Summary of the *Optimal Placement in a Limit Order Book: an analytical approach* by Guo et al. (2016)

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The Optimal Placement Problem

The *Optimal Placement Problem* is closely related to the well studied *Optimal Execution Problem*. However, their distinctions come from the different time scales and order volumes at which algorithmic trading operates. In the Optimal Execution Problem, one wants to slice up big orders on a daily/weekly basis in order to minimize their price impact on the market while in the Optimal Placement Problem, one is interested in optimally placing smaller orders within seconds. The problem can be formulated as follows:

- N shares are to be bought before a certain time T where $T \approx 20s$
- one must find the best way (meaning, the policy that minimizes the expected cost) to split the shares into $(N_0^t, ..., N_k^t, ...)$ for $t = 1, ..., T$, where N_k^t represents the number of shares to buy at price level k at time t. Price level $k = 0$ represents the market order, $k = 1$ the best bid, $k = 2$, the second best bid, etc…
- If N shares have not been purchased by time T, one has to buy the remainder at market price at time T
- when one share of the limit order is executed, the market gives a rebate $r > 0$ (but there is no guarantee of execution) and when one share is bought at market price, the market charges a fee $f > 0$ (in exchange for guaranteed immediate execution)

- no intermediate selling is allowed but one can cancel any non-executed order and replace it with a new order at a later time

The Optimal Placement Problem is also closely related to the *market making problem*, in which one plays with the bid-ask spread to maximize profit (simultaneously placing limit orders and market orders to buy and sell), while controlling inventory risk and execution risk. This boils down to the Optimal Placement Problem with the added possibility of intermediate selling, from which it is more difficult to derive explicit strategies.

The model

The correlated random walk

The bid-ask spread dynamics is modeled by a correlated random walk, which increases or decreases one tick at each time step. Let $\{A_t\}_{t\geq 0}$ be the **best ask price** at time t, measured in ticks compared to $A_0 = 0$. This allows for negative values. We have:

$$A_t = \sum_{i=1}^{t} X_i$$

where $\{X_i\}_{i\geq 1}$ a Markov Chain on state space $\mathcal{S} = \{+1, -1\}$ with initial distribution:

$$\mu_0 = \begin{pmatrix} \mathbb{P}(X_1 = 1) \\ \mathbb{P}(X_1 = -1) \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ 1 - \tilde{p} \end{pmatrix}$$

and one-step transition matrix:

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

where $p < \frac{1}{2}$.

The motivation for fixing a $p < \frac{1}{2}$is empirical. According to *Cont and de Larrard* (2013), it is observed that high frequency price movements have a negative autocorrelation at the first lag. In our model $Cov(X_k, X_{k+1}) < 0$ if and only if $p < 1/2$. Indeed, we have:

$$\begin{aligned} \mathbb{P}(X_k = 1) &= \mathbb{P}(X_k = 1 | X_{k-1} = 1)\mathbb{P}(X_{k-1} = 1) \\ &\quad + \mathbb{P}(X_k = 1 | X_{k-1} = -1)\mathbb{P}(X_{k-1} = 1) \\ \Leftrightarrow u_k &= p u_{k-1} + (1-p)(1 - u_{k-1}) \end{aligned}$$

Denoting $u_k = \mathbb{P}(X_k = 1)$
Writing the corresponding equation for $\mathbb{P}(X_k = -1) = 1 - u_k$, we have the following system:

$$\begin{aligned}
\begin{pmatrix} u_k \\ 1-u_k \end{pmatrix} &= \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \begin{pmatrix} u_{k-1} \\ 1-u_{k-1} \end{pmatrix} && (1) \\
&= \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}^{k-1} \begin{pmatrix} u_1 \\ 1-u_1 \end{pmatrix} && (2) \\
& && (3)
\end{aligned}$$

This matrix can be diagonalized into:

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2p-1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} u_k \\ 1-u_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (2p-1)^{k-1} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} \tilde{p} \\ 1-\tilde{p} \end{pmatrix}$$

That is:

$$\mathbb{P}(X_k = 1) = \frac{1 + (2p-1)^{k-1}(2\tilde{p}-1)}{2}$$

We can now compute the mean and covariance of $\{X_k\}_{k\geq 1}$:

$$\begin{aligned}
\mathbb{E}X_k &= \mathbb{P}(X_k = 1) - \mathbb{P}(X_k = -1) \\
&= u_k - (1 - u_k) \\
&= 2u_k - 1 \\
&= (2p-1)^{k-1}(2\tilde{p}-1)
\end{aligned}$$

An alternative method to the one given in *Cont and de Ladarre* would make use of the martingale property:

$$\mathbb{E}[X_k|X_{k-1}, ..., X_1] \underbrace{=}_{\text{Markov Property}} \mathbb{E}[X_k|X_{k-1}] = \begin{cases} 2p-1, & \text{if } X_{k-1} = 1 \\ 1-2p, & \text{if } X_{k-1} = -1 \end{cases} = X_{k-1}(2p-1)$$

Therefore, given that $p \neq 1/2$, we define the process $Y_k = \frac{X_k}{(2p-1)^{k-1}}$.
$\{Y_k\}_{k\geq 1}$ is a Markov Chain and:

$$\mathbb{E}[Y_k|Y_{k-1}] = Y_{k-1}$$

Therefore, it is a martingale and if we denote $\mathcal{F}_i = \sigma(Y_k, k \leq i)$, by the Tower property we have:

$$\mathbb{E}[Y_k] = \mathbb{E}_{\mathcal{F}_{k-1}}[\mathbb{E}_{Y_k}[Y_k|\mathcal{F}_{k-1}]] = \mathbb{E}[Y_{k-1}]$$

i.e.:

$$\mathbb{E}[Y_k] = \mathbb{E}[Y_{k-1}] = ... = \mathbb{E}[Y_1] = (2\tilde{p} - 1)$$

That is:

$$\mathbb{E}[X_k] = (2\tilde{p} - 1)(2p - 1)^{k-1}$$

When $p < \frac{1}{2}$, we can see $\mathbb{E}X_k$ oscillates around 0 and unless $p = 1/2$, $\mathbb{E}[X_i] \underset{i \neq j}{\neq} \mathbb{E}[X_j]$ and $\{X_t\}_{t \geq 1}$ is not weakly stationary.

Moreover, assuming $j > i$, we can compute the auto-correlation of the Markov Chain:

$$\begin{aligned} Corr(X_i, X_j) &= \mathbb{E}X_iX_j && (4) \\ &= \sum_{r \in \mathcal{S}} \mathbb{E}[X_iX_j|X_i = r]\mathbb{P}(X_i = r) && (5) \\ &= \sum_{r \in \int} \mathbb{P}(X_i = r) \sum_{s \in \mathcal{S}} sr \times \mathbb{P}(X_j = s|X_i = r) && (6) \\ & && (7) \end{aligned}$$

And:

$$\begin{aligned} \mathbb{P}(X_j = 1|X_i = 1) &= \mathbb{P}(X_{j-i+1} = 1|X_1 = 1) && (8) \\ \Leftrightarrow u_{j-i+1} &= pu_{j-i} + (1-p)(1-u_{j-i}) && (9) \end{aligned}$$

Denoting $u_k = \mathbb{P}(X_k = 1|X_1 = 1)$, with boundary condition $u_1 = 1$. This amount to the same previous problem, only changing the initial distribution $\tilde{p}$ to 1 or 0.

Therefore:

$$\mathbb{P}(X_{j-i+1} = 1|X_1 = 1) = \frac{1 + (2p-1)^{j-i}}{2}$$

and:

$$\mathbb{P}(X_{j-i+1} = 1|X_1 = -1) = \frac{1 - (2p-1)^{j-i}}{2}$$

Finally:

$$\begin{aligned} Corr(X_i, X_j) &= \frac{1 + (2p-1)^{i-1}(2\tilde{p}-1)}{2}(2p-1)^{j-i} \\ &+ \frac{1 - (2p-1)^{i-1}(2\tilde{p}-1)}{2}(2p-1)^{j-i} \\ &= (2p-1)^{j-i} \end{aligned}$$

Which only depends on the time difference.
However, as the expectation is not time-invariant, the covariance function won't be, as well.
We can conclude:

$$\begin{aligned} Corr(X_k, X_{k+1}) = (2p-1)^{2k+1} &< 0 \\ \Leftrightarrow 2p - 1 &< 0 \\ \Leftrightarrow p &< 1/2 \end{aligned}$$

As stated in *Cont and de Larrard*. This result motivates our choice for the transition probability p.
The *remark 2* gives the limit of the rescaled random walk:

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i$$

$$\begin{aligned} \mathbb{E}(\sum_{i=1}^{n} X_i)^2 &= \mathbb{E}[X_1^2 + 2X_1(X_2 + ... + X_n) + (\sum_{i=2}^{n} X_i)^2] \\ &= \mathbb{E}[X_1^2] + 2\sum_{i=1}^{n-1} \mathbb{E}[X_1 X_{i+1}] + \mathbb{E}(\sum_{i=2}^{n} X_i)^2 \end{aligned}$$

Let us denote $u_k = \mathbb{E}(\sum_{i=k}^{n} X_i)^2$ with the boundary condition $u_n = \mathbb{E}X_n^2$. We previously showed that:

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_{j-i+1} X_1]$$

Therefore, without making any assumption on the stationarity of $\{X_k\}_{k\geq 1}$:

$$u_k - u_{k+1} = \mathbb{E}X_k^2 + 2\sum_{j=1}^{n-k} \mathbb{E}X_1X_{j+1}$$

Note that $\forall k \geq 1, \mathbb{E}X_k^2 = \mathbb{P}(X_k = 1) + \mathbb{P}(X_k = -1) = 1 = \mathbb{E}X_1^2$.
By telescoping the sum:

$$\begin{aligned}
\frac{1}{n}\ \mathbb{E}(\sum_{i=1}^{n} X_i)^2 &= \frac{1}{n}[u_1 - u_n + u_n] \\
&= \frac{1}{n}[u_n + \sum_{k=1}^{n-1} u_k - u_{k+1}] \\
&= \frac{n-1}{n}\mathbb{E}X_1^2 + \frac{2}{n}(\sum_{k=1}^{n-1}\sum_{j=1}^{n-k} \mathbb{E}X_1X_{j+1}) + \frac{1}{n}\mathbb{E}X_n^2 \\
&= \frac{n-1}{n}\mathbb{E}X_1^2 + 2\sum_{j=1}^{n-1}\frac{n-j}{n}\mathbb{E}[X_1X_{j+1}] + \frac{1}{n}\mathbb{E}X_n^2 \\
&\underset{n\to+\infty}{\to} \mathbb{E}X_1^2 + 2\sum_{j=1}^{\infty}\mathbb{E}X_1X_{j+1} \\
&= 1 + 2\sum_{j=1}^{\infty}(2p-1)^j
\end{aligned}$$

The series converges by the alternating series test, and denoting $\delta = |2p - 1| = 1 - 2p$ for $p < \frac{1}{2}$, its limit is given by:

$$
\begin{aligned}
\sum_{j=1}^{\infty}(2p-1)^j &= \sum_{j=1}^{\infty}(-1)^j\delta^j \\
&= -\delta + \delta^2 - \delta^3 + ... \\
&= (\delta^2 + \delta^4 + ...) - (\delta + \delta^3 + ...) \\
&= (\delta^2 + \delta^4 + ...)(1 - \frac{1}{\delta}) \\
&= (1 - \frac{1}{\delta})\sum_{j=1}^{\infty}\delta^{2j} \\
&= (1 - \frac{1}{\delta})\frac{\delta^2}{1-\delta^2} \\
&= \frac{\delta^2 - \delta}{1 - \delta^2} \\
&= \frac{1 - 4p + 4p^2 - 1 + 2p}{1 - 1 + 4p - 4p^2} \\
&= \frac{4p^2 - 2p}{4p - 4p^2} \\
&= \frac{2p-1}{2-2p} \\
\Rightarrow 1 + 2\sum_{j=1}^{\infty}(2p-1)^j &= 1 + \frac{2p-1}{1-p} \\
&= \frac{p}{1-p} \\
&= \mathbb{E}X_1^2 + 2\sum_{j=1}^{\infty}\mathbb{E}X_1X_{j+1}
\end{aligned}
$$

However, since the $X_i's$ are not identically distributed and do not have the same mean, nor the same variance, I believe we cannot use the Central Limit Theorem to show their scaled sum converges to a Brownian Motion, although I could be wrong.

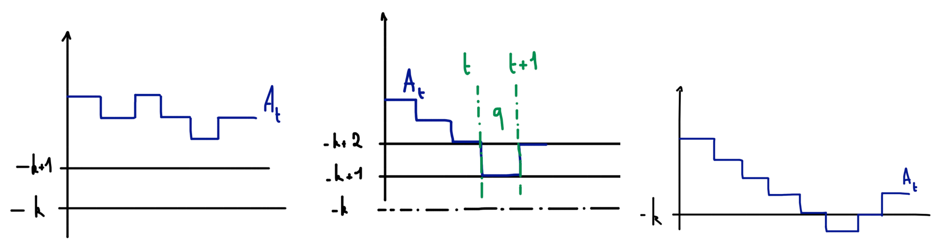
The Limit Order Book

The limit order book follows a first-in first-out (FIFO) policy, and instead of modeling its entire dynamics, the paper assumes that, if we place a limit order at price $-k$ (i.e. the ask price is k ticks lower than at the beginning), the goal being that it is executed between $t = 0$ and $t = T$, we have:

- If $\forall t \leq T,\ A_t > -(k-1)$, the probability of our limit order being executed is 0
- If $\exists t \in [0, T]$ such that $A_t \leq -k$, the order will be executed with probability 1
- If $\exists t \in [0, T-1]$ such that $A_t = -(k-1)$ and $A_{t+1} = -(k-2)$, the probability of execution is some q between time t and $t+1$

The paper further assumes **no impact of large trades on the price dynamics**. Under this assumption, the number of shares to buy won't affect the optimal strategy, therefore, we can assume without loss of generality that $N = 1$ (number of shares to buy).

The following figure summarizes the three previous statements. In the first case, the order is not executed, in the second, it has a probability q of being executed between t and $t+1$ and in the third, it is executed with probability 1.



The paper then considers two cases: the static case and the dynamic case.

The static case

In the static case, the investor needs to decide at which level of the L.O.B. she wants to place her buy order, **only at time** $t = 0$. If the order is not filled before $t < T$, then she has to buy it at market price at $t = T$.

The quantity to minimize is the expected cost of the order. We want to select the price level k (i.e. k ticks lower than initial best ask price $A_0 = 0$), that minimizes the cost C, i.e.:

$$k = \arg \min_{0 \leq k \leq T} C(k, q, T, \bar{p})$$

According to our model for the price dynamics $\{A_t\}_{t\geq 0}$, the best ask price can only move up and down **one tick at a time**. Therefore, in T timesteps starting from 0, it will be contained in $\{-T, ..., T\}$. Since we are not going to pay more than the initial best ask price, the price levels cannot be less than 0. Moreover, we denote $\bar{p} = \mathbb{P}(X_1 = 1)$.

Probability of execution for a given sample path

Let us denote:

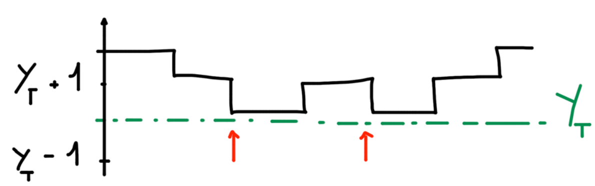
$$Y_t = \min_{0\leq s\leq t} A_s$$

For a given sample path ω, if we place a buy limit order at $Y_T(\omega) - 1$ (at time $t = 0$), each time the best ask price $A_t(\omega)$ reaches $Y_T(\omega)$, there is a probability q that the order will be executed since the price at the next timestep can only go to $Y_T(\omega) + 1$ (cf. third point of the LOB dynamics we stated earlier). Therefore, if we denote $n(\omega)$ the number of times $A_t(\omega)$ reaches $Y_T(\omega)$ for the sample path ω, i.e.:

$$n(\omega) = |\{t \leq T - 1 : A_t(\omega) = Y_T(\omega)\}|$$

We have the following probability of execution:

$$Q(\omega) = 1 - \mathbb{P}(\text{order is never executed each time } A_t(\omega) = Y_T(\omega))$$
$$Q(\omega) = 1 - (1 - q)^{n(\omega)}$$



We can see that $Q(\omega)$ is an increasing function of $n(\omega)$ and of q as well. Indeed, for a limit order placed at k, we can define three distinct scenarios for the sample path of $\{A\}_{t\geq 0}$, denoted ω:

- the minimum price for the whole period, Y_T is $-k + 1$, i.e. $\omega \in \{Y_T = -k + 1\}$
- the minimum price for the whole period, Y_T is lower than $-k + 1$, i.e. $\omega \in \{Y_T \leq -k\}$ (thus, the order will be executed with probability 1)

- the minimum price for the whole period, Y_T is higher than $-k+1$, i.e. $\omega \in \{Y_T \geq -k+2\}$ (thus, the order will never be executed, with probability 1)

Therefore, we can decompose the expected cost in three distinct scenarios:

$$\begin{aligned}
C(k,q,T,\bar{p}) &= \mathbb{E}_{\omega \in \Omega}[cost] \\
&= \mathbb{E}_{\omega \in \{Y_T=-k+1\}}[cost]\mathbb{P}(Y_T=-k+1) \\
&+ \mathbb{E}_{\omega \in \{Y_T<-k+1\}}[cost]\mathbb{P}(Y_T<-k+1) \\
&+ \mathbb{E}_{\omega \in \{Y_T>-k+1\}}[cost]\mathbb{P}(Y_T>-k+1)
\end{aligned}$$

Now:

$$\begin{aligned}
\mathbb{E}_{\omega \in \{Y_T=-k+1\}}[cost] = &\sum_{\omega \in \{Y_T=-k+1\}} \mathbb{P}(\omega)[(cost|\text{order is executed})\mathbb{P}(\text{order is executed}) \\
&+ (cost|\text{order is not executed})(1-\mathbb{P}(\text{order is executed}))] \\
&= \sum_{\omega \in \{Y_T=-k+1\}} \mathbb{P}(\omega)[\underbrace{(-k-r)}_{\text{pay L.O. price k and get rebate r}} Q(\omega) \\
&+ \underbrace{(A_T(\omega)+f)}_{\text{pay market price at time T and fee f}} (1-Q(\omega))]
\end{aligned}$$

With r, the rebate one receives when executing a limit order and f the fee of using a market order. Moreover:

$$\mathbb{E}_{\omega \in \{Y_T<-k+1\}}[cost] = 1 \times (-k-r)$$

And:

$$\mathbb{E}_{\omega \in \{Y_T>-k+1\}}[cost] = \mathbb{E}_{\omega}[A_T|Y_T > -k+1] + f$$

Therefore, we can show that:

$$C(k,q,T,\bar{p}) = \sum_{\omega \in \{Y_T=-k+1\}} \mathbb{P}(\omega)Q(\omega)(-k-r-A_T(\omega)-f) + \underbrace{C(k,0,T,\bar{p})}_{=f}$$

With:

$$-k-r-A_T(\omega)-f < 0$$

since $A_T(\omega) \geq Y_T(\omega) = -k+1$ for $\omega \in \{Y_T = -k+1\}$

Therefore the authors showed that: as the probability of execution increases, the expected cost decreases. I.e., given $q_1 < q_2$ two probabilities of execution (cf. third point of LOB dynamics):

$$C(k, q_1, T, \bar{p}) > C(k, q_2, T, \bar{p})$$

This leads to a really interesting result in the paper: if $q \neq 0$, the expected cost increases with the price level, starting from level 2. Keeping all other parameters fixed:

$$\forall q \neq 0, C(2, q, T, \bar{p}) < ... < C(T+1, q, T\bar{p})$$

One consequence of that is that an optimal strategy **only needs to compare the expected costs at the first three levels** (0 the market order, 1 and 2). When $q = 0$ or $\bar{p} \geq 1 - p$, we can further reduce the comparison between the market order 0 and the best bid order 1:

$$C(1, q, T, \bar{p}) < C(2, q, T, \bar{p}) < ... < C(T+1, q, T\bar{p})$$

Now, focusing on the parameter $\bar{p}$, the paper shows that $C(1, q, T, \bar{p})$ and $C(2, q, T, \bar{p})$ are both increasing functions of $\bar{p}$ and the optimal placement strategy uses threshold values of $\bar{p}$ to decide which price level is best. The thresholds $\bar{p}_1^*$, $\bar{p}_2^*$ and $\bar{p}_3^*$ are given by the intersections of the three costs:

$$C(0, q, T, \bar{p}_1^*) = C(1, q, T, \bar{p}_1^*)$$

$$C(0, q, T, \bar{p}_1^*) = C(2, q, T, \bar{p}_2^*)$$

$$C(1, q, T, \bar{p}_1^*) = C(1, q, T, \bar{p}_3^*)$$

For every scenario, the goal is to choose, the price level which minimizes the cost, given a certain $\bar{p}$.

The dynamic case

In the dynamic case, trades are allowed at any discrete time step t for $t \in (0, T)$. The investor needs to get **one share of stock by time T** and at any time step, she can:

- place an order at any level

- modify a previous limit order (that was not executed), i.e. cancel it or change it into a market order (at zero cost)
- At time T, an unexecuted limit order will be automatically changed to a market order

According to the static case, at any time step t, the choice actually boils down to the market order, the best bid and the second best bid. However, as the best ask price only goes up or down one tick at a time, if we place an order at the second best bid at time t, it will not be executed by time $t+1$ (assuming the volume of the order is less than the first best bid). Therefore, choosing the second best bid is equivalent to not placing an order. Our choice is thus limited to **the market order and the best bid or no order at all**.

We can view the optimal placement problem as a **Markov Decision problem**, considering the Markov structure of $\{X_t\}_{t\geq 1}$. The paper points out that, as transactions take place in a very small timeframe, we won't take any discount factor into account.

Considering the action space:

$$\mathcal{A} = \{ \underbrace{Act^N}_{\text{No order}}, \underbrace{Act^L}_{\text{Best bid in LOB}}, \underbrace{Act^M}_{\text{Market Order}} \}$$

Looking at the future, starting t, the cost should only depend on whether the best ask price is going up or down and should not depend on the present price A_{t-1}. Therefore, we shall consider $V(X_t, \alpha^t)$ the expected cost of taking policy α at time t.

The dynamic programming principle applies and a policy that is optimal for $t \in \{s, ..., T\}$ must be contained in the optimal policy for $t \in \{r, ..., T\}$ where $r \leq s$. Let α^* be the optimal policy. We have:

$$\begin{aligned}
V_t^* = V(X_t, \alpha^*) &= \min_a V(X_t, \{a, \alpha^*_{t+1}, ..., \alpha^*_T\}) \\
&= \min_a \mathbb{E}[\text{cost of taking actions } \{a, \alpha^*_{t+1}, ..., \alpha^*_T\}] \\
&= \sum_{x\in\mathcal{S}} \mathbb{P}(X_t = x)\{(\text{cost of taking action } a \text{ given that } X_t = x) \\
&+ \mathbb{E}[\text{cost of taking actions } \{\alpha^*_{t+1}, ..., \alpha^*_T\} | X_t = x]\} \\
&= \sum_{x\in\mathcal{S}} \mathbb{P}(X_t = x)\{(\text{cost of taking action } a \text{ given that } X_t = x) + V^*_{t+1}\}
\end{aligned}$$

The optimal policy is obtained by taking the arg max of the above problem.

And the boundary conditions are:

$$V_T^* = f \text{ the fee of taking a market order}$$

$$a_T^* = Act^M$$

The authors then derive an explicit optimal strategy for $t \geq 1$ based on thresholds:

- if $t < t_1^*$, $\alpha_t^*(-1) = Act^L$ and $\alpha_t^*(1) = Act^N$
- if $t_1^* \leq t < t_2^*$, $\alpha_t^*(-1) = \alpha_t^*(1) = Act^L$
- if $t_2^* \leq t$, $\alpha_t^*(-1) = Act^M$, $\alpha_t^*(1) = Act^L$

For the case $t = 0$, the authors derive an optimal strategy in terms of $\bar{p}$ (the initial distribution of $\{X_t\}_{t\geq 1}$

Takaways

One can observe that the policy the authors derived is sensitive to the remaining time before T as well as the market momentum, due to the price's mean-reversion property we saw in the first part. In fact, the policy becomes more conservative as we approach T. Moreover, if $X_t = 1$, the price is likely to go down next and we must choose between the market order or the best bid, i.e. we have to take advantage of the opportunity. On the contrary, if $X_t = -1$, the price is likely to go up next and we must choose between the best bid and waiting.