VE203 Mid 1 Logic and Sets

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Outline

Proof by Truth Table

Example

"$\oplus$" is called **exclusive or** (xor). $A \oplus B$ is True if and only if A differs from B. Prove that $A \oplus B \equiv (\neg A \wedge B) \vee (A \wedge \neg B)$

Proof.

A	B	$\neg A \wedge B$	$A \wedge \neg B$	$(\neg A \wedge B) \vee (A \wedge \neg B)$	$A \oplus B$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	F	F	F	F

$\square$

Reminders for your exam

- If you want to use truth table, make sure your steps are clear. **Do not** only list two columns of LHS and RHS. **Do not** make your truth table look like two copied columns.
- Do not use **01** in truth table; do not use **Set Notation** in truth table. Make sure the first row consists of propositional logics.

Question 4 in Assignment I

Example

Suppose that a truth table in n propositional variables is specified. Write the *disjunctive normal form*.

- In the assignment, as long as you correctly and completely describe the process to get a *disjunctive normal form*, you get full marks.
- But it would be better if you could explain why it's a correct transformation.

Solution

1. Pick the rows with true values.
2. For each row, take the conjunction of true variables and the negation of false variables.
3. Take the disjunction of these conjunctions.

Disjunctive Normal Form

- First you should know why we need DNF: to transform a truth table to a logical expression. Thus you could use the logical expression for further calculation.

Example

A	B	$P(A, B)$
T	T	T
T	F	T
F	T	F
F	F	T

Solution

$$(A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge \neg B)$$

Some Important Logical Equivalences

Properties

Implication	$A \Rightarrow B \equiv \neg A \vee B \equiv \neg B \Rightarrow \neg A$ **Important!**
Distributivity	$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
	$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$
Absorption	$A \vee (A \wedge B) \equiv A$
	$A \wedge (A \vee B) \equiv A$
De Morgan's	$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$
	$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$
Contraposition	$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

Rules of Inference

- These rules have fancy latin names, but they are not hard to understand.
- You don't need to remember the names of these rules.

Rule of Inference	Name
$\begin{array}{rl} & A \Rightarrow B \\ & A \\ \hline \therefore & B \end{array}$	Modus (Ponendo) Ponens *Mode that affirms (by affirming)*
$\begin{array}{rl} & A \Rightarrow B \\ & \neg B \\ \hline \therefore & \neg A \end{array}$	Modus (Tollendo) Tollens *Mode that denies (by denying)*
$\begin{array}{rl} & A \Rightarrow B \\ & B \Rightarrow C \\ \hline \therefore & A \Rightarrow C \end{array}$	Transitive Hypothetical Syllogism

Rule of Inference	Name
$\begin{array}{rl} & A \vee B \\ & \neg A \\ \hline \therefore & B \end{array}$	Modus Tollendo Ponens *Mode that affirms by denying*
$\begin{array}{rl} & \neg(A \wedge B) \\ & A \\ \hline \therefore & \neg B \end{array}$	Modus Ponendo Tollens *Mode that denies by affirming*
$\begin{array}{rl} & A \vee B \\ & \neg A \vee C \\ \hline \therefore & B \vee C \end{array}$	Resolution

and others...

Arguments in propositional logic

Definition

An argument is a finite sequence of propositions. All propositions except for the final statement are called **premises** while the final statement is called the **conclusion**. We say that an argument is **valid** if the truth of all premises implies the truth of the conclusion.

$$\begin{array}{rr} & P_1 \\ & \vdots \\ & P_n \\ \hline \therefore & C \end{array}$$

- i.e. $(P_1 \wedge P_2 \cdots \wedge P_n) \Rightarrow C$ is a tautology. Note that it does not mean that C is always true!
- In addition, the only possible situation where an argument wrong is that all of the premises are true but the conclusion is wrong.
- To disprove the validness, providing a counterexample is a wiser strategy than long proofs.

Arguments in propositional logic

Definition

If, in addition to being valid, an argument has only true premises, we say that the argument is **sound**. In that case, its conclusion is **true**.

- Hence, to disprove the soundness, you only need to find a false premise.
- To make a valid argument sound, make all the premises true by assigning the variables properly.

Predicate Logic

- When using quantifiers $\forall$ and $\exists$, pay attention to the order.
- $\forall x \exists y P(x, y)$ is different from $\exists y \forall x P(x, y)$
- $\forall x P(x) \vee \forall x Q(x)$ is different from $\forall x(P(x) \vee Q(x))$
- If you are creating a counterexample, do not involve other variables than A in $P(A)$. For example, do not define $P(x)$ as $x > 1$ and $y < 1$.
- You cannot change the **domain of disclosure** after you define it.

Some mistakes:

Example

Let the domain of $Q(x)$ be $x \in \mathbb{N}$. $Q(x)$ is true if $x \geq 0$. Thus $\exists x Q(x)$ is true (so far everything is fine).
$\cdots$ Then on the right hand side, let $w < 0$. Thus $\exists w Q(w)$ is false (no!).

Vacous Truth

Vacuous Truth

If the domain of the universal quantifier $\forall$ is the empty set $M = \emptyset$, then the statement $(\forall x \in M)A(x)$ is defined to be true.

- Remind the definition of **Chain Complete**. Why a chain complete poset has the least element?
- Empty chain, due to vacuous truth, is also a chain.

Tips for your exam

- If you replace symbols like $\Leftrightarrow$ and $\Rightarrow$ with $\vee$, $\wedge$ or $\neg$, the expression might be terribly longer.
- So before you do terrible calculations, try proving by rules of inference. "Maybe" you can avoid tedious calculations.

Outline

Subsets

- $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

Definition

We say that X is a proper subset of Y if $X \subseteq Y$ but $X \neq Y$. In that case we write $X \subset Y$.

- Pease use $\subset$ to denote proper subset in your assignments and exams rather than $\subsetneq$.

Powerset and Cardinality

Definition

If a set has a finite number of elements, we define the **cardinality** of X to be this number, denoted by $|X|$.

Definition

If X is a set, then the **power set** of X, denoted $\mathscr{P}(X)$, is the set of all subsets of X. I.e.

$$\mathscr{P}(X) = \{A|A \subseteq X\}$$

This means the expressions $A \in \mathscr{P}(X)$ and $A \subseteq X$ are equivalent.

Operations on Sets

- Proof by Venn Diagram will earn 0 point.
- If you want to use $\overline{A}$ to denote A^c, define this notation and denote the M at first.
- Do not misuse $\wedge$ and $\cap$; $\vee$ and $\cup$.
- Before you replace $\backslash$ with $\wedge$ and ^{c}, try using the properties below. Make your calculations as simple as possible.

Properties

- $(A \cup B)\backslash C = (A\backslash C) \cup (B\backslash C)$
- $(A \cap B)\backslash C = (A\backslash C) \cap (B\backslash C)$
- $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$
- $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$
- $A\backslash B = B^c \cap A$
- $(A\backslash B)^c = A^c \cup B$

Operations on Sets

Definition

$$\bigcup_{k=0}^{n} A_k := A_0 \cup \cdots \cup A_n, \quad \bigcap_{k=0}^{n} A_k := A_0 \cap \cdots \cap A_n$$

More generally, if A is a set and $X \subseteq \mathscr{P}(A)$,

$$\bigcup X = \{x \in A | (\exists y \in X)(x \in y)\}, \quad \bigcap X = \{x \in A | (\forall y \in X)(x \in y)\}$$

- I.e., X is a set of some subsets of A.
- $\bigcup X$ is the union of all the sets **in** X.
- $\bigcap X$ is the intersection of all the sets **in** X.
- $\bigcup X$ and $\bigcap X$ are both subsets of A.

Cartesian Product of Sets

Definition

$$A \times B := \{(a, b) | a \in A \land b \in B\}$$

$A \times B$ is called the cartesian product of A and B.

- It's easy to define ordered n-tuples $A_1 \times A_2 \times \cdots \times A_n$
- A^n is short for $A \times \cdots \times A$
- The cartesian product of some sets is still a set

Cartesian Product of Sets

Some mistakes:

- Any subset of $\mathbb{N} \times \mathbb{N}$ can be written as $A \times B$ (no!)

Russell's Paradox

Theorem

The set of all sets that are not members of themselves is not a set. I.e.

$$R := \{x | x \notin x\} \textit{ is not a set}$$

Proof.

If $R \in R$, then R should satisfy $R \notin R$ by definition.
If $R \notin R$, then it should be put in R by definition. Both of the assumptions lead to contradiction. □

- Similar proof is useful in other questions (*Cantor's Theorem*)
- To prove something related to sets, proof by contradiction is useful. You may construct a "set" with particular constraints, and substitute this "set" back to our hypothesis.