MADS-MMS – Mathematics and Multivariate Statistics

Linear Algebra

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SCAN ME

Moodle (SoSe 2025)

Agenda

Outline

Motivation

- Vector space operations are an integral part of various algorithms

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- PCA (next chapter) relies heavily on matrix operations in vector spaces

Chapter Goals

- understand mathematical foundations linear independence and dimensionality

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- understand and apply matrix computations

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- understand and apply matrix computations
- preparation for further algorithms (PCA, SVMs)

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Subvectorspaces

Linear Combinations

Linear Mappings and Matrixes

Vector-Subspaces

Definition 1 (Subspace)

Let $\mathcal{V}$ be a vector space on V with $+$ and $\cdot$ and $U \subseteq V$. Then U with $+$ and $\cdot$ is a subspace of $\mathcal{V}$ if $(U, +, \cdot)$ is a vector space.

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Remarks:

- for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U : \boldsymbol{u}_1 + \boldsymbol{u}_2 \in U$
- for all $r \in \mathbb{R}$, $\boldsymbol{u} \in U : r \cdot \boldsymbol{u} \in U$
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Example:

- in $\mathbb{R}^n$, each line/plane/hyperplane through the origin is a subspace

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Linear Combinations

Definition 2 (Linear Combination)

Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v_n}$ in a vector space $\mathcal{V}$, a **linear combination** of these vectors is every sum

$$\sum_{i=1}^{n} r_i \cdot \boldsymbol{v_i}$$

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Notebook 07_1_linear_algebra_in_python, Cells 1–3

Dimension of a Vector Space

Definition 4 (Base, Dimension)

- A set of vectors $U \subseteq V$ in a vector space $\mathcal{V}$ is called a generating set, if each $v \in V$ is a linear combination of vectors in U.
- U is called a **base** of $\mathcal{V}$ if U is minimal w.r.t. the above property.
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- $\mathbb{R}^3$ is generated by the base $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ as well as by the base $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$

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- the dimension of $\mathbb{R}^n$ is n

Linear Independence

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A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v_n}$ in a vector space $\mathcal{V}$ is called **linear independent**, if and only if

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- $(1,0,1)^T, (2,1,1)^T$, and $(3,1,1)^T$ are linear independent
- $(1,1,2)^T, (3,7,2)^T$, and $(2,10,-4)^T$ are linear dependent

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Generally: If with at least one $r_j \neq 0$

$$0 = \sum_{i=1}^{n} r_i \cdot \boldsymbol{v_i} \quad \text{then} \quad \boldsymbol{v_j} = \sum_{i=1, i \neq j}^{n} -\frac{r_i}{r_j} \cdot \boldsymbol{v_i}$$

Checking for Linear Independence in $\mathbb{R}^n$

Let's check the example from before:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = r_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} + r_3 \begin{pmatrix} 2 \\ 10 \\ -4 \end{pmatrix}$$

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Observation: Checking for solutions r_1, r_2, r_3 is the same as solving:

$$\begin{aligned}0 &= 1r_1 + 3r_2 + 2r_3\\0 &= 1r_1 + 7r_2 + 10r_3\\0 &= 2r_1 + 2r_2 + (-4)r_3\end{aligned}$$

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- linear equation systems can be solved approximately using efficient heuristics
- linear equation systems can be written with matrixes

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 7 & 10 \\ 2 & 2 & (-4) \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Matrix Multiplication

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  - elementwise product of row i of A and column j of B
  - sum up these products

Matrix Multiplication – Example

Example 1:

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 10 \\ 4 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 2 & 3 \\ 4 & 1 \end{pmatrix} =$$

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$$= \begin{pmatrix} 16 & 13 \\ 58 & 35 \\ 24 & 17 \end{pmatrix}$$

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Exercises 1–3

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Linear mappings, matrixes and systems of linear equations are related!

Linear Mappings

Definition 6 (Linear Mapping)

A **linear mapping** between two real valued vector spaces $\mathcal{V} = (V, +, \cdot)$ and $\mathcal{W} = (W, +, \cdot)$ is a mapping $\phi : V \to W$ that preserves the structure of the vector space, i.e. $\phi(\boldsymbol{x} + \boldsymbol{y}) = \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ and $\phi(s\boldsymbol{x}) = s\phi(\boldsymbol{x})$.

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- $\phi : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ✔

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- $\phi : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ✓
- $\phi : \mathbb{R}^2 \to \mathbb{R}^1 : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 + v_2 \end{pmatrix}$ ✓

Linear Mappings and Matrixes

Theorem 7

- *For each linear mapping* $\phi : \mathbb{R}^n \to \mathbb{R}^m$ *there is an* $m \times n$ *matrix* A_ϕ*, such that* $\phi(\boldsymbol{v}) = A_\phi \cdot \boldsymbol{v}$*.*
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Remarks:

- With matrix multiplication, we have all possible linear mappings.
- It is the basic operation in linear regression, logistic regression, support vector machines, neural networks, ...

Linear Mappings and Matrixes – Examples

Examples from before:

- $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{v} \mapsto 2\boldsymbol{v}$ ➔ $A_\phi = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{v} \mapsto \boldsymbol{v} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ☠ ➔ no matrix can represent this (non-linear) mapping
- $\phi : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ➔ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
- $\phi : \mathbb{R}^2 \to \mathbb{R}^1 : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 + v_2 \end{pmatrix}$ ➔ $\begin{pmatrix} 1 & 1 \end{pmatrix}$

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4. Obviously $0 \leq \text{rank}(A) \leq \min(m, n)$
5. If $\text{rank}(A) = \min(m, n)$, then A has **full** rank.

Rank of a Matrix – Examples

- $\operatorname{rank}(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}) = 2$
- $\operatorname{rank}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) = 2$
- $\operatorname{rank}(\begin{pmatrix} 1 & 1 \end{pmatrix}) = 1$

Inverse of a Matrix 1/2

Definition 8

Let A be an $m \times m$ matrix. A matrix B is called the **inverse** of A if and only if

$$A \cdot B = I \quad \text{and} \quad B \cdot A = I.$$

- Here I denotes the identity matrix: $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

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- If a matrix has an inverse, it is unique. It is denoted as A^{-1}.
- The inverse of a matrix gives rise to a linear mapping with $\phi_A(\phi_{A^{-1}}(\boldsymbol{v})) = \boldsymbol{v}$ and $\phi_{A^{-1}}(\phi_A(\boldsymbol{v})) = \boldsymbol{v}$

Inverse of a Matrix 2/2

Theorem 9

Let A be an $m \times m$ matrix. The inverse of A exists if and only if A has full rank (i.e. $\operatorname{rank}(A) = m$).

Inverse of a Matrix 2/2

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Notebook 07_1_linear_algebra_in_python, Cells 10–23

Linear Equation Systems and Invertible Matrixes

A linear equation system, a matrix equation, or an equation with a linear mapping express the same task.

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Remark: On slide 8, we wanted to check for linear independence by solving an equation system. Alternatively, we can check if the respective matrix is invertible, which is the same as checking if the matrix has full rank.

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Notebook 07_1_linear_algebra_in_python, Cells 24–26

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Exercises 4–5