MATH7501 Practical 4, Semester 1-2021

Topic: Matrices, Sets Counting and Cardinality, Logic

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Date: 19-03-2021

Pre-Tutorial Activity

- Students must have familiarised themselves with units 1 to 3 contents (Matrices, Sets, Logic) of the reading materials for MATH7501

Resources

- Chapters 1 to of reading material

Q 1. Properties of Sets

Consider the universal set, $U = \{x \in \mathbb{Z}^+ : x \leq 10\}$ and the sets $A = \{x \in U : x > 7\}$ and $B = \{1, 2, 3\}$

(a) Write out all the elements of U explicitly

(b) Write out all of the elements of A explicitly

(c) What is $A \cap B$? (set of common elements from A and B)

(d) What is $A \cup B$? (set of all elements from A and B)

(e) what is $A^c \cap B$?

(f) What is $A \times B$? (Cartesian Product, which are set of ordered pairs from A and B)

(g) Write the elements of $\mathcal{P}(B)$? (Power set of B, which contains all subsets of B)

(h) What is $|\mathcal{P}(B)|$? (number of elements (or cardinality) in the power set of B)

Q 2. Properties of Sets

Consider the set $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Determine the following:

(a) $A \cup B$

$A \cup B = \{\}$

(b) $A \cap B$

(c) $A \times B$

(d) $2^{|A|}$

(e) $\{C \subset A : |C| = 3\} \setminus \{C \in 2^A : |C| = 3\} =$

(f) $|\{R \subset A \times B : \text{where } R \text{ is a function}\}| =$

Q 3. Logic

Consider the following logical expression $(A \lor B) \land \neg(A \land B)$

Here, ($\lor$ = OR, $\land$ = AND $\neg$ = NOT)

(a) Write the truth table for the above expression

A	B	$A \lor B$	$A \land B$	$\neg (A \land B)$	$(A \lor B) \land \neg (A \land B)$
T	T	□	□	□	□
T	F	□	□	□	□
F	T	□	□	□	□
F	F	□	□	□	□

(b) Write an expression using only ANDs, ORs and NOTs that is logically equivalent to the above expression

Lots of possible answers! It’s equivalent to an XOR operation (last column has the order F, T, T, F). Say

$\neg\neg ((A \lor B) \land \neg (A \land B))$, or

$(A \land \neg B) \lor (\neg A \land B)$.

Verify with a truth table to show equivalence, e.g

A	B	$A \land \neg B$	$\neg A \land B$	$(A \land \neg B) \lor (\neg A \land B)$
T	T	□	□	□
T	F	□	□	□
F	T	□	□	□
F	F	□	□	□

Q 4. Proof

Prove $\sum_{i=1}^{n} (2i - 1) = n^2$

Method 1 : Use mathematical induction

Step 1 : Show that the stament is true for $n = 1$

Step 2 : Assume the statement is true for $n = k$. That is

Step3 : Show the statement is true for $n = k + 1$

Method 2 : Expanding the sum

Q 5. Proof

Prove $\sum_{k=1}^{n} k^2 = \frac{n (n+1) (2n+1)}{6}$

Q 6. Counting

Three identical dice are rolled,
each with 6 sides labelled 1, 2, 3, 4, 5, 6. How many possible outcomes are there?

Q 7. Linear Algebra

Let α and β be two real numbers and consider the matrices,

$A = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \beta & -\alpha\beta \\ -\beta & \beta \end{pmatrix}$

(i) Set $\alpha = -1$ and determine x and y in the system of equations

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) Determine the product A A

(c) Determine the product A B

(d) Given a value of α, with α not equal to 1,
for what value of β does $B = A^{-1}$?

(e) Set $\alpha = \frac{1}{2}$ and $\beta = 2$. Determine $A^9 B^8$

In[17]:=
```
A = (1   1/2
     1   1  );
B = ( 2  -1
     -2   2 );
MatrixPower[A, 2] // MatrixForm;
```

In[•]:=
```
Anine = MatrixPower[A, 9];
Beight = MatrixPower[B, 8];
Anine.Beight // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$$

In[48]:=
```
Athree = MatrixPower[A, 3];
MatrixPower[Athree, 3];
Btwo = MatrixPower[B, 2];
MatrixPower[Btwo, 4]
```

Out[•]=
```
{{9232, -6528}, {-13056, 9232}}
```

Q 8. Linear Algebra

Set

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Define the sequence of vectors V_1, V_2, . . .via $V_{n+1} = AV_n$.

That is $V_2 = AV_1$, $V_3 = AV_2$, etc.

Consider now the sequence of Fibonacci numbers $x_0 = 1$, $x_1 = 1$ and $x_{n=1} = x_n + x_{n-1}$ for $n = 1, 2, \ldots.$

Determine a matrix B such that

$$x_n = BV_n$$

Q9. Matrices and Trigonometric Identities

For any angle θ in $[0, 2\pi]$, consider the reflection matrix:

$$A_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

You may use the following trigonometric identities for the following tasks:

- (T1) $\cos^2\theta + \sin^2\theta = 1$
- (T2) $\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$

- (T3) $\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2$

(a) Find the determinant of A_θ

(b) Find the determinant of $A_{\theta_1} A_{\theta_2}$

(c) Given some A_θ, , find the inverse matrix of A_θ