1 Stuff dump

Cambell Baker Hausdorf Für alle $t \in \mathbb{R}$

$$\exp(tA)\exp(tB) = \exp\left(tA + tB + \frac{t^2}{2}[A,B] + \frac{t^3}{12}[A,[A,B]] + \frac{t^3}{12}[B,[B,A]] + \mathscr{O}\left(t^4\right)\right)$$

Creation-/Annihilation Operators in second quantisation:

$$\left[\hat{a}_i^*, \hat{a}_j^*\right] = 0, \quad \left[\hat{a}_i, \hat{a}_j\right] = 0, \quad \left[\hat{a}_i, \hat{a}_j^*\right] = \delta_{i,j}\hat{\mathrm{id}}$$

Bernoulli Trial Probability of k successes in a bernoulli experiment $B(n,p)$:

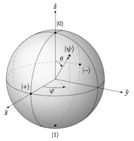
$$P(k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k}$$

Bayes' rule

$$\Pr[A \mid B \wedge C] = \frac{\Pr[B \mid A \wedge C]}{\Pr[B \mid C]} \Pr[A \mid C]$$

Maxwell equations we have
$\nabla E = \frac{\rho}{\varepsilon_0}$ (Gauss law)
$\nabla B = 0$
$\nabla \times E = -\frac{\partial B}{\partial t}$ (Faraday's law of induction)
$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} + \mu_0 J + \varepsilon_0 \frac{\partial P}{\partial t}$ (Ampere's law)

Bloch sphere $\hat{n}(\theta,\phi)$ on Bloch sphere with $\theta \in (0,\pi)$, $\phi \in (0,2\pi)$. For $-\hat{n}$ we have $(\theta,\phi) \to (\pi-\theta, \phi+\pi)$.



Monte Carlo Methods Class of algorithms that rely on random sampling to obtain numerical results

Delta distribution

- $\int dx e^{ik\cdot x} = (2\pi)\delta(k)$
- $\int_{-\infty}^{+\infty} dx \delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}$

Fourier transform For $\varphi \in \mathscr{S}(\mathbb{R}^n)$:
(i) $(\partial_j\varphi)^\wedge(k) = ik_j\hat{\varphi}(k)$
(ii) $\partial_j\hat{\varphi}(k) = \frac{\partial}{\partial k_j}\hat{\varphi}(k) = (-ix_j\varphi)^\wedge(k)$
(iii) $(\partial_j\varphi)^\vee(k) = -ik_j\check{\varphi}(k)$
(iv) $\partial_j\check{\varphi}(k) = (ix_j\varphi)^\vee(k)$
(v) $\mathcal{F}_x\left[e^{-ax^2}\right](k) = \sqrt{\frac{\pi}{a}}e^{k^2a}$ Normalsation 1; osc factor 1

2 Quantum Mechanics I & II

Propagator characterised through
(i) $U(t,t) = \mathbf{I}$.
(ii) additivity/ unitar: $U(t,s)U(s,r) = U(t,r)$.
(iii) The operator $U(t,s)$ satisfies the differential equation

$$i\hbar\partial_t U(t,s) = HU(t,s)$$

For H time independent: $U(t,s) = \exp\left(-iH\frac{(t-s)}{\hbar}\right)$
General: $U(t,s) = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_s^t dt' H(t')\right]$ For $[H(t), H(s)] = 0\ \forall t,s$ we can omit the time order operator $\mathcal{T}$

Heisenberg picture Time dependecy is shifted from states to operators:

$$\Psi_H = \Psi(t_0) = U(t_0,t)\Psi_S(t) \qquad A_H = U(t_0,t)A_S U(t,t_0)$$

The equation of motion in the Heisenberg picture:

$$i\hbar\frac{d}{dt}A_H(t) = [A_H, H_H] + i\hbar\partial_t A_H$$

For $\partial_t H = 0$ we have $H_H = H_S$ and $A_H(t) = e^{iH(t-t_0)/\hbar}Ae^{-iH(t-t_0)/\hbar}$
The Heisenberg picture shows the similaritiy to classical mechanics where we have $\frac{dA}{dt} = \{A,H\} + \partial_t A$. *Replacing the Poisson braket with commutators and imposing the canonical commutator relations gives rise to qunatisation.*

Interaction (Dirac) picture For $H = H_0 + H'(t)$. Idea is to shift (trivial) time dependence of states originating from H_0 on to operators: $\Psi_D(t) = U_D(t,t_0)\Psi_D(t_0)$ with $U_D(t,t_0) = U_0(t_0,t)U(t,t_0)$ where U is the Propagator for $H = H_0 + H'$. We have

$$i\hbar\partial_t U_D == H_D' U_D$$

3 Quantum Information Theory

Quantum probability $Pr[|\Lambda] = \mathrm{Tr}\,\Lambda\rho$
probability density: $\rho \in \mathrm{Lin}(\mathcal{H})$, $\rho \geq 0$, $\mathrm{Tr}[p] = 1$
effect / measuremnt: $\Lambda \in \mathrm{Lin}(\mathcal{H})$, $\Lambda \geq 0$, $\Lambda \leq \mathbb{1}$

positivity of operators: $S \geq 0$ if $\langle v|S|v\rangle \geq 0$ for all $v \in \mathcal{H}$

POVM: postive operator valued measure set of effects $\{\Lambda(x)\}_{x=1}^n$ such that $\Lambda(x) \in \mathrm{Lin}(\mathcal{H}) : \Lambda(x) \geq 0\ \forall x$, $\sum_x \Lambda(x) = 1$

Trace (abstract) $\mathrm{Tr}\,|\Phi\rangle\langle\Psi| := \langle\Psi|\Phi\rangle$, then extend linearly. Then we have further $\mathrm{Tr}[ABC] = \mathrm{Tr}[CAB]$ and for basis transformations we habe $\mathrm{Tr}[U\rho U^*] = \mathrm{Tr}[\rho]$

3.1 Composite Systems

$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
$|+\rangle_A \otimes |+\rangle_B = |{+}{+}\rangle_{AB} = \frac{1}{2}(\,|00\rangle_{AB} + |01\rangle_{AB} + |10\rangle_{AB} + |11\rangle_{AB})$
product state: $|\Psi\rangle \otimes |\phi\rangle = |\Psi\rangle_{AB}$ entagled state: $|\Psi\rangle_{AB}$ such that it cannot be written as productstate.
Partial trace $\mathrm{Tr}_{AB}[M_{AB}] = \mathrm{Tr}_A[\mathrm{Tr}_B[M_{AB}]]$

Technical stuff

- Pauli Operators: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$
- porbability density is pure state iff: $\mathrm{Tr}\left[p^2\right] = 1$
- positivity of operators: $S \geq 0$ if $\langle v|S|v\rangle \geq 0$ for all $v \in \mathcal{H}$. $\longrightarrow$ S is hermitian

4 Quantum Field Theory I

Basis transformation $\{\,|i\rangle \to |\lambda\rangle\}$ for orthonormal Basis:
$|\lambda\rangle = \sum_i |i\rangle\, |i\rangle\langle\lambda| \quad \Longrightarrow \quad$ if $\hat{a}_i^\dagger\,|0\rangle = |i\rangle$ then $\hat{a}_\lambda^\dagger\,|0\rangle = \sum_i |i\rangle\langle\lambda|\,\hat{a}_i^\dagger\,|0\rangle = |\lambda\rangle$

Like this any Hamitonian of the form $H = T + U + V$ (e.g.)
$H = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} - \sum_{i=1}^N \frac{Ze^2}{|\mathbf{x}_i|} + \sum_{i>j} \frac{e^2}{|\mathbf{x}_i - \mathbf{x}_j|}$ can be written as:
$H = \sum_{i,j} a_i^\dagger \langle i|T|j\rangle a_j + \sum_{i,j} a_i^\dagger \langle i|U|j\rangle a_j + \frac{1}{2}\sum_{ijkm} \langle i,j|V|k,m\rangle a_i^\dagger a_j^\dagger a_k a_m$

Klein Gordan equation for real scalar fields. $\varphi(x) = \bar{\varphi}(\bar{x})$ implies that the equations of motion are the same:

$$(-\partial^2 + m^2)\phi(x)$$

with $\hbar$ and $c = 1$ General solution given by $\varphi(x) = \int \widetilde{dk}\left[a(\mathbf{k})e^{ikx} + a^*(\mathbf{k})e^{-ikx}\right]$
with $\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2\omega}$ and $a(\mathbf{k}$ arbitrary function of $\mathbf{k}$. Only quatization and the canonical commutation relations unveil $a(\mathbf{k})$ as annhilation operator. We imposed that $\varphi(x)$ is real and introduced a Lorentz invariant differential for convience. $kx = \mathbf{k}\cdot\mathbf{x} - \omega t$ is the Lorentz four product.

4.1 Lorentzinvariance

Lorentstransformations $\left(\Lambda^{-1}\right)^\rho{}_\nu = \Lambda_\nu{}^\rho$

Lorentz stuff dump

- invaraince integration measure: $d^4\bar{x} = |\det\Lambda|d^4x = d^4x$
- inverse Lorentz transformation: $\left(\Lambda^{-1}\right)^\rho{}_\nu = \Lambda_\nu{}^\rho$
- $K^\mu K_\mu = \left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2 - k_z^2 = \left(\frac{\omega_o}{c}\right)^2 = \left(\frac{m_o c}{\hbar}\right)^2$ with $K^\mu = \left(\frac{\omega}{c}, \boldsymbol{k}\right)$