MADS-MMS – Mathematics and Multivariate Statistics

Linear Algebra

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Moodle (WiSe 24/25)

Agenda

Motivation

Subvectorspaces

Linear Combinations

Linear Mappings and Matrixes

Outline

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- Vector space operations are an integral part of various algorithms

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- PCA (next chapter) relies heavily on matrix operations in vector spaces

Chapter Goals

- understand mathematical foundations linear independence and dimensionality

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- understand and apply matrix computations

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- understand mathematical foundations linear independence and dimensionality
- understand and apply matrix computations
- preparation for further algorithms (PCA, SVMs)

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Vector-Subspaces

Definition 1 (Subspace)

Let $\mathcal{V}$ be a vector space on V with $+$ and $\cdot$ and $U \subseteq V$. Then U with $+$ and $\cdot$ is a subspace of $\mathcal{V}$ if $(U, +, \cdot)$ is a vector space.

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Remarks:

- for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U : \boldsymbol{u}_1 + \boldsymbol{u}_2 \in U$
- for all $r \in \mathbb{R}$, $\boldsymbol{u} \in U : r \cdot \boldsymbol{u} \in U$
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Example:

- in $\mathbb{R}^n$, each line/plane/hyperplane through the origin is a subspace

Outline

Linear Combinations

Definition 2

Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v_n}$ in a vector space $\mathcal{V}$, a **linear combination** of these vectors is every sum

$$\sum_{i=1}^{n} r_i \cdot \boldsymbol{v_i}$$

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Theorem 3

Given vectors as above, then the set of all linear combinations of these vectors is a subspace of $\mathcal{V}$.

Notebook 07_1_linear_algebra_in_python, Cells 1–3

Dimension of a Vector Space

Definition 4 (Base, Dimension)

- A set of vectors $U \subseteq V$ in a vector space $\mathcal{V}$ is called a generating set, if each $v \in V$ is a linear combination of vectors in U.
- U is called a **base** of $\mathcal{V}$ if U is minimal w.r.t. the above property.
- If U is a base of $\mathcal{V}$ then, $\dim(V) = |U|$ is called the **dimension** of $\mathcal{V}$.

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- $\mathbb{R}^3$ is generated by the base $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ as well as by the base $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$

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- the dimension of $\mathbb{R}^n$ is n

Linear Independence

Definition 5

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- $(1,0,1)^T, (2,1,1)^T$, and $(3,1,1)^T$ are linear independent
- $(1,1,2)^T, (3,7,2)^T$, and $(2,10,-4)^T$ are linear dependent

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Generally: If with at least one $r_j \neq 0$

$$0 = \sum_{i=1}^{n} r_i \cdot \boldsymbol{v_i} \quad \text{then} \quad \boldsymbol{v_j} = \sum_{i=1, i \neq j}^{n} -\frac{r_i}{r_j} \cdot \boldsymbol{v_i}$$

Exercises 1

Checking for Linear Independence in $\mathbb{R}^n$

Let's check the example from before:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = r_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} + r_3 \begin{pmatrix} 2 \\ 10 \\ -4 \end{pmatrix}$$

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Observation: Checking for solutions r_1, r_2, r_3 is the same as solving:

$$\begin{aligned} 0 &= 1r_1 + 3r_2 + 2r_3 \\ 0 &= 1r_1 + 7r_2 + 10r_3 \\ 0 &= 2r_1 + 2r_2 + (-4)r_3 \end{aligned}$$

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- linear equation systems can be solved approximately using efficient heuristics
- linear equation systems can be written with matrixes

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 7 & 10 \\ 2 & 2 & (-4) \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

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  - elementwise product of row i of A and column j of B
  - sum up these products

Matrix Multiplication – Example

Example 1:

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 10 \\ 4 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 2 & 3 \\ 4 & 1 \end{pmatrix} =$$

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$$= \begin{pmatrix} 16 & 13 \\ 58 & 35 \\ 24 & 17 \end{pmatrix}$$

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Notebook 07_1_linear_algebra_in_python, Cells 4–9

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Linear mappings, matrixes and systems of linear equations are related!

Linear Mappings

Definition 6 (Linear Mapping)

A **linear mapping** between two real valued vector spaces $\mathcal{V} = (V, +, \cdot)$ and $\mathcal{W} = (W, +, \cdot)$ is a mapping $\phi : V \to W$ that preserves the structure of the vector space, i.e. $\phi(\boldsymbol{x} + \boldsymbol{y}) = \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ and $\phi(s\boldsymbol{x}) = s\phi(\boldsymbol{x})$.

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- $\phi : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ✔
- $\phi : \mathbb{R}^2 \to \mathbb{R}^1 : \begin{pmatrix} v_1 \\ v_2, \end{pmatrix} \mapsto \begin{pmatrix} v_1 + v_2 \end{pmatrix}$ ✔

Linear Mappings and Matrixes

Theorem 7

- *For each linear mapping* $\phi : \mathbb{R}^n \to \mathbb{R}^m$ *there is an* $m \times n$ *matrix* A_ϕ*, such that* $\phi(\boldsymbol{v}) = A_\phi \cdot \boldsymbol{v}$.
- *Each matrix* $m \times n$ *matrix* A *gives rise to a linear mapping* $\phi_A : \boldsymbol{v} \mapsto A \cdot \boldsymbol{v}$.

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Remarks:

- With matrix multiplication, we have all possible linear mappings.
- It is the basic operation in linear regression, logistic regression, support vector machines, neural networks, ...

Linear Mappings and Matrixes – Examples

Examples from before:

- $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{v} \mapsto 2\boldsymbol{v}$ ➔ $A_\phi = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- $\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{v} \mapsto \boldsymbol{v} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ➔ no matrix can represent this (non-linear) mapping
- $\phi : \mathbb{R}^3 \to \mathbb{R}^2 : \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ ➔ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
- $\phi : \mathbb{R}^2 \to \mathbb{R}^1 : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 + v_2 \end{pmatrix}$ ➔ $\begin{pmatrix} 1 & 1 \end{pmatrix}$

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4. Obviously $0 \leq \text{rank}(A) \leq \min(m, n)$
5. If $\text{rank}(A) = \min(m, n)$, then A has **full** rank.

Rank of a Matrix – Examples

- $\text{rank}(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}) = 2$
- $\text{rank}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}) = 2$
- $\text{rank}(\begin{pmatrix} 1 & 1 \end{pmatrix}) = 1$

Inverse of a Matrix 1/2

Definition 8

Let A be an $m \times m$ matrix. A matrix B is called the **inverse** of A if and only if

$$A \cdot B = I \quad \text{and} \quad B \cdot A = I.$$

- Here I denotes the identity matrix: $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

Inverse of a Matrix 1/2

Definition 8

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- The inverse of a matrix gives rise to a linear mapping with $\phi_A(\phi_{A^{-1}}(\boldsymbol{v})) = \boldsymbol{v}$ and $\phi_{A^{-1}}(\phi_A(\boldsymbol{v})) = \boldsymbol{v}$

Inverse of a Matrix 2/2

Theorem 9

Let A be an $m \times m$ matrix. The inverse of A exists if and only if A has full rank (i.e. $\text{rank}(A) = m$).

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Notebook 07_1_linear_algebra_in_python, Cells 10–23

Linear Equation Systems and Invertible Matrixes

A linear equation system, a matrix equation, or an equation with a linear mapping express the same task.

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Notebook 07_1_linear_algebra_in_python, Cells 24–26

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Exercises 4–5