Chapter III

PENTAGONAL NUMBERS

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Chapter III

PENTAGONAL NUMBERS

The theory of numbers has been called the Queen of Mathematics because of its rich varieties of fascinating problems. Many numbers exhibit fascinating properties, they form sequences, they form patterns and so on.

A pentagonal number is a figurate number that extends the concept of triangular and square numbers to the pentagon, but, unlike the first two, the patterns involved in the construction of pentagonal numbers are not rotationally symmetrical. The n^{th} pentagonal number p_n is the number of distinct dots in a pattern of dots consisting of the outlines of regular pentagons whose sides contain 1 to n dots, overlaid so that they share one vertex. For instance, the third one is formed from outlines comprising 1, 5 and 10 dots, but the 1, and 3 of the 5, coincide with 3 of the 10 – leaving 12 distinct dots, 10 in the form of a pentagon, and 2 inside...

p_n is given by the formula:

$$p_n = \frac{1}{2}n(3n-1)$$

for $n \geq 1$. The first few pentagonal numbers are:

1, 5, 12, 22, 35,...

The generating function for the pentagonal numbers is

$$\frac{x(2x+1)}{(1-x)^3} = x + 5x^2 + 12x^3 + 22x^4 + \ldots..$$

Every pentagonal number is $\frac{1}{3}$ of a triangular number. There are conjectured to be exactly 210 positive integers that can not be represented using three pentagonal numbers, namely 4, 8, 9, 16, 19, 20, 21, 26, 30, 31, 33, 38, 42, 43, 50, 54,..., 20250, 33066.

There six positive integers that can not be expressed using four pentagonal numbers: 9, 21, 31, 43, 55 and 89.

All positive integers can be expressed using five pentagonal numbers. Letting x_i be the set of numbers relatively prime to 6, the generalized pentagonal number is given by $(x_i^2 - 1)/24$. Also, letting y_i be the subset of the x_i for which $x_i \equiv 5 \pmod 6$ the usual pentagonal numbers are given by $(y_i^2 - 1)/24$.

Generalized pentagonal numbers are obtained from the formula given above, but with n taking values in the sequence 0, 1, -1, 2, -2 ,3,-3,4,-4... producing the sequence:

0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77, 92, 100, 117,126, 145, 155, 176, 187, 210, 222, 247, 260, 287, 301, 330, 345, 376, 392, 425, 442....

Generalized pentagonal numbers are important to Euler's theory of partitions, as expressed in his pentagonal number theorem.

The number of dots inside the outermost pentagon of a pattern forming a pentagonal number is itself a generalized pentagonal number.

Pentagonal numbers should not be confused with centered pentagonal numbers.

Tests for pentagonal numbers

One can test whether a positive integer x is a pentagonal number by computing

$$n = \frac{1}{6}\sqrt{24x+1} + 1.$$

If n is an integer, then x is the n^{th} pentagonal number. If n is not an integer, then x is not pentagonal .

In this chapter, explicit formulas for the ranks of pentagonal numbers which are simultaneously equal to hexagonal, heptagonal and octagonal numbers in turn are presented, and determines pairs of pentagonal numbers whose ratios are non-square integers. A few interesting relations among pentagonal numbers are also presented [26].

First prove the following theorems relating to explicit formula for pentagonal numbers.

Equality of the pentagonal numbers with the hexagonal numbers

Theorem 3.1

Explicit formula for the ranks of pentagonal numbers which are simultaneously equal to hexagonal numbers are given by

$$m = \frac{1}{2^{n+3}}\ 3\ [(1+\sqrt{3})^{2n+3} + (1-\sqrt{3})^{2n+3} + 2^{n+2}],$$

$$k = \frac{1}{\sqrt{3}}\ 2^{n}\text{[illegible]}[(1+\sqrt{3})^{2n+3} - (1-\sqrt{3})^{2n+3} + \sqrt{3}\ 2^{n+2}].$$

Proof

It is well known that the pentagonal and hexagonal numbers are given by

$$P_m = \frac{m}{2}(3m-1), \qquad \ldots(3.1)$$

$$H_k = k(2k-1), \qquad \ldots(3.2)$$

where m and k are the corresponding ranks.

The identity $P_m = H_k$, leads to the equation

$$\frac{m}{2}(3m-1) = k(2k-1).$$

Multiplying both sides by 24, we get

$$36m^2 - 12m = 3(16k^2 - 8k).$$

By writing complete squares, we get

$$(6m-1)^2 - 1 = 3[(4k-1)^2 - 1].$$

If we take Y=6m-1, ...(3.3)

X=4k-1, ...(3.4)

then we get

$$Y^2 - 3X^2 = -2, \qquad \ldots(3.5)$$

which is the famous Pell's equation.

If $y_0 + \sqrt{3}x_0$ is the fundamental solution of $Y^2 - 3X^2 = 1$, then the sequence of solutions can be obtained by

$$y_n + \sqrt{3}x_n = (y_0 + \sqrt{3}x_0)^{n+1}, \text{ n = 0, 1, 2 ...}$$

$$= (2+\sqrt{3})^{n+1}, \text{ n = 0, 1, 2...} \qquad ...(3.6)$$

Further, we have

$$y_n - \sqrt{3}x_n = (2-\sqrt{3})^{n+1}, \text{ n= 0, 1, 2...} \qquad ...(3.7)$$

Adding and subtracting the above equations (3.6-3.7), we get

$$y_n = \frac{1}{2}[(2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}], \qquad ...(3.8)$$

$$x_n = \frac{1}{2\sqrt{3}}[(2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}]. \qquad ...(3.9)$$

The general solutions of the equation (3.5) are given by

$$X_n = X_0 y_n + Y_0 x_n = y_n + x_n, \qquad ...(3.10)$$

$$Y_n = Y_0 + DX_0 x_n = y_n + 3x_n. \qquad ...(3.11)$$

Using the equations (3.8), (3.9) in (3.10), (3.11), we get

$$Y_n = \frac{1}{2}[(2+\sqrt{3})^{n+1}(\sqrt{3}+1) - (2-\sqrt{3})^{n+1}(\sqrt{3}-1)]$$

$$= \frac{1}{2^{n+2}}[(\sqrt{3}+1)^{2n+3} - (\sqrt{3}-1)^{2n+3}], \qquad ...(3.12)$$

$$X_n = \frac{1}{2\sqrt{3}}[(2+\sqrt{3})^{n+1}(\sqrt{3}+1) + (2-\sqrt{3})^{n+1}(\sqrt{3}-1)]$$

$$= \frac{1}{2\sqrt{3}}[\frac{(\sqrt{3}+1)^{2n+3}}{2^{n+1}} + \frac{(\sqrt{3}-1)^{2n+3}}{2^{n+1}}]$$

$$= \frac{1}{2^{n+2}\sqrt{3}}[(\sqrt{3}+1)^{2n+3} + (\sqrt{3}-1)^{2n+3}] \qquad ...(3.13)$$

In view of (3.3) and (3.4), the ranks of pentagonal and hexagonal numbers are given by

$$m = \frac{1}{2^{n+2}} 3[(1+\sqrt{3})^{2n+3} + (1-\sqrt{3})^{2n+3} + 2^{n+2}],$$

$$k = \frac{1}{\sqrt{3}} 2^{n+4}[(1+\sqrt{3})^{2n+3} - (1-\sqrt{3})^{2n+3} + \sqrt{3}2^{n+2}].$$

The values of ranks m and k will be in integers only when n = 4(r–1), r =1, 2, 3....

Some examples are presented in the table 3.1.

Table 3.1

S. No	Values of n	Rank m	Rank k	Pentagonal number P_m	Hexagonal number H_k
1	0	1	1	1	1
2	4	165	143	40755	40755
3	8	31977	27693	1533776805	1533776805
4	12	6203341	5372251	5722156241751	5722156241751

Equality of the pentagonal numbers with the heptagonal numbers

Theorem 3.2

Explicit formula for the ranks of pentagonal numbers which are simultaneously equal to heptagonal are given by

$$m = \frac{1}{12\sqrt{15}}[(4+\sqrt{15})^n(21+5\sqrt{15}) - (4-\sqrt{15})^n(21-5\sqrt{15}) + 2\sqrt{15}],$$

$$k = \frac{1}{12\sqrt{15}}[(4+\sqrt{15})^n(25+7\sqrt{15}) - (4-\sqrt{15})^n(25-7\sqrt{15}) + 6\sqrt{15}],$$

where r = 4(r-1), r =1,2,3...

Proof

It is well known that the pentagonal and heptagonal numbers are given by

$$P_m = \frac{m}{2}(3m-1), \qquad \ldots(3.14)$$

$$H_K' = \frac{k}{2}(5k-3), \qquad \ldots(3.15)$$

where m and k are the corresponding ranks.

The identity $P_m \equiv H_k'$, leads to the equation

$$\frac{m}{2}(3m-1) = \frac{k}{2}(5m-3)$$

$$3m^2 - m = 5k^2 - 3k.$$

$$3(m^2 - \frac{m}{3}) = 5(k^2 - \frac{3}{5}k).$$

By writing complete squares, we get

$$3[(m-\frac{1}{6})^2 - \frac{1}{36}] = 5[(k-\frac{3}{10})^2 - \frac{9}{100}].$$

$$\frac{3}{36}[(6m-1)^2 - 1] = \frac{5}{100}[(10k-3)^2 - 9].$$

$$20[(6m-1)^2 - 1] = 12[(10k-3)^2 - 9].$$

$$5[(6m-1)^2 - 1] = 3[(10k-3)^2 - 9].$$

If we take X =10k-1, ...(3.16)

Y = 6m-1, ...(3.17)

then we get

$$5(Y^2 - 1) = 3(X^2 - 9).$$

$$5Y^5 - 5 = 3X^2 - 27.$$

$$3X^2 - 5Y^2 = 22. \qquad \ldots(3.18)$$

By introducing the linear transformations

$$X=u+5t, \qquad \text{...(3.19)}$$

$$Y=u+3t, \qquad \text{...(3.20)}$$

the equation (3.18) leads to the equation

$$3(u+5t)^2-5(u+3t)^2=22.$$

$$3(u^2+10ut+25t^2)-5(u^2+6ut+9t^2)=22.$$

$$3u^2+30ut+75t^2-5u^2-30ut-45t^2=22.$$

Simplifying we get

$$-2u^2+30t^2=22.$$

Dividing by 2, we get

$$-u^2+15t^2=11.$$

(or) $u^2-15t^2=-11,$ ...(3.21)

which is the famous Pell's equation.

The general solutions of the above equation (3.21) are given by

$$t_n=t_0U_n+u_0T_n=U_n+2T_n,$$

$$u_n=u_0U_n+Dt_0T_n=2U_n+15T_n,$$

where $u_0+\sqrt{15}t_0=2+\sqrt{15}$ is the fundamental solution of (3.21) and $U_0+\sqrt{15}T_0=4+\sqrt{15}$ is the fundamental solution of the Pellian

$$u^2-15t^2=1. \qquad \text{...(3.22)}$$

The infinite number of solutions of (3.22) can be obtained by

$$U_n+\sqrt{15}T_n=(U_0+\sqrt{15}T_0)^{n+1}=(4+\sqrt{15})^n, \qquad \text{...(3.23)}$$

where n = 1, 2...

Further, we have

$$U_n-\sqrt{15}T_n=(U_0-\sqrt{15}T_0)^{n+1}=(4-\sqrt{15})^n, \qquad \text{...(3.24)}$$

where n = 1, 2...

Adding and subtracting the above two equations (3.23-3.24), we get

$$U_n = \frac{1}{2}[(4+\sqrt{15})^n + (4-\sqrt{15})^n], \qquad \ldots(3.25)$$

$$T_n = \frac{1}{2\sqrt{15}}[(4+\sqrt{15})^n - (4-\sqrt{15})^n]. \qquad \ldots(3.26)$$

Using (3.25), (3.26) in the general solutions of (3.21), we get

$$u_n = \frac{1}{12}[(4+\sqrt{15})^n(2+\sqrt{15}) - (4-\sqrt{15})^n(2-\sqrt{15})], \qquad \ldots(3.27)$$

$$t_n = \frac{1}{12\sqrt{15}}[(4+\sqrt{15})^n(2+\sqrt{15}) - (4-\sqrt{15})^n(\sqrt{15}-2)]. \qquad \ldots(3.28)$$

In view of (3.16), (3.17), (3.19) and (3.20), the ranks of pentagonal and heptagonal numbers are given by

$$m = \frac{1}{12\sqrt{15}}[(4+\sqrt{15})^n(21+5\sqrt{15}) - (4-\sqrt{15})^n(21-5\sqrt{15}) + 2\sqrt{15}],$$

$$k = \frac{1}{12\sqrt{15}}[(4+\sqrt{15})^n(25+7\sqrt{15}) - (4-\sqrt{15})^n(25-7\sqrt{15}) + 6\sqrt{15}].$$

The values of ranks m and k will be in integers only when n = 4(r–1), where r = 1,2,3...

Some examples are presented in the table 3.2.

Table 3.2

S. No	Values of n	Rank m	Rank k	Pentagonal number P_m	Hexagonal number H'_k
1	0	1	1	1	1
2	2	54	42	4347	4347
3	4	3337	2585	16701685	16701685
4	6	206830	160210	64167869935	64167869935

Equality of the pentagonal numbers with the octagonal numbers

Theorem 3.3

Explicit formula for the ranks of pentagonal numbers which are simultaneously equal to octagonal numbers are given by

$$m = \frac{1}{12}\{(3 + 2\sqrt{2})^{n+1} (1 + 2\sqrt{2}) + (3 - 2\sqrt{2})^{n+1} (1 - 2\sqrt{2}) + 2\}$$

$$k = \frac{1}{12\sqrt{2}}\{(3 + 2\sqrt{2})^{n+1}(2\sqrt{2} + 1) + (3 - 2\sqrt{2})^{n+1}(2\sqrt{2}-1)+4\sqrt{2}\}.$$

Proof

It is well known that the pentagonal and octagonal numbers are given by

$$P_m = \frac{1}{2} (3m - 1), \qquad \text{...(3.29)}$$

$$O_k = k (3k - 2) \qquad \text{...(3.30)}$$

where m and k are the corresponding ranks.

The identity $P_m \equiv O_k$, leads to the equation

$$\frac{m}{2}(3m-1) = k(3k-2).$$

Multiplying both sides by 24, we get

$$\begin{aligned} 36m^2 - 12m &= 24(3k^2 - 2k) \\ &= 2(36k^2 - 24k) \end{aligned}$$

By writing complete squares, we get

$$(6m-1)^2 - 1 = 2\,[(6r-2)^2 - 4].$$

If we take $Y = 6m - 1$, ...(3.31)

$X = 6r - 2$, ...(3.32)

then we get

$$Y^2 - 1 = 2X^2 - 8$$

$$Y^2 - 2X^2 = -7, \qquad \ldots(3.33)$$

which is the famous Pell's equation.

The general solutions of the equation (3.33) are given by

$X_n = X_0 y_n + Y_0 x_n = 2y_n + x_n$,

$Y_n = Y_0 y_n + 2X_0 x_n = y_n + 4x_n$,

where $Y_0 + \sqrt{2}X_0 = 1 + 2\sqrt{2}$ is the fundamental solution of (3.33), $y_0 + \sqrt{2}x_0 = 3 + 2\sqrt{2}$ is the fundamental solution of the Pellian

$$Y^2 - 2X^2 = 1. \qquad \ldots(3.34)$$

The infinite number of solutions of (3.34) can be obtained by

$$y_n + \sqrt{2}\,x_n = (3 + 2\sqrt{2})^{n+1}, n = 0,1,2,....... \qquad \ldots(3.35)$$

Further, we have

$$y_n - \sqrt{2}\,x_n = (3 - 2\sqrt{2})^{n+1}, n = 0,1,2,....... \qquad \ldots(3.36)$$

Adding and subtracting the above two equations (3.35) and (3.36), we get

$$y_n = \frac{1}{2}[(3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}], \qquad \ldots(3.37)$$

$$x_n = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}]. \qquad \ldots(3.38)$$

Using (3.37), (3.3.8) in the general solutions of (3.33), we get

$$Y_n = \frac{1}{2}[(3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}] + \frac{4}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}]$$

$$= \frac{1}{2}[(3+2\sqrt{2})^{n+1}(1+2\sqrt{2}) + (3-2\sqrt{2})^{n+1}(1-2\sqrt{2})], \qquad \ldots(3.39)$$

$$X_n = \frac{1}{2}[(3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}] + \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}]$$

$$= \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1}(2\sqrt{2}+1) + (3-2\sqrt{2})^{n+1}(2\sqrt{2}-1)] \qquad \ldots(3.40)$$

In view of (3.31) and (3.32), the ranks of the pentagonal and octagonal numbers are given by

$$6m-1 = \frac{1}{2}[(3+2\sqrt{2})^{n+1}(1+2\sqrt{2})+(3-2\sqrt{2})^{n+1}(1-2\sqrt{2})]$$

$$6m = \frac{1}{2}[(3+2\sqrt{2})^{n+1}(1+2\sqrt{2})+(3-2\sqrt{2})^{n+1}(1-2\sqrt{2})]+1$$

Hence $m = \frac{1}{12}[(3+2\sqrt{2})^{n+1}(1+2\sqrt{2})+(3-2\sqrt{2})^{n+1}(1-2\sqrt{2})+2]$

$$6r-2 = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1}(2\sqrt{2}+1)+(3-2\sqrt{2})^{n+1}(2\sqrt{2}-1)]$$

$$6r = \frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1}(2\sqrt{2}+1)+(3-2\sqrt{2})^{n+1}(2\sqrt{2}-1)]+2$$

Hence $r = \frac{1}{12\sqrt{2}}[(3+2\sqrt{2})^{n+1}(2\sqrt{2}+1)+(3-2\sqrt{2})^{n+1}(2\sqrt{2}-1)+4\sqrt{2}]$.

The values of ranks m and r will be in integers only when $n = 4r-3$, $r = 1, 2, 3, \ldots$

Some examples are presented in the table 3.3.

Table 3.3

S. No	Values of n	Rank m	Rank r	Pentagonal number p_m	Octagonal number o_r
1	1	11	8	176	176
2	5	12507	8844	23463132	23463132
3	9	14432875	10205584	312461813932000 (15 digits)	312461813932000 (15 digits)
4	13	16655525051	11777234708	416109772078405066376 (21 digits)	416109772078405066376 (21 digits)

Also obtain the following observations.

Observation

Pairs of pentagonal numbers whose ratios are non-square integers

Proof

If P_m and P_K are denoting pentagonal numbers, then the identity $P_m=\alpha P_K$, where $\alpha >1$, a square–free number, leads to the equation

$$\frac{m}{2}(3m-1)=\alpha\frac{k}{2}(3k-1).$$

$$3m^2-m=\alpha(3k^2-k).$$

By writing complete squares, we get

$$(6m-1)^2-1=\alpha[(6k-1)^2-1].$$

If we take X = 6K-1, ...(3.41)

Y = 6m-1, ...(3.42)

then we get

$$Y^2-1=\alpha(X^2-1)=\alpha X^2-\alpha.$$

$$Y^2=\alpha X^2-(\alpha-1)=\alpha X^2-N, where\, N=\alpha-1. \quad ...(3.43)$$

$$Y^2-\alpha X^2=-N \quad ...(3.44)$$

The general solutions of the equation (3.44) are given by

$X_n=X_0y_n+Y_0x_n$,

$Y_n=Y_0y_n+\ \alpha X_0x_n$, *where* $Y_0+\sqrt{\alpha}X_0$ is the fundamental solution of (3.44), $y_0+\sqrt{\alpha}x_0$ is the fundamental solution of the Pellian

$$Y^2-\alpha X^2=1. \quad ...(3.45)$$

The infinite number of solutions of (3.45) are given by

$$y_n + \sqrt{\alpha}\, x_n = (y_0 + x_0\sqrt{\alpha})^{n+1}, n = 0,1,2....... \quad ...(3.46)$$

Further we have,

$$y_n - \sqrt{\alpha}\, x_n = (y_0 - x_0\sqrt{\alpha})^{n+1}, n = 0,1,2....... \quad ...(3.47)$$

Adding and subtracting the above equations (3.46) and (3.47) we get

$$y_n = \frac{1}{2}[(y_0 + \sqrt{\alpha}\, x_0)^{n+1} + (y_0 - \sqrt{\alpha} x_0)^{n+1}] \ , \quad ...(3.48)$$

$$x_n = \frac{1}{2\sqrt{\alpha}}[(y_0 + \sqrt{\alpha}\, x_0)^{n+1} - (y_0 - \sqrt{\alpha} x_0)^{n+1}]. \quad ...(3.49)$$

Using (3.48), (3.49) in the general solutions of (3.44), we get

$$X_n = \frac{X_0}{2}[(y_0 + \sqrt{\alpha}\, x_0)^{n+1} + (y_0 - \sqrt{\alpha} x_0)^{n+1}] + \frac{Y_0}{2\sqrt{\alpha}}[(y_0 + \sqrt{\alpha} x_0)^{n+1} - (y_0 - \sqrt{\alpha} x_0)^{n+1}]$$

$$= \frac{1}{2\sqrt{\alpha}}[(y_0 + \sqrt{\alpha} x_0)^{n+1}(X_0\sqrt{\alpha} + Y_0) + (y_0 - \sqrt{\alpha} x_0)^{n+1}(X_0\sqrt{\alpha} - Y_0)]$$

$$...(3.50)$$

$$Y_n = \frac{Y_0}{2}[(y_0 + \sqrt{\alpha}\, x_0)^{n+1} + (y_0 - \sqrt{\alpha} x_0)^{n+1}] + \frac{\alpha X_0}{2\sqrt{\alpha}}[(y_0 + \sqrt{\alpha} x_0)^{n+1} - (y_0 - \sqrt{\alpha} x_0)^{n+1}]$$

$$= \frac{1}{2}[(y_0 + \sqrt{\alpha} x_0)^{n+1}(X_0\sqrt{\alpha} + Y_0) + (y_0 - \sqrt{\alpha} x_0)^{n+1}(Y_0 - X_0\sqrt{\alpha})]$$

$$...(3.51)$$

In view of (3.41) and (3.42), the ranks of the pentagonal numbers are obtained as

$$m = \frac{1}{12}[(y_0 + \sqrt{\alpha} x_0)^{n+1}(X_0\sqrt{\alpha} + Y_0) + (y_0 - \sqrt{\alpha} x_0)^{n+1}(Y_0 - X_0\sqrt{\alpha}) + 2], \quad ...(3.52)$$

$$k = \frac{1}{12\sqrt{\alpha}}[(y_0 + \sqrt{\alpha} x_0)^{n+1}(X_0\sqrt{\alpha} + Y_0) + (y_0 - \sqrt{\alpha} x_0)^{n+1}(X_0\sqrt{\alpha} - Y_0) + 2\sqrt{\alpha}] \ ,$$

$$...(3.53)$$

where n = 0,1,2 ....

Some particular cases

When $\alpha = 2$, the equation (3.44) leads to the Pell's equation

$$Y^2 - 2X^2 = -1.$$

In view of (3.52) and (3.53), the ranks of the pentagonal numbers are given by

$$m = \frac{1}{12}\left[(3+2\sqrt{2})^{n+1}(1+\sqrt{2}) + (3-2\sqrt{2})^{n+1}(1-\sqrt{2}) + 2\right],$$

$$k = \frac{1}{12\sqrt{2}}\left[(3+2\sqrt{2})^{n+1}(1+\sqrt{2}) + (3-2\sqrt{2})^{n+1}(\sqrt{2}-1) + 2\sqrt{2}\right].$$

The values of ranks m and k will be in integers only when $n = 4r + 1$, $r = 0, 1, 2, 3\ldots$

Some examples are presented in the table 3.4.

Table 3.4

S. No	Values of n	Rank m	Rank k	Pentagonal number P_m	Pentagonal number P_k
1	1	7	5	70	35
2	5	7887	5577	93303210	46651605

When $\alpha = 3$, the equation (3.44) leads to the Pell's equation

$$Y^2 - 3X^2 = -2.$$

In view of (3.52) and (3.53), the ranks of the pentagonal numbers are given by

$$m = \frac{1}{12}\left[(2+\sqrt{3})^{n+1}(5+3\sqrt{3}) + (2-\sqrt{3})^{n+1}(5-3\sqrt{3}) + 2\right],$$

$$k = \frac{1}{12\sqrt{3}}\left[(2+\sqrt{3})^{n+1}(5+3\sqrt{3}) + (2-\sqrt{3})^{n+1}(3\sqrt{3}-5) + 2\sqrt{3}\right].$$

The values of ranks m and k will be in integers only when $n = 6r + 1, r = 0, 1, 2...$

Some examples are presented in the table 3.5.

Table 3.5

S. No	Values of n	Rank m	Rank k	Pentagonal number P_m	Pentagonal number P_k
1	1	12	7	210	70
2	5	31977	18462	1533776805	511258935

When $\alpha = 5$, the equation (3.44) leads to the Pell's equation

$$Y^2 - 5X^2 = -4.$$

In view of (3.52) and (3.53), the ranks of the pentagonal numbers are obtained by

$$m = \frac{1}{12}\left[(9+4\sqrt{5})^{n+1}(1+\sqrt{5}) + (9-4\sqrt{5})^{n+1}(1-\sqrt{5}) + 2\right],$$

$$k = \frac{1}{12\sqrt{5}}\left[(9+4\sqrt{5})^{n+1}(1+\sqrt{5}) + (9-4\sqrt{5})^{n+1}(\sqrt{5}-1) + 2\sqrt{5}\right].$$

The values of ranks m and k will be in integers only when $n = (4r + 1), r = 0, 1, 2...$

Some examples are presented in the table 3.6.

Table 3.6

S. No	Values of n	Rank m	Rank k	Pentagonal number P_m	Pentagonal number P_k
1	1	87	39	11310	2262
2	5	9003087	4026303	121583358792810	24316671758562

Observations

A few interesting relations among pentagonal numbers

It presents below a few interesting relations among pentagonal numbers. P_m and T_S are denoting the pentagonal and the triangular numbers. First derives the recurrence relation of pentagonal numbers which satisfy the following recurrence relations.

Recurrence relations 3.1

$$P_{m+1} + P_{m-1} - 2P_m = 3.$$

Proof

By the definition of the pentagonal number,

$$P_m = \frac{m}{2}(3m-1). \quad \text{...(3.54)}$$

Thus $P_{m-1} = \frac{(m-1)}{2}[3(m-1)-1]$

$$= \frac{(m-1)}{2}[3m-4], \quad \text{...(3.55)}$$

$$P_{m+1} = \frac{(m+1)}{2}[3(m+1)-1]$$

$$= \frac{(m+1)}{2}[3m+3-1]$$

$$= \frac{(m+1)}{2}[3m+2] \quad \text{...(3.56)}$$

From the equation (3.55), we get

$$P_{m-1} = \frac{(m-1)}{2}[(3m-1)-3]$$

$$= \frac{m(3m-1)}{2} - \frac{(3m-1)}{2} - \frac{3}{2}(m-1)$$

$$= P_m - \frac{3m}{2} + \frac{1}{2} - \frac{3}{2}m + \frac{3}{2}$$

$$= P_m - 3m + 2. \quad \text{...(3.57)}$$

From the equation (3.56), we get

$$P_{m+1} = \frac{(m+1)}{2}[(3m-1)+3]$$

$$= \frac{m(3m-1)}{2} + \frac{1}{2}(3m-1) + \frac{3}{2}(m+1)$$

$$= P_m + \frac{3}{2}m - \frac{1}{2} + \frac{3}{2}m + \frac{3}{2}$$

$$= P_m + 3m + 1 \qquad \ldots(3.58)$$

Adding the equations (3.57), (3.58), we get

$$P_{m+1} + P_{m-1} = P_m - 3m + 2 + P_m + 3m + 1$$

$$= 2P_m + 3$$

that is, $P_{m+1} + P_{m-1} - 2P_m = 3.$ ...(3.59)

This is the recurrence relation for the pentagonal numbers.

Example 3.1

If we take m = 2, then $T_1 = 1$, the first pentagonal number, $T_2 = 5$, the second pentagonal number, $T_3 = 12$, the third pentagonal number.

Thus $P_{m+1} + P_{m-1} - 2P_m = 12 + 1 - 2(5) = 13 - 10 = 3$. Hence the equation (3.59) is satisfied.

Identity 3.1

The sum of the pentagonal numbers can be expressed as the polynomial in s.

that is, $\sum_{m=1}^{s} P_m = \frac{1}{2}s^2(s+1) = sT_S, where\ T_S\ is\ the\ s^{th}\ triangular\ number$.

Proof

The m^{th} pentagonal number is given by

$$P_m = \frac{m}{2}(3m-1).$$

Hence $\sum_{m=1}^{s} P_m = \sum_{m=1}^{s}[\frac{m}{2}(3m-1)]$

$$= \frac{1}{2}\sum_{m=1}^{s}(3m^2 - m)$$

$$= \frac{1}{2}[3\sum_{m=1}^{s} m^2 - \sum_{m=1}^{s} m]$$

$$= \frac{1}{2}[\frac{3s(s+1)(2s+1)}{6} - \frac{s(s+1)}{2}]$$

$$= \frac{1}{4}s(s+1)[2s+1-1]$$

$$= \frac{1}{2}s^2(s+1)$$

$= s[\frac{s(s+1)}{2}] = sT_S$, where T_S is the s^{th} triangular number.

Hence $\sum_{m=1}^{s} P_m = s^2\frac{(s+1)}{2} = sT_S$.

Identity 3.2

The sum of squares of the pentagonal numbers can be expressed as the polynomial in s.

That is, $\sum_{m=1}^{s} P_m{}^2 = \frac{s(s+1)}{60}(27s^3 + 18s^2 - 13s - 2)$.

$= \frac{T_S}{30}(27s^3 + 18s^2 - 13s - 2)$, where T_S is the s^{th} triangular number.

Proof

The m^{th} pentagonal number is given by

$$P_m = \frac{m}{2}(3m-1).$$

Hence $P_m{}^2 = [\frac{m}{2}(3m-1)]^2 = \frac{1}{4}[m^2(3m-1)^2]$.

$$= \frac{1}{4}[m^2(9m^2 - 6m + 1)].$$

$$= \frac{9}{4}m^4 - \frac{3}{2}m^3 + \frac{1}{4}m^2.$$

Hence $\sum_{m=1}^{s} P_m^{\ 2} = \frac{9}{4}\sum_{m=1}^{s} m^4 - \frac{3}{2}\sum_{m=1}^{s} m^3 + \frac{1}{4}\sum_{m=1}^{s} m^2.$

$$= \frac{9}{4}[s(s+1)(6s^3+9s^2+s-1)] - \frac{3}{2}[\frac{s^2(s+1)^2}{4}] + \frac{1}{4}\frac{s(s+1)(2s+1)}{6}$$

$$= \frac{1}{8}s(s+1)[\frac{3}{5}(6s^3+9s^2+s-1) - 3s(s+1) + \frac{1}{3}(2s+1)]$$

$$= \frac{s(s+1)}{120}[9(6s^3+9s^2+s-1) - 45s(s+1) + 5(2s+1)].$$

$$= \frac{s(s+1)}{120}[54s^3+81s^2+9s-9-45s^2-45s+10s+5].$$

$$= \frac{s(s+1)}{120}[54s^3+36s^2-26s-4].$$

$$= \frac{T_S}{30}[27s^3+18s^2-13s-2].$$

Thus, $\sum_{m=1}^{s} P_m^{\ 2} = \frac{s(s+1)}{60}(27s^3+18s^2-13s-2).$

$$= \frac{T_S}{30}(27s^3+18s^2-13s-2).$$

Identity 3.3

The sum of cubes of the pentagonal numbers can be expressed as the polynomial in s.

That is $\sum_{m=1}^{s} P_m^{\ 3} = \frac{s(s+1)}{280}[135s^5+180s^4-117s^3-128s^2+58s+12].$

$= \frac{T_s}{140}[135s^5+180s^4-117s^3-128s^2+58s+12]$, where T_S is the s^{th} triangular number.

Proof

The m^{th} pentagonal number is given by

$$P_m = \frac{m}{2}(3m-1).$$

Hence $P_m{}^3 = \frac{m^3}{8}(3m-1)^3.$

$$= \frac{m^3}{8}(27m^3 + 9m - 27m^2 - 1).$$

$$= \frac{1}{8}(27m^6 + 9m^4 - 27m^5 - m^3)$$

Thus $\sum_{m=1}^{s} P_m{}^3 = \frac{1}{8}[27\sum_{m=1}^{s} m^6 - 27\sum_{m=1}^{s} m^5 + 9\sum_{m=1}^{s} m^4 - \sum_{m=1}^{s} m^3].$

$$= \frac{1}{8}\left[27\frac{s(s+1)(6s^5+15s^4+6s^3-6s^2-s+1)}{42} - 27\frac{s^2(s+1)^2(2s^2+2s-1)}{12}\right.$$

$$\left. + \frac{9s(s+1)(6s^3+9s^2+s-1)}{30} - \frac{s^2(s+1)^2}{4}\right].$$

$$= \frac{s(s+1)}{1120}\left[90(6s^5+15s^4+6s^3-6s^2-s+1) - 315s(s+1)(2s^2+2s-1)\right.$$

$$\left. + 42(6s^3+9s^2+s-1) - 35s(s+1)\right].$$

$$= \frac{s(s+1)}{1120}\left[540s^5 + 720s^4 - 468s^3 - 512s^2 + 232s + 48\right].$$

$$= \frac{s(s+1)}{280}\left[135s^5 + 180s^4 - 117s^3 - 128s^2 + 58s + 12\right].$$

$= \frac{T_S}{140}\left[135s^5 + 180s^4 - 117s^3 - 128s^2 + 58s + 12\right]$ where T_S is the s^{th} triangular number.

Hence $\sum_{m=1}^{s} P_m^3 = \frac{s(s+1)}{280}[135s^5 + 180s^4 - 117s^3 - 128s^2 + 58s + 12]$.

$$= \frac{T_S}{140}[135s^5 + 180s^4 - 117s^3 - 128s^2 + 58s + 12], \text{ where}$$

T_S is the s^{th} triangular number.

Identity 3.4

$$Pm = a\,square + (m-1)^{th}\, triangular\, number.$$

Proof

The m^{th} pentagonal number is given by

$$Pm = \frac{m}{2}(3m-1).$$

$$= \frac{3m^2}{2} - \frac{m}{2}.$$

$$= m^2 + \frac{m^2}{2} - \frac{m}{2}.$$

$$= m^2 + \frac{m}{2}(m-1) = a\,square + (m-1)^{th}\, triangular \quad number.$$

It has seen that the explicit formulas for the ranks of pentagonal numbers which are simultaneously equal to hexagonal, heptagonal and octagonal numbers. One may search for such hexagonal, heptagonal and octagonal numbers. Further, it has observed that there exist pairs of pentagonal numbers whose ratios are non-square integers. One may search for such numbers whose ratios are non-square integers.