FVM2: Error analysis of the finite volume discretization

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Introduction

Goal

Main goal: Error bounds for the FVM in H^1- and L^2-norm:

- $\|u - u_h\|_{H^1} \leq \quad ??$
- $\|u - u_h\|_{L^2} \leq \quad ??$

The finite-volume method

- discretization method that preserves conservation laws
- Model problem:

$$
\begin{aligned}
\nabla \cdot (-A\nabla u + bu) cu &= f && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

➔ Bilinear- and linearform for continous and discrete problem

Approach

Abstract results

Approach

Abstract results

$\downarrow$

Grid conditions

Approach

Abstract results

$\downarrow$

Grid conditions

$\downarrow$

Approximate single terms

Approach

Abstract results

$\downarrow$

Grid conditions

$\downarrow$

Approximate single terms

$\downarrow$

Global error estimates

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Abstract results

Grid conditions

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Global error estimations

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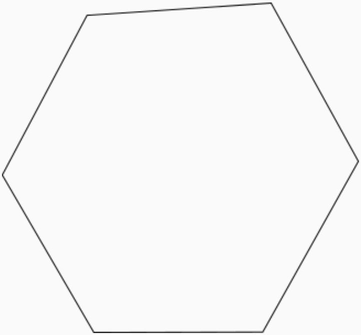
Definitions and Notations

Notations

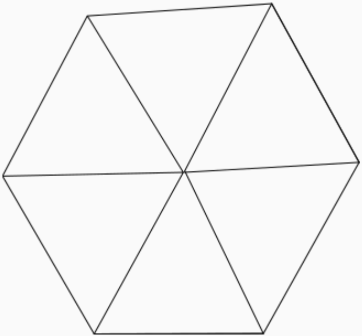
- $\Omega \subset \mathbb{R}^n$ denotes a polygonal domain
- $\mathcal{T}_h$ denotes a consistent triangulation
  - ➔ $\text{vol}(T) > 0 \qquad \forall T \in \mathcal{T}_h$;
  - ➔ $\bigcup_{T \in \mathcal{T}_h} = \overline{\Omega}$;
  - ➔ $\text{int}(T_i) \cap \text{int}(T_j) = \emptyset \qquad \forall T_i, T_j \in \mathcal{T}_h, i \neq j$
- $\mathcal{B}_h$ is a dual box grid

Triangulations



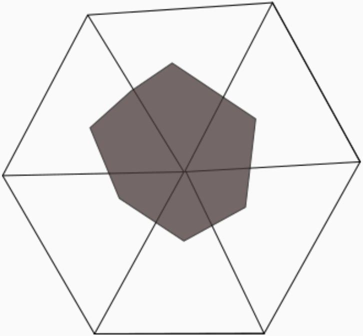
Domain

Triangulations



Domain + Mesh

Triangulations



Domain + Mesh + Dual box grid

Ansatz-spaces

- $\mathcal{P}_1(\mathcal{T}_h) := \{u_h \in C(\overline{\Omega}) \ : \ u_h|_T \in P^1(T) \quad \forall T \in \mathcal{T}_h\}$
  ➔ continuous, on $\mathcal{T}_h$ piecewise linear functions

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- $\mathcal{P}_{1,D}(\mathcal{T}_h) := \{u_h \in \mathcal{P}_1(\mathcal{T}_h) \ : \ u_h = 0 \text{ on } \partial\Omega\}$
  ➔ functions from $\mathcal{P}_1(\mathcal{T}_h)$ that vanish on $\partial\Omega$

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  ➔ functions from $\mathcal{P}_0(\mathcal{B}_h)$ that vanish on boxes that correspond to a vertex on the boundary

Continuous & Coercive

Definition

Let V be a Hilbert space and $A(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a bilinear form. $A(\cdot, \cdot)$ is called

- coercive, if there exits $\alpha \in \mathbb{R}_+$ s.t.
  $A(u, u) \geq \alpha \|u\|^2 \quad \forall u \in V$.
- continuous, if there exits $M \in \mathbb{R}_+$ s.t.
  $A(u, v) \leq M \|u\| \|v\| \quad \forall u, v \in V$.

Abstract results

Intro

The H^1- and L^2 error estimation rest upon two abstract results:

- first lemma of Strang
  - ➔ Generalization of Céa's lemma
- Generalized Aubin-Nitsche lemma

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First lemma of Strang - Statement

Lemma (Strang's first lemma)

With the assumptions from before, there holds:

$$\begin{aligned}\|u - u_h\| \le \frac{2M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\| \\ + \frac{1}{\alpha} \sup_{v_h \in V_h} \frac{|A(u_h, v_h) - A_h(u_h, v_h)|}{\|v_h\|} \\ + \frac{1}{\alpha} \sup_{v_h \in V_h} \frac{|f(v_h) - f_h(v_h)|}{\|v_h\|}.\end{aligned}$$

First lemma of Strang - Intuition

Consider standard Galerkin-discretization:

$$\text{Find } u_h \in V_h \text{ s.t. } A(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h$$

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(C) everything stays the same

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First lemma of Strang - Proof sketch i

Proof.

Use coercivity:

$$
\begin{aligned}
\alpha \left\|u_h - v_h\right\|^2 &\le A(u_h - v_h, u_h - v_h) \\
&= A(u - v_h, u_h - v_h) + A(u_h - u, u_h - v_h) \\
&= A(u - v_h, u_h - v_h) \\
&\quad + \{A(u_h, u_h - v_h) - A_h(u_h, u_h - v_h)\} \\
&\quad + \{f_h(u_h - v_h) - f(u_h - v_h)\}
\end{aligned}
$$

and continuity:

$$
A(u - v_h, u_h - v_h) \le M \left\|u - v_h\right\| \left\|u_h - v_h\right\|
$$

First lemma of Strang - Proof sketch ii

Proof.

Together:

$$\alpha \|u_h - v_h\| \leq M \|u - v_h\| + \frac{|A(u_h, u_h - v_h) - A_h(u_h, u_h - v_h)|}{\|u_h - v_h\|} + \frac{|f^*(u_h - v_h) - f_h^*(u_h - v_h)|}{\|u_h - v_h\|}$$

$\square$

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- u_h solution to the discrete problem

Generalized Aubin-Nitsche lemma - Statement

Lemma (Generalized Aubin-Nitsche lemma)

Under the Asssumptions from before, there holds:

$$\begin{aligned}\|u - u_h\|_U \leq \sup_{g \in U} \frac{1}{\|g\|_U} \inf_{v_h \in V_h} \{ & M \|u - u_h\|_V \|u_g - v_h\|_V \\ & + |A(u_h, v_h) - A_h(u_h, v_h)| \\ & + |f(v_h) - f_h(v_h)| \}.\end{aligned}$$

Abstract results ✓

$\downarrow$

Grid conditions

$\downarrow$

Approximate single terms

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Global error estimates

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Preliminaries i

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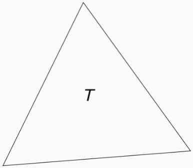

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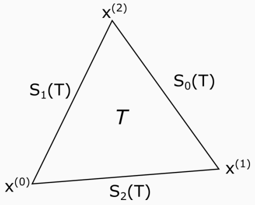
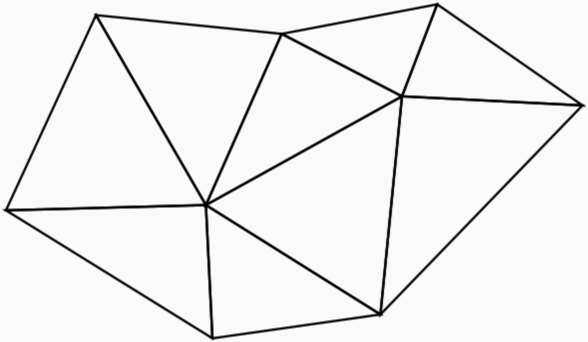
- $S_k(T)$: facet that oppose the vertex x_T^k



Preliminairies ii



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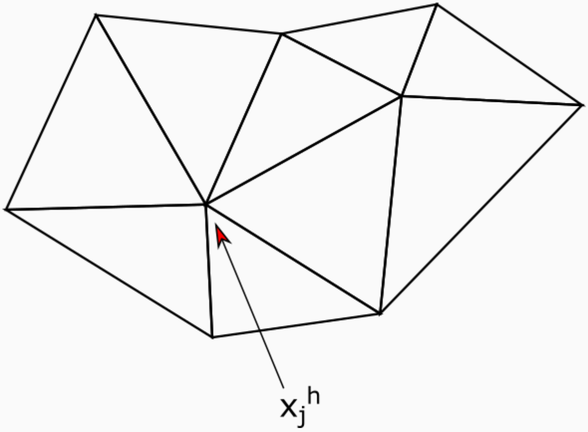

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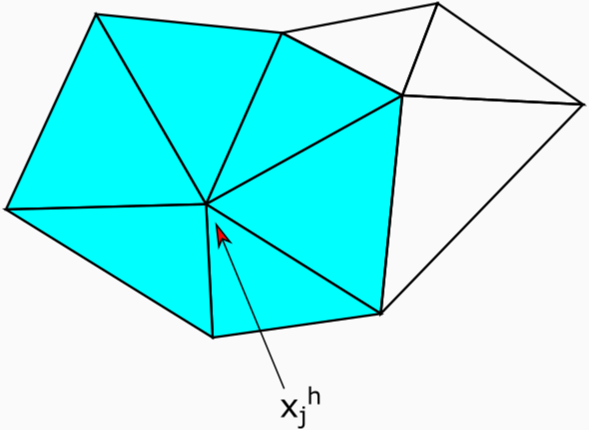

Neighborhood triangulation $\mathcal{T}_j^h$

Preliminairies ii

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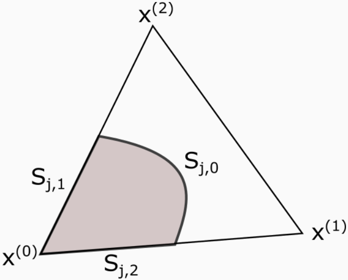

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Conditions of equilibrium

A dual box grid satisfies

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$$\mathrm{vol}(S_{j,k}(T)) = \mathrm{vol}(B_j^h \cap S_k(T)) = \frac{\mathrm{vol}(S_k(T))}{n}.$$

Visualization

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➔ (B) is what (G2) means!

$$\mathrm{vol}(S_{j,k}(T)) = \frac{\mathrm{vol}(S_k(T))}{n}$$

Conditions of regularity

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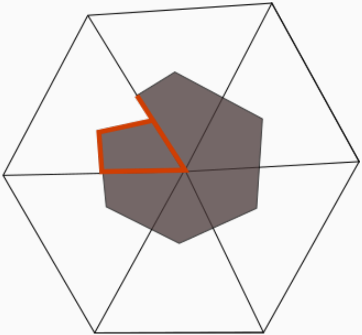
Conditions of regularity

A dual box grid satisfies

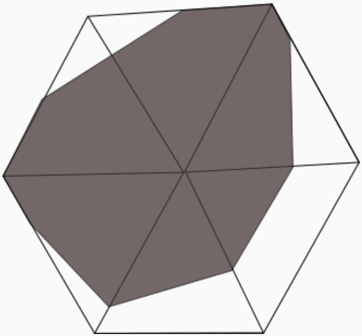
- (R1), if $\forall T \in \mathcal{T}_j^h \forall 1 \leq j \leq N_h$ $B_j^h \cap T$ is a Lipschitz domain
- (R2), if $\forall T \in \mathcal{T}_j^h \forall 1 \leq j \leq N_h$ $S_{j,i_{j,T}}(T) \cap S_{i_{j,T}}(T) = \emptyset$

Visualization



➔ Red part does not satisfy (R1)!

Visualization



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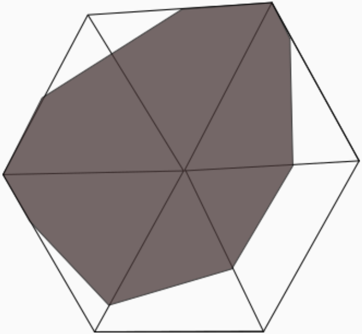

Question: Does this box grid satisfy condition (R2)?

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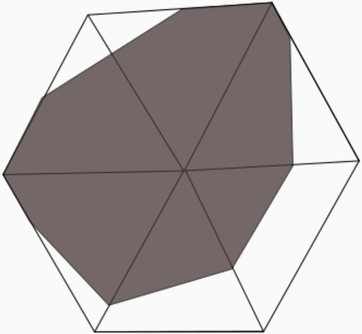

Question: Does this box grid satisfy condition (R2)?

$$S_{j,i_{j,T}}(T) \cap S_{i_{j,T}}(T) = \emptyset$$

➔ No!

Abstract results ✓

↓

Grid conditions ✓

↓

Approximate single terms

↓

Global error estimates

Recap: FVM

Continuous problem

We defined the bilinear form $A(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ for $u, v \in H_0^1(\Omega)$:

$$A(u,v) = \int_\Omega \nabla v \cdot A \nabla u \,\mathrm{d}x - \int_\Omega u b \cdot \nabla v \,\mathrm{d}x + \int_\Omega c u v \,\mathrm{d}x$$

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and the linearform $f^*(\cdot) : H_0^1(\Omega) \to \mathbb{R}$:

$$f^*(v) = \int_\Omega f v \,\mathrm{d}x, \quad v \in H_0^1(\Omega).$$

Continuous problem

We defined the bilinear form $A(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ for $u, v \in H_0^1(\Omega)$:

$$A(u,v) = \int_\Omega \nabla v \cdot A \nabla u \,\mathrm{d}x - \int_\Omega ub \cdot \nabla v \,\mathrm{d}x + \int_\Omega cuv \,\mathrm{d}x$$

and the linearform $f^*(\cdot) : H_0^1(\Omega) \to \mathbb{R}$:

$$f^*(v) = \int_\Omega fv \,\mathrm{d}x, \quad v \in H_0^1(\Omega).$$

Hence, the continous problem is given by

$$\text{Find } u \in H_0^1(\Omega) \text{ s.t. } A(u,v) = f^*(v), \quad v \in H_0^1(\Omega). \tag{CP}$$

Discrete problem

We defined the discrete bilinear form $A_h(\cdot,\cdot) : \mathcal{P}_{1,D}(\mathcal{T}_h) \times \mathcal{P}_{0,D}(\mathcal{B}_h) \to \mathbb{R}$ by

$$A_h(u_h, \overline{v}_h) = \sum_{B \in \mathcal{B}_h} \left(\int_B c u_h \overline{v}_h \, \mathrm{d}x - \int_{\partial B} \overline{v}_h \underline{A} \nabla u_h \, \mathrm{d}\sigma + \int_{\partial B} b u_h \overline{v}_h \, \mathrm{d}\sigma \right).$$

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Therefore, the discrete problem for $\overline{v}_h \in \mathcal{P}_{0,D}(\mathcal{B}_h)$ is given by

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Note, that we consider $A_h(u_h, \overline{v}_h) = A_h(u_h, Gv_h)$ as bilinearform on $\mathcal{P}_{1,D}(\mathcal{T}_h) \times \mathcal{P}_{1,D}(\mathcal{T}_h)$ and $f^*(\overline{v}_h)$ as a corresponding disturbed function on $\mathcal{P}_{1,D}(\mathcal{T}_h)$.

Global error estimations

Motivation

We want to derive bounds for the consistency errors of the bilinearform $A(\cdot,\cdot)$ and $A_h(\cdot,\cdot)$ from before:

$$|A(u_h, v_h) - A_h(u_h, \overline{v}_h)|; \qquad |f^*(v_h - \overline{v}_h)|.$$

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➔ Split the consistency error into its single terms:

- diffusion terms
- convection terms
- reaction terms

➔ Derive boundaries for those

$$
\begin{aligned}
&A(u_h, v_h) - A_h(u_h, \overline{v}_h) \\
&= \left(\int_\Omega \nabla v_h \cdot A \nabla u_h \,\mathrm{d}x - \int_\Omega u_h b \cdot \nabla v_h \,\mathrm{d}x + \int_\Omega c u_h v_h \,\mathrm{d}x \right) \\
&- \sum_{B \in \mathcal{B}_h} \left(\int_B c u_h \overline{v}_h \,\mathrm{d}x - \int_{\partial B} \overline{v}_h \underline{A} \nabla u_h \,\mathrm{d}\sigma + \int_{\partial B} b u_h \overline{v}_h \,\mathrm{d}\sigma \right)
\end{aligned}
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&A(u_h, v_h) - A_h(u_h, \overline{v}_h) \\
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&- \sum_{B \in \mathcal{B}_h} \left(\int_B c u_h \overline{v}_h \,\mathrm{d}x - \int_{\partial B} \overline{v}_h \underline{A} \nabla u_h \,\mathrm{d}\sigma + \int_{\partial B} b u_h \overline{v}_h \,\mathrm{d}\sigma \right) \\
&= \left(\int_\Omega \nabla v_h \cdot A \nabla u_h \,\mathrm{d}x + \sum_{B \in \mathcal{B}_h} \int_{\partial B} \overline{v}_h \underline{A} \nabla u_h \,\mathrm{d}\sigma \right)
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&= \left(\int_\Omega \nabla v_h \cdot A \nabla u_h \,\mathrm{d}x + \sum_{B \in \mathcal{B}_h} \int_{\partial B} \overline{v}_h \underline{A} \nabla u_h \,\mathrm{d}\sigma \right) \\
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&= \left(\int_\Omega \nabla v_h \cdot A \nabla u_h \,\mathrm{d}x + \sum_{B \in \mathcal{B}_h} \int_{\partial B} \overline{v}_h \underline{A} \nabla u_h \,\mathrm{d}\sigma \right) \\
&- \left(\int_\Omega u_h b \cdot \nabla v_h \,\mathrm{d}x + \sum_{B \in \mathcal{B}_h} \int_{\partial B} b u_h \overline{v}_h \,\mathrm{d}\sigma \right) \\
&+ \int_\Omega c u_h (v_h - \overline{v}_h) \,\mathrm{d}x.
\end{aligned}
$$

With Green's identity we can simplify:

$$
\begin{aligned}
&- \int_\Omega u_h b \cdot \nabla v_h \,\mathrm{d}x - \sum_{B \in \mathcal{B}_h} \int_{\partial B} b u_h \overline{v}_h \,\mathrm{d}\sigma \\
&= \int_\Omega v_h \nabla (b u_h) \,\mathrm{d}x - \sum_{B \in \mathcal{B}_h} \int_B \overline{v}_h \nabla (b u_h) \,\mathrm{d}x \\
&= \int_\Omega (v_h - \overline{v}_h) \nabla (b u_h) \,\mathrm{d}x.
\end{aligned}
$$

Diffusion terms

Lemma

Let the dual box grid $\mathcal{B}_h$ satisfy (G2), (R1) and (R2). Then, there holds for arbitrary $u_h, v_h \in \mathcal{P}_{1,D}(\mathcal{T}_h)$:

$$\int_\Omega \nabla v_h \cdot A\nabla u_h \,\mathrm{d}x = -\sum_{B\in\mathcal{B}_h} \int_{\partial B} \overline{v}_h \underline{A}\nabla u_h \,\mathrm{d}\sigma.$$

➔The diffusion term vanish!

Consistency error reduces to:

$$\begin{aligned} &A(u_h, v_h) - A_h(u_h, \overline{v}_h) \\ &\quad = \int_\Omega (v_h - \overline{v}_h)\nabla(bu_h)\,\mathrm{d}x + \int_\Omega cu_h(v_h - \overline{v}_h)\,\mathrm{d}x. \end{aligned}$$

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For the source term, we obtain:

$$f^*(v_h - \overline{v}_h) = \int_\Omega f(v_h - \overline{v}_h)\,\mathrm{d}x.$$

Source terms

Lemma

Let $f \in L^2(\Omega)$. Then, there holds:

$$\left| \int_\Omega f(v_h - \overline{v}_h)\,\mathrm{d}x \right| \leq h \, \|f\|_{L^2} \, |v_h|_{H^1}, \quad v_h \in \mathcal{P}_1(\mathcal{T}_h).$$

Further, if $f \in H^1(\mathcal{T}_h)$ and $\mathcal{B}_h$ satisfies condition (G1), there holds the improved estimation:

$$\left| \int_\Omega f(v_h - \overline{v}_h)\,\mathrm{d}x \right| \leq Ch^2 \, \|f\|_{H^1(\mathcal{T}_h)} \, |v_h|_{H^1}, \quad v_h \in \mathcal{P}_1(\mathcal{T}_h),$$

where C is independent of f, v_h and h.

Convection terms

Lemma

Let $b \in L^\infty(\Omega)^n$ be a vector field with $\nabla b \in L^\infty(\Omega)$. Then, there holds for $u_h, v_h \in \mathcal{P}_1(\mathcal{T}_h)$:

$$\left| \int_\Omega (v_h - \overline{v}_h) \nabla (b u_h) \, \mathrm{d}x \right| \leq Ch \left(\|b\|_{L^\infty} + \|\nabla b\|_{L^\infty} \right) \|u_h\|_{H^1} |v_h|_{H^1}.$$

Further, if $b \in W^{1,\infty}(\mathcal{T}_h)^n$, $\nabla b \in W^{1,\infty}(\mathcal{T}_h)$ and $\mathcal{B}_h$ satisfies condition (G1), there holds the following improved estimation:

$$\begin{aligned} &\left| \int_\Omega (v_h - \overline{v}_h) \nabla (b u_h) \, \mathrm{d}x \right| \\ &\leq Ch^2 \left(\|\nabla b\|_{W^{1,\infty}(\mathcal{T}_h)} + \max_{1 \leq i \leq n} \|b_i\|_{W^{1,\infty}(\mathcal{T}_h)} \right) \|u_h\|_{H^1} |v_h|_{H^1}. \end{aligned}$$

In both cases, the constant C does only depend on n.

Reaction terms

Lemma

Let $c \in L^\infty(\Omega)$. Then, there holds for $u_h, v_h \in \mathcal{P}_1(\mathcal{T}_h)$:

$$\left| \int_\Omega c u_h (v_h - \overline{v}_h) \,\mathrm{d}x \right| \leq h \left\| c \right\|_{L^\infty} \left\| u_h \right\|_{L^2} \left| v_h \right|_{H^1}.$$

Moreover, if $c \in W^{1,\infty}(\mathcal{T}_h)$ and $\mathcal{B}_h$ satisfies (G1), there holds the improved estimation

$$\left| \int_\Omega c u_h (v_h - \overline{v}_h) \,\mathrm{d}x \right| \leq C h^2 \left\| c \right\|_{W^{1,\infty}(\mathcal{T}_h)} \left\| u_h \right\|_{H^1} \left| v_h \right|_{H^1},$$

where C does not depend on c, u_h, v_h and h.

Abstract results ✓

↓

Grid conditions ✓

↓

Approximate single terms ✓

↓

Global error estimates

Assumptions

We formulate our general assumptions for the FVM error analysis:

- $\Omega \subset \mathbb{R}^n$ polygonal Lipschitz domain

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- $f \in L^2(\Omega)$ s.t. $f^*(\cdot)$ is a bounded linear functional
- $u \in H_0^1(\Omega)$ solution to (CP).

Notation

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- $M_0 := \max\{1, M, \|c\|_{L^\infty}, \|b\|_{L^\infty}, \|\nabla b\|_{L^\infty}\}$,
- $M_1 := \max\{1, M, \|c\|_{W^{1,\infty}(\mathcal{T}_h)}, \|\nabla b\|_{W^{1,\infty}(\mathcal{T}_h)},$
  $\max_{1\le j\le n} \|b_j\|_{W^{1,\infty}(\mathcal{T}_h)}\}$.

H^1-error estimate - Statement

Theorem

Let $A(\cdot,\cdot)$ be coercive and let $\mathcal{B}_h$ satisfy (G2),(R1) and (R2). Suppose $u \in H^2(\Omega)$.

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Suppose $u \in H^2(\Omega)$.
Then there exits a $h_0 = h_0(n, \frac{M_0}{\alpha}) > 0$ s.t. (DP) has an unique solution $u_h \in \mathcal{P}_{1,D}(\mathcal{T}_h)$ for $h \leq h_0$. Further, there holds

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$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{C^* M_0}{\alpha} \left(h \|u\|_{H^2} + h \|f\|_{L^2} \right),$$

where C^* is independent of h, u and f.

Proof.

Applying Strang's first lemma yields:

$$\begin{aligned}\|u-u_h\|_{H^1} \le \frac{2M}{\alpha} &\inf_{v_h\in\mathcal{P}_{1,D}(\mathcal{T}_h)} \|u-v_h\|_{H^1} \\ &+ \frac{1}{\alpha}\sup_{v_h\in\mathcal{P}_{1,D}(\mathcal{T}_h)} \frac{|A(u_h,v_h)-A_h(u_h,v_h)|}{\|v_h\|_{H^1}} \\ &+ \frac{1}{\alpha}\sup_{v_h\in\mathcal{P}_{1,D}(\mathcal{T}_h)} \frac{|f^*(v_h-\overline{v}_h)|}{\|v_h\|_{H^1}}.\end{aligned}$$

Proof sketch ii

Proof.

With the single term boundaries we obtain:

$$
\begin{aligned}
&|A(u_h, v_h) - A_h(u_h, \overline{v}_h)| \\
&\leq \left| \int_\Omega (v_h - \overline{v}_h) \nabla (b u_h) \,\mathrm{d}x \right| + \left| \int_\Omega c u_h (v_h - \overline{v}_h) \,\mathrm{d}x \right| \\
&\leq Ch \left(\|b\|_{L^\infty} + \|\nabla b\|_{L^\infty} \right) \|u_h\|_{H^1} |v_h|_{H^1} + \left| \int_\Omega c u_h (v_h - \overline{v}_h) \,\mathrm{d}x \right| \\
&\leq Ch \left(\|b\|_{L^\infty} + \|\nabla b\|_{L^\infty} \right) \|u_h\|_{H^1} |v_h|_{H^1} + h \|c\|_{L^\infty} \|u_h\|_{L^2} |v_h|_{H^1} \\
&\leq \|u_h\|_{H^1} \|v_h\|_{H^1} h (C (\|b\|_{L^\infty} + \|\nabla b\|_{L^\infty}) + \|c\|_{L^\infty}) \\
&\leq (2C + 1) M_0 h \|u_h\|_{H^1} \|v_h\|_{H^1}
\end{aligned}
$$

Proof.

Hence, with $C^* := (2C+1)$:

$$\sup_{v_h \in \mathcal{P}_{1,D}(\mathcal{T}_h)} \frac{|A(u_h, v_h) - A_h(u_h, v_h)|}{\|v_h\|_{H^1}} \leq C^* M_0 h \|u_h\|_{H^1}$$

For the source term:

$$\sup_{v_h \in \mathcal{P}_{1,D}(\mathcal{T}_h)} \frac{|f^*(v_h - \overline{v}_h)|}{\|v_h\|_{H^1}} \leq \sup_{v_h \in \mathcal{P}_{1,D}(\mathcal{T}_h)} \frac{h \|f\|_{L^2} |v_h|_{H^1}}{\|v_h\|_{H^1}} \leq h \|f\|_{L^2}.$$

□

L^2-error estimate - statement

Theorem

Let the assumptions from before be true. Additionally, let every box grid $\mathcal{B}_h$ satisfy condition (G1) and let $b \in W^{1,\infty}(\mathcal{T}_h)^n, \nabla b \in W^{1,\infty}(\mathcal{T}_h), c \in W^{1,\infty}(\mathcal{T}_h)$ and $f \in H^1(\mathcal{T}_h)$ for the coefficients of (CP).

Further, let the dual problem be H^2-regular with constant C_R.

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Further, let the dual problem be H^2-regular with constant C_R.
Then, there exits a $h_0 = h_0(n, \frac{M_0}{\alpha}) > 0$ s.t. the discrete problem has an unique solution $u_h \in \mathcal{P}_{1,D}(\mathcal{T}_h)$ for all $h \leq h_0$.
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Then, there exits a $h_0 = h_0(n, \frac{M_0}{\alpha}) > 0$ s.t. the discrete problem has an unique solution $u_h \in \mathcal{P}_{1,D}(\mathcal{T}_h)$ for all $h \leq h_0$.
Further, there holds

$$\|u - u_h\|_{L^2} \leq \frac{C^* M_0 M_1}{\alpha}(h^2 \|u\|_{H^2} + h^2 \|f\|_{H^1(\mathcal{T}_h)}),$$

where C^* is independent of h, u and f.

Abstract results ✓

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Grid conditions ✓

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Approximate single terms ✓

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References

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