GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

On a Discontinuous Galerkin discretization for a degenerate diffusion equation

Bachelor's thesis

Tim van Beeck

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Institut für Numerische und Angewandte Mathematik

Contents

Motivation

$$
\begin{aligned}
\rho(-i\omega + u\cdot\nabla + \Omega\times)^2\xi - \nabla(\rho c_s^2\nabla\cdot\xi) + (\nabla\cdot\xi)\nabla p - \nabla(\nabla p\cdot\xi) \\
+ (\mathrm{Hess}(p)\xi - \rho\mathrm{Hess}(\phi))\xi - i\gamma\rho\omega\xi + \rho\nabla\varphi = s \text{ in } D, \\
-\frac{1}{4\pi G}\Delta\varphi + \nabla\cdot(\rho\xi) = 0 \text{ in } \mathbb{R}^3.
\end{aligned}
$$

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\rho(-i\omega + u\cdot\nabla + \Omega\times)^2\xi - \nabla(\rho c_s^2\nabla\cdot\xi) + (\nabla\cdot\xi)\nabla p - \nabla(\nabla p\cdot\xi)& \\
+ (\mathrm{Hess}(p)\xi - \rho\mathrm{Hess}(\phi))\xi - i\gamma\rho\omega\xi + \rho\nabla\varphi = s \text{ in } D&, \\
-\frac{1}{4\pi G}\Delta\varphi + \nabla\cdot(\rho\xi) = 0 \text{ in } \mathbb{R}^3&.
\end{aligned}
$$

$$\Downarrow$$

$$-\nabla\cdot(\rho(u\otimes u)\cdot\nabla w) = f \text{ in } D$$

How do we get there?

Well, lots of assumptions:

- Cowling approximation $\varphi = 0$

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- restrict to $\mathbb{R}^d$
- remove all order zero terms
- There still remains

$$a(\xi, \xi') = \langle \rho c^2 \nabla \cdot \xi, \nabla \cdot \xi \rangle - \langle \rho \partial_u \xi, \partial_u \xi' \rangle, \quad \xi, \xi' \in \mathsf{X}$$

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Well, lots of assumptions:

- Cowling approximation $\varphi = 0$
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- restrict to $\mathbb{R}^d$
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$$a(\xi, \xi') = \langle \rho c^2 \nabla \cdot \xi, \nabla \cdot \xi \rangle - \langle \rho \partial_u \xi, \partial_u \xi' \rangle, \quad \xi, \xi' \in X$$

- use space decomposition $\xi_h = v_h + w_h$

Model Problem

Model problem

For a smooth velocity field u and a density $\rho : D \to \mathbb{R}_{>0}$ that is sufficiently smooth, we consider:
Find $w : D \to \mathbb{R}$ such that

$$\begin{aligned} \rho w - \nabla \cdot (\rho(u \otimes u) \cdot \nabla w) &= f \text{ in } D, \\ w = 0 \text{ on } \underbrace{\partial D \cap \{x \in D \mid u \cdot n \neq 0\}}_{=:\partial D^{\mathrm{in}}}. \end{aligned}$$

Continuous problem

Define

$$W := \{ v \in L^2(D) \mid \rho \partial_u v \in L^2(D), v = 0 \text{ on } \partial D^{\text{in}} \},$$

$$|||w|||_W := \|\rho^{\frac{1}{2}} w\|_D + \|\rho^{\frac{1}{2}} \partial_u w\|_D.$$

Continuous problem

Define

$$W := \big\{ v \in L^2(D) \mid \rho \partial_u v \in L^2(D), v = 0 \text{ on } \partial D^{\text{in}} \big\},$$

$$|||w|||_W := \|\rho^{\frac{1}{2}} w\|_D + \|\rho^{\frac{1}{2}} \partial_u w\|_D.$$

Find $w : D \to \mathbb{R}$ such that

$$\mathcal{B}(w, w') = \langle f, w' \rangle_D \quad \forall w' \in W,$$

where

$$\mathcal{B}(w, w') := \langle \rho w, w' \rangle_D + \langle \rho \partial_u w, \partial_u w' \rangle_D.$$

Discrete problem I

Find $w_h \in W_h = V_h^{k,d}$ such that

$$\mathcal{B}_h(w_h, w_h') = \langle f, w_h' \rangle_D \qquad \forall w_h' \in W_h,$$

where

$$\mathcal{B}_h(w_h, w_h') = \sum_{T \in \mathcal{T}_h} \langle \rho w_h, w_h' \rangle_T + b_h(w_h, w_h').$$

Derivation I

- Multiply with test function $w_h' \in W_h$ and integrate over the domain

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- partial integration and using $a(b \cdot c) = (a \otimes b) \cdot c$:

$$\sum_{T \in \mathcal{T}_h} \int_T \rho \partial_u w_h \partial_u w_h' \, \mathrm{d}x + \int_{\partial T} (-\rho \partial_u w_h \cdot u) w_h' \cdot \nu \, \mathrm{d}s.$$

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- Rewrite as sum over all facets, write $u_\nu = u \cdot \nu$:

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} (-\rho \partial_u w_h \cdot u) w_h' \cdot \nu \ \mathrm{d}s = \sum_{F \in \mathcal{F}_h} \int_F u_\nu [\![-\rho \partial_u w_h w_h']\!] \ \mathrm{d}s.$$

Derivation II

- DG magic formula:

$$\sum_{F\in\mathcal{F}_h}\int_F u_\nu(\{\!\{-\rho\partial_u w_h\}\!\}[\![w_h']\!] + \{\!\{w_h'\}\!\}[\![-\rho\partial_u w_h]\!])\ \mathrm{d}s.$$

Derivation II

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- can drop the second term due to consistency

Derivation II

- DG magic formula:

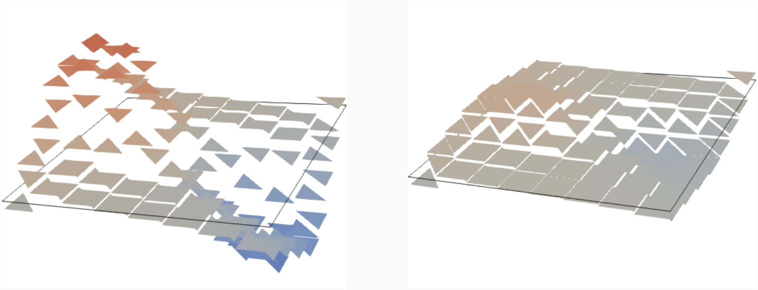
$$\sum_{F\in\mathcal{F}_h}\int_F u_\nu(\{\!\{-\rho\partial_u w_h\}\!\}[\![w_h']\!] + \{\!\{w_h'\}\!\}[\![-\rho\partial_u w_h]\!])\ \mathrm{d}s.$$

- can drop the second term due to consistency
- add term for symmetry

$$\sum_{F\in\mathcal{F}_h}\langle u_\nu\{\!\{-\rho\partial_u w_h'\}\!\},[\![w_h]\!]\rangle_F.$$

Derivation III

- Add stabilization term $\sum_{F\in\mathcal{F}_h}\langle\frac{\rho\lambda}{h}|u_\nu|^2[\![w_h]\!],[\![w_h']\!]\rangle_F$



(a) Discrete solution with $\lambda = 2$ **(b)** Discrete solution with $\lambda = 10$

Figure 1: Penalization effect

Discrete Problem II

Find $w_h \in W_h = V_h^{k,d}$ such that

$$\mathcal{B}_h(w_h, w_h') = \langle f, w_h' \rangle_D \qquad \forall w_h' \in W_h,$$

where

$$\begin{aligned}
\mathcal{B}_h(w_h, w_h') = \sum_{T \in \mathcal{T}_h} & \langle \rho w_h, w_h' \rangle_T + \langle \rho \partial_u w_h, \partial_u w_h' \rangle_T \\
+ \sum_{F \in \mathcal{F}_h} & \Big\{ \langle u_\nu \{\!\{ -\rho \partial_u w_h \}\!\}, [\![w_h']\!] \rangle_F \\
& + \langle u_\nu \{\!\{ -\rho \partial_u w_h' \}\!\}, [\![w_h]\!] \rangle_F \\
& + \langle \frac{\rho \lambda}{h} |u_\nu|^2 [\![w_h]\!], [\![w_h']\!] \rangle_F \Big\}.
\end{aligned}$$

Analysis

Approach

Coercivity

Approach

Coercivity

$\downarrow$

Continuity

Approach

Coercivity

$\downarrow$

Continuity

$\downarrow$

Well-posedness

Approach

Coercivity

$\downarrow$

Continuity

$\downarrow$

Well-posedness

$\downarrow$

Ceá-type error estimate

Discrete norms

For all $w_h \in W_h$ we define:

$$\|w_h\|_\rho^2 := \sum_{T \in \mathcal{T}_h} \left\{ \|\rho^{\frac{1}{2}} w_h\|_T^2 + \|\rho^{\frac{1}{2}} \partial_u w_h\|_T^2 + h^{-1} |u_\nu|^2 \|\rho^{\frac{1}{2}} [\![w_h]\!] \|_{\partial T}^2 \right\},$$

Discrete norms

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$$\|w_h\|_{\rho,*}^2 := \|w_h\|_\rho^2 + \sum_{T\in\mathcal{T}_h} h\|\rho^{\frac{1}{2}} \partial_u w_h\|_{\partial T}^2 .$$

Inverse Inequality

Lemma

For any $w_h \in \mathcal{P}^k(T)$ there holds

$$h\|\rho^{\frac{1}{2}}\partial_u w_h \cdot \nu\|_{\partial T}^2 \leq c_{tr}^2\|\rho^{\frac{1}{2}}\partial_u w_h\|_T^2,$$

where c_{tr} depends on the shape regularity, the polynomial degree k and on the density ρ.

Coercivity and Continuity

Theorem

For all $w_h \in W_h$ and $\lambda \geq 1 + 4c_{tr}^2$ there holds

$$\mathcal{B}_h(w_h, w_h) \geq \frac{1}{2}\|w_h\|_\rho^2 \geq \alpha_{\mathcal{B}_h}\|w_h\|_{\rho,*}^2.$$

For all $w, w' \in W_* := W + W_h$ holds

$$\mathcal{B}_h(w, w') \leq \beta_{\mathcal{B}_h}\|w\|_{\rho,*}\|w'\|_{\rho,*}.$$

The constants $\alpha_{\mathcal{B}_h}$ and $\beta_{\mathcal{B}_h}$ are independent of h.

Error estimate

Theorem

There holds the following Ceá-type estimate

$$\|w - w_h\|_\rho \le \|w - w_h\|_{\rho,*} \lesssim \inf_{v_h \in W_h} \|w - v_h\|_{\rho,*}.$$

Furthermore, if $w \in W \cap H^l(D)$ *for* $2 \le l \le k+1$ *there holds*

$$\inf_{v_h \in W_h} \|w - v_h\|_{\rho,*} \lesssim \bar{\rho}^{\frac{1}{2}} h^{l-1} \|w\|_{H^l(D)}.$$

L^2-error estimates

- With Poincaré-inequality:

$$\|w - w_h\|_D \leq \|w - w_h\|_{\rho,*} \lesssim \bar{\rho}^{\frac{1}{2}} h^k \|w\|_{H^{k+1}(D)}.$$

L^2-error estimates

- With Poincaré-inequality:

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- Usually, we use an Aubin-Nitsche trick to get order $k + 1$
  ➔ Needs L^2-H^2-regularity !

L^2-error estimates

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$$\|w - w_h\|_D \leq \|w - w_h\|_{\rho,*} \lesssim \bar{\rho}^{\frac{1}{2}} h^k \|w\|_{H^{k+1}(D)}.$$

- Usually, we use an Aubin-Nitsche trick to get order $k + 1$
  ➔ Needs L^2-H^2-regularity !
- we cannot apply this ➔ no smoothing orthogonal to the velocity field!

Numerical experiments

Example problem

- $D = [-1, 1]^2 \subset \mathbb{R}^2$

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- exact solution:

$$w := \exp(-6((x+0.5)^2 + y^2)) - \exp(-6((x-0.5)^2 + y^2))$$

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- velocity fields

$$\begin{aligned} u_1 &= (1, 1), \\ u_2 &= (-0.75y, 0.75x) \\ u_3 &= (2y(1 - x^2), -2x(1 - y^2)) \end{aligned}$$

Exact Solution

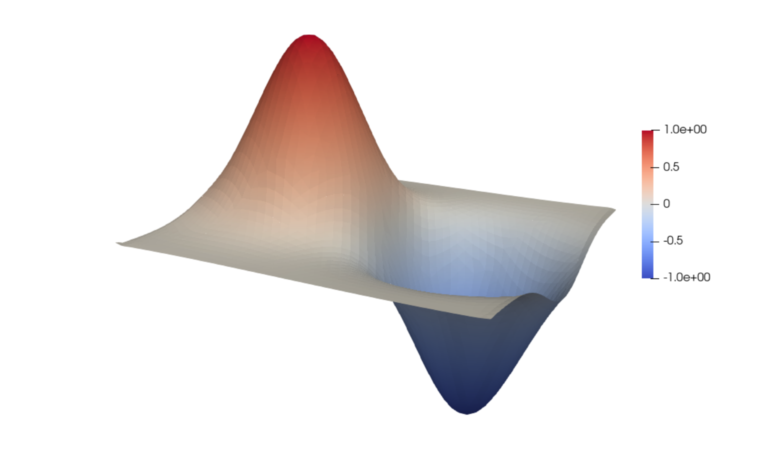


Figure 2: Exact solution w

Velocity field u_1

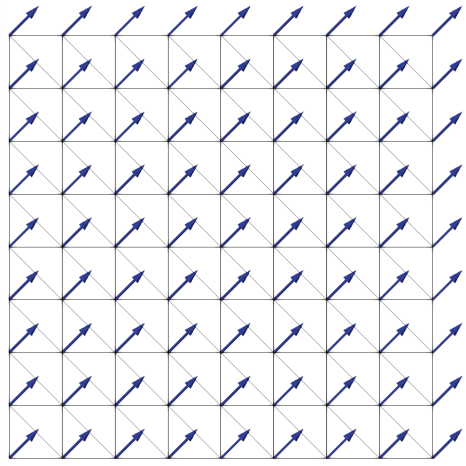


Figure 3: Constant velocity field u_1

Velocity field u_2

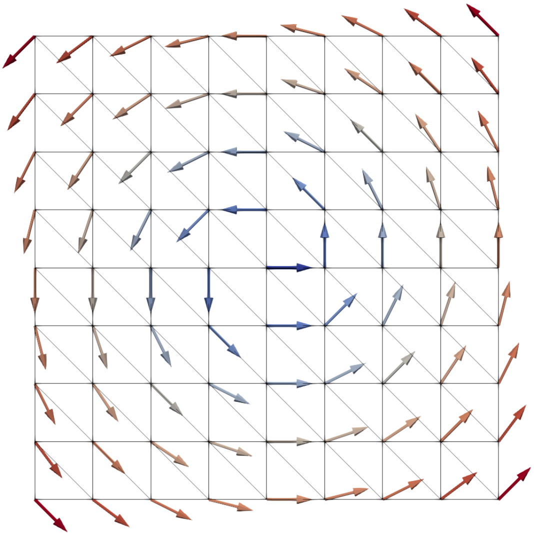


Figure 4: Rotation velocity field u_2

Velocity field u_3

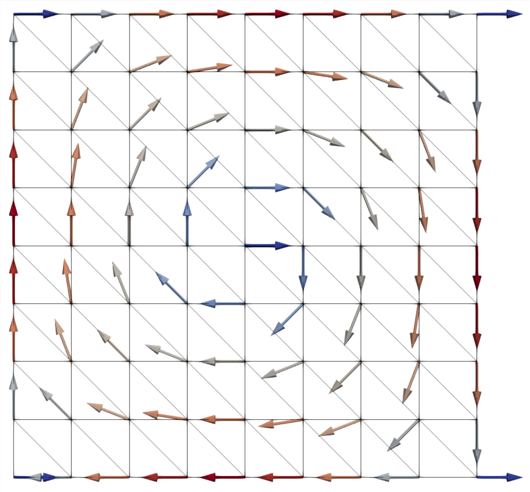
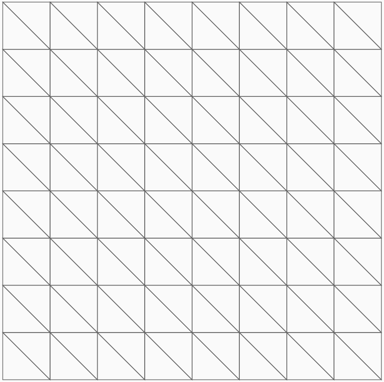
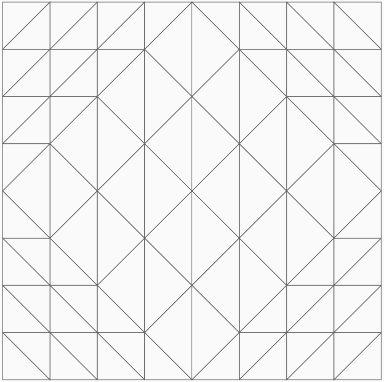


Figure 5: Vortex velocity field u_3

Structured vs. Unstructured Meshes

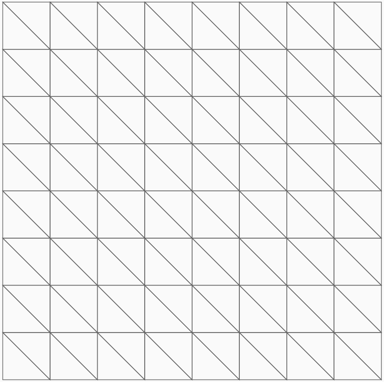


(a) Structured Mesh

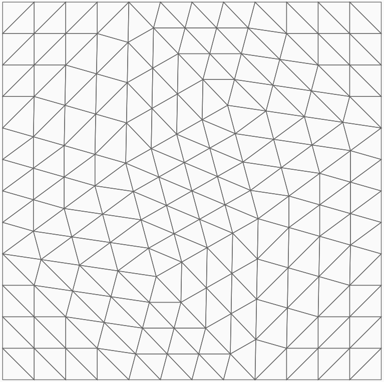


(b) Unstructured Mesh with initial mesh size of 1

Structured vs. Unstructured Meshes



(a) Structured Mesh



(b) Unstructured Mesh with initial mesh size of 0.7

Mesh refinements	e_{L^2}	(eoc)	e_W	(eoc)
$k = 1$				
1	$5.43 \cdot 10^{-1}$		$2.99 \cdot 10^{0}$	
2	$2.63 \cdot 10^{-1}$	(1.05)	$1.43 \cdot 10^{0}$	(1.07)
3	$1.64 \cdot 10^{-1}$	(0.68)	$9.72 \cdot 10^{-1}$	(0.55)
4	$7.22 \cdot 10^{-2}$	(1.18)	$5.30 \cdot 10^{-1}$	(0.88)
5	$2.50 \cdot 10^{-2}$	(1.53)	$2.69 \cdot 10^{-1}$	(0.98)
6	$7.16 \cdot 10^{-3}$	(1.8)	$1.32 \cdot 10^{-1}$	(1.03)
7	$1.87 \cdot 10^{-3}$	(1.94)	$6.47 \cdot 10^{-2}$	(1.03)
8	$4.74 \cdot 10^{-4}$	(1.98)	$3.19 \cdot 10^{-2}$	(1.02)

Table 1: Rates of converges for u_1 with an structured mesh

Mesh refinements	e_{L^2}	(eoc)	e_W	(eoc)
$k = 1$				
1	$1.21 \cdot 10^{-1}$		$1.18 \cdot 10^{0}$	
2	$4.11 \cdot 10^{-2}$	(1.56)	$6.06 \cdot 10^{-1}$	(0.96)
3	$1.24 \cdot 10^{-2}$	(1.73)	$3.03 \cdot 10^{-1}$	(1.0)
4	$3.59 \cdot 10^{-3}$	(1.78)	$1.50 \cdot 10^{-1}$	(1.01)
5	$1.11 \cdot 10^{-3}$	(1.7)	$7.47 \cdot 10^{-2}$	(1.01)
6	$4.03 \cdot 10^{-4}$	(1.46)	$3.73 \cdot 10^{-2}$	(1.0)
7	$1.75 \cdot 10^{-4}$	(1.2)	$1.86 \cdot 10^{-2}$	(1.0)
8	$8.35 \cdot 10^{-5}$	(1.07)	$9.31 \cdot 10^{-3}$	(1.0)

Table 2: Rates of converges for u_1 with an unstructured mesh

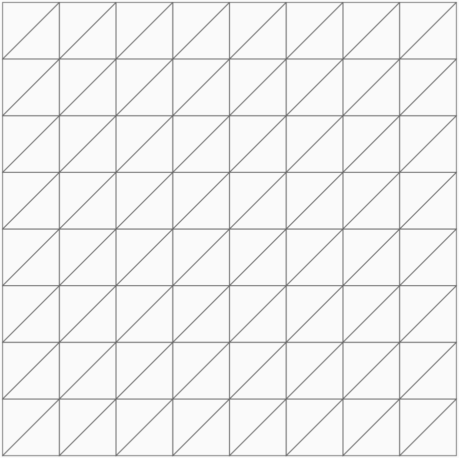


Figure 7: Structured mesh with flipped triangles

Mesh refinements	e_{L^2}	(eoc)	e_W	(eoc)
$k = 1$				
1	$4.83 \cdot 10^{-1}$		$2.67 \cdot 10^{0}$	
2	$1.52 \cdot 10^{-1}$	(1.66)	$1.47 \cdot 10^{0}$	(0.86)
3	$6.44 \cdot 10^{-2}$	(1.24)	$8.60 \cdot 10^{-1}$	(0.77)
4	$2.55 \cdot 10^{-2}$	(1.34)	$4.33 \cdot 10^{-1}$	(0.99)
5	$1.17 \cdot 10^{-2}$	(1.12)	$2.17 \cdot 10^{-1}$	(1.0)
6	$5.72 \cdot 10^{-3}$	(1.03)	$1.08 \cdot 10^{-1}$	(1.0)
7	$2.84 \cdot 10^{-3}$	(1.01)	$5.42 \cdot 10^{-2}$	(1.0)
8	$1.42 \cdot 10^{-3}$	(1.0)	$2.71 \cdot 10^{-2}$	(1.0)

Table 3: Rates of converges for u_1 with an structured mesh with flipped triangles

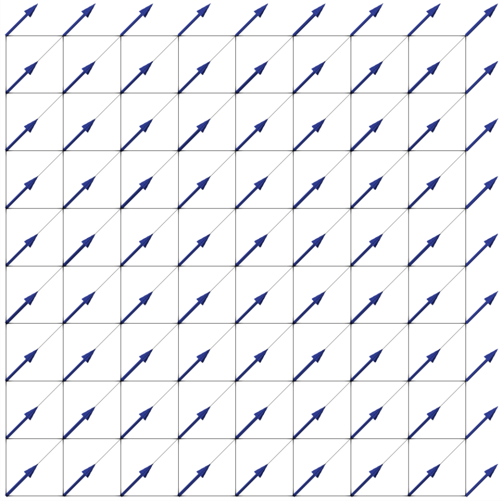


Figure 8: u_1 on a structured mesh with flipped triangles

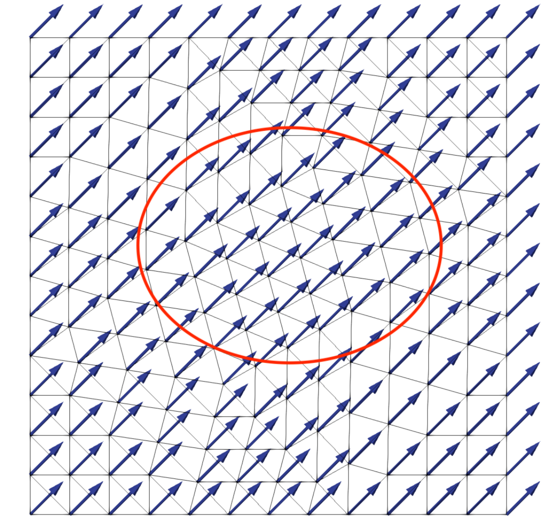
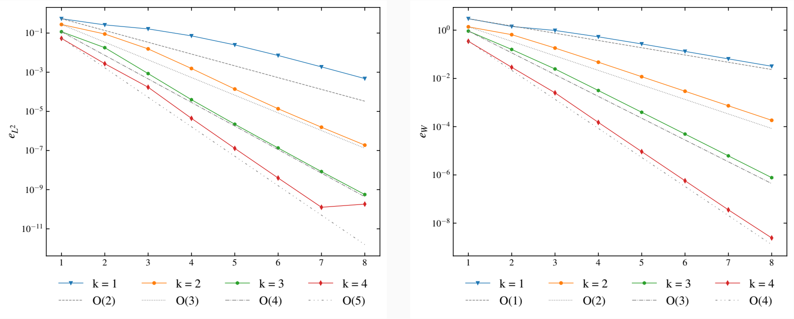


Figure 9: u_1 on an unstructured mesh with initial mesh size 0.7

Code Example

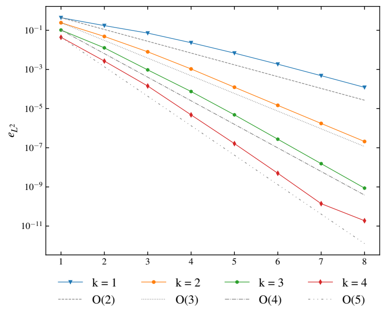


(a) Errors in the L^2-norm.

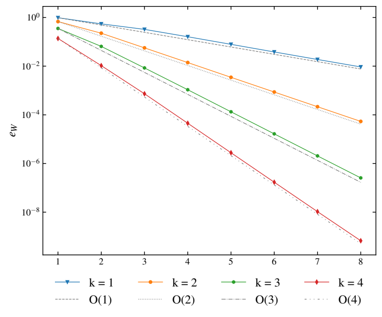
(b) Errors in the W-norm.

Figure 10: Numerical errors for u_1.

Convergence studies II



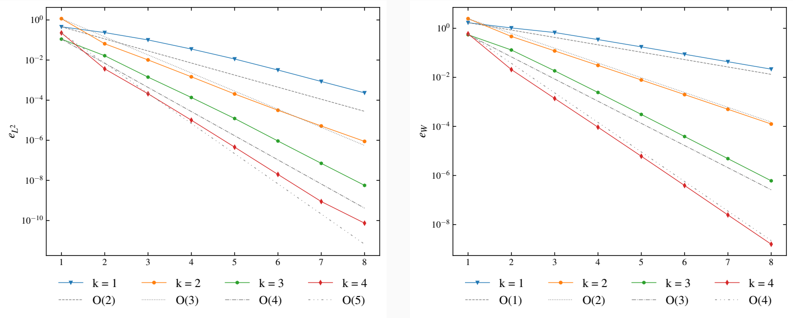
(a) Errors in the L^2-norm.



(b) Errors in the W-norm.

Figure 11: Numerical errors for u_2.

Convergence studies III



(a) Errors in the L^2-norm. **(b)** Errors in the W-norm.

Figure 12: Numerical errors for u_3.

Convergence studies IV

- optimal convergence in the W-norm

Convergence studies IV

- optimal convergence in the W-norm
- L^2-rates are close to optimal
  - ➔ on a structured Mesh!
  - ➔ depend on the velocity field!

Condition numbers

velocity field	u_1	
λ	$\kappa(\mathbb{B})$	$\kappa(J\mathbb{B})$
1	(-1.96)	(-1.01)
2	(-5.91)	(-4.43)
4	8218.48	8601.06
8	12291.3	12798.07
16	18322.75	17842.76
32	25880.81	22168.92
64	32346.54	24467.45
128	36733.82	31598.81
256	43054.67	34331.81
512	74125.53	41153.27
1024	117410.63	70580.26

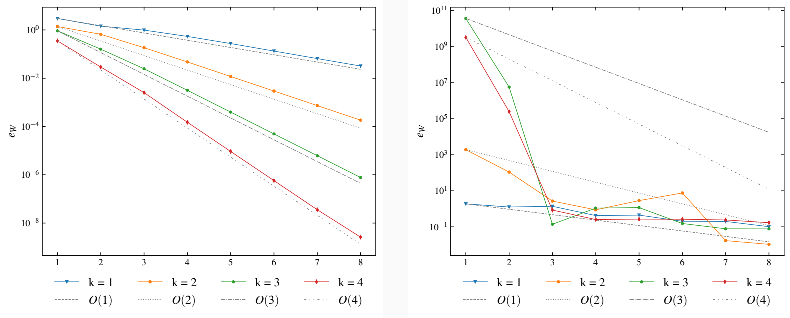
Table 4: Condition numbers with different λ for $k = 1$

Diffusion only I

Consider the problem without the volume term:
Find $w_h \in W_h$ such that

$$-\nabla \cdot (\rho(u \otimes u) \cdot \nabla w_h) = f \text{ in } D.$$

Diffusion only II



(a) u_1 **(b)** u_3

Figure 13: Numerical errors in the W-norm.

Non-constant density I

Consider a circle geometry $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

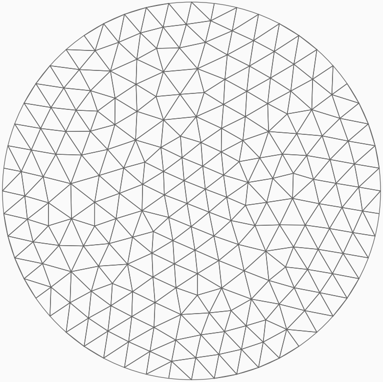
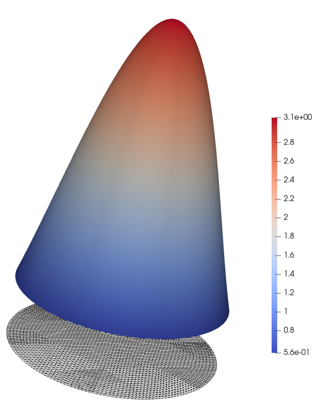


Figure 14: Circle geometry triangulated with an unstructured mesh.

Non-constant density II

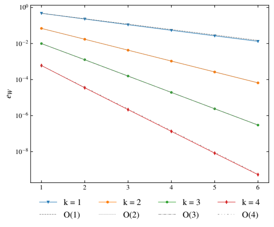
Define $\rho_n(x,y) = (1.75 - (x^2+y^2))^n, n \in \mathbb{N}$.



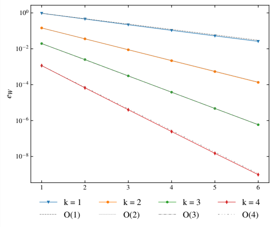
n	$\max \rho_n$	$\min \rho_n$
2	3.0625	0.5625
4	9.3789	0.3164
8	87.9639	0.1001
12	825.005	0.0317
16	7737.6446	0.01

Code Example

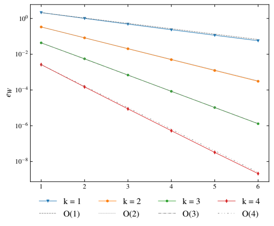
Optimal convergence in the W-norm:



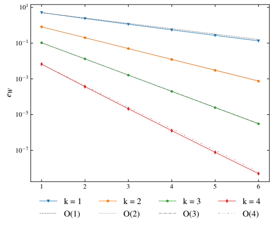
(a) ρ_4



(b) ρ_8



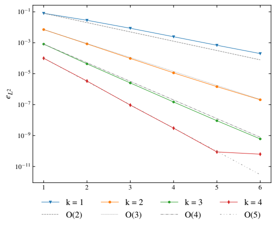
(c) ρ_{12}



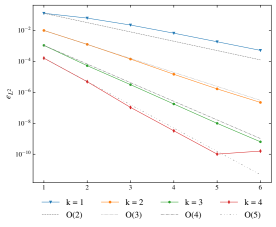
(d) ρ_{16}

Non-constant density IV

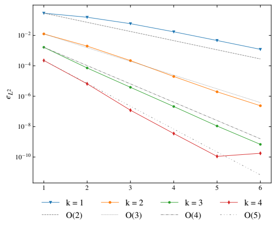
Convergence rates in the L^2-norm for u_1:



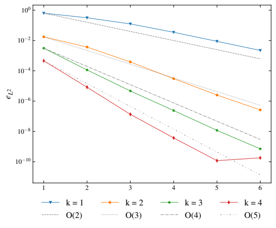
(a) ρ_4



(b) ρ_8



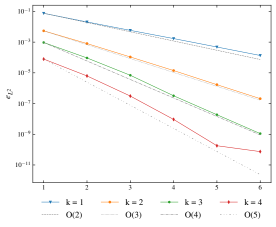
(c) ρ_{12}



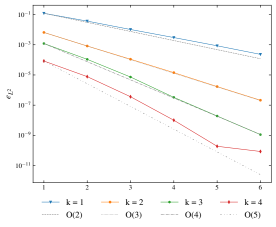
(d) ρ_{16}

Non-constant density V

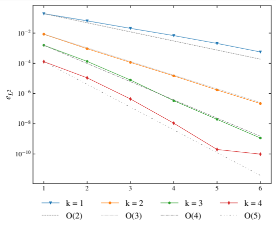
Convergence rates in the L^2-norm for u_3:



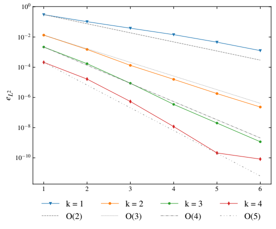
(a) ρ_4



(b) ρ_8



(c) ρ_{12}



(d) ρ_{16}

Conclusion and Outlook

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- modifications, e.g. HDG

Thank you for your attention!