Monte Carlo Methods
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Problem Sheet 4

Problem 9: A simple bremsstrahlung cascade

We want to simulate a simple bremsstrahlung cascade – a Markov process with a variable q, which we associate to a typical energy scale at which an emission occurs, and a variable z which is the momentum fraction left to the emitter after the emission. In the general language of Markov processes, our space of states thus consists of the variable z, and the time variable is q. We shall consider transition rates which read

$$P(z'|z,q') = \alpha \frac{1}{q'} \frac{1}{z-z'} \theta(z - z' - z\ q')\theta(z') \qquad \alpha > 0\ . \tag{1}$$

Simulate the resulting Markov process using the methods derived in the lecture, starting from $z = 1$ at $q = 1$ and evolve down to small values $\mu \sim 0.01$: Generate subsequent emission steps, and terminate the cascade if the time μ has been reached, *i.e.* if the process has performed the last emission at time q with momentum fraction z (or has started at these values), determine the next time q' and state z' from

$$\frac{\mathrm{d}P}{\mathrm{d}q'\mathrm{d}z'} = \delta(\mu - q')\delta(z - z') \exp\left(-\int_\mu^q \mathrm{d}k \int \mathrm{d}\xi P(\xi|z,k)\right) + P(z'|z,q') \exp\left(-\int_{q'}^q \mathrm{d}k \int \mathrm{d}\xi P(\xi|z,k)\right) \theta(q - q')\theta(q' - \mu)\ . \tag{2}$$

until a time $q' = \mu$ has been generated. Interesting questions to be asked are:

- What is the average momentum fraction at the end of the cascade, what is its variance?
- What is the distribution of multiplicities of radiated particles?
- How do these quantities depend on the choice of μ?

Problem 10: The Ising model

We want to simulate the Ising model in two dimensions, *i.e.* we consider a lattice of 'spins' which can take values $\phi_i = \pm 1$ and which are governed by a Hamiltonian

$$H = -\kappa \sum_{\text{nearest neighbours } i,j} \phi_i \phi_j - h \sum_i \phi_i\ , \tag{3}$$

in the sense of a thermal average for which we consider distributions of configurations according to (we set $k_B = 1$)

$$\frac{\exp\left(-\frac{1}{T} H(\vec{\phi})\right)}{\int \prod_i \mathrm{d}\phi_i \exp\left(-\frac{1}{T} H(\vec{\phi})\right)}\ . \tag{4}$$

Here, i indexes the lattice sites – for a lattice of $n \times n$ sites, you might choose this as

$$i = n_x + n_y\ n \tag{5}$$

where (n_x, n_y) labels the individual lattice sites ranging from 0 to $n-1$, and i ranges from 0 to $n^2 - 1$. We also want to impose periodic boundary conditions, such that lattice site indices are always given by $n_x \mod n$ and $n_y \mod n$. You will need a data structure to store the configuration, which now can be just a simple vector, and a way to address the nearest neighbours (often referred to as a 'hopping' function or table).

a) Provide all of the data structures.

b) Use the Metropolis algorithm to implement local updates of a single lattice site.

c) Provide updates of the entire lattice (a so-called 'sweep') by looping over local updates.

d) Monitor the energy, *i.e.* measure H itself as a thermal average.