SIMPLICIAL COMPLEXES ASSOCIATED WITH POINT CLOUDS II

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Voronoi diagrams

Let $S \subseteq \mathbb{R}^d$ be a finite set.

Definition (Voronoi diagram). The Voronoi cell of a point $u \in S$ is given by

$$V_u = \{x \in \mathbb{R}^d : \|x - u\| \leq \|x - v\|, v \in S\}.$$

We call the collection of Voronoi cells of points $u \in S$ the Voronoi diagram.

Proposition. Each V_u is a convex polyhedron.

We can define the weighted Voronoi diagram using the power distance:

Definition (Power distance). For a weighted point (u, w_u), $u \in S$, we define

$$\pi_u(x) = \|x - u\|^2 - w_u,$$

and set $V_u = \{x \in \mathbb{R}^d : \pi_u(x) \leq \pi_v(x), v \in S\}$.

We can lift the cells to one higher dimension:

- Draw $S \subset \mathbb{R}^d$ in $\mathbb{R}^{d+1}$ by adding zeros as the $(d+1)$-th coordinates;
- Map each $u \in S$ to $\mathbb{S}^d$ with inverse of the stereographic projection;
- Denote by Π_u the d-plane tangent to $\mathbb{S}^d$ at the point $\zeta^{-1}(u)$;
- Using inversion, map each d-plane Π_u to the d-sphere $\Sigma_u := \iota(\Pi_u)$.

Proposition (First plane lemma). $x \in \mathbb{R}^d$ belongs to $V_u \Leftrightarrow$ the first intersection of the directed line segment from N to x with the d-planes is with Π_u.

Proposition (First sphere lemma). $x \in \mathbb{R}^d$ belongs to $V_u \Leftrightarrow$ the first intersection of the directed line segment from x to N with the d-spheres is with Σ_u.

Delaunay triangulations

Definition (Delaunay complex) The Delaunay complex of S is isomorphic to the nerve of the Voronoi diagram:

$$\text{Delaunay} = \{\sigma \subset S : \bigcap_{u \in \sigma} V_u \neq \emptyset\}.$$

Assumption. S is in general position, i.e. no $d+2$ points lie on a common $(d-1)$ sphere.

Proposition. The above assumption implies that

- no $d+2$ Voronoi cells have a non-empty common intersection;
- the dimension of any simplex is at most d;
- there exists a geometric realization (called the Delaunay triangulation).

We can generalize the Delaunay complex to the weighted case by considering the weighted Voronoi diagram.

Alpha complexes and Filtrations

For $r \in \mathbb{R} > 0$, $u \in S$ we set $B_u(r) = u + r\mathbb{B}^d$.

Definition (Alpha complex). Set $R_u(r) := B_u(r) \cap V_u$. The alpha complex is defined as

$$\text{Alpha}(r) = \{\sigma \subset S : \bigcap_{u \in \sigma} R_u(r) \neq \emptyset\}.$$

If S is in general position, there exists a geometric realization.

Properties.

- $\text{Alpha}(r) \subseteq \text{Delaunay}$;
- $\text{Alpha}(r) \subseteq \text{Cech}(r)$;
- $\bigcup_{u \in S} B_u(r) \simeq |\text{Alpha}(r)|$.

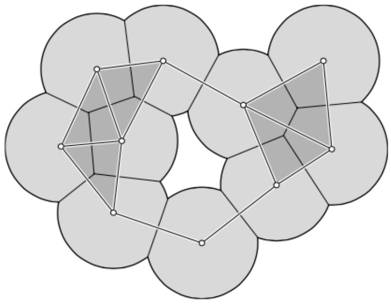

Figure 1: An alpha complex.

Definition (weighted Alpha complex). Let B_u be the ball with center u and radius w_u. Setting $R_u = B_u \cap V_u$, where V_u are the weighted Voronoi cells, we define the weighted alpha complex as the set of simplices $\sigma \subset S$ such that $\bigcap_{u \in \sigma} R_u \neq \emptyset$.

As before, assuming the weighted points are in general position, we obtain a geometric realization.

We can obtain a stepwise assembly by continuously increasing the radius r. Denoting by K_i the i-th complex in the assembly, we get a sequence of nested unions:

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = \text{Delaunay}.$$

Every set in between is a simplicial complex and only finitely many of them are distinct.

Collapses

Definition (elementary collapse). We call the removal of a pair $\tau_0 < \tau$, where τ is the only remaining proper coface of τ_0, an elementary collapse. With $k = \dim \tau_0$, we alternatively call it a $(k, k+1)$-collapse.

Elementary collapses preserve the homotopy type as there exits a deformation retraction between $|K|$ and $|L|$, where K is the original simplicial complex and L is the result of the collapse.

Definition (collapsible). We call a simplicial complex collapsible if there is a sequence of elementary collapses that reduces K to a single vertex.

Proposition. K can only be collapsible, if $|K|$ is contractible. However, $|K|$ being contractible does not imply that K is collapsible.

We can extend the notion of collapse to pairs of simplices $\tau < \upsilon$ whose dimension differ by more than one. Therefore, we require that all cofaces of τ are faces of υ. With $k = \dim \tau$ and $\ell = \dim \upsilon$, a (k, ℓ)-collapse consists of a sequence of $2^{\ell-k-1}$ elementary collapses.

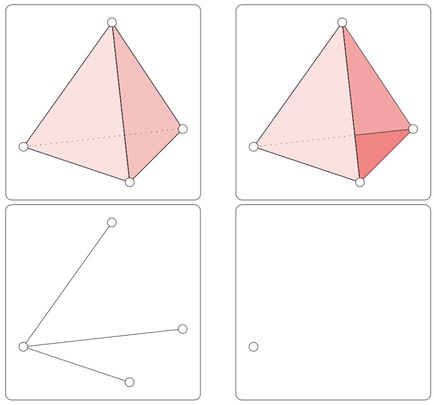

Figure 2: A $(2, 3)$-collapse, three $(1, 2)$-collapses and three $(0, 1)$-collapses.

Critical and regular events

Let r_i be the smallest radius such that $K_i = \text{Alpha}(r_i)$. A simplex τ belongs to K_{i+1} but not to K_i if the balls with radius r_{i+1} have a non-empty common intersection with the corresponding intersection of Voronoi cells, but the balls with radius r_i do not.

Definition (regular event). The addition of τ is called a regular event, if K_i can be obtained from K_{i+1} with a (k, ℓ)-collapse, where k is the dimension of the simplex with the lowest and ℓ the dimension of the simplex with the highest dimension in $K_{i+1} - K_i$.

Definition (critical event). The addition of τ is called a critical event, if τ is the only simplex in $K_{i+1} - K_i$, i.e. the addition of τ changes the homotopy type of the complex.

References

[1] H. Edelsbrunner and J.L. Harer. *Computational Topology: An Introduction*. Miscellaneous Bks. American Mathematical Society, 2022.