Parameterizing Curves

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1 Preliminaries

1.1 Vectors

A vector is a quantity described by its magnitude and direction and can be written in the following form using its scalar components:

$$\overrightarrow{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \overrightarrow{u} = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \tag{1}$$

$\overrightarrow{v}$ is a 3×1 vector or equivalently a 3×1 matrix, comprised of three rows and one column. $\overrightarrow{u}$ is similarly a 1×3 vector or matrix (one row and three columns). Aside from the form they are written in, $\overrightarrow{v}$ and $\overrightarrow{u}$ are equivalent. Vectors are closed under addition:

$$\overrightarrow{v} = \overrightarrow{v_1} + \overrightarrow{v_2} \tag{2}$$

and scalar multiplication for some constant c:

$$\overrightarrow{v} = c\overrightarrow{v_1} \tag{3}$$

which means that both the sum of two vectors and a vector multiplied by a scalar will result in a new vector.

1.2 Unit Vectors

Unit vectors are vectors with lengths or magnitudes of 1. To convert a vector into a unit vector, we simply normalize the vector by its length or magnitude:

$$\hat{\overrightarrow{v}} = \frac{\overrightarrow{v}}{||\overrightarrow{v}||} \text{ for } ||\overrightarrow{v}|| \text{ is length or magnitude (norm) of } \overrightarrow{v} \tag{4}$$

1.3 Vector Transpose

The transpose of a vector is trivially flipping its rows and columns and denoted with the superscript $\intercal$:

$$\overrightarrow{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} , \overrightarrow{v}^{\intercal} = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} , (\overrightarrow{v}^{\intercal})^{\intercal} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \tag{5}$$

The transpose of a transposed vector $\overrightarrow{v}$ gives the original $\overrightarrow{v}$. $\overrightarrow{u}$ from Eqn. 1 is in fact $\overrightarrow{v}^{\intercal}$. Note that the transpose of a matrix is slightly more complicated, but is no more than flipping a matrix across its diagonal.

1.4 Dot Product (Inner Product)

The dot product of two vectors $\overrightarrow{v}$ and $\overrightarrow{u}$ is the summation over the component-wise multiplication of the two vectors:

$$\overrightarrow{v} \cdot \overrightarrow{u} = v_0 \cdot u_0 + v_1 \cdot u_1 + ...v_N \cdot u_N \text{ for } N \text{ elements} \tag{6}$$

which can be equivalently written as:

$$\overrightarrow{v} \cdot \overrightarrow{u} = \sum_{i=0}^{N} v_i u_i \tag{7}$$

Hence, the dot product of two vectors results in a scalar.

Let $\overrightarrow{v}$ and $\overrightarrow{u}$ be:

$$\overrightarrow{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \overrightarrow{u} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \tag{8}$$

Then the dot product of $\overrightarrow{v}$ and $\overrightarrow{u}$ is:

$$\overrightarrow{v} \cdot \overrightarrow{u} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32 \tag{9}$$

Note that the dot product of two vector written in this form (each element in gives some insight on matrix multiplication as they are now both performing row element multiplied by column element).

2 Parameterizing a Line

Suppose we have a point p_0 and a vector $\overrightarrow{v}$, we can form a new point p_1 by simply adding p_0 with $\overrightarrow{v}$:

$$p_1 = p_0 + \overrightarrow{v} \tag{10}$$

Consequently, we can subtract p_1 by p_0 to form the vector $\overrightarrow{v}$:

$$\overrightarrow{v} = p_1 - p_0 \tag{11}$$

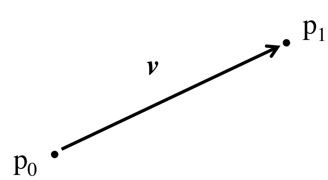

Figure 1: Points p_0, p_1 and the vector $\overrightarrow{v}$.

Now suppose that we would like to parameterize the line formed by p_0 and p_1 using a variable t such that given p_0, p_1 and t we will be able to know every point p along the line formed by $p_1 - p_0$.

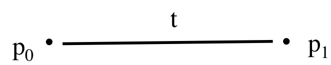

Figure 2: Points p_0, p_1 and the parameter t define a line.

If we treat t as a scale or ratio between p_0, p_1 then we can write this as

$$p = p_0 + t(p_1 - p_0) \text{ for } t \in [0, 1] \tag{12}$$

If $t = 0$ then $p = p_0$ and if $t = 1$ then $p = p_1$. We naturally get the midpoint between the p_0 and p_1 if we were to select $t = 0.5$, $p = 0.5p_0 + 0.5p_1$. If we were to sample (or select different values of) t, then we can interpolate all points along the line. Note, we must have the constraint that $t \in [0, 1]$ or the interpolated points will go outside of the line defined by p_0 and p_1.

We will now rewrite Eqn. 12 in a slightly different form such that we can then motivate using coefficients in front of our points.

$$p = (1 - t)p_0 + tp_1 \text{ for } t \in [0, 1] \tag{13}$$

We can now in fact substitute $(1 - t)$ and t with t_0 and t_1 to write:

$$p = t_0 p_0 + t_1 p_1 \text{ for } t_0 + t_1 = 1 \tag{14}$$

with the new constraint that the sum of the coefficients must sum up to 1. The constraint that we have set explicitly restricts any points we produce using the above equations to live within the space between the two points.

This is known as an affine combination and the space that these points exists in (or is defined by the vectors) is known as an affine space, which in summary:

$$p = p_0 + t\overrightarrow{v} = p_0 + t(p_1 - p_0) = (1 - t)p_0 + tp_1 = t_0 p_0 + t_1 p_1 \tag{15}$$

for $t \in [0, 1]$ and $t_0 + t_1 = 1$.

3 Parameterizing a Curve

Thus far, we know that a by parameterizing a line using two points and a parameter that scales between the two points given some constraint, we are able to obtain every point on the line. Since a line can be parameterized by two points, we can extend the concept to a parameterize a plane using three points:

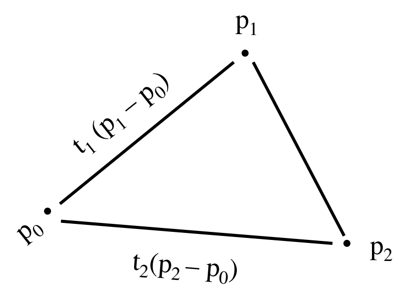

Figure 3: Points p_0, p_1, p_2 define a plane (a triangle)

This allows us to write the following to describe any point within the triangle:

$$p = p_0 + t_1(p_1 - p_0) + t_2(p_2 - p_0) \tag{16}$$

and equivalently:

$$p = (1 - t_1 - t_2)p_0 + t_1 p_1 + t_2 p_2 \tag{17}$$

We then substitute $t_0 = (1 - t_1 - t_2)$ to arrive at the same form we have written for our equation of a line with the constraint that t_0, t_1, t_2 must sum to 1:

$$p = t_0 p_0 + t_1 p_1 + t_2 p_2 \text{ for } t_0 + t_1 + t_2 = 1 \tag{18}$$

We can view each t_0, t_1, t_2 as the coordinates of the plane (in this case it is a triangle) contained within p_0, p_1, p_2. By varying the coordinates we can obtain a new point on the plane. These coordinates are known Barycentric coordinates, which are specific to a triangle – meaning we can parameterize any triangle using the above form. We can utilize this knowledge to form a curved line.

Thus far, we have seen first order polynomials (which we know are linear and hence a line) as coefficients (coordinates) to the space defined by the give points. To form curves (e.g. a parabola), we know that we will need at least second order polynomials (quadratics). Let's consider the following polynomials:

$$\begin{aligned}(1-t)^2 &= 1 - 2t + t^2 \\ 2t(1-t) &= 0 + 2t - 2t^2 \\ t^2 &= 0 + 0 + t^2\end{aligned} \tag{19}$$

Based on their expansions, we see that $(1 - 2t + t^2) + (2t - 2t^2) + t^2 = 1$, which suggests that we can use these polynomials in forming our affine combination for three points (a plane), which allows for curvature.

$$p = (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2 \text{ for } t \in [0,1] \tag{20}$$

We substitute $t_0 = (1-t)^2$, $t_1 = 2t(1-t)$, $t_2 = t^2$ with the constraint that $t \in [0,1]$. We can see that if $t = 0$ then $p = p_0$ and if $t = 1$ then $p = p_2$ – guaranteeing that our point will live within the triangle. The second order terms further tell us that we will have a curve (barring co-linear points).

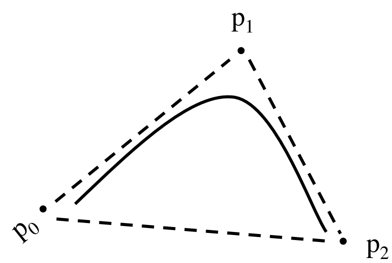

Figure 4: Illustration of the curve formed by $p = (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2$. Dashed lines mark the bounds of the triangle defined by p_0, p_1, and p_2.

Note that such a curve is bounded within p_0, p_1, and p_2 meaning that t must satisfy the constraint $t \in [0,1]$. Not all polynomials satisfy this constraint. A polynomial satisfying this constraint is known as a Bernstein Polynomial. Such polynomials exist in higher orders as well, allowing us to draw curves with multiple inflection points (which generally correspond to the one less than the order).

Now let's suppose we have the following Bernstein polynomial:

$$\begin{aligned}(1-t)^3 &= 1 - 3t + 3t^2 - t^3\\ 3t(1-t)^2 &= 0 + 3t - 6t^2 + 3t^3\\ 3t^2(1-t) &= 0 + 0 + 3t^2 - 3t^3\\ t^3 &= 0 + 0 + 0 + t^3\end{aligned} \tag{21}$$

This again sums to one and satisfies our constraint; hence, we can use these polynomials as our coefficients for an equation with four points p_0, p_1, p_2, and p_3:

$$p = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)p_2 + t^3 p_3 \text{ for } t \in [0,1] \tag{22}$$

We again observe that if $t = 0$ then $p = p_0$ and if $t = 1$ then $p = p_3$. As the expansion of these polynomials can become quite messy, we will now modify our notation by writing the expansion into vector form.

$$p = \begin{bmatrix}1 - 3t + 3t^2 - t^3 & 3t - 6t^2 + 3t^3 & 3t^2 - 3t^3 & t^3\end{bmatrix} \cdot \begin{bmatrix}p_0\\ p_1\\ p_2\\ p_3\end{bmatrix} \tag{23}$$

We can see that the dot product of the two vectors would give us the expanded version of Eqn. 22. Let's further decompose the coefficients by separating each scalar from the parameter t, where each column corresponds one polynomial (e.g. column one contains the coefficients of $(1-t)^3 = 1 - 3t + 3t^2 - t^3$:

$$p = \begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix} \begin{bmatrix}1 & 0 & 0 & 0\\ -3 & 3 & 0 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{bmatrix} \cdot \begin{bmatrix}p_0\\ p_1\\ p_2\\ p_3\end{bmatrix} \tag{24}$$

We have just written our vector space as a matrix. The first (left-most) 1×4 vector (matrix) contains the orders of t, which gives us a sense of the number of inflection points in the curve. The center 4×4 matrix is known as a transformation matrix as it will tell us how points on the curve will deform (i.e. stretch in certain directions). The final 4×1 matrix gives us the space that the curve exists.