

VE203 Mid 1 Logic and Sets

Tianyi Ge

Fall 2018

Outline

- 1 Logic and Sets
 - Logic
 - Set Theory

Proof by Truth Table

Example

" \oplus " is called **exclusive or** (xor). $A \oplus B$ is True if and only if A differs from B . Prove that $A \oplus B \equiv (\neg A \wedge B) \vee (A \wedge \neg B)$

Proof.

A	B	$\neg A \wedge B$	$A \wedge \neg B$	$(\neg A \wedge B) \vee (A \wedge \neg B)$	$A \oplus B$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	F	F	F	F



Reminders for your exam

- If you want to use truth table, make sure your steps are clear. **Do not** only list two columns of LHS and RHS. **Do not** make your truth table look like two copied columns.
- Do not use **01** in truth table; do not use **Set Notation** in truth table. Make sure the first row consists of propositional logics.

Question 4 in Assignment I

Example

Suppose that a truth table in n propositional variables is specified. Write the *disjunctive normal form*.

- In the assignment, as long as you correctly and completely describe the process to get a *disjunctive normal form*, you get full marks.
- But it would be better if you could explain why it's a correct transformation.

Solution

- 1 Pick the rows with true values.
- 2 For each row, take the conjunction of true variables and the negation of false variables.
- 3 Take the disjunction of these conjunctions.

Disjunctive Normal Form

- First you should know **why** we need DNF: **to transform a truth table to a logic expression**. Thus you could use the logic expression for further calculation.

Example

A	B	$P(A, B)$
T	T	T
T	F	T
F	T	F
F	F	T

Solution

$$(A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge \neg B)$$

Some Important Logical Equivalences

Properties

Implication $A \Rightarrow B \equiv \neg A \vee B \equiv \neg B \Rightarrow \neg A$ **Important!**

Distributivity $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$

Absorption $A \vee (A \wedge B) \equiv A$

$$A \wedge (A \vee B) \equiv A$$

De Morgan's $\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$

$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

Contraposition $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

Rules of Inference

- These rules have fancy latin names, but they are not hard to understand.
- You don't need to remember the names of these rules.

Rule of Inference	Name
$\frac{A \Rightarrow B \quad A}{\therefore B}$	Modus (Ponendo) Ponens <i>Mode that affirms (by affirming)</i>
$\frac{A \Rightarrow B \quad \neg B}{\therefore \neg A}$	Modus (Tollendo) Tollens <i>Mode that denies (by denying)</i>
$\frac{A \Rightarrow B \quad B \Rightarrow C}{\therefore A \Rightarrow C}$	Transitive Hypothetical Syllogism

Rule of Inference	Name
$\frac{A \vee B \quad \neg A}{\therefore B}$	Modus Tollendo Ponens <i>Mode that affirms by denying</i>
$\frac{\neg(A \wedge B) \quad A}{\therefore \neg B}$	Modus Ponendo Tollens <i>Mode that denies by affirming</i>
$\frac{A \vee B \quad \neg A \vee C}{\therefore B \vee C}$	Resolution

and others...

Arguments in propositional logic

Definition

An argument is a finite sequence of propositions. All propositions except for the final statement are called **premises** while the final statement is called the **conclusion**. We say that an argument is **valid** if the truth of all premises implies the truth of the conclusion.

$$\begin{array}{c} P_1 \\ \vdots \\ P_n \\ \hline \therefore C \end{array}$$

- i.e. $(P_1 \wedge P_2 \cdots \wedge P_n) \Rightarrow C$ is a tautology. Note that it **does not** mean that C is always true!
- To disprove the validness, make all of the premises true but the conclusion wrong.
- To disprove the validness, providing a counterexample is a wiser strategy than long proofs.

Arguments in propositional logic

Definition

If, in addition to being valid, an argument has only true premises, we say that the argument is **sound**. In that case, its conclusion is **true**.

- Hence, to disprove the soundness, you only need to find a **false premise**.
- To make a valid argument sound, make all the premises true by **assigning the variables** properly.

Predicate Logic

- When using quantifiers \forall and \exists , pay attention to the order.
- $\forall x \exists y P(x, y)$ is different from $\exists y \forall x P(x, y)$
- $\forall x P(x) \vee \forall x Q(x)$ is different from $\forall x (P(x) \vee Q(x))$
- If you are creating a counterexample, do not involve other variables than A in $P(A)$. For example, do not define $P(x)$ as $x > 1$ and $y < 1$.
- You cannot change the **domain of discourse** after you define it.

Some mistakes:

Example

Let the domain of $Q(x)$ be $x \in \mathbb{N}$. $Q(x)$ is true if $x \geq 0$. Thus $\exists x Q(x)$ is true (so far everything is fine).

... Then on the right hand side, let $w < 0$. Thus $\exists w Q(w)$ is false (**no!**).

Vacuous Truth

Vacuous Truth

If the domain of the universal quantifier \forall is the empty set $M = \emptyset$, then the statement $(\forall x \in M)A(x)$ is defined to be true.

- Remind the definition of **Chain Complete**. Why a chain complete poset has the least element?
- Empty chain, due to vacuous truth, is also a chain.

Tips for your exam

- If you replace symbols like \Leftrightarrow and \Rightarrow with \vee , \wedge or \neg , the expression might be terribly longer.
- So before you do terrible calculations, try proving by rules of inference. "Maybe" you can avoid tedious calculations.

Outline

- 1 Logic and Sets
 - Logic
 - Set Theory

Subsets

- $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

Definition

We say that X is a proper subset of Y if $X \subseteq Y$ but $X \neq Y$. In that case we write $X \subset Y$.

- Please use \subset to denote proper subset in your assignments and exams rather than \subsetneq .

Powerset and Cardinality

Definition

If a set has a finite number of elements, we define the **cardinality** of X to be this number, denoted by $|X|$.

Definition

If X is a set, then the **power set** of X , denoted $\mathcal{P}(X)$, is the set of all subsets of X . I.e.

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}$$

This means the expressions $A \in \mathcal{P}(X)$ and $A \subseteq X$ are equivalent.

Operations on Sets

- Proof by Venn Diagram will earn 0 point.
- If you want to use \overline{A} to denote A^c , define this notation and denote the M at first.
- Do not misuse \wedge and \cap ; \vee and \cup .
- Before you replace \setminus with \wedge and c , try using the properties below. Make your calculations as simple as possible.

Properties

- $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$
- $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$
- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
- $A \setminus B = B^c \cap A$
- $(A \setminus B)^c = A^c \cup B$

Cartesian Product of Sets

Definition

$$A \times B := \{(a, b) | a \in A \wedge b \in B\}$$

$A \times B$ is called the cartesian product of A and B .

- It's easy to define ordered n-tuples $A_1 \times A_2 \times \cdots \times A_n$
- A^n is short for $A \times \cdots \times A$
- The cartesian product of some sets is still a set

Cartesian Product of Sets

Some mistakes:

Example

Any subset of $\mathbb{N} \times \mathbb{N}$ can be written as $A \times B$ (**no!**)

Russell's Paradox

Theorem

The set of all sets that are not members of themselves is not a set. I.e.

$$R := \{x \mid x \notin x\} \text{ is not a set}$$

Proof.

If $R \in R$, then R should satisfy $R \notin R$ by definition.

If $R \notin R$, then it should be put in R by definition.

Both of the assumptions lead to contradiction. □

- Similar proof is useful in other questions (in *Cantor's Theorem*)
- To prove something related to sets, you may **construct a "set"** with particular constraints, and **substitute this "set" back** to our hypothesis. Finally it leads to contradiction.

Keep Calm and Carry on!



Figure: Zach's little reward for Mid 1 High!