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The (discontinuous) Petrov-Galerkin method

Seminar on advanced Finite Element Methods

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Contents

Introduction

General Idea

Galerkin method:
Find $u \in X$ such that

$$b(u, v) = \ell(v) \qquad \forall v \in X.$$

General Idea

Petrov-Galerkin method:
Find $u \in X$ such that

$$b(u, v) = \ell(v) \qquad \forall v \in Y.$$

- Inf-sup stability:

$$\exists c > 0 : c \left\| u_h \right\|_X \leq \sup_{v_h \in Y_h} \frac{b_h(u_h, v_h)}{\left\| v_h \right\|_Y} \qquad \forall u_h \in X_h$$

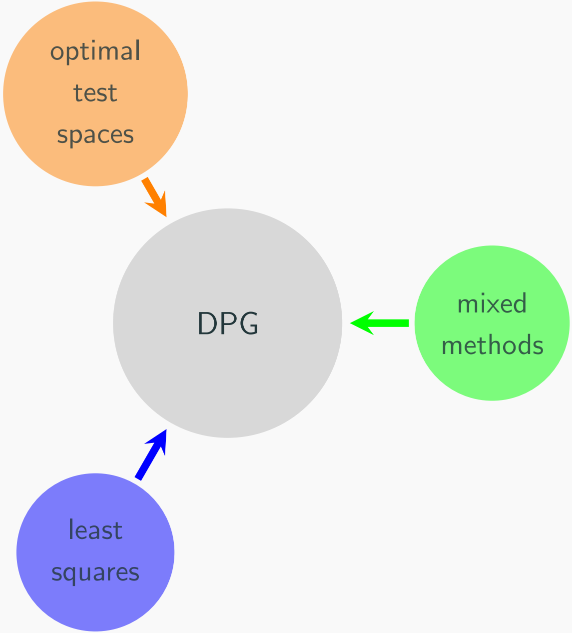
- Error estimate:

$$\left\| u - u_h \right\|_X \lesssim \inf_{v_h \in X_h} \left\| u - v_h \right\|_X$$

- In general: exact inf-sup condition $\not\Rightarrow$ discrete inf-sup condition

Three angles of DPG

Three angles of DPG



Notation

- Dual space: consists of all linear functionals on a vector space; denoted by V^*:

$$\|f\|_{V^*} = \sup_{v \in V, v \neq 0} \frac{|f(v)|}{\|v\|_V}.$$

- Riesz map: $R_V : V \to V^*$ s.t. :

$$(R_V v)(w) = (v, w)_V \quad \forall v, w \in V.$$

- Operator generated by a bilinear form:

$$B : X \to Y^* \text{ s.t. } Bx(y) = b(x, y) \quad \forall x \in X, y \in Y.$$

Ideal DPG method

Definition (iDPG method)

Given $X_h \subset X$: Find $x_h \in X_h$ such that

$$b(x_h, y_h) = \ell(y_h) \qquad \forall y_h \in Y_h^{\mathrm{opt}} := T(X_h),$$

where $T : X \to Y$ is defined through

$$(Tw, y)_Y = b(w, y) \qquad \forall w \in X, y \in Y.$$

T is a supermizer of the inf-sup condition

- we can write $T = R_Y^{-1} \circ B$
- $\| Tx \|_Y = \sup_{y \in Y} \frac{b(x,y)}{\|y\|_Y}$

$$(\leq): \qquad \| Tx \|_Y = \frac{b(x, Tx)}{\| Tx \|_Y} \leq \sup_{y \in Y} \frac{b(x,y)}{\|y\|_Y}$$

$$(\geq): \qquad sup_{y \in Y} \frac{b(x,y)}{\|y\|_Y} = sup_{y \in Y} \frac{(Tx, y)_Y}{\|y\|_Y} \leq \| Tx \|_X$$

- exact inf-sup condition $\Rightarrow$ discrete inf-sup condition
- inf-sup constant is equal to 1

iDPG for Poisson equation

Example

Find $u \in H_0^1(\Omega)$ s.t.

$$\int_\Omega \nabla u \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in H_0^1(\Omega).$$

$$X = Y = H_0^1(\Omega); \quad (v, y)_Y := \int_\Omega \nabla v \nabla y \,\mathrm{d}x;$$

$$(\cdot,\cdot)_Y = b(\cdot,\cdot) \Rightarrow T = \mathrm{id};$$

$$Y_h^{\mathrm{opt}} = X_h.$$

$\Rightarrow$ standard FEM is iDPG method

iDPG is not practically feasible

Q: Generally, can we compute Tx?

➔ Not necessarily! $T = R_Y^{-1} \circ B$

Definition (pratical DPG method)

Given $X_h \subset X$ find $x_h^r \in X_h$ using $Y^r \subset Y$, Y^r finite dimensional s.t.

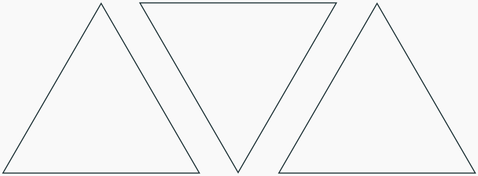
$$b(x_h^r, y) = \ell(y) \quad \forall y \in Y_h^{\text{opt}} = T^r(X_h),$$

where $T^r : X \to Y^r$ is defined through

$$(T^r w, y)_Y = b(w, y) \quad \forall w \in X, y \in Y^r.$$

pDPG method

- T is approximated by T^r (finite rank)
- computation can be localized ➔ DG spaces



➔ $Y = \prod_{T \in \mathcal{T}_h} Y(T)$.

L^2-least squares is iDPG

Example

Given $f \in L^2(\Omega)$ and $A : X \to L^2(\Omega)$ a linear, continuous and bijective operator. Find $u \in X$ s.t.

$$Au = f$$

$$Y = L^2(\Omega), \quad b(x, y) = (Ax, y)_Y, \quad \ell(y) = (f, y)_Y;$$

$$\Rightarrow (Tw, y)_Y = (Aw, y)_Y \Rightarrow T = A;$$

$$\Rightarrow Y_h^{\text{opt}} = AX_h.$$

Hence, the iDPG equations become:

$$(Ax_h, Aw_h)_Y = (f, Aw_h)_Y \quad \forall w_h \in X_h$$

Each DPG method is a least squares method

Theorem

The following statements are equivalent:

1. $x_h \in X_h$ *is the unique solution to the iDPG method*
2. x_h *is the best approximation to* x *in* X_h *in the energy norm* $|||z|||_X := \|Tz\|_Y$*:*

$$|||x - x_h|||_X = \inf_{z_h \in X_h} |||x - z_h|||_X$$

3. x_h *minimizes a residual in the following sense:*

$$x_h = \mathit{argmin}_{z_h \in X_h} \|\ell - Bz_h\|_{Y^*}$$

Each DPG method is a least squares method

Proof.

(i) $\Leftrightarrow$ (ii):

$$\begin{aligned} b(x - x_h, y_h) = 0 \forall y_h \in Y_h^{\text{opt}} &\Leftrightarrow b(x - x_h, T z_h) = 0 \forall z_h \in X_h \\ &\Leftrightarrow (T(x - x_h), T z_h)_Y = 0 \forall z_h \in X_h \end{aligned}$$

(ii) $\Leftrightarrow$ (iii):

$$\begin{aligned} |||x - z_h|||_X = \| T(x - z_h) \|_Y &= \left\| R_Y^{-1} B(x - z_h) \right\|_Y \\ &= \| B(x - z_h) \|_{Y^*} = \| \ell - B z_h \|_{Y^*} \end{aligned}$$

$\square$

DPG methods allow for easy preconditioning

- Arising matrices are symmetric positive definite: $\operatorname{argmin}_{z_h \in X_h} \|\ell - B z_h\|_{Y^*} = \operatorname{argmin}_{z_h \in X_h} \frac{1}{2} \|\ell - B z_h\|_{Y^*}^2$

➔ Derivative in direction $r_h \in X_h$:

$$\underbrace{(B z_h, B r_h)_{Y^*}}_{\text{coercive bilinear form}} \quad -(B r_h, \ell)_{Y^*} = 0$$

- Alternatively: $b(e_j, t_i) = (Te_j, Te_i)_Y = (Te_i, Te_j)_Y = b(e_i, t_j)$
- can use conjugate gradients!

iDPG vs. pDPG in the least-squares sense

- **iDPG:** Find $x_h \in X_h$ such that

$$x_h = \text{argmin}_{z_h \in X_h} \underbrace{\|\ell - Bz_h\|_{Y^*}}_{\text{hard to compute!}}$$

- **pDPG:** Find $x_h^r \in X_h$ using $Y^r \subset Y$ such that

$$x_h = \text{argmin}_{z_h \in X_h} \underbrace{\|\ell - Bz_h\|_{(Y^r)^*}}_{\text{finite dimensional} \rightarrow \text{computable!}}$$

"Built-in" a-posteriori error estimation

- Compute the error without knowing the exact solution:

$$\|\!|x - x_h|\!\|_X = \|T(x - x_h)\|_Y = \left\|R_Y^{-1}B(x - x_h)\right\|_Y$$

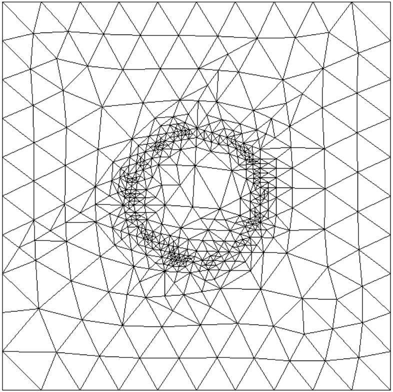
➔ error representation function $\epsilon^r := R_{Y^r}^{-1}(\ell - Bx_h)$

- can be practically computed by

$$(\epsilon^r, y)_Y = \ell(y) - b(x_h, y) \quad \forall y \in Y^r.$$

DPG is predestined for adpative mesh refinements

Error estimator: $\eta = \|\epsilon^r\|_Y$



DPG methods can be viewed as mixed Galerkin methods

Include the error representation function and reinterpret the problem as a saddle-point problem:

Theorem

The following statements are equivalent:

1. $x_h \in X_h$ *solves the pDPG method,*
2. $x_h \in X_h$ *and* $\epsilon^r \in Y^r$ *solve the mixed formulation*

$$\begin{aligned} (\epsilon^r, y)_Y + b(x_h, y) &= \ell(y) \quad \forall y \in Y^r, \\ b(z_h, \epsilon^r) &= 0 \quad \forall z_h \in X_h. \end{aligned}$$

DPG methods can be viewed as mixed Galerkin methods

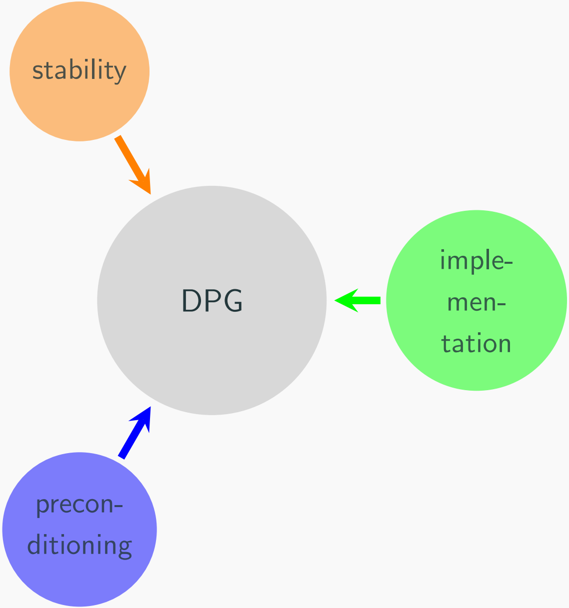
Proof.

(i) $\Rightarrow$ (ii): first equation is just the definition of ϵ^r. Further

$$\begin{aligned} b(z_h, \epsilon^r) = (T^r z_h, \epsilon^r)_Y &= (T^r z_h, R_{Y^r}^{-1}(\ell - B x_h))_Y \\ &= (T^r z_h, T^r(x - x_h))_Y \\ &= b(x - x_h, T^r z_h) \\ &= 0 \end{aligned}$$

(ii) $\Rightarrow$ (i): Analogous. $\square$

In conclusion the three angles yield



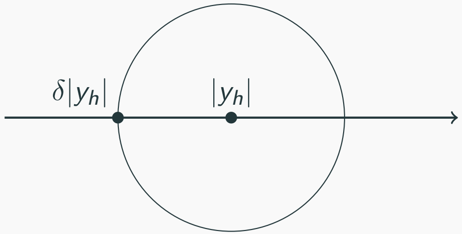
Optimal test spaces

δ-proximal spaces

Definition

For $\delta \in (0,1)$, a subspace $Y_h^\delta \subset Y$ with $\dim Y_h^\delta = N(h) = \dim X_h$ is called δ-proximal if

$$\forall 0 \neq y_h \in Y_h^{\text{opt}} \quad \exists \tilde{y}_h \in Y_h^\delta \text{ s.t. } \|y_h - \tilde{y}_h\|_Y \leq \delta \|y_h\|_Y$$



δ-proximal spaces are inf-sup stable

Lemma

For any $X_h \subset X$ and any δ-proximal $Y_h^\delta \subset Y$ the bilinear form $b(\cdot,\cdot)$ satisfies

$$\inf_{z_h \in X_h} \sup_{y_h \in Y_h^\delta} \frac{b(z_h, y_h)}{\|z_h\|_X \, \|y_h\|_Y} \geq \frac{1-\delta}{1+\delta},$$

i.e. for any $\delta < 1$ the pair X_h, Y_h^δ satisfies a discrete inf-sup condition uniformly in $h > 0$.

Theorem

If Y_h^δ is δ-proximal for X_h, the solution $x_{h,\delta} \in X_h$ to

$$b(x_{h,\delta}, y_h) = \ell(y_h) \qquad \forall y_h \in Y_h^\delta$$

satisfies

$$\|x - x_{h,\delta}\|_X \leq \frac{2}{1-\delta} \inf_{z_h \in X_h} \|x - z_h\|_X$$

Example: Y-orthogonal projection

- $X = X^* = L^2(\Omega) \Rightarrow T = B$
- Let $Z_h \subset Y$ be finite dimensional auxillary space, $\dim Z_h \geq \dim X_h$
- Define the Y-orthogonal projection $P_h : Y \to Z_h$ through

$$(P_h y, z_h)_Y = (y, z_h)_Y \quad \forall z_h \in Z_h.$$

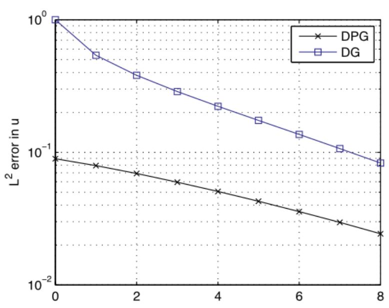
- Natural candidate for δ-proximal subspace for X_h:

$$\tilde{Y}_h := P_h(Y_h^{\mathrm{opt}}) = P_h(BX_h) \subset Z_h \subset Y.$$

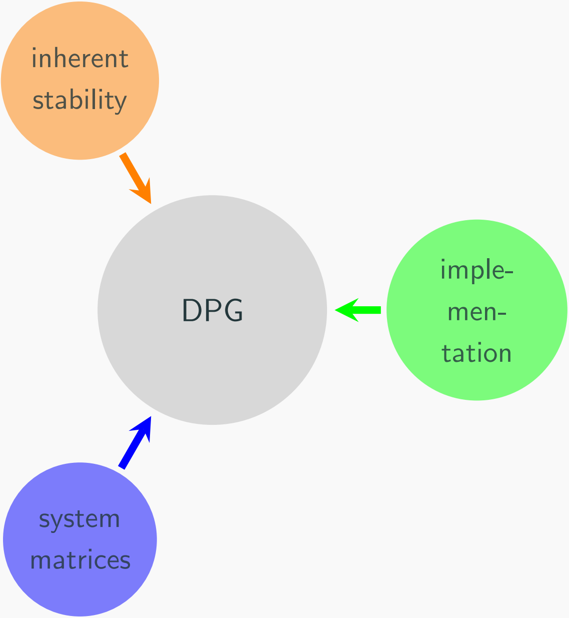
Outlook: transport equation

DPG methods are practically applied for transport equations:

$$\beta \cdot \nabla u + \alpha(x)u = f \text{ in } \Omega,$$
$$u = g \text{ on } \partial\Omega_{\text{in}}.$$



Conclusion



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