Q1. Prove that a set A is (Dedekind) infinite if an only if there exists an injective function $f : \mathbb{N} \longrightarrow A$.
(3 marks)

Q2. Let $(L, \preceq)$ be a lattice and let $a, b \in L$. Prove that the following are equivalent:

(i) $a \preceq b$

(ii) $a \vee b = b$

(iii) $a \wedge b = a$

(3 marks)

Q3. Let $(L, \preceq)$ be a lattice. Prove that for all $a, b, c \in L$,

$$(a \vee b) \vee c = a \vee (b \vee c)$$

(2 marks)

Q4. Without using Cantor's Pairing Function, prove that $\mathbb{N} \times \mathbb{N}$ is countable.
(2 marks)

Q5. For all $n \in \mathbb{N}$, use $[n]$ to denote the set of predecessors of n in the natural numbers with the usual ordering, i.e. $[0] = \emptyset$ and for all $n \geq 1$, $[n] = \{0, \ldots, n-1\}$. Define

$$S = \{f \mid (\exists n \in \mathbb{N})(f \text{ is a function with } \operatorname{dom} f = [n] \text{ and } \operatorname{ran} f \subseteq \mathbb{N})\}$$

(i) Use the fact that every natural number has a unique factorisation into primes to show that S is countable. Is $|S| = |\mathbb{N}|$?

(ii) Define $\preceq_1 \subseteq S \times S$ by: $f \preceq_1 g$ where $f : [n] \longrightarrow \mathbb{N}$ and $g : [m] \longrightarrow \mathbb{N}$, and $A = [n] \cap [m]$ if and only if

$$(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(j) = g(j))) \text{ or } (\forall i \in A)(f(i) = g(i)) \wedge ([n] \subseteq [m]),$$

where $\leq$ and $<$ are the usual orders on $\mathbb{N}$. Is $(S, \preceq_1)$ a partial order? Is $(S, \preceq_1)$ a linear order? Is $(S, \preceq_1)$ chain complete? Is $(S, \preceq_1)$ a lattice? Is $(S, \preceq_1)$ a well-order? Prove your answers.

(iii) Define $\preceq_2 \subseteq S \times S$ by: $f \preceq_2 g$ where $f : [n] \longrightarrow \mathbb{N}$ and $g : [m] \longrightarrow \mathbb{N}$ if and only if

$$[n] \subseteq [m] \text{ and } (\forall i \in [n])(f(i) \leq g(i))$$

Is $(S, \preceq_2)$ a partial order? Is $(S, \preceq_2)$ chain complete? Is $(S, \preceq_2)$ a lattice? Is $(S, \preceq_2)$ a linear order? Prove your answers.

(7 marks)

Q6. Give an explicit formula that defines a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. You do not have to prove that this formula works!
(1 mark)

Q7. Every nonzero rational number has a unique representation in the form $\frac{(-1)^k a}{b}$ where $k \in \{0, 1\}$ and $a, b \in \mathbb{N}$ with $a, b \neq 0$ and the greatest common divisor of a and b is 1. Use this to show that $\mathbb{Q}$ is countable.
(3 marks)

Q8. Let $\pi(x, y)$ be Cantor's pairing function. Find $x, y \in \mathbb{N}$ such that $\pi(x, y) = 223$. You may do this question however you wish.
(1 mark)

Q9. Prove that $|\mathcal{P}(\mathbb{N} \times \mathbb{N})| = |\mathcal{P}(\mathbb{N})|$.
(1 mark)

Q10. Prove that $|\mathbb{N}| < |\mathbb{R}|$ is uncountable.*(Hint: Start with a bijection $f : \mathbb{N} \longrightarrow \mathbb{R}$ and try to construct a real that can not be in the range of f. There may also be a proof that uses continued fractions and Cantor's Theorem, but I have not thought about that too deeply.)*
(3 marks)