Assignment 7 Q3

Find all solutions to the system:

$$2x \equiv 4 (\text{mod } 5)$$
$$3x \equiv 5 (\text{mod } 7)$$
$$7x \equiv 1 (\text{mod } 13)$$

►

Since

$$3 \cdot 2 \equiv 1 (\text{mod } 5)$$
$$5 \cdot 3 \equiv 1 (\text{mod } 7)$$
$$2 \cdot 7 \equiv 1 (\text{mod } 13)$$

then equivalently, we rewrite the system as

$$x \equiv 2 (\text{mod } 5)$$
$$x \equiv 4 (\text{mod } 7)$$
$$x \equiv 4 (\text{mod } 13)$$

Since $m_1 = 5, m_2 = 7, m_3 = 13$ are pairwise relatively prime, then $[M_k]_{m_k} \in (\mathbb{Z}/m_k\mathbb{Z})^*$. Now $m = 5 \cdot 7 \cdot 13 = 455$ and consider

$$M_1 = \frac{455}{5} = 91, M_2 = \frac{455}{7} = 65, M_3 = \frac{455}{13} = 35$$

Note that $y_1 = 1$, $y_2 = 4$, $y_3 = 3$, then

$$x = \sum_{k=1}^{n} a_k M_k y_k = 2 \cdot 91 \cdot 1 + 4 \cdot 65 \cdot 4 + 4 \cdot 35 \cdot 3 = 1642 = 277 (\text{mod } 455)$$

Due to the uniqueness of solutions (mod 455), all the solutions to this system are

$$x = 455k + 277, \quad \forall k \in \mathbb{Z}$$

★★★Get familiar with the procedure! Speed up your calculation!

Assignment 8 Q1

Prove that if (a_n) is a sequence that satisfies a linear homogeneous recurrence relation of degree 2 whose characteristic polynomial has only one real root α, then there exists $q_1, q_2 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$a_n = q_1 \alpha^n + q_2 n \alpha^n$$

►

Proof. Suppose that the characteristic equation of a_n is $(\lambda - \alpha)^2 = 0$, then we have a linear homogeneous recurrence relation

$$a_n = 2\alpha a_{n-1} - \alpha^2 a_{n-2}, \quad n \geq 2$$

with $a_0 = u$ and $a_1 = v$.

Now we only consider that $\alpha \neq 0$. Otherwise, 0^0 does not make sense. Then we prove that there exists $q_1 = u$ and $q_2 = \frac{v-u\alpha}{\alpha}$ such that for all $n \in \mathbb{N}$,

$$a_n = u\alpha^n + \frac{v - u\alpha}{\alpha} n\alpha^n$$

First, it's easy to see that $a_0 = u\alpha^0 + 0 = u$ and $a_1 = u\alpha + \frac{v}{\alpha}\alpha - u\alpha = v$.

For $n \geq 2$,

$$\begin{aligned}
\textbf{LHS} &= a_n = q_1\alpha^n + q_2 n\alpha^n \\
\textbf{RHS} &= 2\alpha a_{n-1} - \alpha^2 a_{n-2} \\
&= 2\alpha(q_1\alpha^{n-1} + q_2(n-1)\alpha^{n-1}) - \alpha^2(q_1\alpha^{n-2} + q_2(n-2)\alpha^{n-2}) \\
&= (2q_1 - q_1)\alpha^n + q_2(2n - 2 - n + 2)\alpha^n \\
&= q_1\alpha^n + q_2 n\alpha^n \\
&= \textbf{LHS}
\end{aligned}$$

□

★★★To prove the existence? Just find it!

Assignment 8 Q4

Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$$

with $a_0 = 3$, $a_1 = 2$, $a_2 = 5$.

▶

The characteristic equation is

$$\begin{aligned}
\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0 \\
(\lambda - 2)^2(\lambda - 3) = 0
\end{aligned}$$

Hence the homogeneous part of (a_n) has the form

$$a_n = q_1 2^n + q_2 n 2^n + q_3 3^n$$

Since $f'(n) = n4^n$, then we guess the particular solution is

$$p(n) = c_0 n 4^n + c_1 4^n$$

This requires

$$
\begin{aligned}
c_0 n4^n + c_1 4^n &= 7(c_0(n-1)4^{n-1} + c_1 4^{n-1}) - 16(c_0(n-2)4^{n-2} + c_1 4^{n-2}) \\
&\quad + 12(c_0(n-3)4^{n-3} + c_1 4^{n-3}) + n4^n \\
&= (16 + 15c_0)n4^{n-2} + (15c_1 - 5c_0)4^{n-2}
\end{aligned}
$$

$$
\begin{cases} c_0 = 1 + \frac{15}{16}c_0 \\ c_1 = \frac{15}{16}c_1 - \frac{5}{16}c_0 \end{cases} \Rightarrow \begin{cases} c_0 = 16 \\ c_1 = -80 \end{cases}
$$

This means that (a_n) has the form

$$
a_n = q_1 2^n + q_2 n 2^n + q_3 3^n + 16n4^n - 80 \cdot 4^n
$$

Thus

$$
\left(\begin{array}{ccc|c} 1 & 0 & 1 & -3-(-80) \\ 2 & 2 & 3 & 2-(-256) \\ 4 & 8 & 9 & 5-(-768) \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 28 \\ 0 & 1 & 0 & \frac{55}{2} \\ 0 & 0 & 1 & 49 \end{array}\right)
$$

Thus

$$
a_n = 28 \cdot 2^n + \frac{55}{2} n 2^n + 49 \cdot 3^n + 16n4^n - 80 \cdot 4^n
$$

★★★For particular solutions, start from the multiplicity of s

Assignment 8 Q5

For all $n \in \mathbb{N} \backslash \{0\}$, let

$$
a_n = \sum_{i=1}^{n} i^4
$$

By finding a recurrence relation that (a_n) satisfies and solving that recurrence relation, find an expression for the terms of the sequence (a_n).

▶

$$
a_n = a_{n-1} + n^4, \quad a_1 = 1
$$

Since $\lambda = 1$, then the homogeneous part of (a_n) is $q_1 1^n = q_1$. On the other hand, since $n^4 = n^4 \times 1^n$, the particular part is

$$
p(n) = c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n
$$

Thus

$$
\begin{aligned}
&c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n \\
=&c_5(n-1)^5 + c_4(n-1)^4 + c_3(n-1)^3 + c_2(n-1)^2 + c_1(n-1) + c_0 + n^4 \\
=&c_5 n^5 + (c_4 - 5c_5 + 1)n^4 + (c_3 - 4c_4 + 10c_5)n^3 + (c_2 - 3c_3 + 6c_4 - 10c_5)n^2 \\
+&(c_1 - 2c_2 + 3c_3 - 4c_4 + 5c_5)n - c_1 + c_2 - c_3 + c_4 - c_5
\end{aligned}
$$

$$
\begin{cases}
c_4 = c_4 - 5c_5 + 1 \\
c_3 = c_3 - 4c_4 + 10c_5 \\
c_2 = c_2 - 3c_3 + 5c_4 - 10c_5 \\
c_1 = c_1 - 2c_2 + 3c_3 - 4c_4 + 5c_5 \\
0 = -c_1 + c_2 - c_3 + c_4 - c_5
\end{cases}
\Rightarrow
\begin{cases}
c_5 = \frac{1}{5} \\
c_4 = \frac{1}{2} \\
c_3 = \frac{1}{3} \\
c_2 = 0 \\
c_1 = -\frac{1}{30}
\end{cases}
$$

This means that (a_n) has the form

$$a_n = q_1 + \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

Since $a_1 = 1$, then $q_1 = 0$ Thus

$$a_n = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

★★★ **When $f'(n)$ is a polynomial, imagine 1^n**

Assignment 8 Q6

Q6. Solve the simultaneous recurrence relations

$$
\begin{cases}
a_n = 3a_{n-1} + 2b_{n-1} \\
b_n = a_{n-1} + 2b_{n-1}
\end{cases}
$$

with $a_0 = 1$ and $b_0 = 2$.

▶

We multiply the second equation by 3

$$
\begin{cases}
a_n = 3a_{n-1} + 2b_{n-1} \\
3b_n = 3a_{n-1} + 6b_{n-1}
\end{cases}
$$

Then we subtract the difference back to the second equation

$$
\begin{aligned}
a_n &= 3b_n - 4b_{n-1} \\
b_n &= (3b_{n-1} - 4b_{n-2}) + 2b_{n-1} \\
&= 5b_{n-1} - 4b_{n-2}
\end{aligned}
$$

Since $a_0 = 1$ and $b_0 = 2$, then $b_1 = a_0 + 2b_0 = 5$.

Thus $b_n = 1 + 4^n$.

Similarly, $a_n = 5a_{n-1} - 4a_{n-2}$ with $a_0 = 1$ and $a_1 = 7$. Thus $a_n = -1 + 2 \cdot 4^n$

Therefore,

$$
\begin{aligned}
a_n &= -1 + 2 \cdot 4^n \\
b_n &= 1 + 4^n
\end{aligned}
$$

★★★ **Substitution!**

Assignment 8 Q11

Prove that if $n \in \mathbb{N}$ with $n \geq 4$, then there exists a 3-regular graph of order n if and only if n is even.

▶

Proof. $\Rightarrow$: Since it's a 3-regular graph, we obtain the total edge number

$$|E| = \frac{3|V|}{2} = \frac{3n}{2}$$

Since $|E|$ has to be an integer, then n is even.

$\Leftarrow$: Since n is even, then we apply induction.

Let $P(n)$ be that for all $n \geq 4$ and n is even, there exists a 3-regular graph.

$P(4)$ holds since K_4.

Assume $P(k)$ holds. For $n = k+2$, here gives the procedure to get a 3-regular graph with order $k+2$ (denoted as G_{k+2}) from a 3-regular graph with order k (denoted as G_k).

1. Consider a pair of new vertices with new edge (v_k, v_{k+1})
2. Break an arbitrary edge (v_0, v_1) from G_k. Build edges (v_0, v_k) and (v_1, v_k)
3. Break an arbitrary edge (v_2, v_3) from G_k. Build edges (v_2, v_k) and (v_3, v_k)

The degree of every vertex is 3. $P(k+2)$ holds.

Thus $P(n)$ holds for $n \geq 4$ and n is even. □

Assignment 8 Q13

Let G be a graph of order 10 and size 15.

1. Is it the case that $\Delta(G) \geq 3$?

   ▶

   Proof. Suppose $\Delta(G) \leq 2$, then

   $$|E| = 15 = \frac{\sum d_G(v)}{2} \leq \frac{|V| \times \Delta(G)}{2} \leq \frac{|V| \times 2}{2} = |V| = 10$$

   which leads to contradiction.

   Hence $\Delta(G) \geq 3$ holds. □

2. Is it the case that $\delta(G) \geq 2$?

   ▶

   Proof. Find a counterexample where $\delta(G) = 0$ □

★★★ Understand $\delta(G)$ and $\Delta(G)$! Use it in inequalities!

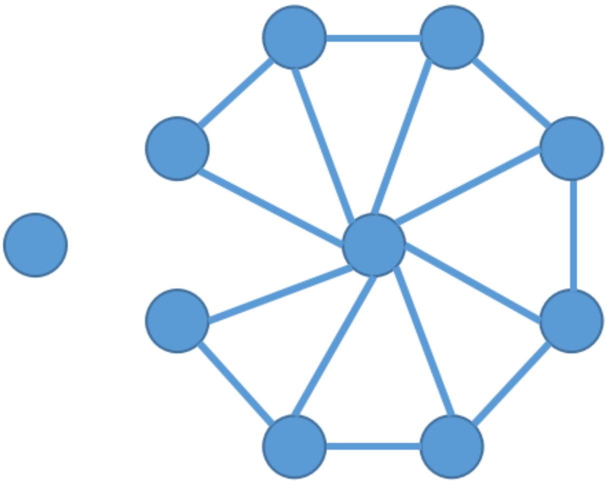

Figure 1: order= 10, size= 15