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Unfitted mixed FEM for Poisson's equation with Dirichlet boundary conditions

CPDE Oberseminar

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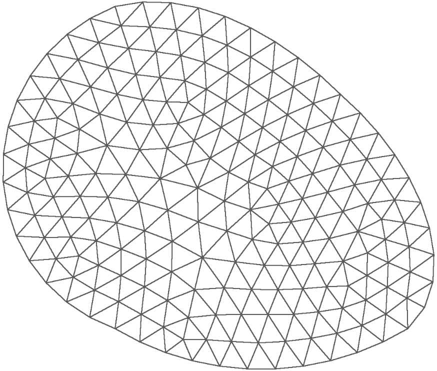
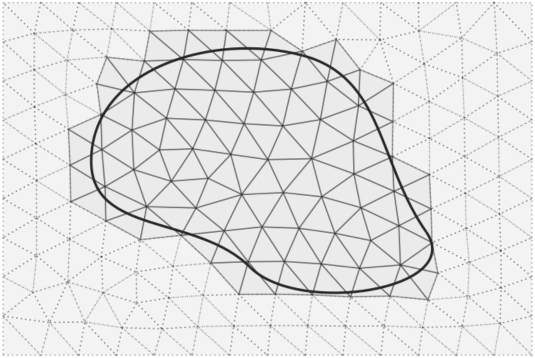
Unfitted FEM - Motivation

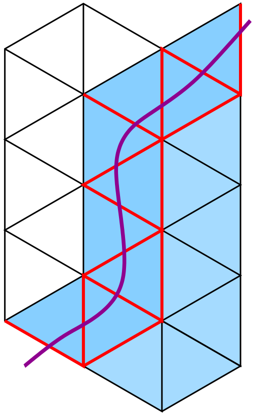




Unfitted FEM - Challenges

- stability is no longer independent of the cut position
- condition numbers depend on the cut position

➔ Stabilization, e.g. ghost penalty



Mixed Poisson formulation

Consider Poisson's equation:

$$-\operatorname{div}(\nabla u) = f \text{ in } \Omega.$$

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Rewrite the equation as a mixed problem

$$\begin{aligned} \sigma - \nabla u &= 0 \text{ in } \Omega, \\ \operatorname{div} \sigma &= -f \text{ in } \Omega. \end{aligned}$$

➔ Mass conservation

Weak formulation

Let $\Sigma = H(\mathrm{div}, \Omega)$ and $Q = L^2(\Omega)$. The weak formulation is given by:
Find $(\sigma, u) \in \Sigma \times Q$ such that

$$
\begin{aligned}
\overbrace{\int_\Omega \sigma \cdot \tau dx}^{a(\sigma,\tau)} + \overbrace{\int_\Omega \mathrm{div}(\tau) u dx}^{b(\tau,u)} &= \int_{\partial\Omega} \tau \cdot n u_D ds \qquad \forall \tau \in \Sigma, \\
\underbrace{\int_\Omega \mathrm{div}(\sigma) v dx}_{b(\sigma,v)} &= -\int_\Omega f v dx \qquad \forall v \in Q.
\end{aligned}
$$

Mass conversation

Lemma

If $f_h \in Q$ *and* $\operatorname{div} \Sigma \subset Q$, *the solution* σ *satisfies*

$$\operatorname{div} \sigma + f_h = 0.$$

Can we put them together?

unfitted FEM $\rightarrow$ **?** $\leftarrow$ mixed FEM

Overview

Unfitted discretization

Error Analysis

Post-processing

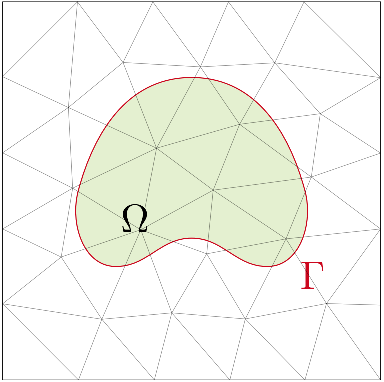
Numerical results

Conclusion and Outlook

Unfitted discretization

Notation

- Ω is the unfitted domain inside of the active mesh $\mathcal{T}_h$,
- $\Omega^{\mathcal{T}}$ is the domain of the active mesh,
- $\mathcal{T}_h^i$ is the set of interior uncut elements,
- $\mathcal{T}_h^\Gamma$ is the set of cut elements.



Discretization

Assume that $f \in L^2(\Omega)$ exists a suitable approximation f_h defined on Ω^τ that is piecewise polynomial. Define the following spaces

- $\Sigma_h \subset H(\mathrm{div}, \Omega^\tau)$ the Raviart-Thomas space of order k,
- $Q_h \subset L^2(Q)$ the space of (discontinous) piecewise polynomials of degree k.

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- $Q_h \subset L^2(Q)$ the space of (discontinous) piecewise polynomials of degree k.

The discrete problem: Find $(\sigma_h, u_h) \in \Sigma_h \times Q_h$ such that

$$\begin{aligned} a(\sigma_h, \tau_h) + b(\tau_h, u_h) &= \int_{\partial\Omega} \tau_h \cdot n u_D ds =: g(\tau_h) \qquad \forall \tau_h \in \Sigma, \\ b(\sigma_h, v_h) &= -\int_\Omega f_h v_h dx =: h(v_h) \qquad \forall v_h \in Q_h. \end{aligned} \tag{1}$$

Well-posedness?

To apply standard saddle-point theory, we need

- coercivity of $a(\cdot, \cdot)$ on the kernel of $b(\cdot, \cdot)$,
- inf-sup stability of $b(\cdot, \cdot)$.

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- inf-sup stability of $b(\cdot,\cdot)$.

Lemma (Kernel-ellipticity)

On the kernel of $b(\cdot,\cdot)$, we have that $a(\sigma_h,\sigma_h) = \|\sigma_h\|_\Sigma^2$.

Inf-sup stability

We have that

$$\inf_{v_h \in Q_h} \sup_{\sigma \in \Sigma_h} \frac{b(\sigma_h, v_h)}{\|\sigma_h\|_\Sigma \|v_h\|_Q} > 0.$$

BUT: We have no lower bound on the inf-sup constant!

Furthermore: condition numbers depend on the cut position!

Our Approach - Motivation I

Split $\sigma_h = \sigma_h^0 \oplus_a \sigma_h^\perp$ with $\sigma_h^0 \in \Sigma_h^0 := \{\sigma_h \in \Sigma_h \mid b(\sigma_h, v_h) = 0 \forall v_h \in Q_h\} = \ker b$.
The problem reduces to three subproblems:

(1) Find $\sigma_h^0 \in \Sigma_h^0$ s.t. $a(\sigma_h^0, \tau_h^0) = g(\tau_h^0) \qquad \forall \tau_h^0 \in \Sigma_h^0,$

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$$(2) \quad \text{Find } \sigma_h^\perp \in \Sigma_h^\perp \text{ s.t. } b(\sigma_h^\perp, v_h) = h(v_h) \qquad \forall v_h \in Q_h,$$

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(2) Find $\sigma_h^\perp \in \Sigma_h^\perp$ s.t. $b(\sigma_h^\perp, v_h) = h(v_h) \qquad \forall v_h \in Q_h$,

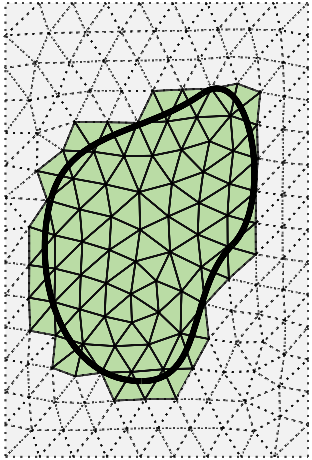
(3) Find $u_h \in Q_h$ s.t. $b(\tau_h^\perp, u_h) = g(\tau_h^\perp) - a(\sigma_h^\perp, \tau_h^\perp) \qquad \forall \tau_h^\perp \in \Sigma_h^\perp$.

Our Approach - Motivation II

Let $\bar{b}(\sigma_h, v_h) := \int_{\Omega^\tau} \operatorname{div} \sigma_h v_h dx$.

Lemma

It holds that $\Sigma_h^0 = \ker \bar{b}$.



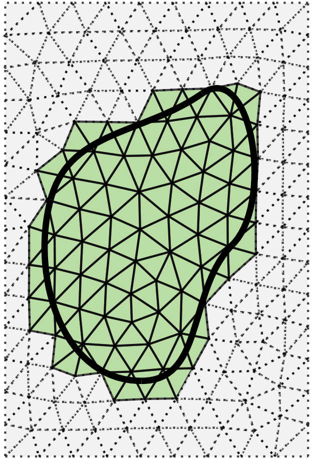
Let $\bar{b}(\sigma_h, v_h) := \int_{\Omega^\mathcal{T}} \operatorname{div} \sigma_h v_h dx$.

Lemma

It holds that $\Sigma_h^0 = \ker \bar{b}$.

Proof.

Choose $v_h = \operatorname{div} \sigma_h$. Then $\int_\Omega (\operatorname{div} \sigma_h)^2 dx = 0 \Rightarrow \operatorname{div} \sigma_h = 0$ on Ω. For cut elements T with $\operatorname{meas}_d T \cap \Omega > 0$, the only polynomial that vanishes is the zero polynomial. Hence $\operatorname{div} \sigma_h = 0$ on $\Omega^\mathcal{T}$. $\square$



Modified discrete problem

Find $(\bar{\sigma}_h, \bar{u}_h) \in \Sigma_h \times Q_h$ such that

$$\begin{aligned} a(\bar{\sigma}_h, \tau_h) + \bar{b}(\tau_h, \bar{u}_h) &= \int_{\partial\Omega} \tau_h \cdot n u_D ds = g(\tau_h) \qquad \forall \tau_h \in \Sigma_h, \\ \bar{b}(\bar{\sigma}_h, v_h) &= -\int_{\Omega^\tau} f_h v_h dx = h(v_h) \qquad \forall v_h \in Q_h. \end{aligned} \tag{2}$$

Inf-sup stability - revisited

Now, we fulfil the inf-sup condition for a constant c not depending on the mesh size or the cut position:

$$\inf_{v_h \in Q_h} \sup_{\sigma \in \Sigma_h} \frac{\bar{b}(\sigma_h, v_h)}{\|\sigma_h\|_\Sigma \|v_h\|_Q} \geq c > 0.$$

$\Rightarrow$ the modified problem (2) has a unique solution $(\bar{\sigma}_h, \bar{u}_h)$.

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$\Rightarrow$ the modified problem (2) has a unique solution $(\bar{\sigma}_h, \bar{u}_h)$.

Lemma

It holds that $\bar{\sigma}_h = \sigma_h$, where (σ_h, u_h) is the solution to (1). Furthermore, for $T \in \mathcal{T}_h^i$, we have that $\bar{u}_h = u_h$.

Error Analysis

Ghost penalty

Introduce ghost penalty term:

$$j_h(\sigma,\tau) := \sum_{F\in\mathcal{F}_h^n} j_{h,F}^{\mathrm{dir}}(\sigma,\tau) \text{ with } j_{h,F}^{\mathrm{dir}}(\sigma,\tau) := \int_{\omega_F} (\sigma_1-\sigma_2)(\tau_1-\tau_2)dx,$$

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For $\gamma \geq 0$, we now consider:

$$\underbrace{a(\sigma_h,\tau_h)+\gamma j_h(\sigma_h,\tau_h)}_{=:A_h(\sigma_h,\tau_h)} + \bar{b}(\tau_h,\bar{u}_h) = \int_{\partial\Omega} \tau_h\cdot n\ u_D\ ds \quad \forall\tau_h\in\Sigma_h,$$

$$\bar{b}(\sigma_h,v_h) = -\int_{\Omega} f_h v_h\ dx \quad \forall v_h\in Q_h.$$

Coercivity

Define the following norm

$$\|\sigma_h\|_{A_h}^2 := \|\sigma_h\|_\Omega^2 + \|\operatorname{div}\sigma_h\|_{\Omega^\tau}^2 + \gamma|\sigma_h|_j^2,$$

where $|\sigma_h|_j^2 := j_h(\sigma_h, \sigma_h)$.

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Lemma (Coercivity revisited)

The bilinear form $A_h(\cdot,\cdot)$ is coercive on Σ_h^0 with respect to the $\|\cdot\|_{A_h}$-norm.

Weak Galerkin orthogonality

Lemma (Weak Galerkin orthogonality)

There holds

$$a(\sigma - \sigma_h, \tau_h) = \gamma j_h(\sigma_h, \tau_h) \qquad \forall \tau_h \in \Sigma_h^0.$$

Proof.

As $\tau_h \in \Sigma_h^0$, we have that $\bar{b}(\tau_h, \bar{u}) = \bar{b}(\tau_h, \bar{u}_h) = 0$. Hence

$$a(\sigma - \sigma_h, \tau_h) = g(\tau_h) - \bar{b}(\tau_h, \bar{u}) - g(\tau_h) + \bar{b}(\tau_h, \bar{u}_h) + \gamma j_h(\sigma_h, \tau_h) = \gamma j_h(\sigma_h, \tau_h).$$

□

σ-error estimates

Lemma

There exists an unique solution $\sigma_h \in \Sigma_h^0$ such that

$$A_h(\sigma_h, \tau_h) = g(\tau_h) \qquad \forall \tau_h \in \Sigma_h^0.$$

Furthermore, there holds

$$\|\sigma - \sigma_h\|_{A_h} \lesssim \inf_{\tau_h \in \Sigma_h^f} \|\sigma - \tau_h\|_{A_h}$$

Existence and uniqueness:

A_h is coercive on $\Sigma_h^0 \Rightarrow \exists$ unique solution σ_h (Lax-Milgram).

Proof of the σ-error estimates

Error estimate:

For $\tau_h \in \Sigma_h^f$, we have that

$$\begin{aligned}\|\sigma_h - \tau_h\|_{A_h}^2 &= A_h(\sigma_h - \tau_h, \sigma_h - \tau_h) \\ &= A_h(\sigma - \tau_h, \sigma_h - \tau_h) + \underbrace{A_h(\sigma_h - \tau_h, \sigma_h - \tau_h)}_{=0 \text{ as } \sigma_h - \tau_h \in \Sigma_h^0} \\ &\lesssim \|\sigma - \tau_h\|_{A_h} \|\sigma_h - \tau_h\|_{A_h}.\end{aligned}$$

Divide by $\|\sigma_h - \tau_h\|_{A_h}$, apply the triangle inequality and take the infimum.

L^2-error estimate and Interpolation

In addition:

- $\|\sigma - \sigma_h\|_{L^2(\Omega^\tau)} \leq \|\sigma - \tau_h\|_{L^2(\Omega^\tau)} + \underbrace{\|\tau_h - \sigma_h\|_{L^2(\Omega^\tau)}}_{\simeq \|\tau_h - \sigma_h\|_{A_h}}$ ➔ L^2 estimate

L^2-error estimate and Interpolation

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- $\|\sigma - \sigma_h\|_{A_h} \lesssim h^{k+1} \|\sigma\|_{H^{k+1}(\Omega)}$ with the help of a BDM interpolator

$\bar{u}_h$ error estimates

Define an interpolation operator $\pi_h : L^2(\Omega) \to Q_h$ s.t.

$$(\pi_h u, q_h)_{\Omega^\tau} = (u, q_h)_\Omega \qquad \forall q_h \in Q_h.$$

Lemma ($\bar{u}_h$ error estimate)

It holds that

$$\|\pi_h u - \bar{u}_h\|_{L^2(\Omega^\tau)} \leq \frac{C_a}{c} \inf_{\tau_h \in \Sigma_h} \|\sigma - \tau_h\|_\Sigma + \left(1 + \frac{C_b}{c}\right) \inf_{\bar{v}_h \in Q_h} \|\pi_h u - \bar{v}_h\|_{L^2(\Omega^\tau)}.$$

Proof of the $\bar{u}_h$ error estimate

Proof.

For $\gamma = 0$, we have due to the inf-sup condition:

$$\begin{aligned}
c\|\bar{u}_h - \bar{v}_h\|_{L^2(\Omega^\tau)}\|\tau_h\|_\Sigma &\leq \bar{b}(\tau_h, \bar{u}_h - \bar{v}_h) = g(\tau_h) - g(\tau_h) + \bar{b}(\tau_h, \bar{u}_h - \bar{v}_h) \\
&= a(\sigma, \tau_h) + \underbrace{b(\tau_h, u)}_{=\bar{b}(\tau_h, \pi_h u)} - a(\sigma_h, \tau_h) - \bar{b}(\tau_h, \bar{u}_h) + \bar{b}(\tau_h, \bar{u}_h - \bar{v}_h) \\
&= a(\sigma - \sigma_h, \tau_h) + \bar{b}(\tau_h, \pi_h u - \bar{v}_h) \\
&\leq C_a\|\sigma - \sigma_h\|_\Sigma\|\tau_h\|_\Sigma + C_b\|\tau_h\|_\Sigma\|\pi_h u - \bar{v}_h\|_{L^2(\Omega^\tau)}
\end{aligned}$$

$\square$

Post-processing

Post-processing scheme

Solve the following local problems: Find $u_h^* \in \mathcal{P}^{k+1}(T)$ s.t.

$$\int_T \nabla u_h^* \cdot \nabla v_h dx = \int_T \sigma_h \cdot \nabla v_h dx \qquad \forall v_h \in \mathcal{P}_0^{k+1}(T),$$
$$\int_T u_h^* dx = \int \bar{u}_h dx \text{ if } T \in \mathcal{T}_h^i,$$
$$\int_{T\cap\partial\Omega} u_h^* ds = \int_{T\cap\partial\Omega} u_D ds \text{ if } T \in \mathcal{T}_h^\Gamma.$$

Superconvergence

Lemma

Post-processing error estimate

$$\|u - u_h^*\|_{L^2(\Omega^\tau)} \lesssim h^{k+2}\|u\|_{H^{k+2}(\Omega^\tau)}.$$

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Main idea: Split the error. For $T \in \mathcal{T}_h^i$:

$$\|u - u_h^*\|_T \leq \|u - \tilde{u}\|_T + \|Q_T(\tilde{u} - u_h^*)\|_T + \|(I - Q_T)(\tilde{u} - u_h^*)\|_T.$$

For $T \in \mathcal{T}_h^\Gamma$:

$$\|u - u_h^*\|_T \leq \|u - \tilde{u}\|_T + \|Q_{T\cap\Omega}(\tilde{u} - u_h^*)\|_T + \|(I - Q_{T\cap\Omega})(\tilde{u} - u_h^*)\|_T.$$

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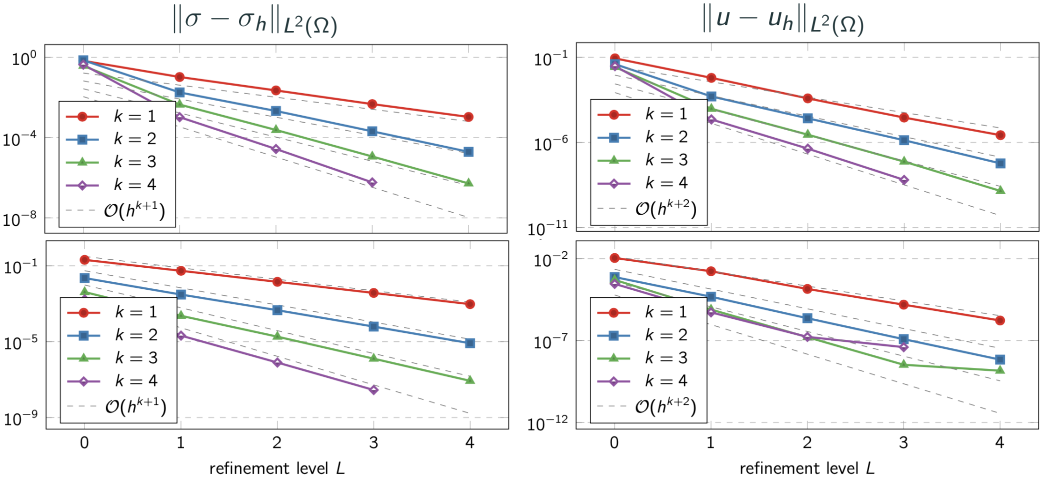
For $T \in \mathcal{T}_h^\Gamma$:

$$\|u - u_h^*\|_T \leq \|u - \tilde{u}\|_T + \|Q_{T\cap\Omega}(\tilde{u} - u_h^*)\|_T + \|(I - Q_{T\cap\Omega})(\tilde{u} - u_h^*)\|_T.$$

For the result, we have to bound $\|\sigma - \sigma_h\|_T$ ➔ Ghost penalty!

Numerical results

Mixed Poisson on a ring



Conclusion and Outlook

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Achievements:

- Circumvent polluting the mass balance
- Recover higher order convergence for u_h with post-processing

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Possible extensions:

- Other boundary conditions
- Stokes problem: Find u, p with $u = u_D$ in $\partial\Omega$, s.t.

$$
\begin{aligned}
-\Delta u + \nabla p &= f \text{ in } \Omega, \\
\operatorname{div} u &= 0 \text{ in } \Omega.
\end{aligned}
$$