22. Производная сложной функции. Производная неявно заданной функции.

Пусть у нас есть функция $u = u(y_1, y_2, y_3 \ldots y_m)$, где $y_1 = y_1(x_1, x_2, x_3 \ldots x_m)$, $y_2 = y_2(x_1, x_2, x_3 \ldots x_m)$ $\ldots$ $y_m = y_m(x_1, x_2, x_3 \ldots x_m)$. Тогда полная производная функции одной переменной будет вычисляться по формуле

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_i} + \frac{\partial u}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_i} + \frac{\partial u}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_i} + \cdots + \frac{\partial u}{\partial y_m} \cdot \frac{\partial y_m}{\partial x_i}$$

Пример 1

Дано $z = x^2 \ln y,\ x = \frac{u}{v},\ y = 3u - 2v$.
Найти $\frac{\partial z}{\partial u};\ \frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial x} = 2x \ln y;\ \frac{\partial z}{\partial y} = \frac{x^2}{y}$$

$$\frac{\partial x}{\partial u} = \frac{1}{v};\ \frac{\partial y}{\partial u} = 3$$

$$\frac{\partial x}{\partial v} = -\frac{u}{v^2};\ \frac{\partial y}{\partial v} = -2$$

$$\frac{\partial z}{\partial u} = 2x \ln y \cdot \frac{1}{v} + \frac{x^2}{y} \cdot 3 = \frac{2u}{v^2} \ln(3u - 2v) + \frac{3u^2}{v^2(3u - 2v)}$$

$$\frac{\partial z}{\partial v} = 2x \ln(-\frac{u}{v^2}) + \frac{x^2}{y^2} \cdot (-2) = -\frac{2u^2}{v^3} \ln(3u - 2v) - \frac{2u^2}{v^2(3u - 2v)}$$

Производная невяно заданной функции

а) Производная от одной переменной

$$F(x, y) = 0$$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial/\partial y} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

б) Производная от двух переменных

$$z = z(x, y)\ F(x, y, z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z};\ \ \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$