VE203 Review Class

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Outline

1. RC Week 1
   - Preliminary information
   - Logic
   - Set Theory

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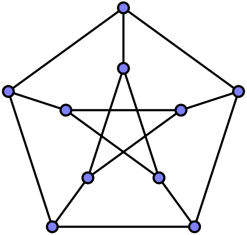
Meet Zach

He's a really nice person!

Figure: Cool Zach!

How to find me

- E-mail: ji_getianyi@sjtu.edu.cn
- Recitation Class: Monday 20:00-22:00 @F103
- Office Hour: Wednesday 19:00-21:00 @326D



Outline

Operator precedence in propositional logic

The operator precedence is important. Parentheses often get you dizzy.

$$\neg > \wedge > \vee > \Rightarrow > \Leftrightarrow$$

Example

See the difference between the following two expressions

$$\exists x\,(\neg P(x) \Rightarrow (\exists y P(y) \vee \exists x Q(x,z)))$$

$$\exists x\,((\neg P(x) \Rightarrow \exists y P(y)) \vee \exists x Q(x,z))$$

Proof by Truth Table

Example

"$\oplus$" is called **exclusive or** (xor). $A \oplus B$ is True if and only if A differs from B. Prove that $A \oplus B \equiv (\neg A \wedge B) \vee (A \wedge \neg B)$

Proof.

A	B	$\neg A \wedge B$	$A \wedge \neg B$	$(\neg A \wedge B) \vee (A \wedge \neg B)$	$A \oplus B$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	F	F	F	F

□

- Also it would be helpful if you know how to transform a truth table back to a symbolic expression (Kmap).

Some Important Logical Equivalences

For more logical equivalences, please refer to Zach's lecture slides. You'd better prove those tiny properties by yourselves.

Properties

Implication	$A \Rightarrow B \equiv \neg A \vee B \equiv \neg B \Rightarrow \neg A$ **Important!**
Distributivity	$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
	$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$
Absorption	$A \vee (A \wedge B) \equiv A$
	$A \wedge (A \vee B) \equiv A$
De Morgan's	$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$
	$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$
Contraposition	$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

Arguments in propositional logic

Definition

An argument is a finite sequence of propositions. All propositions except for the final statement are called **premises** while the final statement is called the **conclusion**. We say that an argument is **valid** if the truth of all premises implies the truth of the conclusion.

$$\begin{array}{rc} & P_1 \\ & \vdots \\ & P_n \\ \hline \therefore & C \end{array}$$

- i.e. $(P_1 \land P_2 \cdots \land P_n) \Rightarrow C$ is a tautology. Note that it does not mean that C is always true!
- In addition, the only possible situation where an argument wrong is that all of the premises are true but the conclusion is wrong.
- Your intuition will be of great help here.

Arguments in propositional logic

Definition

If, in addition to being valid, an argument has only true premises, we say that the argument is **sound**. In that case, its conclusion is **true**.

$$\begin{array}{ll} & \text{Anything that can flow is liquid} \\ & \text{Cats can flow} \\ \hline \therefore & \text{Cats are liquid} \end{array}$$

- It's valid but not sound.
- Both premises are false.
- Hence, to disprove the soundness, you only need to find a false premise.

Predicate Logic

Contraposition of Quantifiers

- $\neg((\forall x \in M)A(x)) \Leftrightarrow (\exists x \in M)\neg A(x)$
- $\neg((\exists x \in M)A(x)) \Leftrightarrow (\forall x \in M)\neg A(x)$

Example

Derive the contraposition of

$$\exists x\,(\neg P(x) \wedge (\exists y P(y) \vee \exists x Q(x, z)))$$

Solution

$$\neg(\forall x(P(x) \vee (\forall y \neg P(y) \wedge \forall x \neg Q(x, z))))$$

Vacous Truth

Vacuous Truth

If the domain of the universal quantifier $\forall$ is the empty set $M = \emptyset$, then the statement $(\forall x \in M)A(x)$ is defined to be true.

- "All pink elephants can fly" is vacuously true.
- If a class in which all the boys are handsome is called a *Dalabengba*, then a class full of girls is a *Dalabengba*!

Tautologies in predicate logic

- To prove that an argument is a tautology, you may prove by contradiction.
- Assume $P_1, P_2, \cdots, P_n$ are all **true** and C is **false**, which will lead to contradiction.
- For example, some variable x is both true and false simultaneously.

Outline

Naive Set Theory

Definition

A **set** is a **collection** of objects that contains no information about order, and in which repetitions of objects are ignored.

- Naive Set Theory is a flawed theory but we assume it's valid in this course.
- In contrast, Axiomatic Set Theory has definite rules for collections.
- In pure set theory every object is itself a set. It's useful

Definition

The empty set $\emptyset = \{x | x \neq x\}$.

Subsets

Definition

X is a subset of Y, writing $X \subseteq Y$; in other words,

$$X \subseteq Y \Leftrightarrow \forall x(x \in X \Rightarrow x \in Y)$$

- $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

Definition

We say that X is a proper subset of Y if $X \subseteq Y$ but $X \neq Y$. In that case we write $X \subset Y$.

- Pease use $\subset$ to denote proper subset in your assignments and exams rather than $\subsetneq$.

Powerset and Cardinality

Definition

If a set has a finite number of elements, we define the **cardinality** of X to be this number, denoted by $|X|$.

Definition

If X is a set, then the **power set** of X, denoted $\mathscr{P}(X)$, is the set of all subsets of X. I.e.

$$\mathscr{P}(X) = \{A | A \subseteq X\}$$

This means the expressions $A \in \mathscr{P}(X)$ and $A \subseteq X$ are equivalent.

- Thus $X \in \mathscr{P}(X)$
- Is it possible that $\mathscr{P}(X) \in X$?

Universal Set

What do we know about the set $U = \{x|x = x\}$?

- $U \in U$
- $\mathscr{P}(U) \in U$
- $\mathscr{P}(U) \subset U$ (because every element in $\mathscr{P}(U)$ is also in U)
- U is known as the **universal set**
- Naive Set Theory allows the existence of the universal set, but it leads to troubles (**Russell's Paradox**).
- In addition, you will encounter **the set of all sets** V very soon.
- Almost every other Axiomatic Set Theory like **ZFC Set Theory** forbids the universal set.

Operations on Sets

Properties

- $(A \cup B)\backslash C = (A\backslash C) \cup (B\backslash C)$
- $(A \cap B)\backslash C = (A\backslash C) \cap (B\backslash C)$
- $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$
- $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$
- $A\backslash B = B^c \cap A$
- $(A\backslash B)^c = A^c \cup B$

- Proof by Venn Diagram is not recommended. Instead, for example, you may indicate $(A \cup B)\backslash C$ as

$$\{x | (x \in A \vee x \in B) \wedge x \notin C\}$$

Operations on Sets

Definition

$$\bigcup_{k=0}^{n} A_k := A_0 \cup \cdots \cup A_n, \quad \bigcap_{k=0}^{n} A_k := A_0 \cap \cdots \cap A_n$$

More generally, if A is a set and $X \subseteq \mathscr{P}(A)$,

$$\bigcup X = \{x \in A | (\exists y \in X)(x \in y)\}, \quad \bigcap X = \{x \in A | (\forall y \in X)(x \in y)\}$$

- I.e., X is a set of some subsets of A.
- $\bigcup X$ is the union of all the sets **in** X.
- $\bigcap X$ is the intersection of all the sets **in** X.
- $\bigcup X$ and $\bigcap X$ are both subsets of A.

Operations on Sets

Example

Let

$$X = \{A \in \mathscr{P}(\mathbb{N}) | (\exists k \in \mathbb{N})(\forall n \in \mathbb{N})(n \in A \vee n = k)\}$$

Then

$$\bigcup X = \mathbb{N}, \quad \bigcap X = \emptyset$$

- X contains $\mathbb{N}$ as well as the subsets of $\mathbb{N}$ that exclude one number from $\mathbb{N}$
- For example, the subsets like $\mathbb{N} \backslash \{0\}$, $\mathbb{N} \backslash \{1\}$ are in X

Cartesian Product of Sets

Definition

$$A \times B := \{(a, b) | a \in A \wedge b \in B\}$$

$A \times B$ is called the cartesian product of A and B.

- It's easy to define ordered n-tuples $A_1 \times A_2 \times \cdots \times A_n$
- A^n is short for $A \times \cdots \times A$
- The cartesian product of some sets is still a set

Russell's Paradox

Theorem

The set of all sets that are not members of themselves is not a set. I.e.

$$R := \{x|x \notin x\} \textit{ is not a set}$$

Proof.

By contradiction, suppose R is a set. if $R \in R$, then R should satisfy $R \notin R$ by definition. If $R \notin R$, then it should be put in R by definition. Both of the assumptions lead to contradiction. $\square$

- It shows the inconsistency of Naive Set Theory.
- However, other set theories are not required in this course.