E : Sightseeing Plan

The problem asks the following:

You are given three rectangular regions A, B, C. Choose a point from each region, and count the number of paths that passes through all chosen points. Compute the sum of these values.

First, we state some easy lemmas. Let $C(x, y) = \frac{(x+y)!}{x!y!}$ be the number of paths from $(0, 0)$ to (x, y).

$\sum_{y=0}^{Y} C(X, y) = C(X + 1, Y)$

Consider a path from $(0, 0)$ to $(X+1, Y)$. It passes through an edge between (X, k) and $(X+1, k)$ for some k, and the number of such paths is $C(X, k)$.

$\sum_{x=0}^{X} \sum_{y=0}^{Y} C(x, y) = C(X + 1, Y + 1) - 1$

Apply the previous lemma twice.

$\sum_{x=X_1}^{X_2} \sum_{y=Y_1}^{Y_2} C(x, y) = C(X_1, Y_1) - C(X_1, Y_2 + 1) - C(X_2 + 1, Y_1) + C(X_2 + 1, Y_2 + 1)$

This is similar to a well-known trick where you compute the sum in a subrectangle by adding/subtracting rectangles four times.

In the original problem, we choose the start from a rectangle region. However, because of this lemma, we can assume that there are only four (weighted) possible positions for the start. In particular,

- $(X_1 - 1, Y_1 - 1)$ with weight $+1$.
- $(X_1 - 1, Y_2)$ with weight -1.
- $(X_2, Y_1 - 1)$ with weight -1.
- (X_2, Y_2) with weight $+1$.

Similarly, the goal is also considered as four points, so we can solve the original problem by solving the following simpler problem $4 \times 4 = 16$ times:

You are given a point A, a rectangular region B, and a point C. Choose a point from B, and count the number of paths that passes through all chosen points. Compute the sum of these values.

We should count the number of paths from A to C that passes through B at least once. However, if the size of the intersection of a path P and the rectangle B is k, this path should be counted with multiplicity k.

The following works:

- For each $X_3 \leq x \leq X_4$, count the number of paths that enters B by $(x, Y_3 - 1) \to (x, Y_3)$ with multiplicity $-(x + Y_3)$.
- For each $Y_3 \leq y \leq Y_4$, count the number of paths that enters B by $(X_3 - 1, y) \to (X_3, y)$ with multiplicity $-(X_3 + y)$.
- For each $X_3 \leq x \leq X_4$, count the number of paths that leaves B by $(x, Y_4) \to (x, Y_4 + 1)$ with multiplicity $x + Y_4 + 1$.
- For each $Y_3 \leq y \leq Y_4$, count the number of paths that leaves B by $(X_4, y) \to (X_4 + 1, y)$ with multiplicity $X_4 + y + 1$.

In total, this solution works in $O(MAXCOORDINATE)$, with pre-computation of factorials and inverses of factorials.