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Research of dark matter using gravitational waves in black hole mergers

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Abstract

In this paper, we investigated the phenomenon of black holes, gravitational waves, and dark matter.

The purpose of the study is to build a model of the motion of a system of binary black holes in the presence of dark matter and to study the influence of dark matter on the dynamics of motion.

The work's hypothesis is that dark matter surrounding a massive black hole can influence the dynamics of motion of a less massive black hole.

The stages of the research include initial processing and review of literature, study of existing detection methods, development of our own method, writing a computer program, debugging the program, and obtaining and processing the results.

The research methods include constructing a theoretical model of the motion of the double black hole system and carrying out a computer calculation.

The relevance of the work is set by a short time interval since the beginning of the study of the nature of gravitational waves, as well as active research on the nature of dark matter, conducted to date.

The work was done completely independently, the supervisor helped with the selection of literature and checking the correctness of the computer program.

Introduction

Modern astrophysics is concerned with the study of objects such as black holes. A black hole is a region of space-time that has a large gravitational attraction due to its large mass. A black hole is a theoretical object obtained by solving Einstein's equation [1, 2].

In 1906, the concept of dark matter was introduced in astrophysics. Dark matter is a hypothetical form of matter that does not participate in electromagnetic interaction, which makes it invisible to direct observation. The existence of dark matter has not yet been experimentally proven, as the dark matter particle has not yet been found. All evidence is circumstantial and no direct way of proving the existence of dark matter has yet been found. The only clue to the practical detection of dark matter is the detection of gravitational waves. Gravitational waves refer to changes in the gravitational field that propagate like waves.

In 2015, the Laser Interferometer Gravitational-Wave Observatory (LIGO) first recorded the presence of gravitational waves in the merger of double black holes with 29 solar masses and 36 solar masses, which gave a new fundamental boost to the study of the Universe [3]. Black hole merger is the process of merging two black holes into one, in which the emission of gravitational waves reaches a maximum. To date, LIGO in collaboration with Virgo has discovered and significantly expanded the catalog of double black holes [4].

The LISA (Laser Interferometer Space Antenna) observatory is considered to be the latest and newest facility for detecting gravitational waves. The LISA project is used to map the sky by studying and measuring gravitational waves using laser interferometry.

It consists of three spacecraft arranged in the shape of an equilateral triangle at a distance of 2.5 million kilometers from each other, rotating in a heliocentric orbit similar to Earth's. The principle of LISA operation is to measure the difference of gravitational wave half-axis lengths using laser interferometry [16].

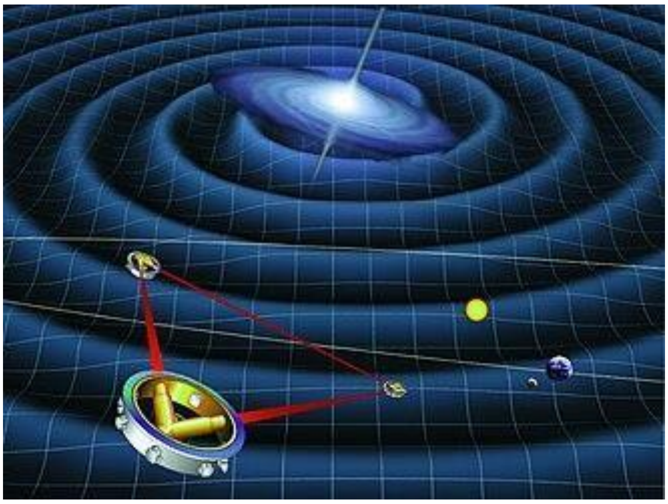

Figure 1: LISA concept. Source: https://upload.wikimedia.org/wikipedia/commons/thumb/f/f5/LISA-waves.jpg/330px-LISA-waves.jpg

LISA can register low-frequency gravitational waves between 0.1 mHz and 1 Hz, which is the limit with high content of strong gravitational wave sources in the Universe [17].

Its main advantage over LIGO and Virgo is that it will not be on the Earth's surface but in space, and its arms will be unfixed and a million times longer, providing large detectable motions.

In this paper, we will study dark matter using gravitational waves in black hole mergers.

Before us, there was a study by scientists such as N. Becker et al. in 2022, where they presented their computer program model [18].

Our model of a computer program is a simplification that gives the same values and results. Also, we supplemented the results of the model with graphs of dependencies, which the work of N. Becker et al. did not include.

Dark Matter Spike is a hypothetical matter surrounding supermassive black holes. We will study it and prove its existence in this paper.

Halo of dark matter is a hypothetical area in space of the black hole system filled with dark matter.

The essence of this work is to study and confirm the influence of dark matter on the dynamics of motion by analyzing the gravitational waves at the merger of the system of double black holes. The system consists of a larger and a smaller black hole that merges into a single black hole over a time interval by emitting gravitational waves. Dark matter causes dynamical friction, slowing down the dynamics of motion of such a system and merging of black holes. Thus, the influence of dark matter can be determined by comparing the motion of a system of double black holes at different dark matter profiles.

1. Model description

Further, we will consider only Schwarzschild black holes. A Schwarzschild black hole is a black hole corresponding to the solution of the Einstein equations in a vacuum with the participation of a point mass and without its own rotation [5,6]. The Schwarzschild radius is a characteristic radius for a black hole on which its event horizon is located. The Schwarzschild radius is described as follows: $r_s = \frac{2GM}{c^2}$ [7,8].

For this model, we assume the values of the gravitational constant G and the speed of light c to be equal:

$$G = c = 1$$

Consequently, in further calculations and formulas, we do not specify the values of these constants.

1.1. Dark matter halo

In this model, we consider the system IMRI (Intermediate mass ratio inspiral). The IMRI system implies a system of two black holes, the larger of which is in the center, and the smaller one moves around it. In such a system, there is a gradual merger of these two black holes. The larger mass m_1 is surrounded by static, spherically symmetric dark matter. Such dark matter can develop at adiabatic growth of the larger central black hole. The existence of dark matter around black holes is ambiguous since it can be interrupted by massive phenomena such as mergers, and the model requires the black hole to be at the center of the dark matter halo.

We imply the density of dark matter around such a system by the following equation:

$$\rho_{dm}(r) = \rho_6 \left(\frac{r_6}{r}\right)^{\alpha_{spike}} , r_{in} < r < r_{spike} \quad (1)$$

where $r_6 = 10^{-6} pc$ and following [12], $r_{in} = 4m_1$, $1 < \alpha_{spike} < 3$.

The different nature of dark matter can give variations in α_{spike} values, for example $\alpha_{spike} = 7/3$ for the Navarro-Frenk-White (NFW) halo profile forming dark matter [9], $\alpha_{spike} = 7/4$ for the self-interacting dark matter (SIDM) (SIDM) [10], $\alpha_{spike} = 9/4$ for the dark matter around the Primary black hole [11].

The radius r_{spike} is the maximum radius of the dark matter region, which can be investigated by comparing the gravitational influence of the central black hole with the total mass of dark matter [9].

1.2. Orbital evolution

The masses of the IMRI system are defined in such a way that m_1 is the mass of the larger black hole, m_2 is the mass of the smaller black hole. We assume that the black hole is Schwarzschild for simplification of calculations as it is indicated in Fig. 2. The smaller object is in a Keplerian orbit around the larger object. During gravitational emission and other dissipative forces, the smaller object loses orbital energy and momentum, leading to the formation of a spiral orbit.

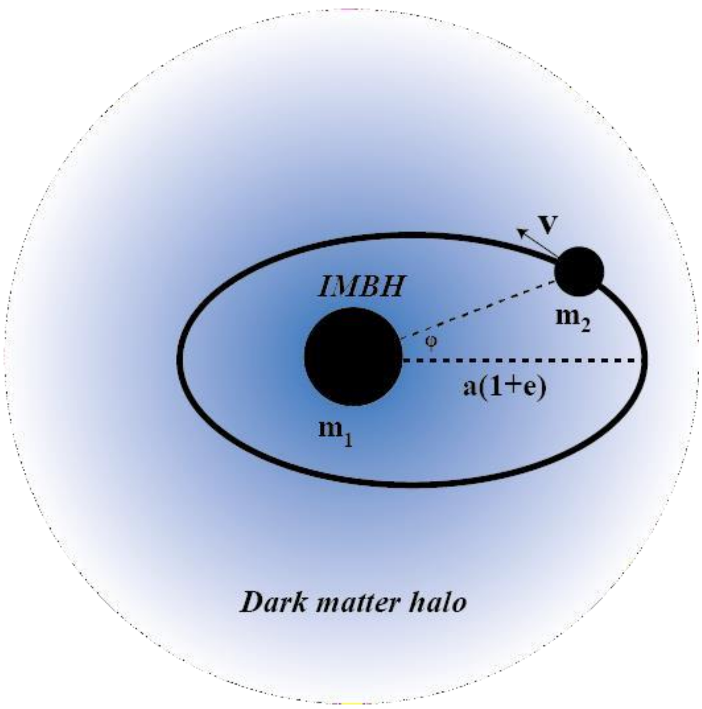

Fig. 2: Keplerian Orbit

In Figure 2, we can see a Keplerian orbit in which the mass $m_1 \gg m_2$, a is the major semi-axis and e is the eccentricity inside the dark matter halo ρ_{dm}.

1.2.1. Keplerian orbit

The secondary (less massive) object is assumed to be in a Kepler orbit around the central (more massive) object. We neglect the additional influence of dark matter on the total mass m and reduced masses μ of the Kepler orbit and assume that they are expressed in a similar way respectively [13].

$$m = m_1 + m_2 \tag{2}$$

$$\mu = \frac{m_1 m_2}{m} \tag{3}$$

This is a great simplification since we consider a system close to circular. It has small orbital distances because the total mass of the contained dark matter is much smaller than the mass of the central black hole $m_{dm}(r = 10^5 r_{\text{isco}}) \ll m_1$, where $r_{\text{isco}} = 6m_1$ for the Schwarzschild black hole.

Any Keplerian orbit is described by two parameters a and e, where a is the semi-major axis e is the eccentricity. The orbit is bounded by $0 \leq e \leq 1$, where $e = 0$ defines a circular orbit.

During one circle of the orbit, its radius can be determined from the following expression [14]:

$$v^2 = m\left(\frac{2}{r} - \frac{1}{a}\right) \tag{4}$$

The orbital energy is given by the following formula [15]:

$$E_{orb} = -\frac{m\mu}{2a} \tag{5}$$

The value of eccentricity e can be derived from the following formula [14]:

$$e^2 - 1 = \frac{2E_{orb}L^2_{orb}}{m^2\mu^3} \tag{6}$$

1.2.2. Dissipative forces

It is assumed that the orbit loses more energy on the long-term time scale than on the orbital time scale. That is, the orbit changes very slowly, which allows us to consider it as a Keplerian orbit for calculating the forces acting on an object.

Specific effects such as gravitational wave emission loss and dynamical friction are considered here. Each effect can be modeled as a force. The former leads to a loss of orbital energy and the latter leads to a loss of momentum on a long-time scale.

The change in orbital energy and momentum is described by the following formulas:

$$\frac{dE_{orb}}{dt} = \langle\frac{dE_{gw}}{dt}\rangle + \langle\frac{dE_{df}}{dt}\rangle \tag{7}$$

$$\frac{dL_{orb}}{dt} = \langle\frac{dL_{gw}}{dt}\rangle + \langle\frac{dL_{df}}{dt}\rangle \tag{8}$$

The emission of gravitational waves is described by the following formulas [15]:

$$\langle\frac{dE_{gw}}{dt}\rangle = -\frac{32}{5}\frac{\mu^2 m^3}{a^5}\frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}} \tag{9}$$

$$\langle\frac{dL_{gw}}{dt}\rangle = -\frac{32}{5}\frac{\mu^2 m^{5/2}}{a^{7/2}}\frac{1 + \frac{7}{8}e^2}{(1-e^2)^2} \tag{10}$$

The Chandrasekhar equation gives us an expression for dynamical friction [19, 20]:

$$F_{DF}(r, v) = \frac{4\pi m_2 \rho_{dm}(r) \log \Lambda}{v^2} \tag{11}$$

The black hole, moving through the dark matter due to gravitational interaction sets in motion the particles of dark matter. Thus, the dark matter takes energy and momentum from the black hole.

1.2.3. Orbital evolution

We are interested in the long-term change of the orbit parameters $a(t)$, $e(t)$ under the influence of dissipative forces.

Using equation (5), we obtain:

$$\frac{\partial E_{orb}}{\partial a} = \frac{m_2 m_1}{2a^2} \tag{12}$$

$$\frac{da}{dt} = \frac{dE_{orb}}{dt} / \frac{\partial E_{orb}}{\partial a} \tag{13}$$

In the same way, we can determine the equation of eccentricity evolution e from equation (6):

$$\frac{de}{dt} = -\frac{1 - e^2}{2e} \left(\frac{dE_{orb}}{dt} / E_{orb} + 2\frac{dL_{orb}}{dt} / L_{orb}\right) \tag{14}$$

Combining equations (7), (8), (13), (14) we obtain a system of differential equations that can be solved numerically.

2. Results

In this part, we demonstrate the results obtained by integrating a system of differential equations. The equations have been solved in Python, numerically worked out, and evaluated.

We have considered the motion of the system of double black holes in the absence of dark matter, and in the presence of dark matter.

a. Model for vacuum

In the first case, the model is created in a vacuum, i.e. $\alpha_{spike} = 7/4$, dark matter density $\rho_{dm} = 0$, initial eccentricity $e_0 = 0.5$, and initial length of the larger semi-major axis $a_0 = 100 r_{isco}$.

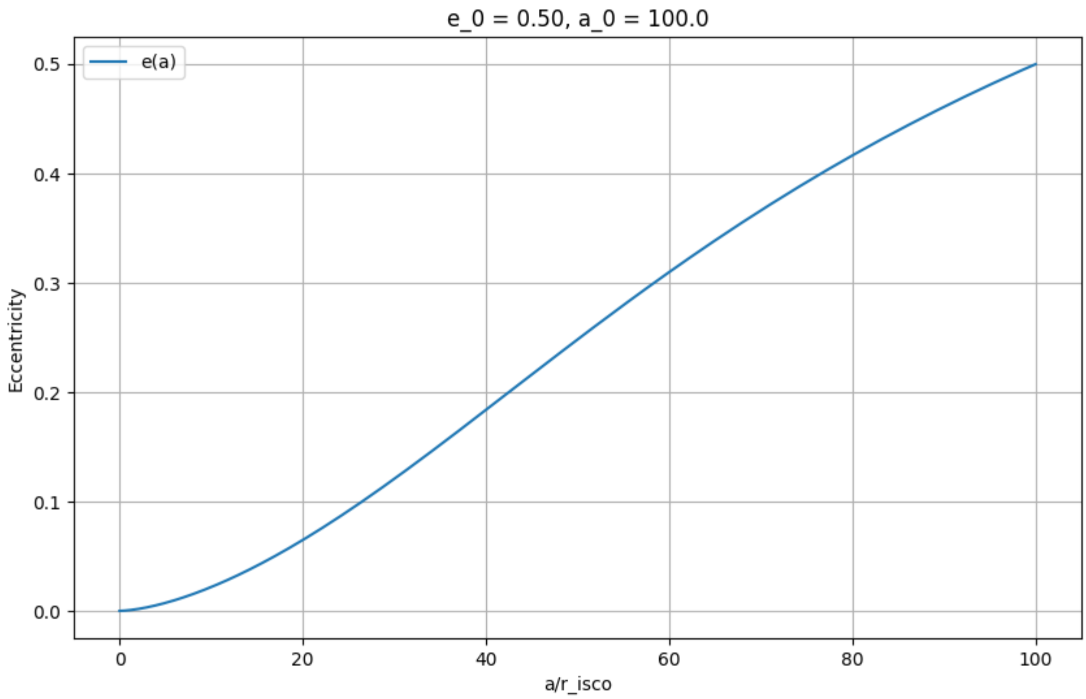

Fig. 3: Dependence of eccentricity e on $^{a}/_{r_{isco}}$

Fig. 3 shows a graph of the dependence of eccentricity e on the semi-major axis $^{a}/_{r_{isco}}$. The graph is read from right to left and it can be seen that as $^{a}/_{r_{isco}}$ decreases, the eccentricity e decreases and tends to zero. With this graph, we want to show that as the minor semi-axis of the orbit of the system decreases, it takes a circular form as its eccentricity tends to and becomes zero.

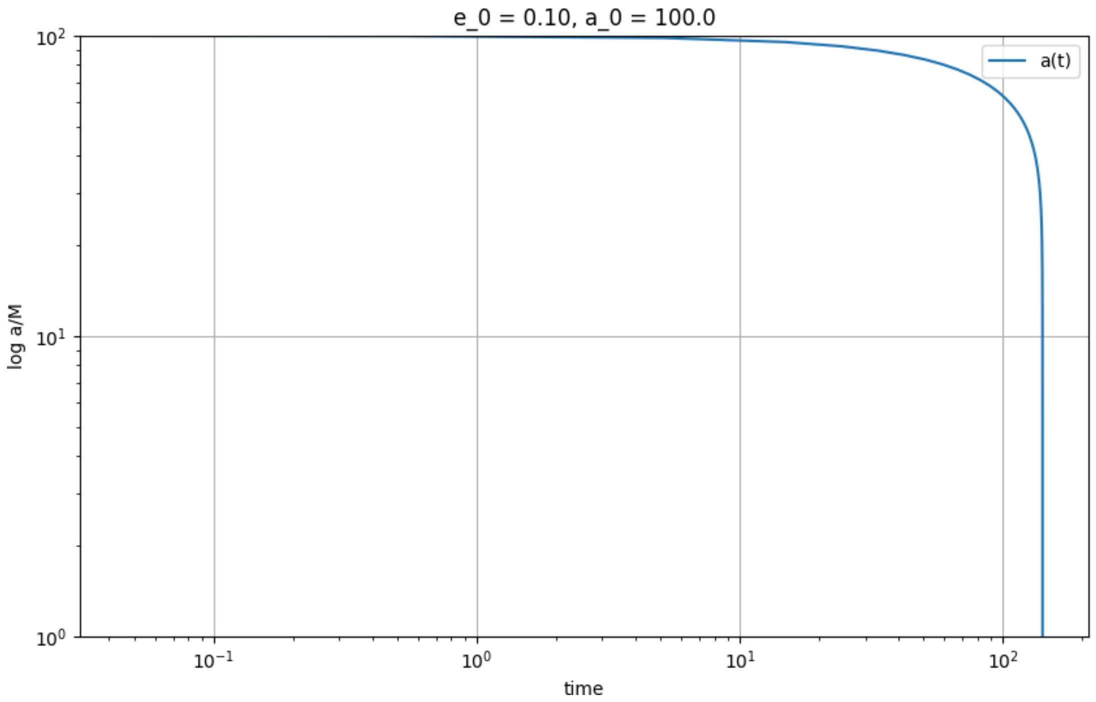

Fig. 4: Dependence of semi-major axis a/r_{isco} on time $time/yr$

Figure 4 shows a graph where we see the dependence of a/r_{isco} on time $time/yr$. This graph corresponds to the case when $\alpha_{spike} = 7/4$, initial eccentricity $e_0 = 0.5$, i.e., the graph was constructed for the model of the system located in a vacuum. The graph is read from left to right. On the graph, we see that as time passes, the minor semi-axis of the orbit of the system first slowly decreases, and then sharply decreases and tends to zero. Thus, the graph shows that the orbit of the system from elongated slowly decreases and becomes circular as time passes, which proves that our orbit is Keplerian.

b. Model for dark matter

In this case, we consider 3 profiles of dark matter: $\alpha_{spike} = [7/4, 9/4, 7/3]$. The initial eccentricity is given as $e_0 = 0.5$, while $\rho_{dm} = 226 M_{\odot}/pc^3$ and $a_0 = 100 r_{isco}$ [18].

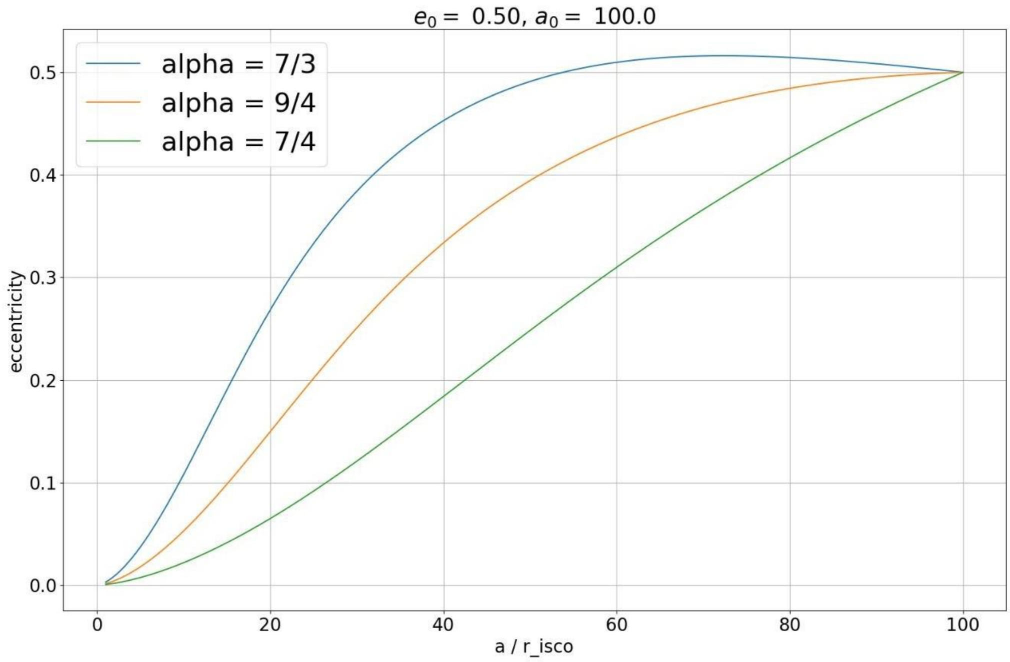

Fig. 5: Dependence of eccentricity e on semi-major axis a/r_{isco}

Figure 4 shows a graph where we see the dependence of eccentricity e on semi-major axis a/r_{isco} for 3 different dark matter profiles α_{spike}. The graph is read from right to left and it can be seen that as a/r_{isco} decreases, the eccentricity e decreases and tends to zero. For $\alpha_{spike} = 7/4$ the decreasing is the fastest, and for $\alpha_{spike} = 7/3$ the slowest. When comparing the plot in Figure 5 with the plot in Figure 3, we can see that the change in the dark matter profile affects the rate of change of the orbital parameters. Thus, the presence of dark matter affects the rate of change of the orbit from elongated to circular.

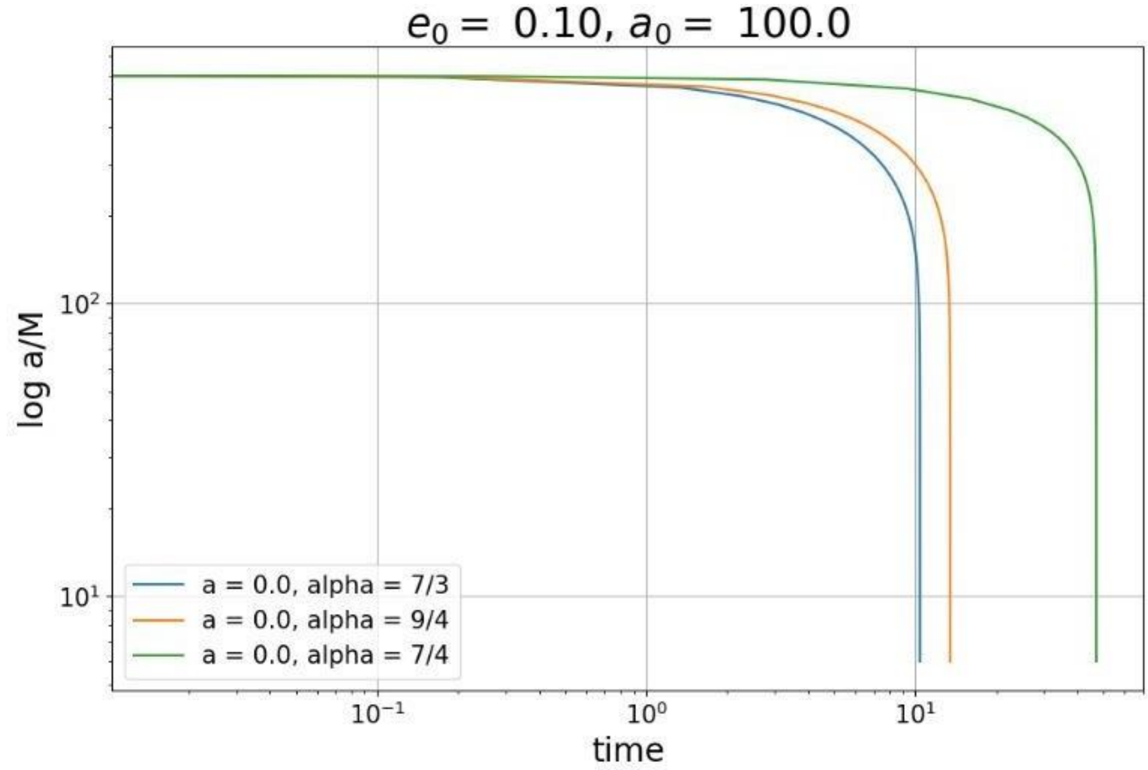

Рис. 6: Dependence of semi-major axis $^{a}/_{r_{isco}}$ on time $^{time}/_{yr}$

Figure 6 shows a graph where we see the dependence of $^{a}/_{r_{isco}}$ on time $^{time}/_{yr}$ for 3 different dark matter profiles. The graph reads from left to right. In the graph, we see that as time passes, the minor semi-axis of the orbit of the system decreases slowly at first and then decreases sharply and tends to zero, but this phenomenon varies for each case and depends on the value of the dark matter profile. When comparing the graph in Fig. 6 with the graph in Fig. 4, we can see that the change in the dark matter profile affects the time of change of the orbit parameters. Thus, the presence of dark matter affects the time of change of the orbit from elongated to circular.

The orbit parameters change in the presence of dark matter due to dynamical friction. By comparing the plots in Fig. 3 and Fig. 5, as well as the plots in Fig. 4 and Fig. 6, we can see that the change in the dark matter profiles affects the rate and time of change of the orbit parameters. Thus, the differences in the dynamics of the system allow us to believe that dark matter exists and directly affects the dynamics of the system at black hole mergers.

3. Conclusion

In the course of this work, we have created a computer model of the motion of the system of double black holes in the presence of dark matter. We have considered the motion of the system for 3 different profiles of dark matter, and have constructed graphs for all specified cases.

The results of the work and the graphs showed that the differences in the values of the dark matter profiles affect the dynamics of the motion of the double black hole system. Detection of changes in the dynamics of the system and characterization of its motion can be done by detecting gravitational waves emitted at the merger of black holes.

To accomplish this task, the design and construction of LISA is underway, which will be a convenient tool in the future and will advance the understanding of the nature of dark matter.

References and acknowledgements

Appendix A: Program code

The code for the computer model was written in the Python programming language using several additional libraries.

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.integrate as integrate
import math

# Constants
parsec = 3.08567758128 * 1e16
year = 31536000 * 299792458 / parsec
mass_to_length = 1.3466 * 1e27

Msun = 1.98992 * 1e30  # Mass of the Sun in kilograms

# Orbital parameters
Msun_parsec = Msun / (mass_to_length * parsec)
mass_big_parsec = Msun_parsec * 1e3
mass_small_parsec = Msun_parsec
mu_parsec = (mass_big_parsec * mass_small_parsec) / (mass_big_parsec +
mass_small_parsec)
m_parsec = mass_big_parsec + mass_small_parsec

# Parameter values
alpha_spike_values = [7/3, 7/4, 9/4]  # Change the value of alpha_spike here
po_6_values = [0, 5.448e15*Msun_parsec / parsec**3, 5.448e17*Msun_parsec /
parsec**3]  # Change the value of dark matter density here

r_isco = 6 * mass_big_parsec

# Dark matter density function
def po_dm(r, alpha_spike, po_6):
    r_6 = 1e-6
    return po_6 * (r_6 / r)**alpha_spike

# Differential equations (gravitational waves and dynamical friction)
def dEdt_gw(e, a):
    return -32. * ((mu_parsec**2 * m_parsec**3) * (1. + 73./24.*(e**2) +
37./96.*(e**4))) / (5. * (a**5) * ((1. - e**2)**(7./2.)))

def dLdt_gw(e, a):
    return -32. * ((mu_parsec**2 * m_parsec**(5./2.)) * (1. + 7./8. * e**2))
/ (5. * a**(7./2.) * (1 - e**2)**2)

def F(r, v, po_6, alpha_spike):
    ln_lambda = 0.5*np.log(mass_big_parsec / mass_small_parsec)
    v_squared = max(v[0]**2 + v[1]**2, 1e-10)  # Avoid division by zero
    return (4.*np.pi * mass_small_parsec**2 * po_dm(r, alpha_spike, po_6) *
ln_lambda) / v_squared

# Main system of equations
def dEdt(e, a, po_6, alpha_spike):
    return dEdt_gw(e, a) + dEdt_df(e, a, po_6, alpha_spike)

def dLdt(e, a, po_6, alpha_spike):
    return dLdt_gw(e, a) + dLdt_df(e, a, po_6, alpha_spike)
```

```
def Lorb(e, m_parsec, mu_parsec, Eorb):
    return np.sqrt(((e**2 - 1.) * (m_parsec**2 * mu_parsec**3)) / (2. *
Eorb))

def dedt(e, a, po_6, alpha_spike):
    Eorb = -(m_parsec * mu_parsec) / (2. * a)
    Lorb_val = Lorb(e, m_parsec, mu_parsec, Eorb)
    return -((1. - e**2) / (2. * e)) * (dEdt(e, a, po_6, alpha_spike) / Eorb
+ 2. * dLdt(e, a, po_6, alpha_spike) / Lorb_val)

def dadt(e, a, po_6, alpha_spike):
    dEda = (mass_big_parsec * mass_small_parsec) / (2. * a**2)
    return ((dEdt(e, a, po_6, alpha_spike)) / dEda)

def system_of_equations(t, y, po_6, alpha_spike):
    e, a = y
    dedt_val = dedt(e, a, po_6, alpha_spike)
    dadt_val = dadt(e, a, po_6, alpha_spike)
    return [dedt_val, dadt_val]

def solve_system(e_initial, a_initial, po_6, alpha_spike):
    Tmax = (5. / 256. * a_initial**4) / (mass_big_parsec * mass_small_parsec
* (mass_small_parsec + mass_big_parsec))
    t_span = (0, Tmax)
    sol = integrate.solve_ivp(system_of_equations, t_span, [e_initial,
a_initial], args=(po_6, alpha_spike), method='RK45', rtol=1e-8, atol=1e-8)
    e_vals, a_vals = sol.y
    return e_vals, a_vals, sol.t

# Insert from code
def get_orbital_elements(e, a, phi):
    r = a*(1. - e**2)/(1. + e*np.cos(phi))
    v = np.sqrt(m_parsec *(2./r - 1./a))
    v_phi = r * np.sqrt(m_parsec * a * (1.-e**2)) / r**2
    v_r = np.sqrt(np.max([v**2 - v_phi**2, 0.]))
    return r, v, v_r, v_phi

def omega_s(r):
    return np.sqrt((mass_big_parsec + mass_small_parsec)/r**3)

def dEdt_df(e, a, po_6, alpha_spike):
    def integrand(phi):
        r, v, v_r, v_phi = get_orbital_elements(e, a, phi)
        return F(r, (v_r, v_phi), po_6, alpha_spike) * v / (1. + e *
np.cos(phi))**2

    result, error = integrate.quad(integrand, 0., 2. * np.pi, limit=1000,
epsabs=1e-8, epsrel=1e-8)
    return -(1. - e**2)**(3. / 2.) / (2. * np.pi) * result

def dLdt_df(e, a, po_6, alpha_spike):
    def integrand(phi):
        r, v, v_r, v_phi = get_orbital_elements(e, a, phi)
        return F(r, (v_r, v_phi), po_6, alpha_spike) / v / (1. + e *
np.cos(phi))**2

    result, error = integrate.quad(integrand, 0., 2. * np.pi, limit=1000,
epsabs=1e-8, epsrel=1e-8)
    return -(1. - e**2)**(3. / 2.) / (2. * np.pi) * np.sqrt(m_parsec * a *
(1. - e**2)) * result

# Plots for three values of alpha_spike and po_6
plt.figure(figsize=(10, 6))
```

```
for alpha_spike in alpha_spike_values:
   po_6 = 5.448e15*Msun_parsec  # Change the value of po_6 here
   e_vals, a_vals, t_vals = solve_system(0.1, 100 * r_isco, po_6,
alpha_spike)

   plt.plot(t_vals, np.log10(a_vals / mass_big_parsec), label=f'alpha =
{alpha_spike:.2f}')

plt.xlabel('Time')
plt.ylabel('log a/M')
plt.xscale('log')  # Set logarithmic scale for time
plt.title(r'$e_0 = 0.10, a_0 = 100.0$')
plt.legend()
plt.grid(True)

plt.show()

po_6 = 5.448e15*Msun_parsec  # Adjust as needed

plt.figure(figsize=(10, 6))

for alpha_spike in alpha_spike_values:  # Loop over different alpha_spike
values
    e_vals, a_vals, t_vals = solve_system(0.5, 100 * r_isco, po_6,
alpha_spike)
    plt.plot(a_vals / r_isco, e_vals, label=f'α = {alpha_spike:.2f}')

plt.ylabel('Eccentricity')
plt.xlabel(r'$a / r_{\rm isco}$')
plt.title(r'$e_0 = 0.50, a_0 = 100.0$')
plt.legend()
plt.grid(True)

plt.show()
```


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