Iterative Closest Pair (ICP) Algorithm

- For two objects, K_1 and K_2 , extract 2 finite sets of points $S \subseteq K_1$ and $R \subseteq K_2$
- For every $r_i \in R$, find the corresponding $s_i \in S$ such that s_i is the closest point to r_i in S
- Translate the two objects by moving their centroids of s_i and r_i to origin
- Compute the rigid motion $\mathcal{M}$ that minimize the error

$$\sum_{i=0}^{n-1} \|\mathcal{M} r_i - s_i\|^2$$

- Repeat until it "converges"

Two Pointsets

$$S = \{\ (-4,\ 4,\ 3),\ (16,\ 4,\ 3)\}$$

$$R = \{\ (-1,\ 11,\ 3),\ (13,-3,3),\ \ (256,7,204),\ (-503,211,905)\}$$

- Which points in R are the closest to S?

i	s_i	r_i
1	(-4, 4, 3)	(-1, 11, 3)
2	(16, 4, 3)	(13,-3,3)

Correspondences

i	s_i	r_i
1	(-4, 4, 3)	(-1, 11, 3)
2	(16, 4, 3)	(13,-3,3)

What are the two centroids of s_i's and r_i's?

Centroids

- ((-4, 4, 3) + (16, 4, 3)) / 2
  - (6,4,3)
- ((-1, 11, 3) + (13,-3,3)) /2
  - (6,4,3)

- In real situations, the two centroids should be different

i	s_i	r_i
1	(-4, 4, 3)	(-1, 11, 3)
2	(16, 4, 3)	(13,-3,3)

After Translating The Two Centroids to O?

i	s_i	r_i
1	(-4, 4, 3)	(-1, 11, 3)
2	(16, 4, 3)	(13,-3,3)

After Translating The Two Centroids to O?

i	s_i	r_i
1	(-10, 0, 0)	(-7, 7, 0)
2	(10, 0, 0)	(7,-7,0)

Compute the Matrices we Need

$S_1 = \begin{pmatrix} \\ \\ \end{pmatrix}$	$\bar{R}_1^T = \begin{pmatrix} \\ \\ \end{pmatrix}$
$S_2 = \begin{pmatrix} \\ \\ \end{pmatrix}$	$\bar{R}_2^T = \begin{pmatrix} \\ \\ \end{pmatrix}$

Quaternion Multiplication

$$rq = \begin{bmatrix} r_0 & -r_x & -r_y & -r_z \\ r_x & r_0 & -r_z & r_y \\ r_y & r_z & r_0 & -r_x \\ r_z & -r_y & r_x & r_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_x \\ q_y \\ q_z \end{bmatrix} = Rq$$

$$qr = \begin{bmatrix} r_0 & -r_x & -r_y & -r_z \\ r_x & r_0 & r_z & -r_y \\ r_y & -r_z & r_0 & r_x \\ r_z & r_y & -r_x & r_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_x \\ q_y \\ q_z \end{bmatrix} = \bar{R}q$$

Compute the Matrices we Need

$$S_1 = \begin{pmatrix} \mathbf{1} & \mathbf{10} & & \cdots & \\ \mathbf{-10} & \mathbf{1} & & & \\ & & & \mathbf{1} & \mathbf{10} \\ & \cdots & & \mathbf{-10} & \mathbf{1} \end{pmatrix}$$

$$\bar{R}_1^T = \begin{pmatrix} \mathbf{1} & \mathbf{-7} & \mathbf{7} & \mathbf{0} \\ \mathbf{7} & \mathbf{1} & \mathbf{0} & \mathbf{7} \\ \mathbf{-7} & \mathbf{0} & \mathbf{1} & \mathbf{7} \\ \mathbf{0} & \mathbf{-7} & \mathbf{-7} & \mathbf{1} \end{pmatrix}$$

$$S_2 = \begin{pmatrix} \mathbf{1} & \mathbf{-10} & & \cdots & \\ \mathbf{10} & \mathbf{1} & & & \\ & & & \mathbf{1} & \mathbf{-10} \\ & \cdots & & \mathbf{10} & \mathbf{1} \end{pmatrix}$$

$$\bar{R}_2^T = \begin{pmatrix} \mathbf{1} & \mathbf{7} & \mathbf{-7} & \mathbf{0} \\ \mathbf{-7} & \mathbf{1} & \mathbf{0} & \mathbf{-7} \\ \mathbf{7} & \mathbf{0} & \mathbf{1} & \mathbf{-7} \\ \mathbf{0} & \mathbf{7} & \mathbf{7} & \mathbf{1} \end{pmatrix}$$

$$\bar{R}_1^T S_1 = \begin{pmatrix} 71 & 3 & 7 & 70 \\ -3 & 71 & -70 & 7 \\ -7 & -70 & -69 & 17 \\ 70 & -7 & -17 & -69 \end{pmatrix}$$

$$\bar{R}_2^T S_2 = \begin{pmatrix} 71 & 3 & 7 & 70 \\ -3 & 71 & -70 & 7 \\ -7 & -70 & -69 & 17 \\ 70 & -7 & -17 & -69 \end{pmatrix}$$

Computing the Eigenvectors, the one with largest Eigenvalue should be approximately

$$(\cos(22.5) , 0, 0, \sin(22.5))$$

- What is the rotational axis and what is the angle rotated?

Rodrigues's Formula

- By Rodrigues's formula, $\bar{Q}^T Q$ is a rotation by θ around the axis defined by a vector $\left(u_x, u_y, u_z\right)$ if
$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(u_x\vec{\imath} + u_y\vec{\jmath} + u_z\vec{k})$$

Computing the Eigenvectors, the one with largest Eigenvalue should be approximately

(cos(22.5) , 0, 0, sin(22.5))

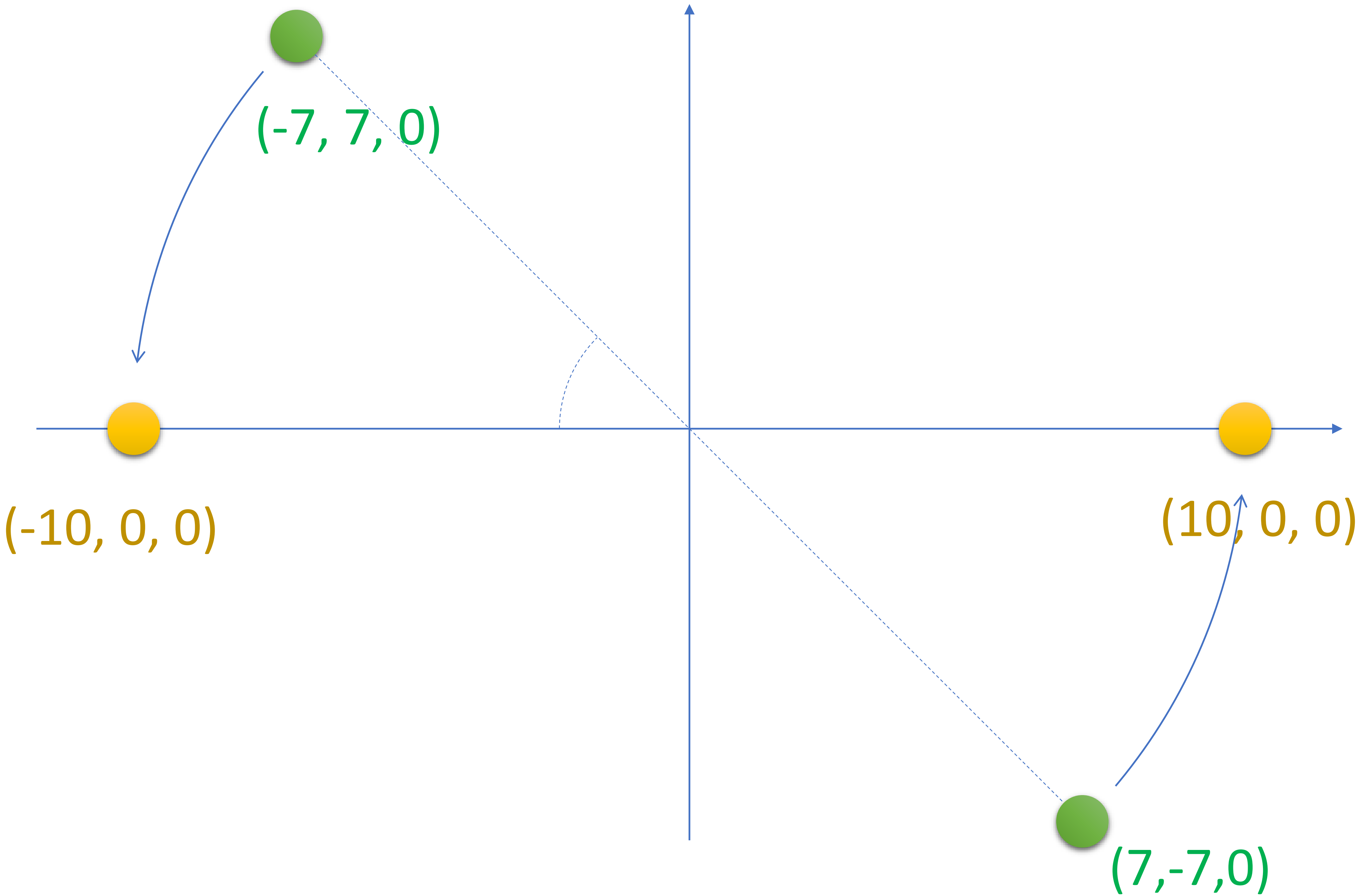
- What is the rotational axis and what is the angle rotated?

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(u_x\vec{\imath} + u_y\vec{\jmath} + u_z\vec{k})$$

- Rotate around the vector (0,0,1) with an angle of 45 degrees

After Translating The Two Centroids to O?

i	s_i	R_i
1	(-10, 0, 0)	(-7, 7, 0)
2	(10, 0, 0)	(7,-7,0)



(-7, 7, 0)
(-10, 0, 0)
(10, 0, 0)
(7,-7,0)

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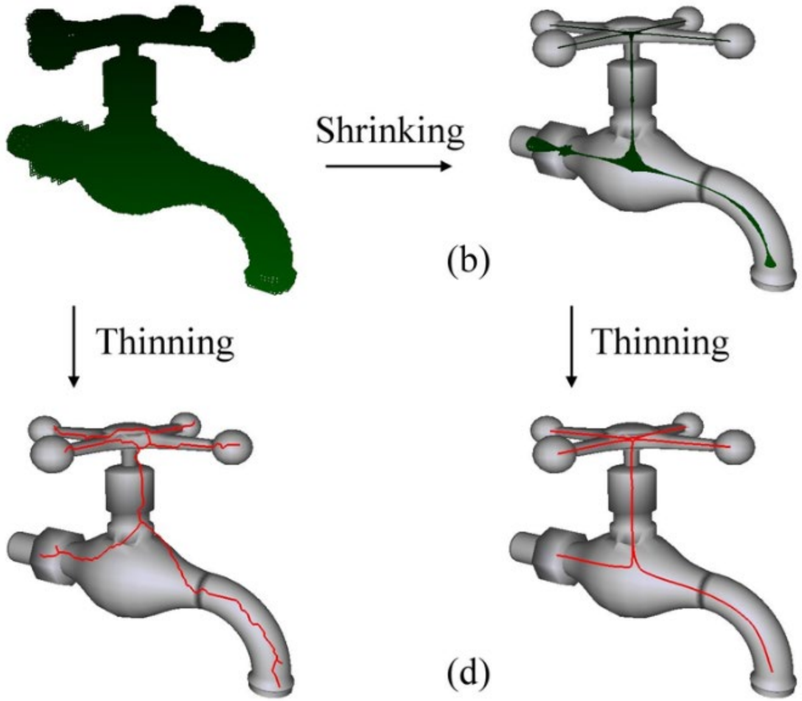
$$\sum_{i=0}^{n-1} \|\mathcal{M} r_i - s_i\|^2$$

- Repeat until it “converges”

"Skeletonization"

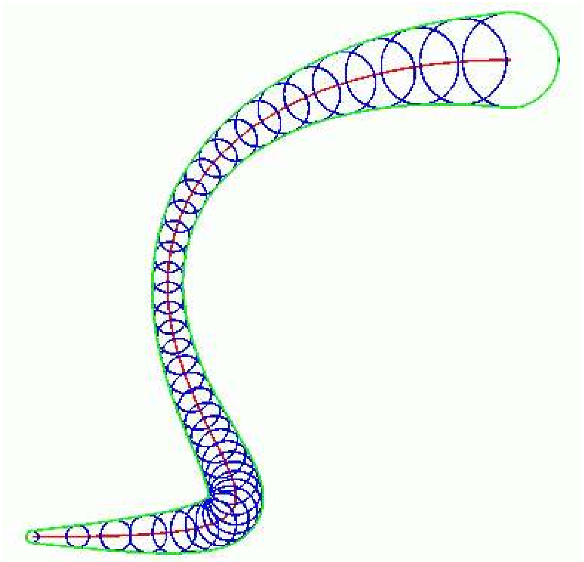
- Medial axis
  - Delaunay/Voronoi

- Volumetric approach
  - Turns into voxels first

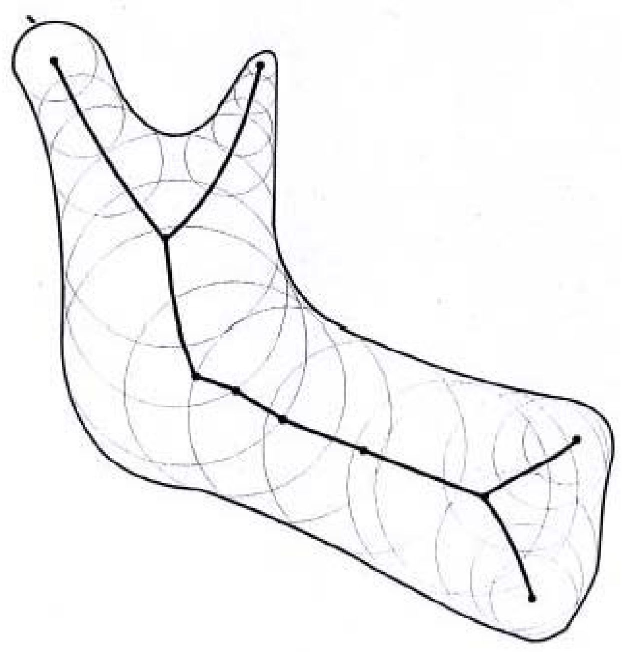


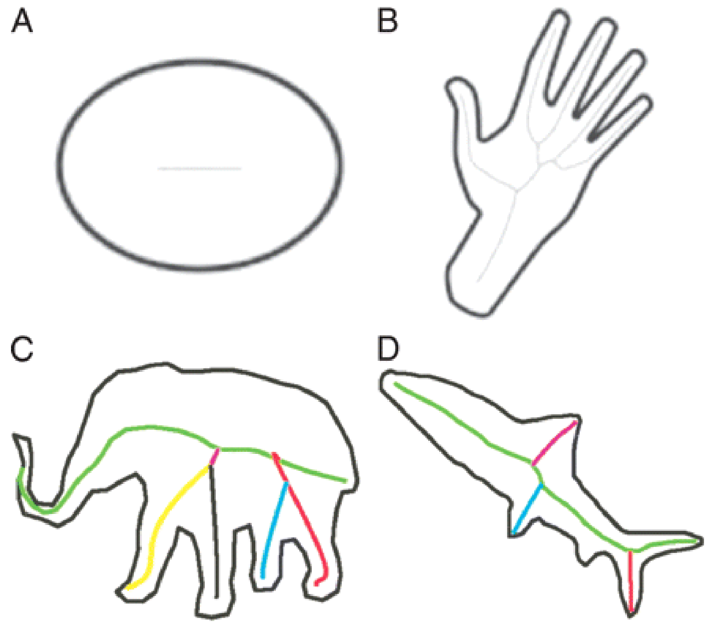
Medial Axis

- In 2D, given a closed curve P
  - There will be a set of circles that touches P in more than two points
  - By "touches" we mean no point of P is inside the circle
- The medial axis is the collection of all the centers of these circles

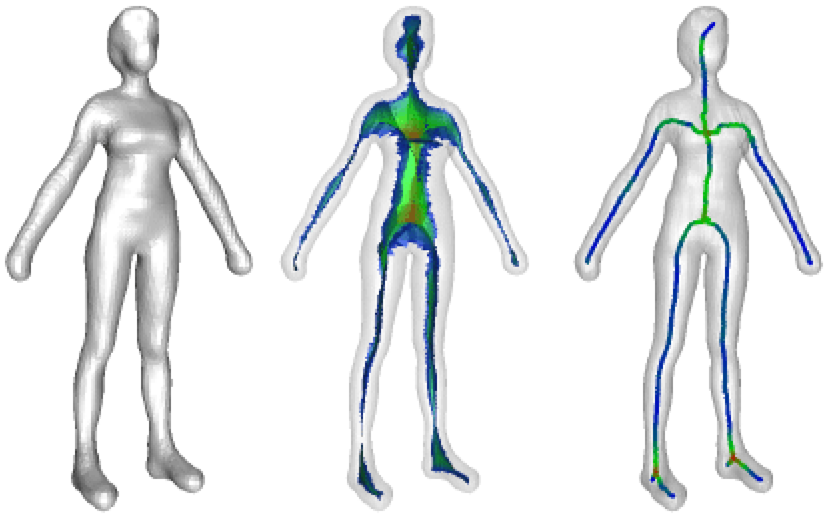


Medial Axis as a Skeleton

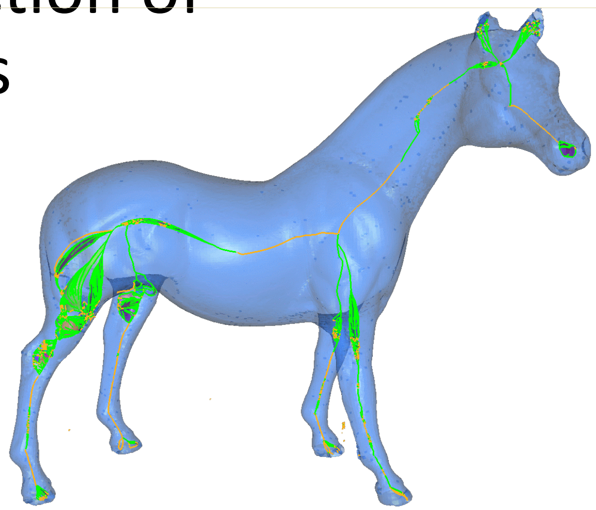


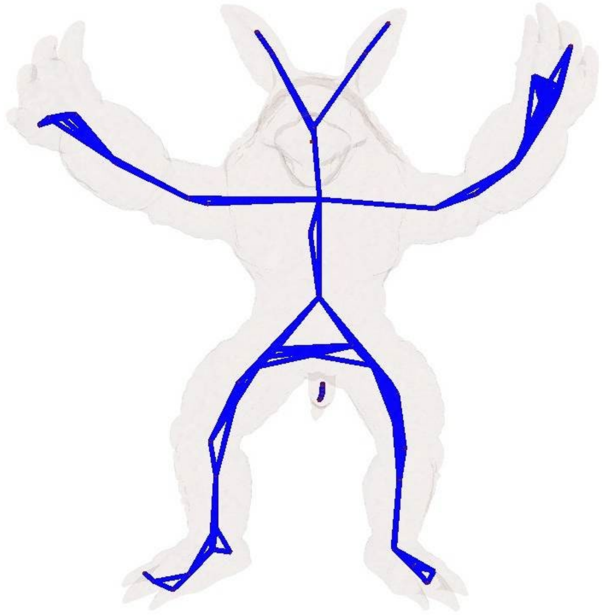


MA of 3D Objects



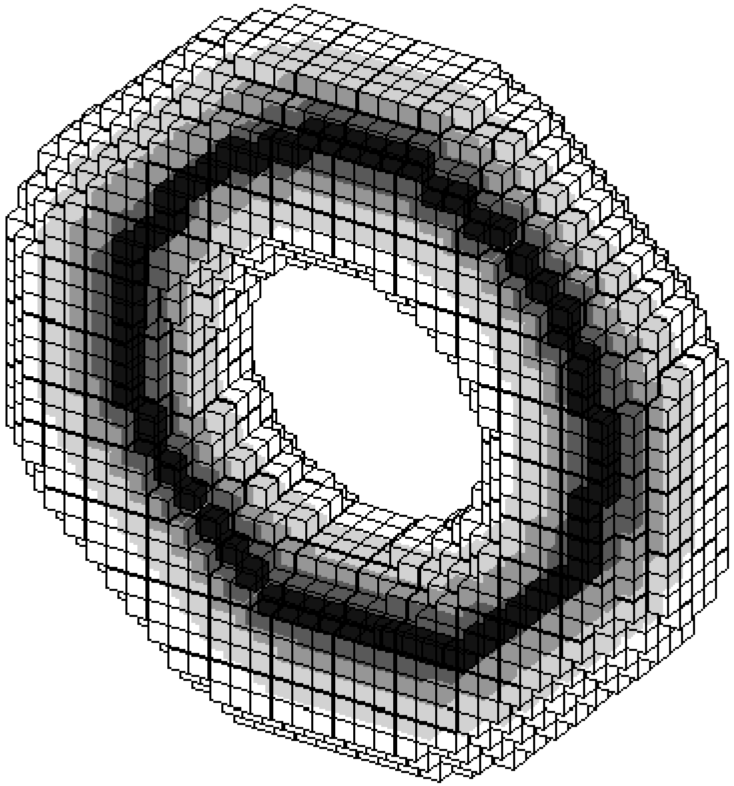
- In 3D, given a closed curve P
  - There will be a set of balls that touches P in more than three points
  - By “touches” we mean no point of P is inside the ball
- The medial axis is the collection of all the centers of these balls





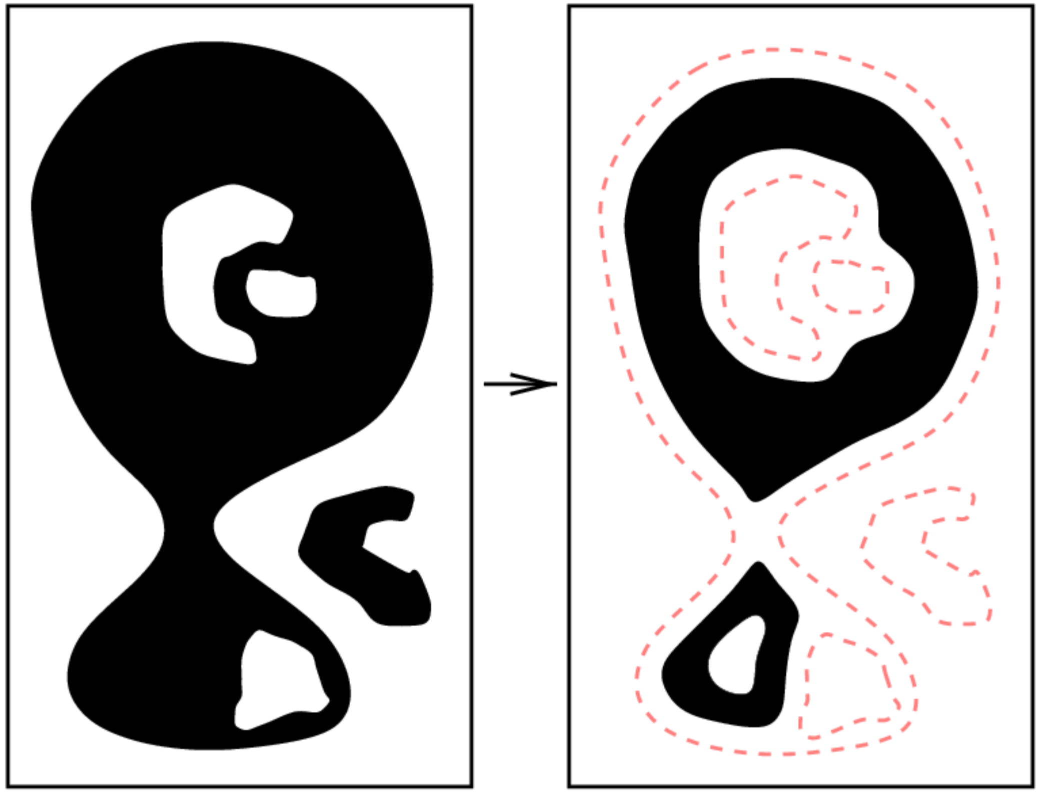
Skeletonization By Shrinking Voxels

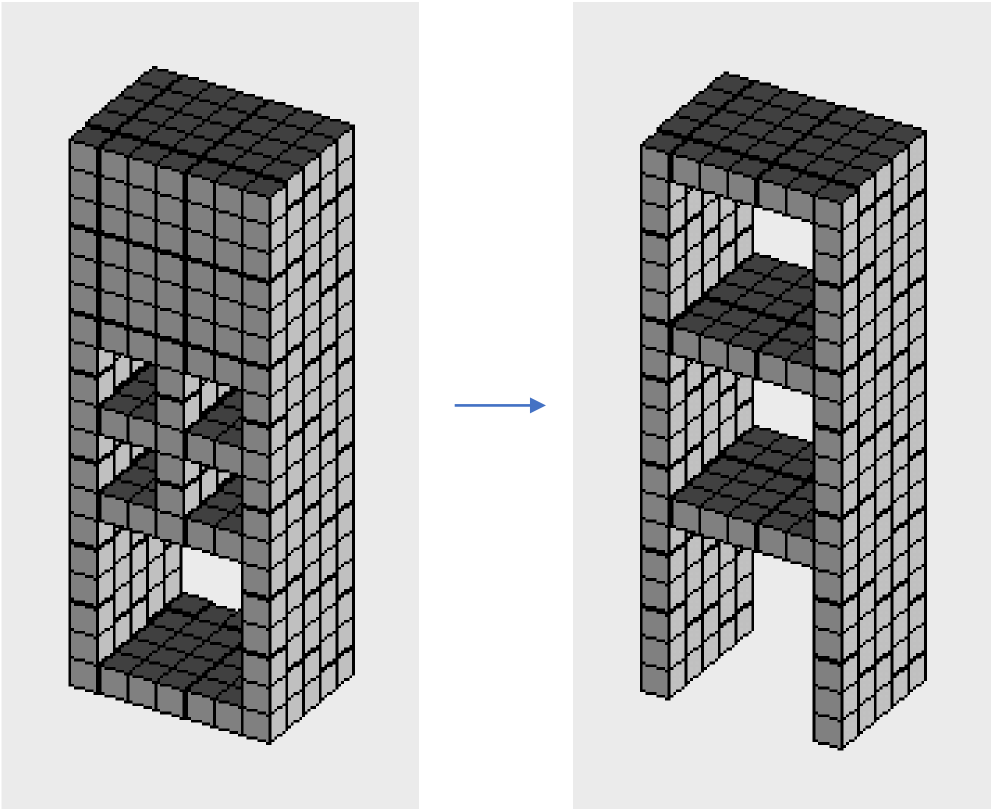
- First, turns the object into voxel representation
- “Eat” up voxels on the “surface” of the object

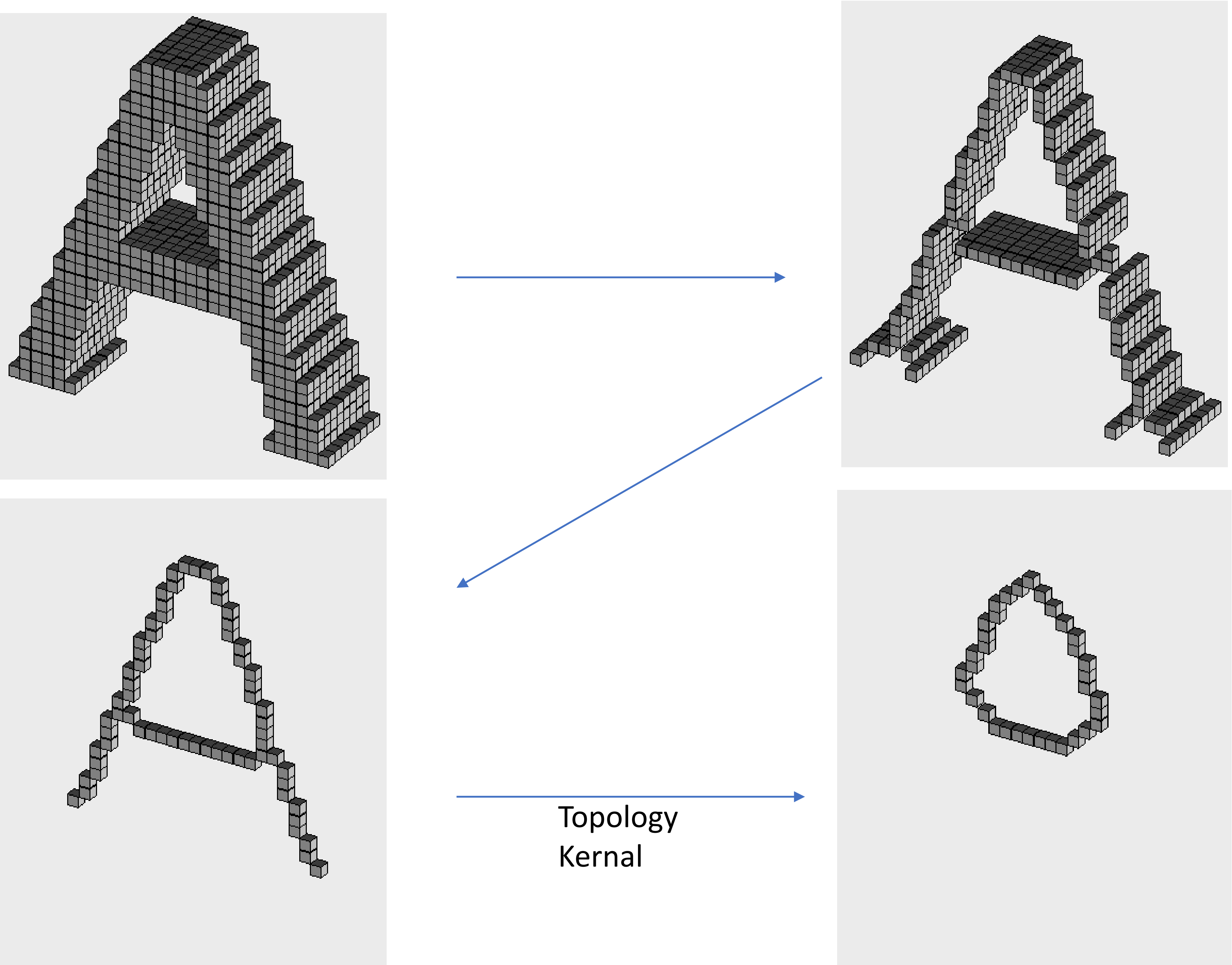


Main Concerns

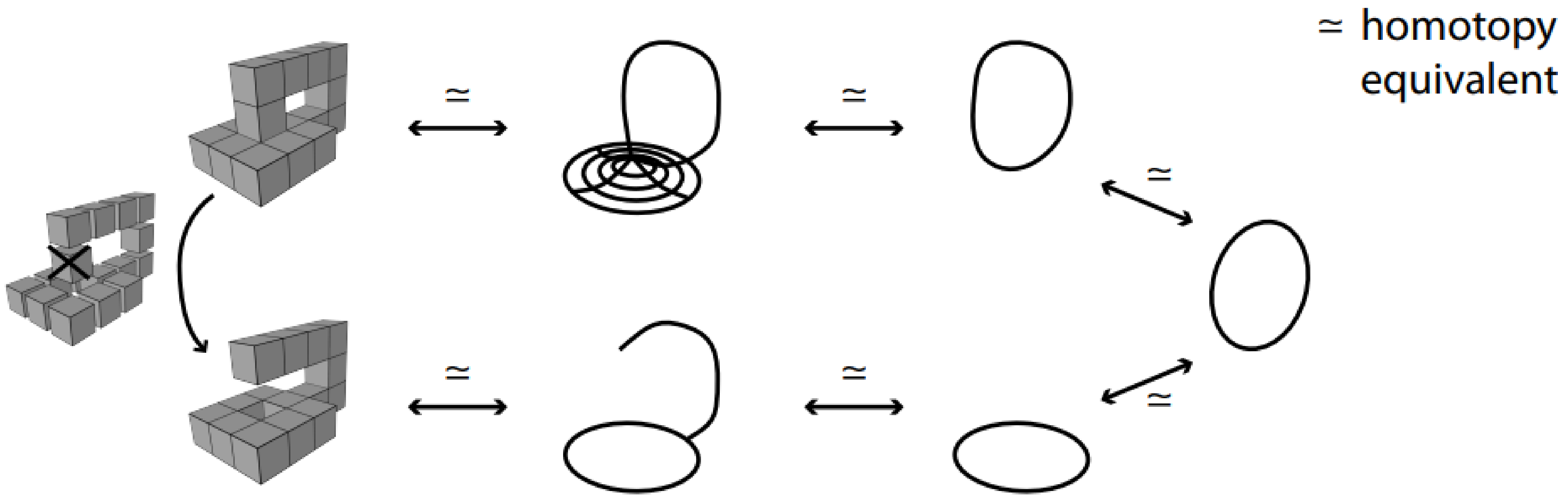
- Preserving Topology





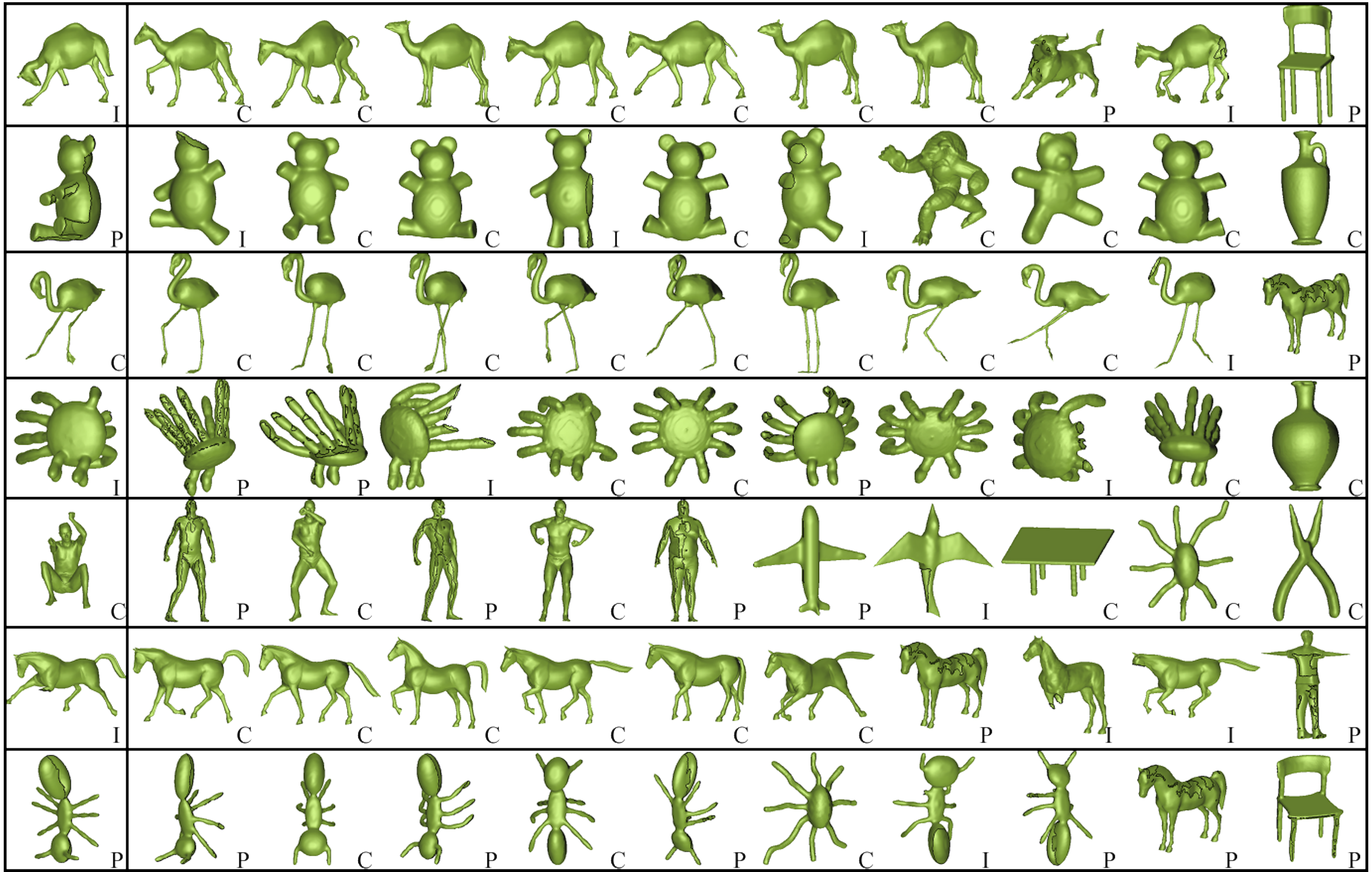

Topology
Kernal

Homotopy Equivalence



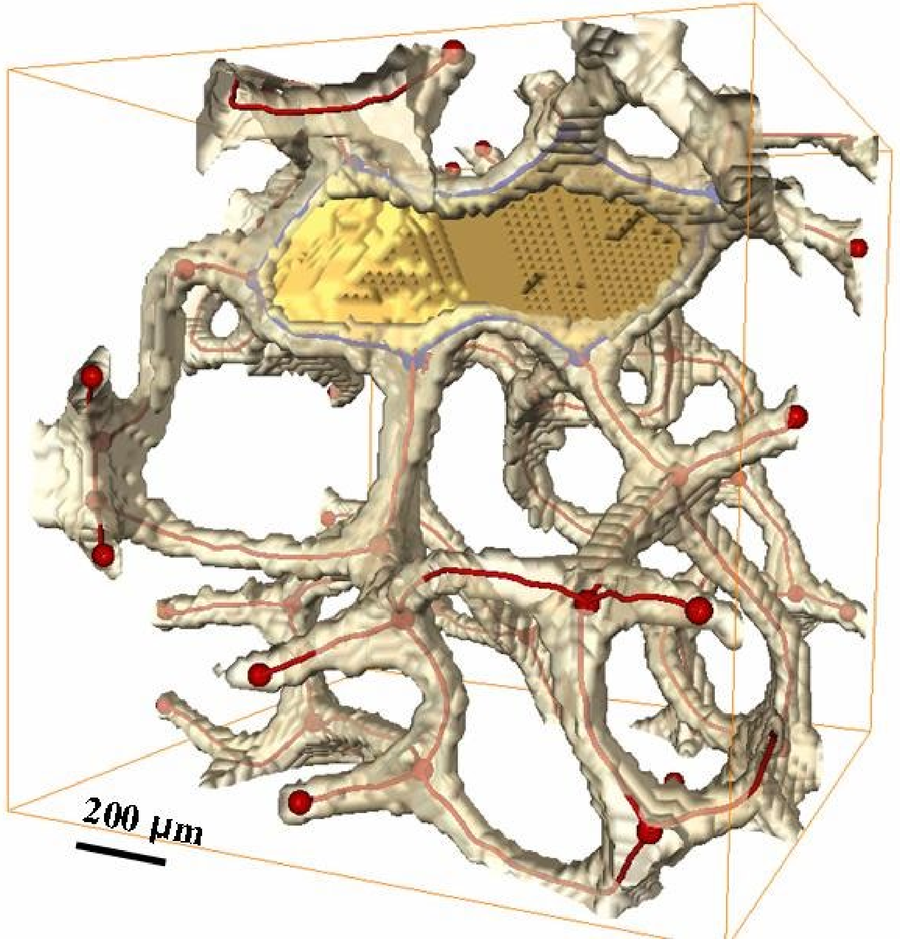
- A weaker condition of Homeomorphism

MA as a Tool for Classification



MA in Medical Field

- 3D rendering of an auxetic foam (in light yellow) on which the corresponding <u>medial axis-based skeleton is superimposed </u>(red curves: edges – red spheres: vertices). An open cell is also closed (depicted by a blue cycle)



However

- Medial Axis is very “sensitive” to “bumps” on the curve