



Unfitted Mixed Finite Element Methods

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Problems

PDEs on embedded surfaces, moving domains, separate geometry description

Challenges

Stability / Robustness w.r.t. arbitrarily small cuts

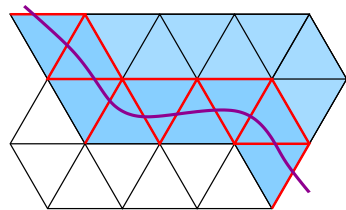
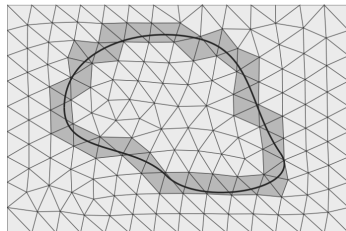


Ghost penalty stabilization¹

$$\mathfrak{G}(u_h, v_h) := \sum_{F \in \mathcal{F}_h^{\partial\Omega}} \sum_{l=0}^k \frac{h_F^{2l+1}}{l!^2} \int_F [[\partial_n^l u_h]] [[\partial_n^l v_h]] ds$$

Requirement: $\|u\|_{H^q(\Omega)} + |u|_{\mathfrak{G}} \simeq \|u\|_{H^q(\Omega^T)}$

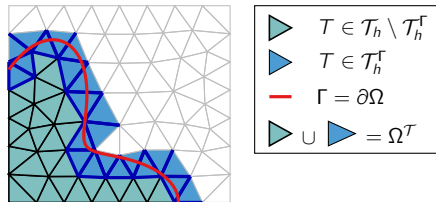
¹E. Burman. Ghost penalty. C.R. Math., 348(21-22):1217–1220, November 2010.



Model problem (Dirichlet case)

Find u, p with $p = p_D$ on $\partial\Omega$ s.t.

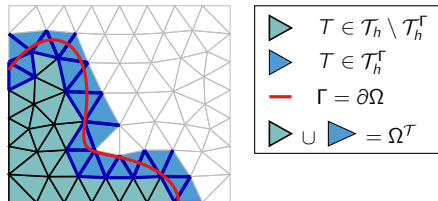
$$\begin{aligned}u - \nabla p &= 0 \quad \text{in } \Omega, \\ \operatorname{div} u &= -f \quad \text{in } \Omega.\end{aligned}$$



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Weak formulation - fitted mixed Poisson

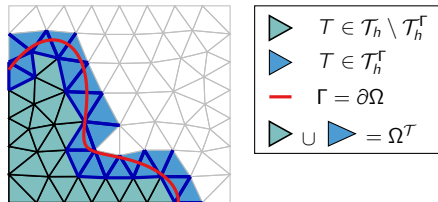
Find $u_h \in \Sigma_h = \mathbb{RT}^k \subset H(\operatorname{div}, \Omega)$, $p_h \in Q_h = \mathbb{P}^k \subset L^2(\Omega)$ s.t.

$$\begin{aligned}(u_h, v_h)_\Omega + (\operatorname{div} v_h, p_h)_\Omega &= (v_h, p_D)_{\partial\Omega} & \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_\Omega &= (-f, q_h)_\Omega & \forall q_h \in Q_h.\end{aligned}$$

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Weak formulation - naive unfitted mixed Poisson

Find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h) \subset H(\operatorname{div}, \Omega^T)$, $p_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h) \subset L^2(\Omega^T)$ s.t.

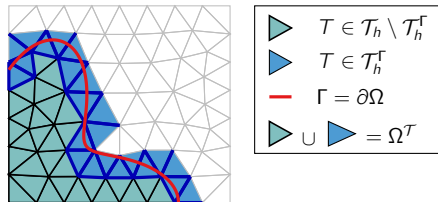
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$$\begin{aligned} (u_h, v_h)_\Omega + (\operatorname{div} v_h, p_h)_\Omega + \text{ghost} &= (v_h, p_D)_{\partial\Omega} && \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_\Omega + \text{ghost} &= (-f, q_h)_\Omega && \forall q_h \in Q_h. \end{aligned}$$

→ Not robust! inf-sup stability depends on cut position

→ Ghost penalty pollutes mass balance!

Poisson / Darcy Problem

Mass balance polluted, stability:

R. Puppi, *A cut finite element method for the Darcy problem*. arXiv: 2111.09922, 2021.

P. Cao, J. Chen, *An extended finite element method for coupled Darcy-Stokes problems*. IJNME, 2022

Mass balance hardly polluted, stability, divergence preserving:

T. Frachon, P. Hansbo, E. Nilsson, S. Zahedi, *A divergence preserving cut finite element method for Darcy flow*. arXiv: 2205.12023, 2022.

Stokes Problem

T. Frachon, E. Nilsson, Sara Zahedi, *Divergence-free cut finite element methods for Stokes flow*. arXiv:2304.14230, 2023.

Similar Method:

E. Burman, P. Hansbo, M. G. Larson, *Cut finite element method for divergence free approximation of incompressible flow: optimal error estimates and pressure independence*. arXiv: 2207.04734, 2022.

Observation

$$\{u_h \in \Sigma_h \mid (\operatorname{div} u_h, q_h)_{\Omega} = 0 \ \forall q_h \in Q_h\} = \{u_h \in \Sigma_h \mid (\operatorname{div} u_h, q_h)_{\Omega^T} = 0 \ \forall q_h \in Q_h\}$$

→ Kernel-coercivity of $(u_h, v_h)_{\Omega}$ -block unaffected

Modification

For $\gamma_{\text{cut}} \geq 0$, find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h)$, $\bar{p}_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h)$ s.t.

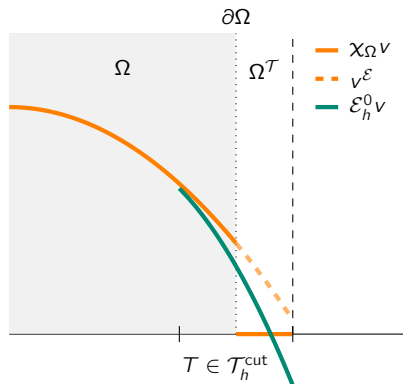
$$\begin{aligned} (u_h, v_h)_{\Omega} + \gamma_{\text{cut}} (u_h, v_h) + (\operatorname{div} v_h, \bar{p}_h)_{\Omega^T} &= (v_h, p_D)_{\partial\Omega} & \forall v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_{\Omega^T} &= (-f_h, q_h)_{\Omega^T} & \forall q_h \in Q_h. \end{aligned}$$

- Inf-sup stable independent of cut position, consistent for u (up to $f_h \approx f$), consistent for p on interior elements
- inconsistent for p on cut elements
- Inf-sup stable independent of cut position, consistent for u (up to $f_h \approx f$), consistent for p on interior elements, but not on cut elements

Define $\mathcal{E}_h^0 : L^2(\Omega) \rightarrow Q_h$, $v \mapsto \Pi^Q(\chi_\Omega v)$ s.t.

$$(\mathcal{E}_h^0 q, r_h)_{\Omega^\mathcal{T}} = (q, r_h)_\Omega \quad \forall r_h \in Q_h.$$

- $(\operatorname{div} v_h, q)_\Omega = (\operatorname{div} v_h, \mathcal{E}_h^0 q)_{\Omega^\mathcal{T}}$
- $\bar{p}_h \approx \mathcal{E}_h^0 p$
- $\bar{p}_h \approx p$ on $T \in \mathcal{T}_h^{\text{interior}}$
- $\bar{p}_h \not\approx p$ on $T \in \mathcal{T}_h^{\text{cut}}$



Norm on $H(\operatorname{div}, \Omega^T)$:

$$\|u\|_{\Sigma}^2 := \|\operatorname{div} u\|_{\Omega^T}^2 + \|u\|_{\Omega_\gamma}^2, \quad \|u\|_{\Omega_\gamma}^2 := \begin{cases} \|u\|_{\Omega^T}^2 & \text{if } \gamma_{\text{out}} > 0, \\ \|u\|_{\Omega}^2 & \text{if } \gamma_{\text{out}} = 0. \end{cases}$$

Theorem (Error estimate for u_h)

For $u \in H^m(\Omega)$ with $m \in \{0, \dots, k+1\}$, there holds

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^T)} + \|u^\mathcal{E} - u_h\|_{\Sigma} + |u_h|_{\text{out}} \lesssim h^m \|u\|_{H^m(\Omega)} + \text{err}_f.$$

Theorem

For Ω smooth enough to assume L^2 - H^2 regularity, there holds

$$\|\bar{p}_h - \mathcal{E}_h^0 p\|_{L^2(\Omega^T)} \lesssim h \|u_h - u\|_{L^2(\Omega)} + h \text{err}_f$$

Motivation

Apply post-processing to

- repair inconsistency on cut elements
- obtain higher order convergence

Idea

Make use of the relation $\nabla p = u$ and exploit the accuracy of u_h .

Two versions

- element-wise post-processing
- patchwise post-processing

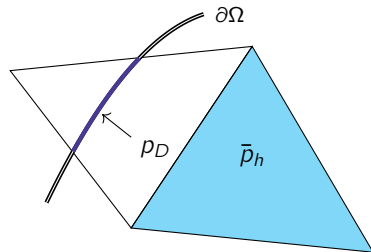
Element-local Scheme

For each $T \in \mathcal{T}_h$, find $p_h^* \in \mathbb{P}^{k+1}(T)$ s.t.

$$(\nabla p_h^*, \nabla q_h^*)_T = (u_h, \nabla q_h^*)_T \quad \forall q_h^* \in \mathbb{P}^{k+1}(T) \setminus \mathbb{R},$$

$$(p_h^*, 1)_T = (\bar{p}_h, 1)_T \quad \text{if } T \in \mathcal{T}_h^{\text{interior}},$$

$$(p_h^*, 1)_{T \cap \partial\Omega} = (p_D, 1)_{T \cap \partial\Omega} \quad \text{if } T \in \mathcal{T}_h^{\text{cut}}.$$



Error estimate

If $\gamma_{\text{min}} > 0$ and $p^\mathcal{E} \in H^{k+2}(\Omega^\mathcal{T})$, then

$$\|p^\mathcal{E} - p_h^*\|_{L^2(\Omega^\mathcal{T})} \lesssim h^{k+2} \|p^\mathcal{E}\|_{H^{k+2}(\Omega^\mathcal{T})}.$$

Patchwise Scheme

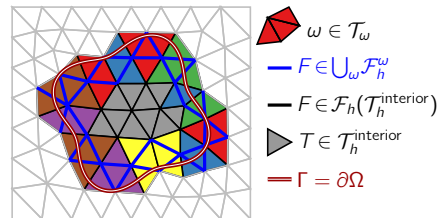
For each $\omega \in \mathcal{T}_\omega$, find $p_h^* \in \mathbb{P}^{k+1}(\omega)$ s.t. for all $q_h^* \in \mathbb{P}^{k+1}(\omega) \setminus \mathbb{R}$:

$$\begin{aligned} (\nabla p_h^*, \nabla q_h^*)_{\Omega \cap \omega} + \text{👤}_\omega(p_h^*, q_h^*) &= (u_h, \nabla q_h^*)_{\Omega \cap \omega}, \\ (p_h^*, 1)_{\Omega^{\text{interior}} \cap \omega} &= (\bar{p}_h, 1)_{\Omega^{\text{interior}} \cap \omega}. \end{aligned}$$

Error estimate

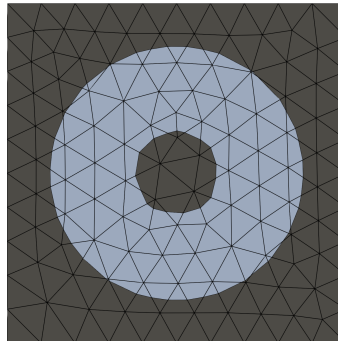
For $p \in H^{k+2}(\Omega)$, it holds

$$\|p - p_h^*\|_{L^2(\Omega)} \lesssim h^{k+2} \|p\|_{H^{k+2}(\Omega)}$$

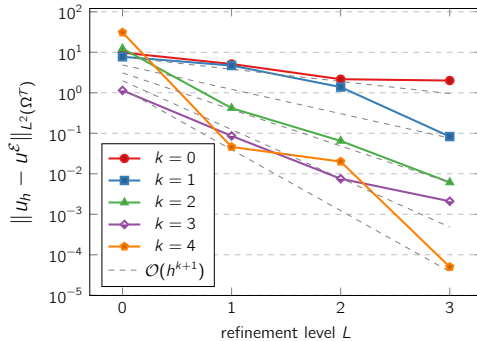
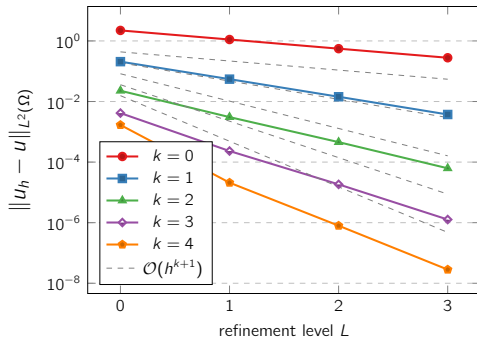


- $\gamma_{\text{👤}} = 0$ allowed (hybridization possible)
- no dependence on Dirichlet boundary data

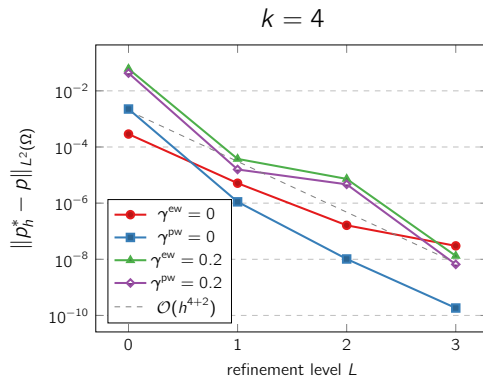
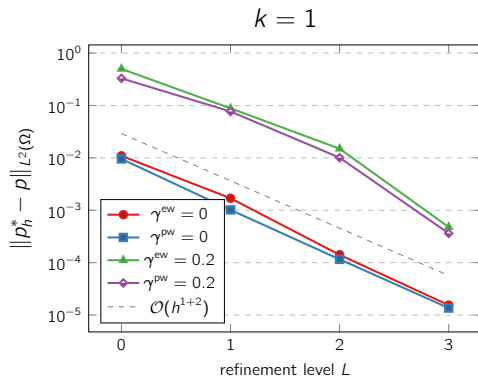
- manufactured solution
- $\mathbb{RT}^k \times \mathbb{P}^k$
- uniform refinements



Convergence of u_h



Elementwise vs. Patchwise Postprocessing



Unfitted mixed FEM

- Inf-sup stability without polluting the mass balance
- Recover higher order convergence of p_h with post processing
 - Element-wise post processing: requires Dirichlet boundary data, $\gamma_{\text{robot}} > 0$ necessary
 - Patchwise post processing: no dependence on boundary data, $\gamma_{\text{robot}} = 0$ allowed
- Hybridization possible
 - Caution: conditioning

Extensions

- Neumann boundary conditions
- Hybridization
- Stokes