MADS-MMS – Mathematics and Multivariate Statistics

Linear Algebra

Prof. Dr. Stephan Doerfel

FACHHOCHSCHULE KIEL
University of Applied Sciences

SCAN ME

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Agenda

Motivation

Subvectorspaces

Linear Combinations

Linear Mappings and Matrixes

Outline

Motivation

- Vector space operations are an integral part of various algorithms
- the notion of dimension is important as high dimensionality often leads to difficulties
- PCA (next chapter) relies heavily on matrix operations in vector spaces

Chapter Goals

- understand mathematical foundations linear independence and dimensionality
- understand and apply matrix computations
- preparation for further algorithms (PCA, SVMs)

Outline

Vector-Subspaces

Definition 1 (Subspace)

Let $\mathcal{V}$ be a vector space on V with $+$ and $\cdot$ and $U \subseteq V$. Then U with $+$ and $\cdot$ is a subspace of $\mathcal{V}$ if $(U, +, \cdot)$ is a vector space.

Remarks:

- for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U : \boldsymbol{u}_1 + \boldsymbol{u}_2 \in U$
- for all $r \in \mathbb{R}$, $\boldsymbol{u} \in U : r \cdot \boldsymbol{u} \in U$
- $0 \in U$

Example:

- in $\mathbb{R}^n$, each line/plane/hyperplane through the origin is a subspace

Outline

Linear Combinations

Definition 2 (Linear Combination)

Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v_n}$ in a vector space $\mathcal{V}$, a **linear combination** of these vectors is every sum

$$\sum_{i=1}^{n} r_i \cdot \boldsymbol{v_i}$$

Theorem 3

Given vectors as above, then the set of all linear combinations of these vectors is a subspace of $\mathcal{V}$.

Notebook 07_1_linear_algebra_in_python, Cells 1–3

Dimension of a Vector Space

Definition 4 (Base, Dimension)

- A set of vectors $U \subseteq V$ in a vector space $\mathcal{V}$ is called a generating set, if each $v \in V$ is a linear combination of vectors in U.
- U is called a **base** of $\mathcal{V}$ if U is minimal w.r.t. the above property.
- If U is a base of $\mathcal{V}$ then, $\dim(V) = |U|$ is called the **dimension** of $\mathcal{V}$.

- $\mathbb{R}^3$ is generated by the base $\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$ as well as by the base $\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \}$
- the dimension of $\mathbb{R}^n$ is n

Linear Independence

Definition 5 (Linear Independence)

A set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v_n}$ in a vector space $\mathcal{V}$ is called **linear independent**, if and only if

$$0 = \sum_{i=1}^{n} r_i \cdot \boldsymbol{v_i}$$

is only true for $r_i = 0 \, (1 \leq i \leq n)$.

Examples in $\mathbb{R}^3$:

- $(1,0,0)^T, (0,1,0)^T$, and $(0,0,1)^T$ are linear independent
- $(1,0,1)^T, (2,1,1)^T$, and $(3,1,1)^T$ are linear independent
- $(1,1,2)^T, (3,7,2)^T$, and $(2,10,-4)^T$ are linear dependent

Linear Independence – Remarks

- Linear independence is a fundamental concept in linear algebra
- linear independent → no redundancy (omit vector → smaller subspace of linear combinations)
- a vector space base is always linear independent
- set of linear dependent vectors → either set contains only 0 or one of the vectors can be described as a linear combination of the others

Example:

- $-4 \cdot (1, 1, 2) + 2 \cdot (3, 7, 2) = (2, 10, -4)$

Generally: If with at least one $r_j \neq 0$

$$0 = \sum_{i=1}^{n} r_i \cdot \boldsymbol{v_i} \quad \text{then} \quad \boldsymbol{v_j} = \sum_{i=1, i \neq j}^{n} -\frac{r_i}{r_j} \cdot \boldsymbol{v_i}$$

Checking for Linear Independence in $\mathbb{R}^n$

Let's check the example from before:

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = r_1 \begin{pmatrix}1\\1\\2\end{pmatrix} + r_2 \begin{pmatrix}3\\7\\2\end{pmatrix} + r_3 \begin{pmatrix}2\\10\\-4\end{pmatrix}$$

Observation: Checking for solutions r_1, r_2, r_3 is the same as solving:

$$\begin{aligned} 0 &= 1r_1 + 3r_2 + 2r_3 \\ 0 &= 1r_1 + 7r_2 + 10r_3 \\ 0 &= 2r_1 + 2r_2 + (-4)r_3 \end{aligned}$$

- linear equation systems can be solved approximately using efficient heuristics
- linear equation systems can be written with matrixes

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1 & 3 & 2\\1 & 7 & 10\\2 & 2 & (-4)\end{pmatrix} \cdot \begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}$$

Matrix Multiplication

Consider two matrixes: $A \in \mathbb{R}^{m \times k}$ and $B \in \mathbb{R}^{k \times n}$.

- the matrix product $A \cdot B$ is another matrix $C \in \mathbb{R}^{m \times n}$
- the second dimension of A must be the same as the first dimension of B
- In C, each element c_{ij} is computed as

$$c_{ij} := \sum_{l=1}^{k} a_{il} b_{lj}$$

  - elementwise product of row i of A and column j of B
  - sum up these products

Matrix Multiplication – Example

Example 1:

$$\begin{pmatrix}1 & 3 & 2\\ 2 & 7 & 10\\ 4 & 2 & 3\end{pmatrix}\cdot\begin{pmatrix}2 & 2\\ 2 & 3\\ 4 & 1\end{pmatrix} = \begin{pmatrix}1\cdot 2+3\cdot 2+2\cdot 4 & 1\cdot 2+3\cdot 3+2\cdot 1\\ 2\cdot 2+7\cdot 2+10\cdot 4 & 2\cdot 2+7\cdot 3+10\cdot 1\\ 4\cdot 2+2\cdot 2+3\cdot 4 & 4\cdot 2+2\cdot 3+3\cdot 1\end{pmatrix}$$

$$= \begin{pmatrix}16 & 13\\ 58 & 35\\ 24 & 17\end{pmatrix}$$

Example 2: $$\begin{pmatrix}1 & 3 & 2\\ 1 & 7 & 10\\ 2 & 2 & (-4)\end{pmatrix}\cdot\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix} = \begin{pmatrix}1r_1+3r_2+2r_3\\ 1r_1+7r_2+10r_3\\ 2r_1+2r_2+(-4)r_3\end{pmatrix}$$

Notebook 07_1_linear_algebra_in_python, Cells 4–9

Exercises 1–3

Outline

Linear Mappings – Motivation

- Linear mappings are the foundation of many data science notions
- SVMs learn linear mappings to separate data into two classes
- The model of Linear Regression is (surprise!) linear
- Pearson's correlation measures a linear correlation
- Linear relations are well understood, often easy to handle
- Linear problems can (despite their simplicity) be computationally complex

Linear mappings, matrixes and systems of linear equations are related!

Linear Mappings

Definition 6 (Linear Mapping)

A linear mapping between two real valued vector spaces $\mathcal{V} = (V, +, \cdot)$ and $\mathcal{W} = (W, +, \cdot)$ is a mapping $\phi : V \to W$ that preserves the structure of the vector space, i.e. $\phi(\boldsymbol{x} + \boldsymbol{y}) = \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ and $\phi(s\boldsymbol{x}) = s\phi(\boldsymbol{x})$.

Examples:

- $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{v} \mapsto 2\boldsymbol{v}$ ✔
- $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{v} \mapsto \boldsymbol{v} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ☠
- $\phi : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ✔
- $\phi : \mathbb{R}^2 \to \mathbb{R}^1 : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 + v_2 \end{pmatrix}$ ✔

Linear Mappings and Matrixes

Theorem 7

- *For each linear mapping* $\phi : \mathbb{R}^n \to \mathbb{R}^m$ *there is an* $m \times n$ *matrix* A_ϕ*, such that* $\phi(\boldsymbol{v}) = A_\phi \cdot \boldsymbol{v}$*.*
- *Each matrix* $m \times n$ *matrix* A *gives rise to a linear mapping* $\phi_A : \boldsymbol{v} \mapsto A \cdot \boldsymbol{v}$*.*

Remarks:

- With matrix multiplication, we have all possible linear mappings.
- It is the basic operation in linear regression, logistic regression, support vector machines, neural networks, . . .

Linear Mappings and Matrixes – Examples

Examples from before:

- $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{v} \mapsto 2\boldsymbol{v}$ ➔ $A_\phi = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{v} \mapsto \boldsymbol{v} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ➔ no matrix can represent this (non-linear) mapping
- $\phi : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ➔ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
- $\phi : \mathbb{R}^2 \to \mathbb{R}^1 : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 + v_2 \end{pmatrix}$ ➔ $\begin{pmatrix} 1 & 1 \end{pmatrix}$

Rank of a Matrix

Let A be a $m \times n$ matrix.

1. The columns of A are n vectors in $\mathbb{R}^m$, the rows of A are m vectors in $\mathbb{R}^n$.
2. the maximum number of linear independent row vectors is equal to the maximum number of linear independent column vectors
3. this number is called the **rank** of A
4. Obviously $0 \leq \text{rank}(A) \leq \min(m, n)$
5. If $\text{rank}(A) = \min(m, n)$, then A has **full** rank.

Rank of a Matrix – Examples

- $\operatorname{rank}(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}) = 2$
- $\operatorname{rank}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) = 2$
- $\operatorname{rank}(\begin{pmatrix} 1 & 1 \end{pmatrix}) = 1$

Inverse of a Matrix 1/2

Definition 8

Let A be an $m \times m$ matrix. A matrix B is called the **inverse** of A if and only if

$$A \cdot B = I \quad \text{and} \quad B \cdot A = I.$$

- Here I denotes the identity matrix: $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$
- For I holds $I \cdot \boldsymbol{v} = \boldsymbol{v}$, and $I \cdot M = M$ for compatible vectors $\boldsymbol{v}$ and matrixes M.
- If a matrix has an inverse, it is unique. It is denoted as A^{-1}.
- The inverse of a matrix gives rise to a linear mapping with $\phi_A(\phi_{A^{-1}}(\boldsymbol{v})) = \boldsymbol{v}$ and $\phi_{A^{-1}}(\phi_A(\boldsymbol{v})) = \boldsymbol{v}$

Inverse of a Matrix 2/2

Theorem 9

Let A be an $m \times m$ matrix. The inverse of A exists if and only if A has full rank (i.e. $\mathrm{rank}(A) = m$).

Notebook 07_1_linear_algebra_in_python, Cells 10–23

Linear Equation Systems and Invertible Matrixes

A linear equation system, a matrix equation, or an equation with a linear mapping express the same task.

- consider $\boldsymbol{u} = A \cdot \boldsymbol{v}$ (given $\boldsymbol{u}$ and A, what is $\boldsymbol{v}$)
- the according linear equation system is spelling out the matrix multiplication and looking at the result componentwise
- the according linear mapping is ϕ_A such that $\boldsymbol{u} = \phi_A(\boldsymbol{v})$
- if A is invertible, then the solution to the equation system is $\boldsymbol{v} = A^{-1}\boldsymbol{u}$

Remark: On slide 8, we wanted to check for linear independence by solving an equation system. Alternatively, we can check if the respective matrix is invertible, which is the same as checking if the matrix has full rank.

Notebook 07_1_linear_algebra_in_python, Cells 24–26

Exercises 4–5