MADS-MMS – Mathematics and Multivariate Statistics

Powers and Logarithms

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Moodle (SoSe 2025)

This chapter is largely repeating (hopefully) known mathematics. We will therefore only very quickly go through the first sections on powers, roots, and logarithms and focus more on the applications.

Use Exercises 1–3 as homework, to test yourself and to (re-)familiarize yourself with the notions.

Agenda

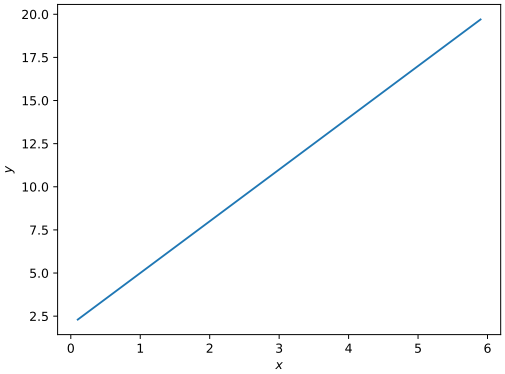
Outline

Chapter Goals

- recall basic notions on powers, roots, and logarithms
- prepare for their use in many ML algorithms (a.o. probabilistic learning, deep learning!)
- awareness of non-linear nature of many phenomena
- visualization and pitfalls using logarithms

Relations Between Two Quantities 1/3

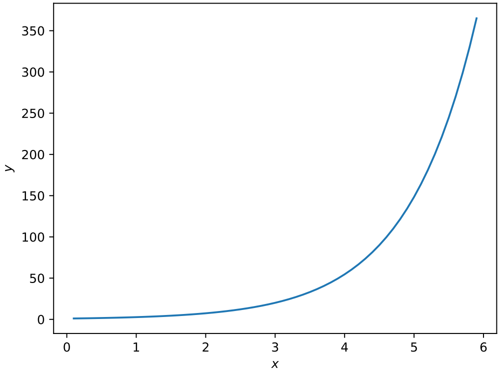
When we think of relations between two quantities x and y, we often subconsciously imagine it like this:



- we usually think in linear relationships
- a change of x has a proportionally large effect on y

Relations Between Two Quantities 2/3

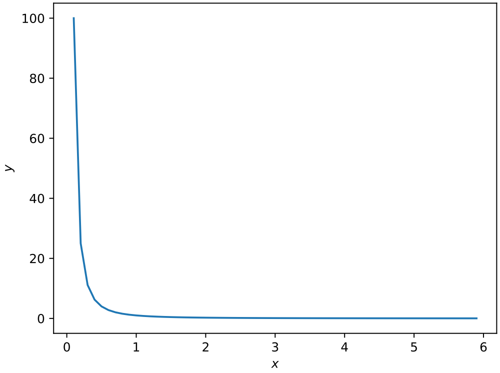
Often, relationships look completely different, e.g. like this:



- exponential growth
- e.g. number of subsets of a set (clustering!), information vs. bit-length, Corona infections

Relations Between Two Quantities 3/3

Sometimes, they look like this:



- the example shows a **power law**
- number of activities per user (90-9-1 rule), number of responses to tweets, number of citations to articles, frequency of words in texts, ...

Outline

Powers

Definition 1

Let n be a natural number and $a \in \mathbb{R}$ with $a > 0$, then a **to the power of** n

$$a^n := \prod_{i=1}^{n} a.$$

a is called the **base** and n the **exponent**.

Examples: $a^0 = 1$
$a^1 = a$
$a^2 = a \cdot a$
$a^3 = a \cdot a \cdot a$

Powers – Examples

$2^2 = 2 \cdot 2 = 4$ $\qquad$ $2^{10} = 1024$ $\qquad$ $2^0 = 1$

$1^n = 1$ for $n \in \mathbb{N}$

$7^5 = 16,807$

Observation:

$2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64,$

$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16,$

$2^2 = 2 \cdot 2 = 4$

$$2^{6-4} = 2^2 = 4 \quad \text{and} \quad \frac{2^6}{2^4} = \frac{64}{16} = 4$$

This is a result of a general rule, and gives rise to defining powers for negative exponents.

Adding / Subtracting Exponents

Definition 2

Let n be a natural number and $a \in \mathbb{R}$ with $a > 0$, then

$$a^{-n} := \frac{1}{a^n}$$

Examples: $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
$2^{-10} = \frac{1}{1024}$

Theorem 3

Let n, m be integers and $a \in \mathbb{R}$ with $a > 0$, then

$$a^{m+n} = a^m \cdot a^n \quad \textit{and} \quad a^{m-n} = \frac{a^m}{a^n}$$

Products / Quotients in the Base

Observation

$$(2\cdot 3)^3 = \prod_{i=1}^{3}(2\cdot 3) = \prod_{i=1}^{3} 2 \cdot \prod_{i=1}^{3} 3 = 2^3\cdot 3^3$$

This is again an example of a general law:

Theorem 4

Let n be an integer and $a, b \in \mathbb{R}$ with $a, b > 0$, then

$$(a\cdot b)^n = a^n\cdot b^n \quad \textit{and} \quad (\frac{a}{b})^n = \frac{a^n}{b^n}$$

Outline

Reverse Operation – Root

$$a^n = b$$

Given b and n, what is the value of a?

Definition 5

Let n be an integer and $a, b \in \mathbb{R}$ with $a, b > 0$, then a is the **n-th root** of b, denoted

$$\sqrt[n]{b} := a \iff a^n = b$$

Examples:
$\sqrt[2]{4} = 2$ $\sqrt[2]{9} = 3$ $\sqrt[2]{2.25} = 1.5$ $\sqrt[3]{8} = 2$

Observation

$$\sqrt[3]{2^6} = \sqrt[3]{64} = 4 = 2^2 = 2^{\frac{6}{3}}$$

This is again an example of a general law.

Products and Quotient Exponents

Definition 6

Let n be a natural number and $a \in \mathbb{R}$ with $a > 0$, then

$$a^{\frac{1}{n}} := \sqrt[n]{a}$$

$4^{\frac{1}{2}} = 2 \qquad 9^{\frac{1}{2}} = 3 \quad 2.25^{\frac{1}{2}} = 1.5 \qquad 8^{\frac{1}{3}} = 2$

Theorem 7

Let $a \in \mathbb{R}$ and $a > 0$. Let $n, m \in \mathbb{N}$. Then

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \quad \text{and} \quad (a^n)^m = a^{n \cdot m}$$

Extension of Definition and Exponent Rules

The previous definitions and laws also work for exponents in $\mathbb{R}$. Exceptions occur when

- base and exponent are zero: 0^0 cannot be defined smoothly
- when the base is negative: $(-1)^2 = 1^2 = 1$, thus, both 1 and -1 are square roots of 1.

Outline

Reverse Operation Logarithm

$$a^c = b$$

Given a and b, what is the value of c?

Definition 8

Let $a, b, c \in \mathbb{R}$ with $a, b > 0$, then c is the **logarithm** of b to **base** a:

$$\log_a b = c$$

Examples:

$\log_2 4 = 2$ $\log_3 9 = 2$ $\log_{1.5} 2.25 = 2$ $\log_2 8 = 3$

Logarithm Rules 1/2

The rules for calculating powers can be transformed into rules for calculating logarithms.

$$a^{m+n} = a^m \cdot a^n \quad \text{and} \quad a^{m-n} = \frac{a^m}{a^n}$$

➔

$$\log_a b + \log_a c = \log_a(b \cdot c) \quad \text{and} \quad \log_a b - \log_a c = \log_a \frac{b}{c}$$

Logarithm Rules 2/2

$$(a^n)^m = a^{n \cdot m}$$

➔

$$\log_a(b^c) = c \cdot \log_a b$$

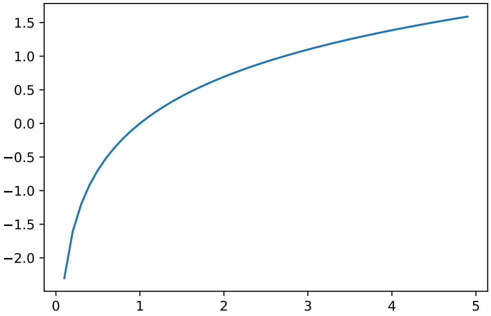
and for $c \neq 1$

$$\log_c b = \frac{\log_a b}{\log_a c}$$

Exercises 1–3

Outline

Logarithm function

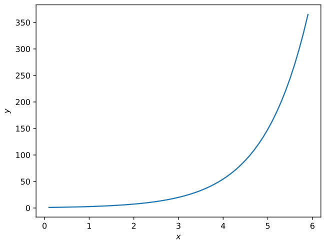


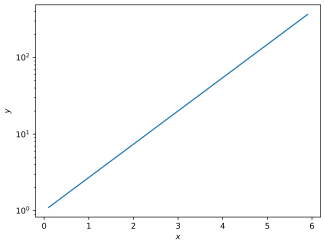
- defined on $\mathbb{R}_{>0}$
- goes quickly to $-\infty$ ($x \to 0$) and slowly to ∞ ($x \to \infty$)

Agenda

Importance of $\log$: Exponential $\rightarrow$ linear 1/2

Remember?



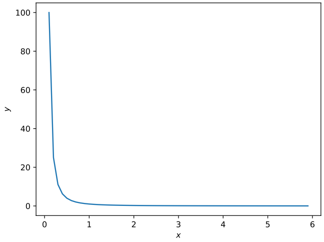


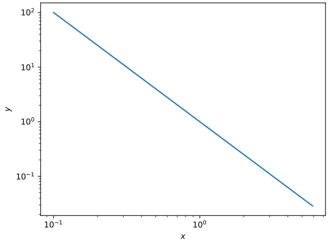
$$y = e^x$$

$$\log_e y = \log_e e^x = x$$

Importance of $\log$: Exponential $\rightarrow$ linear 2/2

Remember?

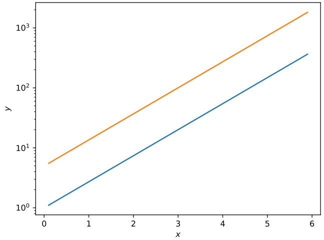




$y = x^{-2}$

$\log_e y = \log_e x^{-2} = -2 \cdot \log_e x$

Scaling Visuals



Scaling in diagrams never changes the actual relationship.

In the diagram (with logarithmic scaling on the y axis) the space between the two lines is constant. The actual difference between the quantities (y_1 and y_2) in the unscaled data

- are small for x near 0
- get bigger with bigger x (it grows exponentially)

$$y_1 = e^x \quad \text{and} \quad y_2 = 5e^x \quad \Rightarrow y_2 - y_1 = 5e^x - e^x = 4e^x$$

Agenda

Logarithms in Python

Implemented in module `math` or `numpy`

- `log(b,a)` computes $\log_a b$
- `log(b)` computes the **natural logarithm** $log_e b$, where e is Euler's number.
- the difference between the packages is that the numpy logarithm works with arrays, applying the logarithm componentwise (to each element of the array separately)

Exercises 4

A Simple Model of Epidemic Spreading

Exercises 5

Agenda

Importance of log: Numeric Stability

In data science, we often compute products of small numbers. E.g. products of small probabilities, i.e. numbers $<< 1$.

- when numbers get too small, computers do no longer compute them correctly
- multiplying many small numbers can yield zero even for very small rounding errors, overflows and underflows can occur

Computing the logarithm of the product instead

- turns very small numbers into (negative) numbers with high absolute value.
- turns the product of numbers into a sum of numbers

Notebook 03_1_numeric_stability