline{\gamma}\left(0\right)\right) + \frac{b}{a+b} \varphi\left(\overline{\gamma}\left(\tau\right)\right) \\
&= \frac{\tau - t}{\tau} \varphi\left(\overline{\gamma}\left(0\right)\right) + \frac{t}{\tau} \varphi\left(\overline{\gamma}\left(\tau\right)\right).
\end{aligned}$$

Proposition 31 and Remark 32 allow us to resolve a question posed by Gross-Hacking-Keel-Kontsevich in [GHKK18, Remark 8.5], and we take a slight detour to do so here.

Proposition 33. *A piecewise linear function $\varphi : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ is convex with respect to broken lines if and only if it is decreasing in the sense of [GHKK18, Definition 8.3].*

Proof. First, suppose φ is convex with respect to broken lines. Let s_1, s_2, and r be in $U^{\mathrm{trop}}(\mathbb{Z})$ and satisfy $\alpha^{r}_{s_1,s_2} \neq 0$. We need to show that $\varphi(r) \geq \varphi(s_1) + \varphi(s_2)$. Comparing to Proposition 31, we have $a_1 = a_2 = 1$, and $r = 2s$. Then

$$\varphi(s) \geq \frac{1}{2}\varphi(s_1) + \frac{1}{2}\varphi(s_2)$$

and

$$\varphi(r) = \varphi(2s) = 2\varphi(s) \geq \varphi(s_1) + \varphi(s_2).$$

That is, φ is decreasing.

For the other direction, we use an induction argument very similar to the one used in Proposition 31. Suppose φ is decreasing. Let $s_1, \ldots, s_d, s \in U^{\mathrm{trop}}(\mathbb{Q})$ and $a_1, \ldots, a_d \in \mathbb{Q}_{\geq 0}$, with $a_1\,s_1, \ldots, a_d\,s_d$, and $(a_1 + \cdots + a_d)s$ all integral and $\alpha^{(a_1+\cdots+a_d)s}_{a_1 s_1, \ldots, a_d s_d} \neq 0$. We need to show that (22) holds for φ. For the $d = 1$ case, (22) trivially reduces to an equality. For $d = 2$, since φ is decreasing we have

$$\varphi((a_1 + a_2)s) \geq \varphi(a_1 s_1) + \varphi(a_2 s_2),$$

which implies

$$\varphi(s) \geq \frac{a_1}{a_1 + a_2}\varphi(s_1) + \frac{a_2}{a_1 + a_2}\varphi(s_2)$$

by piecewise linearity of φ. So (22) holds for $d = 2$. Now assume it holds for $d = k$, and consider the case $d = k + 1$. As we argued in Proposition 31,

$$\vartheta_{(a_1+\cdots+a_{k+1})s} \text{ must be a non-zero summand of } \vartheta_{(a_1+\cdots+a_k)s'}\vartheta_{a_{k+1}\, s_{k+1}}$$

for some $(a_1 + \cdots + a_k)\, s'$ with $\alpha^{(a_1+\cdots+a_k)s'}_{a_1\, s_1,\ldots,a_k\, s_k} \neq 0$. So, by the induction hypothesis we have

$$\begin{aligned}
\varphi(s) &\geq \frac{a_1 + \cdots + a_k}{a_1 + \cdots + a_{k+1}}\varphi(s') + \frac{a_{k+1}}{a_1 + \cdots + a_{k+1}}\varphi(s_{k+1}) \\
&\geq \frac{a_1 + \cdots + a_k}{a_1 + \cdots + a_{k+1}}\left(\sum_{i=1}^{k} \frac{a_i}{a_1 + \cdots + a_k}\varphi(s_i)\right) + \frac{a_{k+1}}{a_1 + \cdots + a_{k+1}}\varphi(s_{k+1}) \\
&= \sum_{i=1}^{k+1} \frac{a_i}{a_1 + \cdots + a_{k+1}}\varphi(s_i),
\end{aligned}$$

which proves the claim. □

Proposition 34. *A piecewise linear function $\varphi : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ is convex with respect to broken lines if and only if it is min-convex in the sense of [GHKK18, Definition 8.2].*

Proof. First, suppose φ is convex with respect to broken lines. We need to verify that $d\varphi$ is decreasing on $\dot{\gamma}$ for all broken lines γ. Suppose γ crosses a wall at time τ. Then for sufficiently small $\epsilon > 0$, we have $\varphi(\gamma(\tau \pm \epsilon)) = \varphi(\gamma(\tau)) \pm \epsilon d\varphi_{\gamma(\tau\pm\epsilon)}(\dot{\gamma}(\tau \pm \epsilon))$ and

$$\varphi(\tau) \geq \frac{1}{2}\left(\varphi(\tau) - \epsilon d\varphi_{\gamma(\tau-\epsilon)}(\dot{\gamma})\right) + \frac{1}{2}\left(\varphi(\tau) + \epsilon d\varphi_{\gamma(\tau+\epsilon)}(\dot{\gamma})\right).$$

Simplifying, we find $d\varphi_{\gamma(\tau-\epsilon)}(\dot{\gamma}) \geq d\varphi_{\gamma(\tau+\epsilon)}(\dot{\gamma})$ as desired.

The other direction follows from [GHKK18, Lemma 8.4] and Proposition 33. □

Taken together, Propositions 33 and 34 resolve a question posed in [GHKK18, Remark 8.5]:

Corollary 35. *The notions "min-convex" and "decreasing" of [GHKK18, Definitions 8.2 & 8.3] are equivalent.*

We now state and prove some basic results about functions which are convex with respect to broken lines.

Lemma 36. *Let φ_1, $\varphi_2 : S \to \mathbb{Q}$ be convex with respect to broken lines. Then $\varphi_1 + \varphi_2$ is convex with respect to broken lines.*

Proof. This follows immediately from Definition 30. □

Proposition 37. *Let $\varphi : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ be convex with respect to broken lines. Then*

$$\Xi_{\varphi,r} := \left\{x \in U^{\mathrm{trop}}(\mathbb{Q}) \,:\, \varphi(x) \geq -r\right\}$$

is broken line convex.

Proof. By Proposition 17, this holds if and only if $\Xi_{\varphi,r} = t\,\Xi_{\varphi,r} +_{\vartheta} (1-t)\Xi_{\varphi,r}$. We always have the inclusion $\Xi_{\varphi,r} \subset t\,\Xi_{\varphi,r} +_{\vartheta} (1-t)\Xi_{\varphi,r}$, so we just need to show the opposite inclusion. Let $z \in t\,\Xi_{\varphi,r} +_{\vartheta} (1-t)\Xi_{\varphi,r}$. Then there exists $x \in t\,\Xi_{\varphi,r}$, $y \in (1-t)\Xi_{\varphi,r}$, and $a \in \mathbb{Z}_{>0}$ such that ax, ay, and az are all integral and $\alpha^{az}_{ax,ay} \neq 0$. Define x', $y' \in \Xi_{\varphi,r}$ by $x = tx'$, $y = (1-t)y'$. Now, $a = ta + (1-t)a$, so $0 \neq \alpha^{az}_{ax,ay} = \alpha^{az}_{atx',a(1-t)y'}$. Then by Proposition 31,

$$\begin{aligned}
\varphi(z) &\geq t\varphi(x') + (1-t)\varphi(y') \\
&\geq t(-r) + (1-t)(-r) \\
&= -r.
\end{aligned}$$

That is, $z \in \Xi_{\varphi,r}$. □

Lemma 38. *Let $\varphi : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ be convex with respect to broken lines, and let $\gamma : [t_1, t_2] \to U^{\mathrm{trop}}(\mathbb{Q})$ be a broken line segment satisfying*

$$\varphi(\gamma(t)) = \left(\frac{t_2 - t}{t_2 - t_1}\right)\varphi(\gamma(t_1)) + \left(\frac{t - t_1}{t_2 - t_1}\right)\varphi(\gamma(t_2))$$

for some $t \in (t_1, t_2)$. Then

$$\varphi(\gamma(t)) = \left(\frac{t_2 - t}{t_2 - t_1}\right)\varphi(\gamma(t_1)) + \left(\frac{t - t_1}{t_2 - t_1}\right)\varphi(\gamma(t_2))$$

for all $t \in [t_1, t_2]$.

Proof. Suppose not. Then there is some $t' \in (t_1, t_2)$[4], $t' \neq t$, with

$$\varphi(\gamma(t')) > \left(\frac{t_2 - t'}{t_2 - t_1}\right)\varphi(\gamma(t_1)) + \left(\frac{t' - t_1}{t_2 - t_1}\right)\varphi(\gamma(t_2)). \tag{23}$$

The argument is identical for $t' < t$ and $t' > t$, so without loss of generality, take $t' < t$. Since φ is convex with respect to broken lines, by restricting γ to $[t', t_2]$ we find

$$\varphi(\gamma(t)) \geq \left(\frac{t_2 - t}{t_2 - t'}\right)\varphi(\gamma(t')) + \left(\frac{t - t'}{t_2 - t'}\right)\varphi(\gamma(t_2)).$$

That is,

$$\left(\frac{t_2 - t}{t_2 - t_1}\right)\varphi(\gamma(t_1)) + \left(\frac{t - t_1}{t_2 - t_1}\right)\varphi(\gamma(t_2)) \geq \left(\frac{t_2 - t}{t_2 - t'}\right)\varphi(\gamma(t')) + \left(\frac{t - t'}{t_2 - t'}\right)\varphi(\gamma(t_2)),$$

which upon simplifying yields

$$\left(\frac{t_2 - t'}{t_2 - t_1}\right)\varphi(\gamma(t_1)) + \left(\frac{t' - t_1}{t_2 - t_1}\right)\varphi(\gamma(t_2)) \geq \varphi(\gamma(t')).$$

This contradicts the strict inequality (23). $\square$

One type of function that will come up frequently in the remainder of the paper is simply given by evaluation: $\langle\,\cdot\,, y\rangle : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$. For this reason, we introduce the following terminology.

Definition 39. We say a function $\varphi : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ is *tropically linear* if $\varphi = \langle\,\cdot\,, y\rangle$ for some $y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$. We also use the terminology for a function ψ on a $\mathbb{Q}_{\geq 0}$-invariant subset σ in $U^{\mathrm{trop}}(\mathbb{Q})$ if there exists an extension of ψ from σ to $U^{\mathrm{trop}}(\mathbb{Q})$ which is tropically linear.

The following results are a corollaries of Theorem 9.

Corollary 40. *Let $\varphi : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ be tropically linear, and consider a collection of integral tropical points $x_1, \ldots, x_d \in U^{\mathrm{trop}}(\mathbb{Z})$. Then*

$$\sum_{i=1}^{d} \varphi(x_i) = \min\left\{\varphi(x) : x \in U^{\mathrm{trop}}(\mathbb{Z}), \alpha^x_{x_1,\ldots,x_d} \neq 0\right\}.$$

Proof. Let $\varphi = \langle\,\cdot\,, y_\varphi\rangle$, and let ay_φ be integral for some $a > 0$. Then

$$ay_\varphi\left(\vartheta_{x_1}\right) + \cdots + ay_\varphi\left(\vartheta_{x_d}\right) = ay_\varphi\left(\vartheta_{x_1}\cdots\vartheta_{x_d}\right) = \min\left\{ay_\varphi(\vartheta_x) : x \in U^{\mathrm{trop}}(\mathbb{Z}), \alpha^x_{x_1,\ldots,x_d} \neq 0\right\}.$$

But then

$$\sum_{i=1}^{d} \varphi(x_i) = \min\left\{\varphi(x) : x \in U^{\mathrm{trop}}(\mathbb{Z}), \alpha^x_{x_1,\ldots,x_d} \neq 0\right\}$$

as claimed. $\square$

[4]We take the open interval here since equality is clear for the endpoints t_1 and t_2.

Corollary 41. *A tropically linear function is convex with respect to broken lines.*

Proof. Let $\varphi : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ be tropically linear, and consider any $x_1, \dots, x_d, x \in U^{\mathrm{trop}}(\mathbb{Q})$, $a_1, \dots, a_d \in \mathbb{Q}_{\geq 0}$ with $a_1 x_1, \dots, a_d x_d$, and $(a_1 + \cdots + a_d)x$ all integral, and $\alpha^{(a_1+\cdots+a_d)x}_{a_1x_1,\dots,a_dx_d} \neq 0$. By Corollary 40,

$$\sum_{i=1}^{d} \varphi(a_i x_i) = \min \left\{ \varphi(s) : s \in U^{\mathrm{trop}}(\mathbb{Z}), \alpha^{s}_{a_1x_1,\dots,a_dx_d} \neq 0 \right\}.$$

So, we have that

$$\varphi\left((a_1 + \cdots + a_d)x\right) \geq \sum_{i=1}^{d} \varphi(a_i x_i),$$

and

$$\varphi\left(x\right) \geq \sum_{i=1}^{d} \frac{a_i}{a_1 + \cdots + a_d} \varphi(x_i).$$

By Remark 32, φ is convex with respect to broken lines. □

5 Broken line convex polyhedral geometry

Let U be a cluster variety for which the full Fock-Goncharov conjecture holds and let $U^\vee$ be its Fock-Goncharov dual.

We borrow some notation from [Brø83].

Definition 42. For $y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ and $r \in \mathbb{Q}$ denote by $K(y,r)$ the set $\{x \in U^{\mathrm{trop}}(\mathbb{Q}) : \langle x, y \rangle \geq -r\}$. We call $K(y,r)$ a *tropical half-space*, and we call its boundary $H(y,r) := \{x \in U^{\mathrm{trop}}(\mathbb{Q}) : \langle x, y \rangle = -r\}$ a *tropical hyperplane*. For $S \subset U^{\mathrm{trop}}(\mathbb{Q})$, we say $K(y,r)$ is a *supporting tropical half-space for* S and $H(y,r)$ is a *supporting tropical hyperplane for* S if $S \subset K(y,r)$ and $S \cap H(y,r) \neq \varnothing$. We define tropical half-spaces and hyperplanes in $(U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ analogously.

Remark 43. As $\langle \,\cdot\,, ay \rangle = a \langle \,\cdot\,, y \rangle$ for all $a > 0$, we have that $K(y,r) = K(ay, ar)$ for all $a > 0$.

Lemma 44. *A tropical half-space is broken line convex.*

Proof. By Corollary 41, a tropically linear function is convex with respect to broken lines. Then the claim follows from Proposition 37. □

Definition 45. A subset $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ is *polyhedral* if

$$S = \bigcap_{i \in I} K(y_i, r_i)$$

for some finite indexing set I. We will always take $y_i \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ and $r_i \in \mathbb{Q}$.[5] If additionally S is bounded, we say it is *polytopal*.

5.1 Faces

Definition 46. Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be broken line convex. We say that a subset F of ∂S is a *face* of S if there is a tropical half-space $K(y,r) \supset S$ with $F = S \cap H(y,r)$. We say this face F is a *proper* face if $K(y,r)$ is a supporting tropical half-space. We call 0-dimensional faces *vertices*, 1-dimensional faces *edges*, and codimension 1 faces *facets*. By convention, we view $\varnothing$ as a -1-dimensional face. We denote the set of faces of S by $\mathcal{F}_S$.

[5] In usual convex geometry, this reduces to the notion of "rational polyhedral". As we only work in the rational setting in this paper, we drop the "rational" descriptor from our terminology here.

Remark 47. We will typically discuss faces of polyhedral sets rather than arbitrary broken line convex sets. However, the definition makes sense for arbitrary broken line convex sets, and we will want to use the face terminology for certain sets prior to proving that they are in fact polyhedral.

Warning: *Unlike in usual convex geometry, faces in broken line convex geometry need not be broken line convex.*

For an example of this phenomenon, see Figure 1.

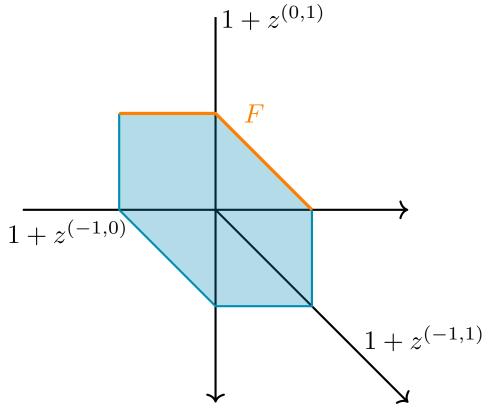

Figure 1: A polytopal set $S \subset (\mathcal{A}^\vee)^{\mathrm{trop}}(\mathbb{Q})$ for the $\mathcal{A}$ cluster variety of type A_2. The indicated face F is not broken line convex. As is standard, to draw this picture we identify $(\mathcal{A}^\vee)^{\mathrm{trop}}(\mathbb{Q})$ with $\mathbb{Q}^2$ via a choice of seed.

However, faces do satisfy some weaker notion of convexity. To motivate this weaker convexity notion, we make an observation about tropical hyperplanes.

Proposition 48. *Every pair of points x_1, x_2 in a tropical hyperplane $H(y, r)$ is connected by a broken line segment γ whose support is contained in $H(y, r)$.*

Proof. This is a simple corollary of Theorem 9 and Lemma 38. By the Theorem 9, there exists a broken line segment $\gamma : [t_1, t_2] \to U^{\mathrm{trop}}(\mathbb{Q})$ with endpoints x_1 and x_2 such that

$$\begin{aligned} \left\langle \gamma\left(\frac{t_1+t_2}{2}\right), y \right\rangle &= \frac{1}{2}\left\langle \gamma\left(t_1\right), y\right\rangle + \frac{1}{2}\left\langle \gamma\left(t_2\right), y\right\rangle \\ &= \frac{1}{2}\left\langle x_1, y\right\rangle + \frac{1}{2}\left\langle x_2, y\right\rangle \\ &= -r. \end{aligned}$$

Then by Lemma 38,

$$\begin{aligned} \varphi(\gamma(t)) &= -\left(\frac{t_2-t}{t_2-t_1}\right) r - \left(\frac{t-t_1}{t_2-t_1}\right) r \\ &= -r \end{aligned}$$

for all $t \in [t_1, t_2]$. That is, the support of γ is contained in $H(y, r)$. $\square$

In light of Proposition 48, we make the following definition.

Definition 49. We say a subset $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ is *weakly convex* if for every pair of points s_1, $s_2 \in S$, there exists a broken line segment with endpoints s_1 and s_2 whose support is contained in S.

Clearly, the notions *broken line convexity* and *weak convexity* coincide in usual convex geometry. They are very different notions in $U^{\mathrm{trop}}(\mathbb{Q})$, but both play important roles in in the theory of broken line convex geometry. In fact, the two convexity notions interact with each other:

Proposition 50. *Let S and S' be subsets of $U^{\mathrm{trop}}(\mathbb{Q})$ with S broken line convex and S' weakly convex. Then $S \cap S'$ is weakly convex.*

Proof. Let s_1, $s_2 \in S \cap S'$. Since S' is weakly convex, there exists a broken line segment γ with endpoints s_1 and s_2 whose support is contained in S'. Since S is broken line convex, the support of γ must be contained in S as well. Hence the support of γ is contained in $S \cap S'$, proving the claim. $\square$

Corollary 51. *Every face of a polyhedral set is weakly convex.*

Proof. By definition, a face F of a polyhedral set $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ is of the form

$$F = S \cap H(y, r)$$

for $H(y, r)$ a tropical hyperplane at the boundary of a tropical half-space $K(y, r)$ which contains S. The polyhedral set S is broken line convex, and by Proposition 48, $H(y, r)$ is weakly convex. $\square$

In usual convex geometry, the set of faces of a polyhedron forms a polyhedral complex. Unfortunately, in general the faces of a polyhedral set in $U^{\mathrm{trop}}(\mathbb{Q})$ will not form such a complex. For instance, if we consider the bigon of Figure 2, the intersection of the pair of facets is a pair of vertices– so in this instance the intersection of two faces is *not* a face, but rather a union of faces.

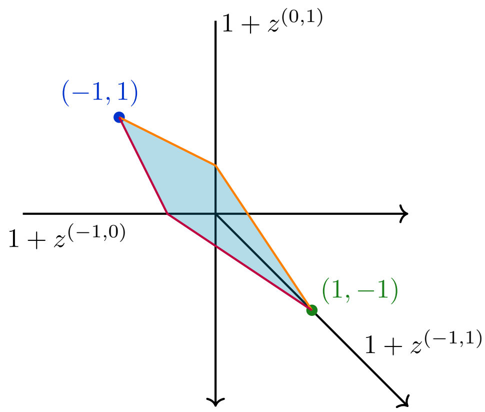

Figure 2: A bigon S in $(\mathcal{A}^{\vee})^{\mathrm{trop}}(\mathbb{Q})$ together with its faces $\mathcal{F}_S$ for the $\mathcal{A}$ cluster variety of type A_2. Note that the intersection of the facets is a pair of vertices rather than a single face.

Nevertheless, the set of faces of a polyhedral set has a structure very reminiscent of a polyhedral complex. To make this precise, we introduce the following definition:

Definition 52. Let $\mathcal{P}$ be a set of subsets of $U^{\mathrm{trop}}(\mathbb{Q})$. We say that $\mathcal{P}$ is a pseudo-complex if it has the following properties:

1. If $P \in \mathcal{P}$, then there is a subset $\mathcal{A}$ of $\mathcal{P}$ with
$$\partial P = \bigcup_{P' \in \mathcal{A}} P'.$$
2. If P_1, $P_2 \in \mathcal{P}$, then there is a subset $\mathcal{B}$ of $\mathcal{P}$ with
$$P_1 \cap P_2 = \bigcup_{P \in \mathcal{B}} P.$$
3. If P_1, $P_2 \in \mathcal{P}$ and $P_1 \cap P_2 \subsetneq P_1$, then
$$P_1 \cap P_2 \subset \partial P_1.$$

Proposition 53. *The set of faces $\mathcal{F}_S$ of a polyhedral set $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ forms a pseudo-complex.*

To establish Proposition 53, we will need a pair of lemmas:

Lemma 54. *Let $S = \bigcap_{i\in I} K(y_i, r_i)$, and let $y \in \sum_{i\in I}{}_{\vartheta}\, y_i$ and $r = \sum_{i\in I} r_i$. Then $S \subset K(y,r)$ and*

$$S \cap H(y,r) \subset \left(S \cap \bigcap_{i\in I} H(y_i, r_i) \right).$$

Proof. Let $x \in S$, then we have that $\langle x, y_i \rangle \geq -r_i$. By Proposition 31,

$$\langle x, y \rangle \geq \sum_{i\in I} \langle x, y_i \rangle \geq -\sum_{i\in I} r_i = -r.$$

So $x \in K(y,r)$.

Now let $x \in S \cap H(y,r)$. Then

$$r = \langle x, y \rangle \geq \sum_{i\in I} \langle x, y_i \rangle \geq -\sum_{i\in I} r_i = -r,$$

and we must have equality throughout. That is, $\langle x, y_i \rangle = -r_i$ and $x \in H(y_i, r_i)$ for all $i \in I$. $\square$

Lemma 55. *Let $\gamma : [t_1, t_2] \to U^{\mathrm{trop}}(\mathbb{Q})$ be a broken line segment whose support is contained in $H(y_1, r_1) \cap K(y_2, r_2)$. If $\mathrm{supp}(\gamma) \cap H(y_2, r_2)$ is one dimensional, then $\mathrm{supp}(\gamma) \subset H(y_2, r_2)$.*

Proof. By restricting the domain of γ and reversing the direction of γ as needed, we may reduce to the case in which $\gamma([t_1, \tau]) \subset H(y_1, r_1) \cap H(y_2, r_2)$ for some $\tau \in (t_1, t_2)$. Without hitting a wall, there is no way for γ to leave the intersection $H(y_1, r_1) \cap H(y_2, r_2)$, so suppose γ crosses the wall $(\mathfrak{d}, f_{\mathfrak{d}}(z^{m_{\mathfrak{d}}}))$, with $\mathfrak{d} \subset n_{\mathfrak{d}}^{\perp}$, at time τ. There are three possibilities for $\gamma(\tau + \epsilon)$ for small $\epsilon > 0$.

1. $\gamma(\tau + \epsilon) \in H(y_1, r_1) \cap H(y_2, r_2)$
2. $\langle \gamma(\tau + \epsilon), y_2 \rangle < -r_2$
3. $\langle \gamma(\tau + \epsilon), y_2 \rangle > -r_2$

We want eliminate Items 2 and 3. We immediately note that if $\langle \gamma(\tau + \epsilon), y_2 \rangle < -r_2$, then $\mathrm{supp}(\gamma) \not\subset K(y_2, r_2)$, eliminating Item 2.

Next, suppose $\langle \gamma(\tau + \epsilon), y_2 \rangle > -r_2$. Denote the velocity of γ immediately prior to crossing $(\mathfrak{d}, f_{\mathfrak{d}}(z^{m_{\mathfrak{d}}}))$ by $\dot{\gamma}_-$ and the velocity immediately after crossing by $\dot{\gamma}_+$. For some $k \geq 0$, we have $\dot{\gamma}_+ = \dot{\gamma}_- - k m_{\mathfrak{d}}$. We can give a new broken line segment $\gamma' : [t_1', t_2'] \to U^{\mathrm{trop}}(\mathbb{Q})$ crossing $(\mathfrak{d}, f_{\mathfrak{d}}(z^{m_{\mathfrak{d}}}))$ such that for some $\lambda > 0$ and some small $\delta > 0$

- $\gamma'(t_1') = \gamma(t_1)$,
- $\dot{\gamma}'_- = \lambda \left(\dot{\gamma}_- - \delta m_{\mathfrak{d}} \right)$,
- $\dot{\gamma}_+ = \lambda \left(\dot{\gamma}_+ - \delta m_{\mathfrak{d}} \right)$, and
- $\langle \gamma'(t_2'), y_2 \rangle > -r_2$.

The factor of λ above is simply to ensure we can make exponent vectors integral. With this in mind, since $\dot{\gamma}_+ = \dot{\gamma}_- - k m_{\mathfrak{d}}$ pertains to an allowed bend and $\langle n_{\mathfrak{d}}, m_{\mathfrak{d}} \rangle = 0$, for some $\lambda > 0$ we have that

$$\dot{\gamma}'_+ = \lambda \left(\dot{\gamma}_+ - \delta m_{\mathfrak{d}} \right) = \lambda \left((\dot{\gamma}_- - k m_{\mathfrak{d}}) - \delta m_{\mathfrak{d}} \right) = \lambda \left((\dot{\gamma}_- - \delta m_{\mathfrak{d}}) - k m_{\mathfrak{d}} \right) = \dot{\gamma}'_- - \lambda k m_{\mathfrak{d}}$$

is also an allowed bend. See Figure 3 for an illustration of this scenario.

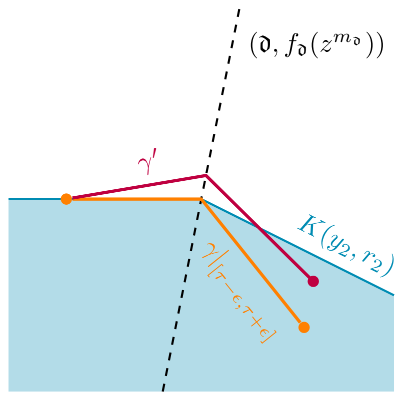

Figure 3: Schematic of $\gamma|_{[\tau-\epsilon,\tau+\epsilon]}$ and γ' as detailed above.

However, this broken line segment γ' has endpoints in $K(y_2, r_2)$, while having support not contained in $K(y_2, r_2)$. In particular, the point at which γ' crosses $(\mathfrak{d}, f_{\mathfrak{d}}(z^{m_{\mathfrak{d}}}))$ does not lie in $K(y_2, r_2)$. This contradicts broken line convexity of $K(y_2, r_2)$. □

Proof of Proposition 53. We begin with Item 1 of Definition 52. For the face $F = \varnothing$, the statement is vacuous. So consider a face $F \neq \varnothing$. Then F is of the form $F = H(y, r) \cap S$ for some supporting tropical hyperplane $H(y, r)$. Let

$$S = \bigcap_{i \in I} K(y_i, r_i)$$

be a presentation of S. Necessarily, the boundary of F is obtained by intersection with some of the tropical hyperplanes $H(y_i, r_i)$. Precisely, define

$$\mathcal{I}_F := \left\{ J \subset I \,:\, \left(F \cap \bigcap_{j \in J} H(y_j, r_j) \right) \subsetneq F \right\}.$$

Then

$$\partial F = F \cap \bigcup_{J \in \mathcal{I}_F} \bigcap_{j \in J} H(y_j, r_j).$$

For shorthand, write

$$F_J := \left(F \cap \bigcap_{j \in J} H(y_j, r_j) \right),$$

so $\partial F = \bigcup_{J \in \mathcal{I}_F} F_J$. Observe that

$$F_J = \left(H(y, r) \cap \bigcap_{j \in J} H(y_j, r_j) \right) \cap S.$$

Now set $r' = r + \sum_{j \in J} r_j$ and let $y' \in y +_{\vartheta} \sum_{\vartheta} y_j$. Then by Lemma 54,

$$\left(K(y, r) \cap \bigcap_{j \in J} K(y_j, r_j) \right) \subset K(y', r')$$

and

$$S \cap H(y', r') \subset F_J.$$

Moreover, by Corollary 40, for each $x \in F_J$, there exists some such $y'_x \in y +_\vartheta \sum_\vartheta y_j$ with $H(y'_x, r')$ a supporting tropical hyperplane containing x. That is, $F_x := H(y'_x, r') \cap S$ is a face of S containing x. This establishes Item 1.

Now we turn our attention to Item 2 of Definition 52, whose proof is very similar to the one above. As before, if $F_1 \cap F_2 = \varnothing$, the claim trivially holds. Suppose $F_1 \cap F_2 \neq \varnothing$, and let $F_i = H(y_i, r_i) \cap S$, where $H(y_i, r_i)$ is a supporting tropical hyperplane. Now Lemma 54 and Corollary 40 imply that for each $x \in F_1 \cap F_2$, there is a supporting tropical hyperplane $H(y_x, r_1 + r_2)$ containing x such that $K(y_1, r_1) \cap K(y_2, r_2) \subset K(y_x, r_1 + r_2)$ and $F_x := S \cap H(y_x, r_1 + r_2) \subset (S \cap H(y_1, r_1) \cap H(y_2, r_2)) = F_1 \cap F_2$, establishing Item 2.

Finally, for Item 3 of Definition 52, let $F_i = H(y_i, r_i) \cap S$. Since $F_1 \cap F_2$ is properly contained in F_1, for some $x \in F_1$ we have $\langle x, y_2 \rangle > -r_2$. However, tropically linear functions are continuous so this implies $\langle \,\cdot\, , y_2 \rangle > -r_2$ on an open neighborhood of x. Now consider a broken line segment contained in F_1 which begins at x and proceeds to some x' with $\langle x', y_2 \rangle = -r_2$. (If no such broken line segment exists, then $F_1 \cap F_2 = \varnothing$, and we are done.) By Lemma 55, this broken line segment *cannot* be extended in such a way that a positive length subsegment lies in $H(y_1, r) \cap H(y_2, r)$. Then x' must in fact lie at the boundary of F_1. $\square$

Proposition 56. *Let F be a face of a polyhedral subset $S \subset U^{\mathrm{trop}}(\mathbb{Q})$. Then F is not contained in $\mathrm{conv}_{\mathrm{BL}}(S \setminus F)$.*

Proof. F is of the form $F = H(y, r) \cap S$ for some supporting tropical hyperplane $H(y, r)$ for S. The open tropical half-space $K(y, r) \setminus H(y, r)$ is broken line convex, so its intersection with S is as well. But F is not contained in this intersection. $\square$

5.2 The weak face fan

The other vitally important polyhedral complex in the theory of toric varieties is the fan. To pursue our goal of a cluster version of Batyrev-Borisov duality, we will primarily be interested in a particular sort of fan, namely a face fan. So, we now turn our attention to defining the broken line convex geometry analogue of a face fan, and showing that it also forms a pseudo-complex.

Proposition 57. *If $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ is weakly convex, then so is $\mathbb{Q}_{\geq 0} \cdot S$.*

Proof. Consider an arbitrary pair of points $\lambda_1 s_1,\ \lambda_2 s_2 \in \mathbb{Q}_{\geq 0} \cdot S$. Let $\gamma : [0, T] \to U^{\mathrm{trop}}(\mathbb{Q})$ be a broken line segment with endpoints s_1 and s_2 whose support is contained in S. We will show that there is a broken line segment $\widetilde{\gamma} : [0, \widetilde{T}] \to U^{\mathrm{trop}}(\mathbb{Q})$ with endpoints $\lambda_1 s_1$ and $\lambda_2 s_2$ whose support is contained in $\mathbb{Q}_{\geq 0} \cdot S$. Let us address a few trivial cases before turning our attention to the generic setting. If $\lambda_1 = \lambda_2 =: \lambda$, then we can simply rescale the support of γ by λ while leaving the decoration monomials unchanged. The particular cases of $\lambda = 0$ and $\lambda = 1$ are the constant broken line segment with image the origin and the original broken line segment γ respectively. Next, if $\lambda_i \neq \lambda_j = 0$, we may take a straight segment between the origin and $\lambda_i s_i$.

The remaining cases are less obvious, but follow from results of [CMN21]. Assume λ_1 and λ_2 are both non-zero. As in [CMN21], denote the initial exponent vector of a broken line η by $I(\eta)$ and the exponent vector of near the endpoint of η by $\mathfrak{m}_0(\eta)$. Define $\tau := \frac{\lambda_2}{\lambda_1 + \lambda_2} T$. Then the algorithm of [CMN21, §5] produces a balanced pair of broken lines $(\gamma^{(1)}, \gamma^{(2)})$ where, for some $\mu > 0$,

- $I(\gamma^{(i)}) = \mu \lambda_i s_i$,
- $\mathfrak{m}_0(\gamma^{(1)}) + \mathfrak{m}_0(\gamma^{(2)}) = \mu(\lambda_1 + \lambda_2)\gamma(\tau)$, and
- $\gamma^{(1)}(0) = \gamma^{(2)}(0) = \mu(\lambda_1 + \lambda_2)\gamma(\tau)$.

Moreover, in this algorithm the bending points of the broken line segment γ are positively proportional to the bending points of the pair $(\gamma^{(1)}, \gamma^{(2)})$. (In the non-generic case in which $\gamma(\tau)$ is a bending point, the corresponding bend for the pair is by convention recorded in $\gamma^{(2)}$ in the algorithm.)

Next, we take this pair of broken lines $(\gamma^{(1)}, \gamma^{(2)})$, together with the pair of integers $a = b = 1$, as input for the algorithm of [CMN21, §4]. The result is a broken line segment $\overline{\gamma} : [0, \overline{T}] \to U^{\mathrm{trop}}(\mathbb{Q})$ with

$\overline{\gamma}(0) = I(\gamma^{(1)}) = \mu\lambda_1 s_1$ and $\overline{\gamma}(\overline{T}) = I(\gamma^{(2)}) = \mu\lambda_2 s_2$, passing through $\frac{\mu}{2}(\lambda_1+\lambda_2)\gamma(\tau)$ at time $\frac{1}{2}\overline{T}$. As before, in this algorithm the bending points of the broken line segment $\overline{\gamma}$ are positively proportional to the bending points of the pair $(\gamma^{(1)}, \gamma^{(2)})$, and thus positively proportional to the bending points of the broken line segment γ. Then the endpoints of each straight segment $\overline{L}$ of $\overline{\gamma}$ are positively proportional to the endpoints of the corresponding straight segment L of γ. As a result, each such $\overline{L}$ is in $\mathbb{Q}_{\geq 0} \cdot L \subset \mathbb{Q}_{\geq 0} \cdot S$.

Finally, we obtain the desired $\widetilde{\gamma}$ by rescaling the support (and elapsed time) of $\overline{\gamma}$ by $\frac{1}{\mu}$. □

Remark 58. Heuristically, in the proof of Proposition 57, we are translating between different tropical representations of the statement:

$$\vartheta_{\mu(\lambda_1+\lambda_2)\gamma(\tau)} \text{ is a non-zero summand of the product } \vartheta_{\mu\lambda_1 s_1}\vartheta_{\mu\lambda_2 s_2}.$$

In particular, if we consider the original input– the broken line segment γ and time τ– the tropical point $\gamma(\tau)$ is viewed as a weighted average along γ of the tropical points $\gamma(0) = s_1$ (with weight λ_1) and $\gamma(T) = s_2$ (with weight λ_2). Meanwhile, for the broken line segment $\overline{\gamma}$, the tropical point $\frac{\mu}{2}(\lambda_1+\lambda_2)\gamma(\tau)$ is interpreted as the (unweighted) average along γ of the tropical points $\mu\lambda_1 s_1$ and $\mu\lambda_2 s_2$.

Definition 59. If S is weakly convex, we call $\mathbb{Q}_{\geq 0} \cdot S$ the *weak cone of* S.

Definition 60. Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be a full-dimensional polytopal set containing 0 in the interior. The *weak face fan of* S, denoted $\Sigma[S]$, is the following collection of weak cones in $U^{\mathrm{trop}}(\mathbb{Q})$:

$$\Sigma[S] := \{\sigma_F := \mathbb{Q}_{\geq 0} \cdot F : F \in \mathcal{F}_S\} \cup \{0\}.$$

Proposition 61. *Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be a full-dimensional polytopal set containing 0 in the interior. Then the weak face fan of S forms a pseudo-complex.*

Proof. This follows almost immediately from Proposition 53. We will simply illustrate that Item 1 of Definition 52 holds for $\Sigma[S]$. The remaining items are recovered similarly. The weak cone $\{0\}$ has empty boundary, so there is nothing to do in this case. Now let $F \in \mathcal{F}_S$. By Proposition 53 there is a subset $\mathcal{A}_F$ of $\mathcal{F}_S$ such that

$$\partial F = \bigcup_{F' \in \mathcal{A}_F} F'.$$

Let $\mathcal{A}_{\sigma_F} := \{\sigma_{F'} \in \Sigma[S] : F' \in \mathcal{A}_F\} \cup \{0\}$. Then

$$\partial \sigma_F = \bigcup_{\tau \in \mathcal{A}_{\sigma_F}} \tau.$$

□

We illustrate the weak face fan of the bigon from Figure 2 in Figure 4 below.

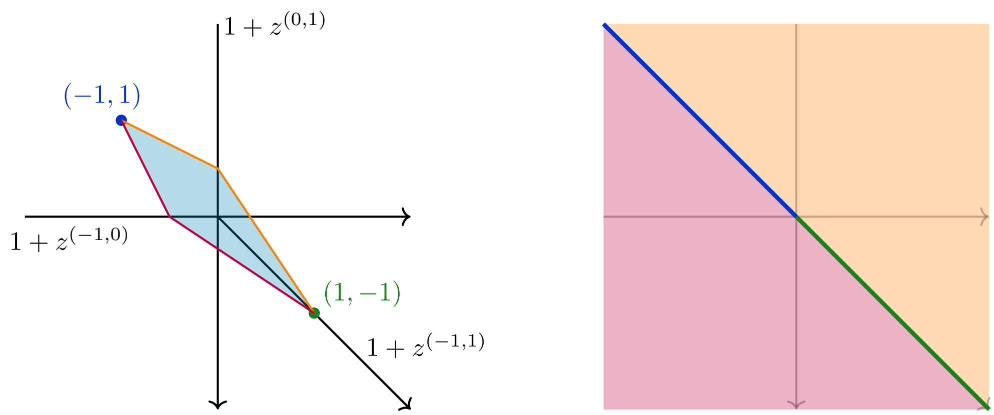

Figure 4: On the left, the bigon S of Figure 2. On the right, the weak face fan $\Sigma[S]$. Note that for F either facet, σ_F is only weakly convex, not broken line convex. In fact, $\mathrm{conv}_{\mathrm{BL}}(\sigma_F) = (\mathcal{A}^\vee)^{\mathrm{trop}}(\mathbb{Q})$ for both facets.

In order to generalize the convex geometry duality of [Bor93] in §6, it will be convenient to have a notion of support functions on weak face fans.

Definition 62. Let $\Sigma[S]$ be the weak face fan of a full-dimensional polytopal set $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ containing 0 in the interior. A function $\varphi : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ is a *support function* for $\Sigma[S]$ if for each weak cone $\sigma \in \Sigma[S]$ there is some $y_\sigma \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ such that $\varphi|_\sigma = \langle \,\cdot\, , y_\sigma \rangle$. A support function φ is *integral* if each y_σ may be taken to lie in $(U^\vee)^{\mathrm{trop}}(\mathbb{Z})$.

5.3 Duality for polyhedral sets and faces

Definition 63. Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$. We define the *polar of S* to be

$$S^\circ := \left\{ y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q}) : \langle s, y \rangle \geq -1 \text{ for all } s \in S \right\}.$$

We define the polar of subsets of $(U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ analogously.

Proposition 64. *Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$. Then*

$$(S^\circ)^\circ = \mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right).$$

Proof. This is proved just like the classical version for polytopes in $\mathbb{Q}^n$. We follow the proof given in [Brø83, Theorem 6.2].

By Lemma 44, a tropical half-space is broken line convex. The polar of a set is by definition an intersection of tropical half-spaces, and the intersection of broken line convex sets is broken line convex. So, $(S^\circ)^\circ$ is broken line convex. Moreover, if $x \in S$ then $\langle x, y \rangle \geq -1$ for all y in S° by definition of S°. So $S \subset (S^\circ)^\circ$, and obviously $\{0\} \subset (S^\circ)^\circ$ as well. That is, $(S^\circ)^\circ$ is a broken line convex set containing $S \cup \{0\}$, and $(S^\circ)^\circ \supset \mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right)$.

Next observe that

$$y \in S^\circ \iff \langle x, y \rangle \geq -1 \text{ for all } x \in S \iff S \subset K(y, 1).$$

So,

$$(S^\circ)^\circ = \bigcap_{y \in S^\circ} K(y,1) = \bigcap_{K(y,1) \supset S} K(y,1).$$

Now take a point $z \notin \mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right)$. There exists a supporting tropical half-space $K(y,r)$ of $\mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right)$ with $z \notin K(y,r)$. So,

$$\min \left\{ \langle x, y \rangle : x \in \mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right) \right\} = -r > \langle z, y \rangle.$$

Then there exists $t \in \mathbb{Q}_{>0}$ such that

$$\min \left\{ \langle x, y \rangle : x \in \mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right) \right\} \geq -t > \langle z, y \rangle.$$

Set $u := t^{-1} y$. Then

$$\min \left\{ \langle x, u \rangle : x \in \mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right) \right\} \geq -1 > \langle z, u \rangle.$$

So $K(u,1) \supset S$, which implies $(S^\circ)^\circ \subset K(u,1)$. But $z \notin K(u,1)$, so $z \notin (S^\circ)^\circ$. That is, $z \notin \mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right)$ implies $z \notin (S^\circ)^\circ$. We conclude that

$$(S^\circ)^\circ = \mathrm{conv}_{\mathrm{BL}}\left(S \cup \{0\} \right).$$

□

Now fix some $r \in \mathbb{Q}_{\geq 0}$ and for $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ write $S^{\vee_r} := \{ y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q}) : \langle x, y \rangle \geq -r \text{ for all } x \in S \}$. (In particular, if $r = 1$, $S^{\vee_r} = S^\circ$.)

Proposition 65. *Let S and T be subsets of $U^{\mathrm{trop}}(\mathbb{Q})$. Then*

$$\left(S^{\vee_r} \cap T^{\vee_r}\right) = (S \cup T)^{\vee_r}.$$

Proof. Let $y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$. Then

$$y \in \left(S^{\vee_r} \cap T^{\vee_r}\right) \Longleftrightarrow \langle x, y\rangle \geq -r \text{ for all } x \in (S \cup T) \Longleftrightarrow y \in (S \cup T)^{\vee_r}.$$

$\square$

🚲Tim: [In background section, along with the definition of broken line, define a *doubly infinite broken line* by taking $\gamma : (-\infty, \infty) \to U^{\mathrm{trop}}(\mathbb{Q})$ rather than $\gamma : (-\infty, 0] \to U^{\mathrm{trop}}(\mathbb{Q})$. Hopefully this would just require minor adjustments to the proof, but I haven't thought it through yet.]

Definition 66. A broken line convex set $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ is *strongly* broken line convex if no doubly infinite broken line has support contained in S.

🚲Tim: [Let's replace S° with $S^{\vee_r}$ if we can. That way we'll be able to apply the result to cones of a fan later if we need to. With that in mind, it may be worth restructuring the theorem a bit, if something like the following is true: Let S be weakly convex. Then $S^{\vee_r}$ is strongly broken line convex if and only if S is full-dimensional.]

Proposition 67. *Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be broken line convex. Then $S^\circ \subset (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ is full-dimensional if and only if S is* strongly *broken line convex.*

Proof. Suppose S is *not* strongly broken line convex. Then there exists a doubly infinite broken line $\gamma : \mathbb{Q} \to U^{\mathrm{trop}}(\mathbb{Q})$ with support contained in S. We will choose a particular seed $\mathbf{s}$ to identify $U^{\mathrm{trop}}(\mathbb{Q})$ and $(U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ with dual $\mathbb{Q}$-vector spaces V and V^* via the maps $\mathfrak{r}_{\mathbf{s}}$ and $\mathfrak{r}_{\mathbf{s}}^\vee$ as in Notation 10. Specifically, we choose $\mathbf{s}$ such that the support of γ intersects the $\mathfrak{D}_{\mathbf{s}}^{U^\vee}$ chamber associated to $\mathbf{s}$ at a non-bending point $\mathfrak{r}_{\mathbf{s}}(x_0)$ of $\mathfrak{r}_{\mathbf{s}}(\mathrm{supp}(\gamma))$. Reparametrizing γ if necessary, we can take $x_0 = \gamma(0)$. Now define

$$\begin{aligned} \eta_- : \mathbb{Q}_{\leq 0} &\to V \\ t &\mapsto \mathfrak{r}_{\mathbf{s}}(\gamma(t)) \end{aligned}$$

and

$$\begin{aligned} \eta_+ : \mathbb{Q}_{\leq 0} &\to V \\ t &\mapsto \mathfrak{r}_{\mathbf{s}}(\gamma(-t)). \end{aligned}$$

Since $\mathfrak{r}_{\mathbf{s}}(x_0)$ is a non-bending point, for sufficiently small $\epsilon > 0$, we have

$$\dot{\eta}_-(-\epsilon) = -\dot{\eta}_+(-\epsilon). \tag{24}$$

Write $v_\pm := \lim_{t \to -\infty} \dot{\eta}_\pm(t)$.

As $\eta_\pm$ has only finitely many bends, there exists some $R > 0$ such that for all $t \in \mathbb{Q}_{\leq 0}$, $\eta_\pm(t)$ is contained in $B(R, tv_\pm)$, the ball of radius R centered at $tv_\pm$. That is, for all $t \in \mathbb{Q}_{\leq 0}$, we can write $\eta_\pm(t) = tv_\pm + b$ for some $b \in B(R, 0)$. Now suppose $\langle \mathfrak{r}_{\mathbf{s}}(-v_\pm), y\rangle < 0$ for some $y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$. Then

$$\lim_{t \to -\infty} \langle \mathfrak{r}_{\mathbf{s}}(tv_\pm), y\rangle = -\infty. \tag{25}$$

The tropically linear function $\langle \cdot\, , y\rangle : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ defines a piecewise linear function on V by $\left(\mathfrak{r}_{\mathbf{s}}^{-1}\right)^* \langle \cdot\, , y\rangle$, and this piecewise linear function has the form

$$\left(\mathfrak{r}_{\mathbf{s}}^{-1}\right)^* \langle \cdot\, , y\rangle = \min_{\ell \in L} \{\ell(\,\cdot\,)\}$$

for some finite set L of linear functions on V. Then

$$\left\langle \mathfrak{r}_{\mathbf{s}}^{-1}\left(\eta_\pm(t)\right), y\right\rangle = \min_{\ell \in L}\{\ell(\eta_\pm(t))\} = \min_{\ell \in L}\{\ell(tv_\pm) + \ell(b)\}$$

for some $b \in B(R,0)$. However, $\ell|_{B(R,0)}$ is bounded for all $\ell \in L$, while $\lim_{t\to-\infty} \min_{\ell\in L} \ell(tv_\pm) = -\infty$ by (25). So, $\lim_{t\to-\infty} \left\langle \mathfrak{r}_{\mathbf{s}}^{-1}\left(\eta_\pm(t)\right), y\right\rangle = -\infty$ as well, and $y \notin \operatorname{supp}(\gamma)^\circ$. In other words, if $y \in \operatorname{supp}(\gamma)^\circ$, then $\left\langle \mathfrak{r}_{\mathbf{s}}^{-1}(v_\pm), y\right\rangle \geq 0$.

Next,

$$
\begin{aligned}
\left\langle \mathfrak{r}_{\mathbf{s}}^{-1}(v_\pm), y\right\rangle = \vartheta^{\mathrm{trop}}_{\mathfrak{r}_{\mathbf{s}}^{-1}(v_\pm)}(y) &= \left(\mathfrak{r}_{\mathbf{s}}^\vee(y)\right)\left(\vartheta_{v_\pm,\mathfrak{r}_{\mathbf{s}}(x_0)}\right)\\
&= \min\left\{m(\mathfrak{r}_{\mathbf{s}}^\vee(y)) \,:\, z^m \text{ is a non-zero summand of } \vartheta_{v_\pm,\mathfrak{r}_{\mathbf{s}}(x_0)}\right\}\\
&\leq \left(-\dot{\eta}_\pm(-\epsilon)\right)\left(\mathfrak{r}_{\mathbf{s}}^\vee(y)\right) \text{ for small } \epsilon > 0.
\end{aligned}
$$

That is, if $y \in \operatorname{supp}(\gamma)^\circ$, then $0 \leq \left\langle \mathfrak{r}_{\mathbf{s}}^{-1}(v_\pm), y\right\rangle \leq \left(-\dot{\eta}_\pm(-\epsilon)\right)\left(\mathfrak{r}_{\mathbf{s}}^\vee(y)\right)$ for small $\epsilon > 0$. Then (24) implies $\mathfrak{r}_{\mathbf{s}}^\vee(y) \in \dot{\eta}_\pm(-\epsilon)^\perp$, and $\operatorname{supp}(\gamma)^\circ$ is not full-dimensional. But $\operatorname{supp}(\gamma) \subset S$, so $S^\circ \subset \operatorname{supp}(\gamma)^\circ$, and S° is also not full-dimensional.

Now suppose S° is not full-dimensional. Say $d := \dim(S^\circ)$. Choose a seed $\mathbf{s}$ such that the chamber $\sigma_{\mathbf{s}}$ of $\mathfrak{D}^U_{\mathbf{s}}$ associated to the seed $\mathbf{s}$ intersects $\mathfrak{r}_{\mathbf{s}}^\vee(S^\circ)$ in a d-dimensional subset. Note that $\mathfrak{r}_{\mathbf{s}}^\vee(S^\circ)$ is contained in some hyperplane through the origin, say $m^\perp$ for some integral m. Let x_+ and x_- to be the points of $U^{\mathrm{trop}}(\mathbb{Z})$ with

$$
\left(\mathfrak{r}_{\mathbf{s}}^{\vee -1}\right)^* \left\langle x_\pm, \,\cdot\,\right\rangle\Big|_{\sigma_{\mathbf{s}}} = \pm m(\,\cdot\,). \tag{26}
$$

Then clearly

$$
S^\circ \cap \mathfrak{r}_{\mathbf{s}}^\vee(\sigma_{\mathbf{s}}) \subset \left(H(x_+,0) \cap H(x_-,0)\right).
$$

Moreover, since S° is broken line convex, Lemma 55 implies that in fact

$$
S^\circ \subset \left(H(x_+,0) \cap H(x_-,0)\right). \tag{27}
$$

Note that (26) implies there is a pair of broken lines η_-, η_+ in V with initial exponent vectors $\mathfrak{r}_{\mathbf{s}}(x_-), \mathfrak{r}_{\mathbf{s}}(x_+)$ and final exponent vectors $-m, +m$ sharing the same basepoint. This pair of broken lines indicates that the product $\vartheta_{x_-}\vartheta_{x_+}$ has non-zero constant (ϑ_0) term. Explicitly, we can dilate the supports of the pair of broken lines to bring the basepoint arbitrarily close to the origin. Then this pair of broken lines precisely describes a contribution to the product $\vartheta_{x_-}\vartheta_{x_+}$ as described in [GHKK18, Definition-Lemma 6.2]. However, by [GHKK18, Proposition 6.4.(3)], we can compute the structure constant $\alpha^0_{x_+,x_-}$ of this multiplication using any basepoint near the origin. In particular, we may choose a basepoint $\mathfrak{r}_{\mathbf{s}}(x_b)$ such that λx_b is in the relative interior of S for some $\lambda > 0$. Then we obtain a pair of broken lines in V with basepoint $\mathfrak{r}_{\mathbf{s}}(x_b)$, initial exponent vectors $\mathfrak{r}_{\mathbf{s}}(x_-)$ and $\mathfrak{r}_{\mathbf{s}}(x_+)$, and final exponent vectors summing to 0. Dilating the supports of these broken lines by λ, we obtain a such a pair with basepoint in the relative interior of S. We may reverse the direction of one of the broken lines to obtain a doubly infinite broken line γ passing through the previous basepoint and having

$$
\lim_{t\to\pm\infty} \dot{\gamma}(t) = -\mathfrak{r}_{\mathbf{s}}(x_\pm). \tag{28}
$$

Next, as

$$
S^\circ = \bigcap_{s\in S} K(s,1)
$$

and the origin is in the interior of each $K(s,1)$, there must be a sequence of points $\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots$ in S with asymptotic directions x_+ as $n \to \infty$ and x_- as $n \to -\infty$. More precisely,

$$
\lim_{n\to\pm\infty} \|\mathfrak{r}_{\mathbf{s}}(s_n) - \mathfrak{r}_{\mathbf{s}}(s_0)\| = \infty
$$

and

$$
\lim_{n\to\pm\infty} \frac{\mathfrak{r}_{\mathbf{s}}(s_n) - \mathfrak{r}_{\mathbf{s}}(s_0)}{\|\mathfrak{r}_{\mathbf{s}}(s_n) - \mathfrak{r}_{\mathbf{s}}(s_0)\|} = \mathfrak{r}_{\mathbf{s}}(x_\pm).
$$

Now suppose $\operatorname{supp}(\gamma) \not\subset \mathfrak{r}_{\mathbf{s}}(S)$. Then at some point γ must leave $\mathfrak{r}_{\mathbf{s}}(S)$. As γ passes through the relative interior of $\mathfrak{r}_{\mathbf{s}}(S)$, Lemma 55 prevents γ from simply entering and remaining in the boundary of the closure of $\mathfrak{r}_{\mathbf{s}}(S)$ in the event that $\mathfrak{r}_{\mathbf{s}}(S)$ is not closed. Then points of γ must eventually be a positive distance from $\mathfrak{r}_{\mathbf{s}}(S)$. However, since $\mathfrak{r}_{\mathbf{s}}(S)$ contains the asymptotic directions $\mathfrak{r}_{\mathbf{s}}(x_{\pm})$, (28) implies this positive distance is bounded. Then as argued in Lemma 55, we may take a small perturbation γ' of a segment of γ, this time adding a small contribution to the velocity at λx_b so that the first bend for γ' after leaving $\mathfrak{r}_{\mathbf{s}}(S)$ is slightly closer to $\mathfrak{r}_{\mathbf{s}}(S)$ the corresponding bend of γ. Keeping all wall contributions the same (up to a multiplicative constant to maintain integrality of exponent vectors) as in Lemma 55, we obtain a broken line segment which must eventually re-enter $\mathfrak{r}_{\mathbf{s}}(S)$ as the direction after the last bend will have a small contribution directed toward $\mathfrak{r}_{\mathbf{s}}(S)$, much like the situation illustrated in Figure 3. This contradicts the assumption that S is broken line convex. As a result, we conclude that $\mathfrak{r}_{\mathbf{s}}^{-1}(\operatorname{supp}(\gamma))$ is contained in S and so S is not strongly broken line convex.

□

Proposition 68. *Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be broken line convex. Then $S^\circ \subset (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ is bounded if and only if S is full-dimensional and contains the origin in its interior.*

Proof. Suppose S is full-dimensional and contains the origin in its interior. Choose a seed $\mathbf{s}$ to identify $U^{\mathrm{trop}}(\mathbb{Q})$ and $(U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ with dual $\mathbb{Q}$-vector spaces V and V^* via $\mathfrak{r}_{\mathbf{s}}$ and $\mathfrak{r}_{\mathbf{s}}^\vee$ as in Notation 10. Then $\mathfrak{r}_{\mathbf{s}}(S)$ contains the ball $B(R,0)$ for some sufficiently small $R > 0$. For any $x \in U^{\mathrm{trop}}(\mathbb{Q})$, we have

$$\left(\mathfrak{r}_{\mathbf{s}}^{\vee\,-1}\right)^* \langle x,\ \cdot\ \rangle = \min_{\ell \in L} \{\ell(\ \cdot\ )\}$$

for some finite set of linear maps L containing $\mathfrak{r}_{\mathbf{s}}(x)$. Then

$$\mathfrak{r}_{\mathbf{s}}^\vee\left(K(x,1)\right) \subset K(\mathfrak{r}_{\mathbf{s}}(x),1),$$

where $K(\mathfrak{r}_{\mathbf{s}}(x),1) := \{v \in V^* : (\mathfrak{r}_{\mathbf{s}}(x))\,(v) \geq -1\}$.

Now note that the polar of $B(R,0) \subset V$ is $B\left(\frac{1}{R},0\right) \subset V^*$. Then we have

$$\mathfrak{r}_{\mathbf{s}}^\vee\left(S^\circ\right) = \mathfrak{r}_{\mathbf{s}}^\vee\left(\bigcap_{s \in S} K(s,1)\right) \subset \mathfrak{r}_{\mathbf{s}}^\vee\left(\bigcap_{\mathfrak{r}_{\mathbf{s}}(s) \in B(R,0)} K(s,1)\right) \subset B(R,0)^\circ = B\left(\frac{1}{R},0\right),$$

and S° is bounded.

Next, if S is not full-dimensional, S° is not strongly convex by Proposition 67, and *a fortiori* not bounded. If S does not contain the origin in the interior, then it is contained in $K(y,0)$ for some non-zero $y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$. So $\mathbb{Q}_{\geq 0} \cdot y \in S^\circ$ and S° is not bounded. □

We have the following immediate corollary:

Corollary 69. *If S is a bounded, full-dimensional set containing the origin in its interior, then so is S°.*

Definition 70. Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$. We say S is *integral* if $S = \mathrm{conv}_{\mathrm{BL}}(A)$ for some finite subset A of $U^{\mathrm{trop}}(\mathbb{Z})$. We define integral subsets of $(U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ analogously, and we say S is *reflexive* if both S and S° are integral.

Definition 71. Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be a full-dimensional polytopal subset and containing 0 in the interior. We define the *dual of a face* F of S to be

$$\breve{F} := \{y \in S^\circ : \langle x, y\rangle = -1 \text{ for all } x \in F\}.$$

Proposition 72. *Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be a full-dimensional polytopal subset and containing 0 in the interior, and let F be a proper face of S. Then $\breve{F}$ is a proper face of S°. Precisely, if x is in the relative interior of F, then $\breve{F} = S^\circ \cap H(x,1)$. Moreover, if $F' \subsetneq F$ then $\breve{F} \subsetneq \breve{F'}$ and $\breve{\breve{F}} = F$. This gives a bijective, containment-reversing correspondence between proper faces of S and S°.*

Proof. For any $x \in F$, define $F_x := S^\circ \cap H(x,1)$. Observe that

$$\check{F} = \bigcap_{x \in F} F_x,$$

and $\check{F} \subset F_x$ for all $x \in F$. Now suppose x is in the relative interior of F. Then by Definition 52 Item 3 and Proposition 53, a supporting tropical hyperplane $H(y,r)$ for S which contains x must in fact contain F. So if $y \in F_x$, then $F \subset H(y,1)$, and $y \in \check{F}$. That is, $\check{F} = F_x$ for any x in the relative interior of F.

Next, we claim that every proper face of S° is the dual of a face of S. By definition, every face of S° is of the form $S^\circ \cap H(x,r)$ for some $x \in U^{\mathrm{trop}}(\mathbb{Q})$, and using Remark 43 we can choose x such that $r = 1$. But such an x is necessarily in the boundary of S since $(S^\circ)^\circ = S$. Every boundary point is contained in some face, and moreover by Proposition 53, contained in the relative interior of a face. Then the previous argument implies that every proper face of S° is the dual of a face of S.

It follows immediately from the definition of the dual of a face that $F' \subset F$ implies $\check{F} \subset \widecheck{F'}$. If moreover $F' \subsetneq F$, then there is some $y' \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$ and $r' \in \mathbb{Q}_{>0}$ with $F' = S \cap H(y', r')$ and $F \subset S \subset K(y', r')$. As argued above, we may take $r' = 1$ and $y' \in S^\circ$. Then $y' \in \widecheck{F'} \setminus \check{F}$, so $\check{F} \subsetneq \widecheck{F'}$.

Finally, S and S° play completely interchangeable roles here. So, every proper face of S is the dual of a proper face of S° as well. We have automatically that $F \subset \check{\check{F}}$. Suppose $F \subsetneq \check{\check{F}}$. Since proper containments are reversed by duality of faces, we must also have $\check{\check{\check{F}}} \subsetneq \check{F}$. But this violates the automatic containment $\check{F} \subset \check{\check{\check{F}}}$. We conclude that $F = \check{\check{F}}$.

□

5.4 Half-space and vertex representations

Proposition 73. *Let $A \subset U^{\mathrm{trop}}(\mathbb{Q})$ be a finite collection of points and let $S = \mathrm{conv}_{\mathrm{BL}}(A)$. Then necessarily $V(S) \subset A$ and $S = \mathrm{conv}_{\mathrm{BL}}(V(S))$.*

Proof. Let $v \in A$, but $v \notin \mathrm{conv}_{\mathrm{BL}}(A \setminus \{v\}) =: S_v$. Then there is some supporting tropical half-space $K(y,r)$ for S_v with $x \notin K(y,r)$. For all $r' > r$, we have that S_v is contained in the interior of $K(y,r')$. Moreover, for some such r', we have that $v \in H(y,r')$.

Now let $s \in S$. By Proposition 29,

$$s \in \sum_{x \in A}{}_{\vartheta}\, a_x x$$

for some collection of non-negative a_x which sum to 1. Next, Proposition 31 implies

$$\begin{aligned}
\langle s, y \rangle &\geq \sum_{x \in A} a_x \langle x, y \rangle \\
&\geq -a_v r' - \sum_{x \in A \setminus \{v\}} a_x r \\
&\geq -r',
\end{aligned}$$

with equality if and only if $a_v = 1$, $a_{x \neq v} = 0$. That is, if $s \in S$ is in $H(y,r')$, then $s = v$. Thus v is a vertex of S. □

Proposition 74. *Every polytopal set $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ is the broken line convex hull of a finite set.*

Proof. This follows directly from the usual convex geometry statement by choosing a seed– the only subtlety being that the finite set of points obtained in this way will generally not be minimal.

Let $d = \dim(U)$. Then a choice of seed identifies $U^{\mathrm{trop}}(\mathbb{Q})$ with $\mathbb{Q}^d$, and S with a polytope P_S in $\mathbb{Q}^d$. Specifically, each tropical half-space defining S is identified with an intersection of a finite number of half-spaces in $\mathbb{Q}^d$. Thus P_S is the intersection of finitely many half-spaces in $\mathbb{Q}^d$; it is a rational polyhedron, and in fact a polytope as it is bounded. Then P_S is the (usual) convex hull of a finite collection of points in $\mathbb{Q}^d$, namely the vertices of P_S in the usual convex geometry sense. Denote by A the collection of tropical points associated to this set of vertices of P_S. The equality $S = \mathrm{conv}_{\mathrm{BL}}(A)$ is clear. □

We can combine Propositions 73 and 74 to obtain

Corollary 75. *If $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ is polytopal, then $V(S)$ is finite and $S = \mathrm{conv}_{\mathrm{BL}}(V(S))$.*

So, we have defined polytopal sets using a broken line convex geometry version of the "half-space representation". Corollary 75 indicates that a polytopal set always has a broken line convex geometry version of the "vertex representation" as well. We may also perform this translation in the opposite direction:

Proposition 76. *Let $A \subset U^{\mathrm{trop}}(\mathbb{Q})$ be a finite set such that $S := \mathrm{conv}_{\mathrm{BL}}(A)$ is full-dimensional and contains the origin. Then S is polytopal.*

Proof. First note that S° is given by

$$S^\circ = \bigcap_{v \in V(S)} K(v, 1).$$

By Proposition 73 $V(S)$ is a finite set contained in A, and so S° is polyhedral. Using Corollary 69, we see that S° is in fact polytopal. But then Corollary 75 implies $V(S^\circ)$ is finite and $S^\circ = \mathrm{conv}_{\mathrm{BL}}(V(S^\circ))$. So, we have

$$(S^\circ)^\circ = \bigcap_{y \in V(S^\circ)} K(y, 1).$$

But since S is a broken line convex set containing the origin, we have $(S^\circ)^\circ = S$ by Proposition 64. $\square$

6 Broken line convex geometry of Batyrev-Borisov duality

Definition 77. Let $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ be reflexive, let $V(S)$ be the vertices of S, and consider a decomposition of $V(S)$ as a disjoint union $V(S) = \cup_{i \in I} V_i(S)$. Write

$$T_i := \bigcap_{j \in I} \bigcap_{v \in V_j(S)} K(v, \delta_{ij}).$$

We call the decomposition of $V(S)$ a *Batyrev-Borisov partition* if for each $i \in I$, the function $\varphi_i : U^{\mathrm{trop}}(\mathbb{Q}) \to \mathbb{Q}$ defined by

$$\varphi_i := \min_{y \in T_i} \{\langle \,\cdot\, , y \rangle\}$$

is an integral support function for $\Sigma[S]$.

In the toric case, what we call Batyrev-Borisov data is referred to as a *nef-partition*. (See [Bor93, Definition 2.5].) This is because the partition of $V(S)$ defines a decomposition of $-K_{X_{S^\circ}}$ as a sum of nef Cartier divisors $-K_{X_{S^\circ}} = \sum_{i \in I} D_i$. See [Bor93] for details. We do not use this terminology here (or similar terminology rooted in algebraic geometry) since we do not yet know what analogous algebro-geometric statement holds in the cluster setting. That said, a careful comparison of Definition 77 and [Bor93, Definition 2.5] will reveal a few subtle differences. First, Borisov does not make reference to the set we call T_i in his definition. Instead, if we rephrase his definition of nef-partition using the conventions of this paper, he asks for the existence of a collection of convex integral support functions $\{\varphi_i : i \in I\}$ for $\Sigma[S]$ with

$$\varphi_i(v) = \begin{cases} -1 & \text{for } v \in V_i(S) \\ 0 & \text{for } v \in V(S) \setminus V_i(S). \end{cases}$$

He later uses φ_i to define the set T_i, interpreted as the polytope of the nef summand D_i of $-K_{X_{S^\circ}}$. (See [Bor93, Definition 2.9].) In the broken line convex geometry setting however, the values of φ_i at vertices of S may not fully determine φ_i and we are forced to be a bit more explicit when defining this function. For instance, if we take the trivial partition $V(S) = V_1(S)$ for S the bigon of Figure 2, as we will see in Example

🚲Tim: [. . .] there are three different integral support functions for $\Sigma[S]$ which are convex with respect to broken lines and evaluate to -1 on both vertices. Only one of these can be used to recover $T_1 = S^\circ$ as in [Bor93, Definition 2.9]. Finally, there is a sign difference between our φ_i and that appearing in [Bor93, Definition 2.5]. This is due to opposite inequality conventions for convex functions. The convention we have chosen matches that of [CLS11] and [GHKK18]. The change is more substantive here than in the toric case as negation isn't a meaningful operation on $U^{\mathrm{trop}}(\mathbb{Q})$.

While our definition of T_i looks somewhat different from [Bor93, Definition 2.9], it is easy to recover Borisov's description:

Proposition 78. *The set T_i is precisely*

$$T_i = \left\{ y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q}) \,:\, \langle x, y\rangle \geq \varphi_i(x) \text{ for all } x \in U^{\mathrm{trop}}(\mathbb{Q}) \right\}.$$

Proof. First suppose $y \in T_i$. Then for all $x \in U^{\mathrm{trop}}(\mathbb{Q})$,

$$\langle x, y\rangle \geq \min_{y' \in T_i} \{\langle x, y'\rangle\} = \varphi_i(x).$$

Next, suppose $y \notin T_i$. Then for some $v \in V(S)$, say with v in the set $V_j(S)$, we have $\langle v, y\rangle < -\delta_{ij}$. But $\varphi_i(v) = -\delta_{ij}$. □

In fact, considering the above description of T_i as its definition (as in [Bor93, Definition 2.9]), Borisov obtained as a corollary that the function φ_i is precisely $\min_{y \in T_i} \{\langle \,\cdot\, , y\rangle\}$. (See [Bor93, Corollary 2.12].) So, in the end our description aligns quite closely with Borisov's. We simply avoid the problem that the most naïve generalization of his definition of φ_i to the broken line convex geometry setting may not be uniquely defined.

For the remainder of the section we fix a reflexive subset $S \subset U^{\mathrm{trop}}(\mathbb{Q})$ and a Batyrev-Borisov partition $V(S) = \cup_{i\in I} V_i(S)$ as in Definition 77. We will now use the following notation:

$$S_i := \mathrm{conv}_{\mathrm{BL}}\left(\{0\} \cup V_i(S)\right) \tag{29}$$

Proposition 79. *For all $x \in S_i$ and $y \in T_j$ we have $\langle x, y\rangle \geq -\delta_{ij}$.*

Proof. If $x \in S_i$, Proposition 29 implies that for some collection of non-negative a_v summing to 1, with $v \in \{0\} \cup V_i(S)$, we have

$$x \in {\sum_v}_\vartheta a_v v.$$

Then Proposition 31 implies that for all $y \in (U^\vee)^{\mathrm{trop}}(\mathbb{Q})$,

$$\langle x, y\rangle \geq \sum_v a_v \langle v, y\rangle.$$

In particular, if

$$y \in T_j = \bigcap_{i\in I} \bigcap_{v \in V_i(S)} K(v, \delta_{ij}),$$

then

$$\langle x, y\rangle \geq \sum_v -a_v \delta_{ij} = -\delta_{ij}.$$

□

Lemma 80. *Let $\varphi := \sum_{i\in I} \varphi_i$. Then $S = \Xi_{\varphi,1}$.*

OR:

Lemma 81. *Let*

$$\varphi := \min\left\{ \langle \,\cdot\, , y\rangle \,:\, y \in {\sum_{i\in I}}_\vartheta T_i \right\}.$$

Then $S = \Xi_{\varphi,1}$.

🚲Tim: [More coming soon...]

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