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On stable discontinuous Galerkin discretizations for Galbrun's equation

submitted by

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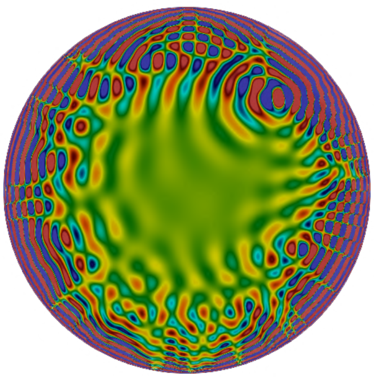

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Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst habe. Wörtlich oder sinngemäß aus anderen Werken entnommene Stellen habe ich unter Angabe der Quellen kenntlich gemacht.

Göttingen, den 05.12.2023.

Foreword & Acknowledgements

> *”The interplay between generality and individuality, deduction and construction, logic and imagination – this is the profound essence of live mathematics. Any one or another of these aspects can be at the center of a given achievement. In a far-reaching development all of them will be involved. Generally speaking, such a development will start from the "concrete" ground, then discard ballast by abstraction and rise to lofty layers of thin air where navigation and observation are easy; after this flight comes the crucial test of landing and reaching specific goals in the newly surveyed low plains of individual "reality". In brief, the flight into abstract generality must start from and return to the concrete and specific.”*
>
> — Richard Courant, Mathematics in the Modern World, Scientific American, Sep. 1964

This thesis has exceeded my initial expectations in terms of length; nevertheless, I hope that the reader will discover it to be a worthwhile endeavor. To me, a thesis serves not only as a documentation of the outcomes of a research project but also as a documentation of the journey of learning and discovery inherent in the process. As such, I have tried to incorporate enough details about the theoretical arguments to make this thesis self-contained, while maintaining the conciseness expected of a mathematical work. Courant’s words above eloquently capture the essence of my approach to this thesis. We start with a problem that is motivated by a concrete physical problem and delve deeply into abstraction to analyze it. Ultimately, we circle back, albeit partially, to the concrete problem that motivated our explorations.

Throughout my work on this project, I have greatly benefited from the support of other people. First of all, I want to thank Prof. Christoph Lehrenfeld for his unwavering guidance and support throughout this thesis and the past three years. I feel privileged to have been given the opportunity to engage in research, attend conferences, and participate in workshops at this point in my career. I am certain that without the impromptu meetings, planning and feedback sessions, and coding advice, this thesis would not have taken its present form.
Furthermore, I want to express my gratitude to my second supervisor, Dr. Martin Halla, who not only proposed this captivating topic but also provided invaluable insights into the theoretical background and his previous work on the project. His intuition and experience played a pivotal role in shaping the direction of this thesis and his feedback greatly improved my work. I also want to thank Dr. Paul Stocker, who was always willing to help me clarify my confusion, ease my frustration or listen to my ideas. The insights I gained from our coding session proved to be invaluable for the numerical experiments in this thesis. Finally, I want to extend my appreciation to the other members of the CPDE working group for providing a productive and enjoyable working environment. The little breaks, mensa lunches, and the occasional game night were always a welcome distraction from work.

Last but not least, I want to thank my family and friends for their support and encouragement throughout my studies. In particular, I want to thank my parents for their unconditional support and for always believing in me.

Abstract

We consider the damped time-harmonic Galbrun's equation which is used to model solar- and stellar oscillations. We introduce a fully discontinuous Galerkin finite element discretization that is nonconforming with respect to the convection and the diffusion operators and is robust with respect to the severe changes in the magnitude of the density and sound speed that occur in the interior of the sun. Further, we establish the stability of the method and derive convergence estimates. The analysis is based on the concepts of discrete approximation schemes, weak T-coercivity and weak T-compatibility. In addition, we present numerical results validating our theoretical findings. Furthermore, we explore the possibility of hybridizing the method to enhance the computational feasibility of the method, although we do not provide a rigorous analysis of the hybridized method.

Keywords: Galbrun's equation, stellar oscillations, discontinuous Galerkin, (weak) T-coercivity, (weak) T-compatibility, discrete approximation schemes.

Contents

Introduction

In this thesis, we consider the damped time-harmonic Galbrun's equation, which models time-harmonic acoustic waves in the presence of a steady background flow, in a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^3$ with boundary $\partial\mathcal{O}$. The equation reads as follows:

$$-\nabla(\rho c_s^2 \operatorname{div} \boldsymbol{u}) + (\operatorname{div} \boldsymbol{u})\nabla p - \nabla(\nabla p \cdot \boldsymbol{u}) - \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u}$$
$$+ (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u} + \gamma\rho(-i\omega)\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathcal{O}, \qquad (0.1a)$$
$$\boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \quad \text{on } \partial\mathcal{O}. \qquad (0.1b)$$

Here ρ, p, c_s, ϕ, $\boldsymbol{b}$, Ω, and $\boldsymbol{f}$ denote the density, the pressure, the sound speed, the gravitational background potential, the background velocity, the angular velocity of the frame and the source term. The damping is modeled by $-i\omega\gamma\rho\boldsymbol{u}$ with damping coefficient γ. Furthermore, $\partial_{\boldsymbol{b}} := \sum_{l=1}^{3} \boldsymbol{b}_l \partial_{\boldsymbol{x}_l}$ denotes the directional derivative in the direction of the flow $\boldsymbol{b}$ and $\operatorname{Hess}(\cdot)$ the Hessian. The unknown $\boldsymbol{u}$ describes the Lagrangian perturbations of displacement. Studying Galbrun's equation in this form is motivated by the equations of solar and stellar oscillation, which were first derived in [LO67]. With an additional unknown ψ describing the scaled Eulerian perturbations of the gravitational potential, the equations of solar and stellar oscillation read as follows:

$$-\rho(\omega + i\partial_b + i\Omega\times)^2 \boldsymbol{u} - \nabla(\rho c_s^2 \operatorname{div} \boldsymbol{u}) + (\operatorname{div} \boldsymbol{u})\nabla p - \nabla(\nabla p \cdot \boldsymbol{u})$$
$$+ (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u} + \gamma\rho(-i\omega)\boldsymbol{u} - \rho\nabla\psi = f \text{ in } \mathcal{O}, \qquad (0.2a)$$
$$-\frac{1}{4\pi G}\Delta\psi + \operatorname{div}(\rho\boldsymbol{u}) = 0 \text{ in } \mathbb{R}^3. \qquad (0.2b)$$

Starting from (0.2), we recover Galbrun's equation (0.1a) with the Cowling-approximation [Cow41] which sets $\psi = 0$. The main motivation to study solar oscillations comes from the field of (local) *Helioseismology*, the study of the solar interior through solar oscillations [GBS10]. The oscillations are excited by turbulent convection in the outer convection zone of the sun and can be measured on earth with Doppler shift measurements. Techniques to interpret this data, like time-distance helioseismology [GB02], involve solving both, the inverse problem of determining the solar interior from the measured data and the forward problem of predicting the measured data from a given model of the solar interior. For the latter, reliable numerical discretizations of Galbrun's equation (and ultimately, the equations of solar and stellar oscillation) are required. This task is one of the main goals of the project C04 of the Collaborative Research Center CRC1456 "Mathematics of Experiment" at the University of Göttingen.

When discretizing and analyzing (0.1) we face multiple challenges. First of all, equation (0.1a) involves two competing second-order differential operators that lead to different signs in the weak formulation of the problem which makes it challenging to apply standard techniques for proving well-posedness. Furthermore, we have to deal with a non-standard differential operator $\partial_{\boldsymbol{b}}$ involving the background flow $\boldsymbol{b}$ while also avoiding too restrictive assumptions on the Mach number $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}$. We also have to be meticulous about constants involving the sound speed c_s and the density ρ which vary drastically in the sun, see also Fig. 7.14. Thus, we want to avoid estimates where constants involve the ratios $\frac{\overline{\overline{c_s \rho}}}{\underline{\underline{c_s \rho}}}$. Finally, physically realistic computations can become very expensive, in particular, if we want to consider the full three-dimensional problem. Thus, it is highly desirable to consider techniques, for instance, Hybridization, that reduce the computational costs.

The well-posedness of Galbrun's equation (0.1) and the equations of solar and stellar oscillation (0.2) in a continuous setting has been proven recently by Halla and Hohage [HH21]. Afterwards, the work on discretizations of Galbrun's equation started. In a previous Masterthesis by Tilman Alemán [Ale22] and a resulting proceedings paper [Ale+22], a simplified vectorial PDE with a similar structure as Galbrun's equation was considered. There the authors considered different robust finite element discretizations, in particular an H^1-conforming, an $H(\mathrm{div})$-conforming discontinuous Galerkin and a fully discontinuous Galerkin method. Based on this guideline for further research on robust discretizations for Galbrun's equations, an H^1-conforming discretization of Galbrun's equation was introduced and analyzed by Halla et al [HLS22]. Furthermore, Halla [Hal23] introduced and analyzed an $H(\mathrm{div})$-conforming discontinuous Galerkin scheme for (0.1), which is nonconforming with respect to the convection term involving the directional derivative $\partial_{\boldsymbol{b}}$. The main research goal of this thesis is to extend this work to the fully discontinuous case, which is nonconforming with respect to the convection term and the diffusion term. Furthermore, we start to investigate hybridization techniques that ease the computational burden for both the $H(\mathrm{div})$-conforming- and the fully discontinuous Galerkin discretization. From an educational point of view, this thesis further aims to provide a comprehensive introduction to the techniques employed to analyze the aforementioned discretizations of Galbrun's equation.

Organization of the thesis

This thesis is divided into two parts. Part I introduces the theoretical framework that we will use to analyze the proposed discretization of Galbrun's equation in Part II. To be precise, the first part is structured as follows:

- **Chapter 1** reviews abstract well-posedness results for variational problems. We introduce the concepts of T-coercivity and weak T-coercivity, discuss their connection with well-known well-posedness results such as the BNB-Theorem, and show how they can be used to analyze variational problems.

- **Chapter 2** introduces the abstract framework of discrete approximation schemes which allows us to analyze approximations of variational problems. Furthermore, we introduce the concept of (weak) T-compatibility to connect these techniques with the notion of (weak) T-coercivity.

- In **Chapter 3**, we apply the techniques developed in the previous chapters to a simpler model problem, the Helmholtz equation with homogeneous Dirichlet boundary conditions. We consider the continuous problem, a conforming Galerkin discretization and a discontinuous Galerkin discretization. Furthermore, we introduce the concept of hybridization.

The second part of the thesis is structured as follows:

- In **Chapter 4**, we briefly introduce Galbrun's equation and review the analysis of the continuous problem from [HH21].

- **Chapter 5** reviews existing discretizations for Galbrun's equation, in particular, the H^1-conforming scheme from [HLS22] and the $H(\mathrm{div})$-conforming discretization introduced in [Hal23]. For the latter, we review the analysis extensively since it serves as a basis for the analysis of the fully discontinuous Galerkin scheme. We briefly introduce (without analysis) a hybridized formulation of the $H(\mathrm{div})$-conforming scheme.
- In **Chapter 6**, we formulate and analyze a fully discontinuous Galerkin scheme for Galbrun's equation using the techniques introduced in **Part I**. We show that the scheme can be considered as a discrete approximation scheme and that we can apply the weak T-compatibility conditions to prove that the proposed scheme is stable. In the last part of the chapter, we briefly introduce a hybridized formulation of the fully discontinuous Galerkin scheme.
- **Chapter 7** presents numerical experiments to validate the theoretical results from Chapters 5 and 6. In particular, we discuss the implementation of the methods and consider a manufactured solution to verify the convergence rates. Furthermore, we consider an example with physically realistic parameters from the sun. We conclude with a brief discussion on computational challenges arising from the application of the discretizations to Helioseismology.
- We conclude in **Chapter 8** with a summary of the results and an outlook on future work.

Furthermore, we introduce some theoretical concepts that are applied throughout this thesis in the appendix in Chapter A.

Software and replication

The numerical examples are implemented with the open source finite element software `Netgen/NGSolve` [Sch97; Sch14] which is available at

`https://ngsolve.org/`

We use the Python packages `pandas` and `numpy` to collect the data and use the LaTeX-package `tikz` for visualization. The numerical examples can be replicated with the provided reproduction files [Bee23].

Notation

Throughout this thesis, we will usually consider D or $\mathcal{O}$ as the default domain for all function spaces and write e.g. $L^2 := L^2(\mathcal{O})$. Furthermore, we denote scalar function spaces as X and use the boldface notation for its vectorial variant, that is $\boldsymbol{X} := (X)^d$, and their elements. Unless specified otherwise, all function spaces are considered over $\mathbb{C}$. We will use the notation $\langle \cdot, \cdot \rangle_X$ or $(\cdot, \cdot)_X$ for scalar products on a space X and use the notation without index $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ for both the scalar L^2- and the vectorial $\boldsymbol{L}^2$-inner product. For any space $X \subset L^2$, we set $X_* := \{u \in X : \langle u, 1 \rangle = 0\}$. In particular, we denote $L^2_0 := L^2_*$. Furthermore, we use the notation $A \lesssim B$ for quantities A and B if there exists a constant $C > 0$ such that $A \leq CB$, where C may depend on the domain, the parameters, and the sequence of approximation spaces $(X_n)_{n \in \mathbb{N}}$, but not on the index n and functions involved in A and B. Note that the constant may change at each occurrence. Finally, we will usually use lowercase letters, e.g., a or a_n, for sesquilinear forms and uppercase letters, e.g. A or A_n for linear operators. We note that unless specified otherwise, the sesquilinear forms and linear operators are redefined in each chapter. For mesh elements $\tau \in \mathcal{T}_n$, we denote by $\mathcal{P}^k(\tau)$ and $\boldsymbol{\mathcal{P}}^k(\tau)$ the spaces of scalar and vectorial piecewise polynomials up to degree k on τ. Furthermore, we denote by $\mathbb{P}^k(\mathcal{T}_n) := \{v \in L^2 : v|_\tau \in \mathcal{P}^k(\tau)\ \forall \tau \in \mathcal{T}_n\}$ the space of piecewise polynomials up to degree k.

Part I

Theoretical framework

Chapter 1

Abstract well-posedness results

In this chapter, we review abstract results from functional analysis to prove the well-posedness of variational problems. We briefly discuss the classical and well-known notions of coercivity and inf-sup stability. Afterwards, we introduce and discuss the concept of (weak) T-coercivity. We expect the reader to be familiar with the concepts of operator theory and Fredholm operators discussed in the Appendix A. The first part of this chapter is partially based on [EG21b, Chap. 25].

Contents of the chapter

1.1 Preliminaries

In the following, let X and Y be complex Banach spaces with associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Furthermore, let $A \in L(X, Y')$ be a bounded linear operator and $f \in Y'$, where Y' denotes the dual space of Y. We want to study the existence and uniqueness of solutions to the following equation in operator form:

$$\text{Find } u \in X \text{ s.t. } Au = f \text{ in } Y'. \tag{1.1}$$

In practice, X and Y are usually Hilbert spaces and we are given a bounded sesquilinear form $a : X \times Y \to \mathbb{C}$ and a bounded antilinear form $f : Y \to \mathbb{C}$. Then, due to the Riesz isomorphism [EG21b, Thm. C.24], we can identify Y' with Y and further identify[1] the sesquilinear form as a bounded linear operator $A \in L(X, Y)$ via the relation

$$(Au, v)_Y = a(u, v) \text{ for all } u \in X, v \in Y. \tag{1.2}$$

Because of this identification, we will use the notation $a(\cdot, \cdot)$ for a sesquilinear form and $A \in L(X, Y)$ for the associated bounded linear operator interchangeably.
Therefore, studying the well-posedness of the problem

$$\text{Find } u \in X \text{ s.t. } a(u, v) = f(v) \text{ for all } v \in Y, \tag{1.3}$$

[1] We note that this identification is not only possible in the Hilbert case. If $a : X \times Y \to \mathbb{C}$ is a bounded sesquilinear form on Banach spaces X, Y, we can identify $a(\cdot, \cdot)$ with a bounded linear operator $A \in L(X, Y')$ via the duality pairing: $\langle Au, v\rangle_{Y',Y} := a(u, v)$. However, the Hilbert case is more convenient, since we can use the inner product $(\cdot, \cdot)_Y$ due to the Risz isomorphism.

amounts to studying the well-posedness of an operator equation of the form (1.1). As the dual space of a Banach space is Banach itself, we will write $A \in L(X, Y)$ and $f \in Y$ for ease of presentation. The following sections will summarize conditions to guarantee the bijectivity of the operator A and hence the well-posedness of problems (1.1) and (1.3).

1.2 Characterization of injective and bijective operators

First of all, we state two abstract results from functional analysis that serve as a basis for the well-posedness results in the following section.

Lemma 1.1. *Let $A \in L(X, Y)$. Then, the following statements are equivalent:*

(i) $A^* : Y' \to X'$ is surjective,

(ii) $A : X \to Y$ is injective and $\operatorname{ran}(A)$ is closed in Y,

(iii) There exists $\alpha > 0$ such that

$$\|Au\|_Y \geq \alpha \|u\|_X \qquad \forall u \in X.$$

Proof. We refer to [EG21b, Lem. C.39]. □

Theorem 1.2. *Let $A \in L(X, Y)$. Then, the following statements are equivalent:*

(i) $A : X \to Y$ is bijective,

(ii) A is injective, $\operatorname{ran}(A)$ is closed in Y and $A^* : Y' \to X'$ is injective,

(iii) A^* is injective and there exists $\alpha > 0$ such that

$$\|Au\|_Y \geq \alpha \|u\|_X \qquad \forall u \in X.$$

Proof. See [EG21b, Thm. C.49]. □

1.3 Classical well-posedness results

In this section, we will review classical well-posedness results. First, we introduce the concept of coercivity and state the Lax-Milgram Lemma, which can only be applied when test- and trialspaces are identical. Afterwards, we define inf-sup stability and state the Banach-Nečas-Babuška (BNB) Theorem, which can also be applied when test- and trialspaces are different and is, therefore, more general than the Lax-Milgram Lemma.

Definition 1.3 (Coercivity)**.** Let X be a Hilbert space. We call an operator $A \in L(X, X)$ *coercive*, if there exists a constant $\alpha > 0$ such that

$$|(Au, u)_X| \geq \alpha \|u\|_X^2 \quad \forall u \in X. \tag{1.4}$$

Equivalently[2], A is coercive if there exists a real number $\alpha > 0$ and $\xi \in \mathbb{C}$, $|\xi| = 1$, s.t.

$$\Re\big(\xi (Au, u)_X\big) \geq \alpha \|u\|_X^2 \quad \forall u \in X. \tag{1.5}$$

[2] For a proof of the equivalence, we refer to [EG21b, Lem. C.58] and references therein.

Lemma 1.4 (Lax-Milgram Lemma). *Let X be a Hilbert space and $A \in L(X, X)$ be a bounded linear operator. If A is coercive in the sense of Definition 1.3, then A is bijective.*

Proof. We follow the proof of [EG21b, Lem. 25.2]. Due to (1.5) and $\xi a(u, u) = a(u, \overline{\xi} u)$, we have that

$$\alpha \|u\|_X \leq \frac{\Re(a(u, \overline{\xi} u))}{\|u\|_X} \leq \sup_{v \in X \setminus \{0\}} \frac{\Re(a(u, \overline{\xi} v))}{\|v\|_X} \leq \sup_{v \in X \setminus \{0\}} \frac{|a(u, v)|}{\|v\|_X} = \|Au\|_X.$$

Therefore we conclude with Lemma 1.1 that A is injective and $\operatorname{ran} A$ is closed. If we show that A^* is injective as well, we can apply Theorem 1.2 to conclude the bijectivity of A. To this end, let $u \in X$ be such that $A^* u = 0$. Then, we have that $0 = \overline{0} = \overline{(A^* u, \xi u)_X} = \xi a(u, u)$. But (1.5) implies that $\alpha \|u\|_X^2 \leq \Re(\xi a(u, u)) = 0$ and hence $u = 0$. Thus, A^* is injective and therefore A is bijective. $\square$

Remark 1.5 (Coercivity and Hilbert space structure). *The notion of coercivity is intimately tied to the Hilbert space structure of the underlying space. It can be shown that a Banach space can be equipped with a Hilbert space structure with the same topology if and only if there exists a coercive operator on the space [EG21b, Prop. C.59]. Consequently, the Lax-Milgram Lemma is only applicable on Hilbert spaces.*

In the following, we assume once more that X and Y are complex Banach spaces. In addition, we require Y to be *reflexive*, i.e. the canonical isometry $Y \to Y''$ is an isomorphism, cf. [EG21b, Def. C.18]. Note that every Hilbert space is reflexive.

Definition 1.6 (Inf-sup condition). The operator $A \in L(X, Y)$ fulfills the *inf-sup condition*, if there exists a constant $\beta > 0$ such that

$$\inf_{u \in X \setminus \{0\}} \sup_{v \in Y \setminus \{0\}} \frac{|\langle Au, v \rangle_{Y', Y}|}{\|u\|_X \|v\|_Y} \geq \beta > 0. \tag{1.6}$$

Theorem 1.7 (BNB-theorem). *Let X and Y be Banach spaces and let Y be reflexive. Further, let $A \in L(X, Y)$ be a bounded linear operator. If*

(i) A fulfills the inf-sup condition (1.6),

(ii) $\forall v \in Y$, $[\forall u \in X, \langle Au, v \rangle_{Y', Y} = 0] \Rightarrow [v = 0]$,

then the operator A is bijective.

Proof. By Lemma 1.1, the inf-sup conditions (1.6) is equivalent to A being injective and $\operatorname{ran}(A)$ being closed. Thus, it suffices to show that A^* is injective by Theorem 1.2, which indeed follows from (ii). For more details, we refer to [EG21b, Thm. 25.9]. $\square$

Remark 1.8. *The BNB Theorem is more general than the Lax-Milgram Lemma since it can be applied on Banach spaces and when test- and trialspaces are different. If test- and trialspaces coincide and X is a Hilbert space, then coercivity implies the inf-sup condition. The converse is not true in general. Thus, coercivity is only a sufficient criterion for well-posedness, while the BNB-conditions are necessary and sufficient.*

Remark 1.9 (Discrete coercivity and discrete inf-sup stability). *For conforming discretizations, i.e. $X_h \subset X$, $Y_h \subset Y$ and $A_h := A|_{X_h \times X_h}$, the coercivity property (1.4) is inherited onto the discrete level. In contrast, the inf-sup condition (1.6) is not directly inherited onto the discrete* *level. To conclude the stability of the discrete problem, one has to show*[3] *that the sesquilinear form* $A_h : X_h \times Y_h \to \mathbb{C}$ *fulfills the uniform discrete inf-sup condition*

$$\inf_{u_h \in X_h} \sup_{v_h \in Y_h} \frac{|(A_h u_h, v_h)_Y|}{\|u_h\|_X \|v_h\|_Y} \geq \beta > 0,$$

where $\beta > 0$ *is a constant independent of* h*. It can be shown that the continuous inf-sup condition is inherited onto the discrete level if and only if there exists an operator* $\Pi_h : Y \to Y_h$ *such that* $(Au_h, \Pi_h v - v)_Y = 0$ *for all* $(u_n, v) \in X_h \times Y$ *and* $\|\Pi_h v\|_Y \lesssim \|v\|_Y$ *for all* $v \in Y$*, cf. [EG21b, Lem. 26.9]. The operator* Π_h *is called Fortin-operator.*

Remark 1.10 (Inf-sup stable spaces for the divergence operator and the de Rham complex)**.** *The construction of finite element spaces that fulfill the discrete inf-sup condition for the divergence operator deserves some attention here, in particular since the construction of stable discretizations for Galbrun's equation is related to the construction for the Stokes problem, cf. [Ale+22; HLS22]. A popular approach for constructing inf-sup stable finite element pairs is the de Rham complex; c.f., for example, [Joh+17; BBF13]. On a simply connected domain* $\Omega \subset \mathbb{R}^2$*, the following sequence is exact*

$$\mathbb{R} \longrightarrow H^1(\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0,$$

i.e., the range of each operator is the kernel of the subsequent one. If we construct finite element spaces[4] $Y_h \subset H^1(\Omega)$*,* $\boldsymbol{W}_h \subset \boldsymbol{H}(\text{div}, \Omega)$*, and* $Q_h \subset L^2(\Omega)$ *such that the following subcomplex*

$$\mathbb{R} \longrightarrow Y_h \xrightarrow{\text{curl}} \boldsymbol{W}_h \xrightarrow{\text{div}} Q_h \longrightarrow 0$$

forms an exact sequence, then the discrete inf-sup condition for the divergence operator is automatically fulfilled for $\boldsymbol{W}_h \times Q_h$ *[Joh+17, Sec. 4.3]. Classical choices for* $H(\text{div})$*-conforming subspaces are Raviart-Thomas or Brezzi-Douglas-Marini elements [BBF13, Sec. 2.3], which together with* Q_h *being the space of discontinuous piecewise polynomials of degree* k*, or* $k-1$*, respectively, form inf-sup stable pairs for the divergence oeperator.*

1.4 T-coercivity and weak T-coercivity

In this section, we review the concept of T-coercivity. In a sense, T-coercivity can be interpreted as a notion of coercivity when the test- and trialspaces do not coincide. The term T-coercivity was introduced in [BCZ10], but the concept goes back at least to [BCS02]. So far, the concept has been applied mainly to problems involving compact perturbations of bijective operators, for example, the Helmholtz equation [Cia12], and to problems involving sign-changing coefficients [BCC18; BCC14; Hal21]. From now on, we assume that X and Y are Hilbert spaces, which means, in particular, that the respective dual spaces are isomorphic to the respective spaces themselves.

Definition 1.11 (T-coercivity)**.** A bounded linear operator $A \in L(X, Y)$ is called T*-coercive*, if there exists a bijective operator $T \in L(X, Y)$ such that $T^*A \in L(X, X)$ is coercive, that is there exists $\alpha > 0$ s.t.

$$|(T^*Au, u)_X| = |(Au, Tu)_Y| \geq \alpha \|u\|_X^2 \quad \forall u \in X.$$

[3]Additionally, we usually require that $\dim(X_h) = \dim(Y_h)$, which is equivalent to the second condition in the BNB Theorem.

[4]we emphasize again that we are in the two-dimensional case here.

Remark 1.12 (Left and right T-coercivity)**.** *To be precise, we call the above notion of T-coercivity left T-coercivity. In contrast, we say that an operator $A \in L(X, Y)$ is right T-coercive, if there exists a bijective operator $T \in L(Y, X)$ such that $AT \in L(Y, Y)$ is coercive, that is there exists $\alpha > 0$ s.t.*

$$|(ATv, v)_Y| \geq \alpha \|v\|_Y^2 \quad \forall v \in Y.$$

We note that right T-coercivity can be useful to use because it avoids the adjoint operator T^*. Both concepts are visualized in Fig. 1.1.

$$\begin{tikzcd} \mathbf{X} \arrow[loop left, "T^*A"] \arrow[r, "A"] \arrow[r, bend left, "T"] & Y \arrow[l, bend left, "T^*"] \end{tikzcd}$$

(a) Left T-coercivity

$$\begin{tikzcd} X \arrow[r, "A"] & \mathbf{Y} \arrow[loop right, "AT"] \arrow[l, bend left, "T"] \end{tikzcd}$$

(b) Right T-coercivity

Figure 1.1: Illustration of left and right T-coercivity.

We note that the notion of T-coercivity also requires a Hilbert space structure, as is the case for coercivity, cf. Remark 1.5. If we are working on Hilbert spaces, the following Lemma shows that T-coercivity is indeed equivalent to the BNB-conditions from Thm. 1.7. Thus, it suffices to show T-coercivity to conclude the well-posedness of the problem.

Lemma 1.13. *Let X and Y be Hilbert spaces and $A \in L(X, Y)$ be a bounded linear operator. Then, A is T-coercive if and only if A fulfills the BNB-conditions from Thm. 1.7.*

Proof. We follow [EG21b, Exercise 25.10]. Suppose there exists a bijective operator $T \in L(X, Y)$ such that A is T-coercive. Then, we have that

$$\alpha \|u\|_X^2 \leq \frac{|(Au, Tu)_Y|}{\|Tu\|_Y} \|Tu\|_Y \leq \sup_{v \in Y \setminus \{0\}} \frac{|(Au, v)_Y|}{\|v\|_Y} \|T\|_{L(X,Y)} \|u\|_X.$$

Thus, (1.6) is fulfilled. To consider the second BNB-condition, let $v \in Y$ be such that $(Au, v)_Y = 0$ for all $u \in X$. As T is bijective, there exists $\tilde{u} \in X$ such that $T\tilde{u} = v$. Therefore, we have that

$$0 = (A\tilde{u}, v)_Y = (A\tilde{u}, T\tilde{u})_Y \geq \alpha \|\tilde{u}\|_X^2.$$

Hence, it holds that $\tilde{u} = 0$ and by bijective $v = T\tilde{u} = 0$. Conversely, if the BNB-conditions hold true, then $A \in L(X, Y)$ is an isomorphism and $A^{-1} \in L(Y, X)$ exists. Then, we set $T := J_Y^{-1}(A^{-1})^* J_X^{\mathrm{RF}}$, where $J_Y : Y \to Y''$ is the canonical isomorphism from Y to Y'' and $J_X^{\mathrm{RF}} : X \to X'$ is the Riesz-Fréchet isomorphism. Then $T \in L(X, Y)$ is an isomorphism and for all $u \in X$, we have that

$$(Au, Tu)_X = \overline{\langle (A^{-1})^*(J_X^{\mathrm{RF}}(u)), Au \rangle_{Y'', Y}} = \overline{\langle J_X^{\mathrm{RF}}(u), u \rangle_{X', X}} = \|u\|_X^2.$$

Thus, A is T-coercive. □

We can weaken the notion of T-coercivity by introducing compact perturbations. This leads to the following definition.

Definition 1.14 (weak T-coercivity)**.** Let $A \in L(X, Y)$ be a bounded linear operator and $T \in L(X, Y)$ be bijective. The operator A is called *weakly T-coercive*, if there exists a compact operator $K \in L(X, X)$ such that $T^*A + K$ is coercive.

Remark 1.15 (Discrete (weak) T-coercivity)**.** *As is the case for inf-sup stability, cf. Remark 1.9, the property of (weak) T-coercivity is in general not inherited onto the discrete level. As for T-coercivity, we can show that discrete T-coercivity is equivalent to the discrete inf-sup condition [Cia12, Thm. 2]. We recall from Remark 1.9 that the inf-sup condition is inherited onto the discrete level if and only if there exists a Fortin operator. Naturally, we have a similar relationship for T-coercivity. For instance, in the case of conforming discretizations, we can construct a discrete T_n-operator through $T_n = \Pi_n T$ such that discrete T_n-coercivity holds if a Fortin operator $\Pi_n : Y \to Y_n$ exists. This relationship is explored in more detail in a recent preprint by Barré and Ciarlet [BC22], where the authors showed that for the mixed Stokes problem, T-coercivity is inherited onto the discrete level if a Fortin operator exits.*

We note that in the spirit of Remark 1.12, we also speak of left and right weak T-coercivity. The following Lemma shows that the notion of weak T-coercivity is useful to prove well-posedness since it implies that the operator A is Fredholm with index zero. Thus, to show the bijectivity of A, it suffices to show injectivity.

Lemma 1.16. *Let $A \in L(X,Y)$ be a bounded linear operator and $T \in L(X,Y)$ be bijective. If A is weakly T-coercive, then A is Fredholm with index zero.*

Proof. By the Lax-Milgram Lemma 1.4, the operator $T^*A + K$ is bijective and therefore Fredholm with index zero. Consequently, T^*A is Fredholm with $\operatorname{ind} T^*A = \operatorname{ind}(T^*A + K) = 0$ by thm. A.23. Since T is bijective, T^{-*} is Fredholm with index zero and thus thm. A.22 yields that $A = T^{-*}T^*A$ is Fredholm with index zero as a composition of Fredholm operators with index zero. □

Corollary 1.17. *Let $A \in L(X,Y)$ be a bounded linear operator. If A is weakly T-coercive and injective, then A is bijective.*

Proof. By the previous Lemma, A is a Fredholm operator with index zero. Since A is injective, it follows by Lemma A.26 that A is bijective. □

CHAPTER 2

Discrete Approximation Schemes

In this chapter, we will introduce an abstract framework for analyzing approximations of operator equations in Banach spaces. These types of problems have been studied intensively by, for example, Anselone [AM64; AT85], Grigorieff [Gri69], Jeggle [Jeg72], Karma [Kar96a; Kar96b], Petryshyn [Pet93], Stummel [Stu70; Stu71] and Vainikko [Vai76]. We will also explore the connection with T-coercivity via the T-compatibility criterion introduced by Halla [Hal21] and review a weaker notion of T-compatibility [HLS22], which we will apply to Galbrun's equation in Part II of this thesis; see also [HLS22; Hal23]. In particular, we will follow the roadmap provided in Section 2.5 to show the well-posedness of the equation and the stability of the approximation. This chapter mainly follows the presentation of Vainikko [Vai76]. In order to be able to draw from a variety of results, we will introduce the concepts in a general setting and restrict the framework afterwards to study approximations of PDE problems. On occasion, we also draw inspiration from [Zei90b, Chap. 34].

Contents of the chapter

2.1 Introduction

Let X and Y be Banach spaces, $A \in L(X, Y)$ be a bounded linear operator and $f \in Y$. In the previous chapter, we considered the unique solvability of the operator equation

$$\text{Find } u \in X \text{ s.t. } Au = f \text{ in } Y. \tag{2.1}$$

Now, we want to study approximations of (2.1) by sequences of Banach spaces $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ and operators $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n, Y_n)$. Therefore, for $f_n \in Y_n$, we consider the discrete problems

$$\text{Find } u_n \in X_n \text{ s.t. } A_n u_n = f_n \text{ in } Y_n, \quad n = 1, 2, \ldots \tag{2.2}$$

In particular, we want to answer the following question:

Question 2.1. *What conditions on X, Y, X_n, Y_n, A, A_n, f and f_n are sufficient such that if the continuous problem (2.1) has a unique solution $u \in X$, then the discrete problems (2.2) have unique solutions u_n which converge to u?*

To this end, we first have to introduce a notion of convergence between elements from X and X_n, since we do not require the spaces X_n to be subspaces of X. Instead, we assume that there exists a system of operators $P = (p_n)_{n\in\mathbb{N}}$, $p_n : X \to X_n$, such that

$$\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X, \tag{2.3}$$

$$\lim_{n\to\infty} \|p_n(\alpha u - \beta v) - (\alpha p_n u + \beta p_n v)\|_{X_n} = 0, \tag{2.4}$$

for all $\alpha, \beta \in \mathbb{K}$ and for all $u, v \in X$. We call the second condition (2.4) *asymptotic linearity*. For ease of presentation, we will restrict ourselves to the case where $p_n \in L(X, X_n)$, which trivially satisfies condition (2.4). The setup is visualized in Fig. 2.1.

$$\begin{tikzcd}
& (X, \|\cdot\|_X) \arrow[dl, "p_{n_1}"'] \arrow[dr, "p_{n_m}"] \arrow[drr, "\dots"] & & \\
(X_{n_1}, \|\cdot\|_{X_{n_1}}) & \dots & (X_{n_m}, \|\cdot\|_{X_{n_m}}) & \dots
\end{tikzcd}$$

Figure 2.1: Set up.

Remark 2.1 (Boundedness of p_n)**.** *If $P = (p_n)_{n\in\mathbb{N}}$, $p_n \in L(X, X_n)$ is a system of linear bounded operators such that* (2.3) *holds true, then the operator norm of p_n is bounded uniformly as*

$$\|p_n\|_{L(X,X_n)} = \sup_{u\in X\setminus\{0\}} \frac{\|p_n u\|_{X_n}}{\|u\|_X} \stackrel{n\to\infty}{\longrightarrow} 1.$$

In this setting, we can define the following notions of convergence and compactness.

Definition 2.2 (P-convergence)**.** We call a sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in X_n$ *P-convergent* to $u \in X$ and write $u_n \xrightarrow{P} u$ if

$$\|p_n u - u_n\|_{X_n} \to 0 \text{ as } n \to \infty. \tag{2.5}$$

Note that in the following, we will often simply speak of convergence, if the context is clear.

Remark 2.3. *The notion of P-convergence fulfills the usual properties of limits. Straightforward calculations yield*

- *For all $u \in X$, it holds that $p_n u \xrightarrow{P} u$.*
- *If $u_n \xrightarrow{P} u$ and $u_n \xrightarrow{P} u'$, then $u = u'$ (Uniqueness).*
- *If $u_n \xrightarrow{P} u$ and $v_n \xrightarrow{P} v$, $a, b \in \mathbb{C}$, then $au_n + bv_n \xrightarrow{P} au + bv$ (Linearity).*
- *If $u_n \xrightarrow{P} u$, then $\|u_n\|_{X_n} \stackrel{n\to\infty}{\longrightarrow} \|u\|_X$.*
- *$u_n \xrightarrow{P} 0$ if and only if $\|u_n\|_{X_n} \stackrel{n\to\infty}{\longrightarrow} 0$.*

Recall that we call a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ *sequentially compact* if for every sequence $(u_n)_{n\in\mathbb{N}}$ there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $u \in X$ such that $u_n \to u$. In the following definition, we will adapt this concept to fit our notion of P-convergence.

Definition 2.4 (P-compactness)**.** A sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in X_n$ is called *P-compact*, if for every subsequence $\mathbb{N}' \subset \mathbb{N}$ there exists a subsubsequence $\mathbb{N}'' \subset \mathbb{N}'$ and $u \in X$ such that $u_n \xrightarrow{P} u$, $n \in \mathbb{N}''$.

Remark 2.5 (Generalization to normed spaces). *We follow the assumption from Vainikko [Vai76] that X and $(X_n)_{n\in\mathbb{N}}$ are Banach spaces. Stummel [Stu70] derives the setting on normed spaces, that is, without the assumption of completeness. While this setting is more general than what we consider here, the definitions are more technical due to the fact that we have to consider representation classes of sequences.*

2.2 Discrete convergence of linear operators

To answer the motivating Question 2.1, we are interested in the convergence of a sequence of linear bounded operators $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n, Y_n)$, towards an operator $A \in L(X, Y)$, where X,Y are Banach spaces and $(X_n)_{n\in\mathbb{N}}$,$(Y_n)_{n\in\mathbb{N}}$ are sequences of Banach spaces. As before, we do not require the spaces X_n and Y_n to be subspaces of X and Y. In view of the previously introduced notion of P-convergence, we assume that there exist sequences $P = (p_n)_{n\in\mathbb{N}}$, $p_n \in L(X, X_n)$ and $Q = (q_n)_{n\in\mathbb{N}}$, $q_n \in L(Y, Y_n)$ such that

$$\|p_n u\|_{X_n} \to \|u\|_X \text{ as } n \to \infty, \tag{2.6a}$$
$$\|q_n v\|_{Y_n} \to \|v\|_Y \text{ as } n \to \infty. \tag{2.6b}$$

The diagram in Fig. 2.2 visualizes these relationships.

$$\begin{tikzcd}
(X, \|\cdot\|_X) \arrow[r, "A"] \arrow[d, "p_n"] & (Y, \|\cdot\|_Y) \arrow[d, "q_n"] \\
(X_n, \|\cdot\|_{X_n}) \arrow[r, "A_n"] & (Y_n, \|\cdot\|_{Y_n})
\end{tikzcd}$$

Figure 2.2: Set up, see [Vai76, p. 27]

The following definition allows us to speak of the convergence of a sequence of operators $A_n \in L(X_n, Y_n)$ to an operator $A \in L(X, Y)$.

Definition 2.6 (PQ-convergence). We call a sequence of operators $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n, Y_n)$ *PQ-convergent* to $A \in L(X, Y)$ and write $A_n \xrightarrow{PQ} A$, if for every P-convergent sequence $(u_n)_{n\in\mathbb{N}}$ we have that

$$u_n \xrightarrow{P} u \Rightarrow A_n u_n \xrightarrow{Q} Au, \tag{2.7}$$

which means that

$$\|p_n u - u\|_{X_n} \to 0 \Rightarrow \|q_n Au - A_n u_n\|_{Y_n} \to 0 \text{ as } n \to \infty.$$

In the case that the spaces X and Y, as well as the spaces X_n and Y_n, coincide, we simply write $A_n \xrightarrow{P} A$. Throughout this thesis, we will often speak of a *discrete approximation scheme*. To be precise, we make the following definition.

Definition 2.7 (Discrete approximation scheme). Let X be a Banach space and $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Furthermore, let $A \in L(X)$ be a bounded linear operator and $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, be a sequence of bounded linear operators. We call (X_n, p_n, A_n) a *discrete approximation scheme (DAS)* of (X, A), if there exists a sequence of bounded linear operators $P = (p_n)_{n\in\mathbb{N}}$ such that (2.3) is fulfilled and $A_n \xrightarrow{P} A$.

Lemma 2.8. *The following statements are equivalent:*

(i) $A_n \xrightarrow{PQ} A$.

(ii) *The exists $C > 0$ such that $\|A_n\|_{L(X_n,Y_n)} \le C$ for all $n \in \mathbb{N}$ and for all $u \in X$*

$$\|A_n p_n u - q_n A u\|_{Y_n} \to 0. \tag{2.8}$$

(iii) *The exists $C > 0$ such that $\|A_n\|_{L(X_n,Y_n)} \le C$ for all $n \in \mathbb{N}$ and a dense subset $X' \subset X$ such that for all $u' \in X'$*

$$\|A_n p_n u' - q_n A u'\|_{Y_n} \to 0. \tag{2.9}$$

Proof. We follow the argumentation of [Vai76, Chap. 2, Thm. (8)].

$(i) \Rightarrow (ii)$: Assume that $A_n \overset{PQ}{\to} A$. Suppose theres does not exists a constant $C > 0$ such that $\|A_n\|_{L(X_n,Y_n)} \le C$ for all $n \in \mathbb{N}$, i.e. $\|A_n\|_{L(X_n,Y_n)} \to \infty$ as $n \to \infty$. Then there exists $u_n' \in X_n$ with $\|u_n'\|_{X_n} = 1$ such that $\|A_n u_n'\|_{Y_n} \to \infty$. Set $u_n := u_n'/\|A_n u_n'\|_{Y_n}$. Then $u_n \overset{P}{\to} 0$, but since $\|A_n u_n\|_{Y_n} = 1$ by definition, $A_n u_n \overset{Q}{\not\to} A(0) = 0$. This contradicts (i).
The second assertion follows, since $p_n u \overset{P}{\to} u$ for all $u \in X$, which implies $A_n p_n u \overset{Q}{\to} Au$ by (i).

$(ii) \Rightarrow (iii)$: This direction follows immediately.

$(iii) \Rightarrow (i)$: Let $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, be given such that $u_n \overset{P}{\to} u$ for some $u \in X$ and assume that (iii) holds true. We have to show that $A_n u_n \overset{Q}{\to} Au$. Let $\epsilon > 0$ and choose $u' \in X'$ such that $\|u' - u\|_X < \epsilon$. Then, we have that

$$\begin{aligned}\|A_n u_n - q_n A u\|_{Y_n} \le \underbrace{\|A_n\|_{L(X_n,Y_n)}}_{\le C} \underbrace{\|u_n - p_n u\|_{X_n}}_{\to 0} + \underbrace{\|A_n\|_{L(X_n,Y_n)}}_{\le C} \underbrace{\|p_n u - p_n u'\|_{X_n}}_{\to \|u-u'\|_X < \epsilon} \\ + \underbrace{\|A_n p_n u' - q_n A u'\|_{Y_n}}_{\to 0} + \underbrace{\|q_n A u' - q_n A u\|_{Y_n}}_{\to \|Au' - Au\|_Y < \|A\|_{L(X,Y)} \cdot \epsilon}.\end{aligned}$$

Thus we have that

$$\lim_{n\to\infty} \|A_n u_n - q_n A u\|_{Y_n} \le (C + \|A\|_{L(X,Y)})\epsilon,$$

and since $A \in L(X,Y)$ is bounded and $\epsilon > 0$ was chosen arbitrary, it follows that $A_n u_n \overset{Q}{\to} Au$. Thus, we have that $A_n \overset{PQ}{\to} A$ by definition. □

Remark 2.9 (On different terminologies)**.** *The notion of discrete convergence of linear operators $A_n \overset{P}{\to} A$ follows the work of Stummel [Stu70]. In the literature, one also finds the terminology A_n approximates A, which is defined to be the case if condition (ii) of Lemma 2.8 is fulfilled:*

$$\|A_n p_n u - q_n A u\|_{Y_n} \to 0 \text{ as } n \to \infty.$$

Thus, both terminologies are equivalent if $\|A_n\|_L(X_n, Y_n) \le C$, which is satisfied when X_n and Y_n are finite dimensional. We also note that in the context of finite element methods, we would rather call the above property asymptotic consistency.

In the following, we will define the notions of *stable* and *regular* convergence. For this, we assume that $A_n \overset{PQ}{\to} A$ for an operator $A \in L(X,Y)$.

Definition 2.10. The sequence $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n, Y_n)$ is called *stable*, if there exist constants $C, n_0 > 0$ such that the inverse operators $A_n^{-1} \in L(Y_n, X_n)$ exist and $\|A_n^{-1}\|_{L(Y_n,X_n)} \le C$ for all $n \ge n_0$.

Definition 2.11. The sequence $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n, Y_n)$ is called *regular*, if $\|u_n\|_{X_n} \le C$ and the Q-compactness of $(A_n u_n)_{n\in\mathbb{N}}$ imply that $(u_n)_{n\in\mathbb{N}}$ is P-compact.

The notion of regularity is more general than stability. For instance, we will show in Lemma 2.13 that for a bijective operator A, the regularity of approximations $(A_n)_{n\in\mathbb{N}}$ implies their stability. However, we might also want to consider a non-bijective operator A, for example, if we are approximating eigenvalue problems [Hal21]. The following lemma shows that the stability of a sequence $(A_n)_{n\in\mathbb{N}}$ which approximates an operator A, immediately implies the injectivity of A. This means, in particular, that an approximation of an operator A that is not injective cannot be stable. As such, the notion of stability is not meaningful when considering non-bijective operators, while the notion of regularity still is.

Lemma 2.12. *If $A_n \overset{PQ}{\to} A$ and $(A_n)_{n\in\mathbb{N}}$ is stable, then there exits $\gamma > 0$ such that*

$$\|Au\|_Y \geq \gamma \|u\|_X \qquad \forall u \in X. \tag{2.10}$$

In particular, this means that A is semifredholm with $\ker(A) = \{0\}$ and ind $A \leq 0$.

Proof. We follow the argumentation of [Vai76, Chap. 2, Thm. (14)].
Since $p_n u \overset{P}{\to} u$ and $A_n p_n u \overset{Q}{\to} Au$, we have that

$$\|Au\|_Y = \lim_{n\to\infty} \|A_n p_n u\|_{Y_n} \geq \overline{\lim_{n\to\infty}} \frac{1}{\|A_n^{-1}\|_{L(Y_n,X_n)}} \|p_n u\|_{X_n} \geq \gamma \lim_{n\to\infty} \|p_n u\|_{X_n} = \gamma \|u\|_X, \tag{2.11}$$

where $\gamma = \overline{\lim}_{n\to\infty} \frac{1}{\|A_n^{-1}\|_{L(Y_n,X_n)}} > 0$ since $(A_n)_{n\in\mathbb{N}}$ is stable. Thus, by Lemma 1.1 it follows that $\ker(A) = \{0\}$ and $\operatorname{ran}(A)$ is closed. Hence, A is semifredholm and $\operatorname{ind}(A) = -\dim \operatorname{coker}(A) \leq 0$. □

The following lemma shows the relationship between regularity and stability and the injectivity and surjectivity of the limit operator A. In particular, we will conclude that if the sequence $(A_n)_{n\in\mathbb{N}}$ is both stable and regular, then A is bijective. In other words, this means that if we find a discretization such that $(A_n)_{n\in\mathbb{N}}$ converges stable and regular to A, then the continuous problem (2.1) is already well-posed.

Lemma 2.13. *The following statements are equivalent:*

(i) $\operatorname{ran}(A) = Y$, $A_n \overset{PQ}{\to} A$, $(A_n)_{n\in\mathbb{N}}$ *stable;*

(ii) $\ker(A) = \{0\}$, $A_n \overset{PQ}{\to} A$, $(A_n)_{n\in\mathbb{N}}$ *regular, A_n are Fredholm with index 0 ($n \geq n_0$);*

(iii) $A_n \overset{PQ}{\to} A$, $(A_n)_{n\in\mathbb{N}}$ *is stable and regular.*

Proof. We adapt the argumentation from [Vai76, Chap. 2, Thm. (60)].

$(i) \Rightarrow (ii)$: Lemma 2.12 implies that $\ker(A) = \{0\}$ and since $A_n^{-1} \in L(Y_n, X_n)$ exist, the operators A_n are Fredholm with index zero. It is left to show that $(A_n)_{n\in\mathbb{N}}$ is regular. Let $(u_n)_{n\in\mathbb{N}}$ be given such that $\|u_n\|_{X_n} \leq C$ and $(A_n u_n)_{n\in\mathbb{N}}$ is Q-compact, i.e. there exists a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $A_n u_n \overset{Q}{\to} y$ for some $y \in Y$. We have to show that $(u_n)_{n\in\mathbb{N}}$ is P-compact. As $\operatorname{ran}(A) = Y$ and $\ker(A) = \{0\}$, $A^{-1} \in L(Y, X)$ exists and we can set $u := A^{-1}y$. We calculate

$$\begin{aligned}
\|u_n - p_n u\|_{X_n} &\leq \|A_n^{-1}\|_{L(Y_n,X_n)} \|A_n u_n - q_n y\|_{Y_n} + \|A_n^{-1}\|_{L(Y_n,X_n)} \|A_n u_n - q_n y\|_{Y_n} \\
&\leq C\big(\|A_n u_n - q_n y\|_{Y_n} + \|A_n u_n - q_n y\|_{Y_n}\big) \\
&\overset{n\in\mathbb{N}'}{\longrightarrow} 0,
\end{aligned}$$

since $A_n u_n \overset{Q}{\to} y$, $A_n \overset{PQ}{\to} A$. Thus, we have that $u_n \overset{P}{\to} u$, $n \in \mathbb{N}'$ and conclude that $(A_n)_{n\in\mathbb{N}}$ is regular.

$(ii) \Rightarrow (iii)$: Since A_n are Fredholm with index 0, it suffices to show that $\ker(A_n) = \{0\}$ for all $n \in \mathbb{N}$ since injectivity implies bijectivity. Suppose the operators A_n are not injective. Then, we can find a sequence $(u_n)_{n\in\mathbb{N}}$ with $\|u_n\|_{X_n} = 1$ and $A_n u_n \overset{Q}{\to} 0$. Since $(A_n)_{n\in\mathbb{N}}$ is regular, $(u_n)_{n\in\mathbb{N}}$ is P-compact, i.e. there exists a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $u_n \overset{P}{\to} u$ for some $u \in X$ with $\|u\|_X = 1$. However, since $A_n \overset{PQ}{\to} A$ we have that $A_n u_n \overset{Q}{\to} Au$ and hence $Au = 0$ which contradicts the assumption that $\ker(A) = \{0\}$.

$(iii) \Rightarrow (i)$: Let $y \in Y$ be arbitrary. We have to show that $y \in \operatorname{ran}(A)$. Choose a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \overset{Q}{\to} y$. With $u_n := A_n^{-1} y_n$, we have that $\|u_n\|_{X_n} \leq C$ and $A_n x_n \overset{Q}{\to} y$. Since $(A_n)_{n\in\mathbb{N}}$ is regular, $(u_n)_{n\in\mathbb{N}}$ is P-compact and hence there exists a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $u_n \overset{P}{\to} u$ for some $u \in X$. Thus $A_n u_n \overset{Q}{\to} Au$ and by uniqueness of limits, we conclude that $Au = y$. □

Corollary 2.14. *If $A_n \overset{PQ}{\to} A$ stable and regular, than there exists $A^{-1} \in L(Y, X)$.*

Proof. By the previous Lemma 2.13, A is bijective and hence invertible. □

Recall from Definition A.3 that we call an operator $A \in L(X, Y)$ *compact*, if it maps bounded sets in X to precompact sets in Y. Below, we will define what it means for the $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n, Y_n)$, to be compact and show that this already implies the compactness of the limit operator $A \in L(X, Y)$ under mild conditions.

Definition 2.15 (Compactness)**.** The sequence $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n, Y_n)$ is called *compact*, if for every bounded sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, $\|u_n\|_{X_n} \leq C$, the sequence $(A_n u_n)_{n\in\mathbb{N}}$ is Q-compact.

Theorem 2.16. *Let $A_n \overset{PQ}{\to} A$ and $(A_n)_{n\in\mathbb{N}}$ be compact. If Y is separable, then the operator A is compact.*

Proof. We refer to [Vai76, Chap. 2, Thm. (49) and (26)]. □

At this point, we return to the motivating Question 2.1. Recall that we want to identify conditions on X, Y, X_n, Y_n, A, A_n, f and f_n such that if the continuous problem (2.1) has a unique solution $u \in X$, the discrete problems (2.2) have unique solutions u_n and $(u_n)_{n\in\mathbb{N}}$ converges to u. As before, we assume that X and Y are Banach spaces and $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are sequences of Banach spaces such that there exist systems of operators $P = (p_n)_{n\in\mathbb{N}}$ and $Q = (Q_n)_{n\in\mathbb{N}}$ such that (2.6) holds true. Furthermore, we assume that $A \in L(X, Y)$ is a bounded linear operator and $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n, Y_n)$, is a sequence of bounded linear operators and that $f \in Y$, $f_n \in Y_n$ are such that $f_n \overset{P}{\to} f$. Then, we can formulate the following theorem to answer Question 2.1.

Theorem 2.17. *Suppose that $\ker(A) = \{0\}$ and that the sequence $(A_n)_{n\in\mathbb{N}}$ consists of Fredholm operators with index zero. If $A_n \overset{PQ}{\to} A$ and $(A_n)_{n\in\mathbb{N}}$ is regular, then there exists a unique solution $u \in X$ to (2.1). Furthermore, there exists $n_0 \in \mathbb{N}$ and constants $C_1, C_2 > 0$ such that there exists a unique solution $u_n \in X_n$ to (2.2) for all $n \geq n_0$ and $u_n \overset{P}{\to} u$. There holds the estimate*

$$C_1 \|A_n p_n u - f_n\|_{Y_n} \leq \|u_n - p_n u\|_{X_n} \leq C_2 \|A_n p_n u - f_n\|_{Y_n}. \tag{2.12}$$

Proof. We follow the argumentation of [Vai76, Chap. 3, Thm. (3)].
By Lemma 2.13, the inverse operator $A^{-1} \in L(Y, X)$ exists and for n sufficiently large, $A_n^{-1} \in L(Y_n, X_n)$ exist and $\|A_n^{-1}\|_{L(Y_n,X_n)} \leq C_2$ for some constant $C_2 > 0$. Thus, the problems (2.1) and (2.2) are uniquely solvable. Since $A_n \overset{PQ}{\to} A$, Lemma 2.8 implies that

$$\|A_n\|_{L(X_n,Y_n)} \leq \frac{1}{C_1} \tag{2.13}$$

for some constant $C_1 > 0$. Furthermore, as $A_n(u_n - p_n u) = f_n - A_n p_n u$, we have that

$$\begin{aligned} \|A_n p_n u - f_n\|_{Y_n} &= \|A_n(p_n u - u_n)\|_{Y_n} \leq \frac{1}{C_1}\|u_n - p_n u\|_{X_n}, \\ \|u_n - p_n u\|_{X_n} &= \|A_n^{-1} A_n(u_n - p_n u)\|_{X_n} \leq C_2\|A_n p_n u - f_n\|_{Y_n}. \end{aligned} \tag{2.14}$$

Finally, as $A_n \overset{PQ}{\to} A$, we have that $A_n p_n u \overset{Q}{\to} Au = f$. By assumption, we have that $f_n \overset{Q}{\to} f$ and therefore $A_n p_n u - f_n \overset{Q}{\to} 0$. Consequently, we have that

$$\|u_n - p_n u\|_{X_n} \leq C_2\|A_n p_n u - f_n\|_{Y_n} \to 0 \text{ as } n \to \infty, \tag{2.15}$$

and conclude that $u_n \overset{P}{\to} u$. □

Remark 2.18. *By means of Lemma 2.13, we can formulate the previous theorem analogously with the assumptions that* $\operatorname{ran}(A) = Y$ *and that* $(A_n)_{n\in\mathbb{N}}$ *is stable. However, we recall that regularity is a more general condition than stability, and showing stability is more difficult. Furthermore, showing the injectivity of the operator* A *is usually easier than showing its surjectivity. Thus, we prefer the previous formulation.*

2.3 Conforming Galerkin schemes

In this section, we want to show how the previously developed framework can be directly applied to conforming Galerkin schemes. Abstractly, a *Galerkin scheme* in a Banach space X is a sequence of nonzero finite dimensional subspaces $(X_n)_{n\in\mathbb{N}}$, $X_n \subset X$, such that

$$\lim_{n\to\infty} \inf_{u_n \in X_n} \|u - u_n\|_X = 0 \quad \text{for all } u \in X. \tag{2.16}$$

For more details, we refer to [Zei90a, Chap. 21.13]. In the context of Finite Element Spaces, the property (2.16) is usually called *approximability property*. In practice, one usually considers X to be a Hilbert space. Thus, we restrict the framework from the previous sections to this case and assume that $X = Y$. Our main goal is to approximate a linear bounded operator $A \in L(X)$ by a sequence of operators $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, where $(X_n)_{n\in\mathbb{N}}$ is a Galerkin scheme. This setting is visualized in the diagram in Fig. 2.3.

$$\begin{tikzcd} (X, (\cdot,\cdot)_X) \arrow[loop left, "A"] \arrow[r, "p_n"] & (X_n, (\cdot,\cdot)_{X_n}) \arrow[loop right, "A_n"] \end{tikzcd}$$

Figure 2.3: Setting for conforming Galerkin Approximations

Lemma 2.19. *Let* $p_n \in L(X, X_n)$ *be the orthogonal projection from* X *to* X_n*. Then, we have that*

$$\lim_{n\to\infty} \|p_n u\|_X = \|u\|_X \quad \textit{for all } u \in X.$$

Proof. Let $u \in X$ be arbitrary. We use the characterization of the orthogonal projection p_n as the best approximation of u [Zei90a, Prop. 21.44]:

$$\|u - p_n u\|_X = \inf_{v_n \in X_n} \|u - v_n\|_X.$$

Thus, the approximability property (2.16) yields $\lim_{n\to\infty} \|u - p_n u\|_X = 0$. $\square$

Corollary 2.20. *Let X be Hilbert, $A \in L(X)$ and $(X_n)_{n\in\mathbb{N}}$ be a conforming Galerkin scheme. Then, the sequence $(A_n)_{n\in\mathbb{N}}$, $A_n := p_n A|_{X_n} \in L(X_n)$, where $p_n \in L(X, X_n)$ is the orthogonal projection, fulfills $A_n \xrightarrow{P} A$.*

Proof. Since A_n and p_n are bounded operators, the statement immediately follows from the previous lemma, since we have for $n \to \infty$ that

$$\|A_n p_n u - p_n A u\|_X = \|p_n A p_n u - p_n A u\|_X \leq \|p_n\|_{L(X,X_n)} \|A\|_{L(X)} \|p_n u - u\|_X \to 0.$$

$\square$

The previous two statements show that the framework developed in the previous sections can be directly applied to conforming Galerkin schemes.

Corollary 2.21. *Let $(X_n)_{n\in\mathbb{N}}$, $X_n \subset X$, be a conforming Galerkin scheme such that (2.16) holds true. Further, let $A \in L(X)$ and $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, be defined through $A_n := p_n A|_{X_n}$. Then (X_n, p_n, A_n) is a discrete approximation scheme of (X, A).*

Example 2.1 (H^1-conforming Finite Element Space)**.** *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of shape regular triangulations of D. For $k \geq 1$, we define*

$$X_n := \{v \in L^2(D) : v|_\tau \in \mathcal{P}^k(\tau) \quad \forall \tau \in \mathcal{T}_n\} \cap H^1(D) \subset H^1(D).$$

Then X_n fulfills the approximability property (2.16) [EG21b, Sec. 26.3.3] and we can apply the previous corollary.

To summarize, the previous results show that whenever we are considering a conforming Galerkin scheme with finite-dimensional subspaces that fulfill the approximation property (2.16), then $(X_n, p_n, p_n A_n|_{X_n})$, where $p_n \in L(X, X_n)$ is the orthogonal projection onto X_n, is always a discrete approximation scheme of (X, A). Furthermore, we note that since the spaces X_n are finite-dimensional, the operators A_n are Fredholm with index zero, cf. Lemma A.21. Thus, we can apply the convergence theorem 2.17 derived in the previous section to show the following lemma.

Lemma 2.22. *Let X be Hilbert and $(X_n)_{n\in\mathbb{N}}$ be a conforming Galerkin scheme. Furthermore, let $p_n \in L(X, X_n)$ denote the orthogonal projection from X to X_n and let $A \in L(X)$ be injective. Define $A_n := p_n A_n|_{X_n}$ such that $A_n \in L(X_n)$. If $(A_n)_{n\in\mathbb{N}}$ is regular, then the continuous problem (2.1) has a unique solution $u \in X$ for all $f \in X$. Furthermore, with setting $f_n := p_n f \in X_n$, there exists an index $n_0 > 0$ such that the discrete problem (2.2) has a unqiue solution $u_n \in X_n$ for all $n > n_0$ and $u_n \xrightarrow{P} u$. There holds the estimate*

$$\|u - u_n\|_X \lesssim \inf_{v_n \in X_n} \|u - v_n\|_X. \tag{2.17}$$

Proof. From Thm. 2.17, the existence of a unique solution $u \in X$ to (2.1) and unique solutions $u_n \in X_n$ to (2.2) for $n > n_0$ such that $u_n \xrightarrow{P} u$ immediately follow. For the estimate, we consider as in [HLS22, Lem. 2]

$$\begin{aligned}
\|u - u_n\|_X &\leq \|u - p_n u\|_X + \|p_n u - u_n\|_X \leq \|u - p_n u\|_X + \|A_n^{-1}(A_n p_n u - A_n u_n)\|_X \\
&\leq \|u - p_n u\|_X + \sup_{m>n_0} \|A_m^{-1}\|_{L(X_m)} \|p_n A p_n u - p_n A u\|_X \\
&\leq \Big(1 + \sup_{m>n_0} \|A_m^{-1}\|_{L(X_m)} \|A\|_{L(X)}\Big) \|u - p_n u\|_X \\
&= C \inf_{v_n \in X_n} \|u - v_n\|_X .
\end{aligned}$$

In the third line, we use that $A_n u_n = p_n f = p_n A u$, in the fourth that $\|p_n\|_{L(X_n)} \leq 1$ and in the final line, we use the characterization of the orthogonal projection as the best approximation. □

Remark 2.23 (Convergence rates)**.** *When we consider conforming finite element approximations, the approximability property* (2.16) *usually follows from the existence of a suitable interpolation operator* $\mathcal{I} : X \to X_n$ *such that*

$$\|u - \mathcal{I}u\|_X \lesssim h^\alpha \|u\|_X ,$$

where $\alpha > 0$ *and* h *is the mesh size. In this case, Lemma 2.22 immediately yields yields a convergence rate of* $\mathcal{O}(h^\alpha)$*. For instance, if we consider* H^1*-conforming finite elements as in Example 2.1, we have that for all* $u \in X \cap H^{1+s}(D)$

$$\|u - \mathcal{I}u\|_{H^1(D)} \lesssim h^s \|u\|_{H^{1+s}(D)} .$$

Altogether, we conclude that the framework of discrete approximation schemes can be conveniently applied to conforming Galerkin approximations. We have shown that with $p_n \in L(X, X_n)$ being the orthogonal projection onto X_n, $(X_n, p_n, p_n A|_{X_n})$ always constitutes a discrete approximation scheme of (X, A). Furthermore, to conclude both the continuous and the discrete well-posedness, as well as the convergence of the discrete solution to the continuous one, we only have to show injectivity of A and regularity of $(A_n)_{n\in\mathbb{N}}$. Let us stress, however, that proving regularity is not a trivial task. In the next section, we will therefore specify conditions that allow us to infer regularity.

Remark 2.24 (Nonconforming schemes)**.** *For nonconforming schemes where* $X_n \not\subset X$*, the situation is more complicated since we cannot choose* p_n *to be the orthogonal projection. Consequently, the properties* (2.3) *and* $A_n \xrightarrow{P} A$ *do not immediately follow and have to be established individually. Nevertheless, the framework is still applicable, as we will demonstrate in sections 3.4.1, 5.2.2, or 6.2.*

2.4 T-compatibility

In this section, we review the concept of T-compatibility [Hal21], which connects discrete approximation schemes with the concept of weak T-coercivity. Recall from Section 1.4 that we call an operator $A \in L(X)$ T-coercive if there exists a bijective operator $T \in L(X)$ such that T^*A is coercive and weakly T-coercive if there exists a compact operator $K \in L(X)$ such that $T^*A + K$ is coercive.

2.4.1 Conforming Galerkin approximations

First of all, we consider conforming Galerkin approximations. Let $(X_n)_{n\in\mathbb{N}}$, $X_n \subset X$, be a sequence of closed subspaces of X and $p_n \in L(X, X_n)$ be the orthogonal projection from X to X_n. We define the following discrete norm on $L(X_n)$ that can be applied to both $T \in L(X)$ and $T \in L(X_n)$ through

$$\|T\|_n := \sup_{u_n \in X_n \setminus \{0\}} \frac{\|Tu_n\|_X}{\|u_n\|_X} = \|T\|_{L(X_n, X)} = \|TP_n\|_{L(X)},$$

In the case that the operator $A \in L(X)$ is T-coercive, the condition that $\|T - T_n\|_n \to 0$ implies that the sequence of Galerkin approximations $(A_n)_{n\in\mathbb{N}}$ is uniformly T_n-coercive as Ciarlet [Cia12, Corollary 1] showed with the following lemma.

Lemma 2.25. *Let $A \in L(X)$ be T-coercive and $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, be a sequence of conforming Galerkin approximations s.t. $X_n \subset X$. Assume that there exists a sequence of index zero Fredholm operators $T_n \in L(X_n)$ such that*

$$\lim_{n\to\infty} \|T - T_n\|_n = 0.$$

Then the sequence $(A_n)_{n\in\mathbb{N}}$ is uniformly T_n-coercive.

Proof. Adapted from [Cia12, Corollary 1]. Let $u_n \in X$. Then, we have that

$$\begin{aligned} |\langle A_n u_n, T_n u_n\rangle_X| &= |\langle A u_n, T_n u_n\rangle_X + \langle (A_n - A) u_n, T_n u_n\rangle_X| \\ &= |\langle A u_n, T u_n\rangle_X - \langle A u_n, (T - T_n) u_n\rangle_X + \langle (A_n - A) u_n, T_n u_n\rangle_X| \\ &\geq |\langle A u_n, T u_n\rangle_X| - |\langle A u_n, (T - T_n) u_n\rangle_X| - |\langle (A_n - A) u_n, T_n u_n\rangle_X| \\ &\geq (\alpha - \|A\|_{L(X)} \|T - T_n\|_n - \|A_n - A\|_{L(X)} \|T_n\|_{L(X_n)}) \|u_n\|_{X_n}^2, \end{aligned}$$

where $\alpha > 0$ is the T-coercivity constant of A. □

In the following, we define T-compatibility and show that T-compatible approximations of weakly T-coercive operators are regular, which enables us to apply Thm. 2.17.

Definition 2.26 (T-compatibility)**.** Let $A \in L(X)$ be weakly T-coercive. We call the sequence of Galerkin approximations $(A_n)_{n\in\mathbb{N}}$ *T-compatible* if $(A_n)_{n\in\mathbb{N}}$ is a sequence of index zero Fredholm operators and there exists a sequence of index zero Fredholm operators $(T_n)_{n\in\mathbb{N}}$, $T_n \in L(X_n)$ such that $\|T - T_n\|_n \to 0$ as $n \to \infty$.

Theorem 2.27. *Let $A \in L(X)$ be weakly T-coercive and $(A_n)_{n\in\mathbb{N}}$ be a T-compatible Galerkin approximation. Then $(A_n)_{n\in\mathbb{N}}$ is regular.*

Proof. We will briefly sketch the proof. For more details, we refer to the proof of [Hal21, Thm. 1]. First, we note that it can be shown that there exists a constant $\tilde{C} > 0$ such that $\|T_n\|_{L(X_n)}, \|T_n^{-1}\|_{L(X_n)} \leq \tilde{C}$. Now, let $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, be such that $\|u_n\|_{X_n} \leq C$ for some constant C independent of n and $(A_n u_n)_{n\in\mathbb{N}}$ is compact. For an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$, we choose $\mathbb{N}'' \subset \mathbb{N}'$ and $f \in X$ such that $\lim_{n\in\mathbb{N}''} \|A_n u_n - f\|_{X_n} = 0$. Let $K \in L(X)$ be compact such that $AT + K$ is coercive. Then, we compute with $P_n : X \to X_n$ being the orthogonal projection onto X_n that

$$\begin{aligned} A_n u_n &= A_n T_n T_n^{-1} u_n = P_n A T_n T_n^{-1} u_n \\ &= P_n A T T_n^{-1} u_n + P_n A (T_n - T) T_n^{-1} u_n \\ &= P_n (AT + K) T_n^{-1} u_n - P_n K T_n^{-1} u_n + P_n A (T_n - T) T_n^{-1} u_n. \end{aligned}$$

We further estimate that

$$\|P_nA(T_n - T)T_n^{-1}\|_{L(X_n)} \le \tilde{C}C\|A\|_{L(X)}\|T - T_n\|_{L(X_n)} \overset{n\to\infty}{\longrightarrow} 0.$$

As K is compact, there exists $\mathbb{N}''' \subset \mathbb{N}''$ and $g \in X$ such that $\lim_{n\to\infty} \|P_nKT_n^{-1}u_n - g\|_X = 0$ and since $AT + K$ is coercive, it follows that $\lim_{n\to\infty} \|T_n^{-1}u_n - (AT + K)^{-1}(f + g)\|_X = 0$. Thus, since $(T_n)_{n\in\mathbb{N}}$ is stable, it follows that $\lim_{n\to\infty} \|u_n - T(AT + K)^{-1}(f + g)\|_X = 0$ and we conclude that $(u_n)_{n\in\mathbb{N}}$ is compact. □

2.4.2 Weak T-compatibility

The T-compatibility condition introduced in the previous section is not applicable when we consider nonconforming Galerkin approximations where for instance $X_n \not\subset X$. In this case, we cannot evaluate the operators T and T_n for functions in X_n and X, respectively, and, in general, the norm $\|\cdot\|_{X_n}$ is not well-defined on X. To solve such issues, a weaker T-compatibility condition was introduced by Halla et al. [HLS22]. The authors showed that instead of requiring $\|T - T_n\|_{L(X_n,X)} \to 0$ as $n \to \infty$, it suffices to ask for $T_n \xrightarrow{P} T$ with $(T_n)_{n\in\mathbb{N}}$ being stable, $B_n \xrightarrow{P} B$ with $(B_n)_{n\in\mathbb{N}}$ is stable and B bijective, and $A_nT_n = B_n + K_n$. We note that the weak T-compatibility condition can also be applied to conforming discretizations where Definition 2.26 is too strong, as is the case for the H^1-conforming discretization in [HLS22].

Theorem 2.28. *Assume there exists a constant $C > 0$, sequences $(A_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ and $B \in L(X)$ such that the following holds: for each $n \in \mathbb{N}$, $A_n, T_n, B_n, K_n \in L(X_n)$, $\|T_n\|_{L(X_n)}$, $\|T_n^{-1}\|_{L(X_n)}$, $\|B_n\|_{L(X_n)}$, $\|B_n^{-1}\|_{L(X_n)} \le C$, B bijective, $(K_n)_{n\in\mathbb{N}}$ compact and*

$$\lim_{n\to\infty} \|T_np_nu - p_nTu\|_{X_n} = 0 \text{ and } \lim_{n\to\infty} \|B_np_nu - p_nBu\|_{X_n} = 0 \text{ for each } u \in X,$$

and

$$A_nT_n = B_n + K_n.$$

Then $(A_n)_{n\in\mathbb{N}}$ is regular.

Proof. We briefly sketch the argument from [HLS22, Thm. 3]. Let $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, be uniformly bounded and $(A_nu_n)_{n\in\mathbb{N}}$ be compact. For an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$, we choose $\mathbb{N}'' \subset \mathbb{N}'$ and $f \in X$ such that $A_nu_n \xrightarrow{P} f$. Since $(K_n)_{n\in\mathbb{N}}$ is compact and T_n^{-1} is bounded, we can choose $\mathbb{N}''' \subset \mathbb{N}''$ and $g \in X$ such that $K_nT_n^{-1}u_n \xrightarrow{P} g$. We want to show that $u_n \xrightarrow{P} TB^{-1}(f - g)$. To this end, we compute that

$$\|u_n - p_nTB^{-1}(f - g)\|_{X_n} \le \|u_n - T_nB_n^{-1}(f - g)\|_{X_n} + \|p_nTB^{-1}(f - g) - T_nB_n^{-1}p_n(f - g)\|_{X_n}.$$

Since $\|T_n\|_{L(X_n)}, \|B_n^{-1}\|_{L(X_n)} \le C$ and $B_n = A_nT_n - K_n$, we estimate that

$$\begin{aligned}
\|p_nTB^{-1}(f - g) - T_nB_n^{-1}(f - g)\|_{X_n} &\le C^2\|B_nT_n^{-1}u_n - p_n(f - g)\|_{X_n} \\
&= C^2\|A_nu_n - K_nT_n^{-1}u_n - p_n(f - g)\|_{X_n} \\
&\le C^2(\|A_nu_n - p_nf\|_{X_n} + \|K_nT_n^{-1}u_n - p_ng\|_{X_n}),
\end{aligned}$$

where the right-hand side tends to zero as $A_nu_n \xrightarrow{P} f$ and $K_nT_n^{-1}u_n \xrightarrow{P} g$. Furthermore, we have that

$$\begin{aligned}
\|p_nTB^{-1}(f - g) - T_nB_n^{-1}p_n(f - g)\|_{X_n} &\le \|p_nTB^{-1}(f - g) - T_np_nB^{-1}(f - g)\|_{X_n} \\
&\quad + \|T_np_nB^{-1}(f - g) - T_nB_n^{-1}p_n(f - g)\|_{X_n} \\
&\le \|p_nTB^{-1}(f - g) - T_np_nB^{-1}(f - g)\|_{X_n} \\
&\quad + C^2\|B_np_nB^{-1}(f - g) - p_n(f - g)\|_{X_n},
\end{aligned}$$

where the right-hand side converges to zero as $T_n \xrightarrow{P} T$ and $B_n \xrightarrow{P} B$ by assumptions. Thus, it follows that $\lim_{n\in\mathbb{N}'''} \|u_n - p_n T B^{-1}(f-g)\|_{X_n} = 0$ which shows that $(A_n)_{n\in\mathbb{N}}$ is regular. □

2.5 Summary

To conclude this chapter, let us briefly summarize the strategy to show continuous and discrete well-posedness based on the previously discussed concepts. The first step is to show that the assumptions required to apply the framework are fulfilled. In particular, one has to show the existence of a bounded linear $p_n \in L(X, X_n)$ such that $\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X$ for all $u \in X$ and that $A_n \xrightarrow{P} A$ and $f_n \xrightarrow{P} f$. Then, as displayed in Fig. 2.4, it suffices to show that A is either T-coercive or weakly T-coercive and injective to establish well-posedness on the continuous level.

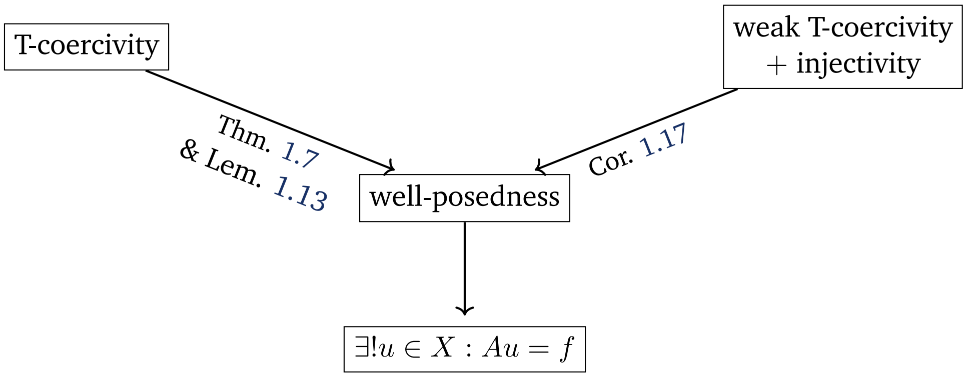


Figure 2.4: Overview of strategies to show well-posedness on the continuous level.

To show stability on the discrete level, we have to show that the sequence of approximations $(A_n)_{n\in\mathbb{N}}$ is regular. To this end, we can either apply the T-compatibility condition or the weak T-compatibility condition described in the previous section, cf. Fig. 2.5.

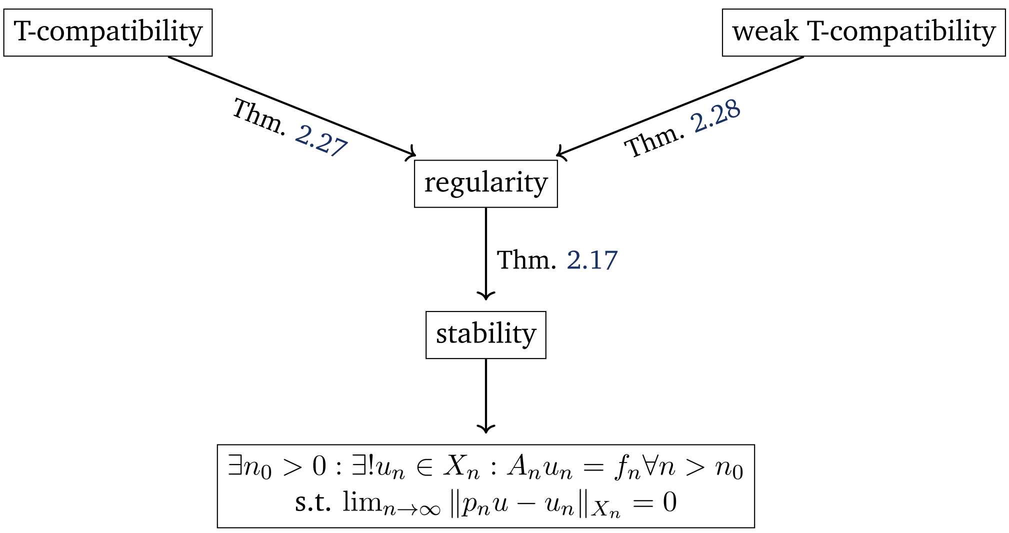


Figure 2.5: Overview of strategies to show stability on the discrete level.

Chapter 3

Application to the Helmholtz problem

In this section, we want to apply the previously introduced framework to the Helmholtz problem with Dirichlet boundary conditions, which is a simple model problem that involves a noncoercive bilinear form. Here, we will briefly introduce the model problem and discuss the noncoerciveness of the bilinear form in more detail. We show that the problem is T-coercive [Cia12] and analyze a conforming Galerkin approximation using discrete approximation schemes. We repeat the latter for a discontinuous Galerkin discretization. Finally, we introduce the concept of hybridization and perform numerical experiments.

Contents of the chapter

3.1 Model problem & Non-coerciveness

The Helmholtz equation is a common model problem for wave propagation. In the following, let $D \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain. For a source term $f \in L^2(D)$ and a wave number $\kappa \in \mathbb{R}_{>0}$, we want to find $u : D \to \mathbb{R}$ such that

$$-\Delta u - \kappa^2 u = f \text{ in } D, \qquad u = 0 \text{ on } \partial D. \tag{3.1}$$

Note that we consider Dirichlet boundary conditions here instead of the more natural Robin boundary conditions, which are used when truncating an unbounded domain where radiation conditions at infinity, for example, the Sommerfeld radiation condition, are imposed. The weak formulation of (3.1) is given by

$$\text{Find } u \in H_0^1(D) \text{ s.t. } a(u, v) = f(v) \text{ for all } v \in H_0^1(D), \tag{3.2}$$

where we define the bilinear form $a(\cdot, \cdot)$ and the linear form $f(\cdot)$ by

$$a(u, v) := \int_D (\nabla u \cdot \nabla v - \kappa^2 uv)dx \qquad \text{and} \qquad f(v) := \int_D fv dx. \tag{3.3}$$

for $u, v \in H_0^1(D)$. While $a(\cdot, \cdot)$ is bounded by the Cauchy-Schwarz inequality, the Helmholtz problem is a classical example of a noncoercive problem [EG21b, Chap. 35]. Consequently, the

Lax-Milgram Lemma 1.4 is not applicable. In the next section, we will demonstrate how the well-posedness of the problem can be proved using T-coercivity. Before we delve deeper into the analysis, let us first discuss the noncoerciveness of the bilinear form $a(\cdot,\cdot)$ in more detail. By means of Remark A.17 to Thm. A.16, we know that there exists an orthonormal basis $(e_\ell)_{\ell\in\mathbb{N}}$ of $L^2(D)$ consisting of eigenvectors of the negative Laplace operator with corresponding eigenvalues $(\lambda_\ell)_{\ell\in\mathbb{N}}$. We order the eigenpairs by increasing values of the eigenvalues λ_ℓ. Thus, for every $u \in H_0^1(D)$, we can find a unique representation $u = \sum_{\ell\in\mathbb{N}} u_\ell e_\ell$, where $u_\ell := (u, e_\ell)_{L^2(D)}$. Furthermore, we can express

$$\|u\|^2_{L^2(D)} = \sum_{\ell\in\mathbb{N}} u_\ell^2 \qquad \text{and} \qquad \|\nabla u\|^2_{L^2(D)} = \sum_{\ell\in\mathbb{N}} \lambda_\ell u_\ell^2.$$

In this basis, we can write the bilinear form $a(\cdot,\cdot)$ defined by (3.3) as

$$a(u, v) = \sum_{\ell\in\mathbb{N}} (\lambda_\ell - \kappa^2) u_\ell v_\ell,$$

where $u, v \in H_0^1(D)$ are represented as $u = \sum_{\ell\in\mathbb{N}} u_\ell e_\ell$ and $v = \sum_{\ell\in\mathbb{N}} v_\ell e_\ell$. Testing the bilinear form with an eigenvector e_ℓ for some $\ell \in \mathbb{N}$ yields $a(e_\ell, e_\ell) = (\lambda_\ell - \kappa^2)$. Thus, for $\ell \in \mathbb{N}$ such that $\lambda_1 < \kappa^2 < \lambda_\ell$, we have that

$$(\lambda_\ell - \kappa^2) = a(e_\ell, e_\ell) > 0 > a(e_1, e_1) = (\lambda_1 - \kappa^2), \tag{3.4}$$

and thus the bilinear form cannot be coercive. In particular, if $\kappa^2 = \lambda_\ell$ for some $\ell \in \mathbb{N}$, the kernel of $a(\cdot,\cdot)$ is non-trivial. Therefore, we will assume that $\kappa^2 \neq \lambda_\ell$, $\ell \in \mathbb{N}$, in the following.

3.2 T-coercivity

In this section, we will discuss how the concept of T-coercivity, cf. Definition 1.11, can be used to prove the well-posedness of the Helmholtz problem (3.2). Before we elaborate, let us stress that the well-posedness of the Helmholtz problem can also be shown with other techniques, e.g., through a compact perturbation analysis [SBH19, Sec. 8.2] or by utilizing the Fredholm alternative. However, the Helmholtz problem is a convenient example to demonstrate the concepts developed in the previous chapters, and the construction of the T-operator itself demonstrates the idea behind T-coercivity quite well. This and the following section are based on [Cia12] with some adaptations to fit into the framework of Chapter 2.
In the following, we denote by $\ell_{\max}$ the largest index $\ell \geq 0$ such that $\lambda_\ell < \kappa^2$. We introduce the following finite-dimensional vector space

$$W := \operatorname{span}_{0\leq\ell\leq\ell_{\max}}(e_\ell) \subset H_0^1(D).$$

Furthermore, let $V = W^\perp$ such that $H_0^1(D) = V \oplus W$. We denote by $P_V \in L(H_0^1(D), V)$, $P_W \in L(H_0^1(D), W)$ the orthogonal projections from $H_0^1(D)$ to V and W respectively. Note that by construction, P_W is of finite rank.

Remark 3.1. *If $\kappa^2 < \lambda_0$, we set $\ell_{\max} = -1$, $W = \{0\}$ and $P_W = 0$.*

Then, we define the operator $T \in L(H_0^1(D))$ through

$$T := P_V - P_W, \text{ i.e. } Tu = v - w \text{ for } u = v + w \in H_0^1(D). \tag{3.5}$$

On basis vectors $(e_\ell)_{\ell\in\mathbb{N}}$, the operator T acts in the following way:

$$Te_\ell = \begin{cases} -e_\ell \text{ if } 0 \leq \ell \leq \ell_{\max}, \\ +e_\ell \text{ if } \ell > \ell_{\max}. \end{cases} \tag{3.6}$$

Intuitively, this construction of the T-operator immediately tackles the origin of the noncoerciveness of the bilinear form $a(\cdot,\cdot)$ as discussed in (3.4), because it flips the sign of the problematic eigenvectors. In the forthcoming analysis, we will see that this construction indeed makes the bilinear form T-coercive. Before, let us briefly note that T is bijective by definition since $T^2 = \mathrm{Id}_{H_0^1(D)}$.

Lemma 3.2. *The operator $A \in L(H_0^1(D))$ induced by the bilinear form $a(\cdot,\cdot)$ is T-coercive.*

Proof. For $u \in H_0^1(D)$, we have by linearity of T

$$Tu = T\big(\sum_{\ell\in\mathbb{N}} u_\ell e_\ell\big) = \sum_{\ell\in\mathbb{N}} u_\ell T e_\ell \Rightarrow (Tu)_\ell = (Tu, e_\ell)_{L^2(D)} = \sum_{\ell\in\mathbb{N}} u_\ell (Te_\ell, e_\ell)_{L^2(D)}.$$

Since T swaps the sign of all e_ℓ with $0 \leq \ell \leq \ell_{\max}$, we have with $\alpha := \min_{\ell\in\mathbb{N}}\{|\lambda_\ell - \kappa^2|/\lambda_\ell\}$

$$\begin{aligned} a(u, Tu) &= \sum_{i\in\mathbb{N}} (\lambda_i - \kappa^2) u_i (Tu)_i \\ &= \sum_{0\leq\ell\leq\ell_{\max}} (\kappa^2 - \lambda_\ell) u_\ell^2 + \sum_{\ell>\ell_{\max}} (\lambda_\ell - \kappa^2) u_\ell^2 \\ &\geq \alpha \sum_{\ell\in\mathbb{N}} \lambda_\ell u_\ell^2 = \alpha \|\nabla u\|_{L^2(D)}^2. \end{aligned}$$

Since $\|\nabla v\|_{L^2(D)} \simeq \|v\|_{H^1(D)}$ on $H_0^1(D)$ due to the Poincaré inequality, cf. Lemma A.32, the operator T^*A is coercive on $H_0^1(D)$, i.e. A is T-coercive. □

Therefore, we can conclude with Lemma 1.13 and Thm. 1.7 that the Helmholtz problem (3.2) is well-posed if and only if $\kappa^2 \notin \{\lambda_\ell\}_{\ell\in\mathbb{N}}$.

Remark 3.3 (Robin boundary conditions)**.** *As mentioned before, we consider Dirichlet boundary conditions for simplicity here. However, we can also apply the frameworks discussed in Chapters 1 and 2 to Robin boundary conditions of the form*

$$\frac{\partial u}{\partial n} - i\kappa u = g \text{ on } \partial D.$$

In this case, the bilinear form $a(\cdot,\cdot)$ has an additional boundary term, i.e.

$$a(u, v) = \int_D (\nabla u \cdot \nabla v - \kappa^2 uv) dx - i\kappa \int_{\partial D} \mathrm{tr}(u)\,\mathrm{tr}(v) ds,$$

where $\mathrm{tr} : H^1(D) \to L^2(\partial D)$ is the trace operator. Since the trace operator is compact on bounded Lipschitz domains [SBH19, Prop. 8.3] as an operator from $H^1(D)$ to $L^2(\partial D)$, the operator induced by the additional boundary term is compact. Thus, with the same argument as before, we can show that the operator A induced by $a(\cdot,\cdot)$ is weakly T-coercive. Furthermore, it can be shown that the operator is injective [EG21b, Thm. 35.5] and thus, the Helmholtz problem with Robin boundary conditions is well-posed by Corollary 1.17.

3.3 Conforming Galerkin Approximation

In this section, we want to introduce and analyze a conforming Galerkin discretization of (3.1) using discrete approximation schemes. Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be sequence of shape regular triangulations of D such that $h_n \to 0$ as $n \to \infty$, where $h_n := \max_{\tau \in \mathcal{T}_n} h_\tau$ and $h_\tau := \text{diam}(\tau)$, $\tau \in \mathcal{T}_n$. For $k \geq 1$ we define the H^1-conforming Finite Element Space

$$X_n := \{v \in L^2(D) : v|_\tau \in \mathcal{P}^k(\tau)\ \forall \tau \in \mathcal{T}_n, v|_{\partial D} = 0\} \cap H^1(D) \subset H_0^1(D).$$

Then, the Galerkin approximation to (3.1) is given by

$$\text{Find } u_n \in X_n \text{ such that } a(u_n, v_n) = f(v_n) \text{ for all } v_n \in X_n. \tag{3.7}$$

As before, we denote by $A \in L(X)$ the operator induced by the bilinear form $a(\cdot,\cdot)$. Let $P_{X_n} \in L(X, X_n)$ be the orthogonal projection from X to X_n. Then, we define $A_n := P_{X_n} A|_{X_n} \in L(X_n)$. Since the H^1-conforming finite element space fulfills the approximability property, cf. Example 2.1, we can immediately apply Corollary 2.21 to conclude that (X_n, P_{X_n}, A_n) is a discrete approximation scheme of (X, A). That is, we know that

$$\lim_{n\to\infty} \|P_{X_n} u\|_X = \|u\|_X \quad \text{and} \quad A_n \xrightarrow{P} A.$$

Now, we want to construct a sequence of discrete operators $(T_n)_{n\in\mathbb{N}}$, $T_n \in L(X_n)$, that mirrors the construction of T from (3.5). To this end, we approximate the continuous eigenpairs $(\lambda_\ell, e_\ell)_{\ell\in\mathbb{N}}$ with discrete eigenpairs $(\lambda_{\ell,n}, e_{\ell,n})_{\ell\in\mathbb{N}}$, $e_{\ell,n} \in X_n$, i.e., we solve the discrete eigenvalue problem. By the approximation property (2.16) of X_n, we can find for all $\ell \in \mathbb{N}$ such that $\lambda_{\ell_{\max},n} < \kappa^2$ an index $n^* > 0$ such that for all $n > n^*$

$$\|e_\ell - e_{\ell,n}\|_X \leq \delta(n), \tag{3.8}$$

where $\delta(n)$ only depends on $\ell_{\max}$ and $\lim_{n\to\infty} \delta(n) = 0$ [Cia12, Sec. 3.2]. We note that with orthonormalization, we can form a discrete basis of X_n with the approximate eigenfunctions which we will still denote by $(e_{\ell,n})_{\ell\in\mathbb{N}}$. Then, we define the discrete space

$$W_n := \text{span}_{0\leq\ell\leq\ell_{\max}}(e_{\ell,n}) \tag{3.9}$$

and denote by $P_{W_n} \in L(X_n, W_n)$ the orthogonal projection from X_n onto W_n. Furthermore, we set $V_n = W_n^\perp$ such that $X_n = V_n \oplus W_n$ and denote by P_{V_n} the orthogonal projection onto V_n. Then, as in (3.5), we define

$$T_n := P_{V_n} - P_{W_n}, \quad \text{i.e.} \quad T_n u_n = v_n - w_n \text{ for all } u_n = v_n + w_n \in X_n. \tag{3.10}$$

To conclude the well-posedness of the discrete problem, we want to show that the sequence $(A_n)_{n\in\mathbb{N}}$ is T-compatible. The following lemma serves as a preparation.

Lemma 3.4. *It holds that*

$$\lim_{n\to\infty} \|P_W - P_{W_n}\|_{L(X)} = 0. \tag{3.11}$$

Proof. Let $u_n \in X_n$. Since $X_n \subset X$, we can write in the continuous eigenbasis $(e_\ell)_{\ell\in\mathbb{N}}$ and in the discrete eigenbasis $(e_{\ell,n})_{\ell\in\mathbb{N}}$, i.e.

$$u_n = \sum_{\ell\in\mathbb{N}} (u_n, e_\ell)_{L^2(D)} e_\ell = \sum_{\ell\in\mathbb{N}} (u_n, e_{\ell,n})_{L^2(D)} e_{\ell,n}.$$

Since P_W and P_{W_n} are orthogonal projections onto W and W_n, respectively, we have that

$$P_W u_n = \sum_{0\leq\ell\leq\ell_{\max}} (u_n, e_\ell)_{L^2(D)} e_\ell \qquad \text{and} \qquad P_{W_n} u_n = \sum_{0\leq\ell\leq\ell_{\max}} (u_n, e_{\ell,n})_{L^2(D)} e_{\ell,n}.$$

(Note that the smallness assumption on h ensures that $\ell_{\max} = \ell_{\max,n}$.) Then,

$$\begin{aligned}\|(P_W - P_{W_n})u_n\|_X &= \sum_{0\le\ell\le\ell_{\max}} \|(u_n, e_\ell)_{L^2(D)}e_\ell - (u_n, e_{\ell,n})_{L^2(D)}e_{\ell,n}\|_X \\ &= \sum_{0\le\ell\le\ell_{\max}} \|(u_n, e_\ell)_{L^2(D)}(e_\ell - e_{\ell,n}) - ((u_n, e_{\ell,n})_{L^2(D)} - (u_n, e_\ell)_{L^2(D)})e_{\ell,n}\|_X \\ &\lesssim \sum_{0\le\ell\le\ell_{\max}} \|e_\ell - e_{\ell,n}\|_X.\end{aligned}$$

Since $\|e_\ell - e_{\ell,n}\|_X \to 0$ as $n \to \infty$, we have that

$$\lim_{n\to\infty} \|P_W - P_{W_n}\|_{L(X_n)} = \lim_{n\to\infty} \sup_{u_n\in X_n, \|u_n\|_X=1} \|(P_W - P_{W_n})u_n\|_X = 0.$$

□

Lemma 3.5. *The sequence $(A_n)_{n\in\mathbb{N}_{>n^*}}$ of Galerkin approximations is T-compatible.*

Proof. By Lemma A.21, the operators $A_n \in L(X_n)$ are Fredholm with index zero and so is T_n, since it is bijective for all $n \in \mathbb{N}$. We note that we can write

$$T = \mathrm{Id}_{H_0^1(D)} - 2P_W \qquad \text{and} \qquad T_n = \mathrm{Id}_{X_n} - 2P_{W_h}.$$

Therefore, we have for $u_n \in X_n$

$$(T - T_n)u_n = u_n - 2P_W u_n - u_n + 2P_{W_h}u_n = 2(P_{W_h} - P_W)u_n.$$

Thus, we have that by Lemma 3.4

$$\|T - T_n\|_n \lesssim \|P_W - P_{W_h}\|_{L(X_n)} \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Furthermore, due to Lemma 2.25, the sequence $(A_n)_{n\in\mathbb{N}}$ is uniformly T-coercive and thus Fredholm with index zero. □

Therefore, we can invoke Thm. 2.27 to conclude that the sequence $(A_n)_{n\in\mathbb{N}}$ is regular. Applying Lemma 2.22 yields the following result.

Theorem 3.6. *Let $\kappa^2 \notin \{\lambda_\ell\}_{\ell\in\mathbb{N}}$ and let u be the solution to (3.2). Then there exists an index $n_0 > 0$ such that for all $n > max(n_0, n^*)$, a unique solution $u_n \in X_n$ to (3.7) exists and*

$$\|u - u_n\|_X \lesssim \inf_{v_n\in X_n} \|u - v_n\|_X \textit{ for all } n > n_0.$$

Remark 3.7 (On the smallness assumption on h)**.** *A drawback of the previous T-coercivity argument is that the smallness assumption on the mesh size h is not explicit in terms of the wave number κ. However, this restriction is practically relevant, as demonstrated in Fig. 3.1. Intuitively, it makes sense that the smallness assumption on the mesh size depends on the wave number because the dimension of the discrete space W_n depends on the index $\ell_{\max}$, which depends on κ. Using other approaches, for instance, a classical Schatz argument [Sch74; Mel95], this requirement can be made explicit. For example, Melenk and Sauter [MS11] showed that for the Helmholtz problem with Robin boundary conditions, cf. Rem. 3.3, on domains with analytic boundaries, it suffices to assume that*

$$\frac{\kappa h}{k} \le C_1 \text{ and } k \ge C_2 \log \kappa$$

to guarantee the quasi-optimality of the Galerkin approximation. It should be possible to derive similar mesh size requirements for the T-coercivity argument through a more detailed analysis of the eigenvalue problem.

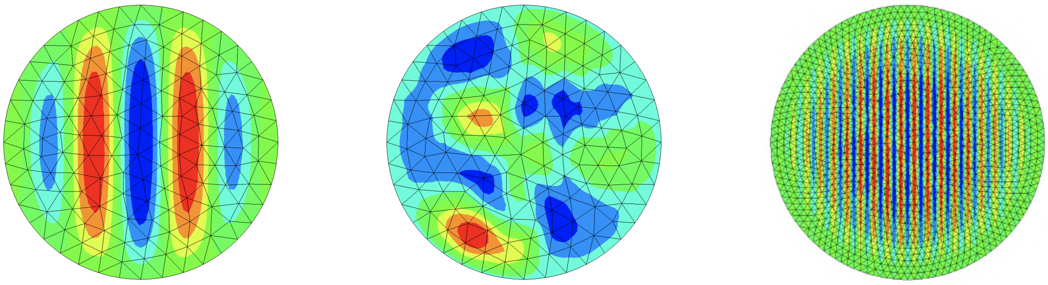


Figure 3.1: Conforming Galerkin approximation of the Helmholtz problem for different wave numbers κ and mesh sizes h: (left) $\kappa = 2$ and $h = 0.15$, (middle) $\kappa = 8$ and $h = 0.15$, (right) $\kappa = 8$ and $h = 0.05$. In the middle, the mesh size is not small enough to capture the oscillations.

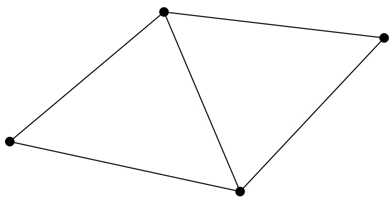
To conclude the discussion on the H^1-conforming Galerkin approximation of (3.1), we apply standard interpolation results to infer the following convergence rates.

Corollary 3.8 (Convergence rates). *Let $\kappa^2 \notin \{\lambda_\ell\}_{\ell\in\mathbb{N}}$ and let $u \in X \cap H^{1+s}(D)$, $s > 0$, be the unique solution to* (3.2)*. Then, we have that*

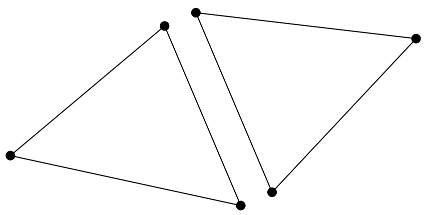
$$\|u - u_n\|_X \lesssim h^s \|u\|_{H^{1+s}(D)}.$$

3.4 Discontinuous Galerkin approximation

Having analyzed a conforming discretization of (3.1), we want to shift our attention to the nonconforming case. To this end, we introduce a *discontinuous Galerkin (DG)* formulation of (3.1). The main idea behind the DG method is to allow for discontinuities across element interfaces, which offers multiple benefits compared with continuous Galerkin methods (CG). DG methods are known to be more flexible with regard to underlying mesh, and due to its local character, the DG method is well-suited for parallelization and adaptive mesh refinements. Furthermore, DG methods are known to be more stable for convective and diffusive operators than CG methods, and they preserve conservation properties. However, we note that DG methods introduce more degrees of freedom than CG methods, cf. Fig. 3.2, which leads to larger and less sparse linear systems [DE12; Leh10].



(a) Continuous Galerkin



(b) Discontinuous Galerkin

Figure 3.2: Degrees of freedom for linear CG and DG FEM in 2D; inspired by [Leh10, Fig. 1.2.1].

In the following, we set $X := H_0^2(D)$, where we note that the additional regularity assumptions ensure that the terms $\nabla u \cdot \nu$ are well-defined. In addition to the notation from the previous section, we denote by $\mathcal{F}_n$ the collection of facets in $\mathcal{T}_n$ and by $\mathcal{F}_n^{\text{int}}$ the interior facets. For $k \geq 1$, we set

$$X_n := \mathbb{P}^{k,d}(\mathcal{T}_n) := \{v \in L^2(D) : v|_\tau \in \mathcal{P}^k(\tau)\ \forall \tau \in \mathcal{T}_n\}. \tag{3.13}$$

Furthermore, we denote by $H^1(\mathcal{T}_n)$ the broken Sobolev space given by

$$H^1(\mathcal{T}_n) := \{u \in L^1(D) : u \in H^1(\tau) \text{ for all } \tau \in \mathcal{T}_n\}.$$

Remark 3.9 (Interpretation of ∇ on $H^1(\mathcal{T}_n)$)**.** *As detailed in [DE12, Def. 1.21], we define the broken gradient $\nabla_h : H^1(\mathcal{T}_n) \to \boldsymbol{L}^2(D)$ element-wise through*

$$(\nabla_h v)|_\tau := \nabla(v|_\tau) \text{ for all } \tau \in \mathcal{T}_n.$$

For ease of presentation, we drop the index h and denote the broken gradient as ∇.

In contrast to before, functions in X_n do not have to be continuous across element interfaces and thus $X_n \not\subset X$. For two elements $\tau_1, \tau_2 \in \mathcal{T}_h$ such that $\partial\tau_1 \cap \partial\tau_2 = F \in \mathcal{F}_h$, we define the *jump* and *average* operator as follows

$$[\![u]\!] := u_1 - u_2, \qquad \{\!\{u\}\!\} := \frac{1}{2}(u_1 + u_2),$$

where $u_i := \operatorname{tr} u|_{\tau_i}$, $i \in \{1, 2\}$. We note that there holds $[\![uv]\!] = \{\!\{u\}\!\}[\![v]\!] + \{\!\{v\}\!\}[\![u]\!]$.
To derive a DG formulation, we multiply (3.1) by a test function $v \in X_n$. Then, we apply partial integration on each element $\tau \in \mathcal{T}_n$ and sum over all elements to obtain

$$\sum_{\tau\in\mathcal{T}_n} \Big(\int_{\partial\tau} (-\nabla u \cdot \nu) v ds + \int_\tau \nabla u \cdot \nabla v dx - \int_\tau \kappa^2 uv dx \Big) = \sum_{\tau\in\mathcal{T}_n} \int_\tau f v dx$$

Since every facet appears twice in the sum over all element boundaries, we get with the definition of the jump operator that

$$-\sum_{\tau\in\mathcal{T}_n} \int_{\partial\tau} (\nabla u \cdot \nu) v ds = -\sum_{F\in\mathcal{F}_h} \int_F [\![\nabla u \cdot \nu v]\!] ds = -\sum_{F\in\mathcal{F}_h} \int_F (\{\!\{\nabla u \cdot \nu\}\!\}[\![v]\!] + [\![\nabla u \cdot \nu]\!]\{\!\{v\}\!\}) ds$$

The second term can be dropped without affecting consistency since it vanishes for the continuous solution u. Usually, one adds the consistent term $-\int_F [\![\nabla v \cdot \nu]\!][\![u]\!] ds$ to symmetrize the formulation. If we add a stabilization term $s_h(\cdot,\cdot)$ defined as

$$s_h^{\text{SIP}}(u, v) = \sum_{F\in\mathcal{F}_n} \frac{\alpha}{h_F} \int_F [\![u]\!][\![v]\!] ds, \tag{3.14}$$

where $\alpha > 0$ is a penalty parameter that has to be chosen appropriately, we would obtain the well-known *symmetric interior penalty (SIP)* penalty method:

$$\begin{aligned} a_n^{\text{DG,SIP}}(u, v) := &\sum_{\tau\in\mathcal{T}_n} \int_\tau \nabla u \cdot \nabla v - \kappa^2 uv dx \\ &- \sum_{F\in\mathcal{F}_n} \Big(\int_F [\![u]\!]\{\!\{\nabla v \cdot \nu\}\!\} + \{\!\{\nabla u \cdot \nu\}\!\}[\![v]\!] ds \Big) + s_h^{\text{SIP}}(u, v). \end{aligned} \tag{3.15}$$

In the perspective of treating Galbrun's equation, however, we want to avoid choosing the penalty parameter α. Therefore, we use a different stabilization term based on *lifting operators*, which was introduced by Bassi, Rebay, and co-authors [BR97; Bas+97]. We refer to [DE12, Chapter 4.3] for an overview, but note that we will follow the opposite sign convention by Buffa and Ortner [BO08]. For a facet $F \in \mathcal{F}_h$ and an integer $l \in \mathbb{N}$, we define the local lifting operator $r_F^l : X_n \to [\mathbb{P}^l(\mathcal{T}_n)]^d$ through

$$\int_D r_F^l u_n \psi_n dx = -\int_F [\![u_n]\!] \cdot \nu \{\!\{\psi_n\}\!\} ds \text{ for all } \psi_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d.$$

Note that the support of r_F^l is local in the sense that

$$\operatorname{supp}(r_F^l) = \overline{\tau_1 \cup \tau_2}, \text{ where } \tau_1, \tau_2 \text{ are s.t. } \tau_1 \cap \tau_2 = F.$$

We define the global lifiting operator $R_n^l : L^2(\mathcal{F}_n) \to [\mathbb{P}^l(\mathcal{T}_n)]^d$ through $R_n^l(u_n) = \sum_{F\in\mathcal{F}_n} r_F^l(u_n)$.

Lemma 3.10 (Bound on the global lifting operator). *For all $u \in H^1(\mathcal{T}_n)$, it holds that*

$$\|R_n^l(u)\|_{\boldsymbol{L}^2(D)} \lesssim \sum_{F\in\mathcal{F}_n} h_F^{-1/2} \|[\![u]\!]\|_{L^2(F)}.$$

Proof. The statement follows directly from the discrete trace inequality A.14, see e.g. [BO08, Lem. 7]. □

Furthermore, we define a discrete gradient $G_n^l : H^1(\mathcal{T}_n) \to \boldsymbol{L}^2(D)$ through

$$G_n^l(v) = \nabla v + R_n^l(v). \tag{3.16}$$

Then, we define the lifting stabilized (LS) DG-bilinear form $a_n^{\mathrm{DG}}(\cdot,\cdot)$ in the following way

$$\begin{aligned} a_n^{\mathrm{DG}}(u,v) &:= \sum_{\tau\in\mathcal{T}_n} \int_\tau G_n^l(u) G_n^l(v) - \kappa^2 uv dx \\ &= \sum_{\tau\in\mathcal{T}_n} \int_\tau \nabla u \cdot \nabla v dx + \int_\tau \nabla v \cdot R_n^l(u) dx + \int_\tau \nabla u \cdot R_n^l(v) dx + s_h(u,v), \end{aligned} \tag{3.17}$$

where the stabilization term $s_h(u,v)$ is given by

$$s_h(u,v) = \sum_{\tau\in\mathcal{T}_n} \int_\tau R_n^l(u) \cdot R_n^l(v) dx. \tag{3.18}$$

Then, the LS-DG scheme reads as: Find $u_n \in X_n$ s.t.

$$a_n^{\mathrm{DG}}(u_n, v_n) = f(v_n) \text{ for all } v_n \in X_n. \tag{3.19}$$

Remark 3.11. *The lifting stabilization term (3.18) is in general weaker than the SIP stabilization term (3.14). For the local lifting operator, it can be shown that $\|[\![u_n]\!]\|_{L^2(F)}^2 \lesssim h^{1/2} \|r_F^l(u_n)\|_{\boldsymbol{L}^2(D)}^2$ for all $u_n \in X_n$ and $F \in \mathcal{F}_n$, cf. [Lew+04, Lem. 3.1]. This estimate, however, does not necessarily hold for the global lifting operator R_n^l, see for instance the counterexample in [JNS16].*

Remark 3.12. *The discrete gradient defined by (3.16) can be interpreted as a distributional gradient on the broken Sobolev space $H^1(\mathcal{T}_n)$, which acts on $\varphi \in C_c^\infty(D)^d$ through (cf. [BO08, Sec. 5])*

$$\langle Du, \varphi\rangle = \int_D \nabla u \cdot \varphi dx - \int_{\Gamma^{int}} [\![u]\!] \cdot \varphi ds.$$

3.4.1 Interpretation as discrete approximation scheme

In this section, we will show how the DG scheme defined in the previous section can be interpreted as a discrete approximation scheme. To this end, we define the following scalar product on the discrete space X_n:

$$(u_n, v_n)_{X_n} := (u_n, v_n)_{L^2(D)} + (G_n^l u_n, G_h^l v_n)_{\boldsymbol{L}^2(D)}. \tag{3.20}$$

Furthermore, we denote by $\|\cdot\|_{X_n} := (\cdot,\cdot)_{X_n}^{1/2}$ the norm induced by the scalar product. Since the trace of H^1 functions is well-defined, we can apply the jump operator $[\![\cdot]\!]$ to functions in X. In particular, we have that $[\![u]\!] = 0$ for all $u \in X$ and thus $G_n^l(u) = \nabla u$. Therefore, the scalar product $(\cdot,\cdot)_{X_n}$ and the norm $\|\cdot\|_{X_n}$ can be applied to functions in X, even though we have that $X_n \not\subset X$.
Now, we define a projection operator $p_n \in L(X, X_n)$. For $u \in X$, let $p_n u \in X_n$ be the solution to

$$(p_n u, v_n)_{X_n} = (u, v_n)_{L^2(D)} + (\nabla u, G_n^l v_n)_{\boldsymbol{L}^2(D)} = (u, v_n)_{X_n} \quad \text{for all } v_n \in X_n. \tag{3.21}$$

We note that p_n is essentially an orthogonal projection onto X_n. Indeed, we have that $p_n \in L(X, X_n)$ with $\|p_n\|_{L(X,X_n)} \leq 1$, since

$$\|p_n u\|_{X_n}^2 = (u, p_n u)_{X_n} \leq \|u\|_X \|p_n u\|_{X_n} \Longrightarrow \|p_n\|_{L(X,X_n)} = \sup_{u \in X\setminus\{0\}} \frac{\|p_n u\|_{X_n}}{\|u\|_X} \leq 1.$$

Furthermore, we denote by $A \in L(X)$ the operator induced by the continuous bilinear for $a(\cdot,\cdot)$ and by $A_n^{\mathrm{DG}} \in L(X_n)$ the operator induced by the discrete bilinear form $a_n^{\mathrm{DG}}(\cdot,\cdot)$. The setup is visualized in Fig. 3.3.

$$\begin{tikzcd}
X = H_0^1(D) \arrow[loop left, "A"] \arrow[r, "p_n"] & X_n = \mathbb{P}^k(\mathcal{T}_n) \arrow[loop right, "A_n^{\mathrm{DG}}"] \\
\|\cdot\|_X, \|\cdot\|_{X_n} & \|\cdot\|_{X_n}
\end{tikzcd}$$

Figure 3.3: Set up.

In the following, we will show that $(X_n, p_n, A_n^{\mathrm{DG}})$ constitutes a discrete approximation scheme of (X, A), i.e., that for all $u \in X$, $\|p_n u\|_{X_n} \to \|u\|_X$ as $n \to \infty$ and $A_n^{\mathrm{DG}} \xrightarrow{P} A$. In the following, we denote by $\pi_\tau^k : H^s(\tau) \to \mathcal{P}^k(\tau)$, $s > 1/2$, the elementwise L^2-interpolation operator and by $\pi_n : H^s(D) \to X_n$, $s > 1/2$, its global version defined by $\pi_n|_\tau := \pi_\tau^k$, $\tau \in \mathcal{T}_n$.

Lemma 3.13. *For all $u \in X$, we have that $\|u - p_n u\|_{X_n} \leq \|u - \pi_n u\|_{X_n}$.*

Proof. Since the definition of p_n yields

$$\|\pi_n u - p_n u\|_{X_n}^2 = (\pi_n u - u, \pi_n u - p_n u)_{X_n} \leq \|u - \pi_n u\|_{X_n} \|\pi_n u - p_n u\|_{X_n},$$

the triangle inequality gives the claim. □

Corollary 3.14. *For each $u \in H^{1+s}(D)$, $s > 0$, it holds that $\|u - \pi_n u\|_{X_n} \lesssim h_n^s \|u\|_{H^{1+s}(D)}$.*

Proof. The claim follows from

$$\|G_n^l(u - \pi_n u)\|_{\boldsymbol{L}^2(D)} \leq \|\nabla(u - \pi_n u)\|_{\boldsymbol{L}^2(D)} + \|R_n^l \pi_n u\|_{\boldsymbol{L}^2(D)}$$

together with standard interpolation results, Lemma 3.10, and $[\![u]\!] = 0$. □

Lemma 3.15. *For $u \in X$, we have that $\lim_{n\to\infty} \|u - p_n u\|_{X_n} = 0$.*

Proof. Let $u \in X$. Since $X = H_0^2(D) = W_0^{2,2}(D) := \overline{C_c^\infty(D)}^{\|\cdot\|_{W^{2,2}}}$, we can find for each $\epsilon > 0$ a $\tilde{u} \in C_c^\infty(D)$ such that $\|u - \tilde{u}\|_X < \epsilon$. The triangle inequality yields

$$\begin{aligned}\|u - p_n u\|_{X_n} &\leq \|u - p_n\tilde{u}\|_{X_n} + \|p_n u - p_n\tilde{u}\|_{X_n} \\ &\leq \|\tilde{u} - p_n\tilde{u}\|_{X_n} + \|p_n u - p_n\tilde{u}\|_{X_n} + \|u - \tilde{u}\|_X \\ &\leq \|\tilde{u} - p_n\tilde{u}\|_{X_n} + 2\epsilon,\end{aligned}$$

since the definition of p_n and an application of Cauchy-Schwarz yields $\|p_n(u - \tilde{u})\|_{X_n} \leq \|u - \tilde{u}\|_X$. Using the Lemma 3.13 and Corollary 3.14, we have that $\lim_{n\to\infty} \|\tilde{u} - p_n\tilde{u}\|_{X_n} = 0$. Therefore, we conclude that

$$\lim_{n\to\infty} \sup \|u - p_n u\|_{X_n} \leq 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, the claim follows. $\square$

Lemma 3.16. *For all $u \in X$, we have that $\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X$.*

Proof. Since $G_n^l u = \nabla u$ for $u \in X$, we have that $\|u\|_{X_n} = \|u\|_X$. Thus, we have that

$$\|p_n u\|_{X_n}^2 = (p_n u, p_n u)_{X_n} = (u, p_n u)_{X_n} = \|u\|_X^2 + (u, p_n u - u)_{X_n}. \tag{3.22}$$

Apply Cauchy-Schwarz on the second term and using that $\lim_{n\to\infty} \|u - p_n u\|_{X_n} = 0$ yields the claim. $\square$

To show that $A_n^{\mathrm{DG}} \xrightarrow{P} A$, we require the following compactness result for the discrete gradient.

Lemma 3.17. *Let $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, be such that $\sup_{n\in\mathbb{N}} \|u_n\|_{X_n} < \infty$. Then, there exists $u \in X$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $u_n \overset{L^2}{\rightharpoonup} u$ and $G_n^l u_n \overset{L^2}{\rightharpoonup} \nabla u$.*

Proof. See [BO08, Thm. 5.2]. $\square$

Lemma 3.18. *For each $u \in X$, we have that $\lim_{n\to\infty} \|A_n^{DG} p_n u - p_n A u\|_{X_n} = 0$.*

Proof. Since X_n is a Hilbert space, we can find for each $u_n \in X_n$ an element $\tilde{u}_n$ such that

$$\|u_n\|_{X_n} = \sup_{\substack{u_n' \in X_n \\ \|u_n'\|_{X_n = 1}}} |(u_n, u_n')_{X_n}| \leq |(u, \tilde{u}_n)_{X_n}| + 1/n.$$

Thus, for every $u \in X$ we can choose $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$ with $\|u_n\|_{X_n} = 1$ such that

$$\|A_n^{\mathrm{DG}} p_n u - p_n A u\|_{X_n} \leq |(A_n^{\mathrm{DG}} p_n u - p_n A u, u_n)_{X_n}| + 1/n.$$

For an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$, we utilize Lemma 3.17 to find a subsubsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $u_n \overset{L^2}{\rightharpoonup} u'$, $G_n^l u_n \overset{L^2}{\rightharpoonup} \nabla u'$ for some $u' \in X$. The definition of p_n yields

$$\lim_{n\in\mathbb{N}''} (p_n A u, u_n)_{X_n} = \lim_{n\in\mathbb{N}''} (Au, u_n)_{X_n} = (Au, u')_X.$$

Furthermore, using the definition of A_n^{DG}, we compute

$$\begin{aligned}(A_n^{\mathrm{DG}} p_n u, u_n)_{X_n} &= a_n^{\mathrm{DG}}(p_n u, u_n) = (G_n^l p_n u, G_n^l u_n)_{\boldsymbol{L}^2(D)} - \kappa^2 (p_n u, u_n)_{L^2(D)} \\ &= (p_n u, u_n)_{X_n} - (\kappa^2 + 1)(p_n u, u_n)_{L^2(D)} \\ &= (u, u_n)_{L^2(D)} + (\nabla u, G_n^l u_n)_{\boldsymbol{L}^2(D)} - (\kappa^2 + 1)(p_n u, u_n)_{L^2(D)} \\ &= (\nabla u, G_n^l u_n)_{\boldsymbol{L}^2(D)} - \kappa^2 (u, u_n)_{L^2(D)} + (1 + \kappa^2)(u - p_n u, u_n)_{L^2(D)}.\end{aligned}$$

Since $\|(1+\kappa^2)(u - p_n u, u_n)_{L^2(D)}| \le (1+\kappa^2)\|u - p_n u\|_{X_n} \overset{n\in\mathbb{N}''}{\longrightarrow} 0$ by Lemma 3.15, we have that $\lim_{n\in\mathbb{N}''}(A_n^{\mathrm{DG}} p_n u, u_n)_{X_n} = (Au, u')$ and therefore

$$\lim_{n\in\mathbb{N}''} \|A_n^{\mathrm{DG}} p_n u - p_n Au\|_{X_n} = 0.$$

Since $\mathbb{N}' \subset \mathbb{N}$ was arbitrary, the claim is proven. □

Thus, we have shown that $(X_n, p_n, A_n^{\mathrm{DG}})$ indeed constitutes a discrete approximation scheme of (X, A). In the following section, we use the weak T-compatibility conditions from Thm. 2.28 to show that the approximation scheme is stable.

3.4.2 Convergence analysis

As for the conforming case, we want to mirror the construction of the operator T in (3.5). To this end, we consider a DG approximation[1] of the continuous Laplace eigenpairs $(\lambda_\ell, e_\ell)_{\ell\in\mathbb{N}}$, i.e. for each $\ell \in \mathbb{N}$ we want to find $\lambda_{\ell,n} \in \mathbb{K}$ and $e_{\ell,n} \in X_n$ such that

$$(G_n^l e_{\ell,n}, G_n^l v_n)_{\boldsymbol{L}^2(D)} = \lambda_{\ell,n}(e_{\ell,n}, v_n)_{L^2(D)} \quad \text{for all } v_n \in X_n.$$

It has been shown [ABP06] that these DG approximations of the eigenpairs converge and thus, there exists an $n^* > 0$ such that

$$\|e_\ell - e_{\ell,n}\|_{X_n} \le \delta(n), \qquad \lim_{n\to\infty} \delta(n) = 0$$

for all $\ell \in \mathbb{N}$ such that $\lambda_{\ell_{\max},n} < \kappa^2$. As before, we set

$$W_n := \mathrm{span}_{0\le\ell\le\ell_{\max}}(e_{\ell,n})$$

and $V_n := W_n^\perp$ such that $X_n := W_n \oplus V_n$. We define the projection $P_{W_n} : X_n \to W_n$ to be the orthogonal projection onto W_n and set

$$T_n := P_{V_n} - P_{W_n}.$$

In the following, we show that with this construction, we have that $T_n \xrightarrow{P} T$ and that A_n^{DG} is uniformly T_n-coercive, which allows us to utilize Thm. 2.28 to show that the sequence $(A_n^{\mathrm{DG}})_{n\in\mathbb{N}}$ is regular. Thus, we can apply the theory of discrete approximation schemes to show the convergence of the method.

Lemma 3.19. *For each $u \in X$, we have that $\lim_{n\to\infty} \|p_n P_W u - P_{W_n} p_n u\|_{X_n} = 0$.*

Proof. We use the continuous and discrete eigenbases to write

$$u = \sum_{\ell\in\mathbb{N}} (u, e_\ell)_{L^2(D)} e_\ell \quad \text{and} \quad p_n u = \sum_{\ell\in\mathbb{N}} (p_n u, e_{\ell,n})_{L^2(D)} e_{\ell,n}.$$

Since P_W and P_{W_n} are the orthogonal projections onto W and W_n, respectively, we have that

$$P_W u = \sum_{0\le\ell\le\ell_{\max}} (u, e_\ell)_{L^2(D)} e_\ell \quad \text{and} \quad P_{W_n} p_n u = \sum_{0\le\ell\le\ell_{\max}} (p_n u, e_{\ell,n})_{L^2(D)} e_{\ell,n}.$$

[1] For ease of presentation, we will assume that all eigenvalues have algebraic multiplicity one. Otherwise, we would have to introduce more complex notation, see e.g. [EG21b, Chap. 48].

The triangle inequality yields

$$\|p_n P_W u - P_{W_n} p_n u\|_{X_n} \le \|p_n P_W u - P_W u\|_{X_n} + \|P_W u - P_{W_n} p_n u\|_{X_n},$$

where the first term converges to zero by Lemma 3.15. For the second, we have that

$$\begin{aligned}\|P_W u - P_{W_n} p_n u\|_{X_n} &= \sum_{0\le \ell\le \ell_{\max}} \|(u, e_\ell)_{L^2(D)} e_\ell - (p_n u, e_{\ell,n})_{L^2(D)} e_{\ell,n}\|_{X_n} \\ &= \sum_{0\le \ell\le \ell_{\max}} \|(u, e_\ell)_{L^2(D)}(e_\ell - e_{\ell,n}) + ((p_n u, e_{\ell,n})_{L^2(D)} - (u, e_\ell)_{L^2(D)}) e_{\ell,n}\|_{X_n}\end{aligned}$$

Since $\|e_\ell - e_{\ell,n}\|_{X_n} \to 0$ as $n \to \infty$, the first term goes to zero. For the second, consider

$$\begin{aligned}(p_n u, e_{\ell,n})_{L^2(D)} - (u, e_\ell)_{L^2(D)} &= (p_n u - u, e_{\ell,n})_{L^2(D)} + (u, e_{\ell,n} - e_\ell)_{L^2(D)} \\ &\le \|u - p_n u\|_{X_n} \|e_\ell\|_{L^2(D)} + \|u\|_X \|e_l - e_{\ell,n}\|_{X_n}\end{aligned}$$

Thus, as $\|e_\ell - e_{\ell,n}\|_{X_n} \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} \|u - p_n u\|_{X_n} = 0$, we have that

$$\lim_{n\to\infty} \|p_n P_W u - P_{W_n} p_n u\|_{X_n} = 0.$$

□

Lemma 3.20. *For each $u \in X$, it holds that* $\lim_{n\to\infty} \|T_n p_n u - p_n T u\|_{X_n} = 0$.

Proof. It holds that

$$\begin{aligned}\|T_n p_n u - p_n T u\|_{X_n} &= \|(\mathrm{Id}_{X_n} - 2P_{W_n}) p_n u - p_n (I_X - 2P_W) u\|_{X_n} \\ &= \|2(p_n P_W u - P_{W_n} p_n u)\|_{X_n}.\end{aligned}$$

Applying Lemma 3.19 yields the claim. □

In the following, we want to apply Thm. 2.28 to show that the sequence of nonconforming DG-approximations $(A_n^{\mathrm{DG}})_{n\in\mathbb{N}}$ is regular. To this end, we have to prove that A_n^{DG} is T_n-coercive.

Theorem 3.21. *The operator A_n^{DG} induced by the bilinear form $a_n^{DG}(\cdot,\cdot)$ is T_n-coercive.*

Proof. Let $u_n \in X_n$. We have to show that there exists a constant $\alpha > 0$ s.t.

$$a_n^{\mathrm{DG}}(u_n, T_n u_n) \ge \alpha \|u_n\|_{X_n}^2. \tag{3.23}$$

Recall that $X_n = V_n \oplus W_n$ s.t. we can write $u_n = v_n + w_n$ and $T_n u_n = v_n - w_n$. Then

$$\begin{aligned}a_n^{\mathrm{DG}}(u_n, T_n u_n) &= a_n^{\mathrm{DG}}(v_n + w_n, v_n - w_n) \\ &= \int_D G_n^l(v_n + w_n) \cdot G_n^l(v_n - w_n) - \kappa^2 (v_n + w_n)(v_n - w_n) dx \\ &= \int_D G_n^l(v_n) \cdot G_n^l(v_n) - \kappa^2 v_n^2 dx - \int_D G_n^l(w_n) \cdot G_n^l(w_n) + \kappa^2 w_n^2 dx\end{aligned}$$

By construction of V_n and W_n, we can write v_n and w_n in the discrete eigenbasis $(e_{\ell,n})_{\ell\in N}$ as

$$v_n = \sum_{\ell>\ell_{\max}} u_{\ell,n} e_{\ell,n} \quad \text{and} \quad w_n = \sum_{0\le\ell\le\ell_{\max}} u_{\ell,n} e_{\ell,n}.$$

Proceeding similar as in the proof of Lemma 3.2,we have with $\alpha := \min_{\ell\in\mathbb{N}}\{|\lambda_{\ell,n} - \kappa^2|/(1 + \lambda_{\ell,n})\}$ that

$$\begin{aligned} a_n^{\mathrm{DG}}(u_n, Tu_n) &= \sum_{0\le\ell\le\ell_{\max}} (\kappa^2 - \lambda_{\ell,n})u_{\ell,n}^2 + \sum_{\ell>\ell_{\max}} (\lambda_{\ell,n} - \kappa^2)u_{\ell,n}^2 \\ &\ge \alpha \sum_{\ell\in\mathbb{N}} (1 + \lambda_{\ell,n})u_{\ell,n}^2 = \alpha\|u_n\|_{X_n}^2. \end{aligned}$$

Thus, A_n^{DG} is indeed T_n-coercive. □

We conclude the analysis of the DG discretization of the Helmholtz problem with the following convergence result.

Theorem 3.22. *Let $\kappa^2 \notin \{\lambda_\ell\}_{\ell\in\mathbb{N}}$ and let $u \in X \cap H^{2+s}(D)$, $s > 0$, be the solution to (3.2). Then there exists an index $n_0 > 0$ such that there exists a unique $u_n \in X_n$ such that $a_n^{DG}(u_n, v_n) = f(v_n)$ for all $v_n \in X_n$ and*

$$\|u - u_n\|_{X_n} \lesssim \left(h_n^{\min(1+s,k)} + h_n^{\min(s,l)}\right)\|u\|_{H^{2+s}(D)}.$$

Proof. Because of $T_n \xrightarrow{P} T$, $A_n^{\mathrm{DG}} \xrightarrow{P} A$, the T-coercivity of A and the T_n-coercivity of A_n^{DG}, we can apply Thm. 2.17 to conclude the existence of a unique discrete solution $u_n \in X_n$. Furthermore, with the triangle inequality, we have that

$$\|u - u_n\|_{X_n} \lesssim \|u - p_n u\|_{X_n} + \|A_n^{\mathrm{DG}}(p_n u - u_n)\|_{X_n}$$

For the first term, we apply Lemma 3.13 and Corollary 3.14 to obtain

$$\|u - p_n u\|_{X_n} \lesssim h_n^{\min(1+s,k)}\|u\|_{H^{2+s}(D)}.$$

For the second term, we compute

$$\begin{aligned} \|A_n^{\mathrm{DG}}(p_n u - u_n)\|_{X_n} &= \sup_{u_n'\in X_n, \|u_n'\|_{X_n}=1} |a_n^{\mathrm{DG}}(p_n u - u_n, u_n')| \\ &= \sup_{u_n'\in X_n, \|u_n'\|_{X_n}=1} |(\nabla(p_n u - u), G_n^l u_n') - \kappa^2(p_n u - u, u_n') \\ &\qquad + (\nabla u, G_n^l u_n') - \kappa^2(u, u_n') - (f, u_n')_{L^2(D)}| \\ &= O(\|p_n u - u\|_{X_n}, n \to \infty) \\ &\quad + \sup_{u_n'\in X_n, \|u_n'\|_{X_n}=1} |(\nabla u, G_n^l u_n') - \kappa^2(u, u_n') - (f, u_n')_{L^2(D)}|. \end{aligned}$$

For the first term, we can again apply Corollary 3.14 to obtain the desired convergence rate. For the second term, we want to utilize that u solves (3.2). In order to apply partial integration on the discrete gradient G_n^l, let $\psi_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d$ be an $\boldsymbol{H}^1$-projection of ∇u. Then, we have that

$$(\nabla u, G_n^l u_n') = (\psi_n, G_n^l u_n') + (\nabla u - \psi_n, G_n^l u_n')$$

and

$$\begin{aligned} (\psi_n, G_n^l u_n') &= \sum_{\tau\in\mathcal{T}_n} (\psi_n, \nabla u_n' + R_n^l u_n')_{\boldsymbol{L}^2(\tau)} = \sum_{\tau\in\mathcal{T}_n} (\psi_n, \nabla u_n')_{\boldsymbol{L}^2(\tau)} - \sum_{F\in\mathcal{F}_n} (\{\!\{\psi_n\}\!\}, [\![u_n']\!] \cdot \nu)_{\boldsymbol{L}^2(F)} \\ &= \sum_{\tau\in\mathcal{T}_n} (\psi_n, \nabla u_n')_{\boldsymbol{L}^2(\tau)} - (\psi_n, u_n' \cdot \nu)_{\boldsymbol{L}^2(\partial\tau)} \\ &= -(\operatorname{div}\psi_n, u_n') = (\operatorname{div}(\nabla u - \psi_n), u_n') - (\Delta u, u_n'). \end{aligned}$$

Thus, since u solves (3.1), we have that

$$\sup_{u_n' \in X_n, \|u_n'\|_{X_n}=1} |(\nabla u, G_n^l u_n') - \kappa^2 (u, u_n') - (f, u_n')_{L^2(D)}| \lesssim \|\nabla u - \psi_n\|_{\boldsymbol{H}(\mathrm{div})} \lesssim h_n^{\min(s,l)} \|u\|_{H^{2+s}(D)}.$$

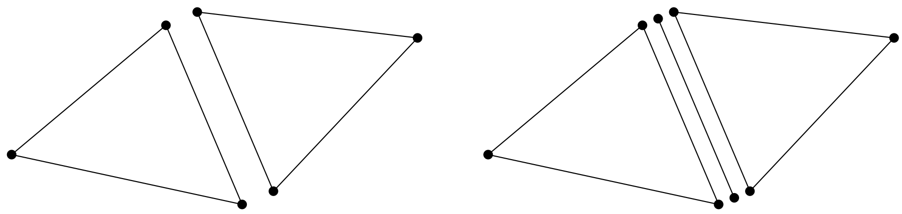
□

Remark 3.23 (Choice of l)**.** *The previous theorem suggests to choose $l = k$ to obtain quasi-optimal convergence rates. However, it also might suffice to choose $l = k-1$, cf. [EG21b, Rem. 38.18] or [DE12, Chap. 4.3], which is also indicated by numerical results. Thus, it might be possible to improve the previous theorem to obtain the following convergence rates:*

$$\|u - u_n\|_{X_n} \lesssim \left(h_n^{\min(1+s,k)} + h_n^{\min(s,l+1)} \right) \|u\|_{H^{2+s}(D)}.$$

3.5 Hybrid discontinuous Galerkin method

In this section, we want to conclude the discussion of the Helmholtz problem by considering a *hybrid discontinuous Galerkin* (HDG) discretization. The main idea behind hybridization [CGL09; Leh10] is to introduce additional unknowns that are defined on the facets of the triangulations, cf. Fig. 3.4. At first glance, this seems to be a counterintuitive approach since the number of degrees of freedom is increased, which makes the linear system larger and thus more expensive to solve. However, the introduction of additional facet unknowns allows us to eliminate the interior degrees of freedom through *static condensation*, cf. Remark 3.25, which makes the overall computation more efficient. Furthermore, in contrast to DG methods, neighboring degrees of freedom do not couple directly, which allows us to assemble the system matrices element-wise.



(a) Discontinuous Galerkin (b) Hybrid Discontinuous Galerkin

Figure 3.4: Degrees of freedom for linear DG and HDG FEM in 2D; inspired by [Leh10, Fig. 1.2.1].

We start by introducing the HDG formulation of the Helmholtz problem (3.1). In addition to the notation introduced in section 3.4, we denote by $L^2(\mathcal{F}_n)$ the space of L^2 functions defined on the facets $F \in \mathcal{F}_n$. We consider the continuous space

$$X := H^2(D) \times L^2(\mathcal{F}_n) \tag{3.24}$$

and the discrete space

$$X_n := \mathbb{P}^k(\mathcal{T}_n) \times \mathbb{P}^k(\mathcal{F}_n). \tag{3.25}$$

We note that it does not suffice to consider H^1 as in the previous section because we require the normal derivative to be well-defined. Thus, we require at least $H^{3/2}$-regularity, but for simplicity, we consider H^2. In the following, we write $u = (u_T, u_F) \in X$. Furthermore, we define the HDG jump operator $[\![\underline{\cdot}]\!]$ through

$$[\![\underline{u}]\!] := u_T - u_F. \tag{3.26}$$

Essentially, we proceed with a similar argumentation as for the derivation of the DG method in the previous section, but use the HDG-jump operator instead of the DG-jump operator and instead of summing over all facets, we stay on the element boundaries. Thus, we do not have average operators involved in the formulation. Consequently, a hybridized SIP method would be implemented through the following bilinear form

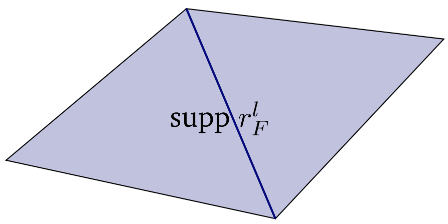
$$
\begin{aligned}
a_n^{\text{HDG,SIP}}(u_n, v_n) := \sum_{\tau\in\mathcal{T}_n} \Big(\int_\tau \nabla u_n \cdot \nabla v_n - \kappa^2 u_n v_n dx \\
- \int_{\partial\tau} \nabla u_n \cdot \nu [\![\underline{v_n}]\!] ds - \int_{\partial\tau} \nabla v_n \cdot \nu [\![\underline{u_n}]\!] ds + \frac{\alpha}{h} \int_{\partial\tau} [\![\underline{u_n}]\!][\![\underline{v_n}]\!] ds \Big).
\end{aligned} \tag{3.27}
$$

However, as in the DG case, we want to avoid choosing the stabilization parameter $\alpha > 0$ in (3.27) explicitly and want to use a lifting stabilization instead. Therefore, we reintroduce the lifting operator defined for the DG scheme in the context of the HDG method. For $\tau \in \mathcal{T}_n$, we define the local HDG lifting operator $\underline{r}_\tau^l : X_n \to [\mathbb{P}^l(\mathcal{T}_n)]^d$ through

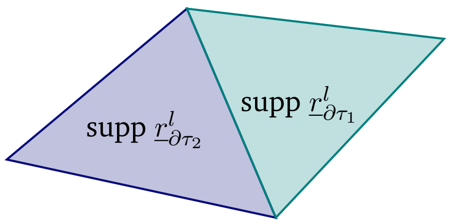
$$
\int_\tau \underline{r}_\tau^l(u_n)\psi_n dx = -\int_{\partial\tau} [\![\underline{u_n}]\!] \cdot \nu \psi_n ds \quad \text{for all } \psi_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d.
$$

One advantage of the HDG lifting operator compared to the DG lifting operator is that it is truly element-local in the sense that $\text{supp}(r_\tau^l) = \tau$, cf. Fig. 3.5. Consequently, the computational costs associated with implementing the lifting operator are reduced. As before, we define the global lifting operator $\underline{R}_n^l = \sum_{\tau\in\mathcal{T}_n} \underline{r}_\tau^l$. Then, we define the discrete gradient through

$$
\underline{G}_n^l u_n := \nabla u_n + \underline{R}_n^l u_n.
$$



(a) DG lifting



(b) HDG lifting

Figure 3.5: Comparison of the support of the DG- and HDG-lifting operators.

With the discrete trace inequality, we have the following result analogously to Lemma 3.10.

Lemma 3.24 (Boundedness of $\underline{R}_n^l$)**.** *For all $u \in H^1(\mathcal{T}_n)$, it holds that*

$$
\|\underline{R}_n^l u\|_{\boldsymbol{L}^2(D)} \lesssim \sum_{F\in\mathcal{F}_n} h_F^{-1/2} \|[\![\underline{u}]\!]\|_{\boldsymbol{L}^2(\partial\tau)}.
$$

Consequently, we define the following lifting stabilized (LS) HDG bilinear form as

$$
a_n^{\text{HDG}}(u_n, v_n) := \sum_{\tau\in\mathcal{T}_n} \int_\tau \underline{G}_n^l(u_n) \cdot \underline{G}_n^l(v_n) dx - \kappa^2 u_n v_n dx, \quad u_n, v_n \in X_n. \tag{3.28}
$$

Then, the LS-HDG scheme reads as: Find $u_n \in X_n$ such that

$$
a_n^{\text{HDG}}(u_n, v_n) = f(v_n) \text{ for all } v_n \in X_n. \tag{3.29}
$$

Remark 3.25 (Static condensation)**.** *We can eliminate interior degrees of freedom through static condensation. In the following, we denote with $\mathcal{A} \in \mathbb{C}^{N\times N}$, $N := \dim X_n$, the stiffness matrix defined through $\mathcal{A}_{ij} := a_n^{HDG}(\varphi_j, \varphi_i)$, $i, j \in \{1, \ldots, N\}$, where $\{\varphi_i\}_{i\in\{1,\ldots,N\}}$ is a basis of X_n. Solving (3.29) amounts to solving the linear system $\mathcal{A}\mathcal{U} = \mathcal{F}$, where $\mathcal{F}_i := f(\varphi_i)$, $i \in \{1, \ldots, N\}$, and $\mathcal{U} \in \mathbb{C}^N$ is the vector corresponding to u. Then, we can split the system in the following way:*

$$\begin{pmatrix} \mathcal{A}_{\mathcal{T}_n\mathcal{T}_n} & \mathcal{A}_{\mathcal{T}_n\mathcal{F}_n} \\ \mathcal{A}_{\mathcal{F}_n\mathcal{T}_n} & \mathcal{A}_{\mathcal{F}_n\mathcal{F}_n} \end{pmatrix} \begin{pmatrix} \mathcal{U}_{\mathcal{T}_n} \\ \mathcal{U}_{\mathcal{F}_n} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{\mathcal{T}_n} \\ \mathcal{F}_{\mathcal{F}_n} \end{pmatrix}. \tag{3.30}$$

Here, $\mathcal{U}_{\mathcal{T}_n}$ corresponds to u_T, $\mathcal{U}_{\mathcal{F}_n}$ to u_F, $\mathcal{A}_{\mathcal{T}_n\mathcal{T}_n}$ to the terms contained in $a(u_T, u_T)$, $\mathcal{A}_{\mathcal{T}_n\mathcal{F}_n}$ to the terms contained in $a(u_T, u_F)$, $\mathcal{A}_{\mathcal{F}_n\mathcal{T}_n}$ to the terms contained in $a(u_F, u_T)$, and $\mathcal{A}_{\mathcal{F}_n\mathcal{F}_n}$ to the terms contained in $a(u_F, u_F)$. Since $\mathcal{A}_{\mathcal{T}_n\mathcal{T}_n}$ is block diagonal, it can be inverted efficiently. Using the Schur complement, we can eliminate the interior degrees of freedom and reduce the computational costs in the following way. First, we observe that

$$\mathcal{U}_{\mathcal{T}_n} = \mathcal{A}_{\mathcal{T}_n\mathcal{T}_n}^{-1}\big(\mathcal{F}_{\mathcal{T}_n} - \mathcal{A}_{\mathcal{T}_n\mathcal{F}_n}\mathcal{U}_{\mathcal{F}_n}\big). \tag{3.31}$$

Inserting this equality into the second equation of (3.30) yields

$$\underbrace{\big(\mathcal{A}_{\mathcal{F}_n\mathcal{F}_n} - \mathcal{A}_{\mathcal{F}_n\mathcal{T}_n}\mathcal{A}_{\mathcal{T}_n\mathcal{T}_n}^{-1}\mathcal{A}_{\mathcal{T}_n\mathcal{F}_n}\big)}_{=:\mathcal{S}}\mathcal{U}_{\mathcal{F}_n} = \mathcal{F}_{\mathcal{F}_n} - \mathcal{A}_{\mathcal{F}_n\mathcal{T}_n}\mathcal{A}_{\mathcal{T}_n\mathcal{T}_n}^{-1}\mathcal{F}_{\mathcal{T}_n}. \tag{3.32}$$

where the matrix $\mathcal{S}$ is called the Schur-complement. Thus, we can first solve for $\mathcal{U}_{\mathcal{F}_n}$ using (3.32) and then reconstruct $\mathcal{U}_{\mathcal{T}_n}$ using (3.31).

Remark 3.26 (Relationship between HDG- and DG schemes)**.** *The HDG scheme can be reformulated to recover an equivalent DG scheme by eliminating the facet unknowns [Leh10; Fu+21]. To this end, we introduce a lifting operator $\mathcal{L}_h$ that maps volume functions to a unique facet function. We define the lifting $\mathcal{L}_h(u_T)$ as the unique function in $\mathbb{P}^k(\mathcal{F}_n)$ such that*

$$a_n^{HDG}((u_T, \mathcal{L}_h(u_T)), (0, v_F)) = 0 \quad \textit{for all } v_F \in \mathbb{P}^k(\mathcal{F}_n). \tag{3.33}$$

Then, the solution $\underline{u}_n = (u_T, u_F) \in \mathbb{P}^k(\mathcal{T}_n) \times \mathbb{P}^k(\mathcal{F}_n)$ of the HDG scheme satisfies $u_F = \mathcal{L}_h(u_T)$ with u_T being the solution to

$$\hat{a}_n(u_T, v_T) = \hat{f}(v_T) \; \textit{for all } v_T \in \mathbb{P}^k(\mathcal{T}_n),$$

where we define the bilinear form $\hat{a}_n(\cdot,\cdot)$ and the linear form $\hat{f}(\cdot)$ on $\mathbb{P}^k(\mathcal{T}_n)$ as

$$\hat{a}_n(u_T, v_T) := a_n^{HDG}((u_T, \mathcal{L}_h(u_T)), (v_T, 0)) \quad \textit{and} \quad \hat{f}(v_T) := f((v_T, 0)).$$

We can also define a corresponding DG norm through

$$\|u_T\|_{DG} := \|(u_T, \mathcal{L}_h(u_T))\|_{HDG}.$$

If we consider the SIP HDG formulation (3.27) with uniform mesh size h, an explicit formula for $\mathcal{L}_h$ would be given by, cf. [Fu+21, Sec. 3.5],

$$\mathcal{L}_h(u_T) = \Pi_F^k(\{\!\{u_T\}\!\} - \frac{h}{2\alpha}[\![\nabla u_T \cdot \nu]\!]), \tag{3.34}$$

where Π_F^k is the $L^2(\mathcal{F}_n)$-projection.

3.5.1 Sketch of Analysis

To analyze problem (3.29), we have to interpret the HDG scheme as a discrete approximation scheme analogously to section 3.4.1. We define the inner product on X_n through

$$(u_n, v_n)_{X_n} := (u_n, v_n)_{L^2(D)} + (\underline{G}_n^l u_n, \underline{G}_n^l v_n)_{\boldsymbol{L}^2(D)}.$$

and a projection operator $p_n \in L(X, X_n)$ through

$$(p_n u, v_n)_{X_n} = (u, v_n)_{L^2(D)} + (\nabla u, \underline{G}_n^l v_n)_{\boldsymbol{L}^2(D)} \quad \text{for all } u \in X, v_n \in X_n.$$

Proceeding as in section 3.4.1, we can show that indeed $\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X$. Furthermore, we can show the following compactness result:

Lemma 3.27. *Let $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, be such that $\sup \|u_n\|_{X_n} < \infty$. Then, there exists $u \in X$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $u_n \overset{L^2}{\rightharpoonup} u$ and $\underline{G}_n^l u_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \nabla u$.*

Proof. The statement was shown in [KCR21, Thm. 4.3] with similar techniques to [BO08, Thm. 5.2]. □

Then, with the same argumentation as in Lemma 3.18, it follows that $A_n^{\mathrm{HDG}} \overset{P}{\to} A$. Therefore, $(X_n, p_n, A_n^{\mathrm{HDG}})$ is a discrete approximation scheme of (X, A).
To analyze the convergence of the method, we have to reconstruct the T-operator once more. As before, we consider approximations of the Laplace eigenpairs $(\lambda_\ell, e_\ell)_{\ell\in\mathbb{N}}$ by solving

$$(\underline{G}_n^l e_{\ell,n}, \underline{G}_n^l v_n)_{\boldsymbol{L}^2(D)} = \lambda_{\ell,n}(e_{\ell,n}, v_n)_{L^2(D)} \quad \text{for all } v_n \in X_n.$$

To our knowledge, the HDG Laplace eigenvalue problem in this form has not yet been analyzed in the literature. However, due to the correspondence between HDG and DG schemes, cf. Remark 3.26, we expect that an analysis of this problem can be carried out analogously to the DG case [ABP06]. Thus, we conjecture at this point that there exists an index $n^* > 0$ such that

$$\|e_\ell - e_{\ell,n}\|_{X_n} \le \delta(n), \qquad \lim_{n\to\infty} \delta(n) = 0,$$

for all $\ell \in \mathbb{N}$ and that $\lambda_{\ell_{\max},n} < \kappa^2$. Then, we set

$$W_n := \mathrm{span}_{0\le\ell\le\ell_{\max}}(e_{\ell,n})$$

and $V_n := W_n^\perp$. Then, we define the T_n-operator through

$$T_n := P_{V_n} - P_{W_n},$$

where P_{V_n}, P_{W_n} are the orthogonal projections from X_n onto V_n and W_n, respectively. As before, we have that $T_n^{-1} = T_n$, i.e. the operator T_n is bijective.
With similar arguments as before, we can show that $T_n \overset{P}{\to} T$ and that the sequence $(A_n^{\mathrm{HDG}})_{n\in\mathbb{N}}$ is T_n-coercive. Consequently, the sequence $(A_n^{\mathrm{HDG}})_{n\in\mathbb{N}}$ is regular and we can apply Thm. 2.17 to conclude the convergence of the HDG scheme:

Theorem 3.28. *Let $\kappa^2 \notin \{\lambda_\ell\}_{\ell\in\mathbb{N}}$ and let $u \in X \cap H^{2+s}(D)$, $s > 0$, be the solution to (3.2). Then, there exists an index $n_0 > 0$ such that there exists a unique $u_n \in X_n$ such that $a_n^{HDG}(u_n, v_n) = f(v_n)$ for all $v_n \in X_n$. Furthermore, it holds that*

$$\|u - u_n\|_{X_n} \lesssim \left(h_n^{\min(1+s,k)} + h_n^{\min(s,l)}\right)\|u\|_{H^{2+s}(D)}.$$

3.6 Numerical Example

To conclude the discussion on the Helmholtz problem, we consider a numerical example. In particular, we compare the DG and HDG lifting methods with each other and their respective SIP counterparts, cf. (3.15) and (3.27). To this end, we consider the unit disk

$$D = \mathbb{S}^1 := \{x \in \mathbb{R}^2 : |x| \leq 1\}$$

and choose the right hand side $f \in L^2(D)$ such that the exact solution of (3.1) is given by

$$u(x,y) := \sin(x^2 + y^2 - 1) \exp\big(i\kappa(x+y)\big). \tag{3.35}$$

Note that the first term enforces homogeneous Dirichlet boundary conditions on the unit disk. Furthermore, we choose a small wavenumber $\kappa = 4$ to keep the mesh size requirements very mild. For both SIP methods, we choose the stabilization parameters $\alpha_{\text{HDG}} = 5k^2$ and $\alpha_{\text{DG}} = 10k^2$. We note that we choose $\alpha_{\text{DG}} = 2\alpha_{\text{HDG}}$ deliberately to compare the methods with each other, because intuitively, the HDG SIP method penalizes the jumps between two elements twice, though indirectly over the facets. More formally, when we considered the equivalent DG formulation in Remark 3.26, we observed a factor $1/2$ in the definition of the lifting operator (3.34), which we can resolve by scaling the stabilization parameter accordingly. For more details, we refer to [Leh10, Sec. 1.2.2.2]. For the implementation of the LS methods, we choose $l = k$ to ensure that we can achieve optimal order convergence. However, numerical experiments indicate that $l = k-1$ would be sufficient, where we note that for the HDG method, we would also have to choose the facet space of order $k-1$. While choosing $l = k-1$ is computationally cheaper, we proceed with $l = k$ to be in line with the theoretical results, cf. also Remark 3.23. The examples are computed on the scientific computing cluster of the GWDG[2]and use 12 threads.

Remark 3.29 (Implementation of the lifting operators). *We implement the lifting operators through a mixed formulation in the following way. Recall, that we have for the bilinear form* $a_n^{DG}(\cdot,\cdot)$ *that*

$$\begin{aligned} a_n^{DG}(u_n, v_n) &:= \sum_{\tau \in \mathcal{T}_n} \int_\tau G_n^l(u_n) \cdot G_n^l(v_n) - \kappa^2 u_n v_n dx \\ &= \sum_{T \in \mathcal{T}_h} \Big(\int_\tau \nabla u_n \cdot \nabla v_n dx + \int_\tau \nabla v_n \cdot R_n^l(u_n) dx + \int_\tau \nabla u_n \cdot R_n^l(v_n) dx \\ &\qquad + \int_\tau R_n^l(u_n) \cdot R_n^l(v_n) dx - \int_\tau \kappa^2 u_n v_n dx \Big) \end{aligned} \tag{3.36}$$

From the definition of the local lifting operator r_F^l*, we have for the second and third term that*

$$\begin{aligned} \sum_{\tau \in \mathcal{T}_n} \int_\tau \nabla v_n \cdot R_n^l(u_n) dx &= - \sum_{F \in \mathcal{F}_n} \int_F [\![u_n]\!] \cdot \nu \{\!\{\nabla v_n\}\!\} ds, \\ \sum_{\tau \in \mathcal{T}_n} \int_\tau \nabla u_n \cdot R_n^l(v_n) dx &= - \sum_{F \in \mathcal{F}_n} \int_F [\![v_n]\!] \cdot \nu \{\!\{\nabla u_n\}\!\} ds. \end{aligned} \tag{3.37}$$

To deal with the fourth term of (3.36)*, we introduce an auxiliary variable* $r := R_n^l(u_n) \in [\mathbb{P}^l(\mathcal{T}_n)]^d$ *such that*

$$\sum_{\tau \in \mathcal{T}_n} \int_\tau R_n^l(u_n) \cdot R_n^l(v_n) dx = \sum_{\tau \in \mathcal{T}_n} \int_\tau r \cdot R_n^l(v_n) dx = - \sum_{F \in \mathcal{F}_n} \int_F \{\!\{r\}\!\} \cdot \nu [\![v_n]\!] ds. \tag{3.38}$$

[2]for more information, we refer to `https://docs.gwdg.de/doku.php?id=en:services:application_services:high_performance_computing:start`.

By definition of the local lifting operator r_F^l, the variable $r \in [\mathbb{P}^l(\mathcal{T}_n)]^d$ solves the variational problem

$$\sum_{\tau \in \mathcal{T}_n} \int_\tau r \cdot s dx = -\sum_{F \in \mathcal{F}_n} \int_F [\![u_n]\!] \cdot \nu \{\!\{s\}\!\} ds \quad \textit{for all } s \in [\mathbb{P}^l(\mathcal{T}_n)]^d. \tag{3.39}$$

Altogether, we can implement the LS method through the following mixed formulation: Find $(u_n, r) \in X_n \times [\mathbb{P}^l(\mathcal{T}_n)]^d$ such that

$$\check{a}_n((u_n, r), (v_n, s)) = f(v_n) \textit{ for all } (v_n, s) \in X_n \times [\mathbb{P}^l(\mathcal{T}_n)]^d,$$

where we define the bilinear form $\check{a}(\cdot, \cdot)$ through

$$\check{a}_n((u_n, r), (v_n, s)) := \sum_{\tau \in \mathcal{T}_n} \Big(\int_\tau \nabla u_n \cdot \nabla v_n - \kappa^2 u_n v_n dx \Big) \tag{3.40a}$$

$$- \sum_{F \in \mathcal{F}_n} \Big(\int_F [\![u_n]\!] \cdot \nu \{\!\{\nabla v_n\}\!\} ds + \int_F \{\!\{\nabla u_n\}\!\} \cdot \nu [\![v_n]\!] ds \Big) \tag{3.40b}$$

$$- \sum_{F \in \mathcal{F}_n} \int_F \{\!\{r\}\!\} \cdot \nu [\![v_n]\!] ds \tag{3.40c}$$

$$- \sum_{\tau \in \mathcal{T}_n} \int_\tau r \cdot s dx - \sum_{F \in \mathcal{F}_n} \int_F [\![u_n]\!] \cdot \nu \{\!\{s\}\!\} ds. \tag{3.40d}$$

The terms in (3.40b) stem from (3.37), the term in (3.40c) from (3.38), and the term (3.40d) accounts for the auxillary problem (3.39). To implement the HDG method, we follow the same steps with the modified definition of the lifting operator. However, we use the HDG-jump $[\![\underline{\cdot}]\!]$ instead of $[\![\cdot]\!]$ and leave out the average operator. We further note that we add the terms corresponding to (3.40c) and (3.40d) twice to ensure that the method is preasymptotically stable.

Fig. 3.6 displays the convergence rates in the L^2- and H^1-norms. We observe that, as expected from Thms. 3.22 and 3.28, the H^1-error converges with order k for all methods. Furthermore, the L^2-error also converges optimally with order $k+1$. Fig. 3.7 compares the computational costs measured in seconds for matrix assembly and solving the linear system for each method. While the HDG method is more efficient for both, the SIP and the LS method, the effect is more profound in the latter case. We observe, in particular, that the cost for solving the linear system is the same for both methods in the HDG case, while the computational costs of the LS-DG method are higher than for the SIP-DG method. This is due to the fact that, after static condensation, the SIP- and LS-HDG methods result in a stiffness matrix with the same sparsity pattern, while the stiffness matrix associated with the LS-DG method is more dense than the one associated with the SIP method, cf. Fig. 3.8.
Altogether, these results clearly demonstrate the efficiency of the HDG method in comparison with the DG method, especially when considering the LS method. We note at this point that this comparison is somewhat naive, as the efficiency of the HDG method is optimized through static condensation, whereas for the DG method no such efforts were made since static condensation is not provided in `NGSolve` for DG-schemes. It is reasonable to assume that the DG method could be further optimized, for instance by building the Schur complement as well. However, this process is more suited for the HDG method due to its local character which is also reflected in the smaller support of the lifting operator, cf. Fig. 3.5. Thus, even if we were to optimize the DG method, the advantage of the HDG method would remain. To conclude the comparison of the different methods, let us note that regardless of whether we use the DG or HDG method, the SIP method is more efficient than the LS method. The true advantage of the latter lies in the fact that we can avoid restrictions on the stabilization parameter α, which becomes more relevant when considering Galbrun's equation in Part II.

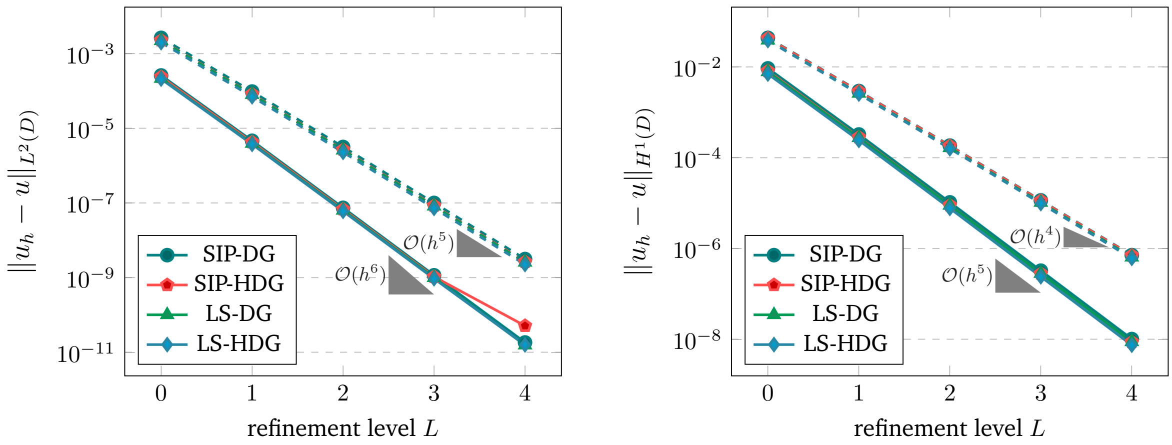


Figure 3.6: Rates of convergence in the L^2-norm (left) and the H^1-norm (right) of SIP-DG,SIP-HDG,LS-DG and LS-HDG for polynomial degrees $k = 4$ (dashed) and $k = 5$.

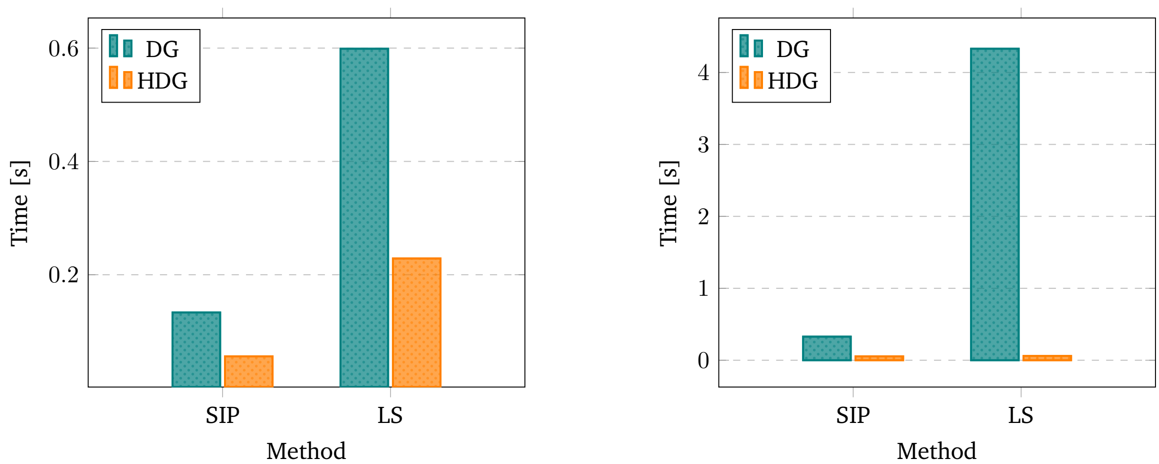
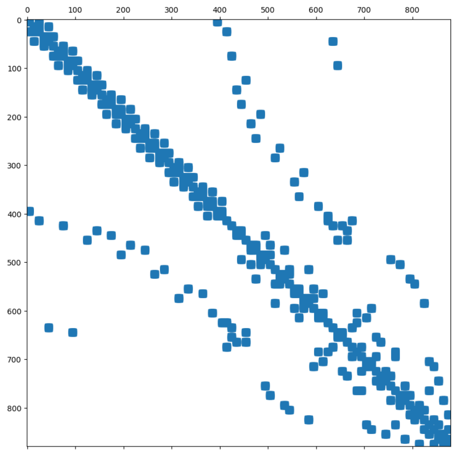
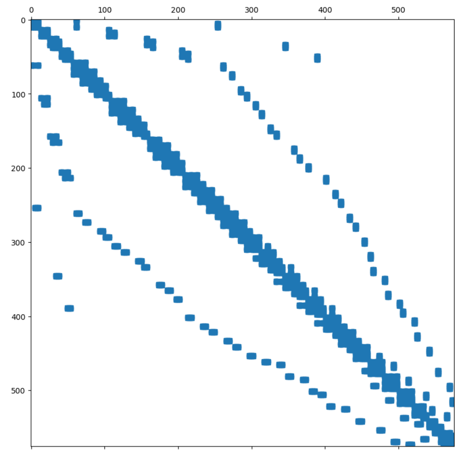


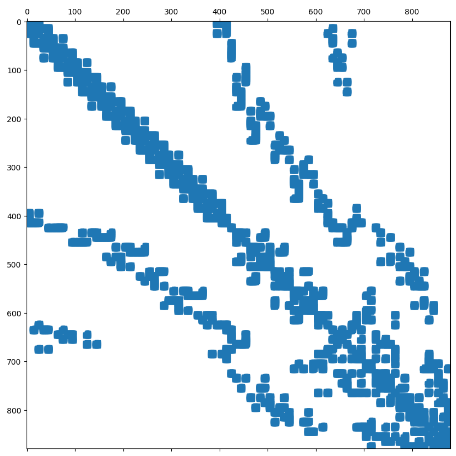
Figure 3.7: Run times for assembly (left) and solve (right) for SIP-DG,SIP-HDG,LS-DG and LS-HDG for polynomial degrees $k = 5$ on the second refinement level.



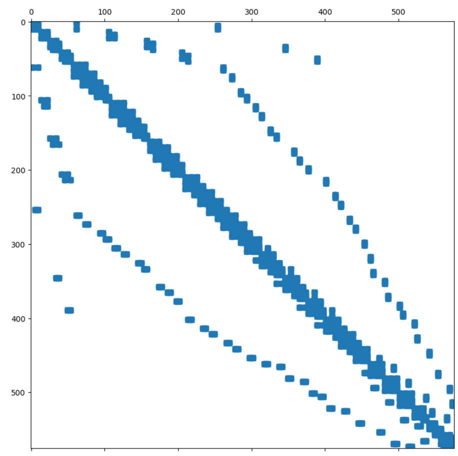
(a) DG SIP



(b) HDG SIP



(c) DG LS



(d) HDG LS

Figure 3.8: Sparsity pattern of the stiffness matrix $\mathcal{A}$ associated with the bilinear forms $a^{\mathrm{DG}}(\cdot,\cdot)$ and $a^{\mathrm{HDG}}(\cdot,\cdot)$ for the SIP and LS methods with polynomial degree $k = 3$ and mesh size $h = 0.25$. For the matrix corresponding to the DG-LS method, the Schur complement was calculated.

Part II

Application for Galbrun's equation

CHAPTER 4

Galbrun's equation

Galbrun's equation was first derived in [Gal31] and is a Lagrangian linearization of the time-dependent nonlinear Euler equations. It is used in aeroacoustics [MGM20] and in its extended form in asteroseismology [LO67] to model stellar oscillation. For more details on its derivation, we refer to [HH21, Sec. 2] and [MGM20, Sec. 2]. In a recent paper, Halla and Hohgage [HH21] proved the well-posedness of Galbrun's equation and the equations of solar and stellar oscillations. In this chapter, we will review the analysis of the former problem, which serves as a basis for the construction of discretizations, and briefly examine the connection of Galbrun's equation with the equations of solar and stellar oscillations. For ease of presentation, we will only present a few selected proofs. For more details, we refer to [HH21].

Contents of the chapter

4.1 Well-posedness of Galbrun's equation

In this section, we consider the well-posedness of Galbrun's equation with homogeneous normal boundary conditions:

$$\begin{aligned} -\nabla(\rho c_s^2 \operatorname{div} \boldsymbol{u}) + (\operatorname{div} \boldsymbol{u})\nabla p - \nabla(\nabla p \cdot \boldsymbol{u}) - \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u} & \\ + (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u} + \gamma\rho(-i\omega)\boldsymbol{u} = \boldsymbol{f} & \quad \text{in } \mathcal{O}, \qquad (4.1a) \\ \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 & \quad \text{on } \partial\mathcal{O}, \qquad (4.1b) \end{aligned}$$

In the following, we assume that $\mathcal{O}$ is a bounded Lipschitz domain and that $\omega \in \mathbb{R}$, $\Omega \in \mathbb{R}^3$. Furthermore, we assume that the functions $c_s, \rho, \gamma : \mathcal{O} \to \mathbb{R}$ are measurable and bounded, i.e. there exists $\underline{c_s}, \overline{c_s}, \underline{\rho}, \overline{\rho}, \underline{\gamma}, \overline{\gamma} \in \mathbb{R}_{>0}$ such that

$$\underline{c_s} \leq c_s \leq \overline{c_s}, \qquad \underline{\rho} \leq \rho \leq \overline{\rho}, \qquad \underline{\gamma} \leq \gamma \leq \overline{\gamma}$$

holds almost everywhere in $\mathcal{O}$. Finally, let $p, \phi \in W^{2,\infty}(\mathcal{O}, \mathbb{R})$ and $\boldsymbol{b} \in \boldsymbol{L}^\infty(\mathcal{O}, \mathbb{R}^3)$ such that $\operatorname{div}(\rho \boldsymbol{b}) \in L^2(\mathcal{O})$. The latter assumption is necessary to define $\partial_{\boldsymbol{b}}$ in a weak sense through

$$\langle \rho \partial_{\boldsymbol{b}} \boldsymbol{u}, \boldsymbol{u}' \rangle := -\langle \rho \boldsymbol{u}, \partial_{\boldsymbol{b}} \boldsymbol{u}' \rangle - \langle \operatorname{div}(\rho \boldsymbol{b}) \boldsymbol{u}, \boldsymbol{u}' \rangle, \qquad (4.2)$$

where $\boldsymbol{u}' \in (C_0^\infty(\mathcal{O}))^3$. We note that this assumption is not particularly problematic as we can expect that mass conservation $\operatorname{div}(\rho \boldsymbol{b}) = 0$ holds.

Remark 4.1 (On the dependence of $\partial_{\boldsymbol{b}}$ on ρ)**.** *Due to the definition* (4.2)*, the weak derivative $\partial_{\boldsymbol{b}}$ depends on the density ρ. Consequently, the space $\boldsymbol{X}$, cf.* (4.3)*, defined below also depends implicitly on ρ. Note, however, that this dependency vanishes for sufficiently smooth $\rho \in W^{1,\infty}(\mathcal{O})$. For more details, we refer to [HH21, Section 2.4].*

Now, we introduce the following space

$$\boldsymbol{X} := \{\boldsymbol{u} \in \boldsymbol{L}^2 : \operatorname{div} \boldsymbol{u} \in L^2, \partial_{\boldsymbol{b}} \boldsymbol{u} \in \mathbf{L}^2, \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \text{ on } \partial\mathcal{O}\} \tag{4.3}$$

together with the corresponding inner product

$$\langle \boldsymbol{u}, \boldsymbol{u}' \rangle_{\boldsymbol{X}} := \langle \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle + \langle \partial_{\boldsymbol{b}} \boldsymbol{u}, \partial_{\boldsymbol{b}} \boldsymbol{u}' \rangle + \langle \boldsymbol{u}, \boldsymbol{u}' \rangle. \tag{4.4}$$

Lemma 4.2 (Lem. 2.1 of [HH21])**.** *With the assumptions on $\boldsymbol{b}$ from above, the space $\boldsymbol{X}$ defined in* (4.3) *is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_{\boldsymbol{X}}$.*

Proof. We focus on completeness. To this end, let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\boldsymbol{X}$. Since $L^2(\mathcal{O})$ and $\boldsymbol{L}^2(\mathcal{O})$ are complete, there exit $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{O})$ and $w \in L^2(\mathcal{O})$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$, $\partial_{\boldsymbol{b}} \boldsymbol{u}_n \to \boldsymbol{v}$ and $\operatorname{div} \boldsymbol{u}_n \to w$ as $n \to \infty$. For each $\boldsymbol{u}' \in C_0^\infty(\mathcal{O})$, we have that

$$\begin{aligned}
\langle \rho \partial_{\boldsymbol{b}} \boldsymbol{u}, \boldsymbol{u}' \rangle &= -\langle \rho \boldsymbol{u}, \partial_{\boldsymbol{b}} \boldsymbol{u}' \rangle - \langle \operatorname{div}(\rho \boldsymbol{b}) \boldsymbol{u}, \boldsymbol{u}' \rangle = -\lim_{n\to\infty} \left(\langle \rho \boldsymbol{u}_n, \partial_{\boldsymbol{b}} \boldsymbol{u}' \rangle + \langle \operatorname{div}(\rho \boldsymbol{b}) \boldsymbol{u}_n, \boldsymbol{u}' \rangle \right) \\
&= \lim_{n\to\infty} \langle \rho \partial_{\boldsymbol{b}} \boldsymbol{u}_n, \boldsymbol{u}' \rangle_{\boldsymbol{X}} = \langle \rho \boldsymbol{v}, \boldsymbol{u}' \rangle,
\end{aligned}$$

and thus $\partial_{\boldsymbol{b}} \boldsymbol{u} = \boldsymbol{v}$. Similarly, it follows that $\operatorname{div} \boldsymbol{u} = w$ and therefore $\boldsymbol{u}_n \xrightarrow{\boldsymbol{X}} \boldsymbol{u}$. □

To derive a variational formulation of (4.1), we multiply with a testfunction $\boldsymbol{u}' \in \boldsymbol{X}$ and integrate over the domain $\mathcal{O}$. Furthermore, we apply partial integration to obtain

$$-\langle \nabla(\rho c_s^2 \operatorname{div} \boldsymbol{u}), \boldsymbol{u}' \rangle = \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle - \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \boldsymbol{u}' \cdot \boldsymbol{\nu} \rangle_{\partial\mathcal{O}}, \tag{4.5a}$$

$$-\langle \nabla(\nabla p \cdot \boldsymbol{u}), \boldsymbol{u}' \rangle = \langle \nabla p \cdot \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle - \langle \nabla p \cdot \boldsymbol{u}, \boldsymbol{u}' \cdot \boldsymbol{\nu} \rangle_{\partial\mathcal{O}}, \tag{4.5b}$$

where the boundary terms are zero due to the $\boldsymbol{u}' \cdot \boldsymbol{\nu} = 0$ on $\partial\mathcal{O}$. Thus, the variational formulation of (4.1) reads as: Find $\boldsymbol{u} \in \boldsymbol{X}$ such that

$$a(\boldsymbol{u}, \boldsymbol{u}') = \langle \boldsymbol{f}, \boldsymbol{u}' \rangle \text{ for all } \boldsymbol{u}' \in \boldsymbol{X}, \tag{4.6}$$

where the sesquilinear form $a(\cdot,\cdot)$ is defined as

$$\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{u}') :=& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}' \rangle \\
&+ \langle \operatorname{div} \boldsymbol{u}, \nabla p \cdot \boldsymbol{u}' \rangle + \langle \nabla p \cdot \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\Phi))\boldsymbol{u}, \boldsymbol{u}' \rangle \\
&- i\omega \langle \gamma \rho \boldsymbol{u}, \boldsymbol{u}' \rangle.
\end{aligned} \tag{4.7}$$

Remark 4.3. *Let us emphasize at this point that we deviate from the notation in [HH21] by calling the sesquilinear form $a(\cdot,\cdot)$ instead of $a_{\text{Cow}}(\cdot,\cdot)$ because the focus of this work is primarily on Galbrun's equation. Therefore, we rather denote the sesquilinear form associated with the equations of solar and stellar oscillation as $a_{\text{ext}}(\cdot,\cdot)$, see Section 4.2.*

Let $A \in L(\boldsymbol{X})$ denote the operator induced by the sesquilinear form $a(\cdot,\cdot)$. To show that the problem (4.6) is well-posed, it suffices to show that the operator A is weakly T-coercive, cf. Definition 1.14, and injective. Then, we can apply Corollary 1.17 to conclude the bijectivity of A. The injectivity of A follows directly from the modeling of the damping term, as the following Lemma shows.

Lemma 4.4 (Injectivity of A, Lem. 3.7 of [HH21])**.** *Additionally to the assumption from above, assume that $\omega \neq 0$. Then the operator A induced by $a(\cdot,\cdot)$ is injective.*

Proof. Let $\boldsymbol{u} \in \ker(A)$. Then, we have

$$0 = |\Im(a(\boldsymbol{u},\boldsymbol{u}))| = |\omega|\langle \gamma\rho\boldsymbol{u},\boldsymbol{u}\rangle \geq |\omega|\underline{\gamma\rho}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2},$$

which implies $\boldsymbol{u} = 0$. □

4.1.1 A generalized Helmholtz decomposition

To construct a bijective operator $T \in L(\boldsymbol{X})$ such that A is indeed weakly T-coercive, a generalized Helmholtz decomposition of the space $\boldsymbol{X}$ is introduced. In the following, we denote

$$\boldsymbol{q} := c_s^{-2}\rho^{-1}\nabla p. \tag{4.8}$$

Theorem 4.5 (Generalized Helmholtz decomposition)**.** *Let ρ, c_s, p and $\boldsymbol{b}$ satisfy the assumptions from above. If $\boldsymbol{b} \neq 0$, assume furthermore that $\mathcal{O}$ is of class $C^{1,1}$ or convex. Then, the space $\boldsymbol{X}$ defined in* (4.3) *admits a topological decomposition*

$$\boldsymbol{X} = \boldsymbol{V} \oplus \boldsymbol{W} \oplus \boldsymbol{Z}, \tag{4.9}$$

where

1. *$\boldsymbol{V} \subset \{\nabla v_0 : v_0 \in H^2(\mathcal{O})$ with $\frac{\partial v_0}{\partial \boldsymbol{\nu}} = 0$ on $\partial\mathcal{O}\}$ is compactly embedded in $\boldsymbol{L}^2$.*
2. *$\boldsymbol{W} = \{\boldsymbol{u} \in \boldsymbol{X} : (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{u} = 0\}$.*
3. *$\boldsymbol{Z}$ is finite dimensional.*

In addition, if the domain $\mathcal{O}$ is of class $C^{1,1}$ or convex, there exists $C_{reg} \in (0,1)$ such that

$$C^2_{reg}\|\nabla \boldsymbol{v}\|^2_{(L^2)^{3\times 3}} - (1 - C^2_{reg})\|\boldsymbol{v}\|^2_{L^2} \leq \|\operatorname{div}\boldsymbol{v}\|^2_{L^2} \tag{4.10}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$. If the domain $\mathcal{O}$ is of class $C^{1,1}$. convex, or piecewise $C^{1,1}$, then for all $\eta \in W^{1.\infty}$ there exists a compact operator $K_\eta \in L(\boldsymbol{X})$ such that

$$\|\eta \operatorname{div}\boldsymbol{v}\|^2_{L^2} = \|\eta\nabla\boldsymbol{v}\|^2_{(L^2)^{3\times 3}} + \langle K_\eta \boldsymbol{v}, \boldsymbol{v}\rangle_{\boldsymbol{X}}. \tag{4.11}$$

Proof. We refer [HH21, Thm. 3.5]. □

Remark 4.6 (Connection to the classical Helmholtz decomposition)**.** *The classical Helmholtz decomposition [Joh+17, Lem. 2.6] asserts that on connected domains, every function $f \in \boldsymbol{L}^2$ can be uniquely decomposed into a gradient function and a divergence-free function, i.e. that there exists $\phi \in H^1 \setminus \mathbb{R}$ and $f_0 \in \boldsymbol{H}(\operatorname{div})$ with $\operatorname{div} f_0 = 0$ such that $f = f_0 + \nabla\phi$ and $(f_0, \nabla\psi) = 0$ for all $\psi \in H^1$. If $\boldsymbol{q} = 0$, for example in the case of constant pressure p, then $\boldsymbol{Z} = \{0\}$ by construction and therefore the decomposition* (4.9) *reduces to the classical Helmholtz decomposition. In contrast, if $\boldsymbol{q} \neq 0$, then $(\operatorname{div} + \boldsymbol{q}\cdot)$ might not be surjective, which is why the space $\boldsymbol{Z}$ is required. We refer to [HH21] for more details.*

4.1.2 T-coercivity

The decomposition of $\boldsymbol{X}$ above allows us to construct the operator T in the following way. For $\boldsymbol{u} \in \boldsymbol{X}$, we introduce the notation

$$\boldsymbol{v} := P_{\boldsymbol{V}}\boldsymbol{u}, \qquad \boldsymbol{w} := P_{\boldsymbol{W}}\boldsymbol{u}, \qquad \boldsymbol{z} := P_{\boldsymbol{Z}}\boldsymbol{u},$$

where $P_{\boldsymbol{V}}$, $P_{\boldsymbol{W}}$ and $P_{\boldsymbol{Z}}$ are the projections from $\boldsymbol{X}$ into the subspace $\boldsymbol{V}$, $\boldsymbol{W}$ and $\boldsymbol{Z}$ from the decomposition (4.9), respectively. Then, $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w} + \boldsymbol{z}$. Now, we define on operator T through

$$T := P_{\boldsymbol{V}} - P_{\boldsymbol{W}} + P_{\boldsymbol{Z}},$$

which switches the sign of $\boldsymbol{w}$. We note that T is selfinverse, that is $T^2 = \mathrm{Id}_{\boldsymbol{X}}$, and therefore bijective. To formulate a weak T-coercivity result, we introduce the following notations. For a matrix $M \in \mathbb{C}^{3\times 3}$, we define its *numerical range* by

$$\operatorname{numran} M := \{\xi^H M \xi : \xi \in \mathbb{C}^3, |\xi|_2 = 1\}, \tag{4.12}$$

where ξ^H denotes the conjugate transpose of ξ. Then, we define

$$M := i\omega\rho\gamma I_{3\times 3} - \mathrm{Hess}(p) + \rho\,\mathrm{Hess}(\phi) + c_s^{-2}\rho^{-1}\nabla p \otimes \nabla p \tag{4.13}$$

$$\theta := \max\{0, \sup_{x\in\mathcal{O}} |\arg\operatorname{numran} M| - \frac{\pi}{2}\}. \tag{4.14}$$

The following theorem asserts the weak T-coercivity of the operator A induced by the sesquilinear form $a(\cdot,\cdot)$ defined in (4.7).

Theorem 4.7. *In addition to the previous assumptions, let $\mathcal{O}$ be of class $C^{1,1}$ or convex and piecewise $C^{1,1}$. Further, let $c_s, \rho \in W^{1,\infty}$ and $\omega \neq 0$. If*

$$\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \leq \frac{1}{1+\tan^2\theta},$$

then the operator A induced by the sesquilinear form $a(\cdot,\cdot)$ is weakly T-coercive.

Proof. We refer to [HH21, Thm. 3.11]. □

4.2 Equations of solar and stellar oscillations

In this section, we will connect Galbrun's equation with the equations of solar and stellar oscillation as described by Lynden-Bell and Ostriker [LO67]:

$$-\rho(\omega + i\partial_b + i\Omega\times)^2\boldsymbol{u} - \nabla(\rho c_s^2 \operatorname{div}\boldsymbol{u}) + (\operatorname{div}\boldsymbol{u})\nabla p - \nabla(\nabla p\cdot\boldsymbol{u}) \tag{4.15a}$$

$$+ (\mathrm{Hess}(p) - \rho\,\mathrm{Hess}(\phi))\boldsymbol{u} + \gamma\rho(-i\omega)\boldsymbol{u} - \rho\nabla\psi = f \text{ in } \mathcal{O}, \tag{4.15b}$$

$$-\frac{1}{4\pi G}\Delta\psi + \operatorname{div}(\rho\boldsymbol{u}) = 0 \text{ in } \mathbb{R}^3. \tag{4.15c}$$

While the Hilbert space $\boldsymbol{X}$ for the Lagrangian pertubations of displacement $\boldsymbol{u}$ was already introduced in (4.3), an appropriate space for the gravitational potential ψ still has to be defined. To this end, let $\boldsymbol{G} := \{\boldsymbol{g} \in \boldsymbol{L}^2(\mathbb{R}^3) : \operatorname{curl}\boldsymbol{g} = 0\}$ which is a closed subspace of $\boldsymbol{L}^2(\mathbb{R}^3)$ by the classical Helmholtz decomposition and thus a Hilbert space with respect to the $L^2(\mathbb{R}^3)$-inner product. Furthermore, for each $\boldsymbol{g} \in \boldsymbol{G}$ there exists a unique gradient potential $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3) \setminus \mathbb{C}$ such that $\boldsymbol{g} = \nabla\psi$. Thus, we define

$$\tilde{H}^1_* := \{\psi : \nabla\psi \in \boldsymbol{G}\}, \qquad \langle\psi,\psi'\rangle_{\tilde{H}^1_*} := \langle\nabla\psi,\nabla\psi'\rangle_{\boldsymbol{L}^2(\mathbb{R}^3)}, \tag{4.16}$$

as an appropriate Hilbert space for the gravitational potential ψ, cf. [HH21, Sec. 2.3]. Then, the weak formulation of the full equations of solar and stellar oscillation (4.15) reads as: Find $(\boldsymbol{u}, \psi) \in \boldsymbol{X} \times \tilde{H}^1_*$ such that

$$a_{\text{ext}}((\boldsymbol{u}, \psi), (\boldsymbol{u}', \psi')) = \langle \boldsymbol{f}, \boldsymbol{u}' \rangle \text{ for all } (\boldsymbol{u}', \psi') \in \boldsymbol{X} \times \tilde{H}^1_*,$$

where the sesquilinear form $a(\cdot, \cdot)$ is given by

$$\begin{aligned} a_{\text{ext}}((\boldsymbol{u}, \psi), (\boldsymbol{u}', \psi')) := {} & \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle + \langle \operatorname{div} \boldsymbol{u}, \nabla p \cdot \boldsymbol{u}' \rangle + \langle \nabla p \cdot \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle \\ & + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi)) \boldsymbol{u}, \boldsymbol{u}' \rangle \\ & - \langle \rho(\omega + i\partial_b + i\Omega \times) \boldsymbol{u}, (\omega + i\partial_b + i\Omega \times) \boldsymbol{u}' \rangle \\ & - i\omega \langle \gamma \rho \boldsymbol{u}, \boldsymbol{u}' \rangle - \langle \nabla \psi, \rho \boldsymbol{u}' \rangle - \langle \rho \boldsymbol{u}, \nabla \psi' \rangle - \frac{1}{4\pi G} \langle \nabla \psi, \nabla \psi' \rangle_{\boldsymbol{L}^2(\mathbb{R}^3)}. \end{aligned}$$

In the following, we denote by $A_{\text{ext}} \in L(\boldsymbol{X} \times \tilde{H}^1_*)$ the operator induced by the sesquilinear form $a_{\text{ext}}(\cdot, \cdot)$. We first note that A_{SSO} is injective [HH21, Lem. 3.3], which can be shown similarly to Lemma 4.4. Furthermore, we note that we can recover Galbrun's equation by setting the gravitational potential $\psi = 0$, which is the so-called *Cowling approximation* [Cow41].
The analysis of A_{ext} is intimately related to the analysis of the operator A associated with the sesquilinearform (4.7). In fact, if $\rho \in W^{1,\infty}$, then A_{ext} is Fredholm if and only if A is Fredholm, cf., [HH21, Sec. 3.2]. Therefore, in this case, it suffices to analyze Galbrun's equation to understand the well-posedness of the full equations of solar and stellar oscillation. In the case that $\rho \notin W^{1,\infty}$, the analysis of A_{ext} can still be connected to A through building a Schur complement. We refer to [HH21, Sec. 3.2] for more details. The theorem below states that the operator A_{ext} is indeed weakly T-coercive. To be precise, we define for $\sigma \in \mathbb{C}$

$$T_1^\sigma := \begin{pmatrix} \overline{\sigma} T & 0 \\ 0 & I_{\tilde{H}^1_*} \end{pmatrix}. \tag{4.17}$$

Theorem 4.8 (Thm. 3.12 of [HH21]). *Assume that the assumptions of Section 4.1 hold true. Furthermore, let $\mathcal{O}$ be of class $C^{1,1}$ or convex and piecewise $C^{1,1}$ and let $c_s, \rho \in W^{1,\infty}$. If $\omega \neq 0$ and*

$$\|c_s^{-1} \boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} < \frac{1}{1 + \tan^2 \theta}, \tag{4.18}$$

then there exists $\sigma \in \mathbb{C}$ such that the operator A_{ext} is weakly T_1^σ-coercive.

Proof. We refer to [HH21, Thm. 3.12]. □

Chapter 5

Existing discretizations for Galbrun's equation

The purpose of this chapter is to review recently developed discretizations for Galbrun's equation. First of all, we will briefly discuss an H^1-conforming discretization introduced in [HLS22]. Then, we will present a $H(\mathrm{div})$-conforming discontinuous Galerkin discretization introduced in [Hal23]. For the latter method, we will discuss the analysis extensively as it serves as a basis for the discretization we aim to introduce in the next chapter and many results will carry over. We will use both discretizations for numerical examples in Chapter 7.

Contents of the chapter

5.1 H^1-conforming finite element discretization

The framework discussed in Chapter 2 has first been applied to Galbrun's equation in [HLS22], where an H^1-conforming finite element discretization was considered. In this section, we will briefly review the method and examine its analysis. Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded Lipschitz domain. For simplicity, we assume that $\mathcal{O}$ is a convex polyhedron. In addition to the assumptions on c_s and ρ from Chapter 4, we assume that $c_s, \rho \in W^{1,\infty}$. Furthermore, let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of shape-regular simplicial triangulations of $\mathcal{O}$ and h_n be the maximal element diameter such that $h_n \to 0$ as $n \to \infty$. For fixed degree $k \in \mathbb{N}$, we define the H^1-conforming finite element space

$$\boldsymbol{X}_n := \{\boldsymbol{u} \in \boldsymbol{H}^1 : \boldsymbol{u}|_\tau \in \boldsymbol{\mathcal{P}}^k(\tau) \quad \forall \tau \in \mathcal{T}_n, \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \text{ on } \partial\mathcal{O}\}. \tag{5.1}$$

Furthermore, we define

$$\begin{aligned} \boldsymbol{H}^1_{\boldsymbol{\nu}0} &:= \{\boldsymbol{u} \in \boldsymbol{H}^1 : \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \text{ on } \partial\mathcal{O}\}, \\ \boldsymbol{H}^1_0 &:= \{\boldsymbol{u} \in \boldsymbol{H}^1 : \boldsymbol{u} = 0 \text{ on } \partial\mathcal{O}\}. \end{aligned}$$

Then, we have that $\boldsymbol{X}_n \subset \boldsymbol{H}^1_{\boldsymbol{\nu}0} \subset \boldsymbol{X}$, where $\boldsymbol{X}$ is the space defined in (4.3). It can be shown that the discrete space $\boldsymbol{X}_n$ fulfills the approximability property.

Lemma 5.1. *For each $\boldsymbol{u} \in \boldsymbol{X}$, it holds that*

$$\lim_{n\to\infty} \inf_{\boldsymbol{u}' \in \boldsymbol{X}_n} \|\boldsymbol{u} - \boldsymbol{u}'\|_{\boldsymbol{X}} = 0. \tag{5.2}$$

Proof. We refer to [HLS22, Lemma 7]. □

In this section, we consider the discrete problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n$ such that

$$a(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}, \boldsymbol{u}_n' \rangle \quad \forall \boldsymbol{u}_n' \in \boldsymbol{X}_n, \tag{5.3}$$

where the sesquilinear form $a(\cdot,\cdot)$ is defined through (4.7).
Let $A \in L(\boldsymbol{X})$ be the associated operator to $a(\cdot,\cdot)$ and $P_{\boldsymbol{X}_n} \in L(\boldsymbol{X}, \boldsymbol{X}_n)$ be the orthogonal projection from $\boldsymbol{X}$ onto $\boldsymbol{X}_n$. We set $A_n := P_{\boldsymbol{X}_n} A|_{\boldsymbol{X}_n}$. Lemma 5.1 allows us to apply Corollary 2.21 to conclude that $(\boldsymbol{X}_n, P_{\boldsymbol{X}_n}, A_n)$ is a discrete approximation scheme of $(\boldsymbol{X}, A)$. In particular, this means that there holds

$$\lim_{n\to\infty} \|\boldsymbol{u} - P_{\boldsymbol{X}_n}\boldsymbol{u}\|_{\boldsymbol{X}} = 0 \quad \text{and} \quad A_n \xrightarrow{P} A.$$

The setup is visualized in Fig. 5.1.

$$A \circlearrowright \boldsymbol{X} \xrightarrow{P_{\boldsymbol{X}_n}} \boldsymbol{X}_n \circlearrowleft P_{\boldsymbol{X}_n} A_{|\boldsymbol{X}_n}$$

Figure 5.1: Set up for the H^1-conforming discretization of (4.1).

Now, we want to apply Lemma 2.22 to conclude that (5.3) has unique solutions $\boldsymbol{u}_n$ for all $n > n_0$, $n_0 > 0$, such that $\boldsymbol{u}_n \xrightarrow{P} \boldsymbol{u}$. To this end, we have to show that the sequence $(A_n)_{n\in\mathbb{N}}$ is regular. We define the spaces

$$\boldsymbol{V} := \{\boldsymbol{u} \in \boldsymbol{H}_0^1 : \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{u}' \rangle = 0 \text{ for all } \boldsymbol{u}' \in \boldsymbol{H}_0^1 \text{ with } \operatorname{div} \boldsymbol{u}' = 0\}, \tag{5.4a}$$

$$\boldsymbol{W} := \{\boldsymbol{u} \in \boldsymbol{X} : \operatorname{div} \boldsymbol{u} = 0\}. \tag{5.4b}$$

We note that the operator $D \in L^2(\boldsymbol{V}, L_0^2)$ defined by $D\boldsymbol{v} := \operatorname{div} \boldsymbol{v}$ is bijective [ADM06, Thm. 4.1] and that the projections onto $\boldsymbol{V}$ and $\boldsymbol{W}$ are given by

$$\boldsymbol{v} := D^{-1} \operatorname{div} \boldsymbol{u}, \qquad \boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}.$$

Therefore, $\boldsymbol{X} = \boldsymbol{V} \oplus \boldsymbol{W}$ is a topological decomposition.
To analyze the discretization (5.3), we want to transfer this topological decomposition onto the discrete level. Therefore we require the divergence operator to be surjective on $\boldsymbol{X}_n$, which means that the space $\boldsymbol{X}_n$ has to fulfill an inf-sup condition, see also Remark 1.10. To be precise, let us define the space

$$Q_n := \{f \in L_0^2 : f|_\tau \in \mathcal{P}^{k-1}(\tau) \text{ for all } \tau \in \mathcal{T}_n\} \tag{5.5}$$

and let $P_{Q_n} \in L(L_0^2, Q_n)$ be the associated orthogonal projection from L_0^2 to Q_n. We pose the following assumption on $\boldsymbol{X}_n$:

Assumption 5.1. *There exists a constant $\beta_h > 0$ such that for all $n \in \mathbb{N}$, it holds*

$$\inf_{f_n \in Q_n\setminus\{0\}} \sup_{\boldsymbol{u}_n \in \boldsymbol{X}_n\setminus\{0\}} \frac{|\langle \operatorname{div} \boldsymbol{u}_n, f_n\rangle|}{\|\nabla \boldsymbol{u}_n\|_{(L^2)^{3\times 3}} \|f_n\|_{L^2}} > \beta_h$$

To ensure that this assumption is fulfilled one has to take special care. Usually, a sufficiently high polynomial degree k is required. In 2D, for instance, we have to assume that $k \geq 4$. If one applies special meshes that use barycentric refinements, this requirement can be relaxed to $k \geq 2$. For more details, we refer to [HLS22] and the references therein.

5.1.1 Homogeneous pressure and gravity

Restricting to the case where both, pressure and gravity, are constant, simplifies the sesquilinear form $a(\cdot,\cdot)$ defined by (4.7) to

$$a(\boldsymbol{u},\boldsymbol{u}') = \langle c_s^2\rho \operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}'\rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}'\rangle - i\omega\langle\gamma\rho\boldsymbol{u},\boldsymbol{u}'\rangle.$$

In the following, let $T := P_{\boldsymbol{V}} - P_{\boldsymbol{W}}$, where $\boldsymbol{V},\boldsymbol{W}$ are defined by (5.4) such that $\boldsymbol{X} = \boldsymbol{V}\oplus\boldsymbol{W}$. Then, we can show that $a(\cdot,\cdot)$ is weakly right T-coercive.

Lemma 5.2. *Let $\beta > 0$ be the inf-sup constant of the divergence on the domain $\mathcal{O}$ and assume that $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty} < \beta\frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}}$. Then the operator A induced by $a(\cdot,\cdot)$ is weakly right T-coercive.*

Proof. See [HLS22, Corollary 10]. ☐

Now we want to transfer the decompositon (5.4) to the discrete level to construct a discrete operator T_n. To this end, we define the spaces

$$\boldsymbol{V}_n := \{\boldsymbol{u}_n \in \boldsymbol{X}_n \cap \boldsymbol{H}_0^1 : \langle\nabla\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle = 0 \text{ for all } \boldsymbol{u}_n' \in \boldsymbol{H}_0^1 \cap \boldsymbol{W}_n\}, \tag{5.6a}$$

$$\boldsymbol{W}_n := \{\boldsymbol{u}_n \in \boldsymbol{X}_n : \operatorname{div}\boldsymbol{u}_n = 0\}. \tag{5.6b}$$

Assumption 5.1 guarantees that the problem

$$\text{Find } \boldsymbol{v}_n \in \boldsymbol{V}_n \text{ s.t. } \operatorname{div}\boldsymbol{v}_n = \operatorname{div}\boldsymbol{u}_n \tag{5.7}$$

has a unique solution $\boldsymbol{v}_n$, which satisfies $\beta_h\|\nabla\boldsymbol{v}_n\|_{(L^2)^{3\times 3}} \leq \|\operatorname{div}\boldsymbol{v}_n\|_{L^2}$. Let $D_n^{-1} \in L(Q_n,\boldsymbol{V}_n)$ be the respective solution operator and define $P_{\boldsymbol{V}_n}\boldsymbol{u}_n := \boldsymbol{v}_n$, where $\boldsymbol{v}_n$ is the solution of (5.7). Then, $P_{\boldsymbol{V}_n} \in L(\boldsymbol{X}_n)$ is uniformly bounded and indeed a projection. Setting $P_{\boldsymbol{W}_n}\boldsymbol{u}_n := \boldsymbol{u}_n - \boldsymbol{v}_n$ yields a decomposition $\boldsymbol{X}_n = \boldsymbol{V}_n \oplus \boldsymbol{W}_n$. Thus, we define $T_n := P_{\boldsymbol{V}_n} - P_{\boldsymbol{W}_n} \in L(\boldsymbol{X}_n)$, which is uniformly bounded with uniformly bounded inverse $T_n^{-1} = T_n$. The following lemmata state that $T_n \xrightarrow{P} T$ and that $(A_n)_{n\in\mathbb{N}}$ is regular.

Lemma 5.3. *For each $\boldsymbol{u} \in \boldsymbol{X}$, we have that $\lim_{n\to\infty}\|T_nP_{\boldsymbol{X}_n}\boldsymbol{u} - P_{\boldsymbol{X}_n}T\boldsymbol{u}\|_{\boldsymbol{X}} = 0$.*

Proof. See [HLS22, Lemma 12]. ☐

Lemma 5.4. *If $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty} < \beta_h\frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}}$, then $(A_n)_{n\in\mathbb{N}}$ is regular.*

Proof. The proof of the Lemma utilizes Theorem 2.28. For more details, we refer to [HLS22, Lemma 13]. ☐

Consequently, we can apply Lemma 2.22 and standard interpolation results to obtain the following result.

Theorem 5.5. *Let p and ϕ be constant and let $\boldsymbol{u}$ solve (0.1a). Furthermore, let assumption 5.1 be satisfied and*

$$\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty} < \beta_h\frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}}.$$

Then there exists an index $n_0 > 0$ such that for all $n > n_0$, the solution $\boldsymbol{u}_n$ to (5.3) exists and $\boldsymbol{u}_n$ converges in the $\boldsymbol{X}$-norm with the estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{X}} \lesssim \inf_{\boldsymbol{u}_n'\in\boldsymbol{X}_n}\|\boldsymbol{u} - \boldsymbol{u}_n'\|_{\boldsymbol{X}}.$$

If $\boldsymbol{u} \in \boldsymbol{H}^{1+s}$, $s > 0$, then $\|\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{X}} \lesssim h^{\min(s,k)}\|\boldsymbol{u}\|_{\boldsymbol{H}^{1+s}}$.

5.1.2 Heterogeneous pressure and gravity

To consider heterogeneous pressure and gravitational potential, we denote by $\boldsymbol{q} := c_s^{-2}\rho^{-1}\nabla p$ as in (4.8), which allows us to express $a(\cdot,\cdot)$ as

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{u}') :=& \langle c_s^2\rho(\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{u}, (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{u}'\rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}'\rangle \\ & - i\omega\langle\gamma\rho\boldsymbol{u},\boldsymbol{u}'\rangle + \langle(\operatorname{Hess}(p) - \operatorname{Hess}(\phi) - c_s^2\rho\boldsymbol{q}\otimes\boldsymbol{q})\boldsymbol{u},\boldsymbol{u}'\rangle \end{aligned} \tag{5.8}$$

The following construction deviates from the analysis in Section 4.1 by avoiding the introduction of a discrete subspace $\boldsymbol{Z}$. Instead, we consider the divergence operator $D \in L(\boldsymbol{V}, L^2_0)$, $D\boldsymbol{v} = \operatorname{div}\boldsymbol{v}$ and its inverse $D^{-1} \in L(L^2_0, \boldsymbol{V})$. We define $\tilde{D} \in L(\boldsymbol{V}, L^2_0)$ by

$$\tilde{D}\boldsymbol{v} := D\boldsymbol{v} + \boldsymbol{q}\cdot\boldsymbol{v} + M\boldsymbol{v} + F\boldsymbol{v}, \tag{5.9}$$

where $M\boldsymbol{v} := -\operatorname{mean}(q\cdot v)$ and $F\boldsymbol{v} := \sum_{n=1}^N \phi_n\langle\operatorname{div}\boldsymbol{v}, \operatorname{div}\psi_n\rangle$ is finite dimensional where N,$\phi_n \in L^2_0$ and $\psi_n \in V$ are specified below. Let us briefly elaborate on this construction. Since we assume that $p \neq \text{const}$, we have that $\boldsymbol{q} \neq 0$, but $(D + \boldsymbol{q}\cdot)$ is not necessarily bijective. In the corollary below, we will show that adding the finite-dimensional operator F indeed makes the operator $\tilde{D}$ bijective. Furthermore, we add the mean value operator M to ensure that $\tilde{D} \in L^2_0$.

Corollary 5.6. *The operator $\tilde{D}$ defined by (5.9) is bijective.*

Proof. Since the embedding $\boldsymbol{H}^1 \hookrightarrow \boldsymbol{L}^2$ is compact, the operator $\boldsymbol{q}\cdot$ is compact and therefore $D + \boldsymbol{q}\cdot + M \in L(\boldsymbol{V}, L^2_0)$ is a compact perturbation of a bijective operator. Therefore, it is Fredholm with index zero by Thm. A.23. Consequently, by definition of a Fredholm operator, its kernel and cokernel are finite-dimensional and by definition of the index, it follows that

$$\dim\ker(D + \boldsymbol{q}\cdot + M) = \dim\operatorname{coker}(D + \boldsymbol{q}\cdot + M) := N. \tag{5.10}$$

Since F is of finite rank, $\tilde{D}$ is Fredholm with index zero. Therefore, to show that $\tilde{D}$ is indeed bijective, it suffices to show that it is injective. To this end, we note that $\langle\operatorname{div}\cdot, \operatorname{div}\cdot\rangle$ is an equivalent scalar product to $\langle\cdot,\cdot\rangle_{\boldsymbol{H}^1}$ on $\boldsymbol{V}$. Let ψ_n, $n = 1,\dots,N$ be an orthonormal basis of $\ker(D + \boldsymbol{q}\cdot + M)$ with respect to $\langle\operatorname{div}\cdot, \operatorname{div}\cdot\rangle$ and ϕ_n, $n = 1,\dots,N$ be an orthonormal basis of $\operatorname{ran}(D + \boldsymbol{q}\cdot + M)^\perp$. Now suppose that $v \in \ker(\tilde{D})$. Then, either

$$v \in \ker(D + \boldsymbol{q}\cdot + M) \cap \ker(F) \qquad \text{or} \qquad Fv \in \operatorname{ran}(D + \boldsymbol{q}\cdot + M)$$

Suppose $v \neq 0$. Then, $Fv \neq 0$ since ψ_n is a basis of $\ker(D + \boldsymbol{q}\cdot + M)$. Thus, the first case would immediately imply $v = 0$. Since ϕ_n is a basis of $\operatorname{ran}(D + \boldsymbol{q}\cdot + M)^\perp$, $Fv \notin \operatorname{ran}(D + \boldsymbol{q}\cdot + M)$. ☐

Remark 5.7. *We note that the existence of a finite rank operator F such that $(D + \boldsymbol{q}\cdot + M) + F$ is bijective immediately follows from Theorem A.25, which states that an operator is Fredholm with index zero if and only if there exists a finite rank operator such that the sum of the two operators is bijective.*

While the operator $\tilde{D}$ is bounded and well-defined as an operator from $\boldsymbol{X}$ to L^2_0, we only consider the inverse $\tilde{D}^{-1}$ as an operator from L^2_0 to $\boldsymbol{V}$. Thus, for $\boldsymbol{u} \in \boldsymbol{X}$, we construct a topological decomposition $\boldsymbol{X} = \boldsymbol{V} \oplus \boldsymbol{W}$ through

$$\boldsymbol{v} := \tilde{D}^{-1}\tilde{D}\boldsymbol{u}, \qquad \boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}. \tag{5.11}$$

This construction is visualized in Fig. 5.2. Then, for $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, we define an operator $T \in L(\boldsymbol{X})$ by

$$T\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{w}. \tag{5.12}$$

We note that $\|\operatorname{div} \boldsymbol{v}\|_{L^2} \geq \beta \|\nabla\|_{(L^2)^{3\times 3}}$, since $\boldsymbol{v} \in \boldsymbol{V}$ and

$$\begin{aligned}(\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{w} &= (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{u} - (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{v} \\ &= (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{u} - (\operatorname{div} + \boldsymbol{q}\cdot + M + F)\boldsymbol{v} + (M + F)\boldsymbol{v} \\ &= (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{u} - \tilde{D}\boldsymbol{u} + (M + F)\boldsymbol{v} \\ &= -(M + F)\boldsymbol{u} + (M + F)\boldsymbol{v} \\ &= -(M + F)\boldsymbol{w},\end{aligned} \tag{5.13}$$

which is a compact operator.

$$\boldsymbol{X} \xrightarrow{\tilde{D}} L^2_0 \xrightarrow{\tilde{D}^{-1}} \boldsymbol{V}$$

$$\boldsymbol{u} \longrightarrow \tilde{D}\boldsymbol{u} \longrightarrow \tilde{D}^{-1}\tilde{D}\boldsymbol{u}$$

Figure 5.2: Construction of $\boldsymbol{v}$ in (5.11).

We want to show that the continuous sesquilinear form $a(\cdot,\cdot)$ is T-coercive with respect to the operator defined in (5.12). Therefore, we denote by $\lambda_-(\underline{\underline{m}}) \in L^\infty$ the smallest eigenvalue of a positive definite matrix and consider $\underline{\underline{m}} := -\rho^{-1}\operatorname{Hess}(p) + \operatorname{Hess}(\phi)$. Furthermore, we set

$$C_M := \max\left\{0, \sup_{x\in\mathcal{O}} \frac{-\lambda_-(\underline{\underline{m}}(x))}{\gamma(x)}\right\} \text{ and } \theta := \arctan(C_M/|\omega|) \in [0, \pi/2), \omega \neq 0. \tag{5.14}$$

Lemma 5.8. *Let* $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty} \leq \beta \frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}} \frac{1}{1+\tan^2\theta}$. *Then A is weakly right T-coercive.*

Proof. See [HLS22, Corollary 16]. □

As before, we want to mimic the topological decomposition on the discrete level. We set

$$\tilde{D}_n := P_{Q_n}\tilde{D}|_{V_n},$$

where P_{Q_n} is the orthogonal projection onto the space Q_n defined in (5.5). As $\boldsymbol{V}_n \not\subset \boldsymbol{V}$, $\tilde{D}_n$ is a nonconforming approximation of $\tilde{D}$. However, by setting $p_n := D_n^{-1}P_{Q_n}\operatorname{div} \in L(\boldsymbol{V}, \boldsymbol{V}_n)$, it follows that $(\boldsymbol{V}_n, p_n, \tilde{D}_n)$ is a discrete approximation scheme of $(\boldsymbol{V}, \tilde{D})$, cf. [HLS22, Lem. 17]. Furthermore, the sequence $(\tilde{D}_n)_{n\in\mathbb{N}}$ is stable.

Lemma 5.9. *The sequence $(\tilde{D}_n)_{n\in\mathbb{N}}$ approximates $\tilde{D}$ and is stable.*

Proof. See [HLS22, Lemma 17]. □

Thus, we can define the discrete operator $T_n \in L(\boldsymbol{X}_n)$ through $T_n\boldsymbol{u}_n := \boldsymbol{v}_n - \boldsymbol{w}_n$, where

$$\boldsymbol{v}_n := \tilde{D}_n^{-1}P_{Q_n}\tilde{D}\boldsymbol{u}, \qquad \boldsymbol{w}_n = \boldsymbol{u}_n - \boldsymbol{v}_n. \tag{5.15}$$

This construction is visualized in Fig. 5.3. Due to the previous lemma, $\boldsymbol{v}_n$ and $\boldsymbol{w}_n$ are well-defined and T_n is uniformly bounded in the operator norm. Furthermore, since $\boldsymbol{v}_n \in \boldsymbol{V}_n$, it holds that $\|\operatorname{div} \boldsymbol{v}_n\|_{L^2} \geq \beta_h \|\nabla \boldsymbol{v}_n\|_{(L^2)^{3\times 3}}$.
Similar to (5.13), we compute that $P_{Q_n}(\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{w}_n = -P_{Q_n}F\boldsymbol{w}_n$ and set

$$T_n\boldsymbol{u}_n := \boldsymbol{v}_n - \boldsymbol{w}_n. \tag{5.16}$$

Note that T_n is self-inverse and hence bijective and bounded due to the previous Lemma. The following Lemma shows that we can apply Theorem 2.28.

$$
\begin{array}{ccccccc}
\boldsymbol{X}_n & \xrightarrow{\tilde{D}} & L^2_0 & \xrightarrow{P_{Q_n}} & Q_n & \xrightarrow{\tilde{D}_n^{-1}} & V_n \\
\boldsymbol{u}_n & \longrightarrow & \tilde{D}\boldsymbol{u}_n & \longrightarrow & P_{Q_n}\tilde{D}_n\boldsymbol{u}_n & \longrightarrow & \tilde{D}_n^{-1}P_{Q_n}\tilde{D}\boldsymbol{u}_n
\end{array}
$$

Figure 5.3: Construction of $\boldsymbol{v}_n$ in (5.15).

Lemma 5.10. *For each $\boldsymbol{u} \in \boldsymbol{X}$, we have that* $\lim_{n\to\infty} \|T_n P_{\boldsymbol{X}_n}\boldsymbol{u} - P_{\boldsymbol{X}_n} T\boldsymbol{u}\|_{\boldsymbol{X}} = 0$.

Proof. See [HLS22, Lemma 18]. □

Lemma 5.11. *Assuming that* $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty} < \frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}}\frac{1}{1+\tan^2\theta}$, *the sequence* $(A_n)_{n\in\mathbb{N}}$ *is regular.*

Proof. As before, the main strategy to show that $(A_n)_{n\in\mathbb{N}}$ is regular lies in applying Theorem 2.28. For details, we refer to [HLS22, Lemma 19]. □

Thus, we can apply Lemma 2.22 together with standard interpolation results to obtain the following result.

Theorem 5.12. *Let Assumption 5.1 be satisfied and* $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty} < \frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}}\frac{1}{1+\tan^2\theta}$. *Then, there exists an index $n_0 > 0$ such that for all $n > n_0$ the solution $\boldsymbol{u}_n$ to (5.3) exists and $\boldsymbol{u}_n$ converges to $\boldsymbol{u}$ in the $\boldsymbol{X}$-norm with the estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{X}} \lesssim \inf_{\boldsymbol{u}'_n \in \boldsymbol{X}_n} \|\boldsymbol{u} - \boldsymbol{u}'_n\|_{\boldsymbol{X}}.$$

If $\boldsymbol{u} \in \boldsymbol{H}^{1+s}$, $s > 0$, then $\|\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{X}} \lesssim h^{\min(s,k)}\|\boldsymbol{u}\|_{\boldsymbol{H}^{1+s}}$.

5.2 $H(\mathrm{div})$-conforming discontinuous Galerkin discretization

In this section, we consider a $H(\mathrm{div})$-conforming finite element discretization for Galbrun's equation (4.1), which is nonconforming with respect to the convection operator. This method, which was derived and analyzed in [Hal23], does not require a minimal assumption on the polynomial degree or the mesh structure as the H^1-conforming method introduced in Section 5.1 does. Furthermore, the method improves the smallness assumption on the Mach number and is robust with respect to changes in density and sound speed.

5.2.1 Formulation of the method

Let $H(\mathrm{div}) := \{\boldsymbol{u} \in \boldsymbol{L}^2 : \mathrm{div}\,\boldsymbol{u} \in L^2\}$. For a polynomial degree $k \in \mathbb{N}, k \geq 1$ we introduce the following finite element spaces

$$\boldsymbol{X}_n := \{\boldsymbol{u} \in H_0(\mathrm{div}) : \boldsymbol{u}|_\tau \in \boldsymbol{\mathcal{P}}^k(\tau) \text{ for all } \tau \in \mathcal{T}_n\}, \tag{5.17a}$$

$$\boldsymbol{X}_n^{\mathrm{wbc}} := \{\boldsymbol{u} \in H(\mathrm{div}) : \boldsymbol{u}|_\tau \in \boldsymbol{\mathcal{P}}^k(\tau) \text{ for all } \tau \in \mathcal{T}_n\}, \tag{5.17b}$$

where $H_0(\mathrm{div}) := \{\boldsymbol{u} \in H(\mathrm{div}) : \boldsymbol{\nu}\cdot\boldsymbol{u} = 0 \text{ on } \partial\mathcal{O}\}$ and

$$Q_n := \{u \in L^2_0 : u|_\tau \in \mathcal{P}^{k-1} \text{ for all } \tau \in \mathcal{T}_n\}, \tag{5.17c}$$

$$Q_n^{\mathrm{wbc}} := \{u \in L^2 : u|_\tau \in \mathcal{P}^{k-1} \text{ for all } \tau \in \mathcal{T}_n\}. \tag{5.17d}$$

The space Q_n is equipped with the standard L^2-scalar product. In the following, we denote by $\pi_\tau^d : \boldsymbol{H}^s(\tau) \to \boldsymbol{\mathcal{P}}^k(\tau)$, $s > 1/2$ and $\pi_\tau^l : L^2(\tau) \to \mathcal{P}^k(\tau)$, $\tau \in \mathcal{T}_n$ the respective standard local interpolation operators, cf. Section A.3.3. Furthermore, we denote by $\pi_n^d : \boldsymbol{H}^s \to \boldsymbol{X}_n^{\text{wbc}}$, $s > 1/2$, $\pi_n^d|_\tau = \pi_\tau^d$, $\tau \in \mathcal{T}_n$ and $\pi_n^l : L^2 \to Q_n^{\text{wbc}}$, $\pi_n^l|_\tau = \pi_\tau^l$, $\tau \in \mathcal{T}_n$ the respective global interpolation operators. We note that if $\boldsymbol{\nu} \cdot \boldsymbol{v} = 0$ on $\partial\mathcal{O}$, then $\pi_n^d \boldsymbol{v} \in \boldsymbol{X}_n$ and $\pi_n^l v \in Q_n$ if $v \in L_0^2$, respectively. In particular, we have that the interpolation operators commute with respect to the divergence operator, i.e., it holds that $\operatorname{div} \pi_n^d = \pi_n^l \operatorname{div}$. Furthermore, we have for all $\boldsymbol{v} \in \boldsymbol{H}^r(\tau)$, $v \in H^r(\tau)$, $r \in [1, k+1]$, $m \in [0, r]$, $\tau \in \mathcal{T}_n$ that

$$|\boldsymbol{v} - \pi_n^d \boldsymbol{v}|_{\boldsymbol{H}^m(\tau)} \leq C_{\text{apr}} h_\tau^{r-m} |\boldsymbol{v}|_{\boldsymbol{H}^r(\tau)}, \tag{5.18a}$$

$$|v - \pi_n^l v|_{H^m(\tau)} \leq C_{\text{apr}} h_\tau^{r-m} |v|_{H^r(\tau)}, \tag{5.18b}$$

and

$$\|\boldsymbol{v} - \pi_n^d \boldsymbol{v}\|_{\boldsymbol{L}^2(\partial\tau)} \leq C_{\text{ab}} h_\tau^{r-1/2} |\boldsymbol{v}|_{\boldsymbol{H}^r(\tau)}. \tag{5.19}$$

As before, let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of shape-regular simplicial triangulations of $\mathcal{O}$. We denote by $\mathcal{F}_n$ and $\mathcal{F}_n^{\text{int}}$ the set of all faces and interior faces of $\mathcal{T}_n$, respectively. For $\tau \in \mathcal{T}_n$ and $F \in \mathcal{F}_n$, we denote by h_τ and h_F their diameters and set $h_n := \max_{\tau\in\mathcal{T}_n} h_\tau$. Furthermore, we define $\mathfrak{h} : \mathcal{F}_n \to \mathbb{R}$ by $\mathfrak{h}|_F := h_F$. We assume that $\lim_{n\to\infty} h_n = 0$. We further define the broken Sobolev $\boldsymbol{H}^1(\mathcal{T}_n)$ by

$$\boldsymbol{H}^1(\mathcal{T}_n) := \left\{\boldsymbol{u} \in \boldsymbol{L}^2 : \boldsymbol{u}|_\tau \in \boldsymbol{H}^1(\tau) \text{ for all } \tau \in \mathcal{T}_n\right\}.$$

For $F \in \mathcal{F}_n^{\text{int}}$ and $\tau_1, \tau_2 \in \mathcal{T}_n$ such that $\tau_1 \cap \tau_2 = F$ and $\boldsymbol{u} \in \boldsymbol{H}^1(\mathcal{T}_n)$ we set

$$\{\!\{\boldsymbol{u}\}\!\} := \frac{1}{2}(\boldsymbol{u}_1 - \boldsymbol{u}_2), \qquad [\![\boldsymbol{u}]\!]_{\boldsymbol{b}} := (\boldsymbol{b}\cdot\boldsymbol{\nu})\boldsymbol{u}_1 + (\boldsymbol{b}\cdot\boldsymbol{\nu})\boldsymbol{u}_2,$$

where $\boldsymbol{u}_i$ is the trace of $\boldsymbol{u}|_{\tau_i}$, $i = 1, 2$. Furthermore, we abbreviate

$$\langle\cdot,\cdot\rangle_{\mathcal{F}_n^{\text{int}}} := \sum_{F\in\mathcal{F}_n^{\text{int}}} \langle\cdot,\cdot\rangle_{\boldsymbol{L}^2(F)}, \qquad \|\cdot\|^2_{\mathcal{F}_n^{\text{int}}} := \langle\cdot,\cdot\rangle_{\mathcal{F}_n^{\text{int}}}.$$

We want to introduce a lifting operator as in Chapter 3 to define a discrete version of the differential operator $\partial_{\boldsymbol{b}}$. To this end, let $l \in \mathbb{N}$ and set

$$\boldsymbol{Q}_n := \{\boldsymbol{\psi}_n \in \boldsymbol{L}^2 : \boldsymbol{\psi}_n|_\tau \in \boldsymbol{\mathcal{P}}^l(\tau) \text{ for all } \tau \in \mathcal{T}_h\}.$$

For $\boldsymbol{u}_n \in \boldsymbol{X}_n$ and $F \in \mathcal{F}_h^{\text{int}}$, we define $\boldsymbol{r}_n^F \boldsymbol{u}_n \in \boldsymbol{Q}_n$ as the solution to

$$\langle \boldsymbol{r}_n^F \boldsymbol{u}_n, \boldsymbol{\psi}_n\rangle = -\langle [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, \{\!\{\boldsymbol{\psi}_n\}\!\}\rangle_{\boldsymbol{L}^2(F)} \text{ for all } \boldsymbol{\psi}_n \in \boldsymbol{Q}_n. \tag{5.20}$$

Then, we define the global lifting operator $\boldsymbol{R}_n^l := \sum_{F\in\mathcal{F}_h^{\text{int}}} \boldsymbol{r}_n^F$ and note that due to the discrete trace inequality (A.14) it holds that

$$\|\boldsymbol{R}_n^l \boldsymbol{u}_n\|_{\boldsymbol{L}^2} \lesssim \sum_{F\in\mathcal{F}_n} h_F^{-1/2} \|[\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}\|_{\boldsymbol{L}^2(F)}. \tag{5.21}$$

Furthermore, we define a linear operator $\boldsymbol{D}_{\boldsymbol{b}}^n : \boldsymbol{X}_n \to \boldsymbol{Q}_n$ through

$$(\boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{u}_n)|_T := \partial_{\boldsymbol{b}}(\boldsymbol{u}_n|_T) + \boldsymbol{R}_n^l \boldsymbol{u}_n \text{ for all } T \in \mathcal{T}_h. \tag{5.22}$$

With this operator, we can define a scalar product on $\boldsymbol{X}_n$ via

$$\langle \boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} := \langle \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n'\rangle + \langle \boldsymbol{u}_n, \boldsymbol{u}_n'\rangle + \langle \boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{u}_n, \boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{u}_n'\rangle$$

and denote the induced norm by $\|\cdot\|_{\boldsymbol{X}_n} = \langle\cdot,\cdot\rangle_{\boldsymbol{X}_n}^{1/2}$.
Now, we introduce the following sesquilinear form. For all $\boldsymbol{u}_n, \boldsymbol{u}_n' \in \boldsymbol{X}_n$, let

$$\begin{aligned} a_n(\boldsymbol{u}_n, \boldsymbol{u}_n') :=& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle \\ &+ \langle \operatorname{div} \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle \\ &+ \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle - i\omega \langle \gamma \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \end{aligned} \tag{5.23}$$

Then, we consider the discrete problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n$ such that

$$a_n(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} \text{ for all } \boldsymbol{u}_n' \in \boldsymbol{X}_n. \tag{5.24}$$

Remark 5.13 (Motivation for the introduction of lifting operators)**.** *The main motivation to introduce the lifting operator* (5.20) *and the discrete differential operator $\boldsymbol{D}_{\boldsymbol{b}}^n$ is to avoid more confining restrictions on the Mach number $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2$. A classical symmetric interior penalty formulation of* (5.23) *would involve the term*

$$-\langle \rho \frac{\alpha_{\boldsymbol{b}}}{h} [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}} \rangle_{\mathcal{F}_n}, \tag{5.25}$$

where the stabilization parameter $\alpha_{\boldsymbol{b}} > 0$ has to be chosen large enough to guarantee the stability of the discrete sesquilinear form $a_n(\cdot,\cdot)$. However, in the subsequent analysis, this would lead to a more restrictive assumption on the Mach number. The difficulties of choosing suitable stabilization parameters are further explored through numerical experiments in Section 7.2.3.

5.2.2 Interpretation as discrete approximation scheme

The interpretation of the DG-scheme (5.24) as a discrete approximation scheme goes along the lines of Section 3.4.1 since it served as a basis for the interpretation developed there. Therefore, we only give a short overview here, with a special focus on the differences between both argumentations. As in Section 3.4.1, we have to define suitable projection operators $p_n \in L(\boldsymbol{X}, \boldsymbol{X}_n)$ to apply the theory developed in Chapter 2. For $\boldsymbol{u} \in \boldsymbol{X}$, let $p_n\boldsymbol{u} \in \boldsymbol{X}_n$ be the solution to

$$\langle p_n\boldsymbol{u}, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} = \langle \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}_n' \rangle_{L^2} + \langle \boldsymbol{u}, \boldsymbol{u}_n' \rangle_{\boldsymbol{L}^2} + \langle \partial_{\boldsymbol{b}} \boldsymbol{u}, D_{\boldsymbol{b}}^n \boldsymbol{u}_n' \rangle_{\boldsymbol{L}^2} \text{ for all } \boldsymbol{u}_n' \in \boldsymbol{X}_n. \tag{5.26}$$

One main differences to the analysis in Section 3.4.1 is that the jump $[\![\cdot]\!]_{\boldsymbol{b}}$ is not necessarily well-defined for $\boldsymbol{u} \in \boldsymbol{X}$. Thus, we introduce the following distance function between $\boldsymbol{u}_n \in \boldsymbol{X}_n$ and $\boldsymbol{u} \in \boldsymbol{X}$:

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n)^2 := \|\operatorname{div} \boldsymbol{u} - \operatorname{div} \boldsymbol{u}_n\|_{L^2}^2 + \|\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{L}^2}^2 + \|\partial_{\boldsymbol{b}} \boldsymbol{u} - D_{\boldsymbol{b}}^n \boldsymbol{u}_n\|_{\boldsymbol{L}^2}^2. \tag{5.27}$$

We note that the distance function $d_n(\cdot,\cdot)$ satisfies the triangle inequalities

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n) \leq d_n(\tilde{\boldsymbol{u}}, \boldsymbol{u}_n) + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{\boldsymbol{X}}, \quad d_n(\boldsymbol{u}, \boldsymbol{u}_n) \leq d_n(\boldsymbol{u}, \tilde{\boldsymbol{u}}_n) + \|\boldsymbol{u}_n - \tilde{\boldsymbol{u}}_n\|_{\boldsymbol{X}_n},$$

for $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in \boldsymbol{X}$ and $\boldsymbol{u}_n, \tilde{\boldsymbol{u}}_n \in \boldsymbol{X}_n$. With this distance function, we can show that $(\boldsymbol{X}_n, p_n, A_n)$ is a discrete approximation scheme of $(\boldsymbol{X}, A)$ in the sense of Definition 2.7. To this end, we have to show analogous lemmeta to those in Section 3.4.1 in terms of the new distance function $d_n(\cdot,\cdot)$.

Lemma 5.14 (Lem. 3 of [Hal23])**.** *For each $\boldsymbol{u} \in \boldsymbol{H}_{\boldsymbol{\nu}0}^1$, it holds that $d_n(\boldsymbol{u}, p_n\boldsymbol{u}) \leq d_n(\boldsymbol{u}, \pi_n^d \boldsymbol{u})$.*

Lemma 5.15 (Lem. 4 of [Hal23])**.** *For each $\boldsymbol{u} \in \boldsymbol{H}_{\boldsymbol{\nu}0}^1 \cap \boldsymbol{H}^{1+s}$, $s > 0$, it holds that $d_n(\boldsymbol{u}, \pi_n^d \boldsymbol{u}) \lesssim h_n^s \|\boldsymbol{u}\|_{\boldsymbol{H}^{1+s}}$.*

Lemma 5.16 (Lem. 5 of [Hal23])**.** *For each $\boldsymbol{u} \in \boldsymbol{X}$ we have that $\lim_{n\to\infty} d_n(\boldsymbol{u}, p_n\boldsymbol{u}) = 0$.*

Lemma 5.17 (Lem. 6 of [Hal23])**.** *For each $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0}$ we have that $\lim_{n\to\infty} d_n(\boldsymbol{u}, \pi_n^d\boldsymbol{u}) = 0$.*

Lemma 5.18 (Lem. 7 of [Hal23])**.** *For each $\boldsymbol{u} \in \boldsymbol{X}$, we have that $\lim_{n\to\infty} \|p_n\boldsymbol{u}\|_{\boldsymbol{X}_n} = \|\boldsymbol{u}\|_{\boldsymbol{X}}$.*

The following lemma essentially gives an analogous result to Lemma 3.17, but accounts for the divergence operator and the nonstandard differential operator $\partial_{\boldsymbol{b}}$.

Lemma 5.19 (Lem. 8 of [Hal23])**.** *Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $\boldsymbol{u}_n \in \boldsymbol{X}_n$ satisfy $\sup_{n\in\mathbb{N}} \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} < \infty$. Then there exists $\boldsymbol{u} \in \boldsymbol{X}$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $\boldsymbol{u}_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \boldsymbol{u}$, $\operatorname{div} \boldsymbol{u}_n \overset{L^2}{\rightharpoonup} \operatorname{div} \boldsymbol{u}$ and $D^n_{\boldsymbol{b}}\boldsymbol{u}_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \partial_{\boldsymbol{b}}\boldsymbol{u}$.*

Proof. Follows with techniques from [BO08, Thm. 5.2]. For more details, we refer to [Hal23, Lem. 8]. □

With the previous lemmata, we can conclude that $A_n \xrightarrow{P} A$.

Theorem 5.20 (Thm. 9 of [Hal23])**.** *The operator A_n associated to the sesquilinear form $a_n^{DG}(\cdot,\cdot)$ approximates A, i.e. for each $\boldsymbol{u} \in \boldsymbol{X}$, it holds that*

$$\lim_{n\to\infty} \|A_n p_n \boldsymbol{u} - p_n A\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0. \tag{5.28}$$

Altogether, we have that $(\boldsymbol{X}_n, p_n, A_n)$ constitutes a discrete approximation scheme of $(\boldsymbol{X}, A)$. We can now work towards applying the results from Chapter 2.

5.2.3 Convergence analysis

To apply the theoretical framework developed in Chapter 2, the goal is to construct a suitable T-operator and show weak T-coercivity of the operator A associated with the continuous bilinear form $a(\cdot,\cdot)$. Afterwards, a discrete operator T_n is constructed such that the weak T-compatibility conditions from Thm. 2.28 are satisfied. In particular, this allows us to conclude that the sequence $(A_n)_{n\in\mathbb{N}}$ is regular and to derive convergence rates.

5.2.3.1 T-coercivity

First of all, we want to construct a topological decomposition of $\boldsymbol{X}$ and an operator $T \in L(\boldsymbol{X})$ such that A is weakly T-coercive. To be precise, we will use the right T-coercivity, cf. Remark 1.12, to avoid introducing the adjoint operator and therefore deviate from the analysis in Section 4.1. For $\boldsymbol{u} \in H_0(\operatorname{div})$, let $v \in H^2$ be the solution to

$$(\operatorname{div} + \boldsymbol{q}\cdot)\nabla v = (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{u} \qquad \text{in } \mathcal{O}, \tag{5.29a}$$

$$\boldsymbol{\nu} \cdot v = 0 \qquad \text{on } \partial\mathcal{O}. \tag{5.29b}$$

Let us consider (5.29) as a variational problem in H^1. First of all, we note that if a solution $v \in H^2$ exists, then the map $\boldsymbol{u} \mapsto v$ is a bounded linear mapping from $H_0(\operatorname{div})$ to H^2 [Amr+98, Thm. 2.17]. However, while the operator associated with the left-hand side of (5.29) is weakly coercive[1], its injectivity is not guaranteed. Whereas the construction in Thm. 4.5 circumvented this issue by introducing an additional discrete subspace into the decomposition of $\boldsymbol{X}$, we remedy this issue by introducing a suitable perturbation of the left-hand side. To this end, we consider the problem on $H^2_* := \{u \in H^2 : \langle u, 1\rangle = 0\}$ and

[1] Since $\boldsymbol{q}\cdot$ is a compact operator, we essentially consider a compact perturbation of the Laplacian.

introduce an operator M that has finite rank. Furthermore, we project $\boldsymbol{q}$ into L^2_0, i.e. we replace $\boldsymbol{q}$ by $P_{L^2_0}\boldsymbol{q}$, where $P_{L^2_0} \in L(L^2, L^2_0)$ is the orthogonal projection. To be precise, we define $H^2_{*,\mathrm{Neu}} := \{\phi \in H^2_* : \boldsymbol{\nu} \cdot \nabla\phi = 0\}$ and

$$M := \sum_{l=1}^{L} \psi_l \langle \operatorname{div} \cdot, \operatorname{div} \nabla\phi_l \rangle, \tag{5.30}$$

where $L \in \mathbb{N}_0$ is the dimension of the kernel of $(\operatorname{div} + P_{L^2_0}\boldsymbol{q}\cdot)\nabla \in L(H^2_{*,\mathrm{Neu}}, L^2_0)$, $\phi_l \in H^2_{*,\mathrm{Neu}}$, $l = 1, \ldots, L$ is an orthonormal basis with respect to the $H^2_{*,\mathrm{Neu}}$-equivalent inner product $\langle \operatorname{div} \cdot, \operatorname{div} \cdot \rangle$ of the kernel space, and $\psi_l \in L^2_0$, $l = 1, \ldots, L$ is an orthonormal basis of the L^2_0-orthogonal complement of $(\operatorname{div} + P_{L^2_0}\boldsymbol{q}\cdot)H^2_{*,\mathrm{Neu}}$. Then, for given $\boldsymbol{u} \in H_0(\mathrm{div})$, we want to find $v \in H^2_*$ such that

$$(\operatorname{div} + P_{L^2_0}\boldsymbol{q} \cdot + M)\nabla v = (\operatorname{div} + P_{L^2_0}\boldsymbol{q} \cdot + M)\boldsymbol{u} \qquad \text{in } \mathcal{O}, \tag{5.31a}$$

$$\boldsymbol{\nu} \cdot v = 0 \qquad \text{on } \partial\mathcal{O}. \tag{5.31b}$$

This problem is well-posed since the operator M ensures the injectivity of the problem, see also the argumentation in Corollary 5.6. For $\boldsymbol{u} \in \boldsymbol{X} \subset H_0(\mathrm{div})$, let $v \in H^2$ be the solution to (5.31). Then, we define the operator $T \in L(\boldsymbol{X}, \boldsymbol{X})$ by setting

$$T\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{w}, \tag{5.32}$$

where $\boldsymbol{v} := P_V\boldsymbol{u} := \nabla v, \boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$. Per construction, we have that $T \in L(\boldsymbol{X})$ and $TT = \mathrm{Id}_{\boldsymbol{X}}$, thus T is bijective. With slight adaptation to the proof of [HH21, Thm. 3.11], it can be shown that this construction indeed makes the operator A weakly T-coercive. For more details, we refer to the proof of Lemma 5.31.

Remark 5.21. *We note that if $\boldsymbol{q} = 0$, for example in the case of constant pressure, the decomposition (5.31) reduces to the usual Helmholtz decomposition, i.e. we decompose $\boldsymbol{u} \in \boldsymbol{X}$ into a gradient potential and a divergence-free function. Furthermore, Remark 5.7 still applies, i.e. the construction of M is an explicit realization of the theoretical result from Thm. A.25.*

5.2.3.2 Discrete weak T-coercivity

Now, we introduce and analyze a discrete counterpart of T. Let us note that while $\boldsymbol{X}_n \not\subset \boldsymbol{X}$ in general, we have that $\boldsymbol{X}_n \subset H_0(\mathrm{div})$ and $\boldsymbol{X} \subset H_0(\mathrm{div})$ which is why we considered $\boldsymbol{u} \in H_0(\mathrm{div})$ in Problem 5.31. Therefore, we consider the following problem: For $\boldsymbol{u} \in H_0(\mathrm{div})$, we define $\tilde{v} \in H^2_*$ to be the solution to

$$(\operatorname{div} + P_{L^2_0}\boldsymbol{q} \cdot + M)\nabla\tilde{v} = (\operatorname{div} + \pi^l_n\boldsymbol{q} \cdot + M)\boldsymbol{u} \qquad \text{in } \mathcal{O}, \tag{5.33a}$$

$$\boldsymbol{\nu} \cdot \tilde{v} = 0 \qquad \text{on } \partial\mathcal{O}. \tag{5.33b}$$

We note that this problem is well-posed with the same arguments as for Problem (5.31). For $\boldsymbol{u}_n \in \boldsymbol{X}_n$, let $\tilde{v}$ be the solution to (5.33). Then, we define the discrete operator T_n by setting

$$T_n\boldsymbol{u}_n := \boldsymbol{v}_n - \boldsymbol{w}_n, \tag{5.34}$$

where $\boldsymbol{v}_n := P_{V_n}\boldsymbol{u}_n := \pi^d_n\nabla\tilde{v}$ and $\boldsymbol{w}_n := \boldsymbol{u}_n - \boldsymbol{v}_n$. We also denote $\tilde{\boldsymbol{v}} := P_{\tilde{V}_n}\boldsymbol{u} := \nabla\tilde{v}$. Next, we will show that this operator is indeed bounded, stable and approximates T.

Lemma 5.22. *There exists a constant $C > 0$ such that $\|T_n\|_{L(\boldsymbol{X}_n)} \leq C$ for all $n \in \mathbb{N}$.*

Proof. Since $T_n = 2P_{V_n} - \mathrm{Id}_{\boldsymbol{X}}$, it suffices to show that P_{V_n} is bounded. Thus, for given $\boldsymbol{u}_n \in \boldsymbol{X}_n$, let $\tilde{v}$ be the solution to (5.33). Then, we have that $\|\tilde{v}\|_{H^2} \lesssim \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}$. As $\nabla\tilde{v} \in \boldsymbol{H}^1$, the function $\pi_n^d \nabla\tilde{v}$ is well-defined and with (5.21), (5.18), and (5.19) we estimate

$$\|\boldsymbol{R}_n^l(\pi_n^d\nabla\tilde{v})\|_{\boldsymbol{L}^2} \lesssim \sum_{F\in\mathcal{F}_n} h_F^{-1/2}\|[\![\pi_n^d\nabla\tilde{v} - \nabla\tilde{v}]\!]_{\boldsymbol{b}}\|_{\boldsymbol{L}^2(F)} \lesssim |\pi_n^d\nabla\tilde{v} - \nabla\tilde{v}|_{H^1} \lesssim |\nabla\tilde{v}|_{H^1},$$

where we exploit that $[\![\nabla\tilde{v}_{\boldsymbol{b}}]\!] = 0$. Thus, with $\operatorname{div}\pi_n^d\nabla\tilde{v} = \pi_n^l \operatorname{div}\nabla\tilde{v}$ and the boundedness of π_n^d, π_n^l, we have that $\|\pi_n^d\nabla\tilde{v}\|_{\boldsymbol{X}_n} \lesssim \|\pi_n^d\nabla\tilde{v}\|_{\boldsymbol{X}} + \|\boldsymbol{R}_n^l(\pi_n^d\nabla\tilde{v})\|_{\boldsymbol{L}^2} \lesssim \|\tilde{v}\|_{H^2}$ and therefore we obtain that $\|P_{V_n}\|_{L(\boldsymbol{X}_n)} \leq C$ for a constant $C > 0$. ☐

The next lemma states that the projection P_{V_n} is asymptotically idempotent.

Lemma 5.23. *Let $O_n := P_{V_n}P_{V_n} - P_{V_n}$. Then, it holds that $\lim_{n\to\infty}\|O_n\|_{L(\boldsymbol{X}_n)} = 0$.*

Proof. Let $\boldsymbol{u}_n \in \boldsymbol{X}_n$ and $\tilde{v}_1$ be the solution to (5.33). Then, we have that $P_{V_n}\boldsymbol{u}_n = \pi_n^d\nabla\tilde{v}_1$. Let $\tilde{v}_2$ be the solution of (5.33) with $\boldsymbol{u}_n$ replaced by $P_{V_n}\boldsymbol{u}_n$ in the right hand side. Then, we compute

$$\begin{aligned}(\operatorname{div} + P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla\tilde{v}_2 &= (\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)P_{V_n}\boldsymbol{u}_n \\ &= \pi_n^l(\operatorname{div} + P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla\tilde{v}_1 + M\pi_n^d\nabla\tilde{v}_1 - \pi_n^l M\nabla\tilde{v}_1 \\ &\quad + \pi_n^l\boldsymbol{q}\cdot(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla\tilde{v}_1 \\ &= \pi_n^l(\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)\boldsymbol{u}_n + M(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla\tilde{v}_1 + (\mathrm{Id}_{L_0^2} - \pi_n^l)M\nabla\tilde{v}_1 \\ &\quad + \pi_n^l\boldsymbol{q}\cdot(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla\tilde{v}_1 \\ &= (\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)\boldsymbol{u}_n + \tilde{O}_n\boldsymbol{u}_n,\end{aligned}$$

where $\tilde{O}_n\boldsymbol{u}_n := M(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla\tilde{v}_1 + (\mathrm{Id}_{L_0^2} - \pi_n^l)M\nabla\tilde{v}_1 + (\pi_n^l - \mathrm{Id}_{L_0^2})M\boldsymbol{u}_n + \pi_n^l\boldsymbol{q}\cdot(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla\tilde{v}_1$. Since M is compact and maps into L_0^2 and $\mathrm{Id}_{L_0^2} - \pi_n^l$ converges pointwise to zero, we have that $(\mathrm{Id}_{L_0^2} - \pi_n^l)M$ and $(\pi_n^l - \mathrm{Id}_{L_0^2})M$ converge to zero in the operator norm due to the Banach-Steinhaus theorem [BS18, Thm. 2.1.5]. Furthermore, we estimate

$$\|M(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})P_{\tilde{V}_n}\|_{L(\boldsymbol{X}_n, L_0^2)} \lesssim \|\operatorname{div}(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})P_{\tilde{V}_n}\|_{L(\boldsymbol{X}_n, L_0^2)} = \|(\pi_n^l - \mathrm{Id}_{L_0^2})\operatorname{div} P_{\tilde{V}_n}\|_{L(\boldsymbol{X}_n, L_0^2)},$$

where the first estimate follows from the definition of M and the second since $\operatorname{div}\pi_n^d = \pi_n^l \operatorname{div}$. We further compute for $\tilde{v}$ solving (5.33) that

$$\begin{aligned}\operatorname{div}\nabla\tilde{v} &= (\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)\boldsymbol{u}_n - (P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla\tilde{v}, \\ \pi_n^l \operatorname{div}\nabla\tilde{v} &= (\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)\boldsymbol{u}_n - \pi_n^l(\boldsymbol{q}\cdot + M)\nabla\tilde{v} + (\pi_n^l - \mathrm{Id})M\boldsymbol{u}_n.\end{aligned}$$

Thus, we have that

$$\|M(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})P_{\tilde{V}_n}\|_{L(\boldsymbol{X}_n, L_0^2)} \lesssim \|(P_{L_0^2} - \pi_n^l)(\boldsymbol{q}\cdot + M)P_{\tilde{V}_n}\|_{L(H_0(\operatorname{div}), L_0^2)} + \|(\pi_n^l - \mathrm{Id})M\|_{L(H_0(\operatorname{div}), L_0^2)}.$$

Since $\mathrm{Id}_{L_0^2} - \pi_n^l$ tends pointwise to zero and M is compact, the second term on the right-hand side tends to zero as well. Furthermore, we have that

$$\|(P_{L_0^2} - \pi_n^l)(\boldsymbol{q}\cdot P_{\tilde{V}_n})\|_{L(H_0(\operatorname{div}), L_0^2)} \lesssim h_n\|\boldsymbol{q}\cdot P_{\tilde{V}_n}\|_{L(H_0(\operatorname{div}), H^1)} \lesssim h_n,$$

and therefore it follows that $\|M(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})P_{\tilde{V}_n}\|_{L(\boldsymbol{X}_n, L_0^2)}$ converges to zero. Finally, we estimate

$$\|\pi_n^l\boldsymbol{q}\cdot(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})P_{\tilde{V}_n}\|_{L(\boldsymbol{X}_n, L_0^2)} \lesssim h_n\|P_{\tilde{V}_n}\|_{L(H_0(\operatorname{div}), H^1)} \lesssim h_n.$$

Therefore, it follows that $\lim_{n\to\infty} \|\tilde{O}_n\|_{L(\boldsymbol{X}_n, L^2_0)} = 0$. Since $(\operatorname{div} + P_{L^2_0}\boldsymbol{q}\cdot + M)\nabla(\tilde{v}_2 - \tilde{v}_1) = \tilde{O}_n \boldsymbol{u}_n$, the claim now follows from

$$\|(P_{V_n}P_{V_n} - P_{V_n})\boldsymbol{u}_n\|_{\boldsymbol{X}_n} \lesssim \|\nabla(\tilde{v}_2 - \tilde{v}_1)\|_{\boldsymbol{H}^1} \lesssim \|\tilde{O}_n\|_{L(\boldsymbol{X}_n, L^2_0)} \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}. \tag{5.35}$$

□

Lemma 5.24. *There exists a constants $n_0, C > 0$ such that T_n is invertible and $\|T_n^{-1}\|_{L(\boldsymbol{X}_n)} \leq C$ for $n > n_0$.*

Proof. From the previous lemma and the definition of T_n, we have that $T_nT_n = 4P_{V_n}P_{V_n} - 4P_{V_n} + \mathrm{Id}_{\boldsymbol{X}_n} = \mathrm{Id}_{\boldsymbol{X}_n} + 4O_n$. Since $\|O_n\|_{L(\boldsymbol{X}_n)} \to 0$ for $n \to \infty$, there exists an index $n_0 > 0$ such that $\|O_n\|_{L(\boldsymbol{X}_n)} < 1/8$ for all $n > n_0$ and hence $\|(\mathrm{Id}_{\boldsymbol{X}} + 4O_n)^{-1}\|_{L(\boldsymbol{X}_n)} \leq 2$ for $n > n_0$. Then, as $T_n^{-1} = (T_nT_n)^{-1}T_n = (\mathrm{Id}_{\boldsymbol{X}} + 4O_n)^{-1}T_n$, we have that $\|T_n^{-1}\|_{L(\boldsymbol{X}_n)} \leq 2\|T_n\|_{L(\boldsymbol{X}_n)}$ for $n > n_0$. The claim now follows from Lemma 5.22. □

Lemma 5.25. *It holds that $\lim_{n\to\infty} \|(T_np_n - p_nT)\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$ for all $\boldsymbol{u} \in \boldsymbol{X}$.*

Proof. By definition of T_n and T, it suffices to show that $\lim_{n\to\infty} \|(P_{V_n}p_n - p_nP_V)\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$ for all $\boldsymbol{u} \in \boldsymbol{X}$. Due to the boundedness of $P_{V_n} = \pi_n^d P_{\tilde{V}_n}$, we have that

$$\begin{aligned}
&\|(P_{V_n}p_n - p_nP_V)\boldsymbol{u}\|_{\boldsymbol{X}_n} \\
&\qquad \leq d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, P_{V_n}p_n\boldsymbol{u}) \\
&\qquad = d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, \pi_n^d P_{\tilde{V}_n}p_n\boldsymbol{u}) \\
&\qquad \leq d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, \pi_n^d P_{\tilde{V}_n}\boldsymbol{u}) + \|\pi_n^d P_{\tilde{V}_n}(\boldsymbol{u} - p_n\boldsymbol{u})\|_{\boldsymbol{X}_n} \\
&\qquad \lesssim d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, \pi_n^d P_V\boldsymbol{u}) + \|P_V\boldsymbol{u} - P_{\tilde{V}_n}\boldsymbol{u}\|_{\boldsymbol{X}} + \|\boldsymbol{u} - p_n\boldsymbol{u}\|_{H(\mathrm{div})} \\
&\qquad \lesssim d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, \pi_n^d P_V\boldsymbol{u}) + \|(P_{L^2_0} - \pi_n^l)(\boldsymbol{q}\cdot\boldsymbol{u})\|_{L^2} + d_n(\boldsymbol{u}, p_n\boldsymbol{u}),
\end{aligned}$$

where the last estimates follows as $\|P_V\boldsymbol{u} - P_{\tilde{V}_n}\boldsymbol{u}\|_{\boldsymbol{X}} \lesssim \|(P_{L^2_0} - \pi_n^l)(\boldsymbol{q}\cdot\boldsymbol{u})\|_{L^2}$ due to (5.31) and (5.33). The claim now follows from Lemma 5.16, Lemma 5.17 and the pointwise convergence of π_n^l to $P_{L^2_0}$. □

Therefore, we are left with showing that the remaining conditions from Thm. 2.28 are satisfied. In preparation, we show that the discrete differential operator $\boldsymbol{D}^n_{\boldsymbol{b}}$ can be bounded by a suitably weighted H^1-seminorm. Before we proceed, let us note some preliminaries. First of all, we define the weighted H^1-seminorm $|\cdot|_{\boldsymbol{H}^1_{c_s^2\rho}}$ on $\boldsymbol{H}^1_{\boldsymbol{\nu}0}$ by setting

$$|\boldsymbol{u}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} := \|c_s\rho^{1/2}\nabla\boldsymbol{u}\|^2_{(L^2)^{3\times 3}}.$$

Furthermore, we note that due to [HH21, Thm. 3.5], see also Thm. 4.5, there exists a compact operator $K_G \in L(\boldsymbol{V})$ such that

$$\langle c_s^2\rho \operatorname{div}\boldsymbol{v}, \operatorname{div}\boldsymbol{v}\rangle = |\boldsymbol{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} + \langle K_G\boldsymbol{v}, \boldsymbol{v}\rangle_{\boldsymbol{V}}, \tag{5.36}$$

where $\boldsymbol{V} := \{\nabla v : v \in H^2_{*,\mathrm{Neu}}\}$, $\|\cdot\|_{\boldsymbol{V}} = |\cdot|_{\boldsymbol{H}^1_{c_s^2\rho}}$. We also note that the calculation in the proof of Lemma 5.23 yields that for $\boldsymbol{u}_n \in \boldsymbol{X}_n$, it holds that

$$(\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot)\boldsymbol{w}_n = -M\boldsymbol{w}_n - \tilde{O}_n\boldsymbol{u}_n, \tag{5.37}$$

where $\tilde{O}_n$ is defined as in Lemma 5.23. Furthermore, recall that the operator $M \in L(H_0(\mathrm{div}), L^2_0)$ is compact and that $\|\tilde{O}_n\|_{L(\boldsymbol{X}_n, L^2_0)}$ converges to zero.

Lemma 5.26 (Lem. 16 of [Hal23]). *For all $\boldsymbol{v} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0}$ and $n \in \mathbb{N}$, it holds that*

$$\|\rho^{1/2} D^n_{\boldsymbol{b}} \pi^d_n \boldsymbol{v}\|^2_{\boldsymbol{L}^2} \leq (C^{\#}_\pi)^2 (1 + h_n^2 \tilde{C}_\pi) \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} |\boldsymbol{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}}, \tag{5.38}$$

with constants $\tilde{C}_\pi > 0$ and

$$(C^{\#}_\pi)^2 := 2\big((C_{ab}C_{sh}C_{dt})^2 + \sup_{n\in\mathbb{N}} \sup_{\tau\in\mathcal{T}_n} \|\pi^d_n\|^2_{L(\boldsymbol{H}^1_*(\tau))}\big), \qquad \|\cdot\|_{\boldsymbol{H}^1_*(\tau)} := |\cdot|_{\boldsymbol{H}^1(\tau)}. \tag{5.39}$$

Proof. For each $\tau \in \mathcal{T}_n$, it holds that

$$\begin{aligned}
\|\rho^{1/2}\partial_{\boldsymbol{b}}\pi^d_n\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\tau)} &\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty(\tau)} \overline{c_{s_\tau}}^2 \overline{\rho_\tau} |\pi^d_n \boldsymbol{v}|^2_{\boldsymbol{H}^1(\tau)} && (5.40a)\\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty(\tau)} \overline{c_{s_\tau}}^2 \overline{\rho_\tau} \|\pi^d_n\|^2_{L(\boldsymbol{H}^1_*(\tau))} |\boldsymbol{v}|^2_{\boldsymbol{H}^1(\tau)} && (5.40b)\\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty(\tau)} \overline{c_{s_\tau}}^2 \overline{\rho_\tau} \|\pi^d_n\|^2_{L(\boldsymbol{H}^1_*(\tau))} \frac{1}{c^2_{s_\tau}\rho_\tau} |\boldsymbol{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}(\tau)} && (5.40c)\\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty(\tau)} \|\pi^d_n\|^2_{L(\boldsymbol{H}^1_*(\tau))} \Big(1 + h_n^2 \frac{1}{\underline{c_{s_\tau}}^2 \underline{\rho_\tau}} (C^L_{c^s\rho^{1/2}})^2\Big)^2 |\boldsymbol{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}(\tau)} && (5.40d)
\end{aligned}$$

For the last line, we utilize $|x-y| \leq h_\tau \leq h_n$ for all $x, y \in \tau$ to obtain

$$\frac{1}{c^2_{s_\tau}\rho_\tau} \leq \frac{1}{\underline{c_{s_\tau}}^2\underline{\rho_\tau}} \leq 1 + \frac{|\overline{c_{s_\tau}}^2\overline{\rho_\tau} - \underline{c_{s_\tau}}^2\underline{\rho_\tau}|}{\underline{c_{s_\tau}}^2\underline{\rho_\tau}} \leq 1 + h_n^2 \frac{1}{\underline{c_{s_\tau}}^2\underline{\rho_\tau}} (C^L_{c^s\rho^{1/2}})^2, \tag{5.41}$$

where $C^L_{c^s\rho^{1/2}}$ is the Lipschitz constant[2]of $c_s\rho^{1/2}$. Furthermore, we compute

$$\begin{aligned}
\|\rho^{1/2}\boldsymbol{R}^l_n\pi^d_n\boldsymbol{v}\|_{\boldsymbol{L}^2(\tau)} = \|\rho^{1/2}\sum_{F\in\mathcal{F}_\tau} r^F_n \pi^d_n \boldsymbol{v}\|_{\boldsymbol{L}^2(\tau)} &\leq C_{\mathrm{dt}}\overline{\rho_\tau}\|\mathfrak{h}^{-1/2}[\![\pi^d_n\boldsymbol{v}]\!]_{\boldsymbol{b}}\|_{\boldsymbol{L}^2(\partial\tau)}\\
&= C_{\mathrm{dt}}\overline{\rho_\tau}\|\mathfrak{h}^{-1/2}[\![\pi^d_n\boldsymbol{v} - \boldsymbol{v}]\!]_{\boldsymbol{b}}\|_{\boldsymbol{L}^2(\partial\tau)}.
\end{aligned}$$

Using (5.19) we further estimate with the same arguments as in (5.41) that

$$\begin{aligned}
C^2_{\mathrm{dt}} \sum_{\tau\in\mathcal{T}_n} \overline{\rho_\tau} \|\mathfrak{h}^{-1/2}[\![\pi^d_n\boldsymbol{v} - \boldsymbol{v}]\!]_{\boldsymbol{b}}\|^2_{\boldsymbol{L}^2(\partial\tau)} &\leq C^2_{\mathrm{dt}} \sum_{\tau\in\mathcal{T}_n} \overline{\rho_\tau} \sum_{F\in\mathcal{F}_\tau} \|\mathfrak{h}^{-1/2}[\![\pi^d_n\boldsymbol{v} - \boldsymbol{v}]\!]_{\boldsymbol{b}}\|^2_{\boldsymbol{L}^2(F)}\\
&\leq C^2_{\mathrm{dt}} \sum_{\tau\in\mathcal{T}_n} \overline{\rho_\tau} \sum_{F\in\mathcal{F}_\tau} \Big(\frac{1}{2}\sum_{j=1}^2 \|\mathfrak{h}^{-1/2}(\boldsymbol{\nu}\cdot\boldsymbol{b})((\pi^d_n\boldsymbol{v})_j - \boldsymbol{v})\|_{\boldsymbol{L}^2(F)}\Big)^2\\
&\leq \frac{C^2_{\mathrm{dt}}}{2} \sum_{\tau\in\mathcal{T}_n} \overline{\rho_\tau} \sum_{F\in\mathcal{F}_\tau} \sum_{j=1}^2 \|\mathfrak{h}^{-1/2}(\boldsymbol{\nu}\cdot\boldsymbol{b})((\pi^d_n\boldsymbol{v})_j - \boldsymbol{v})\|^2_{\boldsymbol{L}^2(F)}\\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} C_{\mathrm{dt}^2} \sum_{\tau\in\mathcal{T}_n} \overline{c_{s_\tau}}^2 \overline{\rho}_\tau \|\mathfrak{h}^{-1/2}((\pi^d_n\boldsymbol{v})|_\tau - \boldsymbol{v})\|^2_{\boldsymbol{L}^2(\partial\tau)}\\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} C^2_{\mathrm{ab}} C^2_{\mathrm{sh}} C^2_{\mathrm{dt}} \sum_{\tau\in\mathcal{T}_n} \overline{c_{s_\tau}}^2 \overline{\rho}_\tau |\boldsymbol{v}|^2_{\boldsymbol{H}^1(\tau)}\\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} C^2_{\mathrm{ab}} C^2_{\mathrm{sh}} C^2_{\mathrm{dt}} \Big(1 + (Ch_n)^2 \frac{1}{\underline{c_s}^2\underline{\rho}} (C^L_{c_s\rho^{1/2}})^2\Big)^2 |\boldsymbol{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}},
\end{aligned}$$

where the constant $C > 0$ only depends on C_{sh}. Combining the estimates for $\|\rho^{1/2}\partial_{\boldsymbol{b}}\pi^d_n\boldsymbol{v}\|_{\boldsymbol{L}^2}$ and $\|\rho^{1/2}\boldsymbol{R}^l_n\pi^d_n\boldsymbol{v}\|_{\boldsymbol{L}^2}$ yields the claim. □

[2]By assumption, we have that $c_s\rho^{1/2} \in W^{1,\infty}$. Since $\mathcal{O}$ is assumed to be bounded and convex, every function in $W^{1,\infty}$ is Lipschitz continuous [EG21a, Rem. 2.12].

Now, we define operators $K_n^{EP_V}, K_n^{\text{mean}}, K_n^{K_G}, K_n^M \in L(\boldsymbol{X}_n)$ by setting for $\boldsymbol{u}_n, \boldsymbol{u}_n' \in \boldsymbol{X}_n$

$$\begin{aligned}
\langle K_n^{EP_V}\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} &:= \langle P_{V_n}\boldsymbol{u}_n, P_{V_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2},\\
\langle K_n^{\text{mean}}\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} &:= \langle \text{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n), \text{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle_{L^2},\\
\langle K_n^{K_G}\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} &:= \langle K_G P_{\tilde{V}_n}\boldsymbol{u}_n, K_G P_{\tilde{V}_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{V}},\\
\langle K_n^{M}\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} &:= \langle M\boldsymbol{u}_n, M\boldsymbol{u}_n'\rangle_{L^2}.
\end{aligned}$$

The following Lemma shows that these operators define P-compact sequences in the sense of Definition 2.15.

Lemma 5.27. *The sequences of operators $(K_n^{EP_V})_{n\in\mathbb{N}}, (K_n^{mean})_{n\in\mathbb{N}}, (K_n^{K_G})_{n\in\mathbb{N}}, (K_n^{M})_{n\in\mathbb{N}}$ are compact in the sense of discrete approximation schemes.*

Proof. Following the argumentation of [Hal23, Lem. 17], we show the statement for $(K_n^{EP_V})_{n\in\mathbb{N}}$ and note that the argumentation for the other sequences goes along the same lines. Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $\boldsymbol{u}_n \in \boldsymbol{X}_n$, be a bounded sequence with $\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} \le 1$ for each $n \in \mathbb{N}$ and $\mathbb{N}' \subset \mathbb{N}$ be an arbitrary subsequence. To show that $(K_n^{EP_V})_{n\in\mathbb{N}}$ is compact, we have to show that $(K_n^{EP_V}\boldsymbol{u}_n)_{n\in\mathbb{N}}$ is P-compact, i.e. that there exists a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $(K_n^{EP_V}\boldsymbol{u}_n)_{n\in\mathbb{N}''}$ converges in $\boldsymbol{X}_n$, cf. Definition 2.15. We recall that the embedding $\boldsymbol{H}^1_{\boldsymbol{\nu}0} \hookrightarrow \boldsymbol{L}^2$ is compact and that the operators $P_{V_n} = \pi_n^d P_{\tilde{V}_n}$, $P_{\tilde{V}_n} \in L(H_0(\mathrm{div}), \boldsymbol{H}^1_{\boldsymbol{\nu}0})$, are uniformly bounded. Therefore, there exists $\boldsymbol{z} \in \boldsymbol{L}^2$ and a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $\lim_{n\to\infty} \|\boldsymbol{z} - P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{L}^2} = 0$. Furthermore, we compute

$$\begin{aligned}
\|\boldsymbol{z} - P_{V_n}\boldsymbol{u}_n\|_{\boldsymbol{L}^2} = \|\boldsymbol{z} - \pi_n^d P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{L}^2} &\le \|\boldsymbol{z} - P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{L}^2} + \|(1-\pi_n^d)P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{L}^2}\\
&\lesssim \|\boldsymbol{z} - P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{L}^2} + h_n\|P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{H}^1} \xrightarrow{n\in\mathbb{N}''} 0.
\end{aligned}$$

In the following, we want to show that there exists a subsequence $\mathbb{N}''' \subset \mathbb{N}''$ such that $\lim_{n\in\mathbb{N}'''} \|p_n P_V^*\boldsymbol{z} - K_n^{EP_V}\boldsymbol{u}_n\|_{\boldsymbol{X}_n} = 0$, where P_V^* is the adjoint operator of P_V. Let $\boldsymbol{u}_n' \in \boldsymbol{X}_n$, $\|u_n\|_{X_n} = 1$, $n \in \mathbb{N}''$ be such that

$$\|p_n P_V^*\boldsymbol{z} - K_n^{EP_V}\boldsymbol{u}_n\|_{\boldsymbol{X}_n} \le |\langle p_n P_V^*\boldsymbol{z} - K_n^{EP_V}\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n}| + 1/n.$$

Due to Thm. 5.19, we can choose $\mathbb{N}''' \subset \mathbb{N}''$ and $\boldsymbol{u}_n \in \boldsymbol{X}$ such that $\boldsymbol{u}_n' \overset{\boldsymbol{L}^2}{\rightharpoonup} \boldsymbol{u}$. On the one hand, we obtain

$$\begin{aligned}
\langle p_n P_V^*\boldsymbol{z}, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} &= \langle \mathrm{div}\, P_V^*\boldsymbol{z}, \mathrm{div}\,\boldsymbol{u}_n'\rangle + \langle P_V^*\boldsymbol{z}, \boldsymbol{u}_n'\rangle + \langle \partial_{\boldsymbol{b}} P_V^*\boldsymbol{z}, D_{\boldsymbol{b}}^n\boldsymbol{u}_n'\rangle\\
&\xrightarrow{n\in\mathbb{N}'''} \langle \mathrm{div}\, P_V^*\boldsymbol{z}, \mathrm{div}\,\boldsymbol{u}\rangle + \langle P_V^*\boldsymbol{z}, \boldsymbol{u}\rangle + \langle \partial_{\boldsymbol{b}} P_V^*\boldsymbol{z}, \partial_{\boldsymbol{b}}\boldsymbol{u}\rangle = \langle P_V^*\boldsymbol{z}, \boldsymbol{u}\rangle_{\boldsymbol{X}} = \langle \boldsymbol{z}, P_V\boldsymbol{u}\rangle.
\end{aligned}$$

On the other hand, it holds that

$$\begin{aligned}
\langle K_n^{EP_V}\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} &= \langle P_{V_n}\boldsymbol{u}_n, P_{V_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}\\
&= \langle P_{V_n}\boldsymbol{u}_n - \boldsymbol{z}, P_{V_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{z}, \pi_n^d P_{\tilde{V}_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}\\
&= \langle P_{V_n}\boldsymbol{u}_n - \boldsymbol{z}, P_{V_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{z}, \pi_n^d P_V\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{z}, \pi_n^d (P_{\tilde{V}_n} - P_V)\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}\\
&= \langle P_{V_n}\boldsymbol{u}_n - \boldsymbol{z}, P_{V_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{z}, P_V\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{z}, (\pi_n^d - 1)P_V\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}\\
&\quad + \langle \boldsymbol{z}, \pi_n^d (P_{\tilde{V}_n} - P_V)\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}\\
&= \langle P_{V_n}\boldsymbol{u}_n - \boldsymbol{z}, P_{V_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{z}, P_V\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{z}, (\pi_n^d - 1)P_V\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}\\
&\quad + \langle \boldsymbol{z}, (\pi_n^d - 1)(P_{\tilde{V}_n} - P_V)\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2} + \langle \boldsymbol{z}, (P_{\tilde{V}_n} - P_V)\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}.
\end{aligned}$$

We estimate

$$\|(1-\pi_n^d)P_V\boldsymbol{u}_n'\|_{\boldsymbol{L}^2}+\|(1-\pi_n^d)(P_{\tilde{V}_n}-P_V)\boldsymbol{u}_n'\|_{\boldsymbol{L}^2}\lesssim h_n(\|P_V\boldsymbol{u}_n'\|_{\boldsymbol{H}^1}+\|P_{\tilde{V}_n}\boldsymbol{u}_n'\|_{\boldsymbol{H}^1})\lesssim h_n.$$

Furthermore, we can write $(P_{\tilde{V}}-P_V)\boldsymbol{u}_n'=S(\pi_n^l-P_{L_0^2})(\boldsymbol{q}\cdot\boldsymbol{u}_n')$ with $S:=\nabla((\operatorname{div}+P_{L_0^2}\boldsymbol{q}\cdot+M)\nabla)^{-1}\in L(L_0^2,\boldsymbol{L}^2)$ and therefore we obtain since π_n^l converges pointwise to $P_{L_0^2}$ that

$$\langle\boldsymbol{z},(P_{\tilde{V}_n}-P_V)\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}=\langle\boldsymbol{z},S(\pi_n^l-P_{L_0^2})(\boldsymbol{q}\cdot\boldsymbol{u}_n')\rangle_{\boldsymbol{L}^2}=\langle(\pi_n^l-P_{L_0^2})S^*\boldsymbol{z},P_{L_0^2}(\boldsymbol{q}\cdot\boldsymbol{u}_n')\rangle_{L_0^2}\stackrel{n\in\mathbb{N}'''}{\longrightarrow}0.$$

Finally, since

$$\langle\boldsymbol{z},P_V\boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}=\langle P_V^*\boldsymbol{z},\boldsymbol{u}_n'\rangle_{H_0(\operatorname{div})}\stackrel{n\in\mathbb{N}'''}{\longrightarrow}\langle P_V^*\boldsymbol{z},\boldsymbol{u}'\rangle_{H_0(\operatorname{div})}=\langle\boldsymbol{z},P_V\boldsymbol{u}'\rangle_{\boldsymbol{L}^2},$$

it follows that $K_n^{EP_V}\boldsymbol{u}_n\stackrel{P}{\to}P_V^*\boldsymbol{z}$, $n\in\mathbb{N}'''$. Thus, $(K_n^{EP_V})_{n\in\mathbb{N}}$ is indeed P-compact. With the same technique, we can show that $(K_n^{\text{mean}})_{n\in\mathbb{N}}$, $(K_n^{K_G})_{n\in\mathbb{N}}$ and $(K_n^M)_{n\in\mathbb{N}}$ are P-compact as well. ☐

Let $\lambda_-(\underline{\underline{m}})\in L^\infty$ be the smallest eigenvalue of a symmetric matrix $\underline{\underline{m}}$. From now on, we set $\underline{\underline{m}}=-\rho^{-1}\operatorname{Hess}(p)+\operatorname{Hess}(\phi)$. Additionally, we define the constants

$$C_{\underline{\underline{m}}}:=\max\left\{0,\sup_{x\in\mathcal{O}}\frac{-\lambda_-(\underline{\underline{m}}(x))}{\gamma(x)}\right\}\qquad\text{and}\qquad\theta:=\arctan(C_{\underline{\underline{m}}}/|\omega|)\in[0,2\pi),\omega\neq0.\quad(5.42)$$

With this notation, we pose the following smallness assumption on the Mach number.

Assumption 5.2. *The background flow $\boldsymbol{b}$ is subsonic in the sense that*

$$\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2<\frac{1}{(C_\pi^\#)^2}\frac{1}{1+C_{\underline{\underline{m}}}^2/|\omega|^2},\tag{5.43}$$

where $C_\pi^\#$ is the constant from Lemma 5.26.

Comparing this smallness assumption with the assumption on the Mach number from Theorem 5.12 from Section 5.1, we notice that we avoid the ratio $\frac{\overline{c_s}^2\overline{\rho}}{\underline{c_s}^2\underline{\rho}}$ which improves the robustness of the method against drastic changes in sound-speed and density.
Now, our main goal is to show that the sequence $(A_n)_{n\in\mathbb{N}}$ is regular by applying Thm. 2.28. Since we have already shown that the sequence $(T_n)_{n\in\mathbb{N}}$ is stable and approximates T, we have to show that we can write $A_nT_n=B_n+K_n$, where $(B_n)_{n\in\mathbb{N}}$ is stable and approximates a bijective operator B and $(K_n)_{n\in\mathbb{N}}$ is compact. To show that this is indeed the case, we follow the proof of [Hal23, Thm. 18]. For ease of presentation, we split the proof into several lemmata. Let $K_G\in L(\boldsymbol{V})$ be the compact operator from (5.36). With constants $C_1,C_2>0$ that will be specified later on, we define for $\boldsymbol{u}_n,\boldsymbol{u}_n'\in\boldsymbol{X}_n$

$$\begin{aligned}
\langle\tilde{B}_n\boldsymbol{u}_n,\boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n}:=&\\
&\langle c_s^2\rho\operatorname{div}\boldsymbol{v}_n,\operatorname{div}\boldsymbol{v}_n'\rangle-\langle\rho iD_{\boldsymbol{b}}^n\boldsymbol{v}_n,iD_{\boldsymbol{b}}^n\boldsymbol{v}_n'\rangle+\langle c_s^2\rho\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n),\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle &&(5.44\text{a})\\
&-\langle\rho iD_{\boldsymbol{b}}^n\boldsymbol{v}_n,(\omega+iD_{\boldsymbol{b}}^n+i\Omega\times)\boldsymbol{w}_n'\rangle+\langle\rho(\omega+iD_{\boldsymbol{b}}^n+i\Omega\times)\boldsymbol{w}_n,iD_{\boldsymbol{b}}^n\boldsymbol{v}_n'\rangle &&(5.44\text{b})\\
&+\langle\rho(\omega+iD_{\boldsymbol{b}}^n+i\Omega\times)\boldsymbol{w}_n,(\omega+iD_{\boldsymbol{b}}^n+i\Omega\times)\boldsymbol{w}_n'\rangle+\langle\rho(i\omega\gamma+\underline{\underline{m}})\boldsymbol{w}_n,\boldsymbol{w}_n'\rangle &&(5.44\text{c})\\
&+\langle\boldsymbol{v}_n,\boldsymbol{v}_n'\rangle+C_1\langle K_GP_{\tilde{V}_n}\boldsymbol{u}_n,K_GP_{\tilde{V}_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{V}}+\langle M\boldsymbol{w}_n,M\boldsymbol{w}_n'\rangle+\langle\tilde{O}_n\boldsymbol{u}_n,\tilde{O}_n\boldsymbol{u}_n'\rangle &&(5.44\text{d})
\end{aligned}$$

and

$$
\begin{aligned}
\langle \tilde{K}_n \boldsymbol{u}_n, &\boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} := \\
&C_2\big(\langle \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle + \langle K_G P_{\tilde{V}_n} \boldsymbol{u}_n, K_G P_{\tilde{V}_n} \boldsymbol{u}_n' \rangle_{\boldsymbol{V}} + \langle \tilde{O}_n \boldsymbol{u}_n, \tilde{O}_n \boldsymbol{u}_n' \rangle && \text{(5.45a)}\\
&+ \langle M \boldsymbol{w}_n, M \boldsymbol{w}_n' \rangle + \langle \mathrm{mean}(\boldsymbol{q} \cdot \boldsymbol{w}_n), \mathrm{mean}(\boldsymbol{q} \cdot \boldsymbol{w}_n') \rangle \big) && \text{(5.45b)}\\
&+ \langle c_s^2 \rho \boldsymbol{q} \cdot \boldsymbol{v}_n, \operatorname{div} \boldsymbol{v}_n' \rangle + \langle c_s^2 \rho \operatorname{div} \boldsymbol{v}_n, \boldsymbol{q} \cdot \boldsymbol{v}_n' \rangle - \langle \rho(\omega + i\Omega \times) \boldsymbol{v}_n, (\omega + i\Omega \times) \boldsymbol{v}_n' \rangle && \text{(5.45c)}\\
&- \langle \rho(\omega + i\Omega \times) \boldsymbol{v}_n, i D_{\boldsymbol{b}}^n \boldsymbol{v}_n' \rangle - \langle \rho i D_{\boldsymbol{b}}^n \boldsymbol{v}_n, (\omega + i\Omega \times) \boldsymbol{v}_n' \rangle - i\omega \langle \gamma \rho \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle && \text{(5.45d)}\\
&- \langle \rho \underline{\underline{m}} \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle && \text{(5.45e)}\\
&- \langle \rho \underline{\underline{m}} \boldsymbol{v}_n, \boldsymbol{w}_n' \rangle - i\omega \langle \gamma \rho \boldsymbol{v}_n, \boldsymbol{w}_n' \rangle - \langle c_s^2 \rho \pi_n^l (\boldsymbol{q} \cdot \boldsymbol{v}_n), \pi_n^l (\boldsymbol{q} \cdot \boldsymbol{w}_n') \rangle && \text{(5.45f)}\\
&- \langle \rho(\omega + i\Omega \times) \boldsymbol{v}_n, (\omega + i D_{\boldsymbol{b}}^n + i\Omega \times) \boldsymbol{w}_n' \rangle && \text{(5.45g)}\\
&- \langle c_s^2 \rho (\operatorname{div} + \pi_n^l \boldsymbol{q} \cdot) \boldsymbol{v}_n, M \boldsymbol{w}_n' + \tilde{O}_n \boldsymbol{u}_n' \rangle + \langle c_s^2 \rho (\mathrm{Id} - \pi_n^l)(\boldsymbol{q} \cdot \boldsymbol{v}_n), \operatorname{div} \boldsymbol{w}_n' \rangle && \text{(5.45h)}\\
&+ \langle \rho \underline{\underline{m}} \boldsymbol{w}_n, \boldsymbol{v}_n' \rangle + i\omega \langle \gamma \rho \boldsymbol{w}_n, \boldsymbol{v}_n' \rangle + \langle c_s^2 \rho \pi_n^l (\boldsymbol{q} \cdot \boldsymbol{w}_n), \pi_n^l (\boldsymbol{q} \cdot \boldsymbol{v}_n') \rangle && \text{(5.45i)}\\
&+ \langle \rho(\omega + i D_{\boldsymbol{b}}^n + i\Omega \times) \boldsymbol{w}_n, (\omega + i\Omega \times) \boldsymbol{v}_n' \rangle && \text{(5.45j)}\\
&+ \langle c_s^2 \rho (M \boldsymbol{w}_n + \tilde{O}_n \boldsymbol{u}_n), (\operatorname{div} + \pi_n^l \boldsymbol{q} \cdot) \boldsymbol{v}_n' \rangle - \langle c_s^2 \rho \operatorname{div} \boldsymbol{w}_n, (\mathrm{Id} - \pi_n^l)(\boldsymbol{q} \cdot \boldsymbol{v}_n') \rangle && \text{(5.45k)}\\
&- \langle c_s^2 \rho (\mathrm{Id} - \mathrm{mean} - \pi_n^l)(\boldsymbol{q} \cdot \boldsymbol{w}_n), \operatorname{div} \boldsymbol{w}_n' \rangle - \langle c_s^2 \rho \mathrm{mean}(\boldsymbol{q} \cdot \boldsymbol{w}_n), \operatorname{div} \boldsymbol{w}_n' \rangle && \text{(5.45l)}\\
&- \langle c_s^2 \rho \operatorname{div} \boldsymbol{w}_n, (\mathrm{Id} - \mathrm{mean} - \pi_n^l)(\boldsymbol{q} \cdot \boldsymbol{w}_n') \rangle - \langle c_s^2 \rho \operatorname{div} \boldsymbol{w}_n, \mathrm{mean}(\boldsymbol{q} \cdot \boldsymbol{w}_n') \rangle && \text{(5.45m)}\\
&- \langle c_s^2 \rho (M \boldsymbol{w}_n + \tilde{O}_n \boldsymbol{u}_n), M \boldsymbol{w}_n' + \tilde{O}_n \boldsymbol{u}_n' \rangle. && \text{(5.45n)}
\end{aligned}
$$

Then, we define $B_n := \tilde{B}_n + \tilde{K}_n$. Furthermore, we define

$$
\begin{aligned}
\langle K_n \boldsymbol{u}_n, &\boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} := \\
&- (1 + C_2) \langle \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle - (C_1 + C_2) \langle K_G P_{\tilde{V}_n} \boldsymbol{u}_n, K_G P_{\tilde{V}_n} \boldsymbol{u}_n' \rangle_{\boldsymbol{V}} && \text{(5.46a)}\\
&- (1 + C_2) \langle M \boldsymbol{w}_n, M \boldsymbol{w}_n' \rangle - C_2 \langle \mathrm{mean}(\boldsymbol{q} \cdot \boldsymbol{w}_n), \mathrm{mean}(\boldsymbol{q} \cdot \boldsymbol{w}_n') \rangle && \text{(5.46b)}\\
&- (1 + C_2) \langle \tilde{O}_n \boldsymbol{u}_n, \tilde{O}_n \boldsymbol{u}_n' \rangle && \text{(5.46c)}
\end{aligned}
$$

We note that the uniform boundedness of B_n, $n \in \mathbb{N}$, follows from straightforward computations.

Lemma 5.28. *There exist sequences of operators $(B_n)_{n\in\mathbb{N}}$, $B_n \in L(\boldsymbol{X}_n)$, and $(K_n)_{n\in\mathbb{N}}$, $K_n \in L(\boldsymbol{X}_n)$, such that $A_n T_n = B_n + K_n$ for all $n \in \mathbb{N}$ and the sequence $(K_n)_{n\in\mathbb{N}}$ is compact.*

Proof. The sequence $(K_n)_{n\in\mathbb{N}}$ is indeed compact in the sense of discrete approximation schemes due to Lemma 5.27 and $\lim_{n\to\infty} \|\tilde{O}_n\|_{L(\boldsymbol{X}_n, L^2)} = 0$. Furthermore, the sequence $(\tilde{K}_n)_{n\in\mathbb{N}}$ is also compact for a sufficiently large constant C_2. In the following, we argue that there holds $A_n T_n = B_n + K_n$. First of all, we note that $\langle A_n T_n \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} = a_n(T_n \boldsymbol{u}_n, \boldsymbol{u}_n') = a_n(\boldsymbol{v}_n - \boldsymbol{w}_n, \boldsymbol{v}_n' + \boldsymbol{w}_n')$. We rewrite the sesquilinear form $a_n(\cdot,\cdot)$ using the previously defined matrix $\underline{\underline{m}}$ and $\boldsymbol{q}$ as in (4.8):

$$\begin{aligned}
a_n(\boldsymbol{v}_n - \boldsymbol{w}_n, \boldsymbol{v}_n' + \boldsymbol{w}_n') =& \langle c_s^2 \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{w}_n), \operatorname{div}(\boldsymbol{v}_n' + \boldsymbol{w}_n')\rangle \\
&- \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)(\boldsymbol{v}_n - \boldsymbol{w}_n), (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)(\boldsymbol{v}_n' + \boldsymbol{w}_n')\rangle \\
&+ \langle c_s^2 \rho \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{w}_n), \boldsymbol{q}\cdot(\boldsymbol{v}_n' + \boldsymbol{w}_n')\rangle \\
&+ \langle c_s^2 \rho \boldsymbol{q}\cdot(\boldsymbol{v}_n - \boldsymbol{w}_n), \operatorname{div}(\boldsymbol{v}_n' + \boldsymbol{w}_n')\rangle \\
&- \langle \rho \underline{\underline{m}}(\boldsymbol{v}_n - \boldsymbol{w}_n), \boldsymbol{v}_n' + \boldsymbol{w}_n'\rangle \\
&- i\omega\langle \gamma\rho\boldsymbol{v}_n - \boldsymbol{w}_n, \boldsymbol{v}_n' + \boldsymbol{w}_n'\rangle.
\end{aligned}$$

Splitting this expression into the terms associated with $(\boldsymbol{v}_n, \boldsymbol{v}_n)$, $(\boldsymbol{v}_n, \boldsymbol{w}_n)$, $(\boldsymbol{w}_n, \boldsymbol{v}_n)$ and $(\boldsymbol{w}_n, \boldsymbol{w}_n)$ gives rise to some of the terms appearing in the definition of B_n and K_n, for example all terms in the lines (5.45c)-(5.45e). For the sake of readability, we will only explain the terms that were added or modified. For stability, we add the terms (5.44d), (5.45a) and (5.45b), which cancel with (5.46a), (5.46b) and (5.46c). Additionally, we add the term $\langle c_s^2\rho\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n), \pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n)\rangle$ in (5.44a). It is subtracted again in (5.45) with the following argument. Since we want to get rid of the term $-\langle c_s^2\rho \operatorname{div}\boldsymbol{w}_n, \operatorname{div}\boldsymbol{w}_n\rangle$ coming from $a_n(\cdot,\cdot)$, we add

$$-\langle c_s^2\rho \operatorname{div}\boldsymbol{w}_n, \pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle - \langle c_s^2\rho\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n), \operatorname{div}\boldsymbol{w}_n'\rangle - \langle c_s^2\rho\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n), \pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle \tag{5.47}$$

such that (5.37) yields

$$-\langle c_s^2\rho(\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot)\boldsymbol{w}_n, (\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot)\boldsymbol{w}_n'\rangle = -\langle c_s^2\rho(M\boldsymbol{w}_n + \tilde{O}_n\boldsymbol{u}_n), M\boldsymbol{w}_n' + \tilde{O}_n\boldsymbol{u}_n'\rangle = (5.45\text{n}).$$

The remaining added terms in (5.47) are balanced in (5.45l) and (5.45m). In these lines, we can also find the naturally appearing terms $-\langle c_s^2\rho\boldsymbol{q}\cdot\boldsymbol{w}_n, \operatorname{div}\boldsymbol{w}_n'\rangle$ and $-\langle c_s^2\rho \operatorname{div}\boldsymbol{w}_n, \boldsymbol{q}\cdot\boldsymbol{w}_n'\rangle$, as well as the terms $\pm\langle c_s^2\rho\text{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n), \operatorname{div}\boldsymbol{w}_n'\rangle$ and $\pm\langle c_s^2\rho \operatorname{div}\boldsymbol{w}_n, \text{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle$ which are used lateron. In the same manner, we reformulate the terms $\langle c_s^2\rho \operatorname{div}\boldsymbol{v}_n, (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{w}_n'\rangle$ and $-\langle c_s^2\rho(\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{w}_n, \operatorname{div}\boldsymbol{v}_n'\rangle$ stemming from $a_n(\cdot,\cdot)$ in (5.45f)-(5.45h) and (5.45i)-(5.45k). □

Lemma 5.29. *Let Assumption 5.2 be satisfied. Then, there exists an index $n_0 > 0$ such that the operator $\tilde{B}_n$ defined in (5.44) is coercive for all $n > n_0$.*

Proof. We recall that the divergence operator and the interpolation operator π_n^d commute in the sense that $\operatorname{div}\pi_n^d = \pi_n^l \operatorname{div}$. Thus, we have with the definition of $\tilde{v}$, cf. (5.33), that

$$\begin{aligned}
\operatorname{div}\boldsymbol{v}_n &= \operatorname{div}\pi_n^d\nabla\tilde{v} = \pi_n^l \operatorname{div}\nabla\tilde{v} = \pi_n^l\Big(-(P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla\tilde{v} + (\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)\boldsymbol{u}_n\Big) \\
&= -(P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla\tilde{v} + (\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)\boldsymbol{u}_n \\
&\quad + (Id - \pi_n^l)(P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla\tilde{v} + (\pi_n^l - Id)M\boldsymbol{u}_n \\
&= \operatorname{div}\nabla\tilde{v} + (Id - \pi_n^l)(P_{L_0^2}\boldsymbol{q}\cdot + M)P_{\tilde{V}_n}\boldsymbol{u}_n + (\pi_n^l - Id)M\boldsymbol{u}_n \\
&=: \Delta\tilde{v} + \hat{O}_n\boldsymbol{u}_n,
\end{aligned}$$

where the second to last line follows from the definitions of $\tilde{v}$ and $P_{\tilde{V}_n}$. Note that similarly to the proof of Lemma 5.23, one can show that $\lim_{n\to\infty}\|\hat{O}_n\|_{L(\boldsymbol{X}_n, L_0^2)} = 0$. Defining

$$\langle\check{O}_n\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle := \langle c_s^2\rho \operatorname{div}\boldsymbol{v}_n, \hat{O}_n\boldsymbol{u}_n'\rangle + \langle c_s^2\rho\hat{O}_n\boldsymbol{u}_n, \operatorname{div}\boldsymbol{v}_n'\rangle + \langle c_s^2\rho\hat{O}_n\boldsymbol{u}_n, \hat{O}_n\boldsymbol{u}_n'\rangle,$$

we can write

$$\langle c_s^2\rho \operatorname{div}\boldsymbol{v}_n, \operatorname{div}\boldsymbol{v}_n\rangle = \langle c_s^2\rho\Delta\tilde{v}, \Delta\tilde{v}\rangle + \langle\check{O}_n\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n}. \tag{5.48}$$

Note that there holds $\lim_{n\to\infty}\|\check{O}_n\|_{L(\boldsymbol{X}_n)}=0$ and using (5.36) we have that

$$\langle c_s^2\rho \operatorname{div}\boldsymbol{v}_n, \operatorname{div}\boldsymbol{v}_n\rangle = |\nabla\tilde{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} + \langle K_G P_{\tilde{V}_n}\boldsymbol{u}_n, P_{\tilde{V}_n}\boldsymbol{u}_n\rangle_{\boldsymbol{V}} + \langle \check{O}_n\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n}. \tag{5.49}$$

Using the smallness assumption 5.2 on the Mach number, we can find $\epsilon\in(0,1)$, $\tau\in(0,\pi/2-\theta)$ and $n_0>0$ such that

$$C_{\theta,\tau,\epsilon,n_0} := 1-(C_\pi^{\#})^2(1+\sup_{n>n_0} h_n^2\tilde{C}_\pi)\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty}(1+\tan^2(\theta+\tau)(1+\epsilon)^{-1}-\epsilon)-\epsilon>0.$$

Now, we can estimate with the weighted Young's inequality[3] and the definition of θ

$$\begin{aligned}
&\frac{1}{\cos(\theta+\tau)}\Re\big(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\big)\\
&\quad= \|c_s\rho^{1/2}\operatorname{div}\boldsymbol{v}_n\|^2_{L^2} - \|\rho^{1/2}\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} + \|\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} + C_1\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|^2_{\boldsymbol{V}} + \|M\boldsymbol{w}_n\|^2_{L^2}\\
&\qquad+\|\tilde{O}_n\boldsymbol{u}_n\|^2_{L^2} + \|c_s\rho^{1/2}\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n)\|^2_{\boldsymbol{L}^2} + \|\rho^{1/2}(\omega+i\boldsymbol{D}^n_{\boldsymbol{b}}+i\Omega\times)\boldsymbol{w}_n\|^2_{\boldsymbol{L}^2} + \langle\rho\underline{\underline{m}}\boldsymbol{w}_n,\boldsymbol{w}_n\rangle_{\boldsymbol{L}^2}\\
&\qquad+2\tan(\theta+\tau)\operatorname{sgn}\omega\Im\Big(\langle\rho(\omega+i\boldsymbol{D}^n_{\boldsymbol{b}}+i\Omega\times)\boldsymbol{w}, i\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n\rangle\Big) - |\omega|\tan(\theta+\tau)\|(\gamma\rho)^{1/2}\boldsymbol{w}_n\|^2_{\boldsymbol{L}^2}\\
&\quad\geq \|c_s\rho^{1/2}\operatorname{div}\boldsymbol{v}_n\|^2_{L^2} - (1+\tan^2(\theta+\tau)(1-\epsilon)^{-1})\|\rho^{1/2}\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} + \|\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2}\\
&\qquad+C_1\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}} + \|M\boldsymbol{w}_n\|^2_{L^2} + \|\tilde{O}_n\boldsymbol{u}_n\|^2_{L^2} + \|c_s\rho^{1/2}\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n)\|^2_{\boldsymbol{L}^2}\\
&\qquad+\epsilon\|\rho^{1/2}(\omega+i\boldsymbol{D}^n_{\boldsymbol{b}}+\Omega\times)\boldsymbol{w}_n\|^2_{\boldsymbol{L}^2} + |\omega|(\tan(\theta+\tau)-\tan(\theta))\|(\gamma\rho)^{1/2}\boldsymbol{w}_n\|^2_{\boldsymbol{L}^2}.
\end{aligned}$$

Here, we use the identities $e^{-ix}=\cos(x)-i\sin(x)$ and $\tan(x)=\sin(x)/\cos(x)$. Furthermore, we use the definition of θ to estimate

$$\langle\rho\underline{\underline{m}}\boldsymbol{w}_n,\boldsymbol{w}_n\rangle \geq \langle\rho\lambda_-(\underline{\underline{m}})\boldsymbol{w}_n,\boldsymbol{w}_n\rangle \geq -|\omega|\tan(\theta)\|(\gamma\rho)^{1/2}\boldsymbol{w}_n\|_{\boldsymbol{L}^2}.$$

Then, we apply Lemma 5.26, (5.49) and Young's inequality to obtain

$$\begin{aligned}
&\|c_s\rho^{1/2}\operatorname{div}\boldsymbol{v}_n\|^2_{L^2} - (1+\tan^2(\theta+\tau)(1-\epsilon)^{-1})\|\rho^{1/2}\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} + \|\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} + C_1\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|^2_{\boldsymbol{V}}\\
&\quad\geq \epsilon\Big(\|c_s\rho^{1/2}\operatorname{div}\boldsymbol{v}_n\|^2_{L^2} + \|\rho^{1/2}\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2}\Big) + C_{\theta,\tau,\epsilon,n_0}|\nabla\tilde{v}|^2_{c_s^2\rho} + \|\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2}\\
&\qquad+\Big(C_1-\frac{1}{\delta}\Big)\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|^2_{\boldsymbol{V}} - (\delta\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})} + \|\check{O}_n\|_{L(\boldsymbol{X}_n)})\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n}\\
&\quad\geq \epsilon\min\{\underline{c_s}^2\underline{\rho},\underline{\rho},1\}\|\boldsymbol{v}_n\|^2_{\boldsymbol{X}_n} + \Big(C_1-\frac{1}{4\delta}\Big)\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|^2_{\boldsymbol{V}}\\
&\qquad-(\delta\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})} + \|\check{O}_n\|_{L(\boldsymbol{X}_n)})\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n}
\end{aligned}$$

From (5.37) we have that

$$4(\|M\boldsymbol{w}_n\|^2_{L^2} + \|\tilde{O}_n\boldsymbol{u}_n\|^2_{L^2} + \|c_s\rho^{1/2}\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n)\|^2_{\boldsymbol{L}^2}) \geq \|\operatorname{div}\boldsymbol{w}_n\|^2_{L^2}.$$

[3] Before applying Young's inequality, we estimate

$$\begin{aligned}
&2\tan(\theta+\tau)\operatorname{sgn}\omega\Im\Big(\langle\rho(\omega+i\boldsymbol{D}^n_{\boldsymbol{b}}+i\Omega\times)\boldsymbol{w}, i\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n\rangle\Big)\\
&\qquad\geq -\Big|2\tan(\theta+\tau)\operatorname{sgn}\omega\Im\Big(\langle\rho(\omega+i\boldsymbol{D}^n_{\boldsymbol{b}}+i\Omega\times)\boldsymbol{w}, i\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n\rangle\Big)\Big|.
\end{aligned}$$

and therefore

$$\begin{aligned}\|M\boldsymbol{w}_n\|_{L^2}^2 + \|\tilde{O}_n\boldsymbol{u}_n\|_{L^2}^2 + \|c_s\rho^{1/2}\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n)\|_{\boldsymbol{L}^2}^2 + \epsilon\|\rho^{1/2}(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2 \\ + |\omega|\big(\tan(\theta+\tau) - \tan(\theta)\big)\|(\gamma\rho)^{1/2}\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2 \gtrsim \|\boldsymbol{w}_n\|_{\boldsymbol{X}_n}^2.\end{aligned}$$

Altogether, we have with $C_{\tilde{B}} > 0$ independent of δ, C_1 and $n > n_0$ that

$$\begin{aligned}&\frac{1}{\cos(\theta+\tau)}\Re\big(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\big) \\ &\geq C_{\tilde{B}}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 + \Big(C_1 - \frac{1}{\delta}\Big)\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2 - (\delta\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})} + \|\check{O}\|_{L(\boldsymbol{X}_n)})\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2,\end{aligned}$$

Therefore, we can choose $\delta > 0$ small enough and $n_1 > n_0$ big enough such that

$$(\delta\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})} + \|\check{O}\|_{L(\boldsymbol{X}_n)}) \leq C_{\tilde{B}}/2,$$

where we recall that $\lim_{n\to\infty}\|\check{O}\|_{L(\boldsymbol{X}_n)} = 0$. Consequently, we have for all $n > n_1$ that

$$\frac{1}{\cos(\theta+\tau)}\Re\big(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\big) \geq \frac{C_{\tilde{B}}}{2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 + \Big(C_1 - \frac{1}{\delta}\Big)\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2.$$

Now, we choose $C_1 > 1/(4\delta)$ to obtain for $n > n_1$

$$\frac{1}{\cos(\theta+\tau)}\Re\big(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\big) \geq \frac{C_{\tilde{B}}}{2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2,$$

which proves the claim. □

Lemma 5.30. *Assume that Assumption 5.2 is fulfilled. Then there exists an index $n_0 > 0$ such that the operator $B_n := \tilde{B}_n + \tilde{K}_n$ is coercive for all $n > n_0$.*

Proof. The previous Lemma already established the statement for $\tilde{B}_n$. To show that $\tilde{K}_n$ is coercive for all $n > n_0$, we first estimate the first terms in (5.45l) and (5.45m) respectively by

$$\begin{aligned}|\langle c_s^2\rho(\operatorname{Id} - \operatorname{mean} - \pi_n^l)(\boldsymbol{q}\cdot\boldsymbol{w}_n), \operatorname{div}\boldsymbol{w}_n'\rangle| &= |\langle\boldsymbol{q}\cdot\boldsymbol{w}_n, (\operatorname{Id} - \operatorname{mean} - \pi_n^l)(c_s^2\rho\operatorname{div}\boldsymbol{w}_n')\rangle| \\ &\leq \|\boldsymbol{q}\|_{\boldsymbol{L}^\infty}\|\boldsymbol{w}_n\|_{\boldsymbol{L}^2}\|(\operatorname{Id} - \operatorname{mean} - \pi_n^l)(c_s^2\rho\operatorname{div}\boldsymbol{w}_n')\|_{L^2}.\end{aligned}$$

For suitable constants c_τ, $\tau \in \mathcal{T}_n$, we estimate

$$\begin{aligned}\|(\operatorname{Id} - \operatorname{mean} - \pi_n^l)(c_s^2\rho\operatorname{div}\boldsymbol{w}_n)\|_{L^2}^2 &= \sum_{\tau\in\mathcal{T}_n}\|(\operatorname{Id} - \operatorname{mean} - \pi_n^l)(c_s^2\rho\operatorname{div}\boldsymbol{w}_n)\|_{L^2(\tau)}^2 \\ &= \sum_{\tau\in\mathcal{T}_n}\|(\operatorname{Id} - \operatorname{mean} - \pi_n^l)((c_s^2\rho - c_\tau)\operatorname{div}\boldsymbol{w}_n)\|_{L^2(\tau)}^2 \\ &\leq \sum_{\tau\in\mathcal{T}_n}\|(c_s^2\rho - c_\tau)\operatorname{div}\boldsymbol{w}_n\|_{L^2(\tau)}^2 \\ &\leq \sum_{\tau\in\mathcal{T}_n}\|(c_s^2\rho - c_\tau)\|_{L^\infty(\tau)}^2\|\operatorname{div}\boldsymbol{w}_n\|_{L^2(\tau)}^2 \\ &\leq (C_{c_s^2\rho}^L)^2 h_n^2\sum_{\tau\in\mathcal{T}_n}\|\operatorname{div}\boldsymbol{w}_n\|_{L^2(\tau)}^2 \\ &= (C_{c_s^2\rho}^L)^2 h_n^2\|\operatorname{div}\boldsymbol{w}_n\|_{L^2}^2,\end{aligned}$$

where the last steps follow from a similar argument as in (5.41). Now, we define the seminorm

$$|\boldsymbol{u}_n|_{Y_n}^2 := \|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + \|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2 + \|\tilde{O}_n\boldsymbol{u}_n\|_{L^2}^2 + \|M\boldsymbol{w}_n\|_{L^2}^2 + \|\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n)\|_{L^2}^2,$$

which allows us to estimate with constants $C_{Y,1}, C_{Y,2} > 0$ with the weighted Young's inequality

$$\frac{1}{\cos(\theta+\tau)}\Re\big(e^{-i(\theta+\tau)\mathrm{sgn}\omega}\langle\tilde{K}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\big) \geq C_2|\boldsymbol{u}_n|_{Y_n}^2 - h_n C_{Y,1}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 - C_{Y,2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}|\boldsymbol{u}_n|_{Y_n}.$$

Together with the previous Lemma, we obtain using the weighted Young's inequality that

$$\begin{aligned}\frac{1}{\cos(\theta+\tau)}&\Re\big(e^{-i(\theta+\tau)\mathrm{sgn}\omega}\langle B_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\big)\\ &\geq \frac{C_{\tilde{B}}}{2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 + C_2|\boldsymbol{u}_n|_{Y_n}^2 - h_nC_{Y,1}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 - C_{Y,2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}|\boldsymbol{u}_n|_{Y_n}\\ &\geq \Big(\frac{C_{\tilde{B}}}{4} - h_nC_{Y,1}\Big)\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 + \Big(C_2 - \frac{C_{Y,2}^2}{C_{\tilde{B}}}\Big)|\boldsymbol{u}_n|_{Y_n}^2.\end{aligned}$$

Choosing $C_2 > C_{Y,2}^2/C_{\tilde{B}}$ yields the uniform coercivity of B_n for n sufficiently large. □

Lemma 5.31. *Let $A \in L(\boldsymbol{X})$ be the operator induced by the continuous sesquilinear form $a(\cdot,\cdot)$. There exists $B, K \in L(X)$ such that B is coercive and $AT = B + K$. Furthermore, it holds that $B_n \xrightarrow{P} B$.*

Proof. Let $\boldsymbol{u}, \boldsymbol{u}' \in \boldsymbol{X}$. Then we define

$$\langle B\boldsymbol{u},\boldsymbol{u}'\rangle_{\boldsymbol{X}} :=$$

$$\langle c_s^2\rho\,\mathrm{div}\,\boldsymbol{v},\mathrm{div}\,\boldsymbol{v}'\rangle - \langle\rho i\partial_{\boldsymbol{b}}\boldsymbol{v}, i\partial_{\boldsymbol{b}}\boldsymbol{v}'\rangle + \langle c_s^2\rho P_{L_0^2}(\boldsymbol{q}\cdot\boldsymbol{w}), P_{L_0^2}(\boldsymbol{q}\cdot\boldsymbol{w}')\rangle \tag{5.50a}$$

$$- \langle\rho i\partial_{\boldsymbol{b}}\boldsymbol{v}, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}'\rangle + \langle\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}, i\partial_{\boldsymbol{b}}\boldsymbol{v}'\rangle \tag{5.50b}$$

$$+ \langle\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}'\rangle + \langle\rho(i\omega\gamma + \underline{\underline{m}})\boldsymbol{w},\boldsymbol{w}'\rangle \tag{5.50c}$$

$$+ \langle\boldsymbol{v},\boldsymbol{v}'\rangle + C_1\rangle K_G\boldsymbol{v}, K_G\boldsymbol{v}'\rangle + \langle M\boldsymbol{w}, M\boldsymbol{w}'\rangle \tag{5.50d}$$

$$+ C_2\Big(\langle\boldsymbol{v},\boldsymbol{v}'\rangle + \langle K_G\boldsymbol{v}, K_G\boldsymbol{v}'\rangle_{\boldsymbol{V}} + \langle M\boldsymbol{w}, M\boldsymbol{w}'\rangle + \langle\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}),\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}')\rangle\Big) \tag{5.50e}$$

$$+ \langle c_s^2\rho\boldsymbol{q}\cdot\boldsymbol{v},\mathrm{div}\,\boldsymbol{v}'\rangle + \langle c_s^2\rho\,\mathrm{div}\,\boldsymbol{v},\boldsymbol{q}\cdot\boldsymbol{v}'\rangle - \langle\rho(\omega + i\Omega\times)\boldsymbol{v},(\omega + i\Omega\times)\boldsymbol{v}'\rangle \tag{5.50f}$$

$$- \langle\rho(\omega + i\Omega\times)\boldsymbol{v}, i\partial_{\boldsymbol{b}}\boldsymbol{v}'\rangle - \langle\rho i\partial_{\boldsymbol{b}}\boldsymbol{v},(\omega + i\Omega\times)\boldsymbol{v}'\rangle - i\omega\langle\gamma\rho\boldsymbol{v},\boldsymbol{v}'\rangle - \langle\rho\underline{\underline{m}}\boldsymbol{v},\boldsymbol{v}'\rangle \tag{5.50g}$$

$$- \langle\rho\underline{\underline{m}}\boldsymbol{v},\boldsymbol{w}'\rangle - i\omega\langle\gamma\rho\boldsymbol{v},\boldsymbol{w}'\rangle - \langle c_s^2\rho P_{L_0^2}(\boldsymbol{q}\cdot\boldsymbol{v}), P_{L_0^2}(\boldsymbol{q}\cdot\boldsymbol{w}')\rangle \tag{5.50h}$$

$$- \langle\rho(\omega + i\Omega\times)\boldsymbol{v},(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}'\rangle - \langle c_s^2\rho(\mathrm{div} + P_{L_0^2}\boldsymbol{q}\cdot)\boldsymbol{v}, M\boldsymbol{w}'\rangle \tag{5.50i}$$

$$+ \langle c_s^2\rho\,\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{v}),\mathrm{div}\,\boldsymbol{w}'\rangle \tag{5.50j}$$

$$+ \langle\rho\underline{\underline{m}}\boldsymbol{w},\boldsymbol{v}'\rangle + i\omega\langle\gamma\rho\boldsymbol{w},\boldsymbol{v}'\rangle + \langle c_s^2\rho P_{L_0^2}(\boldsymbol{q}\cdot\boldsymbol{w}), P_{L_0^2}(\boldsymbol{q}\cdot\boldsymbol{v}')\rangle \tag{5.50k}$$

$$+ \langle\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w},(\omega + i\Omega\times)\boldsymbol{v}'\rangle + \langle c_s^2\rho M\boldsymbol{w},(\mathrm{div} + P_{L_0^2}\boldsymbol{q}\cdot)\boldsymbol{v}'\rangle \tag{5.50l}$$

$$- \langle c_s^2\rho\,\mathrm{div}\,\boldsymbol{w},\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{v}')\rangle \tag{5.50m}$$

$$- \langle c_s^2\rho\,\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}),\mathrm{div}\,\boldsymbol{w}'\rangle - \langle c_s^2\rho\,\mathrm{div}\,\boldsymbol{w},\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}')\rangle - \langle c_s^2\rho M\boldsymbol{w}, M\boldsymbol{w}'\rangle \tag{5.50n}$$

and

$$\begin{aligned}\langle K\boldsymbol{u},\boldsymbol{u}'\rangle_{\boldsymbol{X}} :=& -(1+C_2)\langle \boldsymbol{v},\boldsymbol{v}'\rangle - (C_1+C_2)\langle K_G\boldsymbol{v},K_G\boldsymbol{v}'\rangle_{\boldsymbol{V}} - (1+C_2)\langle M\boldsymbol{w},M\boldsymbol{w}'\rangle \\ &- C_2\langle \operatorname{mean}(\boldsymbol{q}\cdot\boldsymbol{w}),\operatorname{mean}(\boldsymbol{q}\cdot\boldsymbol{w}')\rangle.\end{aligned}$$

Then, using the same argumentation as in the proof of Lemma 5.28 we have that $AT = B + K$. The coercivity of B follows along the lines of Lemma 5.30. It remains to show that $\lim_{n\to\infty}\|(p_nB - B_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$. Since $B_n = A_nT_n - K_n$ and $B = AT - K$, we have that

$$\begin{aligned}&\|(p_nB - B_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} \\ &\quad\le \|(p_nK - K_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} + \|p_nAT - A_nT_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} \\ &\quad\le \|(p_nK - K_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} + \|(p_nA - A_np_n)T\boldsymbol{u}\|_{X_n} + \|A_n\|_{L(\boldsymbol{X}_n)}\|(p_nT - T_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n}.\end{aligned}$$

Since $(A_n)_{n\in\mathbb{N}}$ is uniformly bounded, $T_n \xrightarrow{P} T$ by Lemma 5.25 and $A_n \xrightarrow{P} A$ by Thm. 5.20, we only have to show that $K_n \xrightarrow{P} K$. For given $\boldsymbol{u}\in\boldsymbol{X}$, we choose $\boldsymbol{u}'_n \in \boldsymbol{X}_n$, $\|\boldsymbol{u}'_n\|_{\boldsymbol{X}_n} = 1$, $n\in\mathbb{N}$ such that

$$\|(p_nK - K_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} \le |\langle p_nK\boldsymbol{u} - K_np_n\boldsymbol{u},\boldsymbol{u}'_n\rangle_{\boldsymbol{X}_n}| + 1/n.$$

For an arbitrary subsequence $\mathbb{N}'\subset\mathbb{N}$, we can choose $\mathbb{N}''\subset\mathbb{N}'$ and $\boldsymbol{u}'\in\boldsymbol{X}$ such that $\boldsymbol{u}'_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \boldsymbol{u}'$, $\operatorname{div}\boldsymbol{u}'_n \overset{L^2}{\rightharpoonup} \operatorname{div}\boldsymbol{u}'$ and $D^n_{\boldsymbol{b}}\boldsymbol{u}'_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \partial_{\boldsymbol{b}}\boldsymbol{u}'$ in accordance with Lemma 5.19. On the one hand, we compute that

$$\begin{aligned}\langle p_nK\boldsymbol{u},\boldsymbol{u}'_n\rangle_{\boldsymbol{X}_n} &= \langle \operatorname{div}K\boldsymbol{u},\operatorname{div}\boldsymbol{u}'_n\rangle + \langle K\boldsymbol{u},\boldsymbol{u}'_n\rangle + \langle \partial_{\boldsymbol{b}}K\boldsymbol{u},D^n_{\boldsymbol{b}}\boldsymbol{u}'_n\rangle \\ &\xrightarrow{n\in\mathbb{N}''} \langle \operatorname{div}K\boldsymbol{u},\boldsymbol{u}'\rangle + \langle K\boldsymbol{u},\boldsymbol{u}'\rangle + \langle \partial_{\boldsymbol{b}}K\boldsymbol{u},\partial_{\boldsymbol{b}}\boldsymbol{u}'\rangle = \langle K\boldsymbol{u},\boldsymbol{u}'\rangle_{\boldsymbol{X}}.\end{aligned}$$

On the other hand, we have using the definition of $\boldsymbol{v}$ and P_{V_n} that

$$\begin{aligned}|\langle \boldsymbol{v},\boldsymbol{v}'_n\rangle - \langle P_{V_n}p_n\boldsymbol{u},\boldsymbol{v}'_n\rangle| &= |\langle P_V\boldsymbol{u} - \pi^d_nP_{\tilde{V}_n}p_n\boldsymbol{u},\boldsymbol{v}'_n\rangle| \\ &\lesssim |\langle P_V\boldsymbol{u} - \pi^d_nP_{\tilde{V}_n}\boldsymbol{u},\boldsymbol{v}'_n\rangle| + d_n(\boldsymbol{u},p_n\boldsymbol{u}) \\ &\lesssim |\langle P_V\boldsymbol{u} - \pi^d_nP_V\boldsymbol{u},\boldsymbol{v}'_n\rangle| + d_n(\boldsymbol{u},p_n\boldsymbol{u}) + \|(P_{L^2_0} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{u})\|_{L^2} \\ &\lesssim h_n\|P_V\boldsymbol{u}\|_{\boldsymbol{H}^1} + d_n(\boldsymbol{u},p_n\boldsymbol{u}) + \|(P_{L^2_0} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{u})\|_{L^2},\end{aligned}$$

where we use that $\|P_{\tilde{V}_n}\boldsymbol{u} - P_V\boldsymbol{u}\|_{\boldsymbol{L}^2} = |\tilde{v} - v|_{H^1} \lesssim \|(P_{L^2_0} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{u})\|_{L^2}$ since $\tilde{v}, v$ solve (5.33) and (5.31) respectively. Additionally, we calculate

$$\begin{aligned}|\langle K_G\boldsymbol{v},K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}} - \langle K_GP_{\tilde{V}_n}p_n\boldsymbol{u},K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}| &= |\langle K_G(P_V\boldsymbol{u} - P_{\tilde{V}_n}p_n\boldsymbol{u}),K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}| \\ &\lesssim |\langle K_G(P_V\boldsymbol{u} - P_{\tilde{V}_n}\boldsymbol{u}),K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}| + d_n(\boldsymbol{u},p_n\boldsymbol{u}) \\ &\lesssim \|(P_{L^2_0} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{u})\|_{L^2} + d_n(\boldsymbol{u},p_n\boldsymbol{u})\end{aligned}$$

and as $\boldsymbol{w} = \boldsymbol{u} - P_V\boldsymbol{u}$, $\boldsymbol{w}_n = \boldsymbol{u}_n - P_{V_n}\boldsymbol{u}_n$ we have that

$$\begin{aligned}&|\langle M\boldsymbol{w},M\boldsymbol{w}'_n\rangle - \langle M\boldsymbol{w}_n(p_n\boldsymbol{u}),M\boldsymbol{w}'_n\rangle| \\ &\qquad= |\langle M(\boldsymbol{w} - \boldsymbol{w}_n(p_n\boldsymbol{u})),M\boldsymbol{w}'_n\rangle| \\ &\qquad= |\langle M(\boldsymbol{u} - P_V\boldsymbol{u} - (p_n\boldsymbol{u} - P_{V_n}p_n\boldsymbol{u})),M\boldsymbol{w}'_n\rangle| \\ &\qquad\lesssim \|\boldsymbol{u} - P_V\boldsymbol{u} - (p_n\boldsymbol{u} - P_{V_n}p_n\boldsymbol{u})\|_{H(\operatorname{div})} \\ &\qquad\lesssim \|P_V\boldsymbol{u} - P_{V_n}p_n\boldsymbol{u}\|_{H(\operatorname{div})} + d_n(\boldsymbol{u},p_n\boldsymbol{u}) \\ &\qquad\lesssim \|p_nP_V\boldsymbol{u} - P_{V_n}p_n\boldsymbol{u}\|_{H(\operatorname{div})} + d_n(\boldsymbol{u},p_n\boldsymbol{u}) + d_n(P_V\boldsymbol{u},p_nP_V\boldsymbol{u}).\end{aligned}$$

With a similar argument, we obtain that

$$|\langle \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}), \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle - \langle \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n(p_n\boldsymbol{u})), \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle| \\ \lesssim \|p_n P_V\boldsymbol{u} - P_{V_n}\boldsymbol{u}\|_{H(\mathrm{div})} + d_n(\boldsymbol{u}, p_n\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, p_n P_V\boldsymbol{u}).$$

Thus, we obtain with Lemma 5.16 and the pointwise convergence of π_n^l to $P_{L_0^2}$ that

$$\lim_{n\to\infty} |\langle K_n p_n\boldsymbol{u}, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} + (1+C_2)\langle \boldsymbol{v}, \boldsymbol{v}_n'\rangle + (C_1+C_2)\langle K_G\boldsymbol{v}, K_G P_{\tilde{V}_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{V}} + (1+C_2)\langle M\boldsymbol{w}, M\boldsymbol{w}_n'\rangle \\ + C_2\langle \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}), \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle| = 0.$$

Let $S := \nabla((\mathrm{div} + P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla)^{-1} \in L(L_0^2, \boldsymbol{V})$. Then, we compute

$$\begin{aligned}\langle K_G\boldsymbol{v}, K_G P_{\tilde{V}_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{V}} &= \langle K_G^* K_G\boldsymbol{v}, P_{\tilde{V}_n}\boldsymbol{u}_n'\rangle_{\boldsymbol{V}} \\ &= \langle K_G^* K_G\boldsymbol{v}, P_V\boldsymbol{u}_n'\rangle_{\boldsymbol{V}} + \langle K_G^* K_G\boldsymbol{v}, S(\pi_n^l - P_{L_0^2})(\boldsymbol{q}\cdot\boldsymbol{u}_n')\rangle_{\boldsymbol{V}} \\ &= \langle P_V^* K_G^* K_G\boldsymbol{v}, \boldsymbol{u}_n'\rangle_{H_0(\mathrm{div})} + \langle (\pi_n^l - P_{L_0^2})S^* K_G^* K_G\boldsymbol{v}, \boldsymbol{q}\cdot\boldsymbol{u}_n'\rangle_{L^2} \\ &\xrightarrow{n\in\mathbb{N}''} \langle P_V^* K_G^* K_G\boldsymbol{v}, \boldsymbol{u}'\rangle_{H_0(\mathrm{div})} = \langle K_G\boldsymbol{v}, K_G\boldsymbol{v}'\rangle_{\boldsymbol{V}}\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}&\lim_{n\in\mathbb{N}''}\Big((1+C_2)\langle \boldsymbol{v}, \boldsymbol{v}_n'\rangle + (C_1+C_2)\langle K_G\boldsymbol{v}, K_G P_{\tilde{V}_n}\boldsymbol{u}_n'\rangle + (1+C_2)\langle M\boldsymbol{w}, M\boldsymbol{w}_n'\rangle \\ &\qquad + C_2\langle \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}), \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n')\rangle\Big) \\ &= -(1+C_2)\langle \boldsymbol{v}, \boldsymbol{v}'\rangle + (C_1+C_2)\langle K_G\boldsymbol{v}, K_G\boldsymbol{v}'\rangle_{\boldsymbol{V}} - (1+C_2)\langle M\boldsymbol{w}, M\boldsymbol{w}'\rangle \\ &\quad - C_2\langle \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}), \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}')\rangle \\ &= \langle K\boldsymbol{u}, \boldsymbol{u}'\rangle_{\boldsymbol{X}},\end{aligned}$$

which shows that $\lim_{n\in\mathbb{N}''}\|(p_nK - K_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$ for all $\boldsymbol{u}\in\boldsymbol{X}$. □

With these results, we can apply Theorem 2.28 to obtain the following theorem.

Theorem 5.32. *Under the smallness assumption on the Mach number 5.2, the sequence $(A_n^{DG})_{n\in\mathbb{N}}$ is regular.*

Proof. Using the previous lemmata, we obtain that $A_nT_n = B_n + K_n$, where the sequence $(K_n)_{n\in\mathbb{N}}$, $K_n \in L(\boldsymbol{X}_n)$ is compact and there exists a bijective operator $B\in L(\boldsymbol{X})$ such that $\lim_{n\to\infty}\|B_np_n\boldsymbol{u} - p_nB\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$ for all $\boldsymbol{u}\in\boldsymbol{X}$. Thus, we can apply Thm. 2.28 to conclude that $(A_n^{\mathrm{DG}})_{n\in\mathbb{N}}$ is regular. □

5.2.3.3 Convergence results

Theorem 5.33. *Assume that Assumption 5.2 is satisfied and let $\boldsymbol{f}\in\boldsymbol{L}^2$. Further, let $\boldsymbol{u}\in\boldsymbol{X}$ be the solution to $a(\boldsymbol{u},\boldsymbol{u}') = \langle\boldsymbol{f},\boldsymbol{u}'\rangle$ for all $\boldsymbol{u}'\in\boldsymbol{X}$. Then there exists $n_0>0$ such that for all $n>n_0$ the solution $\boldsymbol{u}_n\in\boldsymbol{X}_n$ to $a_n(\boldsymbol{u}_n,\boldsymbol{u}_n') = \langle\boldsymbol{f},\boldsymbol{u}_n'\rangle$ for all $\boldsymbol{u}_n'\in\boldsymbol{X}_n$ exists and $\lim_{n\to\infty} d_n(\boldsymbol{u},\boldsymbol{u}_n) = 0$. Additionally, if $\boldsymbol{u}\in\boldsymbol{H}^{2+s}$, $\rho\in W^{1+s,\infty}$ and $\boldsymbol{b}\in\boldsymbol{W}^{1+s,\infty}$ with $s>0$, then $d_n(\boldsymbol{u},\boldsymbol{u}_n)\lesssim h_n^{\min(1+s,k)} + h_n^{\min(s,l)}$.*

Proof. A is injective due to Lemma 4.4, $A_n \xrightarrow{P} A$ due to Thm. 5.20 and $(A_n)_{n\in\mathbb{N}}$ is regular due to Thm. 5.32. To apply the convergence theorem 2.17, we have to still show that the right-hand side of the discrete problem converges to the right-hand side of the continuous problem. To be precise, let $\boldsymbol{g} \in \boldsymbol{X}$ be such that $\langle \boldsymbol{g}, \boldsymbol{u}'\rangle_{\boldsymbol{X}} = \langle \boldsymbol{f}, \boldsymbol{u}'\rangle$ for all $\boldsymbol{u}' \in \boldsymbol{X}$ and $\boldsymbol{g}_n \in \boldsymbol{X}_n$ be such that $\langle \boldsymbol{g}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} = \langle \boldsymbol{f}, \boldsymbol{u}_n'\rangle$ for all $\boldsymbol{u}_n' \in \boldsymbol{X}_n$. Then we have to show that $\boldsymbol{g}_n \xrightarrow{P} \boldsymbol{g}$. To this end, we choose $\boldsymbol{u}_n' \in \boldsymbol{X}_n$, $\|\boldsymbol{u}_n'\|_{\boldsymbol{X}_n} = 1$ such that $\|p_n\boldsymbol{g} - \boldsymbol{g}_n\|_{\boldsymbol{X}_n} \leq |\langle p_n\boldsymbol{g} - \boldsymbol{g}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n}| + 1/n$. For an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$, we choose $\boldsymbol{u}' \in \boldsymbol{X}$ and a subsubsequence $\mathbb{N}'' \subset \mathbb{N}'$ as in Lemma 5.19 and obtain with the definition of $\boldsymbol{g}_n$ and p_n that

$$\begin{aligned}\langle p_n\boldsymbol{g} - \boldsymbol{g}_n, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} &= \langle p_n\boldsymbol{g}, \boldsymbol{u}_n'\rangle_{\boldsymbol{X}} - \langle \boldsymbol{f}, \boldsymbol{u}_n'\rangle = \langle \operatorname{div}\boldsymbol{g}, \boldsymbol{u}_n'\rangle + \langle \boldsymbol{g}, \boldsymbol{u}_n'\rangle + \langle \partial_{\boldsymbol{b}}\boldsymbol{g}, D_{\boldsymbol{b}}^n\boldsymbol{u}_n'\rangle - \langle \boldsymbol{f}, \boldsymbol{u}_n'\rangle \\ &\xrightarrow{n\to\infty} \langle \boldsymbol{g}, \boldsymbol{u}'\rangle_{\boldsymbol{X}} - \langle \boldsymbol{f}, \boldsymbol{u}'\rangle = 0.\end{aligned}$$

Thus, $\boldsymbol{g}_n \xrightarrow{P} \boldsymbol{g}$. Therefore Thm. 2.17 yields the existence of an index $n_0 > 0$ such that for all $n > n_0$ there exist discrete solutions $\boldsymbol{u}_n \in \boldsymbol{X}_n$ such that $\boldsymbol{u}_n \xrightarrow{P} \boldsymbol{u}$. We estimate

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n) \leq d_n(\boldsymbol{u}, p_n\boldsymbol{u}) + \|p_n\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{X}_n} \lesssim d_n(\boldsymbol{u}, p_n\boldsymbol{u}) + \|A_n(p_n\boldsymbol{u} - \boldsymbol{u}_n)\|_{\boldsymbol{X}_n},$$

where we exploit that $\|A_n^{-1}\|_{L(\boldsymbol{X}_n)}$ is uniformly bounded for $n > n_0$ due to the stability of $(A_n)_{n\in\mathbb{N}}$. Since $\lim_{n\to\infty} d_n(\boldsymbol{u}, p_n\boldsymbol{u}) = 0$ by Lemma 5.16 and $\lim_{n\to\infty} \|p_n\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{X}_n} = 0$ since $\boldsymbol{u}_n \xrightarrow{P} \boldsymbol{u}$, the first inequality implies that $\lim_{n\to\infty} d_n(\boldsymbol{u}, \boldsymbol{u}_n) = 0$. Furthermore, using Lemma 5.15 for the first term on the right-hand side yields $d_n(\boldsymbol{u}, \boldsymbol{u}_n) \lesssim h_n^{\min(1+s,k)}$. For the second term, we compute similar to the proof of Thm. 3.22 that

$$\begin{aligned}\|A_n(p_n\boldsymbol{u} - \boldsymbol{u}_n)\|_{\boldsymbol{X}_n} &= \sup_{\|\boldsymbol{u}_n'\|_{\boldsymbol{X}_n}=1} |a_n(p_n\boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{u}_n')| \\ &= O(d_n(\boldsymbol{u}, p_n\boldsymbol{u}), n \to \infty) \\ &+ \sup_{\|\boldsymbol{u}_n'\|_{\boldsymbol{X}_n}=1} |\langle c_s^2\rho \operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}_n'\rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + iD_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n'\rangle \\ &+ \langle \operatorname{div}\boldsymbol{u}, \nabla p \cdot \boldsymbol{u}_n'\rangle + \langle \nabla p \cdot \boldsymbol{u}, \operatorname{div}\boldsymbol{u}_n'\rangle + \langle (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\boldsymbol{u}, \boldsymbol{u}_n'\rangle \\ &- i\omega\langle \gamma\rho\boldsymbol{u}, \boldsymbol{u}_n'\rangle - \langle \boldsymbol{f}, \boldsymbol{u}_n'\rangle_{\boldsymbol{L}^2}|.\end{aligned}$$

In the following, we want to use the fact that $\boldsymbol{u} \in \boldsymbol{X}$ solves Galbrun's equation (4.1). With integration by parts, we have that

$$\langle c_s^2\rho \operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}_n'\rangle = -\langle \nabla(c_s^2\rho \operatorname{div}\boldsymbol{u}), \boldsymbol{u}_n'\rangle, \quad \langle \nabla p \cdot \boldsymbol{u}, \operatorname{div}\boldsymbol{u}_n'\rangle = -\langle \nabla(\nabla p \cdot \boldsymbol{u}), \boldsymbol{u}_n'\rangle.$$

We note that due to the assumptions that $\boldsymbol{u} \in \boldsymbol{H}^2$, $c_s, \rho \in W^{1,\infty}$ and $p \in W^{2,\infty}$, we have that $\nabla(c_s^2\rho \operatorname{div}\boldsymbol{u}), \nabla(\nabla p \cdot \boldsymbol{u}) \in \boldsymbol{L}^2$. Now, we also want to apply partial integration to the discrete differential operator $\boldsymbol{D}_{\boldsymbol{b}}^n$. To this end, let $\boldsymbol{\psi}_n \in \boldsymbol{Q}_n$ be a suitable $\boldsymbol{H}^1$ projection of $\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}$, e.g., $\boldsymbol{\psi}_n = \mathcal{J}_n(\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u})$ with $\mathcal{J}_n$ as in [EG16, Eq. (6.4)]. Then, we obtain

$$\langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, D_{\boldsymbol{b}}^n\boldsymbol{u}_n'\rangle = \langle \boldsymbol{\psi}_n, D_{\boldsymbol{b}}^n\boldsymbol{u}_n'\rangle + \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u} - \boldsymbol{\psi}_n, D_{\boldsymbol{b}}^n\boldsymbol{u}_n'\rangle.$$

Furthermore, we compute

$$\begin{aligned}
\langle \boldsymbol{\psi}_n, D^n_{\boldsymbol{b}} \boldsymbol{u}'_n\rangle &= \sum_{\tau\in\mathcal{T}_n} \langle \boldsymbol{\psi}_n, \partial_{\boldsymbol{b}} \boldsymbol{u}'_n + \boldsymbol{R}_n \boldsymbol{u}'_n\rangle_{\boldsymbol{L}^2(\tau)} = \sum_{\tau\in\mathcal{T}_n} \langle \boldsymbol{\psi}_n, \partial_{\boldsymbol{b}} \boldsymbol{u}'_n\rangle_{\boldsymbol{L}^2(\tau)} - \langle [\![\boldsymbol{\psi}_n]\!], [\![\boldsymbol{u}'_n]\!]_{\boldsymbol{b}}\rangle_{\mathcal{F}_n^{\text{int}}} \\
&= \sum_{\tau\in\mathcal{T}_n} \langle \boldsymbol{\psi}_n, \partial_{\boldsymbol{b}} \boldsymbol{u}'_n\rangle_{\boldsymbol{L}^2(\tau)} - \langle \boldsymbol{\psi}_n, (\boldsymbol{\nu}\cdot\boldsymbol{b})\boldsymbol{u}'_n\rangle_{\boldsymbol{L}^2(\partial\tau)} \\
&= -\langle(\partial_{\boldsymbol{b}} + \operatorname{div}(\boldsymbol{b}))\boldsymbol{\psi}_n, \boldsymbol{u}'_n\rangle \\
&= -\langle(\partial_{\boldsymbol{b}} + \operatorname{div}(\boldsymbol{b}))\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, \boldsymbol{u}'_n\rangle \\
&\quad + \langle(\partial_{\boldsymbol{b}} + \operatorname{div}(\boldsymbol{b}))(\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u} - \boldsymbol{\psi}_n), \boldsymbol{u}'_n\rangle \\
&= -\langle\rho\partial_{\boldsymbol{b}}(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, \boldsymbol{u}'_n\rangle - \langle\operatorname{div}(\rho\boldsymbol{b})(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, \boldsymbol{u}'_n\rangle \\
&\quad + \langle(\partial_{\boldsymbol{b}} + \operatorname{div}(\boldsymbol{b}))(\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u} - \boldsymbol{\psi}_n), \boldsymbol{u}'_n\rangle.
\end{aligned}$$

Thus, since $\boldsymbol{u}$ fulfills (4.1), we get due to the properties of $\mathcal{J}_n$ and (5.18) that

$$\begin{aligned}
\sup_{\|\boldsymbol{u}'_n\|_{\boldsymbol{X}_n}=1} &|\langle c_s^2\rho\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}'_n\rangle - \langle\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + iD^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}'_n\rangle + \langle\operatorname{div}\boldsymbol{u}, \nabla p\cdot\boldsymbol{u}'_n\rangle \\
&+ \langle\nabla p\cdot\boldsymbol{u}, \operatorname{div}\boldsymbol{u}'_n\rangle + \langle(\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\boldsymbol{u}, \boldsymbol{u}'_n\rangle - i\omega\langle\gamma\rho\boldsymbol{u}, \boldsymbol{u}'_n\rangle - \langle\boldsymbol{f}, \boldsymbol{u}'_n\rangle_{\boldsymbol{L}^2}| \\
&\lesssim \|\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u} - \boldsymbol{\psi}_n\|_{\boldsymbol{H}^1} \lesssim h_n^{\min(s,l)}.
\end{aligned}$$

Thus, we conclude that $d_n(\boldsymbol{u}, \boldsymbol{u}_n) \lesssim h_n^{\min(1+s,k)} + h_n^{\min(s,l)}$. □

5.2.4 Hybrid $H(\mathrm{div})$-conforming discretization

In Section 3.6 we discussed the computational costs associated with the implementation of the lifting operator and the possibility of reducing the computational costs by hybridization. We recall that hybridization allows us to eliminate interior degrees of freedom through static condensation, see Remark 3.25, which reduces the computational costs. Furthermore, we recall that hybridization reduces the dimension of the system matrices associated with solving the discrete problem involving the lifting operator significantly, see Fig. 3.8. To ease the computational burden of solving (5.24), we want to introduce a hybrid version of the $H(\mathrm{div})$-conforming DG scheme. We note that we will not analyze the hybrid scheme in detail and only discuss the main ideas. We proceed similarly as in Section 3.5 and introduce a hybrid version of the lifting operator. To ensure that the HDG scheme remains $H(\mathrm{div})$-conforming, we use the exactly divergence-free HDG method as proposed in [Leh10; LS16]. The main idea is to distinguish between normal- and tangential continuity and to introduce facet unknowns only for the tangential components, see Fig. 5.4. To this end, we denote by $P_{\text{tang}} := \operatorname{Id} - \boldsymbol{\nu}_F \otimes \boldsymbol{\nu}_F$ the tangential projection and define

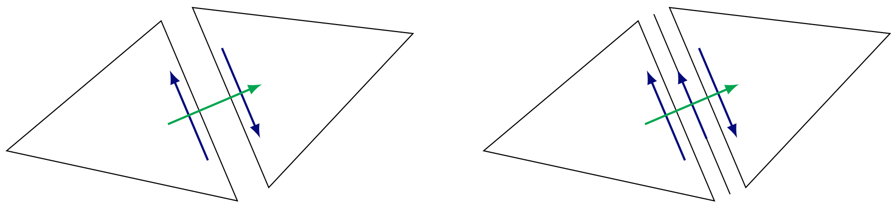
$$\boldsymbol{F}_n^{\text{tang}} := \{\boldsymbol{u}_F \in \boldsymbol{L}^2_{\text{tang}}(\mathcal{F}_n) : \boldsymbol{u}_F \in \boldsymbol{\mathcal{P}}^k(F), \boldsymbol{u}_F\cdot\boldsymbol{\nu}_F = 0\ \forall F\in\mathcal{F}_n\}, \tag{5.52}$$

where $\boldsymbol{L}^2_{\text{tang}}(\mathcal{F}_n) := \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{F}_n) : \boldsymbol{u}\cdot\boldsymbol{\nu}_F = 0\}$. Then, with $\boldsymbol{X}_n$ being defined by (5.17a), we define the discrete space

$$\boldsymbol{X}_n^{\text{HDG}} = \boldsymbol{X}_n \times \boldsymbol{F}_n^{\text{tang}}.$$

In the following, we define the jump operator $[\![\cdot]\!]_{\boldsymbol{b},\text{tang}} := [\![P_{\text{tang}}\boldsymbol{u}_n]\!]_{\boldsymbol{b}}$, where P_{tang} is the tangential projection defined above. Then, for $\tau\in\mathcal{T}_n$, we define the local HDG lifting operator $\underline{\boldsymbol{r}}^l_\tau : \boldsymbol{X}_n \to \boldsymbol{Q}_n$ through

$$\langle\underline{\boldsymbol{r}}^l_\tau\boldsymbol{u}_n, \boldsymbol{\psi}_n\rangle = -\langle[\![\boldsymbol{u}_n]\!]_{\boldsymbol{b},\text{tang}}, \boldsymbol{\psi}_n\rangle_{\boldsymbol{L}^2(\partial\tau)} \quad \forall\boldsymbol{\psi}_n\in\boldsymbol{Q}_n. \tag{5.53}$$



(a) $H(\mathrm{div})$-conforming DG (b) $H(\mathrm{div})$-conforming HDG

Figure 5.4: Comparison $H(\mathrm{div})$-DG and $H(\mathrm{div})$-HDG, see [LS16, Fig. 1] or [Leh10, Fig. 2.1.1]

Then, we set $\underline{\boldsymbol{R}}_n^l := \sum_{\tau\in\mathcal{T}_n} \underline{\boldsymbol{r}}_\tau^l$ and define the discrete differential operator

$$(\underline{\boldsymbol{D}}_{\boldsymbol{b}}^n \boldsymbol{u}_n)|_\tau := \partial_{\boldsymbol{b}}(\boldsymbol{u}_n|_\tau) + \underline{\boldsymbol{R}}_n^l \boldsymbol{u}_n \text{ for all } \tau \in \mathcal{T}_n. \tag{5.54}$$

Then, we consider the following problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n^{\mathrm{HDG}}$ such that

$$a_n^{\mathrm{HDG}}(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}, \boldsymbol{u}_n' \rangle \quad \forall \boldsymbol{u}_n' \in \boldsymbol{X}_n^{\mathrm{HDG}},$$

where the sesquilinear form $a_n^{\mathrm{HDG}}(\cdot,\cdot)$ is defined through

$$\begin{aligned} a_n^{\mathrm{HDG}}(\boldsymbol{u}_n, \boldsymbol{u}_n') :=& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\underline{\boldsymbol{D}}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\underline{\boldsymbol{D}}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle \\ &+ \langle \operatorname{div} \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \\ &- i\omega \langle \gamma \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle. \end{aligned} \tag{5.55}$$

To interpret the HDG scheme as a discrete approximation scheme similar techniques as in Section 5.2.2 should be considered. In particular, we require a suitable compactness result for the discrete differential operator $\underline{\boldsymbol{D}}_{\boldsymbol{b}}^n$, which can be shown with the same techniques as in [KCR21, Thm. 4.3].

Lemma 5.34. *Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $\boldsymbol{u}_n \in \boldsymbol{X}_n$, be such that $\sup_{n\in\mathbb{N}} \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} < \infty$. Then, there exists $\boldsymbol{u} \in \boldsymbol{X}$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $\boldsymbol{u}_n \xrightarrow{\boldsymbol{L}^2} \boldsymbol{u}$, $\operatorname{div} \boldsymbol{u}_n \xrightarrow{L^2} \operatorname{div} \boldsymbol{u}$ and $\underline{\boldsymbol{D}}_{\boldsymbol{b}}^n \boldsymbol{u}_n \xrightarrow{\boldsymbol{L}^2} \partial_{\boldsymbol{b}} \boldsymbol{u}$.*

To establish weak T-coercivity, similar techniques as discussed in Section 5.2.3.2 should be applicable. Note, however, that the facet variable has to be incorporated suitably. We leave further analysis of the $H(\mathrm{div})$-conforming HDG scheme to future work.

CHAPTER 6

A fully discontinuous Galerkin discretization of Galbrun's equation

This section is devoted to the introduction and analysis of a fully discontinuous Galerkin discretization for Galbrun's equation. For the analysis, we make use of the concepts introduced in Part I of the thesis. After introducing the discretization scheme, we show that it can be interpreted as a discrete approximation scheme and that the sequence of approximations is stable. To this end, we utilize the weak T-compatibility conditions from Thm. 2.28. We derive optimal order convergence estimates. Finally, we briefly discuss a hybrid version of the discretization. This chapter builds upon and follows the structure of the analysis of the $H(\mathrm{div})$-conforming discontinuous Galerkin discretization by Halla [Hal23] that we reviewed in Section 5.2.

Contents of the chapter

6.1 Formulation of the scheme

As before, let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded and convex Lipschitz polyhedron. Recall that we want to develop a fully discontinuous Galerkin discretization for the damped time-harmonic Galbrun's equation which reads as

$$\begin{aligned} -\nabla(\rho c_s^2 \operatorname{div} \boldsymbol{u}) + (\operatorname{div} \boldsymbol{u})\nabla p - \nabla(\nabla p \cdot \boldsymbol{u}) - \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u} & \\ + (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u} + \gamma\rho(-i\omega)\boldsymbol{u} = \boldsymbol{f} \quad & \text{in } \mathcal{O}, \qquad (6.1a) \\ \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \quad & \text{on } \partial\mathcal{O}, \qquad (6.1b) \end{aligned}$$

for given density ρ, pressure p, gravitational potential ϕ, sound speed c_s, damping coefficient γ, wave number ω, background velocity $\boldsymbol{b}$, angular velocity of the frame Ω and source term $\boldsymbol{f}$. As in the previous chapters, we assume that $c_s, \rho \in W^{1,\infty}(\mathcal{O}, \mathbb{R})$, $\gamma \in L^\infty(\mathcal{O}, \mathbb{R})$, $p, \phi \in W^{2,\infty}(\mathcal{O}, \mathbb{R})$ and that $\boldsymbol{b} \in \boldsymbol{L}^\infty(\mathcal{O}, \mathbb{R})$ is compactly supported in $\mathcal{O}$. Furthermore, we assume that there exist $\overline{c_s}, \underline{c_s}, \overline{\rho}, \underline{\rho}, \overline{\gamma}, \underline{\gamma} \in \mathbb{R}_{>0}$ such that

$$\underline{c_s} \le c_s(\boldsymbol{x}) \le \overline{c_s}, \quad \underline{\rho} \le \rho(\boldsymbol{x}) \le \overline{\rho}, \quad \underline{\gamma} \le \gamma(\boldsymbol{x}) \le \overline{\gamma} \quad \text{for all } \boldsymbol{x} \in \mathcal{O}.$$

As in the previous chapters, we denote the L^2 and $\boldsymbol{L}^2$-scalar products by $\langle \cdot, \cdot \rangle$.

Further, recall from (4.7) that the continuous sesquilinear form $a(\cdot,\cdot)$ is given by

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{u}') :=& \langle c_s^2\rho \operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{u}'\rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}'\rangle \\ &+ \langle \operatorname{div}\boldsymbol{u}, \nabla p\cdot\boldsymbol{u}'\rangle + \langle \nabla p\cdot\boldsymbol{u}, \operatorname{div}\boldsymbol{u}'\rangle + \langle (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\boldsymbol{u}, \boldsymbol{u}'\rangle \\ &- i\omega\langle\gamma\rho\boldsymbol{u},\boldsymbol{u}'\rangle, \end{aligned} \tag{6.2}$$

and the continuous weak formulation of (6.1) reads as: Find $\boldsymbol{u}\in\boldsymbol{X}$ such that

$$a(\boldsymbol{u},\boldsymbol{u}') = \langle \boldsymbol{f},\boldsymbol{u}'\rangle \qquad \text{for all } \boldsymbol{u}'\in\boldsymbol{X},$$

where $\boldsymbol{X}$ is the Hilbert space defined in (4.3). In the following, let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of shape regular simplicial triangulations of $\mathcal{O}$ and $\mathcal{F}_n$ ($\mathcal{F}_n^{\text{int}}$) be the collection of (interior) facets of $\mathcal{T}_n$. For $\tau\in\mathcal{T}_n$ and $F\in\mathcal{F}_n$, we denote by h_τ and h_F the diameters of τ and F, respectively, and define $\mathfrak{h}:\mathcal{F}_n\to\mathbb{R}$ through $\mathfrak{h}|_F := h_F$. Additionally, we define $h_n := \max_{\tau\in\mathcal{T}_n} h_\tau$ and assume that $\lim_{n\to\infty} h_n = 0$. We introduce the abbreviations

$$\langle\cdot,\cdot\rangle_{\mathcal{T}_n} := \sum_{\tau\in\mathcal{T}_n}\langle\cdot,\cdot\rangle_\tau, \qquad \langle\cdot,\cdot\rangle_{\mathcal{F}_n} := \sum_{F\in\mathcal{F}_n}\langle\cdot,\cdot\rangle_F.$$

Now, we define $\boldsymbol{X}_n$ to be the space of discontinuous polynomials of degree k, $k\in\mathbb{N}$, $k\geq 1$, that is

$$\boldsymbol{X}_n := \{\boldsymbol{u}_n\in\boldsymbol{L}^2(\mathcal{O}) : \boldsymbol{u}_n|_\tau\in\boldsymbol{\mathcal{P}}^k(\tau) \text{ for all } \tau\in\mathcal{T}_n\}.$$

The scalar product on $\boldsymbol{X}_n$ will be specified later. We note that due to the discontinuity of $\boldsymbol{u}_n\in\boldsymbol{X}_n$, we have that $\boldsymbol{X}_n\not\subset\boldsymbol{X}$. In addition to the weighted jump-operator $[\![\cdot]\!]_{\boldsymbol{b}}$ defined in Section 5.2, we also consider the normal jump operator $[\![\cdot]\!]_{\boldsymbol{\nu}}$. To be precise, we define

$$\begin{aligned} [\![\boldsymbol{u}]\!]_{\boldsymbol{b}} &:= (\boldsymbol{b}\cdot\boldsymbol{\nu}_1)\boldsymbol{u}_1 + (\boldsymbol{b}\cdot\boldsymbol{\nu}_2)\boldsymbol{u}_2, \\ [\![\boldsymbol{u}]\!]_{\boldsymbol{\nu}} &:= \boldsymbol{u}_1\cdot\boldsymbol{\nu}_1 + \boldsymbol{u}_2\cdot\boldsymbol{\nu}_2. \end{aligned}$$

Here, $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ denote the outer unit normals of the two elements τ_1,τ_2 sharing the common facet $F\in\mathcal{F}_n^{\text{int}}$ and with $\boldsymbol{u}_i$ we denote the traces of $\boldsymbol{u}|_{\tau_i}$, $i=1,2$. On boundary facets $F\in\mathcal{F}_n\cap\partial\mathcal{O}$, we define $[\![\boldsymbol{u}]\!]_{\boldsymbol{b}} = \operatorname{tr}((\boldsymbol{b}\cdot\boldsymbol{\nu})\boldsymbol{u})$ and $[\![\boldsymbol{u}]\!]_{\boldsymbol{\nu}} = \operatorname{tr}(\boldsymbol{\nu}\cdot\boldsymbol{u})$.
The formulation of a DG scheme for (6.1) follows the standard argumentation as described in Section 3.4. That is, we apply partial integration locally on each element $\tau\in\mathcal{T}_n$ and obtain similar to (4.5) that

$$-\langle\nabla(\rho c_s^2\operatorname{div}\boldsymbol{u}_n),\boldsymbol{u}_n'\rangle_\tau = \langle c_s^2\rho\operatorname{div}\boldsymbol{u}_n,\operatorname{div}\boldsymbol{u}_n'\rangle_\tau - \langle c_s^2\rho\operatorname{div}\boldsymbol{u}_n,\boldsymbol{u}_n'\cdot\boldsymbol{\nu}\rangle_{\partial\tau}, \tag{6.3a}$$

$$-\langle\nabla(\nabla p\cdot\boldsymbol{u}_n),\boldsymbol{u}_n'\rangle_\tau = \langle\nabla p\cdot\boldsymbol{u}_n,\operatorname{div}\boldsymbol{u}_n'\rangle_\tau - \langle\nabla p\cdot\boldsymbol{u}_n,\boldsymbol{u}_n'\cdot\boldsymbol{\nu}\rangle_{\partial\tau}. \tag{6.3b}$$

We note that in (4.5), we applied partial integration on the whole domain $\mathcal{O}$ such that the boundary terms vanished due to the boundary conditions. Here, we use the normal jump-operator $[\![\cdot]\!]_{\boldsymbol{\nu}}$ and the average operator $\{\!\{\cdot\}\!\}$ to reformulate the boundary terms of (6.3). After summing over all elements $\tau\in\mathcal{T}_n$, we obtain the terms

$$\langle c_s^2\rho\operatorname{div}\boldsymbol{u}_n,\operatorname{div}\boldsymbol{u}_n'\rangle_{\mathcal{T}_n} - \langle\rho c_s^2\{\!\{\operatorname{div}\boldsymbol{u}_n\}\!\},[\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n} - \langle\rho c_s^2[\![\operatorname{div}\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}},\{\!\{\boldsymbol{u}_n'\}\!\}\rangle_{\mathcal{F}_n} \tag{6.4}$$

and

$$\langle\nabla p\cdot\boldsymbol{u}_n,\operatorname{div}\boldsymbol{u}_n'\rangle_{\mathcal{T}_n} - \langle\{\!\{\nabla p\cdot\boldsymbol{u}_n\}\!\},[\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n} - \langle[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}},\{\!\{\nabla p\cdot\boldsymbol{u}_n'\}\!\}\rangle_{\mathcal{F}_n}, \tag{6.5}$$

where the last terms in (6.4) and (6.5) are added for symmetry and also correspond intuitively to the term $\langle\nabla p\cdot\boldsymbol{u}_n,\operatorname{div}\boldsymbol{u}_n'\rangle$ arising in $a_n(\cdot,\cdot)$. If we added a stabilization term of the form

$$\langle c_s^2\rho\frac{\alpha_{\boldsymbol{\nu}}}{h}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}},[\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n}$$

to (6.4) and (6.5), we would obtain a classical symmetric interior penalty (SIP) method for the divergence terms. However, we will refrain from doing so at this point and rather reformulate the terms (6.4) and (6.5) with the help of a discrete divergence operator $\operatorname{div}_{\boldsymbol{\nu}}^n$ that we will introduce in the following. We recall from Section 5.2 that we denote for $l_{\boldsymbol{b}} \in \mathbb{N}_{\geq 1}$

$$\boldsymbol{Q}_n := \{\boldsymbol{\psi}_n \in \boldsymbol{L}^2 : \boldsymbol{\psi}_n|_\tau \in \boldsymbol{\mathcal{P}}^{l_{\boldsymbol{b}}}(\tau) \text{ for all } \tau \in \mathcal{T}_n\}. \tag{6.6}$$

In the following, we will denote by $\boldsymbol{R}_n^{l_{\boldsymbol{b}}} \in \boldsymbol{Q}_n$ the lifting operator defined through the local lifting operators (5.20) which acts with respect to the $[\![\cdot]\!]_{\boldsymbol{b}}$-jump operator. We recall the definition of the discrete differential operator $\boldsymbol{D}_{\boldsymbol{b}}^n : \boldsymbol{X}_n \to \boldsymbol{Q}_n$ through

$$(\boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{u}_n)|_\tau := \partial_{\boldsymbol{b}}(\boldsymbol{u}_n|_\tau) + \boldsymbol{R}_n^{l_{\boldsymbol{b}}} \boldsymbol{u}_n \qquad \text{for all } \tau \in \mathcal{T}_n.$$

Furthermore, we recall that by application of the discrete trace inequality (A.14), we have that

$$\|\boldsymbol{R}_n^{l_{\boldsymbol{b}}} \boldsymbol{u}_n\|_{\boldsymbol{L}^2}^2 \lesssim \sum_{F \in \mathcal{F}_n} h_F^{-1} \|[\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}\|_{\boldsymbol{L}^2(F)}^2. \tag{6.7}$$

Let us stress that, in contrast to the notation in [Hal23], we write these operators in bold to emphasize that they are vector-valued. In order to define a discrete version of the divergence, we require a scalar-valued lifting operator associated with the normal jump $[\![\cdot]\!]_{\boldsymbol{\nu}}$. For $l_{\boldsymbol{\nu}} \in \mathbb{N}_{\geq 1}$, we set

$$Q_n := \{\psi_n \in L^2 : \psi_n \in \mathcal{P}^{l_{\boldsymbol{\nu}}}(\tau) \text{ for all } \tau \in \mathcal{T}_n\}.$$

Then, for $\eta \in W^{1,\infty}$ we define the local lifting operator $r_n^F \in Q_n$ to be the solution to

$$\langle \eta r_n^F \boldsymbol{u}_n, \psi_n \rangle = -\langle \eta [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \{\!\{\psi_n\}\!\} \rangle_{L^2(F)} \quad \text{for all } \psi_n \in Q_n \tag{6.8}$$

for all $\boldsymbol{u}_n \in \boldsymbol{X}_n$ and $F \in \mathcal{F}_n$. As before, we define the global version of this operator by summing over all facets, i.e. we define $R_n^{l_{\boldsymbol{\nu}}} := \sum_{F \in \mathcal{F}_n} r_n^F$. We note that the discrete trace inequality (A.14) immediately yields the boundedness of the lifting operator

$$\|\eta^{1/2} R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n\|_{L^2}^2 \leq C_{\mathrm{dt}}^2 N_\partial \sum_{F \in \mathcal{F}_n} h_F^{-1/2} \|\eta^{1/2} [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|_{L^2(F)}^2, \tag{6.9}$$

where $N_\partial := \max_{\tau \in \mathcal{T}_n} \operatorname{card}\{F \in \mathcal{F}_n : F \subset \partial\tau\}$, see also [DE12, Lem. 4.34]. Then, we define the discrete divergence operator $\operatorname{div}_{\boldsymbol{\nu}}^n : \boldsymbol{X}_n \to Q_n$ through

$$(\operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n)|_\tau := \operatorname{div} \boldsymbol{u}_n + R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n \qquad \text{for all } \tau \in \mathcal{T}_n. \tag{6.10}$$

We note that a similar construction for a discrete divergence operator can also be found in [DE10]. Furthermore, let us stress that we differentiate between the degree of the vector-valued and the scalar-valued lifting operator, i.e. we do not necessarily assume that $l_{\boldsymbol{b}} = l_{\boldsymbol{\nu}}$. However, the convergence rates in Thms. 6.26 and 6.38 indicate that the preferred choice should be $l_{\boldsymbol{b}} = l_{\boldsymbol{\nu}} = k$ to obtain an optimal order of convergence.
Then, we consider the discrete problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n$ such that

$$a_n(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}, \boldsymbol{u}_n' \rangle \qquad \text{for all } \boldsymbol{u}_n' \in \boldsymbol{X}_n, \tag{6.11}$$

where the sesquilinear form $a_n : \boldsymbol{X}_n \times \boldsymbol{X}_n \to \mathbb{C}$ is defined through

$$\begin{aligned} a_n(\boldsymbol{u}_n, \boldsymbol{u}_n') :=& \langle c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle \\ &+ \langle \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle \\ &+ \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle - i\omega \langle \gamma\rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle + s_n^\beta(\boldsymbol{u}_n, \boldsymbol{u}_n'). \end{aligned} \tag{6.12}$$

Here, we define the stabilization term $s_n^\beta : \boldsymbol{X}_n \times \boldsymbol{X}_n \to \mathbb{C}$ through

$$s_n^\beta(\boldsymbol{u}_n, \boldsymbol{u}_n') := \langle c_s^2\rho \frac{\alpha_{\boldsymbol{\nu}}}{h_F} [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}} \rangle_{\mathcal{F}_n} - \beta \langle c_s^2 \rho R_n^{l_\nu}(\boldsymbol{u}_n), R_n^{l_\nu}(\boldsymbol{u}_n') \rangle. \tag{6.13}$$

where $\alpha_{\boldsymbol{\nu}} > 0$ is a stabilization parameter and $\beta \in \{0,1\}$. As discussed in Remark 6.1, the choice of β allows us to choose between a standard SIP method ($\beta = 1$) and a lifting stabilized method ($\beta = 0$) similar to a Bassi Rebay formulation [BR97; Bas+97] with an additional jump-jump stabilization term. Recall from that Section 5.2 that the discrete differential operator $\boldsymbol{D}_{\boldsymbol{b}}^n$ was mainly introduced to avoid choosing a stabilization parameter $\alpha_{\boldsymbol{b}}$, which would lead to stronger restrictions on the Mach number $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty}$, see also [Hal23, Rem. 21]. In contrast, the choice of the stabilization parameter $\alpha_{\boldsymbol{\nu}}$ does not depend on the background flow $\boldsymbol{b}$ and therefore does not lead to more restrictive assumptions on the Mach number. We still chose to introduce a lifting operator $R_n^{l_\nu}$ associated with the normal jump $[\![\cdot]\!]_{\boldsymbol{\nu}}$ to stay consistent with the analysis framework applied in Section 5.2. The introduction of the stabilization term $s_n^\beta(\cdot,\cdot)$ with the parameter $\beta \in \{0,1\}$ allows us to conveniently analyze both, a SIP and a lifting stabilized, method for the diffusion operator.
For $\boldsymbol{u}_n, \boldsymbol{u}_n' \in \boldsymbol{X}_n$, we define the following scalar product on $\boldsymbol{X}_n$

$$\langle \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} := \langle \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{u}_n, \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{u}_n' \rangle + \langle \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle + \langle \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{u}_n, \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{u}_n' \rangle + \langle h_F^{-1} [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, [\![\boldsymbol{u}_n']\!] \rangle_{\mathcal{F}_n} \tag{6.14}$$

and denote by $\|\cdot\|_{\boldsymbol{X}_n} := \langle \cdot,\cdot \rangle^{1/2}_{\boldsymbol{X}_n}$ the induced norm. In particular, we denote for $\boldsymbol{u}_n \in \boldsymbol{X}_n$

$$\|\boldsymbol{u}_n\|^2_{\mathcal{F}_n,1/2,\boldsymbol{\nu}} := \sum_{F \in \mathcal{F}_n} h_F^{-1} \|[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(F)}.$$

Due to the boundedness of the lifting operators, cf., (6.7) and (6.9), we have for all $\boldsymbol{u}_n \in \boldsymbol{X}_n$ that

$$\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n} \lesssim \|\boldsymbol{u}_n\|^2_{\boldsymbol{X},\mathcal{T}_n} + \sum_{F \in \mathcal{F}_n} h_F^{-1} \|[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(F)} + \sum_{F \in \mathcal{F}_n} h_F^{-1} \|[\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}\|^2_{\boldsymbol{L}^2(F)}, \tag{6.15}$$

where $\|\boldsymbol{u}_n\|^2_{\boldsymbol{X},\mathcal{T}_n} := \sum_{\tau \in \mathcal{T}_n} (\|\operatorname{div} \boldsymbol{u}_n\|^2_{L^2(\tau)} + \|\boldsymbol{u}_n\|^2_{\boldsymbol{L}^2} + \|\partial_{\boldsymbol{b}} \boldsymbol{u}_n\|^2_{\boldsymbol{L}^2})$ is the broken $\boldsymbol{X}$-norm.

Remark 6.1 (The role of β). *Let us elaborate on the role of $\beta \in \{0,1\}$ in the definition (6.13) of the stabilization term $s_n^\beta(\cdot,\cdot)$. By definition of the lifting operator $R_n^{l_\nu}$, we have that*

$$\begin{aligned}\langle c_s^2\rho \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{u}_n, \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{u}_n' \rangle = &\langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle - \langle c_s^2\rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \{\!\{\operatorname{div} \boldsymbol{u}_n'\}\!\} \rangle_{\mathcal{F}_n} \\ &- \langle c_s^2 \rho \{\!\{\operatorname{div} \boldsymbol{u}_n\}\!\}, [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}} \rangle_{\mathcal{F}_n} + \langle c_s^2\rho R_n^{l_\nu} \boldsymbol{u}_n, R_n^{l_\nu} \boldsymbol{u}_n' \rangle.\end{aligned}$$

Thus, if $\beta = 1$, the lifting term $\langle c_s^2 R_n^{l_\nu} \boldsymbol{u}_n, R_n^{l_\nu} \boldsymbol{u}_n' \rangle$ cancels out and we are left with a standard SIP formulation for the diffusion operator. In contrast, if $\beta = 0$, the lifting term remains. While the latter method relaxes the conditions on the stabilization parameter $\alpha_{\boldsymbol{\nu}}$ slightly, both methods require that $\alpha_{\boldsymbol{\nu}}$ is chosen large enough to guarantee coercivity of the sesquilinear form $a_n(\cdot,\cdot)$. We will compare both methods computationally with the numerical experiments in Chapter 7.

6.2 Interpretation as discrete approximation scheme

The first step to apply the framework introduced in Chapter 2 is to show that we can interprete the proposed discretization as a discrete approximation scheme in the sense of Definition 2.7. Therefore, we will follow the same steps as in the $H(\operatorname{div})$-conforming case from Section 5.2.2, but account for the introduction of the discrete divergence operator.

First of all, we define a suitable projection operator $p_n : \boldsymbol{X} \to \boldsymbol{X}_n$. For $\boldsymbol{u} \in \boldsymbol{X}$, let $p_n\boldsymbol{u} \in \boldsymbol{X}_n$ be the solution to

$$\langle p_n\boldsymbol{u}, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} = \langle \operatorname{div} \boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle_{L^2} + \langle \boldsymbol{u}, \boldsymbol{u}_n' \rangle_{\boldsymbol{L}^2} + \langle \partial_{\boldsymbol{b}}\boldsymbol{u}, \boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{u}_n' \rangle_{\boldsymbol{L}^2} \quad \text{for all } \boldsymbol{u}_n' \in \boldsymbol{X}_n. \tag{6.16}$$

Clearly, p_n is a linear operator and is bounded, since $\|p_n\boldsymbol{u}\|_{\boldsymbol{X}_n}^2 \le \|\boldsymbol{u}\|_{\boldsymbol{X}} \|p_n\boldsymbol{u}_n\|_{\boldsymbol{X}_n}$ implies that $\|p_n\|_{L(\boldsymbol{X},\boldsymbol{X}_n)} \le 1$. Hence, it holds that $p_n \in L(\boldsymbol{X}, \boldsymbol{X}_n)$. Furthermore, for all $\boldsymbol{u}_n' \in \boldsymbol{X}_n$ we have the following Galerkin orthogonality property

$$\begin{aligned} 0 =& \langle \operatorname{div} \boldsymbol{u} - \operatorname{div}_{\boldsymbol{\nu}}^n p_n\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle_{L^2} + \langle \boldsymbol{u} - p_n\boldsymbol{u}, \boldsymbol{u}_n' \rangle_{\boldsymbol{L}^2} \\ &+ \langle \partial_{\boldsymbol{b}}\boldsymbol{u} - D_{\boldsymbol{b}}^n p_n\boldsymbol{u}, D_{\boldsymbol{b}}^n \boldsymbol{u}_n' \rangle_{\boldsymbol{L}^2} + \langle [\![p_n\boldsymbol{u}]\!]_{\boldsymbol{\nu}}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}} \rangle_{\mathcal{F}_n}. \end{aligned} \tag{6.17}$$

We recall from Section 5.2.2 that the $[\![\cdot]\!]_{\boldsymbol{b}}$-jump is not necessarily well-defined for functions $\boldsymbol{u} \in \boldsymbol{X}$ which is why we introduced a suitable distance function $d_n(\cdot,\cdot)$ allowing us to analysis the error $\boldsymbol{u} - \boldsymbol{u}_n$ for $\boldsymbol{u} \in \boldsymbol{X}$, $\boldsymbol{u}_n \in \boldsymbol{X}_n$. Here, we face the same technical issue for the normal jump $[\![\cdot]\!]_{\boldsymbol{\nu}}$, as the $\boldsymbol{L}^2$-trace of $\boldsymbol{u} \in \boldsymbol{X}$ is not necessarily well-defined. Therefore, we define a distance function $d_n : \boldsymbol{X} \times \boldsymbol{X}_n \to \mathbb{R}$ similar to the one introduced in Section 5.2.2 which also takes into account the discrete divergence $\operatorname{div}_{\boldsymbol{\nu}}^n$ and the normal jump contribution. For $\boldsymbol{u} \in \boldsymbol{X}$ and $\boldsymbol{u}_n \in \boldsymbol{X}_n$, we define

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n)^2 := \|\operatorname{div} \boldsymbol{u} - \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\|_{L^2}^2 + \|\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{L}^2}^2 + \|\partial_{\boldsymbol{b}}\boldsymbol{u} - D_{\boldsymbol{b}}^n\boldsymbol{u}_n\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}}^2.$$

We note that $d_n(\cdot,\cdot)$ satisfies the following triangle inequalities

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n) \le d_n(\tilde{\boldsymbol{u}}, \boldsymbol{u}_n) + \|\tilde{\boldsymbol{u}} - \boldsymbol{u}\|_{\boldsymbol{X}}, \qquad d_n(\boldsymbol{u}, \boldsymbol{u}_n) \le d_n(\boldsymbol{u}, \tilde{\boldsymbol{u}}_n) + \|\tilde{\boldsymbol{u}}_n - \boldsymbol{u}_n\|_{\boldsymbol{X}_n}$$

for all $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in \boldsymbol{X}$ and $\boldsymbol{u}_n, \tilde{\boldsymbol{u}}_n \in \boldsymbol{X}_n$.

In the following, let $\pi_n^d : \boldsymbol{H}^s \to [\mathbb{P}^k(\mathcal{T}_n)]^d \cap H(\operatorname{div})$, $s > 1/2$, and $\pi_n^l : L^2 \to \mathbb{P}^{k-1}(\mathcal{T}_n)$ be the interpolation operators defined in Section 5.2 such that $\operatorname{div} \pi_n^d = \pi_n^l \operatorname{div}$. For $s > 1/2$ and $\tau \in \mathcal{T}_n$, we denote by $\pi_\tau : \boldsymbol{H}^s(\tau) \to \boldsymbol{\mathcal{P}}^k(\tau)$ the canonical local interpolation operator and by $\pi_n : \boldsymbol{H}^s \to \boldsymbol{X}_n$, $\pi_n|_\tau := \pi_\tau$, $\tau \in \mathcal{T}_n$, its global version. We recall there hold the following estimates for all $\boldsymbol{v} \in \boldsymbol{H}^r(\tau)$, $r \in [1, k+1]$, $m \in [0, r]$

$$|\boldsymbol{v} - \pi_n\boldsymbol{v}|_{\boldsymbol{H}^m(\tau)} \le C_{\text{apr}} h_\tau^{r-m} |\boldsymbol{v}|_{\boldsymbol{H}^r(\tau)}, \tag{6.18a}$$

$$\|\boldsymbol{v} - \pi_n\boldsymbol{v}\|_{\boldsymbol{L}^2(\partial\tau)} \le C_{\text{ab}} h_\tau^{r-1/2} |\boldsymbol{v}|_{\boldsymbol{H}^r(\tau)}. \tag{6.18b}$$

Lemma 6.2. *For each $\boldsymbol{u} \in \boldsymbol{H}_{\boldsymbol{\nu}0}^1$ it holds that $d_n(\boldsymbol{u}, p_n\boldsymbol{u}) \le d_n(\boldsymbol{u}, \pi_n\boldsymbol{u})$.*

Proof. Using (6.17), we compute with the Cauchy-Schwarz inequality

$$\begin{aligned} d_n(\boldsymbol{u}, p_n\boldsymbol{u})^2 &= \|\operatorname{div} \boldsymbol{u} - \operatorname{div}_{\boldsymbol{\nu}}^n p_n\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{u} - p_n\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 + \|\partial_{\boldsymbol{b}}\boldsymbol{u} - D_{\boldsymbol{b}}^n p_n\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 + \|p_n\boldsymbol{u}\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}}^2 \\ &= \langle \operatorname{div} \boldsymbol{u} - \operatorname{div}_{\boldsymbol{\nu}}^n p_n\boldsymbol{u}, \operatorname{div} \boldsymbol{u} - \operatorname{div}_{\boldsymbol{\nu}}^n \pi_n\boldsymbol{u} \rangle_{L^2} + \langle \boldsymbol{u} - p_n\boldsymbol{u}, \boldsymbol{u} - \pi_n\boldsymbol{u} \rangle_{\boldsymbol{L}^2} \\ &\quad + \langle \partial_{\boldsymbol{b}}\boldsymbol{u} - D_{\boldsymbol{b}}^n p_n\boldsymbol{u}, \partial_{\boldsymbol{b}}\boldsymbol{u} - D_{\boldsymbol{b}}^n \pi_n\boldsymbol{u} \rangle_{\boldsymbol{L}^2} + \langle h_F^{-1} [\![p_n\boldsymbol{u}]\!]_{\boldsymbol{\nu}}, [\![\pi_n\boldsymbol{u}]\!]_{\boldsymbol{\nu}} \rangle_{\mathcal{F}_n} \\ &\le d_n(\boldsymbol{u}, p_n\boldsymbol{u}) d_n(\boldsymbol{u}, \pi_n\boldsymbol{u}). \end{aligned}$$

Dividing by $d_n(\boldsymbol{u}, p_n\boldsymbol{u})$ yields the claim. □

Lemma 6.3. *For each $\boldsymbol{u} \in \boldsymbol{H}_{\boldsymbol{\nu}0}^1 \cap \boldsymbol{H}^{1+s}$, $s > 0$, it holds that $d_n(\boldsymbol{u}, \pi_n\boldsymbol{u}) \lesssim h_n^s \|\boldsymbol{u}\|_{H^{1+s}}$.*

Proof. We estimate $\|\boldsymbol{u}-\pi_n\boldsymbol{u}\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}} \lesssim C_{\mathrm{dt}} h_n^{-1/2} \|\boldsymbol{u}-\pi_n\boldsymbol{u}\|_{\boldsymbol{L}^2}$ with the discrete trace inequality and

$$\begin{aligned}
\|\partial_{\boldsymbol{b}}\boldsymbol{u} - \boldsymbol{D}^n_{\boldsymbol{b}}\pi_n\boldsymbol{u}\|_{\boldsymbol{L}^2(\tau)} &\leq \|\partial_{\boldsymbol{b}}\boldsymbol{u} - \partial_{\boldsymbol{b}}\pi_n\boldsymbol{u}\|_{\boldsymbol{L}^2(\tau)} + \|\boldsymbol{R}^l_n\pi_n\boldsymbol{u}\|_{\boldsymbol{L}^2(\tau)},\\
\|\operatorname{div}\boldsymbol{u} - \operatorname{div}^n_{\boldsymbol{\nu}}\pi_n\boldsymbol{u}\|_{L^2(\tau)} &\leq \|\operatorname{div}\boldsymbol{u} - \operatorname{div}\pi_n\boldsymbol{u}\|_{L^2(\tau)} + \|R^l_n\pi_n\boldsymbol{u}\|_{L^2(\tau)}.
\end{aligned}$$

Therefore the claim follows with (6.9) and (6.18). ☐

Lemma 6.4. *For each $\boldsymbol{u} \in \boldsymbol{X}$, it holds that $\lim_{n\to\infty} d_n(\boldsymbol{u}, p_n\boldsymbol{u}) = 0$.*

Proof. This follows with the same argumentation as in [Hal23, Lem. 5] (or as in Lemma 3.15 with $\|\cdot\|_{X_n}$ replaced by $d_n(\cdot,\cdot)$). The key ingredient is the density of $\boldsymbol{C}_0^\infty$ in $\boldsymbol{X}$ [HLS22, Thm. 6] and an application of the previous two lemmata. ☐

Lemma 6.5. *For each $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0}$, we have that $\lim_{n\to\infty} d_n(\boldsymbol{u}, \pi_n\boldsymbol{u}) = 0$.*

Proof. This follows from the same techniques as in the proof of [Hal23, Lem. 6] with considering $\operatorname{div}^n_{\boldsymbol{\nu}}$ instead of div. ☐

Lemma 6.6. *For each $\boldsymbol{u} \in \boldsymbol{X}$ it holds that $\lim_{n\to\infty} \|p_n\boldsymbol{u}\|_{\boldsymbol{X}_n} = \|\boldsymbol{u}\|_{\boldsymbol{X}}$.*

Proof. As in the proof of [Hal23, Lem. 7], we compute

$$\begin{aligned}
\|p_n\boldsymbol{u}\|^2_{\boldsymbol{X}_n} &= \langle p_n\boldsymbol{u}, p_n\boldsymbol{u}\rangle_{\boldsymbol{X}_n} = \langle \operatorname{div}\boldsymbol{u}, \operatorname{div}^n_{\boldsymbol{\nu}} p_n\boldsymbol{u}\rangle_{L^2} + \langle \boldsymbol{u}, p_n\boldsymbol{u}\rangle_{\boldsymbol{L}^2} + \langle \partial_{\boldsymbol{b}}\boldsymbol{u}, D^n_{\boldsymbol{b}} p_n\boldsymbol{u}\rangle_{\boldsymbol{L}^2}\\
&= \|\boldsymbol{u}\|^2_{\boldsymbol{X}} + \langle \operatorname{div}\boldsymbol{u}, \operatorname{div}^n_{\boldsymbol{\nu}} p_n\boldsymbol{u} - \operatorname{div}\boldsymbol{u}\rangle_{L^2} + \langle \boldsymbol{u}, p_n\boldsymbol{u} - \boldsymbol{u}\rangle_{\boldsymbol{L}^2} + \langle \partial_{\boldsymbol{b}}\boldsymbol{u}, D^n_{\boldsymbol{b}} p_n\boldsymbol{u} - \partial_{\boldsymbol{b}}\boldsymbol{u}\rangle_{\boldsymbol{L}^2}
\end{aligned}$$

Furthermore, we compute that

$$|\langle \operatorname{div}\boldsymbol{u}, \operatorname{div}^n_{\boldsymbol{\nu}} p_n\boldsymbol{u} - \operatorname{div}\boldsymbol{u}\rangle_{L^2} + \langle \boldsymbol{u}, p_n\boldsymbol{u} - \boldsymbol{u}\rangle_{\boldsymbol{L}^2} + \langle \partial_{\boldsymbol{b}}\boldsymbol{u}, D^n_{\boldsymbol{b}} p_n\boldsymbol{u} - \partial_{\boldsymbol{b}}\boldsymbol{u}\rangle_{\boldsymbol{L}^2}| \leq \|\boldsymbol{u}\|_{\boldsymbol{X}} d_n(\boldsymbol{u}, p_n\boldsymbol{u}).$$

Thus, the claim follows, since $\lim_{n\to\infty} d(\boldsymbol{u}, p_n\boldsymbol{u}) = 0$ by Lemma 6.5. ☐

We extend the compactness result from Lemma 5.19, or [Hal23, Lem. 8], to include the discrete divergence operator defined in (6.10).

Lemma 6.7. *Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $\boldsymbol{u}_n \in \boldsymbol{X}_n$, satisfy $\sup_{n\in\mathbb{N}} \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} < \infty$. Then there exists $\boldsymbol{u} \in \boldsymbol{X}$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $\boldsymbol{u}_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \boldsymbol{u}$, $\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n \overset{L^2}{\rightharpoonup} \operatorname{div}\boldsymbol{u}$ and $\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{u}_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \partial_{\boldsymbol{b}}\boldsymbol{u}$.*

Proof. Due to Lemma 5.19, we only have to consider the statement for the discrete divergence operator $\operatorname{div}^n_{\boldsymbol{\nu}}$. Since $\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n$ is a bounded sequence in L^2, there exists a subsequence $\mathbb{N}'$ and $q \in L^2$ such that $\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n \overset{L^2}{\rightharpoonup} q$. It remains to show that $q = \operatorname{div}\boldsymbol{u}$. Let $\psi \in C_0^\infty$ and ψ_n be the lowest order standard H^1-interpolant of ψ on $\mathcal{T}_n$. Then, we compute

$$\begin{aligned}
\langle \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \psi\rangle &= \langle \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \psi - \psi_n\rangle + \langle \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \psi_n\rangle\\
&= \langle \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \psi - \psi_n\rangle + \sum_{\tau\in\mathcal{T}_n} \langle \operatorname{div}\boldsymbol{u}_n, \psi_n\rangle_{\boldsymbol{L}^2(\tau)} - \langle [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \{\!\{\psi_n\}\!\}\rangle_{\mathcal{F}_n}\\
&= \langle \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \psi - \psi_n\rangle + \sum_{\tau\in\mathcal{T}_n} \langle \operatorname{div}\boldsymbol{u}_n, \psi_n\rangle_{\boldsymbol{L}^2(\tau)} - \langle \boldsymbol{\nu}\cdot\boldsymbol{u}_n, \psi_n\rangle_{\boldsymbol{L}^2(\partial\tau)}\\
&= \langle \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \psi - \psi_n\rangle - \langle \boldsymbol{u}_n, \nabla\psi_n\rangle\\
&= -\langle \boldsymbol{u}_n, \nabla\psi\rangle + \langle \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \psi - \psi_n\rangle + \langle \boldsymbol{u}_n, \nabla(\psi - \psi_n)\rangle.
\end{aligned}$$

Since $\|\psi - \psi_n\|_{H^1} \lesssim h_n\|\psi\|_{H^2}$ and $\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} \lesssim 1$, it follows that

$$\langle q, \psi\rangle = \lim_{n\to\infty} \langle \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \psi\rangle = \lim_{n\to\infty} -\langle \boldsymbol{u}_n, \nabla\psi\rangle = -\langle \boldsymbol{u}, \nabla\psi\rangle,$$

and thus $q = \operatorname{div}\boldsymbol{u}$. ☐

In the following, we denote by $A \in L(\boldsymbol{X})$ the operator induced by the continuous sesquilinear form $a(\cdot,\cdot)$ defined in (4.7) and by $A_n \in L(\boldsymbol{X}_n)$ the operator induced by the discrete sesquilinear form $a_n(\cdot,\cdot)$ defined by (6.12). The next theorem shows that $A_n \xrightarrow{P} A$.

Theorem 6.8. *For each $\boldsymbol{u} \in \boldsymbol{X}$, we have that* $\lim_{n\to\infty} \|A_n p_n \boldsymbol{u} - p_n A\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$.

Proof. Let $\boldsymbol{u} \in \boldsymbol{X}$. Since $\boldsymbol{X}_n$ is Hilbert, we can choose a sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $\boldsymbol{u}_n \in \boldsymbol{X}_n$, with $\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} = 1$ such that $\|A_n p_n\boldsymbol{u} - p_n A\boldsymbol{u}\|_{\boldsymbol{X}_n} \leq |\langle A_n p_n \boldsymbol{u} - p_n A\boldsymbol{u}, \boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}| + 1/n$. For an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$, we choose a subsubsequence $\mathbb{N}'' \subset \mathbb{N}'$ and $\boldsymbol{u}' \in \boldsymbol{X}$ in accordance with Lemma 6.7. Then, we compute

$$
\begin{aligned}
\lim_{n\in\mathbb{N}''} \langle p_n A\boldsymbol{u}, \boldsymbol{u}_n\rangle_{\boldsymbol{X}_n} &= \lim_{n\in\mathbb{N}''} \left(\langle \operatorname{div} A\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle_{L^2} + \langle A\boldsymbol{u}, \boldsymbol{u}_n\rangle_{\boldsymbol{L}^2} + \langle \partial_{\boldsymbol{b}} A\boldsymbol{u}, D_{\boldsymbol{b}}^n \boldsymbol{u}_n\rangle_{\boldsymbol{L}^2}\right) \\
&= \langle \operatorname{div} A\boldsymbol{u}, \operatorname{div} \boldsymbol{u}'\rangle_{L^2} + \langle A\boldsymbol{u}, \boldsymbol{u}'\rangle_{\boldsymbol{L}^2} + \langle \partial_{\boldsymbol{b}} A\boldsymbol{u}, \partial_{\boldsymbol{b}} \boldsymbol{u}'\rangle_{\boldsymbol{L}^2} \\
&= \langle A\boldsymbol{u}, \boldsymbol{u}'\rangle_{\boldsymbol{X}} = a(\boldsymbol{u}, \boldsymbol{u}')
\end{aligned}
$$

Furthermore, we have that

$$
\begin{aligned}
\langle A_n p_n \boldsymbol{u}, \boldsymbol{u}_n\rangle_{\boldsymbol{X}_n} &= a_n(p_n\boldsymbol{u}, \boldsymbol{u}_n) \\
=\;& \langle c_s^2 \operatorname{div}_{\boldsymbol{\nu}}^n p_n\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle - \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)p_n\boldsymbol{u}, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n\rangle \\
&+ \langle \operatorname{div}_{\boldsymbol{\nu}}^n p_n\boldsymbol{u}, \nabla p\cdot \boldsymbol{u}_n\rangle + \langle \nabla p \cdot p_n\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))p_n\boldsymbol{u}, \boldsymbol{u}_n\rangle \\
&- i\omega\langle \gamma\rho p_n\boldsymbol{u}, \boldsymbol{u}_n\rangle + s_n^{\beta}(p_n\boldsymbol{u}, \boldsymbol{u}_n) \\
=\;& \left.\begin{aligned}&\langle c_s^2 \operatorname{div} \boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n\rangle \\ &+ \langle \operatorname{div} \boldsymbol{u}, \nabla p\cdot \boldsymbol{u}_n\rangle + \langle \nabla p\cdot\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle + \langle (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\boldsymbol{u}, \boldsymbol{u}_n\rangle \\ &- i\omega\langle\gamma\rho\boldsymbol{u}, \boldsymbol{u}_n\rangle\end{aligned}\right\} \quad (6.19) \\
&\left.\begin{aligned}&+ \langle c_s^2 \operatorname{div}_{\boldsymbol{\nu}}^n p_n\boldsymbol{u} - \operatorname{div}\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle \\ &- \langle \rho((\omega + i\Omega\times)(p_n\boldsymbol{u} - \boldsymbol{u}) + \boldsymbol{D}_{\boldsymbol{b}}^n p_n\boldsymbol{u} - \partial_{\boldsymbol{b}}\boldsymbol{u}), (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n\rangle \\ &+ \langle \operatorname{div}_{\boldsymbol{\nu}}^n p_n\boldsymbol{u} - \operatorname{div}\boldsymbol{u}, \nabla p\cdot\boldsymbol{u}_n\rangle + \langle \nabla p\cdot(p_n\boldsymbol{u} - \boldsymbol{u}), \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle \\ &+ \langle (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))(p_n\boldsymbol{u} - \boldsymbol{u}), \boldsymbol{u}_n\rangle - i\omega\langle\gamma\rho(p_n\boldsymbol{u} - \boldsymbol{u}), \boldsymbol{u}_n\rangle \\ &+ s_n^{\beta}(p_n\boldsymbol{u}, \boldsymbol{u}_n)\end{aligned}\right\} \quad (6.20)
\end{aligned}
$$

By choice of the subsubsequence $\mathbb{N}''$ and Lemma 6.7, it holds that $\lim_{n\in\mathbb{N}''}$ (6.19) $= a(\boldsymbol{u}, \boldsymbol{u}')$. Furthermore, due to (6.9) and $\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} = 1$ we have that $|s_n^{\beta}(p_n\boldsymbol{u}, \boldsymbol{u}_n)| \lesssim \|p_n\boldsymbol{u}\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}}$. Thus, with an application of the Cauchy-Schwarz inequality we have that $|(6.20)| \lesssim d_n(\boldsymbol{u}, p_n\boldsymbol{u})$ and hence $\lim_{n\in\mathbb{N}''}$ (6.20) $= 0$ by Lemma 6.4. Therefore, we conclude that

$$
\lim_{n\to\infty} \|A_n p_n\boldsymbol{u} - p_n A\boldsymbol{u}_n\|_{\boldsymbol{X}_n} = \lim_{n\in\mathbb{N}''} \|A_n p_n\boldsymbol{u} - p_n A\boldsymbol{u}_n\|_{\boldsymbol{X}_n} = 0.
$$

□

The previous results show that $(\boldsymbol{X}_n, p_n, A_n)$ indeed constitutes a discrete approximation scheme of $(\boldsymbol{X}, A)$ in the sense of Definiton 2.7. Therefore, we can apply the theory developed in Chapter 2, and in particular the convergence theorem 2.17 if we show that the sequence $(A_n)_{n\in\mathbb{N}}$ is regular.

6.3 Convergence Analysis

In this section, we want to analyze the convergence of the proposed discrete approximation scheme. To show regularity, we want to utilize the weak T-compatibility condition from

Thm. 2.28 that was introduced in [HLS22]. Let us recall the construction of the continuous T-operator from Section 5.2. For $\boldsymbol{u} \in H_0(\operatorname{div})$, let $v \in H^2$ be the solution to

$$(\operatorname{div} + P_{L_0^2}\boldsymbol{q} \cdot + M)\nabla v = (\operatorname{div} + P_{L_0^2}\boldsymbol{q} \cdot + M)\boldsymbol{u} \text{ in } \mathcal{O}, \tag{6.21a}$$

$$\boldsymbol{\nu} \cdot \nabla v = 0 \text{ on } \partial\mathcal{O}. \tag{6.21b}$$

Then, for $\boldsymbol{u} \in \boldsymbol{X} \subset H_0(\operatorname{div})$, we define $\boldsymbol{v} = P_V \boldsymbol{u} = \nabla v$ and $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$. The operator $T \in L(\boldsymbol{X})$ is then defined through $T\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{w}$. Recall that with this construction, the operator A associated with the continuous bilinear form is weakly T-coercive.
In the following, we want to construct a discrete operator $T_n \in L(\boldsymbol{X}_n)$ such that the weak T-compatibility conditions from Thm. 2.28 are satisfied. To this end, we want to adjust the construction in (5.33) to the fully discontinuous case. The main idea is to decompose the functions $\boldsymbol{v}_n$ into an $H(\operatorname{div})$-conforming part and a remainder that accounts for the jumps. For the latter, we require the following construction from [Ale22, Chap. 4.3.1]. We define the spaces $\mathcal{F}_n^k := \prod_{F \in \mathcal{F}_n} \mathcal{P}^k(F)$ and $Q_n^{k-1} := \{\psi_n \in L^2 : \psi_n|_\tau \in \mathcal{P}^{k-1}(\tau) \text{ for all } \tau \in \mathcal{T}_n\}$. Furthermore, we define for all $\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{u}}_n' \in \boldsymbol{X}_n$ the sesquilinear form $\tilde{a}_n(\cdot,\cdot) : \boldsymbol{X}_n \times \boldsymbol{X}_n \to \mathbb{C}$ by

$$\tilde{a}_n(\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{u}}_n') := \sum_{\tau \in \mathcal{T}_n} \langle \boldsymbol{D}_{\boldsymbol{b}}^n \tilde{\boldsymbol{u}}_n, \boldsymbol{D}_{\boldsymbol{b}}^n \tilde{\boldsymbol{u}}_n' \rangle_{\boldsymbol{L}^2(\tau)} + \langle \tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{u}}_n' \rangle_{\boldsymbol{L}^2(\tau)}$$

Then, we consider the following auxiliary problem: For given $g \in L^2$ and $f \in L^2(\mathcal{F}_n)$, find $(\tilde{\boldsymbol{u}}_n, \tilde{p}_n, \tilde{\sigma}_n) \in \boldsymbol{X}_n \times Q_n^{k-1} \times \mathcal{F}_n^k$ such that

$$\tilde{a}_n(\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{u}_n}') + d_n(\tilde{\boldsymbol{u}}_n', \tilde{p}_n) + e_n(\tilde{\boldsymbol{u}}_n', \tilde{\sigma}_n) = 0 \qquad \forall \tilde{\boldsymbol{u}}_n' \in \boldsymbol{X}_n, \tag{6.22a}$$

$$d_n(\tilde{\boldsymbol{u}}_n, \tilde{p}_n') = \sum_{\tau \in \mathcal{T}_n} \langle g, \tilde{p}_n' \rangle_{L^2(\tau)} \qquad \forall \tilde{p}_n' \in Q_n^{k-1}, \tag{6.22b}$$

$$e_n(\tilde{\boldsymbol{u}}_n, \tilde{\sigma}_n') = \sum_{F \in \mathcal{F}_n} \langle f, \tilde{\sigma}_n' \rangle_{L^2(F)} \qquad \forall \tilde{\sigma}_n' \in \mathcal{F}_n^k, \tag{6.22c}$$

where $d_n(\tilde{\boldsymbol{u}}_n, \tilde{p}_n) := \sum_{\tau \in \mathcal{T}_n} \langle \operatorname{div} \tilde{\boldsymbol{u}}_n, \tilde{p}_n \rangle_{L^2(\tau)}$ and $e_n(\tilde{\boldsymbol{u}}_n, \tilde{\sigma}_n) := \sum_{F \in \mathcal{F}_n} \langle [\![\tilde{\boldsymbol{u}}_n]\!]_{\boldsymbol{\nu}}, \tilde{\sigma}_n \rangle_{L^2(F)}$.

Lemma 6.9. *The problem* (6.22) *is well-posed and there holds the estimate*

$$\|\tilde{\boldsymbol{u}}_n\|_{\boldsymbol{X}_n}^2 + \sum_{\tau \in \mathcal{T}_n} \|\tilde{p}_n\|_{L^2(\tau)}^2 + \sum_{F \in \mathcal{F}_n} h \|\tilde{\sigma}_n\|_{L^2(F)}^2 \lesssim \|g\|_{L^2}^2 + \|f\|_{\mathcal{F}_n, 1/2, \boldsymbol{\nu}}. \tag{6.23}$$

Proof. For $\tilde{\boldsymbol{u}}_n \in \ker d_n \cap \ker e_n$, we have that $\operatorname{div} \tilde{\boldsymbol{u}}_n = 0$, $[\![\tilde{\boldsymbol{u}}_n]\!]_{\boldsymbol{\nu}} = 0$ and consequently $R_n^{l_\nu} \tilde{u}_n = 0$. Thus, we have that $\tilde{a}_n(\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{u}}_n) = \|\tilde{\boldsymbol{u}}_n\|_{\boldsymbol{X}_n}^2$ for $\tilde{\boldsymbol{u}}_n \in \ker d_n \cap \ker e_n$, which makes $\tilde{a}_n(\cdot,\cdot)$ coercive on $\ker d_n \cap \ker e_n$. Since $d_n(\cdot,\cdot)$ is inf-sup stable on $\ker e_n$ [Ale22, Sec. 4.2] and $e_n(\cdot,\cdot)$ is inf-sup stable [Ale22, Prop. 4.3.3], the claim follows from [Ale22, Thm. 2.2.8]. □

This allows us to construct a unique function $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) \in \boldsymbol{X}_n$ for $\boldsymbol{u}_n \in \boldsymbol{X}_n$ such that $[\![\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)]\!]_{\boldsymbol{\nu}} = [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}$ and $\operatorname{div} \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)|_\tau = -R_n^{l_\nu} \boldsymbol{u}_n|_\tau$ for all $\boldsymbol{u}_n \in \boldsymbol{X}_n$ and $\tau \in \mathcal{T}_n$ by choosing $f = [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}} \in \mathcal{F}_n^k$ and $g = -R_n^l \boldsymbol{u}_n \in L^2$. In particular, (6.23) and (6.9) directly imply that there exists a constant $C_{\mathcal{I}} > 0$ such that

$$\|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)\|_{\boldsymbol{X}_n} \leq C_{\mathcal{I}} \|\boldsymbol{u}_n\|_{\mathcal{F}_n, 1/2, \boldsymbol{\nu}}. \tag{6.24}$$

We show the following results for the interpolation operator $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\cdot)$.

Lemma 6.10. *For all $\boldsymbol{u} \in \boldsymbol{X}$, it holds that $\lim_{n\to\infty} \|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n \boldsymbol{u})\|_{\boldsymbol{X}_n} = 0$.*

Proof. From the definition of $d_n(\cdot,\cdot)$ and (6.24), we have that

$$\|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})\|_{\boldsymbol{X}_n} \leq C_{\mathcal{I}}\|p_n\boldsymbol{u}\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}} \lesssim d_n(\boldsymbol{u}, p_n\boldsymbol{u}).$$

Thus, the claim follows with Lemma 6.4. □

Lemma 6.11. *Let* $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$*,* $\boldsymbol{u}_n \in \boldsymbol{X}_n$*, be such that* $\sup_{n\in\mathbb{N}}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} < \infty$*. Let* $\mathbb{N}' \subset \mathbb{N}$ *and* $\boldsymbol{u} \in \boldsymbol{X}$ *be chosen with Lemma 6.7 such that* $\boldsymbol{u}_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \boldsymbol{u}$*,* $\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n \overset{L^2}{\rightharpoonup} \operatorname{div}\boldsymbol{u}$ *and* $\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{u}_n \overset{\boldsymbol{L}^2}{\rightharpoonup} \partial_{\boldsymbol{b}}\boldsymbol{u}$*. Then, it holds that* $\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) \overset{L^2}{\rightharpoonup} \operatorname{div}\boldsymbol{u}$*.*

Proof. Since $\sup_{n\in\mathbb{N}}\|u_n\|_{\boldsymbol{X}_n} < \infty$, $\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))$ is a bounded sequence due to (6.9). Thus, there exists a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and $q \in L^2$ such that $\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) \overset{L^2}{\rightharpoonup} q$. Let $\psi \in C_0^\infty$ and ψ_n be the lowest order standard H^1-interpolant on $\mathcal{T}_n$. Then, we compute

$$\begin{aligned}
\langle \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)), \psi\rangle &= \langle \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)), \psi - \psi_n\rangle + \langle \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)), \psi_n\rangle \\
&= \langle \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)), \psi - \psi_n\rangle \\
&\quad + \sum_{\tau\in\mathcal{T}_n} \langle \operatorname{div}\boldsymbol{u}_n, \psi_n\rangle_\tau - \langle \operatorname{div}\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n), \psi_n\rangle_\tau \\
&= \langle \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)), \psi - \psi_n\rangle + \sum_{\tau\in\mathcal{T}_n} \langle \operatorname{div}\boldsymbol{u}_n, \psi_n\rangle_\tau + \langle R^{l_{\boldsymbol{\nu}}}_n\boldsymbol{u}_n, \psi_n\rangle_\tau \\
&= \langle \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)), \psi - \psi_n\rangle - \langle \boldsymbol{u}_n, \nabla\psi_n\rangle \\
&= -\langle \boldsymbol{u}_n, \nabla\psi\rangle + \langle \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)), \psi - \psi_n\rangle + \langle \boldsymbol{u}_n, \nabla(\psi - \psi_n)\rangle,
\end{aligned}$$

where the last lines follow from the same argumentation as in the proof of Lemma 6.7. Since $\|\psi - \psi_n\|_{H^1} \lesssim h_n\|\psi\|_{H^2}$ and $\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))$ is bounded, it follows that

$$\langle q, \psi\rangle = \lim_{n\to\infty} \langle \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)), \psi\rangle = \lim_{n\to\infty} -\langle \boldsymbol{u}_n, \nabla\psi\rangle = -\langle \boldsymbol{u}, \nabla\psi\rangle,$$

and therefore $q = \operatorname{div}\boldsymbol{u}$. □

In the following, we will proceed to analyze the discrete problem by utilizing the weak T-compatibility condition from Thm. 2.28. For ease of presentation, we will first consider the case of homogeneous pressure and gravity before considering the general case.

6.3.1 Homogeneous pressure and gravity

As in Section 5.1.1, we first consider the case of homogeneous pressure and gravity, i.e. $p = \text{const.}$ and $\phi = \text{const}$, which implies that $\boldsymbol{q} = 0$. Furthermore, we also have that $M = 0$ in (6.21). Recall that in this case, the bilinear form $a_n(\cdot,\cdot)$ reduces to

$$\begin{aligned}
a_n(\boldsymbol{u}_n, \boldsymbol{u}'_n) =& \langle c_s^2\rho \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}_n, \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}'_n\rangle - \langle \rho(\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}'_n\rangle \\
&- i\omega\langle \gamma\rho\boldsymbol{u}_n, \boldsymbol{u}'_n\rangle + s^\beta_n(\boldsymbol{u}_n, \boldsymbol{u}'_n).
\end{aligned}$$

In the following, we want to construct a discrete operator T_n such that the weak T-compatibility conditions from Thm. 2.28 are satisfied. Therefore, we consider the following problem: For given $\boldsymbol{u}_n \in \boldsymbol{X}_n$, let $\tilde{v} \in H^2_*$ be the solution to

$$\operatorname{div}\nabla\tilde{v} = \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) \text{ in } \mathcal{O}, \tag{6.25a}$$

$$\boldsymbol{\nu}\cdot\nabla\tilde{v} = 0 \text{ on } \partial\mathcal{O}. \tag{6.25b}$$

Due to the linearity of $[\![\cdot]\!]_{\boldsymbol{\nu}}$, we have that $[\![\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)]\!]_{\boldsymbol{\nu}} = 0$ and thus $\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) \in \boldsymbol{H}(\operatorname{div})$. Consequently, $\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) \in \boldsymbol{L}^2$. Therefore, the problem (6.25) is well-posed as $\operatorname{div}\nabla = \Delta$. We set

$$\tilde{\boldsymbol{v}} := P_{\tilde{V}_n}\boldsymbol{u}_n := \nabla\tilde{v} \text{ and } \boldsymbol{v}_n := P_{V_n}\boldsymbol{u}_n := \pi_n^d\nabla\tilde{v} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n), \tag{6.26}$$

where we recall that $\pi_n^d : \boldsymbol{H}^s \to [\mathbb{P}^k(\mathcal{T}_n)]^d \cap H(\operatorname{div})$, $s > 1/2$. Further, we set $\boldsymbol{w}_n := \boldsymbol{u}_n - \boldsymbol{v}_n$ and define $T_n : \boldsymbol{X}_n \to \boldsymbol{X}_n$ through

$$T_n\boldsymbol{u}_n := \boldsymbol{v}_n - \boldsymbol{w}_n. \tag{6.27}$$

6.3.1.1 Analysis of T_n

First of all, we want to analyze the operator T_n defined in (6.27). In particular, we want to show that T_n is bounded, and bijective and that $T_n \xrightarrow{P} T$.

Lemma 6.12. *There exists a constant $C > 0$ such that $\|P_{V_n}\|_{L(\boldsymbol{X}_n)} \leq C$ for all $n \in \mathbb{N}$.*

Proof. Let $\boldsymbol{u}_n \in \boldsymbol{X}_n$ be given and $\tilde{v}$ be the solution to (6.25). Then, we have that $\|\tilde{v}\|_{H^2} \lesssim \|\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|_{L^2} \lesssim \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}$. Furthermore, we have that $\nabla\tilde{v} \in \boldsymbol{H}^1$ and therefore the function $\pi_n^d\nabla\tilde{v}$ is well-defined and with the same arguments as in Lemma 5.22 we have that $\|\pi_n^d\nabla\tilde{v}\|_{\boldsymbol{X}_n} \lesssim \|\tilde{v}\|_{H^2} \lesssim \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}$. Thus, we have that

$$\|P_{V_n}\boldsymbol{u}_n\|_{\boldsymbol{X}_n} = \|\pi_n^d\nabla\tilde{v} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)\|_{\boldsymbol{X}_n} \lesssim \|\pi_n^d\nabla\tilde{v}\|_{\boldsymbol{X}_n} + \|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}} \lesssim \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}$$

for all $\boldsymbol{u}_n \in \boldsymbol{X}_n$. Thus, the claim follows. □

Corollary 6.13. *There exists a constant $C > 0$ such that $\|T_n\|_{L(\boldsymbol{X}_n)} \leq C$ for all $n \in \mathbb{N}$.*

Proof. By definition, we have that $T_n = 2P_{V_n} - \operatorname{Id}_{\boldsymbol{X}_n}$. Thus, the claim follows with the previous lemma. □

The following results show that T_n is self-inverse, which implies its bijectivity.

Lemma 6.14. *It holds that $P_{V_n}^2 = P_{V_n}$.*

Proof. Let $\boldsymbol{u}_n \in \boldsymbol{X}_n$ and $\boldsymbol{v}_{1,n}$ be defined through (6.26) and let $\boldsymbol{v}_{2,n}$ be defined through (6.26) with $\boldsymbol{u}_n$ replaced by $\boldsymbol{v}_{1,n}$. It holds that

$$\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{v}_{1,n}) = \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\pi_n^d\nabla\tilde{v} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) = \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) = \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)$$

since $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\tilde{\boldsymbol{v}}_n) = 0$ and $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}^2 = \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}$. Furthermore, we have that

$$\begin{aligned}\operatorname{div}\nabla\tilde{v}_2 &= \operatorname{div}(\boldsymbol{v}_{1,n} - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{v}_{1,n})) = \operatorname{div}(\pi_n^d\nabla\tilde{v}_1) = \pi_n^l\operatorname{div}\nabla\tilde{v}_1\\ &= \pi_n^l\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) = \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)).\end{aligned}$$

Thus $\tilde{v}_1 = \tilde{v}_2$ and consequently $\boldsymbol{v}_{2,n} = \boldsymbol{v}_{1,n}$, which implies that $P_n^2 = P_{V_n}$. □

Corollary 6.15. *It holds that $T_n^2 = \operatorname{Id}_{\boldsymbol{X}_n}$.*

Proof. Follows directly from the previous lemma and $T_n^2 = 4P_{V_n}^2 - 4P_{V_n} + \operatorname{Id}_{\boldsymbol{X}_n}$. □

Lemma 6.16. *For each $\boldsymbol{u} \in \boldsymbol{X}$, we have that* $\lim_{n\to\infty}\|(P_{V_n}p_n - p_nP_V)\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$.

Proof. We estimate

$$
\begin{aligned}
&\|(P_{V_n}p_n - p_nP_V)\boldsymbol{u}\|_{\boldsymbol{X}_n} \\
&\qquad \le d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, P_{V_n}p_n\boldsymbol{u}) \\
&\qquad = d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, \pi_n^d P_{\tilde{V}_n}p_n\boldsymbol{u} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})) \\
&\qquad \le d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, \pi_n^d P_{\tilde{V}_n}\boldsymbol{u}) + \|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})\|_{\boldsymbol{X}_n} + \|\pi_n^d P_{\tilde{V}_n}(\boldsymbol{u} - p_n\boldsymbol{u})\|_{\boldsymbol{X}_n} \\
&\qquad \lesssim d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, \pi_n^d P_V\boldsymbol{u}) + \|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})\|_{\boldsymbol{X}_n} \\
&\qquad\qquad + \|P_V\boldsymbol{u} - P_{\tilde{V}_n}\boldsymbol{u}\|_{\boldsymbol{X}} + \|\boldsymbol{u} - p_n\boldsymbol{u}\|_{H(\mathrm{div})} \\
&\qquad \lesssim d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, \pi_n^d P_V\boldsymbol{u}) + \|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})\|_{\boldsymbol{X}_n} + d_n(\boldsymbol{u}, p_n\boldsymbol{u}),
\end{aligned}
$$

where we use that $P_V\boldsymbol{u} = P_{\tilde{V}_n}\boldsymbol{u}$ by uniqueness when we interprete $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}) := 0$. Now, the claim follows from Lemma 6.4, Lemma 6.5 and Lemma 6.10. ☐

Lemma 6.17. *For each $\boldsymbol{u} \in \boldsymbol{X}$, it holds that $\lim_{n\to\infty} \|(T_np_n - p_nT)\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$.*

Proof. Follows from the definitions of T, T_n and the previous lemma. ☐

6.3.1.2 Discrete weak T_n-coercivity

First of all, let us recall the following definitions from Section 5.2.3.2. For $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0}$, we define the weighted H^1-seminorm $|\cdot|_{\boldsymbol{H}^1_{c_s^2\rho}}$ through

$$
|\boldsymbol{u}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} := \|c_s\rho^{1/2}\nabla\boldsymbol{u}\|^2_{(L^2)^{3\times 3}}.
$$

Additionally, we also define the following weighted jump norm on $\boldsymbol{X}_n$:

$$
\|\boldsymbol{u}_n\|^2_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho} := \sum_{F\in\mathcal{F}_n} h_F^{-1}\|c_s\rho^{1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(F)}.
$$

Furthermore, due to [HH21, Thm. 3.5], see also Thm. 4.5, there exists a compact operator $K_G \in L(\boldsymbol{V})$ such that

$$
\langle c_s^2\rho \operatorname{div}\boldsymbol{v}, \operatorname{div}\boldsymbol{v}\rangle = |\boldsymbol{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} + \langle K_G\boldsymbol{v}, \boldsymbol{v}\rangle_{\boldsymbol{V}}, \tag{6.28}
$$

where $\boldsymbol{V} := \{\nabla v : v \in H^2_{*,\mathrm{Neu}}\}$, $\|\cdot\|_{\boldsymbol{V}} := |\cdot|_{\boldsymbol{H}^1_{c_s^2\rho}}$. Finally, we recall the following result.

Lemma 6.18 (Lem. 16 from [Hal23])**.** *For all $\boldsymbol{v} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0}$ and $n \in \mathbb{N}$, it holds that*

$$
\|\rho^{1/2}\boldsymbol{D}^n_{\boldsymbol{b}}\pi_n^d\boldsymbol{v}\|^2_{\boldsymbol{L}^2} \le (C_\pi^{\#})^2(1 + h_n^2\tilde{C}_\pi)\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty}|\boldsymbol{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}},
$$

with constants $\tilde{C}_\pi > 0$ and

$$
(C_\pi^{\#})^2 := 2((C_{ab}C_{sh}C_{dt}) + \sup_{n\in\mathbb{N}}\sup_{\tau\in\mathcal{T}_n}\|\pi_n^d\|^2_{L^2(\boldsymbol{H}^1_*(\tau))}), \qquad \|\cdot\|_{\boldsymbol{H}^1_*(\tau)} := |\cdot|_{\boldsymbol{H}^1(\tau)}.
$$

We want to show a similar statement for $\boldsymbol{v}_n$ defined in (6.26).

Lemma 6.19. *For $\boldsymbol{u}_n \in \boldsymbol{X}_n$ let $\boldsymbol{v}_n = P_{V_n}\boldsymbol{u}_n = \pi_n^d\nabla\tilde{v} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)$ be as in (6.26). Then, it holds that*

$$
\|\rho^{1/2}\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} \le (\tilde{C}_\pi^{\#})^2(1 + h_n^2\tilde{C}_\pi)\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty}\Big(|\nabla\tilde{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} + \|\boldsymbol{u}_n\|^2_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}\Big), \tag{6.29}
$$

with constants $\tilde{C}_\pi > 0$ and $\tilde{C}_\pi^{\#} := \max(C_\pi^{\#}, (C_{\mathcal{I}}^2 + C_{dt}^2)^{1/2})$, where $C_\pi^{\#}$ is defined as in Lemma 6.18.

Proof. By definition of $\boldsymbol{v}_n$, we have that $\boldsymbol{v}_n = \pi_n^d \nabla \tilde{v} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)$. Thus, we can estimate

$$\|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{v}_n\|_{\boldsymbol{L}^2} \leq \|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n \pi_n^d \nabla \tilde{v}\|_{\boldsymbol{L}^2} + \|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n (\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|_{\boldsymbol{L}^2}. \tag{6.30}$$

Since $\nabla \tilde{v} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0}$, we can apply the Lemma 6.18 to the first term. For the second term, we estimate for each $\tau \in \mathcal{T}_n$ with (6.24) and the argumentation from (5.41) that

$$\begin{aligned}
\|\rho^{1/2}\partial_{\boldsymbol{b}}(\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|^2_{\boldsymbol{L}^2(\tau)} &\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty}\overline{c_{s_\tau}}^2\overline{\rho_\tau}|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)|^2_{\boldsymbol{H}^1(\tau)} \\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} C_{\mathcal{I}}^2 \overline{c_{s_\tau}}^2\overline{\rho_\tau}\|\mathfrak{h}^{-1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(\partial\tau)} \\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} C_{\mathcal{I}}^2 \Big(1 + h_n^2 \frac{1}{\underline{c_{s_\tau}}^2\underline{\rho_\tau}}(C^L_{c_s\rho^{1/2}})^2\Big)^2 \|\mathfrak{h}^{-1/2}c_s\rho^{1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(\partial\tau)},
\end{aligned}$$

where $C^L_{c_s\rho^{1/2}}$ is the Lipschitz constant of $c_s\rho^{1/2}$. Similarly, we have that

$$\begin{aligned}
\|\rho^{1/2}\boldsymbol{R}_n^{l_b}(\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|^2_{\boldsymbol{L}^2(\tau)} &\leq C_{\mathrm{dt}}^2\overline{\rho}_\tau\|\mathfrak{h}^{-1/2}[\![\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)]\!]_{\boldsymbol{b}}\|^2_{\boldsymbol{L}^2(\partial\tau)} \\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} C_{\mathrm{dt}}^2 \overline{c_{s_\tau}}^2\overline{\rho_\tau}\|\mathfrak{h}^{-1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(\partial\tau)} \\
&\leq \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} C_{\mathrm{dt}}^2 \Big(1 + h_n^2 \frac{1}{\underline{c_{s_\tau}}^2\underline{\rho_\tau}}(C^L_{c_s\rho^{1/2}})^2\Big)^2 \|\mathfrak{h}^{-1/2}c_s\rho^{1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(\partial\tau)}.
\end{aligned}$$

Thus, summing over all elements $\tau \in \mathcal{T}_n$ yields that

$$\|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n(\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|^2_{\boldsymbol{L}^2} \leq (C_{\mathcal{I}}^2 + C_{\mathrm{dt}}^2)(1 + h_n^2\tilde{C}_\pi)\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty}\|\boldsymbol{u}_n\|^2_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}, \tag{6.31}$$

which shows the claim. $\square$

Additionally, we also want to have the following lemma.

Lemma 6.20. *For $\boldsymbol{u}_n \in \boldsymbol{X}_n$ let $\boldsymbol{v}_n := P_{V_n}\boldsymbol{u}_n$ be defined by (6.26). Then, for $\delta \in (0,1)$ it holds that*

$$\begin{aligned}
\|c_s\rho^{1/2}\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{v}_n\|^2_{L^2} \geq &\Big(1-\delta\Big)\Big(|\nabla\tilde{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} + \langle K_G P_{\tilde{V}_n}\boldsymbol{u}_n, P_{\tilde{V}_n}\boldsymbol{u}_n\rangle_{\boldsymbol{V}}\Big) \\
&+ \Big(1 - \frac{1}{\delta}\Big)(C_{\mathcal{I}}^2 + C_{\mathrm{dt}}^2)^{1/2}(1 + h_n^2\tilde{C}_\pi)^2\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}.
\end{aligned} \tag{6.32}$$

Proof. From the definition of $\boldsymbol{v}_n$ we have that $\boldsymbol{v}_n = \pi_n^d\nabla\tilde{v} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)$. First, we note that since $\pi_n^d\nabla\tilde{v} \in H(\operatorname{div})$, we have that $\operatorname{div}^n_{\boldsymbol{\nu}}\pi_n^d\nabla\tilde{v} = \operatorname{div}\pi_n^d\nabla\tilde{v}$. On each element $\tau \in \mathcal{T}_n$ we estimate with the weighted Young's inequality

$$\|c_s\rho^{1/2}\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{v}_n\|^2_{L^2(\tau)} \geq (1-\delta)\|c_s\rho^{1/2}\operatorname{div}\pi_n^d\nabla\tilde{v}\|^2_{L^2(\tau)} + (1 - \frac{1}{\delta})\|c_s\rho^{1/2}\operatorname{div}^n_{\boldsymbol{\nu}}\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)\|^2_{L^2(\tau)}.$$

For the first term, we compute with $\operatorname{div}\pi_n^d = \pi_n^l\operatorname{div}$ that

$$\begin{aligned}
\operatorname{div}\pi_n^d\nabla\tilde{v} &= \pi_n^l\operatorname{div}\nabla\tilde{v} = \pi_n^l\Big(\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\Big) &\text{(6.33a)}\\
&= \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) = \Delta\tilde{v}, &\text{(6.33b)}
\end{aligned}$$

since $\operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) \in Q_n$. Thus, we obtain with (6.28) similar to [Hal23, Eq. (28)] that

$$\langle c_s^2\rho\operatorname{div}\pi_n^d\nabla\tilde{v}, \operatorname{div}\pi_n^d\nabla\tilde{v}\rangle = |\nabla\tilde{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} + \langle K_G P_{\tilde{V}_n}\boldsymbol{u}_n, P_{\tilde{V}_n}\boldsymbol{u}_n\rangle_{\boldsymbol{V}}$$

For the remaining term, we estimate on each element $\tau \in \mathcal{T}_n$ with (6.24) and similar arguments as in (5.21)

$$\begin{aligned}\|c_s\rho^{1/2}\operatorname{div}(\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|^2_{L^2(\tau)} &\leq C^2_{\mathcal{I}}\overline{c_{s_\tau}}^2\overline{\rho_\tau}\|\mathfrak{h}^{-1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(\partial\tau)}\\ &\leq C^2_{\mathcal{I}}\Big(1+h_n^2\frac{1}{\underline{c_{s_\tau}}^2\underline{\rho_\tau}}(C^L_{c_s\rho^{1/2}})^2\Big)^2\|\mathfrak{h}^{-1/2}c_s\rho^{1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(\partial\tau)}.\end{aligned}$$

Furthermore, applying (6.9) yields

$$\begin{aligned}\|c_s\rho^{1/2}R^l_n(\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|^2_{L^2(\tau)} &\leq C^2_{\mathrm{dt}}\overline{c_{s_\tau}}^2\overline{\rho_\tau}\|\mathfrak{h}^{-1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(\partial\tau)}\\ &\leq C^2_{\mathrm{dt}}\Big(1+h_n^2\frac{1}{\underline{c_{s_\tau}}^2\underline{\rho_\tau}}(C^L_{c_s\rho^{1/2}})^2\Big)^2\|\mathfrak{h}^{-1/2}c_s\rho^{1/2}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\|^2_{L^2(\partial\tau)}.\end{aligned}$$

Thus, summing over each element $\tau \in \mathcal{T}_n$ yields

$$\|c_s\rho^{1/2}\operatorname{div}^n_{\boldsymbol{\nu}}(\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|^2_{L^2} \leq (C^2_{\mathcal{I}}+C^2_{\mathrm{dt}})^{1/2}(1+h_n^2\tilde{C}_\pi)\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho},$$

for a constant $\tilde{C}_\pi > 0$. Combining the estimates for the two terms yields the claim. □

Lemma 6.21. *For $\boldsymbol{u}_n \in \boldsymbol{X}_n$, we have that*

$$s^\beta_n(\boldsymbol{u}_n,\boldsymbol{u}_n) \geq (\alpha_{\boldsymbol{\nu}} - \beta C^2_{\mathrm{dt}}N_\partial)\|\boldsymbol{u}_n\|^2_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}.$$

Proof. Due to (6.9) we calculate

$$s^\beta_n(\boldsymbol{u}_n,\boldsymbol{u}_n) = \alpha_{\boldsymbol{\nu}}\|\boldsymbol{u}_n\|^2_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho} - \beta\|c_s\rho^{1/2}R^{l_{\boldsymbol{\nu}}}_n(\boldsymbol{u}_n)\|^2_{L^2} \geq (\alpha_{\boldsymbol{\nu}} - \beta C^2_{\mathrm{dt}}N_\partial)\|\boldsymbol{u}_n\|^2_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}.$$

□

Now, we want to show that the requirements from Thm. 2.28 are satisfied. Thus we first show that we can write $A_nT_n = B_n + K_n$, $B_n, K_n \in L(\boldsymbol{X}_n)$, such that $(K_n)_{n\in\mathbb{N}}$ is compact (Lemma 6.22) and B_n is stable (Lemma 6.24). Furthermore, we show that $K_n \xrightarrow{P} K$, $K \in L(\boldsymbol{X})$ and $B_n \xrightarrow{P} B$, where $B \in L(\boldsymbol{X})$ is bijective (Lemma 6.25). We pose the following assumption on the Mach number of the background flow $\boldsymbol{b}$.

Assumption 6.1. *The background flow satisfies*

$$\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} < \frac{1}{\tilde{C}^\#_\pi}, \tag{6.34}$$

where $\tilde{C}^\#_\pi > 0$ is the constant from Lemma 6.19.

For $\boldsymbol{u}_n, \boldsymbol{u}'_n \in \boldsymbol{X}_n$, we now define with constants $C_1, C_2 > 0$ to be specified lateron

$$\begin{aligned}\langle\tilde{B}_n\boldsymbol{u}_n,&\boldsymbol{u}'_n\rangle_{\boldsymbol{X}_n} :=\\ &\langle c_s^2\rho\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{v}_n, \operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{v}'_n\rangle - \langle\rho i\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n, i\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}'_n\rangle &&\text{(6.35a)}\\ &- \langle\rho i\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}_n, (\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}'_n\rangle + \langle\rho(\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n, i\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{v}'_n\rangle &&\text{(6.35b)}\\ &+ \langle\rho(\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n, (\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}'_n\rangle + \langle\rho i\omega\gamma\boldsymbol{w}_n, \boldsymbol{w}'_n\rangle &&\text{(6.35c)}\\ &+ \langle\boldsymbol{v}_n, \boldsymbol{v}'_n\rangle + C_1\langle K_GP_{\tilde{V}_n}\boldsymbol{u}_n, K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}} + s^\beta_n(\boldsymbol{u}_n,\boldsymbol{u}'_n) &&\text{(6.35d)}\end{aligned}$$

and

$$
\begin{aligned}
\langle \tilde{K}_n \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} :=& \\
& C_2(\langle \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle + \langle K_G P_{\tilde{V}_n} \boldsymbol{u}_n, K_G P_{\tilde{V}_n} \boldsymbol{u}_n' \rangle_{\boldsymbol{V}}) && \text{(6.36a)} \\
& - \langle \rho(\omega + i\Omega\times)\boldsymbol{v}_n, (\omega + i\Omega\times)\boldsymbol{v}_n' \rangle - \langle \rho(\omega + i\Omega\times)\boldsymbol{v}_n, i\boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{v}_n' \rangle && \text{(6.36b)} \\
& - \langle \rho i \boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{v}_n, (\omega + i\Omega\times)\boldsymbol{v}_n' \rangle - i\omega \langle \gamma\rho \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle && \text{(6.36c)} \\
& - \langle \rho(\omega + i\Omega\times)\boldsymbol{v}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{w}_n' \rangle - i\omega \langle \gamma\rho \boldsymbol{v}_n, \boldsymbol{w}_n' \rangle && \text{(6.36d)} \\
& + \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{w}_n, (\omega + i\Omega\times)\boldsymbol{v}_n' \rangle + i\omega \langle \gamma\rho \boldsymbol{w}_n, \boldsymbol{v}_n' \rangle. && \text{(6.36e)}
\end{aligned}
$$

We set $B_n := \tilde{B}_n + \tilde{K}_n$ and further define

$$
\langle K_n \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} := -(1 + C_2)\langle \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle - (C_1 + C_2)\langle K_G P_{\tilde{V}_n} \boldsymbol{u}_n, K_G P_{\tilde{V}_n} \boldsymbol{u}_n' \rangle_{\boldsymbol{V}}. \tag{6.37}
$$

The uniform boundedness of B_n, $n \in \mathbb{N}$, follows straightforwardly.

Lemma 6.22. *It holds that $A_n T_n = B_n + K_n$ and $(K_n)_{n\in\mathbb{N}}$ is compact.*

Proof. We note that the operators K_n and $\tilde{K}_n$ only contain terms that are compact due to Lemma 5.27, see also the argumentation in [Hal23, Proof of Thm. 18, 1. step]. Furthermore, (6.25) yields that

$$
\begin{aligned}
\operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{w}_n &= \operatorname{div}_{\boldsymbol{\nu}}^n (\boldsymbol{u}_n - \boldsymbol{v}_n) \\
&= \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n - \underbrace{\operatorname{div} \pi_n^d \nabla \tilde{v}}_{=\pi_n^l \operatorname{div} \nabla \tilde{v}} - \operatorname{div}_{\boldsymbol{\nu}}^n \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) \\
&= \operatorname{div} \boldsymbol{u}_n - \operatorname{div}(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) - \operatorname{div} \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) + R_n^l \boldsymbol{u}_n - R_n^l \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) \\
&= R_n^l \boldsymbol{u}_n - R_n^l \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) = 0,
\end{aligned}
$$

where the last step is due to $[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}} = [\![\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)]\!]_{\boldsymbol{\nu}}$. We note that $\langle A_n T_n \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} = a_n(\boldsymbol{v}_n - \boldsymbol{w}_n, \boldsymbol{v}_n' + \boldsymbol{w}_n)$. By construction of $\boldsymbol{v}_n$, we have that $[\![\boldsymbol{v}_n]\!]_{\boldsymbol{\nu}} = [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}$ and thus $s_n^\beta(T_n \boldsymbol{u}_n, \boldsymbol{u}_n') = s_n^\beta(\boldsymbol{v}_n, \boldsymbol{v}_n') = s_n^\beta(\boldsymbol{u}_n, \boldsymbol{u}_n')$. Writing out the terms and reordering yields that $A_n T_n = B_n + K_n$, where we note that the terms added by K_n cancel out with (6.36a) and (6.35d). $\square$

Lemma 6.23. *Assume that Assumption 6.1 is satisfied. For $\alpha_{\boldsymbol{\nu}}$ sufficiently large, there exists an index $n_0 > 0$ such that the operator $\tilde{B}_n \in L(\boldsymbol{X}_n)$ defined by (6.35) is uniformly coercive for all $n > n_0$.*

Proof. Let $\boldsymbol{u}_n \in \boldsymbol{X}_n$ be arbitrary. In the following, we denote for $\delta \in (0, 1)$

$$
C_\delta := -\Big(1 - \frac{1}{\delta}\Big)(C_{\mathcal{I}}^2 + C_{\mathrm{dt}}^2)^{1/2}(1 + h_n^2 \tilde{C}_\pi) > 0 \tag{6.38}
$$

the constant from the second term of Lemma 6.20. Due to the smallness assumption on the Mach number, we can find $\epsilon, \delta \in (0, 1)$, $\tau \in (0, \pi/2)$ and $n_0 > 0$ such that

$$
C_{\tau,\epsilon,\delta,n_0} := (1-\delta) - (\tilde{C}_\pi^\#)^2 (1 + \sup_{n>n_0} h_n^2 \tilde{C}_\pi) \|c_s^{-1} \boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 (1 + \tan^2(\tau)(1-\epsilon)^{-1} - \epsilon) - \epsilon(1 - \delta + C_\delta) > 0.
$$

We calculate

$$
\begin{aligned}
\frac{1}{\cos(\tau)} \Re\Big(e^{-i\tau \mathrm{sgn}\omega} \langle \tilde{B}_n \boldsymbol{u}_n, \boldsymbol{u}_n \rangle_{\boldsymbol{X}_n}\Big) &= \|c_s \rho^{1/2} \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{v}_n\|_{L^2}^2 - \|\rho^{1/2} \boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 \\
&\quad + C_1 \|K_G P_{\tilde{V}_n} \boldsymbol{u}_n\|_{\boldsymbol{V}}^2 + \|\rho^{1/2}(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2 \\
&\quad + 2\tan(\tau)\mathrm{sgn}\omega \Im\Big(\langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{w}_n, i\boldsymbol{D}_{\boldsymbol{b}}^n \boldsymbol{v}_n \rangle\Big) \\
&\quad + |\omega| \tan(\tau) \|(\gamma\rho)^{1/2} \boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2 + s_n^\beta(\boldsymbol{u}_n, \boldsymbol{u}_n).
\end{aligned}
$$

Due to the Cauchy-Schwarz and a weighted Young's inequality, we estimate

$$
\begin{aligned}
&-|2\tan(\tau)\mathrm{sgn}\omega\Im\Big(\langle\rho(\omega+i\boldsymbol{D}_{\boldsymbol{b}}^n+i\Omega\times)\boldsymbol{w}_n, i\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\rangle\Big)| \\
&\qquad \geq -\tan^2(\tau)(1-\epsilon)^{-1}\|\rho^{1/2}i\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2-(1-\epsilon)\|\rho^{1/2}(\omega+i\boldsymbol{D}_{\boldsymbol{b}}^n+i\Omega\times)\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2
\end{aligned}
$$

and therefore we have that

$$
\begin{aligned}
&\frac{1}{\cos(\tau)}\Re\Big(e^{-i\tau\mathrm{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big) \\
&\qquad\geq \|c_s\rho^{1/2}\operatorname{div}_{\boldsymbol{\nu}}^n\boldsymbol{v}_n\|_{L^2}^2-(1+\tan^2(\tau)(1-\epsilon)^{-1})\|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2+\|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 \\
&\qquad\quad+C_1\|K_GP_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2+\epsilon\|\rho^{1/2}(\omega+i\boldsymbol{D}_{\boldsymbol{b}}^n+i\Omega\times)\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2 \\
&\qquad\quad+|\omega|\tan(\tau)\|(\gamma\rho)^{1/2}\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2+s_n^\beta(\boldsymbol{u}_n,\boldsymbol{u}_n).
\end{aligned}
$$

Then, with Lemma 6.19 and 6.20, we estimate

$$
\begin{aligned}
&\|c_s\rho^{1/2}\operatorname{div}_{\boldsymbol{\nu}}^n\boldsymbol{v}_n\|_{L^2}^2-(1+\tan^2(\tau)(1-\epsilon)^{-1})\|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 \\
&\quad\geq\epsilon\Big(\|c_s\rho^{1/2}\operatorname{div}_{\boldsymbol{\nu}}^n\boldsymbol{v}_n\|_{L^2}^2+\|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2\Big)+C_{\tau,\epsilon,\delta,n_0}\Big(|\nabla\tilde{v}|_{\boldsymbol{H}^1_{c_s^2\rho}}^2+\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}\Big) \\
&\qquad-C_\delta\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}+(1-\epsilon)(1-\delta)\langle K_GP_{\tilde{V}_n}\boldsymbol{u}_n,P_{\tilde{V}_n}\boldsymbol{u}_n\rangle_{\boldsymbol{V}}.
\end{aligned}
$$

Therefore, Lemma 6.21 and a weighted Young's inequality yield for $\alpha_{\boldsymbol{\nu}}\geq\beta C_{\mathrm{tr}}^2N_\partial+C_\delta+\epsilon$

$$
\begin{aligned}
&\|c_s\rho^{1/2}\operatorname{div}_{\boldsymbol{\nu}}^n\boldsymbol{v}_n\|_{L^2}^2-(1+\tan^2(\tau)(1-\epsilon)^{-1})\|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2+\|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 \\
&\qquad+C_1\|K_GP_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2+s_n^\beta(\boldsymbol{u}_n,\boldsymbol{u}_n) \\
&\quad\geq\epsilon\Big(\|c_s\rho^{1/2}\operatorname{div}_{\boldsymbol{\nu}}^n\boldsymbol{v}_n\|_{L^2}^2+\|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2\Big)+C_{\tau,\epsilon,\delta,n_0}\Big(|\nabla\tilde{v}|_{\boldsymbol{H}^1_{c_s^2\rho}}^2+\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}\Big) \\
&\qquad+\|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2+\Big(\alpha_{\boldsymbol{\nu}}-\beta C_{\mathrm{tr}}^2N_\partial-C_\delta\Big)\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}+\Big(C_1-\frac{1}{4\tilde{\delta}}\Big)\|K_GP_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2 \\
&\qquad-\tilde{\delta}\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})}^2\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 \\
&\quad\geq\epsilon\min\{\underline{c_s}^2\underline{\rho},\underline{\rho},1\}\|\boldsymbol{v}_n\|_{\boldsymbol{X}_n}^2+\Big(C_1-\frac{1}{4\tilde{\delta}}\Big)\|K_GP_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2-\tilde{\delta}\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})}^2\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2,
\end{aligned}
$$

where we use that $\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}=\|\boldsymbol{v}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}$ since $[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}=[\![\boldsymbol{v}_n]\!]_{\boldsymbol{\nu}}$. Finally, we note that since $\operatorname{div}_{\boldsymbol{\nu}}^n\boldsymbol{w}_n=0$, we have as in [HH21] due to a weighted Young's inequality that

$$
\epsilon\|\rho^{1/2}(\omega+i\boldsymbol{D}_{\boldsymbol{b}}^n+i\Omega\times)\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2+|\omega|\tan(\tau)\|(\gamma\rho)^{1/2}\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2\gtrsim\|\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2+\|\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2\gtrsim\|\boldsymbol{w}_n\|_{\boldsymbol{X}_n}.
$$

Thus, we obtain with $C_{\tilde{B}}>0$ independent of $\delta,\tilde{\delta},C_1$ and $n>n_0$ that

$$
\begin{aligned}
&\frac{1}{\cos(\tau)}\Re\Big(e^{-i\tau\mathrm{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big) \\
&\qquad\geq C_{\tilde{B}}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2+\Big(C_1-\frac{1}{4\tilde{\delta}}\Big)\|K_GP_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2-\tilde{\delta}\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})}^2\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2.
\end{aligned}
$$

Thus, we can choose $\tilde{\delta}$ small enough and $n_1>n_0$ big enough such that

$$
\tilde{\delta}\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})}^2\leq C_{\tilde{B}}/2.
$$

Then, we obtain for $n>n_1$ that

$$
\frac{1}{\cos(\tau)}\Re\Big(e^{-i\tau\mathrm{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big)\geq\frac{C_{\tilde{B}}}{2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2+\Big(C_1-\frac{1}{4\tilde{\delta}}\Big)\|K_GP_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2.
$$

Finally, choosing $C_1 > 1/(4\tilde{\delta})$ gives for $n > n_1$

$$\frac{1}{\cos(\tau)}\Re\Big(e^{-i\tau\mathrm{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big) \geq \frac{C_{\tilde{B}}}{2}\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n},$$

which proves that $\tilde{B}_n$ is indeed coercive for $n > n_1$ and $\alpha_{\boldsymbol{\nu}}$ sufficiently large. □

Lemma 6.24. *Assume that $\alpha_{\boldsymbol{\nu}} > 0$ is sufficiently large. Under Assumption 6.1, there exist an index $n_0 > 0$ such that the operator $B_n := \tilde{B}_n + \tilde{K}_n$ is uniformly coercive for all $n > n_0$.*

Proof. Let $\boldsymbol{u}_n \in \boldsymbol{X}_n$ be arbitrary. With the weighted Young's inequality, we estimate with a constant $C_{\tau,\epsilon} > 0$ depending on $\epsilon > 0$ and $\tau \in (0,\pi/2)$ that

$$\frac{1}{\cos(\tau)}\Re\Big(e^{-i\tau\mathrm{sgn}\omega}\langle\tilde{K}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big) \geq C_2(\|\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} + \|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|^2_{\boldsymbol{V}}) - C_{\tau,\epsilon}\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n}.$$

Thus, we with Lemma 6.23 that there exists $C_{\tilde{B}} > 0$ and $n_0 > 0$ such that for $\alpha_{\boldsymbol{\nu}}$ sufficiently large

$$\begin{aligned}
&\frac{1}{\cos(\tau)}\Re\Big(e^{-i\tau\mathrm{sgn}\omega}\langle B_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big)\\
&\quad\geq (\frac{C_{\tilde{B}}}{2} - C_{\tau,\epsilon})\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n} + C_2(\|\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2}\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|^2_{\boldsymbol{V}})\\
&\quad\geq (\frac{C_{\tilde{B}}}{2} - C_{\tau,\epsilon})\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n}.
\end{aligned}$$

Choosing $\epsilon > 0$ such that $C_{\tau,\epsilon} < C_{\tilde{B}}/2$ yields the claim. □

Lemma 6.25. *There exist operators $B, K \in L(\boldsymbol{X})$ such that $AT = B + K$ such that $B_n \xrightarrow{P} B$ and $K_n \xrightarrow{P} K$, where B is coercive.*

Proof. For $\boldsymbol{u},\boldsymbol{u}' \in \boldsymbol{X}$, we define

$$\begin{aligned}
&\langle B\boldsymbol{u},\boldsymbol{u}'\rangle_{\boldsymbol{X}} :=\\
&\quad \langle c_s^2\rho\operatorname{div}\boldsymbol{v},\operatorname{div}\boldsymbol{v}'\rangle - \langle\rho i\partial_{\boldsymbol{b}}\boldsymbol{v}, i\partial_{\boldsymbol{b}}\boldsymbol{v}'\rangle - \langle\rho i\partial_{\boldsymbol{b}}\boldsymbol{v},(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}'\rangle && (6.39a)\\
&\quad + \langle\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}, i\partial_{\boldsymbol{b}}\boldsymbol{v}'\rangle + \langle\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w},(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}'\rangle && (6.39b)\\
&\quad + (1 + C_2)\langle\boldsymbol{v},\boldsymbol{v}'\rangle + (C_1 + C_2)\langle K_G\boldsymbol{v}, K_G\boldsymbol{v}'\rangle && (6.39c)\\
&\quad - \langle\rho(\omega + i\Omega\times)\boldsymbol{v},(\omega + i\Omega\times)\boldsymbol{v}'\rangle - \langle\rho(\omega + i\Omega\times)\boldsymbol{v}, i\partial_{\boldsymbol{b}}\boldsymbol{v}'\rangle && (6.39d)\\
&\quad - \langle\rho i\partial_{\boldsymbol{b}}\boldsymbol{v},(\omega + i\Omega\times)\boldsymbol{v}'\rangle - i\omega\langle\gamma\rho\boldsymbol{v},\boldsymbol{v}'\rangle && (6.39e)\\
&\quad - i\omega\langle\gamma\rho\boldsymbol{v},\boldsymbol{w}'\rangle - \langle\rho(\omega + i\Omega\times)\boldsymbol{w},(\omega + i\Omega\times)\boldsymbol{v}'\rangle && (6.39f)\\
&\quad + i\omega\langle\gamma\rho\boldsymbol{w},\boldsymbol{v}'\rangle + \langle\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w},(\omega + i\Omega\times)\boldsymbol{v}'\rangle && (6.39g)
\end{aligned}$$

and

$$\langle K\boldsymbol{u},\boldsymbol{u}'\rangle_{\boldsymbol{X}} := -(1 + C_1)\langle\boldsymbol{v},\boldsymbol{v}'\rangle - (C_1 + C_2)\langle K_G\boldsymbol{v}, K_G\boldsymbol{v}'\rangle. \qquad (6.40)$$

Then it holds that $AT = B + K$ with the same argumentation as in Lemma 6.22. Furthermore, with the same argumentation as in Lemma 6.24, we obtain that B is coercive. It remains to show that $B_n \xrightarrow{P} B$ and $K_n \xrightarrow{P} K$. First, we recall $B_n = A_nT_n - K_n$ and $B = AT - K$ and estimate

$$\begin{aligned}
&\|(p_nB - B_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n}\\
&\quad\leq \|(p_nK - K_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} + \|(p_nAT - A_nT_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n}\\
&\quad\leq \|(p_nK - K_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n} + \|(p_nA - A_np_n)T\boldsymbol{u}\|_{\boldsymbol{X}_n} + \|A_n\|_{L(\boldsymbol{X}_n)}\|(p_nT - T_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n}.
\end{aligned}$$

Due to the uniform boundedness of $(A_n)_{n\in\mathbb{N}}$, Thm. 6.8 and Lemma 6.17, it suffices to show that $\lim_{n\to\infty}\|(p_nK-K_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n}=0$. Let $\boldsymbol{u}'_n\in\boldsymbol{X}_n$, $\|\boldsymbol{u}'_n\|_{\boldsymbol{X}_n}$, $n\in\mathbb{N}$, be such that $\|(p_nK-K_np_n)\boldsymbol{u}_n\|_{\boldsymbol{X}_n}\leq|\langle p_nK\boldsymbol{u}-K_np_n\boldsymbol{u},\boldsymbol{u}'_n\rangle_{\boldsymbol{X}_n}|+1/n$ and let $\mathbb{N}'\subset\mathbb{N}$ be an arbitrary subsequence. Then, we can apply Lemma 6.7 to obtain a subsequence $\mathbb{N}''\subset\mathbb{N}'$ such that $\boldsymbol{u}'_n\overset{\boldsymbol{L}^2}{\rightharpoonup}\boldsymbol{u}'$, $\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}'_n\overset{L^2}{\rightharpoonup}\operatorname{div}\boldsymbol{u}'$ and $\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{u}'_n\overset{\boldsymbol{L}^2}{\rightharpoonup}\partial_{\boldsymbol{b}}\boldsymbol{u}'$. On the one hand, we compute

$$\begin{aligned}\langle p_nK\boldsymbol{u},\boldsymbol{u}'_n\rangle_{\boldsymbol{X}_n}&=\langle\operatorname{div}K\boldsymbol{u},\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}'_n\rangle+\langle K\boldsymbol{u},\boldsymbol{u}'_n\rangle+\langle\partial_{\boldsymbol{b}}K\boldsymbol{u},\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{u}'_n\rangle\\&\overset{n\in\mathbb{N}''}{\to}\langle\operatorname{div}K\boldsymbol{u},\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}'\rangle+\langle K\boldsymbol{u},\boldsymbol{u}'\rangle+\langle\partial_{\boldsymbol{b}}K\boldsymbol{u},\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{u}'\rangle=\langle K\boldsymbol{u},\boldsymbol{u}'\rangle_{\boldsymbol{X}}.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}|\langle\boldsymbol{v},\boldsymbol{v}'_n\rangle-\langle P_{V_n}p_n\boldsymbol{u},\boldsymbol{v}'_n\rangle|&=|\langle P_V\boldsymbol{u}-(\pi^d_nP_{\tilde{V}_n}p_n\boldsymbol{u}+\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})),\boldsymbol{v}'_n\rangle|\\&\lesssim|\langle P_V\boldsymbol{u}-\pi^d_nP_{\tilde{V}_n}\boldsymbol{u},\boldsymbol{v}'_n\rangle|+d_n(\boldsymbol{u},p_n\boldsymbol{u})+\|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})\|_{\boldsymbol{X}_n}\\&\lesssim|\langle P_V\boldsymbol{u}-\pi^d_nP_V\boldsymbol{u},\boldsymbol{v}'_n\rangle|+d_n(\boldsymbol{u},p_n\boldsymbol{u})+\|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})\|_{\boldsymbol{X}_n}\\&\lesssim h_n\|P_V\boldsymbol{u}\|_{\boldsymbol{H}^1}+d_n(\boldsymbol{u},p_n\boldsymbol{u})+\|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})\|_{\boldsymbol{X}_n}\end{aligned}\tag{6.41}$$

and

$$\begin{aligned}&|\langle K_G\boldsymbol{v},K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}-\langle K_GP_{\tilde{V}_n}p_n\boldsymbol{u},K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}|\\&\quad=|\langle K_G(P_V\boldsymbol{u}-P_{\tilde{V}}p_n\boldsymbol{u}),K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}|\\&\quad\lesssim|\langle K_G(P_V\boldsymbol{u}-P_{\tilde{V}}\boldsymbol{u}),K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}|+d_n(\boldsymbol{u},p_n\boldsymbol{u})\\&\quad\lesssim d_n(\boldsymbol{u},p_n\boldsymbol{u}),\end{aligned}\tag{6.42}$$

where the last step follows from $P_V\boldsymbol{u}=P_{\tilde{V}_n}\boldsymbol{u}$. Furthermore, we have that

$$\begin{aligned}\langle K_G\boldsymbol{v},K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}&=\langle K_G^*K_G\boldsymbol{v},P_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}=\langle K_G^*K_G\boldsymbol{v},P_V(\boldsymbol{u}'_n-\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}'_n))\rangle_{\boldsymbol{V}}\\&=\langle P_V^*K_G^*K_G\boldsymbol{v},\boldsymbol{u}'_n-\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}'_n)\rangle_{H_0(\operatorname{div})}\\&\overset{n\in\mathbb{N}''}{\to}\langle P_V^*K_G^*K_G\boldsymbol{v},\boldsymbol{u}'\rangle_{H_0(\operatorname{div})}=\langle K_G\boldsymbol{v},K_G\boldsymbol{v}'\rangle_{\boldsymbol{V}},\end{aligned}$$

where the last step follows since $\operatorname{div}(\boldsymbol{u}'_n-\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)')\overset{L^2}{\rightharpoonup}\operatorname{div}\boldsymbol{u}'$ by Lemma 6.11. Thus, we conclude that

$$\lim_{n\to\infty}\langle K_np_n\boldsymbol{u},\boldsymbol{u}'_n\rangle_{\boldsymbol{X}_n}=(1+C_2)\langle\boldsymbol{v},\boldsymbol{v}'\rangle+(C_1+C_2)\langle K_G\boldsymbol{v},K_G\boldsymbol{v}'\rangle_{\boldsymbol{V}}=\langle K\boldsymbol{u},\boldsymbol{u}'\rangle_{\boldsymbol{X}}.$$

Since $\mathbb{N}'\subset\mathbb{N}$ was chosen arbitrary, we conclude that $\lim_{n\to\infty}\|(p_nK-K_np_n)\boldsymbol{u}\|_{\boldsymbol{X}_n}=0$ for all $\boldsymbol{u}\in\boldsymbol{X}$ which proves the claim. □

Now, we can apply Thm. 2.28 to obtain the following result.

Theorem 6.26. *Assume that Assumption 6.1 is satisfied and $\alpha_{\boldsymbol{\nu}}$ is sufficiently large. Let $\boldsymbol{f}\in\boldsymbol{L}^2$ and $\boldsymbol{u}\in\boldsymbol{X}$ be the solution to $a(\boldsymbol{u},\boldsymbol{u}')=\langle\boldsymbol{f},\boldsymbol{u}'\rangle$ for all $\boldsymbol{u}'\in\boldsymbol{X}$. Then there exists an index $n_0>0$ such that for all $n>n_0$ the solution $\boldsymbol{u}_n\in\boldsymbol{X}_n$ to (6.11) exists and $\lim_{n\to\infty}d_n(\boldsymbol{u},\boldsymbol{u}_n)=0$. Additionally, if $\boldsymbol{u}\in\boldsymbol{X}\cap\boldsymbol{H}^{2+s}$, $s>0$, $\rho\in W^{1+s,\infty}$, and $\boldsymbol{b}\in\boldsymbol{W}^{1+s,\infty}$, then*

$$d_n(\boldsymbol{u},\boldsymbol{u}_n)\lesssim h_n^{\min(1+s,k)}+h_n^{\min(s,l_{\boldsymbol{b}})}+h_n^{\min(s,l_{\boldsymbol{\nu}})}.$$

Proof. Due to the previous results and Thm. 6.8, we can apply Thm. 2.28 to conclude that $(A_n)_{n\in\mathbb{N}}$ is regular. Furthermore, Lemma 4.4 yields the injectivity of A and with the same argumentation as in the proof of Thm. 5.33, we can show that the right-hand side of the discrete problem P-converges towards the right-hand side of the continuous problem. Thus, we can apply Thm. 2.17 to conclude the existence of discrete solutions $\boldsymbol{u}_n \in \boldsymbol{X}_n$ to (6.11) for all $n > n_0$. That $\lim_{n\to\infty} d_n(\boldsymbol{u}, \boldsymbol{u}_n) = 0$ follows from the same argumentation as in [Hal23, Thm. 19], see also Thm. 5.33. To show the desired convergence rates, we estimate as in the proof of Thm. 5.33 that

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n) \leq d_n(\boldsymbol{u}, p_n\boldsymbol{u}) + \|p_n\boldsymbol{u} - \boldsymbol{u}_n\|_{\boldsymbol{X}_n} \lesssim d_n(\boldsymbol{u}, p_n\boldsymbol{u}) + \|A_n(p_n\boldsymbol{u} - \boldsymbol{u}_n)\|_{\boldsymbol{X}_n}.$$

For the first term of the right-hand side, we apply Lemma 6.3 to obtain the convergence rate $d_n(\boldsymbol{u}, p_n\boldsymbol{u}) \lesssim h_n^{\min(1+s,k)}$. For the second term, we compute

$$\begin{aligned}
\|A_n(p_n\boldsymbol{u} - \boldsymbol{u}_n)\|_{\boldsymbol{X}_n} &= \sup_{\|\boldsymbol{u}_n'\|_{\boldsymbol{X}_n}=1} |a_n(p_n\boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{u}_n')| \\
&= O(d_n(\boldsymbol{u}, p_n\boldsymbol{u}), n\to\infty) + \sup_{\|\boldsymbol{u}_n'\|_{\boldsymbol{X}_n}=1} |\langle c_s^2\rho \operatorname{div}\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n'\rangle \\
&\quad - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n'\rangle - i\omega\langle\gamma\rho\boldsymbol{u}, \boldsymbol{u}_n'\rangle - \langle \boldsymbol{f}, \boldsymbol{u}_n'\rangle|,
\end{aligned}$$

where we proceed as in the proof of Thm. 6.8 to obtain the terms of order $O(d_n(\boldsymbol{u}, p_n\boldsymbol{u}), n \to \infty)$. Recall that for $l_{\boldsymbol{b}}, l_{\boldsymbol{\nu}} \in \mathbb{N}_{\geq 1}$, we set

$$\boldsymbol{Q}_n := \{\boldsymbol{\psi}_n \in \boldsymbol{L}^2 : \boldsymbol{\psi}_n|_\tau \in \boldsymbol{\mathcal{P}}^{l_{\boldsymbol{b}}}(\tau)\ \forall \tau \in \mathcal{T}_n\}, \quad Q_n := \{\psi_n \in L^2 : \psi_n|_\tau \in \mathcal{P}^{l_{\boldsymbol{\nu}}}(\tau)\ \forall \tau \in \mathcal{T}_n\}.$$

We note that the calculations from the proof of Thm. 5.33 for the integration by parts of $\boldsymbol{D}_{\boldsymbol{b}}^n$ remain valid. Let $\boldsymbol{\psi}_n \in \boldsymbol{Q}_n$ be the $\boldsymbol{H}^1$-projection of $\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}$ as discussed there. Furthermore, let $\psi_n \in Q_n$ be a H^1-projection of $c_s^2\rho\operatorname{div}\boldsymbol{u}$. Then, we compute

$$\langle c_s^2\rho\operatorname{div}\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n'\rangle = \langle \psi_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n'\rangle + \langle c_s^2\rho\operatorname{div}\boldsymbol{u} - \psi_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle$$

and

$$\begin{aligned}
\langle \psi_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n'\rangle &= \langle \psi_n, \operatorname{div}\boldsymbol{u}_n' + R_n^l \boldsymbol{u}_n'\rangle_{\mathcal{T}_n} = \langle \psi_n, \operatorname{div}\boldsymbol{u}_n'\rangle_{\mathcal{T}_n} - \langle \{\!\{\psi_n\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n} \\
&= \sum_{\tau\in\mathcal{T}_n} \langle \psi_n, \operatorname{div}\boldsymbol{u}_n'\rangle_\tau - \langle \psi_n, \boldsymbol{\nu}\cdot\boldsymbol{u}_n'\rangle_{L^2(\partial\tau)} \\
&= -\langle \nabla\psi_n, \boldsymbol{u}_n'\rangle = -\langle \nabla(c_s^2\rho\operatorname{div}\boldsymbol{u}), \boldsymbol{u}_n'\rangle + \langle \nabla(c_s^2\rho\operatorname{div}\boldsymbol{u} - \psi_n), \boldsymbol{u}_n'\rangle.
\end{aligned} \tag{6.43}$$

Thus, we conclude that due to the properties of $\boldsymbol{\psi}_n$ and ψ_n that

$$\begin{aligned}
&\sup_{\|\boldsymbol{u}_n'\|_{\boldsymbol{X}_n}=1} |\langle c_s^2\rho\operatorname{div}\boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n'\rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n'\rangle \\
&\qquad - i\omega\langle\gamma\rho\boldsymbol{u}, \boldsymbol{u}_n'\rangle - \langle \boldsymbol{f}, \boldsymbol{u}_n'\rangle| \\
&\qquad \lesssim \|\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u} - \boldsymbol{\psi}_n\|_{\boldsymbol{H}^1} + \|c_s^2\rho\operatorname{div}\boldsymbol{u} - \psi_n\|_{H^1} \\
&\qquad \lesssim h_n^{\min(s,l_{\boldsymbol{b}})} + h_n^{\min(s,l_{\boldsymbol{\nu}})}
\end{aligned}$$

Applying Lemma 6.3 again, we conclude that

$$d_n(\boldsymbol{u}, p_n\boldsymbol{u}) \lesssim h_n^{\min(1+s,k)} + h_n^{\min(s,l_{\boldsymbol{b}})} + h_n^{\min(s,l_{\boldsymbol{\nu}})}.$$

□

6.3.2 Heterogeneous pressure and gravity

Now, we want to consider the general case of heterogeneous pressure and gravity. We recall that with $\boldsymbol{q} := c_s^{-2}\rho^{-1}\nabla p$, we can express the sesquilinear form $a_n(\cdot,\cdot)$ as

$$\begin{aligned} a_n(\boldsymbol{u}_n, \boldsymbol{u}_n') :=& \langle c_s^2 \rho(\operatorname{div}_{\boldsymbol{\nu}}^n + \boldsymbol{q}\cdot)\boldsymbol{u}_n, (\operatorname{div}_{\boldsymbol{\nu}}^n + \boldsymbol{q}\cdot)\boldsymbol{u}_n'\rangle \\ &- \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n'\rangle \\ &- i\omega\langle\gamma\rho\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle + \langle(\operatorname{Hess}(p) - \operatorname{Hess}(\phi) - c_s^2\rho\boldsymbol{q}\otimes\boldsymbol{q})\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle + s_n^\beta(\boldsymbol{u}_n, \boldsymbol{u}_n'). \end{aligned}$$

Let $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}} : \boldsymbol{X}_n \to \boldsymbol{X}_n$ be as defined in the previous section. We consider the following problem: For given $\boldsymbol{u}_n \in \boldsymbol{X}_n$, let $\tilde{v} \in H_*^2$ be the solution to

$$(\operatorname{div} + P_{L_0^2}\boldsymbol{q}\cdot + M)\nabla\tilde{v} = (\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) \text{ in } \mathcal{O}, \tag{6.44a}$$

$$\boldsymbol{\nu}\cdot\nabla\tilde{v} = 0 \quad \text{on } \partial\mathcal{O}. \tag{6.44b}$$

This problem is well-posed due to the arguments from Section 5.2.3.2. We define

$$\tilde{\boldsymbol{v}} := P_{\tilde{V}_n\boldsymbol{u}_n} := \nabla\tilde{v} \text{ and } \boldsymbol{v}_n := P_{V_n}\boldsymbol{u}_n := \pi_n^d\nabla\tilde{v} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n), \tag{6.45}$$

where we recall that $\pi_n^d : \boldsymbol{H}^s \to [\mathbb{P}^k(\mathcal{T}_n)]^d \cap H(\operatorname{div})$, $s > 1/2$, and set $\boldsymbol{w}_n := \boldsymbol{u}_n - \boldsymbol{v}_n$. We further note that this construction implies that

$$\begin{aligned} \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{w}_n &= \operatorname{div}_{\boldsymbol{\nu}}^n(\boldsymbol{u}_n - \boldsymbol{v}_n) = \operatorname{div}\boldsymbol{u}_n + R_n^l\boldsymbol{u}_n - \operatorname{div}\boldsymbol{v}_n - R_n^l\boldsymbol{v}_n \\ &= \operatorname{div}\boldsymbol{u}_n + R_n^l\boldsymbol{u}_n - \operatorname{div}\pi_n^d\nabla\tilde{v} - \operatorname{div}\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) - R_n^l\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) = \operatorname{div}\boldsymbol{w}_n, \end{aligned} \tag{6.46}$$

since $R_n^l\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n) = R_n^l\boldsymbol{u}_n$ by definition of $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}$. Now, we define an operator $T_n : \boldsymbol{X}_n \to \boldsymbol{X}_n$ through

$$T_n\boldsymbol{u}_n := \boldsymbol{v}_n - \boldsymbol{w}_n. \tag{6.47}$$

In the following, we will first analyze the operator T_n and then prove that the weak T-compatibility conditions from Thm. 2.28 are fulfilled.

6.3.2.1 Analysis of T_n

Lemma 6.27. *There exists a constant $C > 0$ such that $\|P_{V_n}\|_{L(\boldsymbol{X}_n)} \le C$ for all $n \in \mathbb{N}$.*

Proof. For $\boldsymbol{u}_n \in \boldsymbol{X}_n$ let $\tilde{v}$ be the solution to (6.44). Then, we have that

$$\|\tilde{v}\|_{H^2} \lesssim \|(\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot + M)(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\|_{L^2} \lesssim \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}.$$

With the same argumentation as in Lemma 6.12, the claim follows. □

Corollary 6.28. *There exists a constant $C > 0$ such that $\|T_n\|_{L(\boldsymbol{X}_n)} \le C$ for all $n \in \mathbb{N}$.*

Proof. This follows from the definition of T_n and the previous result. □

Now, we show that the sequence $(T_n)_{n\in\mathbb{N}}$ is stable. To this end, we proceed as in [Hal23] and show that the projection P_{V_n} is asymptotically idempotent as a preliminary result.

Lemma 6.29. *Let $O_n := P_{V_n}P_{V_n} - P_{V_n}$. Then $\lim_{n\to\infty}\|O_n\|_{L(\boldsymbol{X}_n)} = 0$.*

Proof. Let $\boldsymbol{u}_n \in \boldsymbol{X}_n$ and $\boldsymbol{v}_{1,n} := \pi_n^d \nabla \tilde{v}_1 + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)$ be defined by (6.45). Further, let $\boldsymbol{v}_{2,n} := \pi_n^d \nabla \tilde{v}_2 + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{v}_{1,n})$ be defined by (6.45) with $\boldsymbol{u}_n$ being replaced by $\boldsymbol{v}_{1,n}$ in (6.44). Then, with the same argumentation as in the proof of Lemma 6.14, we have that $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{v}_{1,n}) = \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)$. Furthermore, since $\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\pi_n^d \nabla \tilde{v}_1) = 0$ we compute

$$\begin{aligned}
(\operatorname{div} + P_{L_0^2}\boldsymbol{q} \cdot + M)\nabla \tilde{v}_2 &= (\operatorname{div} + \pi_n^l \boldsymbol{q} \cdot + M)\pi_n^d \nabla \tilde{v}_1 \\
&= \pi_n^l (\operatorname{div} + P_{L_0^2}\boldsymbol{q} \cdot + M)\nabla \tilde{v}_1 + M\pi_n^d \nabla \tilde{v}_1 - \pi_n^l M \nabla \tilde{v}_1 \\
&\quad + \pi_n^l \boldsymbol{q} \cdot (\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla \tilde{v}_1 \\
&= \pi_n^l (\operatorname{div} + \pi_n^l \boldsymbol{q} \cdot + M)(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) \\
&\quad + M(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla \tilde{v}_1 + (\mathrm{Id}_{L_0^2} - \pi_n^l) M \nabla \tilde{v}_1 \\
&\quad + \pi_n^l \boldsymbol{q} \cdot (\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla \tilde{v}_1 \\
&= (\operatorname{div} + \pi_n^l \boldsymbol{q} \cdot + M)(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) + \tilde{O}_n \boldsymbol{u}_n,
\end{aligned} \tag{6.48}$$

where $\tilde{O}_n \boldsymbol{u}_n := M(\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla \tilde{v}_1 + (\mathrm{Id}_{L_0^2} - \pi_n^l) M \nabla \tilde{v}_1 + (\pi_n^l - \mathrm{Id}_{L_0^2}) M(\boldsymbol{u}_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)) + \pi_n^l \boldsymbol{q} \cdot (\pi_n^d - \mathrm{Id}_{\boldsymbol{X}})\nabla \tilde{v}_1$. With the same arguments as in [Hal23, Lem. 12], see also Lemma 5.23, we obtain that $\lim_{n\to\infty} \|\tilde{O}_n\|_{L(\boldsymbol{X}_n, L_0^2)} = 0$. This proves the claim since we have that $(\operatorname{div} + P_{L_0^2}\boldsymbol{q} \cdot + M)\nabla(\tilde{v}_2 - \tilde{v}_1) = \tilde{O}_n \boldsymbol{u}_n$ and

$$\|(P_{V_n} P_{V_n} - P_{V_n})\boldsymbol{u}_n\|_{\boldsymbol{X}_n} \lesssim \|\nabla(\tilde{v}_2 - \tilde{v}_1)\|_{\boldsymbol{H}^1} \lesssim \|\tilde{O}_n\|_{L(\boldsymbol{X}_n, L_0^2)} \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}.$$

□

Note that from (6.48) we have that

$$(\operatorname{div} + \pi_n^l \boldsymbol{q} \cdot)\boldsymbol{w}_n = -M\boldsymbol{w}_n - \tilde{O}_n \boldsymbol{u}_n. \tag{6.49}$$

Lemma 6.30. *There exist constants $n_0, C > 0$ such that T_n is invertible and $\|T_n^{-1}\|_{L(\boldsymbol{X}_n)} \leq C$ for all $n > n_0$.*

Proof. This follows with the same argumentation as in the proof of Lemma 5.24 from Lemma 6.28 and Lemma 6.29. □

Lemma 6.31. *For each $\boldsymbol{u} \in \boldsymbol{X}$, it holds that $\lim_{n\to\infty} \|(P_{V_n} p_n - p_n P_V)\boldsymbol{u}\|_{X_n} = 0$.*

Proof. The statement follows from the same argumentation as in the proof of Lemma 6.16 with the difference that

$$\|P_V \boldsymbol{u} - P_{\tilde{V}_n} \boldsymbol{u}\|_{\boldsymbol{X}} \lesssim \|(P_{L_0^2} - \pi_n^l)(\boldsymbol{q} \cdot \boldsymbol{u})\|_{L^2},$$

which converges to zero since π_n^l converges pointwise to $P_{L_0^2}$, cf. also [Hal23, Lem. 14]. □

Lemma 6.32. *For each $\boldsymbol{u} \in \boldsymbol{X}$, it holds that $\lim_{n\to\infty} \|(T_n p_n - p_n T)\boldsymbol{u}\|_{\boldsymbol{X}_n} = 0$.*

Proof. Follows directly from the definitions of T, T_n and the previous lemma. □

6.3.2.2 Discrete weak T_n-coercivity

First of all, we note that the proof of Lemma 6.19 carries over to the heterogeneous case because it does not depend explicitly on the construction of $\nabla\tilde{v}$ in (6.25). In contrast, the proof of Lemma 6.20 has to be adjusted because we make use of the explicit construction of $\nabla\tilde{v}$. Note that Lemma 6.21 also remains unaffected.

Lemma 6.33. *For $\boldsymbol{u}_n \in \boldsymbol{X}_n$ let $\boldsymbol{v}_n = P_{V_n}\boldsymbol{u}_n = \pi_n^d\nabla\tilde{v} + \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)$ be defined by (6.45). Then, for $\delta \in (0,1)$ it holds that*

$$\begin{aligned}\|c_s\rho^{1/2}\operatorname{div}^n_{\boldsymbol{\nu}}\boldsymbol{v}_n\|^2_{L^2} \geq &\Big(1-\delta\Big)\Big(|\nabla\tilde{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} + \langle K_G P_{\tilde{V}_n}\boldsymbol{u}_n, P_{\tilde{V}_n}\boldsymbol{u}_n\rangle_{\boldsymbol{V}} + \langle\check{O}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big)\\ &+\Big(1-\frac{1}{\delta}\Big)(C^2_{\mathcal{I}}+C^2_{\mathrm{dt}})^{1/2}(1+h_n^2\tilde{C}_\pi)\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}.\end{aligned} \tag{6.50}$$

Proof. We only have to modify the proof of Lemma 6.20 by reconsidering (6.33). We compute

$$\begin{aligned}\operatorname{div}\pi_n^d\nabla\tilde{v} &= \pi_n^l\operatorname{div}\nabla\tilde{v} = \pi_n^l(-(P_{L^2_0}\boldsymbol{q}\cdot+M)\nabla\tilde{v} + (\operatorname{div}+\pi_n^l\boldsymbol{q}\cdot+M)(\boldsymbol{u}_n-\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n)))\\ &= -(P_{L^2_0}\boldsymbol{q}\cdot+M)\nabla\tilde{v} + (\operatorname{div}+\pi_n^l\boldsymbol{q}\cdot+M)(\boldsymbol{u}_n-\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\\ &\quad + (\operatorname{Id}-\pi_n^l)(P_{L^2_0}\boldsymbol{q}\cdot+M)\nabla\tilde{v} + (\pi_n^l-\operatorname{Id})M(\boldsymbol{u}_n-\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\\ &= \operatorname{div}\nabla\tilde{v} + (\operatorname{Id}-\pi_n^l)(P_{L^2_0}\boldsymbol{q}\cdot+M)P_{\tilde{V}_n}\boldsymbol{u}_n + (\pi_n^l-\operatorname{Id})M(\boldsymbol{u}_n-\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}_n))\\ &=: \Delta\tilde{v} + \hat{O}_n\boldsymbol{u}_n.\end{aligned}$$

With the same techniques as in the proof of Lemma 5.23, we can show that $\lim_{n\to\infty}\|\hat{O}_n\|_{L(\boldsymbol{X}_n,L^2_0)} = 0$. Thus, we have with

$$\langle\check{O}_n\boldsymbol{u}_n,\boldsymbol{u}_n'\rangle_{\boldsymbol{X}_n} := \langle c_s^2\rho\operatorname{div}\pi_n^d\nabla\tilde{v},\hat{O}_n\boldsymbol{u}_n\rangle + \langle c_s^2\rho\hat{O}_n\boldsymbol{u}_n,\operatorname{div}(\pi_n^d\nabla\tilde{v})'\rangle + \langle c_s^2\rho\hat{O}_n\boldsymbol{u}_n,\hat{O}_n\boldsymbol{u}_n'\rangle$$

that it holds $\lim_{n\to\infty}\|\check{O}_n\|_{L(\boldsymbol{X}_n)} = 0$ and

$$\langle c_s^2\rho\operatorname{div}\pi_n^d\nabla\tilde{v},\operatorname{div}\pi_n^d\nabla\tilde{v}\rangle = \langle c_s^2\Delta\tilde{v},\Delta\tilde{v}\rangle + \langle\check{O}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}. \tag{6.51}$$

Thus, with (6.28) we obtain that

$$\langle c_s^2\rho\operatorname{div}\pi_n^d\tilde{v},\operatorname{div}\pi_n^d\tilde{v}\rangle = |\nabla\tilde{v}|^2_{\boldsymbol{H}^1_{c_s^2\rho}} + \langle K_G P_{\tilde{V}_n}\boldsymbol{u}_n, P_{\tilde{V}_n}\boldsymbol{u}_n\rangle_{\boldsymbol{V}} + \langle\check{O}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}. \tag{6.52}$$

This proves the claim since all other steps from the proof of Lemma 6.20 carry over. $\square$

As in the previous section, we now want to show that we can apply Thm. 2.28. First, let us recall the following definitions from Section 5.2.3.2. For a symmetric matrix $\underline{\underline{m}}$, we denote by $\lambda_-(\underline{\underline{m}}) \in L^\infty$ its smallest eigenvalue. In the following, we set $\underline{\underline{m}} := -\rho^{-1}\operatorname{Hess}(p) + \operatorname{Hess}(\phi)$ and define

$$C_{\underline{\underline{m}}} := \max\Big\{0, \sup_{x\in\mathcal{O}}\frac{-\lambda_-(\underline{\underline{m}}(x))}{\gamma(x)}\Big\} \qquad \text{and} \qquad \theta := \arctan(C_{\underline{\underline{m}}}/|\omega|) \in [0,2\pi), \omega \neq 0. \tag{6.53}$$

Now, we first show that we can write $A_nT_n = B_n + K_n$, $B_n, K_n \in L(\boldsymbol{X}_n)$, such that $(K_n)_{n\in\mathbb{N}}$ is compact (Lemma 6.34) and B_n is stable (Lemma 6.36). Then, we show that $K_n \xrightarrow{P} K$, $K \in L(\boldsymbol{X})$ and $B_n \xrightarrow{P} B$, where $B \in L(\boldsymbol{X})$ is bijective (Lemma 6.37). In preparation, we pose the following assumption on the Mach number of the background flow.

Assumption 6.2. *The background flow $\boldsymbol{b}$ satisfies*

$$\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} < \frac{1}{(\tilde{C}_\pi^\#)^2}\frac{1}{1+C^2_{\underline{\underline{m}}}/|\omega|^2}, \tag{6.54}$$

where $\tilde{C}_\pi^\# > 0$ is the constant from Lemma 6.19.

For $\boldsymbol{u}_n, \boldsymbol{u}_n' \in \boldsymbol{X}_n$ we now define with constants $C_1, C_2 > 0$ to be specified lateron

$$\begin{aligned}
\langle \tilde{B}_n \boldsymbol{u}_n, &\boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} := \\
&\langle c_s^2 \rho \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{v}_n, \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{v}_n' \rangle - \langle \rho i D^n_{\boldsymbol{b}} \boldsymbol{v}_n, i D^n_{\boldsymbol{b}} \boldsymbol{v}_n' \rangle + \langle c_s^2 \rho \pi^l_n(\boldsymbol{q}\cdot\boldsymbol{w}_n), \pi^l_n(\boldsymbol{q}\cdot\boldsymbol{w}_n') \rangle && \text{(6.55a)}\\
&- \langle \rho i D^n_{\boldsymbol{b}} \boldsymbol{v}_n, (\omega + iD^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n' \rangle + \langle \rho(\omega + iD^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n, iD^n_{\boldsymbol{b}} \boldsymbol{v}_n' \rangle && \text{(6.55b)}\\
&+ \langle \rho(\omega + iD^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n, (\omega + iD^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n' \rangle + \langle \rho(i\omega\gamma + \underline{\underline{m}})\boldsymbol{w}_n, \boldsymbol{w}_n' \rangle && \text{(6.55c)}\\
&+ \langle \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle + C_1 \langle K_G P_{\tilde{V}_n} \boldsymbol{u}_n, K_G P_{\tilde{V}_n} \boldsymbol{u}_n' \rangle_{\boldsymbol{V}} + \langle M\boldsymbol{w}_n, M\boldsymbol{w}_n' \rangle + \langle \tilde{O}_n \boldsymbol{u}_n, \tilde{O}_n \boldsymbol{u}_n' \rangle && \text{(6.55d)}\\
&+ s^\beta_n(\boldsymbol{u}_n, \boldsymbol{u}_n') && \text{(6.55e)}
\end{aligned}$$

and

$$\begin{aligned}
\langle \tilde{K}_n \boldsymbol{u}_n, &\boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} := \\
&C_2\big(\langle \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle + \langle K_G P_{\tilde{V}_n} \boldsymbol{u}_n, K_G P_{\tilde{V}_n} \boldsymbol{u}_n' \rangle_{\boldsymbol{V}} + \langle \tilde{O}_n \boldsymbol{u}_n, \tilde{O}_n \boldsymbol{u}_n' \rangle && \text{(6.56a)}\\
&+ \langle M\boldsymbol{w}_n, M\boldsymbol{w}_n' \rangle + \langle \operatorname{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n), \operatorname{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n') \rangle\big) && \text{(6.56b)}\\
\\
&+ \langle c_s^2 \rho \boldsymbol{q}\cdot\boldsymbol{v}_n, \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{v}_n' \rangle + \langle c_s^2 \rho \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{v}_n, \boldsymbol{q}\cdot\boldsymbol{v}_n' \rangle - \langle \rho(\omega + i\Omega\times)\boldsymbol{v}_n, (\omega + i\Omega\times)\boldsymbol{v}_n' \rangle && \text{(6.56c)}\\
&- \langle \rho(\omega + i\Omega\times)\boldsymbol{v}_n, iD^n_{\boldsymbol{b}} \boldsymbol{v}_n' \rangle - \langle \rho i D^n_{\boldsymbol{b}} \boldsymbol{v}_n, (\omega + i\Omega\times)\boldsymbol{v}_n' \rangle - i\omega\langle \gamma\rho\boldsymbol{v}_n, \boldsymbol{v}_n' \rangle && \text{(6.56d)}\\
&- \langle \rho\underline{\underline{m}}\boldsymbol{v}_n, \boldsymbol{v}_n' \rangle && \text{(6.56e)}\\
\\
&- \langle \rho\underline{\underline{m}}\boldsymbol{v}_n, \boldsymbol{w}_n' \rangle - i\omega\langle \gamma\rho\boldsymbol{v}_n, \boldsymbol{w}_n' \rangle - \langle c_s^2 \rho \pi^l_n(\boldsymbol{q}\cdot\boldsymbol{v}_n), \pi^l_n(\boldsymbol{q}\cdot\boldsymbol{w}_n') \rangle && \text{(6.56f)}\\
&- \langle \rho(\omega + i\Omega\times)\boldsymbol{v}_n, (\omega + iD^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n' \rangle && \text{(6.56g)}\\
&- \langle c_s^2 \rho(\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n \boldsymbol{q}\cdot)\boldsymbol{v}_n, M\boldsymbol{w}_n' + \tilde{O}_n \boldsymbol{u}_n' \rangle + \langle c_s^2 \rho(\operatorname{Id} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{v}_n), \operatorname{div} \boldsymbol{w}_n' \rangle && \text{(6.56h)}\\
\\
&+ \langle \rho\underline{\underline{m}}\boldsymbol{w}_n, \boldsymbol{v}_n' \rangle + i\omega\langle \gamma\rho\boldsymbol{w}_n, \boldsymbol{v}_n' \rangle + \langle c_s^2 \rho \pi^l_n(\boldsymbol{q}\cdot\boldsymbol{w}_n), \pi^l_n(\boldsymbol{q}\cdot\boldsymbol{v}_n') \rangle && \text{(6.56i)}\\
&+ \langle \rho(\omega + iD^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n, (\omega + i\Omega\times)\boldsymbol{v}_n' \rangle && \text{(6.56j)}\\
&+ \langle c_s^2 \rho(M\boldsymbol{w}_n + \tilde{O}_n \boldsymbol{u}_n), (\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n \boldsymbol{q}\cdot)\boldsymbol{v}_n' \rangle - \langle c_s^2 \rho \operatorname{div} \boldsymbol{w}_n, (\operatorname{Id} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{v}_n') \rangle && \text{(6.56k)}\\
\\
&- \langle c_s^2 \rho(\operatorname{Id} - \operatorname{mean} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{w}_n), \operatorname{div} \boldsymbol{w}_n' \rangle - \langle c_s^2 \rho \operatorname{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n), \operatorname{div} \boldsymbol{w}_n' \rangle && \text{(6.56l)}\\
&- \langle c_s^2 \rho \operatorname{div} \boldsymbol{w}_n, (\operatorname{Id} - \operatorname{mean} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{w}_n') \rangle - \langle c_s^2 \rho \operatorname{div} \boldsymbol{w}_n, \operatorname{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n') \rangle && \text{(6.56m)}\\
&- \langle c_s^2 \rho(M\boldsymbol{w}_n + \tilde{O}_n \boldsymbol{u}_n), M\boldsymbol{w}_n' + \tilde{O}_n \boldsymbol{u}_n' \rangle. && \text{(6.56n)}
\end{aligned}$$

Then, we set $B_n := \tilde{B}_n + \tilde{K}_n$ and define

$$\begin{aligned}
\langle K_n \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\boldsymbol{X}_n} := &- (1+C_2)\langle \boldsymbol{v}_n, \boldsymbol{v}_n' \rangle - (C_1 + C_2)\langle K_G P_{\tilde{V}_n} \boldsymbol{u}_n, K_G P_{\tilde{V}_n} \boldsymbol{u}_n' \rangle_{\boldsymbol{V}} && \text{(6.57a)}\\
&- (1+C_2)\langle M\boldsymbol{w}_n, M\boldsymbol{w}_n' \rangle - C_2\langle \operatorname{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n), \operatorname{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n') \rangle && \text{(6.57b)}\\
&- (1+C_2)\langle \tilde{O}_n \boldsymbol{u}_n, \tilde{O}_n \boldsymbol{u}_n' \rangle && \text{(6.57c)}
\end{aligned}$$

We note that the uniform boundedness of B_n, $n \in \mathbb{N}$, follows from straightforward computations.

Lemma 6.34. *It holds that $A_n T_n = B_n + K_n$ and $(K_n)_{n\in\mathbb{N}}$ is compact.*

Proof. We note that the operators K_n and $\tilde{K}_n$ give us compact sequences, see also Lemma 5.27 and the argumentation in Lemma 5.28. To see that indeed $A_n T_n = B_n + K_n$, we first recall that $\operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{w}_n = \operatorname{div} \boldsymbol{w}_n$ by construction, see (6.46). Furthermore, since we have that $[\![\boldsymbol{v}_n]\!]_{\boldsymbol{\nu}} = [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}$, it holds that $s^\beta_n(T_n \boldsymbol{u}_n, \boldsymbol{u}'_n) = s^\beta_n(\boldsymbol{v}_n, \boldsymbol{v}'_n) = s^\beta_n(\boldsymbol{u}_n, \boldsymbol{u}'_n)$. Due to (6.49), the same argumentation as in Lemma 5.28 applies, where the term $-\langle c_s^2 \rho \operatorname{div} \boldsymbol{w}_n, \operatorname{div} \boldsymbol{w}'_n\rangle$ is reformulated into (6.56l)-(6.56n). Similarly, the reformulation of the terms $-\langle c_s^2 \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{v}_n, (\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{w}'_n\rangle - \langle c_s^2 \rho(\operatorname{div} + \boldsymbol{q}\cdot)\boldsymbol{w}_n, \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{v}'_n\rangle$ into (6.56f)-(6.56h) and (6.56i)-(6.56k) follows with the same argumentation as in Lemma 5.28. $\square$

Lemma 6.35. *Let Assumption 6.2 be satisfied. Then, for $\alpha_{\boldsymbol{\nu}}$ sufficiently large, there exists an index $n_0 > 0$ such that the operator $\tilde{B}_n \in L(\boldsymbol{X}_n)$ defined by (6.55) is uniformly coercive for all $n > n_0$.*

Proof. We built upon the argumentation in the proof of Lemma 6.23 and estimate the additional terms in (6.55) as done in the proof of Lemma 5.29. Due to the smallness assumption on the Mach number, cf. Assumption 6.2, we can find $\epsilon, \delta \in (0,1)$, $\tau \in (0, \pi/2 - \theta)$ and $n_0 > 0$ such that

$$\begin{aligned} C_{\theta,\tau,\epsilon,\delta,n_0} :=& (1-\delta) - (\tilde{C}^\#_\pi)^2 (1 + \sup_{n>n_0} h_n^2 \tilde{C}_\pi) \|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} (1 + \tan^2(\theta+\tau)(1-\epsilon)^{-1} - \epsilon) \\ &- \epsilon(1-\delta+C_\delta) > 0, \end{aligned}$$

where $C_\delta > 0$ is the constant from Lemma 6.33, see (6.38). We estimate with the definition of θ and a weighted Young's inequality

$$\begin{aligned} &\frac{1}{\cos(\theta+\tau)} \Re\Big(e^{-i(\theta+\tau)\mathrm{sgn}\omega} \langle \tilde{B}_n \boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big) \\ &\quad= \|c_s \rho^{1/2} \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{v}_n\|^2_{L^2} - \|\rho^{1/2} \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} + \|\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} + C_1 \|K_G P_{\tilde{V}_n} \boldsymbol{u}_n\|^2_{\boldsymbol{V}} + \|M \boldsymbol{w}_n\|^2_{L^2} \\ &\qquad + \|\tilde{O} \boldsymbol{u}_n\|^2_{L^2} + \|c_s \rho^{1/2} \pi^l_n (\boldsymbol{q}\cdot\boldsymbol{w}_n)\|^2_{L^2} + \|\rho^{1/2}(\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n\|^2_{\boldsymbol{L}^2} + \langle \rho \underline{\underline{m}} \boldsymbol{w}_n, \boldsymbol{w}_n\rangle_{\boldsymbol{L}^2} \\ &\qquad + 2\tan(\theta+\tau)\mathrm{sgn}\omega \Im\Big(\langle \rho(\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n, i\boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{v}_n\rangle\Big) \\ &\qquad + |\omega| \tan(\theta+\tau) \|(\gamma\rho)^{1/2} \boldsymbol{w}_n\|^2_{\boldsymbol{L}^2} + s^\beta_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \\ &\quad\geq \|c_s^2 \rho \operatorname{div}^n_{\boldsymbol{\nu}} \boldsymbol{v}_n\|^2_{L^2} - (1 + \tan^2(\theta+\tau)(1-\epsilon)^{-1}) \|\rho^{1/2} \boldsymbol{D}^n_{\boldsymbol{b}} \boldsymbol{v}_n\|^2 + \|\boldsymbol{v}_n\|^2_{\boldsymbol{L}^2} \\ &\qquad + C_1 \|K_G P_{\tilde{V}_n} \boldsymbol{u}_n\|^2_{\boldsymbol{V}} + \|M \boldsymbol{w}_n\|^2_{L^2} + \|\tilde{O} \boldsymbol{u}_n\|^2_{L^2} + \|c_s \rho^{1/2} \pi^l_n (\boldsymbol{q}\cdot\boldsymbol{w}_n)\|^2_{L^2} \\ &\qquad + \epsilon \|\rho^{1/2}(\omega + i\boldsymbol{D}^n_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{w}_n\|^2_{\boldsymbol{L}^2} + |\omega|(\tan(\theta+\tau) - \tan(\theta)) \|(\gamma\rho)^{1/2} \boldsymbol{w}_n\|^2_{\boldsymbol{L}^2} + s^\beta_n(\boldsymbol{u}_n, \boldsymbol{u}_n). \end{aligned}$$

With the same arguments as in the proof of Lemma 6.23 and the use of the adapted estimate from Lemma 6.33, we obtain for $\alpha_{\boldsymbol{\nu}} \geq \beta C_{\mathrm{tr}}^2 N_\partial + C_\delta + \epsilon$

$$
\begin{aligned}
&\|c_s^2\rho \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{v}_n\|_{L^2}^2 - (1+\tan^2(\theta+\tau)(1-\epsilon)^{-1})\|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\|^2 + \|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 \\
&\quad + C_1\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2 + s_n^\beta(\boldsymbol{u}_n,\boldsymbol{u}_n) \\
&\geq \epsilon\Big(\|c_s\rho^{1/2}\operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{v}_n\|_{L^2}^2 + \|\rho^{1/2}\boldsymbol{D}_{\boldsymbol{b}}^n\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2\Big) + C_{\tau,\epsilon,\delta,n_0}\Big(|\nabla\tilde{v}|_{\boldsymbol{H}^1_{c_s^2\rho}}^2 + \|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho}\Big) \\
&\quad + \|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + \Big(\alpha_{\boldsymbol{\nu}} - \beta C_{\mathrm{tr}}^2 N_\partial - C_\delta\Big)\|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu},c_s^2\rho} + \Big(C_1 - \frac{1}{4\tilde{\delta}}\Big)\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2 \\
&\quad - (\tilde{\delta}\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})}^2 + \|\check{O}_n\|_{L(\boldsymbol{X}_n)})\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 \\
&\geq \epsilon\min\{\underline{c_s}^2\underline{\rho},\underline{\rho},1\}\|\boldsymbol{v}_n\|_{\boldsymbol{X}_n}^2 + \Big(C_1 - \frac{1}{4\tilde{\delta}}\Big)\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2 \\
&\quad - (\tilde{\delta}\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})}^2 + \|\check{O}_n\|_{L(\boldsymbol{X}_n)})\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2,
\end{aligned}
$$

Furthermore, since $(\operatorname{div} + \pi_n^l\boldsymbol{q}\cdot)\boldsymbol{w}_n = -M\boldsymbol{w}_n - \tilde{O}\boldsymbol{u}_n$, see (6.49), we have that

$$
4(\|M\boldsymbol{w}_n\|_{L^2}^2 + \|\tilde{O}\boldsymbol{u}_n\|_{L^2}^2 + \|c_s\rho^{1/2}\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n)\|_{L^2}^2) \geq \|\operatorname{div}\boldsymbol{w}_n\|_{L^2}^2
$$

and since $\operatorname{div}\boldsymbol{w}_n = \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{w}_n$ this yields

$$
\begin{aligned}
&\|M\boldsymbol{w}_n\|_{L^2}^2 + \|\tilde{O}\boldsymbol{u}_n\|_{L^2}^2 + \|c_s\rho^{1/2}\pi_n^l(\boldsymbol{q}\cdot\boldsymbol{w}_n)\|_{L^2}^2 + \epsilon\|\rho^{1/2}(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2 \\
&\quad + |\omega|(\tan(\theta+\tau) - \tan(\theta))\|(\gamma\rho)^{1/2}\boldsymbol{w}_n\|_{\boldsymbol{L}^2}^2 \gtrsim \|\boldsymbol{w}_n\|_{\boldsymbol{X}_n}^2.
\end{aligned}
$$

Combining both estimates, we obtain with a constant $C_{\tilde{B}} > 0$ independent of $\delta, \tilde{\delta}$ and $n > n_0$ that

$$
\begin{aligned}
&\frac{1}{\cos(\theta+\tau)}\Re\Big(e^{-i(\theta+\tau)\mathrm{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big) \\
&\quad \geq C_{\tilde{B}}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2 + \Big(C_1 - \frac{1}{4\tilde{\delta}}\Big)\|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|_{\boldsymbol{V}}^2 - (\tilde{\delta}\sup_{m\in\mathbb{N}}\|P_{\tilde{V}_m}\|_{L(\boldsymbol{X}_m,\boldsymbol{V})}^2 + \|\check{O}_n\|_{L(\boldsymbol{X}_n)})\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2.
\end{aligned}
$$

Thus, with the same argumentation as in the proof of Lemma 6.23, we can choose $\tilde{\delta}, n_1 > n_0$ and $C_1 > 1/(4\tilde{\delta})$ such that for all $n > n_1$ it holds that

$$
\frac{1}{\cos(\theta+\tau)}\Re\Big(e^{-i(\theta+\tau)\mathrm{sgn}\omega}\langle\tilde{B}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big) \geq \frac{C_{\tilde{B}}}{2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}^2,
$$

which proves the claim. □

Lemma 6.36. *Assume that Assumption 6.2 holds true and that $\alpha_{\boldsymbol{\nu}}$ is sufficiently large. Then, there exists an index $n_0 > 0$ such that the operator $B_n := \tilde{B}_n + \tilde{K}_n$ is uniformly coercive for all $n > n_0$.*

Proof. The previous Lemma established the coercivity of $\tilde{B}$ for $n > n_0$ and $\alpha_{\boldsymbol{\nu}}$ sufficiently large. To show that $\tilde{K}_n$ is coercive, we notice that $\tilde{K}_n$ as defined in (6.56) only differs from $\tilde{K}_n$ as defined in (5.45) by the terms corresponding with the discrete divergence $\operatorname{div}_{\boldsymbol{\nu}}^n$. In particular, we have that $\operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{w}_n = \operatorname{div}\boldsymbol{w}_n$. Due to the boundedness of the lifting operator $R_n^{l_\nu}$, the inclusion of $\|\cdot\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}}$ in the $\|\cdot\|_{\boldsymbol{X}_n}$-norm and $\|\boldsymbol{v}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}} = \|\boldsymbol{u}_n\|_{\mathcal{F}_n,1/2,\boldsymbol{\nu}}$, we can argue analogously to the proof of Lemma 5.30. That means that we can estimate

$$
\begin{aligned}
|\langle c_s^2\rho(\mathrm{Id} - \mathrm{mean} - \pi_n^l)(\boldsymbol{q}\cdot\boldsymbol{w}_n), \operatorname{div}\boldsymbol{w}_n'\rangle &\leq \|\boldsymbol{q}\|_{\boldsymbol{L}^\infty}\|\boldsymbol{w}_n\|_{\boldsymbol{L}^2}\|(\mathrm{Id} - \mathrm{mean} - \pi_n^l)(c_s^2\rho\operatorname{div}\boldsymbol{w}_n')\|_{L^2} \\
&\leq \|\boldsymbol{q}\|_{\boldsymbol{L}^\infty}\|\boldsymbol{w}_n\|_{\boldsymbol{L}^2}(C_{c_s^2\rho}^L)h_n\|\operatorname{div}\boldsymbol{w}_n\|_{L^2}
\end{aligned}
$$

and define the seminorm

$$|\boldsymbol{u}_n|^2_{Y_n} := \|\boldsymbol{v}_n\|^2_{\boldsymbol{X}_n} + \|K_G P_{\tilde{V}_n}\boldsymbol{u}_n\|^2_{\boldsymbol{V}} + \|\tilde{O}_n\boldsymbol{u}_n\|^2_{L^2} + \|M\boldsymbol{w}_n\|^2_{L^2} + \|\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n)\|^2_{L^2}.$$

Then, we can estimate for constants $C_{Y,1}, C_{Y,2} > 0$ with the weighted Young's inequality that

$$\frac{1}{\cos(\theta+\tau)}\Re\Big(e^{-i(\theta+\tau)\mathrm{sgn}\omega}\langle\tilde{K}_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big) \geq C_2|\boldsymbol{u}_n|^2_{Y_n} - h_n C_{Y,1}\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n} - C_{Y,2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}|\boldsymbol{u}_n|_{Y_n}.$$

Thus, with Lemma 6.35 and an application of the weighted Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{\cos(\theta+\tau)}\Re\Big(e^{-i(\theta+\tau)\mathrm{sgn}\omega}\langle B_n\boldsymbol{u}_n,\boldsymbol{u}_n\rangle_{\boldsymbol{X}_n}\Big)\\
&\qquad\geq \frac{C_{\tilde{B}}}{2}\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n} + C_2|\boldsymbol{u}_n|^2_{Y_n} - h_n C_{Y,1}\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n} - C_{Y,2}\|\boldsymbol{u}_n\|_{\boldsymbol{X}_n}|\boldsymbol{u}_n|_{Y_n}\\
&\qquad\geq \Big(\frac{C_{\tilde{B}}}{4} - h_n C_{Y,1}\Big)\|\boldsymbol{u}_n\|^2_{\boldsymbol{X}_n} + \Big(C_2 - \frac{C^2_{Y,2}}{C_{\tilde{B}}}\Big)|\boldsymbol{u}_n|^2_{Y_n}.
\end{aligned}$$

Choosing $C_2 > C^2_{Y,2}/C_{\tilde{B}}$ and n large enough such that $h_n C_{Y,1} < C_{\tilde{B}}/4$ yields the claim. □

Lemma 6.37. *Let $A \in L(\boldsymbol{X})$ be the operator induced by the continuous sesquilinear form $a(\cdot,\cdot)$. Then, there exist operators $B, K \in L(\boldsymbol{X})$ such that B is coercive and $AT = B + K$. Furthermore, we have that $B_n \xrightarrow{P} B$ and $K_n \xrightarrow{P} K$.*

Proof. Let $B, K \in L(\boldsymbol{X})$ be the operators defined by in the proof of Lemma 5.31, that is (5.50) and (5.51) respectively. We recall that B is coercive and that $AT = B + K$. Furthermore, with the same argument as in Lemma 5.31 or Lemma 6.25, it suffices to show that $K_n \xrightarrow{P} K$ to conclude that $B_n \xrightarrow{P} B$. Let $\boldsymbol{u}'_n \in \boldsymbol{X}_n$, $\|\boldsymbol{u}'_n\|_{\boldsymbol{X}_n} = 1$ be such that $\|(p_nK - K_np_n)\boldsymbol{u}_n\|_{\boldsymbol{X}_n} \leq |\langle p_nK\boldsymbol{u} - K_np_n\boldsymbol{u}, \boldsymbol{u}'_n\rangle_{\boldsymbol{X}_n}| + 1/n$ and let $\mathbb{N}' \subset \mathbb{N}$ be an arbitrary subsequence. Due to Lemma 6.7, we can choose a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and $\boldsymbol{u}' \in \boldsymbol{X}$ such that $\boldsymbol{u}'_n \xrightharpoonup{\boldsymbol{L}^2} \boldsymbol{u}'$, $\mathrm{div}^n_{\boldsymbol{\nu}}\boldsymbol{u}'_n \xrightharpoonup{L^2} \mathrm{div}\,\boldsymbol{u}'$ and $\boldsymbol{D}^n_{\boldsymbol{b}}\boldsymbol{u}'_n \xrightharpoonup{\boldsymbol{L}^2} \partial_{\boldsymbol{b}}\boldsymbol{u}'$. We note that we only have to adjust the estimates in (6.41) and (6.42) by taking into consideration that

$$\|P_V\boldsymbol{u} - P_{\tilde{V}_n}\boldsymbol{u}\|^2_{L^2} \lesssim \|(P_{L^2_0} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{u})\|^2_{L^2}.$$

Thus, we have that

$$\begin{aligned}
|\langle\boldsymbol{v},\boldsymbol{v}'_n\rangle - \langle P_{V_n}p_n\boldsymbol{u},\boldsymbol{v}'_n\rangle| \lesssim{}& h_n\|P_V\boldsymbol{u}\|_{\boldsymbol{H}^1} + d_n(\boldsymbol{u},p_n\boldsymbol{u})\\
&+ \|(P_{L^2_0} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{u})\|^2_{L^2} + \|\mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(p_n\boldsymbol{u})\|_{\boldsymbol{X}_n}
\end{aligned}$$

and

$$|\langle K_G\boldsymbol{v}, K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}} - \langle K_GP_{\tilde{V}_n}p_n\boldsymbol{u}, K_GP_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{V}}| \lesssim \|(P_{L^2_0} - \pi^l_n)(\boldsymbol{q}\cdot\boldsymbol{u})\|^2_{L^2} + d_n(\boldsymbol{u},p_n\boldsymbol{u}).$$

With the same argumentation as in the proof of Lemma 5.31, it follows that

$$\begin{aligned}
|\langle M\boldsymbol{w}, M\boldsymbol{w}'_n\rangle - \langle M\boldsymbol{w}_n(p_n\boldsymbol{u}), M\boldsymbol{w}'_n\rangle| \lesssim{}& \|p_nP_V\boldsymbol{u} - P_{V_n}p_n\boldsymbol{u}\|_{H(\mathrm{div})}\\
&+ d_n(\boldsymbol{u},p_n\boldsymbol{u}) + d_n(P_V\boldsymbol{u}, p_nP_V\boldsymbol{u}),
\end{aligned}$$

and

$$\begin{aligned}
&|\langle\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}),\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}'_n)\rangle - \langle\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n(p_n\boldsymbol{u})),\mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}'_n)\rangle|\\
&\qquad\lesssim \|p_nP_V\boldsymbol{u} - P_{V_n}p_n\boldsymbol{u}\|_{H(\mathrm{div})} + d_n(\boldsymbol{u},p_n\boldsymbol{u}) + d_n(P_V\boldsymbol{u},p_nP_V\boldsymbol{u}).
\end{aligned}$$

Finally, we denote by $S := \nabla((\operatorname{div} + P_{L^2_0}\boldsymbol{q} \cdot + M)\nabla)^{-1} \in L(L^2_0, \boldsymbol{V})$ and compute

$$\begin{aligned}
\langle K_G \boldsymbol{v}, &K_G P_{\tilde{V}_n} \boldsymbol{u}'_n\rangle_{\boldsymbol{V}} \\
&= \langle K_G^* K_G \boldsymbol{v}, P_{\tilde{V}_n} \boldsymbol{u}'_n\rangle_{\boldsymbol{V}} \\
&= \langle K_G^* K_G \boldsymbol{v}, P_V(\boldsymbol{u}'_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}'_n))\rangle_{\boldsymbol{V}} + \langle K_G^* K_G \boldsymbol{v}, S(\pi_n^l - P_{L^2_0})(\boldsymbol{q} \cdot (\boldsymbol{u}'_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}'_n)))\rangle_{\boldsymbol{V}} \\
&= \langle P_V^* K_G^* K_G \boldsymbol{v}, \boldsymbol{u}'_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}'_n)\rangle_{H_0(\operatorname{div})} + \langle (\pi_n^l - P_{L^2_0}) S^* K_G^* K_G \boldsymbol{v}, \boldsymbol{q} \cdot (\boldsymbol{u}'_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}'_n))\rangle_{H_0(\operatorname{div})} \\
&\overset{n\in N''}{\longrightarrow} \langle P_V^* K_G^* K_G \boldsymbol{v}, \boldsymbol{u}'\rangle_{\boldsymbol{V}} = \langle K_G \boldsymbol{v}, K_G \boldsymbol{v}'\rangle_{\boldsymbol{V}},
\end{aligned}$$

where the last line follows since π_n^l converges pointwise to $P_{L^2_0}$ and $\operatorname{div}(\boldsymbol{u}'_n - \mathcal{I}_{[\![\cdot]\!]_{\boldsymbol{\nu}}}(\boldsymbol{u}'_n)) \overset{L^2}{\rightharpoonup} \operatorname{div} \boldsymbol{u}'$ by Lemma 6.11. Thus, with Lemma 6.10, Lemma 6.4 and the pointwise convergence of π_n^l towards $P_{L^2_0}$ we obtain that

$$\lim_{n\to\infty} \langle K_n p_n \boldsymbol{u}, \boldsymbol{u}'_n\rangle_{\boldsymbol{X}_n} = \langle K\boldsymbol{u}, \boldsymbol{u}'\rangle_{\boldsymbol{X}}.$$

□

Theorem 6.38. *Assume that Assumption 6.2 is satisfied and let $\alpha_{\boldsymbol{\nu}}$ be sufficiently large. Let $\boldsymbol{f} \in \boldsymbol{L}^2$ and $\boldsymbol{u} \in \boldsymbol{X}$ be the solution to $a(\boldsymbol{u}, \boldsymbol{u}') = \langle \boldsymbol{f}, \boldsymbol{u}'\rangle$ for all $\boldsymbol{u}' \in \boldsymbol{X}$. Then, there exists an index $n_0 > 0$ such that the solution $\boldsymbol{u}_n \in \boldsymbol{X}_n$ to $a_n(\boldsymbol{u}_n, \boldsymbol{u}'_n) = \langle \boldsymbol{f}, \boldsymbol{u}'_n\rangle$ for all $\boldsymbol{u}'_n \in \boldsymbol{X}_n$ exists for all $n > n_0$ and $\lim_{n\to\infty} d_n(\boldsymbol{u}, \boldsymbol{u}_n) = 0$. Furthermore, if $\boldsymbol{u} \in \boldsymbol{X} \cap \boldsymbol{H}^{2+s}$, $s > 0$, $\rho \in W^{1+s,\infty}$ and $\boldsymbol{b} \in \boldsymbol{W}^{1+s,\infty}$, then*

$$d_n(\boldsymbol{u}, \boldsymbol{u}_n) \lesssim h_n^{\min(1+s,k)} + h_n^{\min(s,l_{\boldsymbol{b}})} + h_n^{\min(s,l_{\boldsymbol{\nu}})}.$$

Proof. As in the proof of Thm. 6.26, we note that we can can apply Thm. 2.28 to conclude that $(A_n)_{n\in\mathbb{N}}$ is regular due to the previous results and Thm. 6.8. Furthermore, with the same argumentation as in the proof of Thm. 5.33, we have that the right-hand side of the discrete problem P-converges towards the right-hand side of the continuous problem. Together with the injectivity of A, cf. Lemma 4.4, this allows us to apply Thm. 2.17, which yields the existence of discrete solutions $\boldsymbol{u}_n \in \boldsymbol{X}_n$ to (6.11) for all $n > n_0$. With the same argumentation as in [Hal23, Thm. 19], see also Thm. 5.33, it follows that $\lim_{n\to\infty} d_n(\boldsymbol{u}, \boldsymbol{u}_n) = 0$. For the convergence rates, we notice that the argumentation from the proof of Thm. 6.26 can easily be adapted to the heterogeneous case since we only have to consider the partial integration of the additional pressure term $\langle \nabla p \cdot \boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}'_n\rangle$. Let $\tilde{\psi}_n$ be a suitable H^1-interpolator of $\nabla p \cdot \boldsymbol{u}$. Then, we compute

$$\langle \nabla p \cdot \boldsymbol{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}'_n\rangle = \langle \tilde{\psi}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}'_n\rangle + \langle \nabla p \cdot \boldsymbol{u} - \tilde{\psi}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}'_n\rangle$$

and with the same argumentation as in (6.43)

$$\begin{aligned}
\langle \tilde{\psi}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n\rangle = \langle \tilde{\psi}_n, \operatorname{div} \boldsymbol{u}'_n + R_n^l \boldsymbol{u}'_n\rangle_{\mathcal{T}_n} &= \langle \tilde{\psi}_n, \operatorname{div} \boldsymbol{u}'_n\rangle_{\mathcal{T}_n} - \langle \{\!\{\tilde{\psi}_n\}\!\}, [\![\boldsymbol{u}'_n]\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n} \\
&= -\langle \nabla\tilde{\psi}_n, \boldsymbol{u}'_n\rangle = -\langle \nabla(\nabla p \cdot \boldsymbol{u}), \boldsymbol{u}'_n\rangle + \langle \nabla(\nabla p \cdot \boldsymbol{u} - \tilde{\psi}_n), \boldsymbol{u}'_n\rangle
\end{aligned}$$

Since $\|\nabla p \cdot \boldsymbol{u} - \tilde{\psi}_n\|_{H^1} \lesssim h_n^{\min(s,l_{\boldsymbol{\nu}})}$, the same argumentation as in Thm. 6.26 yields the claim. □

6.4 Hybridization

Similar to Section 5.2.4, we want to discuss the possibility of hybridization for the fully discontinuous scheme to reduce computational costs. As before, we only introduce the necessary modifications to the formulation and do not discuss the analysis in detail. In contrast to Section 5.2.4, where we only introduced facet variables for the tangential component, we now also introdce an additional facet variable for the normal component, see Fig. 6.1. Thus, we define the hybrid space $\boldsymbol{X}_n^{\mathrm{HDG}}$ by

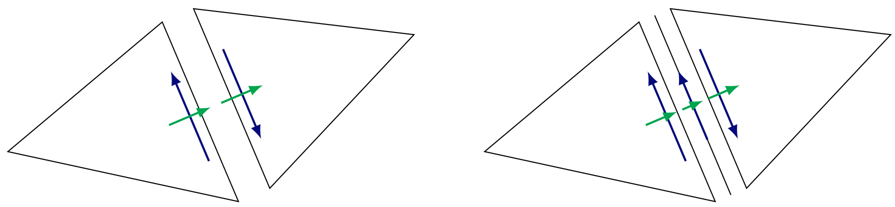
$$\boldsymbol{X}_n^{\mathrm{HDG}} := \boldsymbol{X}_n \times \boldsymbol{L}^2(\mathcal{F}_n).$$

In addition to the introduction of a hybrid version of the vector-valued lifting $\underline{\boldsymbol{R}}_n^{l_{\boldsymbol b}}$ from Section 5.2.4, we introduce a hybrid version of the scalar lifting operator. For $\tau \in \mathcal{T}_n$, let $\underline{r}_n^{\partial\tau} \in Q_n$ be the solution to

$$\langle \underline{r}_n^{\partial\tau} \boldsymbol{u}_n, \psi_n \rangle = -\langle \llbracket \underline{\boldsymbol{u}_n} \rrbracket_{\boldsymbol{\nu}}, \psi_n \rangle_{L^2(\partial\tau)} \text{ for all } \psi_n \in Q_n. \tag{6.58}$$

We set $\underline{R}_n^{l_\nu} := \sum_{\tau\in\mathcal{T}_n} \underline{r}_n^{\partial\tau}$ and define the discrete divergence operator $\underline{\operatorname{div}}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n$ by

$$(\underline{\operatorname{div}}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n)|_\tau := \operatorname{div} \boldsymbol{u}_n|_\tau + \underline{R}_n^{l_\nu} \boldsymbol{u}_n|_\tau, \quad \tau \in \mathcal{T}_n. \tag{6.59}$$



(a) DG (b) HDG

Figure 6.1: Comparison of DG and HDG degrees of freedom.

Then, we consider the problem: Find $\boldsymbol{u}_n^{\mathrm{HDG}} \in \boldsymbol{X}_n^{\mathrm{HDG}}$ such that

$$a_n^{\mathrm{HDG}}(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}, \boldsymbol{u}_n' \rangle \text{ for all } \boldsymbol{u}_n' \in \boldsymbol{X}_n^{\mathrm{HDG}},$$

where the sesquilinear form $a_n^{\mathrm{HDG}}(\cdot,\cdot)$ is defined by

$$\begin{aligned} a_n^{\mathrm{HDG}}(\boldsymbol{u}_n, \boldsymbol{u}_n') :=& \langle c_s^2 \rho \underline{\operatorname{div}}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \underline{\operatorname{div}}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\underline{\boldsymbol{D}}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\underline{\boldsymbol{D}}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle \\ &+ \langle \underline{\operatorname{div}}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \underline{\operatorname{div}}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \\ &- i\omega \langle \gamma\rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle + s_n^\beta(\boldsymbol{u}_n, \boldsymbol{u}_n'). \end{aligned}$$

To analyze the hybrid fully discontinuous Galerkin discretization with the techniques discussed in Part I, we would have to show that the scheme can be interpreted as a discrete approximation scheme. The most notable difference to the analysis in Section 6.2 is that the compactness result from Lemma 6.7 would need to be adjusted to the hybridized setting. With techniques similar to [KCR21, Thm. 4.3], we can show the following result.

Lemma 6.39. *Let $\boldsymbol{u}_n \in \boldsymbol{X}_n := \boldsymbol{X}_n^{HDG}$ be such that $\sup_{n\in\mathbb{N}} \|\boldsymbol{u}_n\|_{\boldsymbol{X}_n} < \infty$. Then there exists a subsequence $\mathbb{N}' \subset \mathbb{N}$ and $\boldsymbol{u} \in \boldsymbol{X}$ such that $\boldsymbol{u}_n \xrightarrow{\boldsymbol{L}^2} \boldsymbol{u}$, $\underline{\operatorname{div}}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n \xrightarrow{L^2} \operatorname{div} \boldsymbol{u}$ and $\underline{\boldsymbol{D}}_{\boldsymbol{b}}^n \boldsymbol{u}_n \xrightarrow{\boldsymbol{L}^2} \partial_{\boldsymbol{b}} \boldsymbol{u}$.*

Afterwards, the construction to establish weak T-compatibility from Section 6.3 would have to be adapted to the hybrid setting as well by suitably incorporating the additional facets variables. We leave a detailed analysis of the hybrid fully discontinuous Galerkin scheme for future work.

CHAPTER 7

Numerical experiments

In this section, we perform numerical experiments to compare the previously introduced methods computationally. First of all, we summarize the methods that we are considering and give remarks on their implementation. Then, we validate the theoretical convergence rates by considering convergence against an artificially constructed exact solution and a reference solution. Afterwards, we explore the possibility of hybridization to reduce the computational costs, consider the effects of the stabilization term introduced in Section 6.1, and investigate the choice of stabilization parameters for the symmetric interior penalty variants of the considered discretization schemes. Finally, we consider a computational example with physically relevant coefficients for the sun. We conclude the section with a discussion of the challenges that arise when considering the application of the developed discretizations to computational helioseismology. The numerical experiments can be replicated with the provided reproduction files [Bee23].

Contents of the chapter

7.1 Methods and implementational remarks

Before we delve into the numerical experiments, we want to provide an overview of the different methods that we are considering. The overview encompasses the discrete spaces $\boldsymbol{X}_n$, the sesquilinear forms $a_n(\cdot,\cdot)$ and remarks on the implementation.

H^1-conforming finite element discretization.

The first method under consideration is the H^1-conforming discretization introduced by Halla et. al. [HLS22], which we discussed in Section 5.1. We consider the following problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n = \{\boldsymbol{u} \in \boldsymbol{H}^1 : \boldsymbol{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\mathcal{O}, \boldsymbol{u}|_\tau \in \boldsymbol{\mathcal{P}}^k(\tau)\ \forall \tau \in \mathcal{T}_n\}$ such that

$$a_n^{\mathrm{H}^1}(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}_n, \boldsymbol{u}_n' \rangle \quad \forall \boldsymbol{u}_n' \in \boldsymbol{X}_n,$$

where the sesquilinear form $a_n^{\mathrm{H}^1}(\cdot,\cdot)$ is given by

$$\begin{aligned} a_n^{\mathrm{H}^1}(\boldsymbol{u}_n, \boldsymbol{u}_n') &= \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n' \rangle \\ &\quad + \langle \operatorname{div} \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \\ &\quad - i\omega \langle \gamma \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle. \end{aligned}$$

To incorporate the boundary conditions $\boldsymbol{\nu} \cdot \boldsymbol{u}_n = 0$ on $\partial\mathcal{O}$, we use Nitsche's method, cf., for example [JS09]. This means, that we consider the finite element space $[\mathbb{P}^k(\mathcal{T}_n)]^d \cap \boldsymbol{H}^1$ and add the following terms to the sesquilinear form:

$$-\langle c_s^2 \rho \boldsymbol{u}_n \cdot \boldsymbol{\nu}, \operatorname{div} \boldsymbol{u}_n' \rangle_{\partial\mathcal{O}} - \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\partial\mathcal{O}} + \langle \frac{\alpha_{\mathrm{N}} k^2}{h} c_s^2 \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle_{\partial\mathcal{O}}, \tag{7.1}$$

where $\alpha_{\mathrm{N}} > 0$ is a stabilization parameter.

<u>$H(\operatorname{div})$**-conforming discontinous Galerkin - Lifting variant.**</u>

The second method that we are considering is the $H(\operatorname{div})$-conforming discontinuous Galerkin discretization, as introduced by Halla [Hal23] and reviewed in Section 5.2. We consider the problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n := \{\boldsymbol{u} \in H_0(\operatorname{div}) : \boldsymbol{u}|_\tau \in \boldsymbol{\mathcal{P}}^k(\tau)\ \forall \tau \in \mathcal{T}_n\}$ such that

$$a_n^{\mathrm{H(div),LS}}(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}_n, \boldsymbol{u}_n' \rangle \quad \forall \boldsymbol{u}_n' \in \boldsymbol{X}_n,$$

where the sesquilinear form $a_n^{\mathrm{H(div),LS}}$ is given by

$$\begin{aligned} a_n^{\mathrm{H(div),LS}}(\boldsymbol{u}_n, \boldsymbol{u}_n') = {} & \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle \\ & + \langle \operatorname{div} \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle \\ & + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle - i\omega \langle \gamma \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle. \end{aligned} \tag{7.2}$$

To implement the method, we follow Remark 3.29. By definition of $\boldsymbol{D}_{\boldsymbol{b}}^n$, we have that

$$\begin{aligned} & -\langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle_{\mathcal{T}_n} \\ = {} & -\langle \rho(\omega + i(\partial_{\boldsymbol{b}} + \boldsymbol{R}_n^l) + i\Omega\times)\boldsymbol{u}_n, (\omega + i(\partial_{\boldsymbol{b}} + \boldsymbol{R}_n^l) + i\Omega\times)\boldsymbol{u}_n' \rangle_{\mathcal{T}_n} \\ = {} & -\langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n' \rangle_{\mathcal{T}_n} - \langle \rho i\boldsymbol{R}_n^l \boldsymbol{u}_n, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n' \rangle_{\mathcal{T}_n} \\ & - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, i\boldsymbol{R}_n^l \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} - \langle \rho i\boldsymbol{R}_n^l \boldsymbol{u}_n, i\boldsymbol{R}_n^l \boldsymbol{u}_n' \rangle_{\mathcal{T}_n}. \end{aligned}$$

Due to the definition[1] of the lifting operator (5.20), $\langle \cdot, i\cdot \rangle = -i\langle \cdot, \cdot \rangle$ and $\langle u, v \rangle = \overline{\langle v, u \rangle}$, we have that

$$\begin{aligned} -\langle \rho i\boldsymbol{R}_n^l \boldsymbol{u}_n, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n' \rangle_{\mathcal{T}_n} &= -i\langle \rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, \{\!\{(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n'\}\!\} \rangle_{\mathcal{F}_n}, \\ -\langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, i\boldsymbol{R}_n^l \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} &= -i\langle \rho \{\!\{(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}} \rangle_{\mathcal{F}_n}. \end{aligned}$$

Furthermore, we introduce a variable $\boldsymbol{r} = \boldsymbol{R}_n^l \boldsymbol{u}_n \in \boldsymbol{Q}_n$ such that

$$-\langle \rho i\boldsymbol{R}_n^l \boldsymbol{u}_n, i\boldsymbol{R}_n^l \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} = -\langle \rho \boldsymbol{r}, \boldsymbol{R}_n^l \boldsymbol{u}_n' \rangle_{\mathcal{T}_n} = \langle \rho \{\!\{\boldsymbol{r}\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}} \rangle_{\mathcal{F}_n}.$$

By definition of the lifting, $\boldsymbol{r} \in \boldsymbol{Q}_n$ solves

$$\langle \rho \boldsymbol{r}, \boldsymbol{s} \rangle_{\mathcal{T}_n} = -\langle \rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, \{\!\{\boldsymbol{s}\}\!\} \rangle_{\mathcal{F}_n} \text{ for all } \boldsymbol{s} \in \boldsymbol{Q}_n.$$

Thus, we implement the term $-\langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle$ in (7.2) through a mixed formulation as in Remark (3.29) through the following terms:

$$\begin{aligned} & \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n' \rangle_{\mathcal{T}_n} - i\langle \rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, \{\!\{(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n'\}\!\} \rangle_{\mathcal{F}_n} \\ & - i\langle \rho \{\!\{(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}} \rangle_{\mathcal{F}_n} + \langle \rho \{\!\{\boldsymbol{r}\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}} \rangle_{\mathcal{F}_n} \\ & - \langle \rho \boldsymbol{r}, \boldsymbol{s} \rangle_{\mathcal{T}_n} - \langle \rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, \{\!\{\boldsymbol{s}\}\!\} \rangle_{\mathcal{F}_n}. \end{aligned}$$

We note that the boundary conditions $\boldsymbol{\nu} \cdot \boldsymbol{u}_n = 0$ on $\partial\mathcal{O}$ can be directly incorporated in to the $H(\operatorname{div})$-conforming finite element space $\boldsymbol{X}_n$. Thus, we do not have to add any additional terms to the sesquilinear form.

[1]Technically, we redefine $\boldsymbol{R}_n^l$ such that for $\eta \in W^{1,\infty}$: $\langle \eta \boldsymbol{R}_n^l \boldsymbol{u}_n, \boldsymbol{\psi}_n \rangle = -\langle \eta [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, \{\!\{\boldsymbol{\psi}_n\}\!\} \rangle_{\mathcal{F}_n}$ for all $\boldsymbol{\psi}_n \in \boldsymbol{Q}_n$.

Remark 7.1 (Preasymptotic stability)**.** *The convergence theorem 5.33 states that there exists an index $n_0 > 0$ such that a unique discrete solution $\boldsymbol{u}_n \in \boldsymbol{X}_n$ exists for all $n > n_0$ and $\boldsymbol{u}_n \xrightarrow{P} \boldsymbol{u}$. That means that the method is asymptotically stable. However, we have no specific knowledge of n_0. To ensure that the method is numerically stable, even pre-asymptotically, we add the terms that implement the lifting operator twice.*

$H(\mathrm{div})$-conforming discontinuous Galerkin - SIP variant.

For the sake of comparison, we will also consider a symmetric interior penalty variant of the $H(\mathrm{div})$-conforming discontinuous Galerkin method. That means, we consider the problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n := \{\boldsymbol{u} \in H_0(\mathrm{div}) : \boldsymbol{u}|_\tau \in \boldsymbol{\mathcal{P}}^k(\tau)\ \forall \tau \in \mathcal{T}_n\}$ such that

$$a_n^{\mathrm{H(div),SIP}}(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}_n, \boldsymbol{u}_n' \rangle \quad \forall \boldsymbol{u}_n' \in \boldsymbol{X}_n,$$

where the sesquilinear form $a_n^{\mathrm{H(div),SIP}}$ is given by

$$\begin{aligned} a_n^{\mathrm{H(div),SIP}}(\boldsymbol{u}_n, \boldsymbol{u}_n') = {}& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n' \rangle \\ & + \langle \rho \{\!\{ (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n \}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}} \rangle_{\mathcal{F}_n} + \langle \rho \{\!\{ (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n' \}\!\}, [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}} \rangle_{\mathcal{F}_n} \\ & - \langle \rho \frac{\alpha_{\boldsymbol{b}}}{h} [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}} \rangle_{\mathcal{F}_n} \\ & + \langle \operatorname{div} \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle + \langle (\mathrm{Hess}(p) - \rho\, \mathrm{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \\ & - i\omega \langle \gamma \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \end{aligned}$$

for a stabilization parameter $\alpha_{\boldsymbol{b}} > 0$. In contrast to the lifting stabilized version from [Hal23], the penalty parameter $\alpha_{\boldsymbol{b}}$ has to be chosen large enough to ensure stability.

Fully discontinuous Galerkin - Lifting variant.

Furthermore, we consider the two fully discontinuous Galerkin discretizations that were introduced and analyzed in Chapter 6. We consider the problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n := \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \boldsymbol{u}|_\tau \in \boldsymbol{\mathcal{P}}^k(\tau)\ \forall \tau \in \mathcal{T}_n\}$ such that

$$a_n^{\mathrm{DG,LS},\beta}(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}_n, \boldsymbol{u}_n' \rangle \quad \forall \boldsymbol{u}_n' \in \boldsymbol{X}_n,$$

where the sesquilinear form $a_n^{\mathrm{DG,LS},\beta}$ is given by

$$\begin{aligned} a_n^{\mathrm{DG,LS},\beta}(\boldsymbol{u}_n, \boldsymbol{u}_n') = {}& \langle c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle \\ & + \langle \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle + \langle (\mathrm{Hess}(p) - \rho\, \mathrm{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \\ & - i\omega \langle \gamma \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle + s_n^\beta(\boldsymbol{u}_n, \boldsymbol{u}_n'). \end{aligned}$$

We recall that from Remark 6.1 that the choice $\beta = 1$ yields a SIP formulation for the diffusion operator, i.e. we have that

$$\begin{aligned} a_n^{\mathrm{DG,LS},1}(\boldsymbol{u}_n, \boldsymbol{u}_n') = {}& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle \\ & - \langle c_s^2 \rho \{\!\{ \operatorname{div} \boldsymbol{u}_n \}\!\}, [\![\boldsymbol{u}_n']\!] \rangle_{\mathcal{F}_n} - \langle c_s^2 \rho \{\!\{ \operatorname{div} \boldsymbol{u}_n' \}\!\}, [\![\boldsymbol{u}_n]\!] \rangle_{\mathcal{F}_n} + \langle c_s^2 \rho \frac{\alpha_{\boldsymbol{\nu}}}{h} [\![\boldsymbol{u}_n]\!], [\![\boldsymbol{u}_n']\!] \rangle_{\mathcal{F}_n} \\ & + \langle \operatorname{div} \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n' \rangle + \langle (\mathrm{Hess}(p) - \rho\, \mathrm{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \\ & - i\omega \langle \gamma \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \end{aligned}$$

For $\beta = 0$, we have that

$$\begin{aligned} a_n^{\mathrm{DG,LS},0}(\boldsymbol{u}_n, \boldsymbol{u}_n') = {}& \langle c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle - \langle \rho(\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n, (\omega + i\boldsymbol{D}_{\boldsymbol{b}}^n + i\Omega\times)\boldsymbol{u}_n' \rangle \\ & + \langle \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n' \rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n' \rangle + \langle (\mathrm{Hess}(p) - \rho\, \mathrm{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n' \rangle \\ & - i\omega \langle \gamma \rho \boldsymbol{u}_n, \boldsymbol{u}_n' \rangle + \langle c_s^2 \rho \frac{\alpha_{\boldsymbol{\nu}}}{h} [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}} \rangle_{\mathcal{F}_n} \end{aligned}$$

For the implementation of $\boldsymbol{D}_{\boldsymbol{b}}^n$, we follow the same approach as for the $H(\operatorname{div})$-conforming lifting method. In the case that $\beta = 0$, we apply the same argumentation for the discrete divergence operator. By definition of $\operatorname{div}_{\boldsymbol{\nu}}^n$, we have that

$$\begin{aligned}\langle c_s^2\rho \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n'\rangle_{\mathcal{T}_n} =& \langle c_s^2\rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n'\rangle_{\mathcal{T}_n} + \langle c_s^2\rho R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n'\rangle_{\mathcal{T}_n}\\ &+ \langle c_s^2\rho \operatorname{div} \boldsymbol{u}_n, R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n'\rangle_{\mathcal{T}_n} + \langle c_s^2\rho R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n, R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n'\rangle_{\mathcal{T}_n}.\end{aligned}$$

Using the definition of the lifting operator (6.8), we obtain

$$\begin{aligned}\langle c_s^2\rho R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n'\rangle_{\mathcal{T}_n} &= -\langle c_s^2\rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \{\!\{\operatorname{div} \boldsymbol{u}_n'\}\!\}\rangle_{\mathcal{F}_n},\\ \langle c_s^2\rho \operatorname{div} \boldsymbol{u}_n, R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n'\rangle_{\mathcal{T}_n} &= -\langle c_s^2\rho \{\!\{\operatorname{div} \boldsymbol{u}_n\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n}.\end{aligned}$$

We now introduce the auxillary variable $r = R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n \in Q_n$ to obtain

$$\langle c_s^2\rho R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n, R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n'\rangle_{\mathcal{T}_n} = \langle c_s^2\rho r, R_n^{l_{\boldsymbol{\nu}}} \boldsymbol{u}_n'\rangle_{\mathcal{T}_n} = -\langle c_s^2\rho \{\!\{r\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n},$$

where $r \in Q_n$ solves

$$\langle c_s^2\rho r, s\rangle_{\mathcal{T}_n} = -\langle c_s^2\rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \{\!\{s\}\!\}\rangle_{\mathcal{F}_n} \text{ for all } s \in Q_n.$$

Altogether, we implement the term $\langle c_s^2\rho \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \boldsymbol{u}_n'\rangle$ through a mixed formulation as in Remark (3.29) through the following terms:

$$\begin{aligned}&\langle c_s^2\rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n'\rangle - \langle c_s^2\rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \{\!\{\operatorname{div} \boldsymbol{u}_n'\}\!\}\rangle_{\mathcal{F}_n} - \langle c_s^2\rho \{\!\{\operatorname{div} \boldsymbol{u}_n\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n}\\ &- \langle c_s^2\rho \{\!\{r\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n} - \langle c_s^2\rho r, s\rangle_{\mathcal{T}_n} - \langle c_s^2\rho [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \{\!\{s\}\!\}\rangle_{\mathcal{F}_n}.\end{aligned}$$

Similar to our approach for the H^1-conforming method, we add a Nitsche boundary term to incorporate the boundary conditions $\boldsymbol{\nu} \cdot \boldsymbol{u}_n = 0$ on $\partial\mathcal{O}$. Thus, we consider the finite element space $[\mathbb{P}^k(\mathcal{T}_n)]^d$ and add the terms

$$- \langle c_s^2\rho \boldsymbol{u}_n \cdot \boldsymbol{\nu}, \operatorname{div} \boldsymbol{u}_n'\rangle_{\partial\mathcal{O}} - \langle c_s^2\rho \operatorname{div} \boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\partial\mathcal{O}} + \langle \frac{\alpha_{\mathrm{N}} k^2}{h} c_s^2\rho \boldsymbol{u}_n, \boldsymbol{u}_n'\rangle_{\partial\mathcal{O}}, \tag{7.3}$$

where $\alpha_{\mathrm{N}} > 0$ is a stabilization parameter.

Fully discontinuous Galerkin - SIP variant.

As for the $H(\operatorname{div})$-conforming method, we will also compare the introduced discontinuous Galerkin methods with a symmetric interior penalty variant. That means, we consider the problem: Find $\boldsymbol{u}_n \in \boldsymbol{X}_n := \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \boldsymbol{u}|_\tau \in \boldsymbol{\mathcal{P}}^k(\tau)\ \forall \tau \in \mathcal{T}_n\}$ such that

$$a_n^{\text{DG,SIP}}(\boldsymbol{u}_n, \boldsymbol{u}_n') = \langle \boldsymbol{f}_n, \boldsymbol{u}_n'\rangle \quad \forall \boldsymbol{u}_n' \in \boldsymbol{X}_n,$$

where the sesquilinear form $a_n^{\text{DG,SIP}}$ is given by

$$\begin{aligned}a_n^{\text{DG,SIP}}(\boldsymbol{u}_n, \boldsymbol{u}_n') =& \langle c_s^2\rho \operatorname{div} \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n'\rangle - \langle \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n, (\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n'\rangle\\ &+ \langle \rho\{\!\{(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}}\rangle_{\mathcal{F}_n} + \langle \rho\{\!\{(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)\boldsymbol{u}_n'\}\!\}, [\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}\rangle_{\mathcal{F}_n}\\ &- \langle \rho\frac{\alpha_{\boldsymbol{b}}}{h}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{b}}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{b}}\rangle_{\mathcal{F}_n}\\ &- \langle c_s^2\rho\{\!\{\operatorname{div} \boldsymbol{u}_n\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n} - \langle c_s^2\rho\{\!\{\operatorname{div} \boldsymbol{u}_n'\}\!\}, [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n} + \langle c_s^2\rho\frac{\alpha_{\boldsymbol{\nu}}}{h}[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\mathcal{F}_n}\\ &+ \langle \operatorname{div} \boldsymbol{u}_n, \nabla p \cdot \boldsymbol{u}_n'\rangle + \langle \nabla p \cdot \boldsymbol{u}_n, \operatorname{div} \boldsymbol{u}_n'\rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle\\ &- \langle [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \{\!\{\nabla p \cdot \boldsymbol{u}_n'\}\!\}\rangle - \langle \{\!\{\nabla p \cdot \boldsymbol{u}_n\}\!\}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}}\rangle\\ &- i\omega\langle \gamma\rho\boldsymbol{u}_n, \boldsymbol{u}_n'\rangle\end{aligned}$$

for stabilization parameters $\alpha_{\boldsymbol{b}}, \alpha_{\boldsymbol{\nu}} > 0$. In contrast to the lifting stabilized method from Chapter 6, we have to choose the penalty parameter $\alpha_{\boldsymbol{b}}$ sufficiently large to guarantee stability. To incorporate the boundary conditions, we add the Nitsche boundary terms (7.3) to the sesquilinear form.

7.2 Convergence studies

First of all, we want to examine the convergence behavior of the different method[2] introduced in Chapters 5 and 6. To this end, we consider the computational examples from [HLS22, Sec. 4.2]. We choose the domain $\mathcal{O} = (-4,4)^2 \subset \mathbb{R}^2$ and use a shape-regular unstructured simplicial mesh with initial mesh size $h = 1$ such that for each refinement level L, we have a mesh size of $h = 2^{-L}$. Furthermore, choose the following parameters:

$$\begin{aligned} &\rho = 1.5 + 0.2\cos(\pi x/4)\sin(\pi y/2), && c_s^2 = 1.44 + 0.16\rho, && \omega = 0.78 \cdot 2\pi,\\ &\gamma = 0.1, && \Omega = (0,0), && p = 1.44\rho + 0.08\rho^2. \end{aligned}$$

The background flow $\boldsymbol{b}$ is chosen as

$$\boldsymbol{b} = \frac{c_{\boldsymbol{b}}}{\rho}\begin{pmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix}, \tag{7.4}$$

where $c_{\boldsymbol{b}} \in \mathbb{R}$ is a constant that scales the background flow. We note that $\boldsymbol{b}\cdot\boldsymbol{\nu} = 0$ on $\partial\mathcal{O}$ and $\operatorname{div}(\rho\boldsymbol{b}) = 0$. The source term $\boldsymbol{f} \in \boldsymbol{L}^2$ is chosen such that the exact solution is given by

$$\boldsymbol{u}_{\text{exact}} = \frac{1}{\rho}\begin{pmatrix} (1+i)g \\ -(1+i)g \end{pmatrix}, \tag{7.5}$$

where $g(x,y)$ is the Gaussian given by

$$g(x,y) = \sqrt{a/\pi}\exp(-a(x^2+y^2)).$$

In this case, we set $a = \log(10^6)$ such that g equals $\sqrt{a/\pi}10^{-6} \approx 2\cdot 10^{-6}$ on the unit circle. Thus, we can assume the boundary condition $\boldsymbol{\nu}\cdot\boldsymbol{u} = 0$ is fulfilled approximately. For this example, we choose the Nitsche parameter as $\alpha_{\mathrm{N}} = 2^{15}$ as in [HLS22]. We measure the error of the discretization methods in the $\|\cdot\|_{\boldsymbol{X}}$-norm, which we interpret for the $H(\operatorname{div})$-conforming DG and full DG methods in a broken way, i.e. for $\boldsymbol{u}_n \in \boldsymbol{X}_n \not\subset \boldsymbol{X}$, we interpret $\|\boldsymbol{u}_n\|_{\boldsymbol{X}} = \|\boldsymbol{u}_n\|_{\boldsymbol{X},\mathcal{T}_n}$ with $\|\cdot\|_{\boldsymbol{X},\mathcal{T}_n}$ as in (6.15). We note that we have $\|\cdot\|_{\boldsymbol{X},\mathcal{T}_n} \leq \|\cdot\|_{\boldsymbol{X}_n}$ for the $\boldsymbol{X}_n$-norms defined in Sections 5.2 and 6.1. Figure 7.1 displays the error in the $\|\cdot\|_{\boldsymbol{X}}$ norm for the H^1-conforming-, the $H(\operatorname{div})$-conforming discontinuous Galerkin-, and the fully discontinuous Galerkin discretization with $\beta = 0$ and $\beta = 1$ for $k \in \{2,3,4,5\}$ and $c_{\boldsymbol{b}} = 0.1$ which leads to a Mach number of $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003$. For the H^1-conforming method, we observe stable convergence of order k for $k \geq 3$, which is in accordance with the theoretical results from [HLS22]. While the error deteriorates for $k = 2$, this is not unexpected since the requirements to satisfy Assumption 5.1 are not satisfied. For the other three methods, we observe stable convergence of order k for all $k \in \{2,3,4,5\}$ as expected from Thm. 5.33 and Thm. 6.38.
For the $H(\operatorname{div})$-conforming DG method, we also consider the second example from [HLS22, Sec. 4.2], where the source term $\boldsymbol{f}$ is chosen as

$$\boldsymbol{f} = (-i\omega + \partial_{\boldsymbol{b}})\begin{pmatrix} g \\ 0 \end{pmatrix}. \tag{7.6}$$

Then we consider the convergence of the methods against a reference solution computed with the respective method with polynomial degree $k = 5$ and mesh size $h = 2^{-5}$. Furthermore, we choose the Nitsche parameter $\alpha_{\mathrm{N}} = 2^{15}$ as in [HLS22]. Figure 7.2 displays the reference

[2] Unless explicitly specified otherwise, we consider the lifting versions of the $H(\operatorname{div})$-conforming DG and the full DG methods.

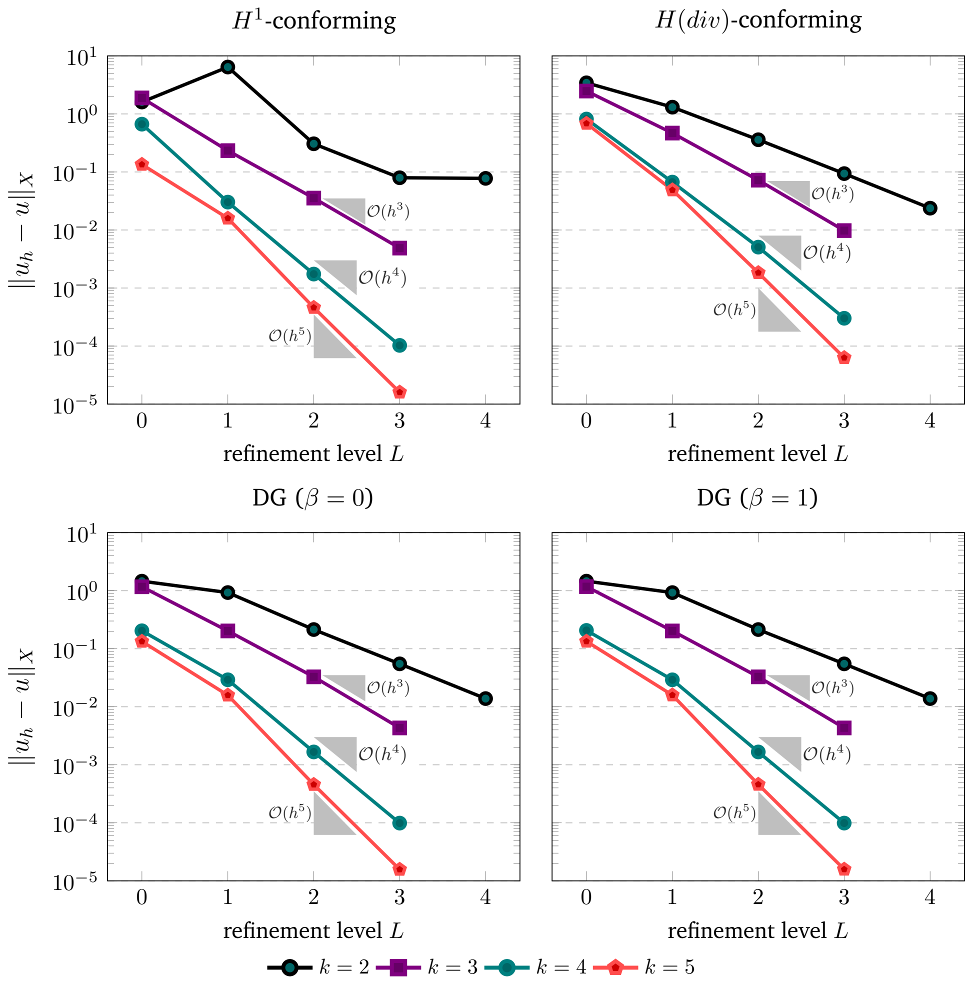


Figure 7.1: Comparison of the H^1-conforming method, the $H(\mathrm{div})$-conforming DG method, and the full DG method with $\beta = 0$ and $\beta = 1$ for Mach number $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003$ and polynomial degree $k \in \{2, 3, 4, 5\}$.

solutions computed with the $H(\operatorname{div})$-conforming DG method for $c_{\boldsymbol{b}} = 0.1$ and $c_{\boldsymbol{b}} = 0.4$ which corresponds to the Mach numbers $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003$ and $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.053$. Furthermore, Fig. 7.3 displays the error in the $\|\cdot\|_{\boldsymbol{X}}$-norm for the H^1-conforming and the $H(\operatorname{div})$-conforming DG method for Mach numbers $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003, 0.013, 0.03, 0.053$. We observe only slight differences between both methods. In particular, we observe a loss of convergence for both methods for increasing Mach number. Due to the high computational costs associated with computing the reference solution, similar experiments with the fully discontinuous Galerkin methods are left for future work.

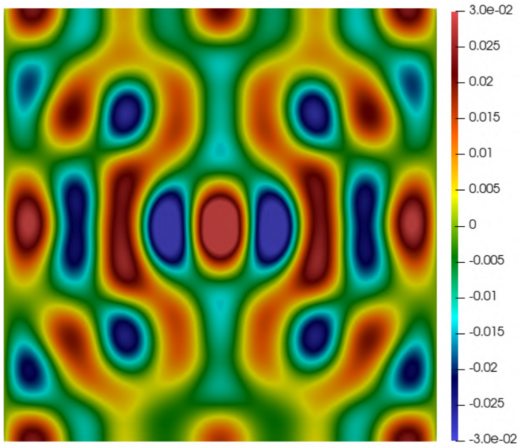


Figure 7.2: Real part of the first entry of the reference solutions for $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003$ (left) and $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.053$ (right) computed with the $H(\operatorname{div})$-conforming DG method with $k = 5$ and $h = 2^{-5}$.

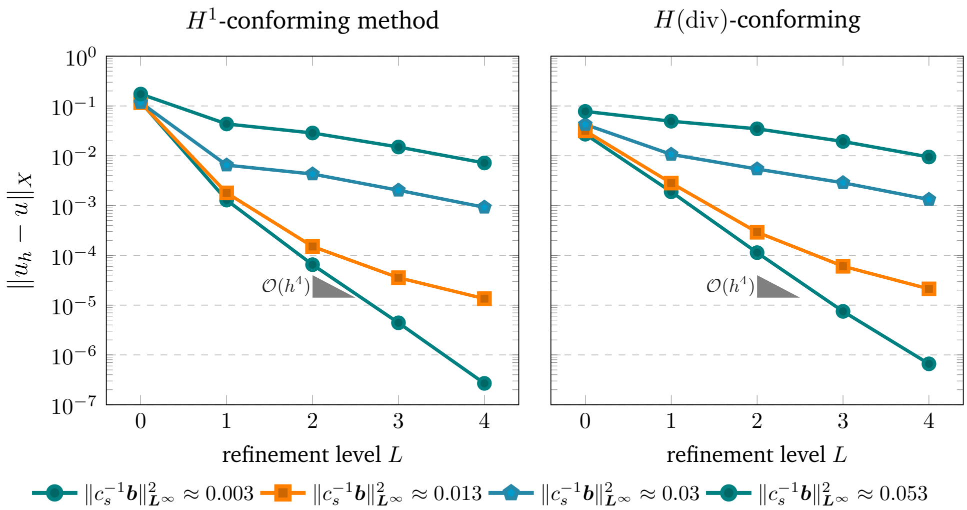


Figure 7.3: $\|\cdot\|_{\boldsymbol{X}}$-norm for the H^1-conforming and the $H(\operatorname{div})$-conforming DG method for Mach numbers $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003, 0.013, 0.03, 0.053$ with $k = 4$ against a reference solution computed with the respective method for $k = 5$ and $h = 2^{-5}$.

7.2.1 Hybridization

In Sections 5.2.4 and 6.4 we briefly introduced hybrid DG versions of the $H(\mathrm{div})$-conforming and the fully discontinuous Galerkin method to reduce the computational costs. While we did not pursue a theoretical analysis of the hybrid versions, we want to explore how they perform numerically. Therefore, we return to the first example presented in the previous section, where the source term $\boldsymbol{f}$ is chosen such that the exact solution is given by (7.5). This allows us to compute the error in the $\|\cdot\|_{\boldsymbol{X}}$ norm. First, we consider the hybrid version of the $H(\mathrm{div})$-conforming DG method introduced in Section 5.2.4. Figure 7.4 displays the error of both, the original DG and the hybrid DG, method and the number of non-zero entries of the associated system matrices as a measure of sparsity. We observe that the hybrid version of the discretization reduces the number of non-zero entries and therefore the computational effort significantly while achieving the same accuracy.

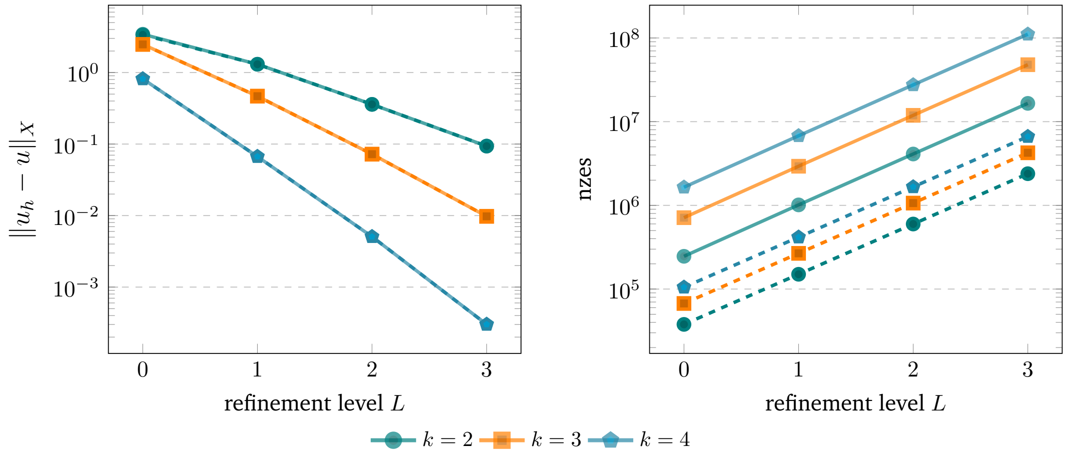


Figure 7.4: $\|\cdot\|_{\boldsymbol{X}}$-error of the $H(\mathrm{div})$-conforming DG- and the hybrid $H(\mathrm{div})$-conforming DG method (dashed) for $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003$ and polynomial degrees $k \in \{2, 3, 4\}$ (left) and the non-zero entries of the associated system matrices (right).

Now, we also want to consider a hybrid version of the fully discontinuous Galerkin method, as briefly introduced in Section 6.4. To this end, we make the same comparison as in Fig. 7.4 for the fully discontinuous case in Fig. 7.5 for $\beta = 0$ and in Fig. 7.6 for $\beta = 1$. We fix the stabilization parameter $\alpha_{\boldsymbol{\nu}} = 1000 \cdot k^2$ for both methods. For $\beta = 0$, the error of the hybrid method agrees with the error of the DG method except for $k = 4$ at the last refinement level, where the chosen stabilization parameter $\alpha_{\boldsymbol{\nu}}$ might be too small to ensure the stability of the HDG method, cf. also the discussion in Section 7.2.2. As before, we observe a significant reduction in the nonzero entries of the matrix, which improves the computational feasibility. For $\beta = 1$, we achieve the same errors with the hybrid DG method as with the DG method, while reducing the number of nonzero entries. In comparison with the casee where $\beta = 0$, the reduction is less pronounced; nevertheless, the reduction is still significant.

Altogether, we observe that the hybrid versions of the $H(\mathrm{div})$-conforming DG and the fully discontinuous Galerkin method reduce the computational costs significantly while achieving the same accuracy as the original methods. Therefore, hybridization seems to be a promising approach to reduce the computational complexity of the methods making them more feasible for physically relevant applications. For the fully discontinuous method with $\beta = 0$, the stabilization parameter $\alpha_{\boldsymbol{\nu}}$ might have to be chosen with special care to ensure that the method is stable.

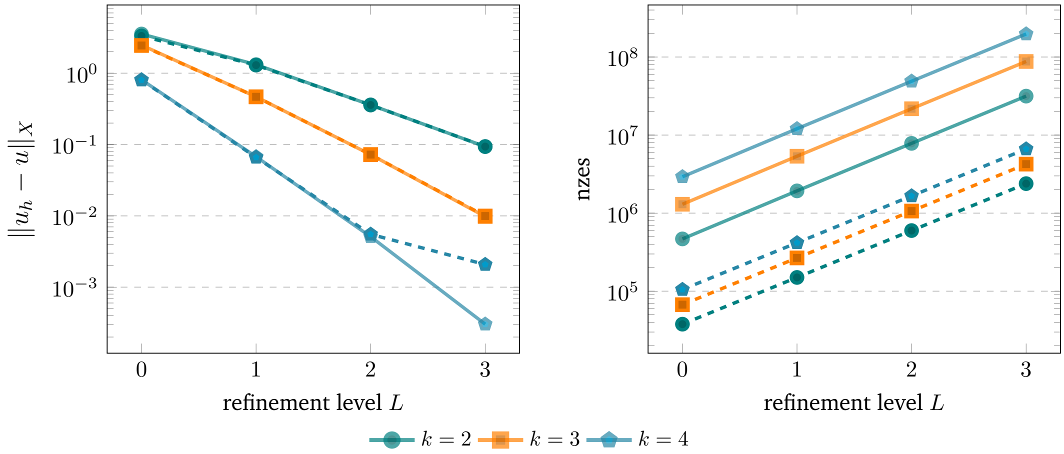


Figure 7.5: $\|\cdot\|_{\boldsymbol{X}}$-error of the fully discontinuous Galerkin- and the hybrid fully discontinuous Galerkin method with $\beta = 0$ (dashed) and $\alpha_{\boldsymbol{\nu}} = 1000 \cdot k^2$ for $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \approx 0.003$ and polynomial degrees $k \in \{2, 3, 4\}$ (left) and the the non-zero entries of the associated system matrices (right).

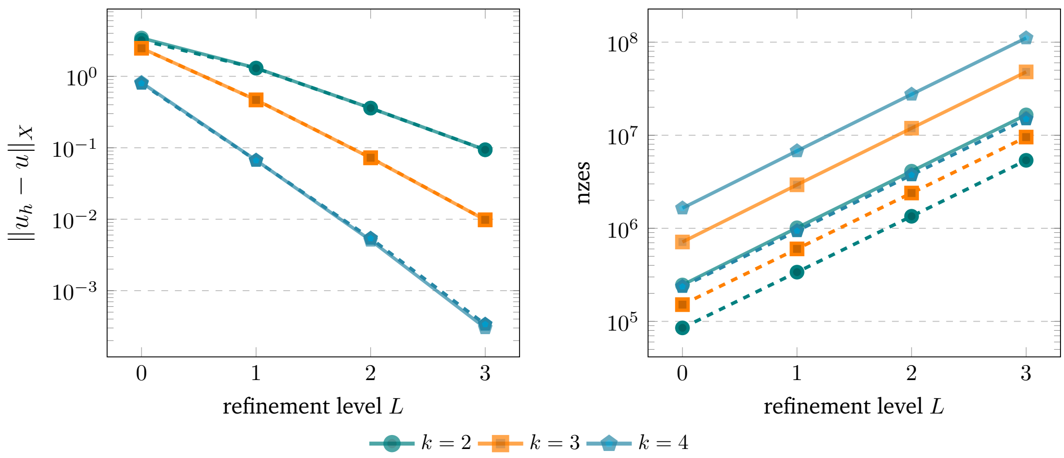


Figure 7.6: $\|\cdot\|_{\boldsymbol{X}}$-error of the fully discontinuous Galerkin- and the hybrid fully discontinuous Galerkin method with $\beta = 1$ (dashed) and $\alpha_{\boldsymbol{\nu}} = 1000 \cdot k^2$ for $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \approx 0.003$ and polynomial degrees $k \in \{2, 3, 4\}$ (left) and the the non-zero entries of the associated system matrices (right).

7.2.2 The role of the stabilization term s_n^β

For the fully discontinuous Galerkin method we introduced a stabilization term s_n^β defined by

$$s_n^\beta(\boldsymbol{u}_n, \boldsymbol{u}_n') := \langle c_s^2 \rho \frac{\alpha_{\boldsymbol{\nu}}}{h} [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, [\![\boldsymbol{u}_n']\!]_{\boldsymbol{\nu}} \rangle_{\mathcal{F}_n} - \beta \langle c_s^2 \rho R_n^{l_{\boldsymbol{\nu}}}(\boldsymbol{u}_n), R_n^{l_{\boldsymbol{\nu}}}(\boldsymbol{u}_n') \rangle,$$

where $\beta \in \{0, 1\}$. For the analysis in Chapter 6 we required that $\alpha_{\boldsymbol{\nu}} > 0$ is chosen large enough to prove that the method is stable. Here, we want to explore numerically what happens if we choose $\alpha_{\boldsymbol{\nu}} = \beta = 0$, i.e. if we do not add any stabilization term. Again, we consider the exact solution (7.5) and measure the error in the $\|\cdot\|_{\boldsymbol{X}}$-norm. Figure 7.7 displays the errors of the fully discontinuous method without the stabilization term s_n^β and its hybrid version. We observe that the DG method still seems to be stable and converges with the expected order of k for $k \in \{2, 3, 4\}$, while the hybrid DG version seems to be unstable for $k \geq 3$.

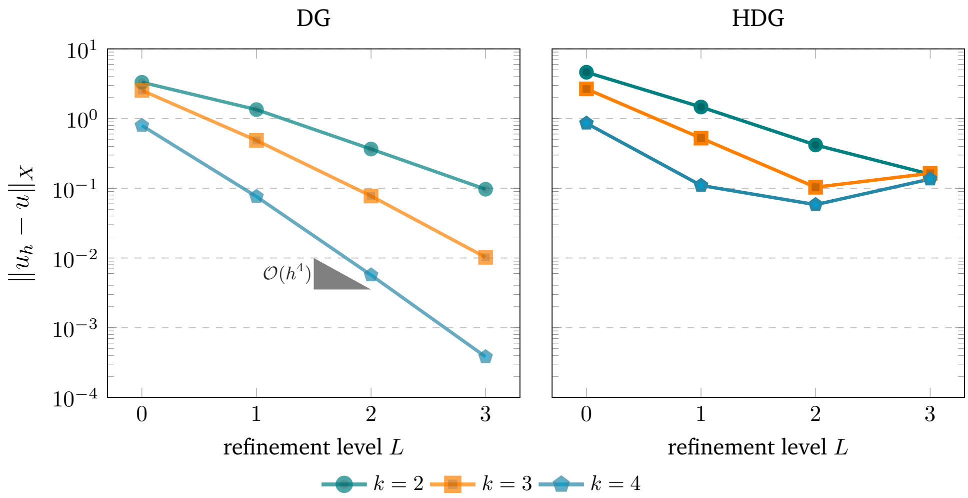


Figure 7.7: $\|\cdot\|_{\boldsymbol{X}}$-error for the fully discontinuous and the hybrid fully discontinuous method with $\alpha_{\boldsymbol{\nu}} = \beta = 0$ for $k \in \{2, 3, 4\}$ and $\|c_s^{-1} \boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \approx 0.003$.

A possible reason for the instability of the hybrid DG method could be that the kernel of the lifting operator is non-trivial. Let us recall the definition of the hybrid local lifting operator from Section 6.4. For each element $\tau \in \mathcal{T}_n$, we define $\underline{r}_n^{\partial\tau}$ as the solution to

$$\langle \underline{r}_n^{\partial\tau} \boldsymbol{u}_n, \psi_n \rangle = -\langle [\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}}, \psi_n \rangle_{\partial\tau} \quad \forall \psi_n \in Q_n = \mathbb{P}^{l_{\boldsymbol{\nu}}}(\tau).$$

Then, if $\boldsymbol{u}_n \in [\mathbb{P}^{k,d}(\mathcal{T}_n)]^d$, we have that $[\![\boldsymbol{u}_n]\!]_{\boldsymbol{\nu}} \in \mathcal{P}^k(\partial\tau)$, and thus $\psi_n \in \mathcal{P}^{l_{\boldsymbol{\nu}}}(\tau)$ might not have enough degrees of freedom on $\partial\tau$ with $l_{\boldsymbol{\nu}} = k$ to ensure that the kernel of the lifting operator is trivial, cf., Fig. 7.8. It is not quite clear why this issue is not present for the DG method, but it could be related to the fact that the definition of the DG lifting operator involves the average of ψ_n. Thus, it might be possible that we have contributions from neighboring elements that prevent a non-trivial kernel. In Fig. 7.9 we compare the DG and the HDG method without the stabilization term s_n^β as in Fig. 7.7, but with an increased polynomial degree of the scalar lifting variable to $l_{\boldsymbol{\nu}} = k + 4$. We observe that the hybrid DG method seems to be more stable for $k = 3$ and only unstable for $k = 4$ at the last refinement level. This might be the case because the higher polynomial degree of the lifting variable prevents a non-trivial kernel. Let us stress that we do not recommend this modification because a higher polynomial degree of the lifting variable increases the computational costs by increasing the dimension of Q_n.

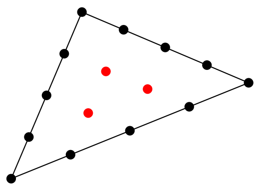


Figure 7.8: Degrees of freedom of $\psi_n \in \mathcal{P}^4(\tau)$. In total, we have 15 degrees of freedom, but only 12 of them are associated with $\partial\tau$.

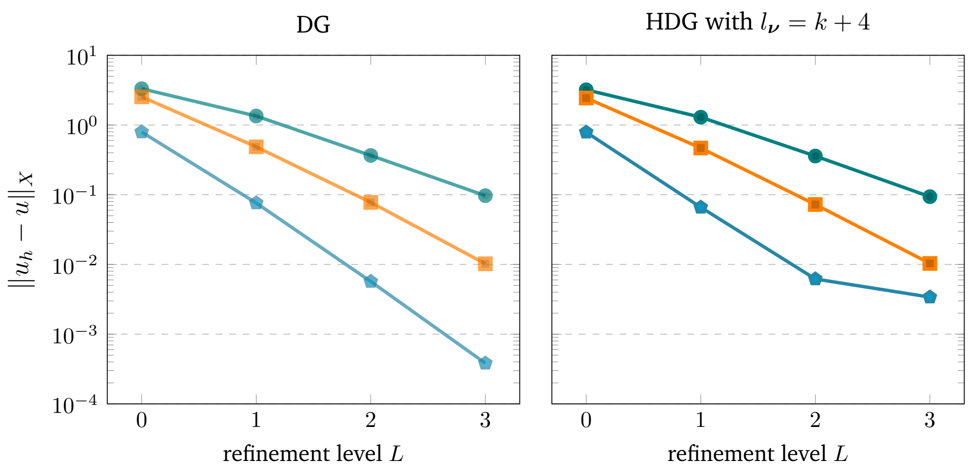


Figure 7.9: $\|\cdot\|_{\boldsymbol{X}}$-error of the DG method (left) and the HDG method (right) without the stabilization term s_n^β and increase polynomial degree $l_{\boldsymbol{\nu}} = k + 4$ for the hybrid lifting operator for $k \in \{2, 3, 4\}$ and $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \approx 0.003$.

Altogether, these results show that the stabilization term s_n^β is indeed necessary to ensure that the method is stable. While the DG method performs adequately without the stabilization term for this example, it is not clear that this is the case for other scenarios. Furthermore, the HDG method seems to be more reliant on the stabilization term s_n^β, and does not appear to be stable without it.

7.2.3 The choice of stabilization parameters for the SIP methods

The main motivation to introduce the lifting operator $\boldsymbol{R}_n^{l_b}$ is to avoid having to choose the stabilization parameter $\alpha_{\boldsymbol{b}}$ that would occur in a symmetric interior penalty formulation for the $H(\mathrm{div})$-conforming and fully discontinuous Galerkin methods, as this choice would lead to a more restrictive assumption on the Mach number $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty}$. However, the implementation of the lifting operator is computationally expensive, and thus, we want to investigate how the symmetric interior penalty variants of both methods perform computationally for different stabilization parameters. We consider the first example from the previous sections where the exact solution was given by (7.5), which allows us to compute the error in the $\|\cdot\|_{\boldsymbol{X}}$-norm. We compare the errors of the SIP versions with the error of the H^1-conforming discretization which does not involve stabilization parameters. To ensure that the H^1-conforming method is stable without using special meshes, we fix the polynomial degree to $k = 4$. Figure 7.10 displays the discretization errors of the $H(\mathrm{div})$-conforming SIP DG method for $c_{\boldsymbol{b}} \in \{0.1, 0.2, 0.5\}$ which lead to Mach numbers $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003, 0.013, 0.083$. To achieve a similar order of convergence as the H^1-conforming method, the stabilization parameter $\alpha_{\boldsymbol{b}}$ has to be chosen quite large[3], e.g. $\alpha_{\boldsymbol{b}} \geq 10^4 \cdot k^2$, which can become problematic as the condition numbers of the system matrices increase with the stabilization parameter. We further observe that for the higher Mach numbers, the error of the method implemented with a lower stabilization parameter seems to be worse. Note, however, that the influence of the background flow $\boldsymbol{b}$ might be mitigated because the right-hand side $\boldsymbol{f}$ is constructed explicitly such that the exact solution is given by (7.5) and therefore also changes with $c_{\boldsymbol{b}}$.

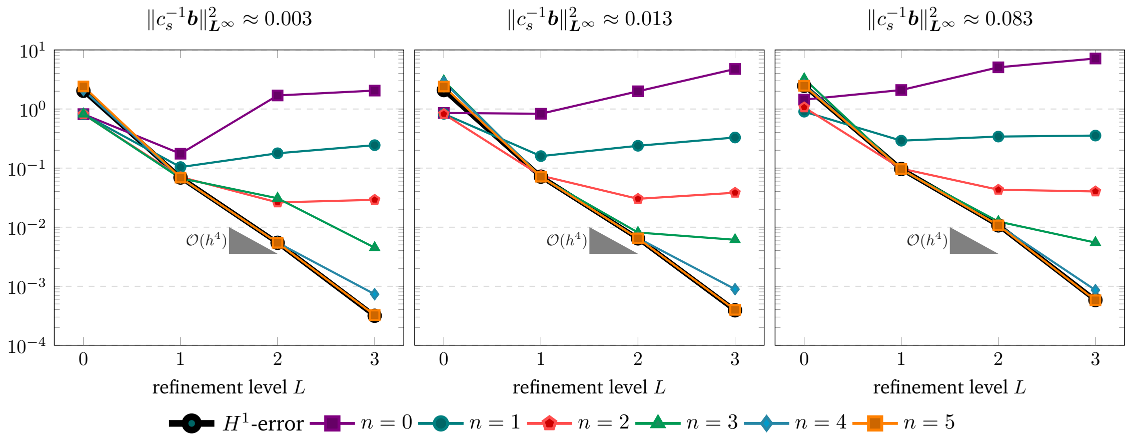


Figure 7.10: Error of the $H(\mathrm{div})$-SIP method for stabilization parameters $\alpha_{\boldsymbol{b}} = 10^n \cdot k^2$, $n \in \{0, 1, 2, 3, 4, 5\}$, and $c_{\boldsymbol{b}} \in \{0.1, 0.2, 0.5\}$ compared to the error of the H^1-conforming discretization for polynomial degree $k = 4$.

Figures 7.11 and 7.12 show the error of the fully discontinuous SIP discretization for $\alpha_{\boldsymbol{\nu}} \in \{1, 10, 100, 1000\}$ with fixed stabilization parameter $\alpha_{\boldsymbol{b}} = 10^4 \cdot k^2$ and $\alpha_{\boldsymbol{b}} = 10^5 \cdot k^2$, respectively.

[3]The required stabilization parameters are unusually large in fact. It is not clear why this is the case; it might be a consequence of the specific choice for $\boldsymbol{b}$ or indicate an error in the implementation.

As before, we consider $c_{\boldsymbol{b}} \in \{0.1, 0.2, 0.5\}$. In contrast to the previous example, the influence of the Mach number is more profound in this example. For $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003$ and $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.053$, the error with $\alpha_{\boldsymbol{\nu}} = 1000 \cdot k^2$ is close to the error of the H^1-conforming method, while the lower choices for $\alpha_{\boldsymbol{\nu}}$ lead to instability for higher refinement levels. In contrast, for $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.083$, all choices for $\alpha_{\boldsymbol{\nu}}$ perform poorly at the last refinement level. For $\alpha_{\boldsymbol{b}} = 10^5 \cdot k^2$ the results differ significantly. For $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.003$ and $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.053$, none of the choices for $\alpha_{\boldsymbol{\nu}}$ leads to an error approaching the error of the H^1-conforming method, and for the latter Mach number the choice $\alpha_{\boldsymbol{\nu}} = 1000 \cdot k^2$ yields the worst results. For $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.083$, the choice $\alpha_{\boldsymbol{\nu}} = 1 \cdot k^2$ yields the best results which closely match the error of the H^1-conforming method. We note that these effects might be severely influenced by the condition numbers of the system matrices which increase with the stabilization parameters.

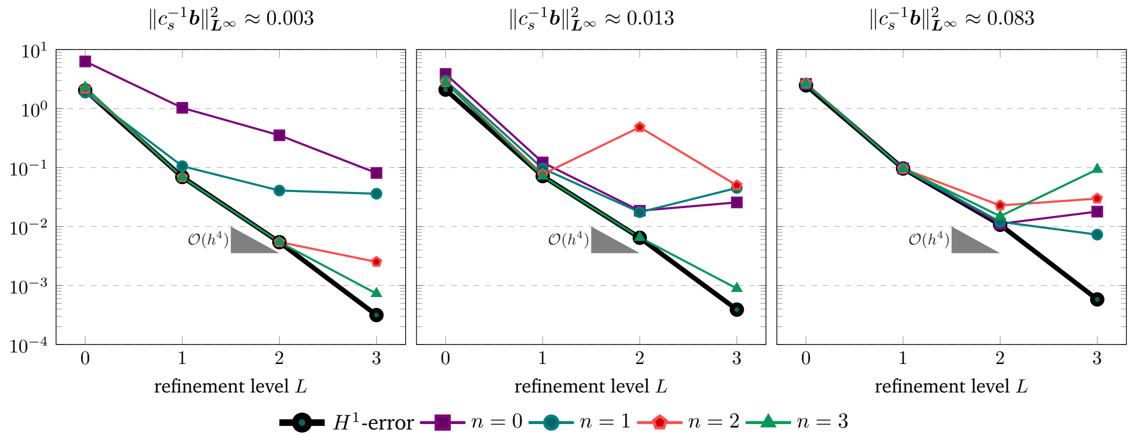


Figure 7.11: Error of the fully discontinuous SIP method for stabilization parameters $\alpha_{\boldsymbol{\nu}} = 10^n \cdot k^2$, $n \in \{0, 1, 2, 3\}$ and $\alpha_{\boldsymbol{b}} = 10000 \cdot k^2$, and $c_{\boldsymbol{b}} \in \{0.1, 0.2, 0.5\}$ compared to the error of the H^1-conforming discretization for polynomial degree $k = 4$.

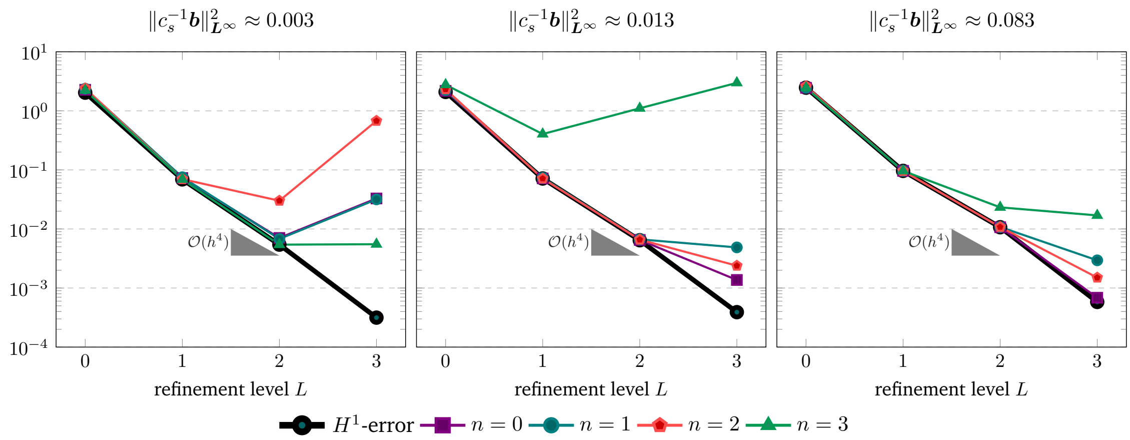


Figure 7.12: Error of the fully discontinuous SIP method for stabilization parameters $\alpha_{\boldsymbol{\nu}} = 10^n \cdot k^2$, $n \in \{0, 1, 2, 3\}$ and $\alpha_{\boldsymbol{b}} = 100000 \cdot k^2$, and $c_{\boldsymbol{b}} \in \{0.1, 0.2, 0.5\}$ compared to the error of the H^1-conforming discretization for polynomial degree $k = 4$.

We consider a final example with the fully discontinuous SIP methods with $\alpha_{\boldsymbol{b}} = 10^n \cdot k^2$ for $n \in \{0, 1, 2, 3, 4, 5\}$ and fixed $\alpha_{\boldsymbol{\nu}} = 100 \cdot k^2$ in Fig. 7.13. As before, the results vary with the Mach number $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty}$. For example $\alpha_{\boldsymbol{b}} = 10^5 \cdot k^2$ yields the best results for

$\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \approx 0.013$ and $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \approx 0.083$, but is not stable for $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \approx 0.003$.

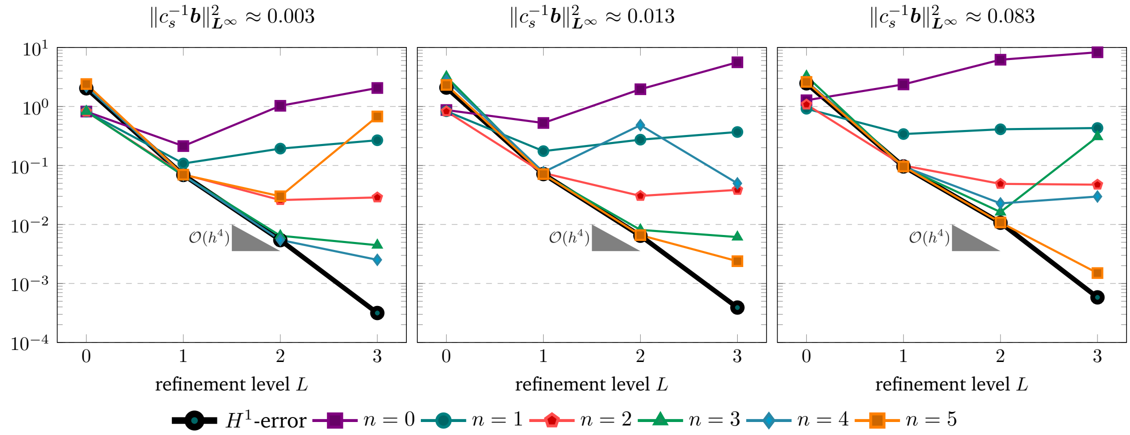


Figure 7.13: Error of the fully discontinuous SIP method for stabilization parameters $\alpha_{\boldsymbol{b}} = 10^n \cdot k^2$, $n \in \{0, 1, 2, 3, 4, 5\}$ and $\alpha_{\boldsymbol{\nu}} = 100 \cdot k^2$, and $c_{\boldsymbol{b}} \in \{0.1, 0.2, 0.5\}$ compared to the error of the H^1-conforming discretization for polynomial degree $k = 4$.

In summary, we notice that the right choice of the stabilization parameters $\alpha_{\boldsymbol{b}}$ and $\alpha_{\boldsymbol{\nu}}$ is very delicate and seems to be dependent on the Mach number. We seem to require unusually large stabilization parameters $\alpha_{\boldsymbol{b}}$ to ensure the stability of the $H(\mathrm{div})$-conforming DG and full DG SIP methods which can lead to a significant increase in the condition number of the system matrices. We note that this phenomenon might be specific to the example under consideration. Additionally, we observe that choosing a combination of $\alpha_{\boldsymbol{b}}$ and $\alpha_{\boldsymbol{\nu}}$ can be tricky, in particular since the Mach number seems to have a significant influence on the results.

7.3 Sun parameters in 2D

In this section, we want to consider a two-dimensional computational example similar to [CD18, Sec. 6] which is motivated by the study of the Sun. In particular, we will use realistic values for the density ρ, the sound-speed c_s and the pressure p provided by the `modelS` [Chr+96] of the solar interior. For the computational examples, we will use a mesh that is fitted to the drastic change of the density and the sound speed in the convection zone. Figure 7.14 visualizes these changes in magnitude and displays an example of the adapted mesh.
We choose the parameters

$$\omega = 2\pi f_{\mathrm{hz}}, \quad \gamma = \frac{\omega}{100},$$

where the frequency f_{hz} is chosen as 3mHz. Furthermore, with $R = 1.0007126$ being the radius of the sun, the background flow is chosen as

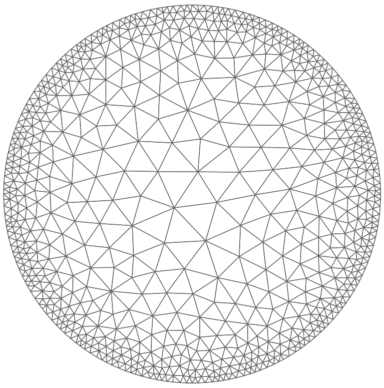
$$\boldsymbol{b} = \frac{c_{\boldsymbol{b}}}{R} c_s \begin{pmatrix} -y \\ x \end{pmatrix}. \tag{7.7}$$

We note that by the construction of the background flow, we have that $\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 \approx c_{\boldsymbol{b}}^2$. The source term $\boldsymbol{f}$ is chosen as

$$\boldsymbol{f} = (-i\omega + \partial_{\boldsymbol{b}}) \begin{pmatrix} g \\ 0 \end{pmatrix}$$

where $g(x, y)$ is a gaussian located at $(0.5, 0.5)$ with radius of distribution equal to 0.1, i.e.

$$g(x, y) = \sqrt{a/\pi} \exp(-a((x - 0.5)^2 + (y - 0.5)^2)), \quad a = \log(10^6)/0.1^2.$$



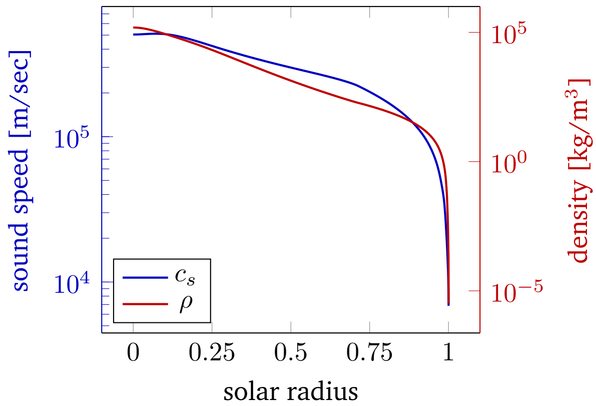


Figure 7.14: Mesh (on the left) fitted to the drastic changes in the density and sound speed (on the right) towards the boundary with mesh size 0.25 in the interior and mesh size 0.025 near the boundary.

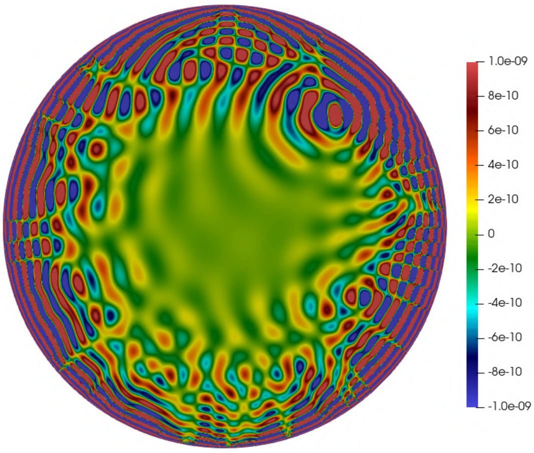
Furthermore, we impose natural boundary conditions of the form $\operatorname{div} \boldsymbol{u} = 0$.
For the implementation of the $H(\operatorname{div})$-conforming DG and the full DG method we use the hybrid DG versions to reduce the computational costs. Figure 7.15 shows the real part of the first entry of the solutions computed with the three methods for $c_{\boldsymbol{b}} = 0.2$, i.e. $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.04$, and $k = 6$. For these parameters, the results are indistinguishable which indicates that all three methods perform reasonably well. In Fig. 7.16 and 7.17, the results for $c_{\boldsymbol{b}} = 1.0$ and $c_{\boldsymbol{b}} = 1.5$ are shown, i.e. for $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 1.0$ and $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 2.25$, which means that the flow is not transsonic anymore and the smallness assumption on the Mach number is violated. In these cases, we observe differences between the H^1-conforming and the other three methods, since both, the $H(\operatorname{div})$-conforming DG and the fully discontinuous DG methods, seem to allow for more oscillation than the H^1-conforming method. This is not necessarily unexpected, in particular since the three nonconforming methods all involve the lifting operator for the $[\![\cdot]\!]_{\boldsymbol{b}}$-jump which penalizes oscillations weaker than a corresponding SIP method would. Finally, let us stress that this is merely a visual comparison and since we have no knowledge of the exact solution, we cannot make precise statements about the accuracy of the methods.

7.4 Towards Computational Helioseismology

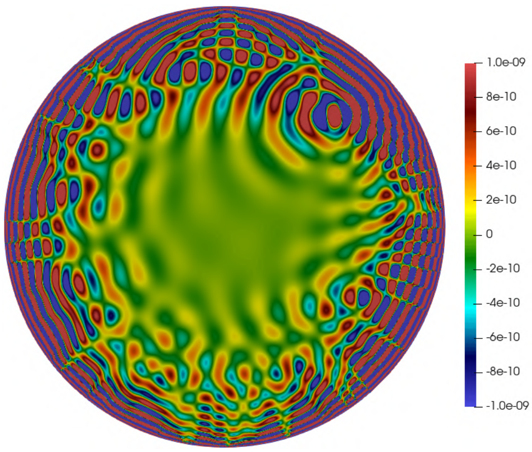
To conclude this section, we want to return to the physical motivation behind studying Galbrun's equation – computational Helioseismology. We will not perform numerical experiments in this Section but rather discuss some aspects of solving the equation in a more realistic setting.

7.4.1 Reducing dimensions for axisymmetrical geometries

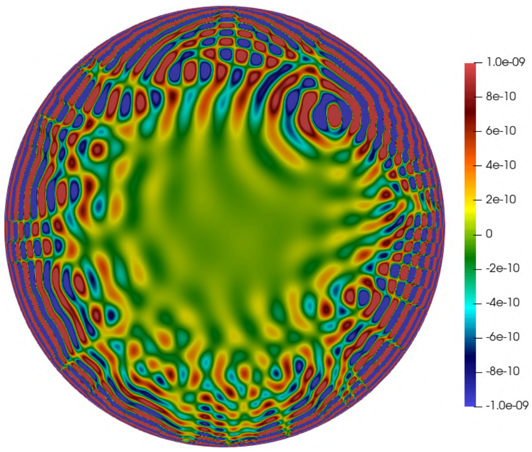
Considering a physically relevant setting, we have to deal with the significant increase in computational costs associated with solving a three-dimensional problem. It is well-known that increasing the dimension increases the computational costs, cf. Fig. 7.18. In particular since solving Galbrun's equation is already computationally expensive in 2D, this can become problematic. In the following, we will briefly discuss how the axisymmetrical geometry of the sun can be exploited to reduce the dimension of the problem.
It has been argued in [Giz+17] that the sun can be considered as axisymmetrical around the z-axis. In spherical coordinates, we can associate a point $\boldsymbol{x} = (r, \theta, \varphi)$, where r is the radius,



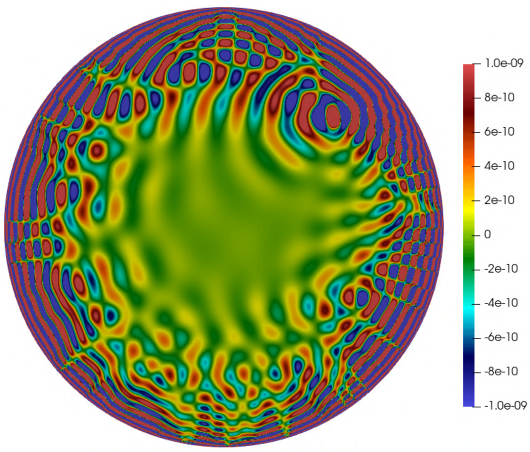
(a) H^1-conforming



(b) Hybrid $H(\mathrm{div})$-conforming

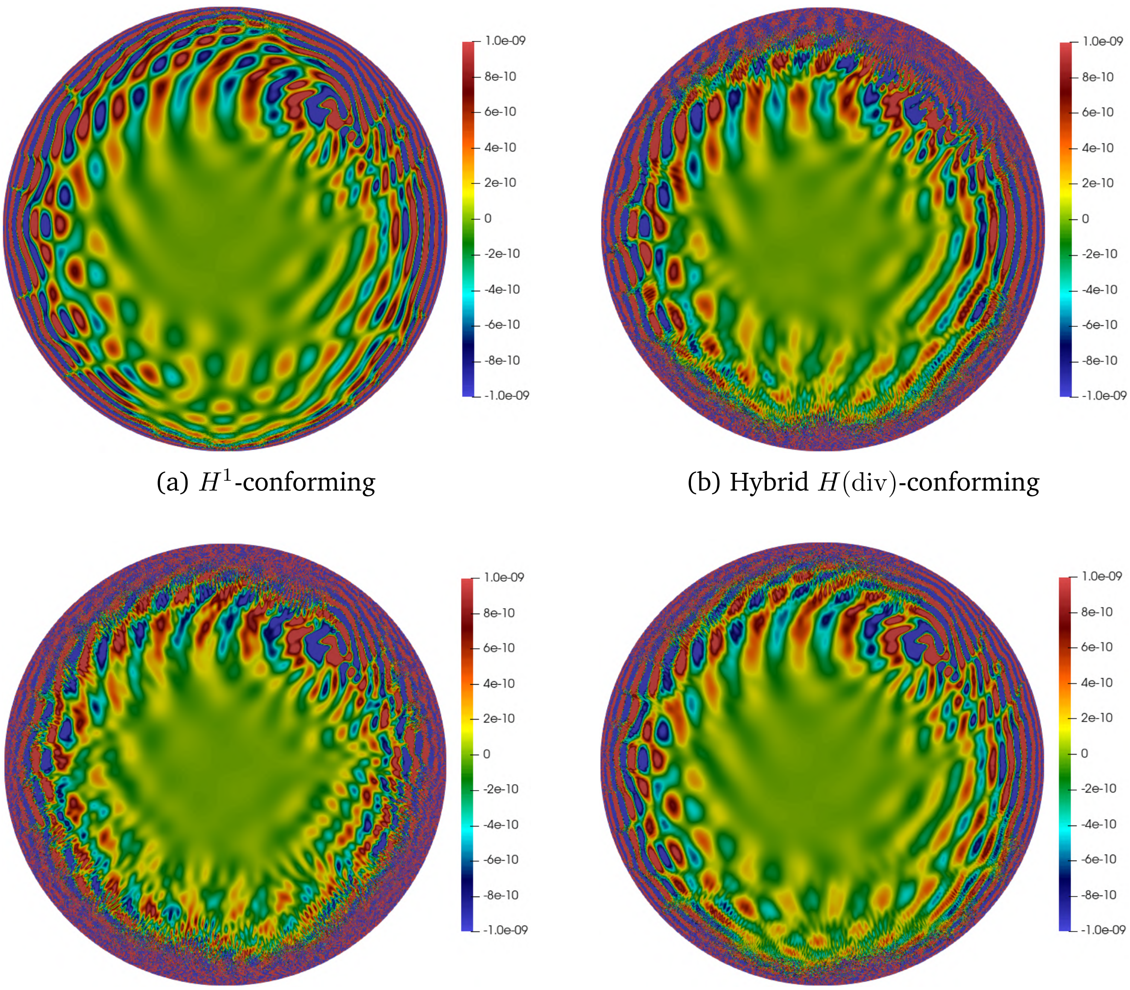


(c) Hybrid DG with $\beta = 0$



(d) Hybrid DG with $\beta = 1$

Figure 7.15: Real part of the first entry of solutions computed with the H^1-conforming, the hybrid $H(\mathrm{div})$-conforming DG, and the hybrid fully discontinuous Galerkin-method with $\beta \in \{0, 1\}$ method for $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 0.04$ and $k = 6$ on a mesh with maximal mesh size 0.025 in the interior and 0.005 on the boundary.



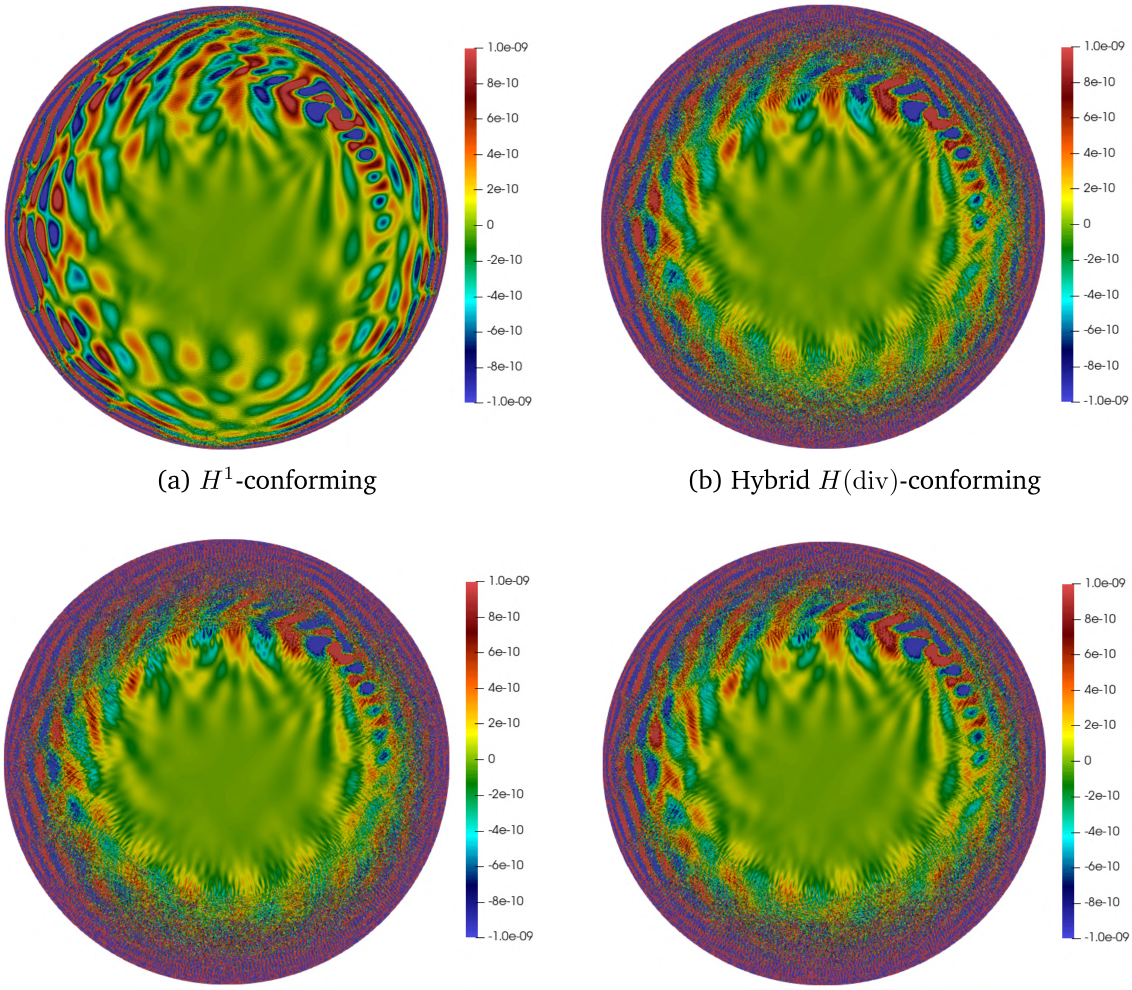
(a) H^1-conforming

(b) Hybrid $H(\mathrm{div})$-conforming

(c) Hybrid DG with $\beta = 0$

(d) Hybrid DG with $\beta = 1$

Figure 7.16: Real part of the first entry of solutions computed with the H^1-conforming, the hybrid $H(\mathrm{div})$-conforming DG, and the hybrid fully discontinuous Galerkin-method with $\beta \in \{0, 1\}$ method for $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 1.0$ and $k = 6$ on a mesh with maximal mesh size 0.025 in the interior and 0.005 on the boundary.



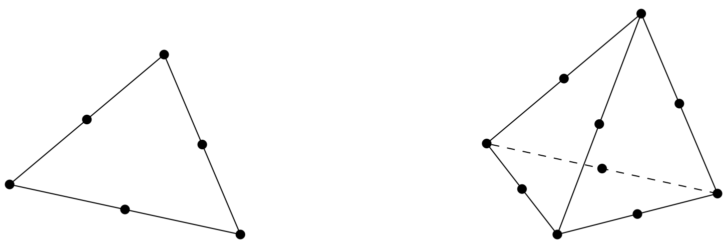
(a) H^1-conforming

(b) Hybrid $H(\mathrm{div})$-conforming

(c) Hybrid DG with $\beta = 0$

(d) Hybrid DG with $\beta = 1$

Figure 7.17: Real part of the first entry of solutions computed with the H^1-conforming, the hybrid $H(\mathrm{div})$-conforming DG, and the hybrid fully discontinuous Galerkin-method with $\beta \in \{0, 1\}$ method for $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} \approx 2.25$ and $k = 6$ on a mesh with maximal mesh size 0.025 in the interior and 0.005 on the boundary.



(a) 2D

(b) 3D

Figure 7.18: Degrees of freedom for $\mathcal{P}^2$-elements on simplices in 2D and 3D for continuous Galerkin methods. In 2D, we have $\frac{1}{2}(k+1)(k+2)$ degrees of freedom per element, while in 3D we have $\frac{1}{6}(k+1)(k+2)(k+3)$ degrees of freedom per element, cf. [EG21a, Sec. 7.3].

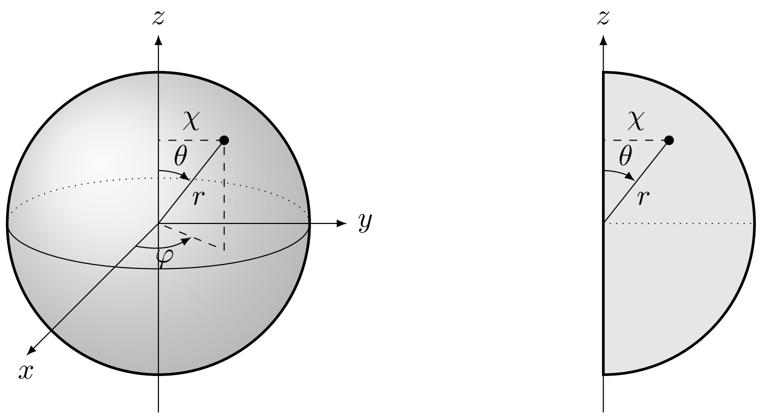


Figure 7.19: A three-dimensional axisymmetrical geometry (on the left) and an associated two-dimensional generating section (on the right).

θ the polar angle and φ the azimuthal angle. Then, the assumption of an axisymmetrical geometry means that the solution is independent of φ which means that we can consider a two-dimensional generating section of the original geometry, cf., Fig. 7.19. In particular, this potentially allows us to decompose the solution of the three-dimensional problem into a set of independent two-dimensional problems. The authors refer to this setup as a 2.5D problem. For a more detailed explanation of the reduction of the dimension in the case of Galbrun's equation, we refer to [CD18, Appendix C].

7.4.2 Wave propagation in the atmosphere

So far, we have only considered Dirichlet boundary conditions of the form $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\mathcal{O}$. However, this boundary condition neglects the propagation of waves in the solar atmosphere. In [Hal22], the following coupled system that takes wave propagation in the atmosphere into account was derived and analyzed:

$$\begin{aligned}
-\nabla(\rho c_s^2 \operatorname{div} \boldsymbol{u} + \nabla p \cdot \boldsymbol{u}) + \nabla p \operatorname{div} \boldsymbol{u} + \operatorname{Hess}(p)\boldsymbol{u}& \\
-\rho \operatorname{Hess}(\phi)\boldsymbol{u} - \rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u} - i\omega\gamma\rho\boldsymbol{u} = \boldsymbol{f} \quad & \text{in } B_{r_2}, && (7.8a)\\
\operatorname{div}\left(\frac{e^{2\eta}}{\rho}(\underline{\underline{m_2}} + i\omega\gamma)^{-1}\nabla v\right) - \frac{e^{2\eta}}{c_s^2\rho} v = 0 \quad & \text{in } B_{r_2}^c, && (7.8b)\\
\boldsymbol{\nu} \cdot e^{\eta}\rho^{-1}(\underline{\underline{m_2}} + i\omega\gamma)^{-1}\nabla v = \boldsymbol{\nu} \cdot \boldsymbol{u} \quad & \text{on } \partial B_{r_2}, && (7.8c)\\
\operatorname{div}\left(e^{\eta}\rho^{-1}(\underline{\underline{m_2}} + i\omega\gamma)^{-1}\nabla v\right) = \operatorname{div} \boldsymbol{u} \quad & \text{on } \partial B_{r_2}. && (7.8d)
\end{aligned}$$

Here B_{r_2} is a ball of radius r_s such that supp$\boldsymbol{f} \subset B_{r_2}$ and $B_{r_2}^c$ is the exterior domain. We further assume that in $B_{r_2}^c$ the parameters c_s, ρ, and p only depend on the radial coordinate $r = |\boldsymbol{x}|$. Furthermore, we define $\underline{\underline{m_2}} := \underline{\underline{m_1}} + (\omega + i\Omega\times)(\omega + i\Omega\times)$ with $\underline{\underline{m_1}} := -\rho^{-1}(\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi) - c_s^2\rho\boldsymbol{q}\boldsymbol{q}^T)$ and $\eta(r) := \int_{r_2}^{r} \frac{\partial_r p(r')}{c_s^2(r')}\rho(r')dr'$. Under the assumption that

$$\|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 < \frac{1}{1 + \tan^2\theta},$$

where θ is defined similarly as in Chapters 4, 5, and 6, the system (7.8) admits a unique solution $(\boldsymbol{u}, v)$ [Hal22, Prop. 5.4]. The implementation of the system, however, remains an open problem for research. To treat wave equations on unbounded domains, one usually introduces an artificial boundary on which one has to impose appropriate boundary conditions, so-called *transparent boundary conditions*. Unfortunately, traditional methods to incorporate transparent boundary conditions, for example Perfectly Matched Layers [Ber94], are not equipped to deal with the strongly varying coefficients in the solar atmosphere, cf., Fig 7.14. To overcome this issue, a new technique – *learned infinite elements* – has been developed recently [Pre21; HLP21] which allows for more flexibility. Applying this technique to incorporate wave propagation in the atmosphere through (7.8) is an objective for further research.

Chapter 8

Conclusion

Summary

In the first part of the thesis, we introduced the abstract framework that serves as a basis for our analysis. Specifically, we introduced the notions of (weak) T-coercivity, discrete approximation schemes and (weak) T-compatibility. Subsequently, we applied these concepts to the Helmholtz equation to explain the general ideas of the analysis within a more accessible context. In the second part, we then applied the abstract framework to Galbrun's equation. We began by examining the well-posedness of the continuous problem and evaluated two existing discretization schemes: an H^1-conforming discretization [HLS22] and an $H(\mathrm{div})$-conforming discontinuous Galerkin scheme [Hal23]. In Chapter 6, we then introduced and analyzed a fully discontinuous Galerkin scheme for Galbrun's equation. Building upon the structures established in [Hal23], we demonstrated that the discretization can be interpreted as a discrete approximation scheme. Then we constructed a discrete T_n-operator such that the discrete problem is weakly T-compatible in the sense of Thm. 2.28. To conclude the analysis, we proved optimal order convergence in the $\|\cdot\|_{\boldsymbol{X}}$-norm provided the exact solution is sufficiently regular and the degree of the lifting operators is chosen equal to the polynomial degree of the discretization. Finally, we performed numerical examples to validate the theoretical results. We considered the convergence behavior of the method against a manufactured solution and a reference solution. Furthermore, we considered hybrid versions of the $H(\mathrm{div})$-conforming DG and the full DG method to reduce computational costs, considered an example with physically relevant parameters, and briefly described further computational challenges when moving toward computational Helioseismology.

Outlook

In Section 7.4 we already discussed some challenges that arise when moving toward computational Helioseismology: the computational costs associated with physically realistic simulations and the need for a more sophisticated treatment of physically realistic boundary conditions. Additionally, we already investigated hybrid versions of the $H(\mathrm{div})$-conforming discontinuous Galerkin and the fully discontinuous Galerkin method computationally. One goal for further research is to analyze these methods with the techniques employed in this thesis by incorporating the facet unknowns in the construction of the T-operators. Considering the numerical results in Sections 7.2.1 and 7.2.2, it makes sense to focus on the analysis of the hybridized fully discontinuous Galerkin method to the case where $\beta = 1$, i.e. where we are considering the combination of a lifting method for the convection operator and a symmetric interior penalty method for the diffusion operator. With this combination, we avoid potential issues with the kernel of the scalar lifting operator, while also avoiding stronger restrictions on the Mach number through the lifting operator for the convection term. In view of Section 7.2.3, it might also be worthwhile to investigate the unusually high stabilization parameters of the SIP method further.

CHAPTER A

Appendix

In this appendix, we collect some broader mathematical theory that is employed through the thesis and is generally expected to be known by the reader. We assume that the reader is familiar with the basic concepts of functional analysis, in particular, the notions of Banach-, Hilbert-, L^p-, and Sobolev spaces. The first section deals with fundamental definitions and results from Operator theory. Then, we discuss Fredholm operators and their properties. In the final section, we present general and FEM-specific inequalities, including approximation results.

A.1 Operator theory

In the following sections, we will denote, unless specified otherwise, by X and Y Banach Spaces and by V and W Hilbert spaces over a field $\mathbb{K}$. We note by X', Y', V' and W' their respective dual spaces. Usually, we think of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We call a linear transformation $A : X \to Y$ a *linear operator* from X to Y.

Definition A.1 (Range, kernel and cokernel)**.** Let $A : X \to Y$ be a linear operator. Then we define its *range*, *kernel* and *cokernel* as

$$\begin{aligned} \operatorname{ran}(A) &:= \{Ax : x \in X\} \subseteq Y, \\ \ker(A) &:= \{x \in X : Ax = 0\} \subseteq X, \\ \operatorname{coker}(A) &:= Y / \operatorname{ran}(A) \subseteq Y. \end{aligned}$$

Definition A.2 (Bounded Operators)**.** We call a linear operator $A : X \to Y$ *bounded*, if there exists a constant $C > 0$ such that

$$\|Ax\|_Y \leq C\|x\|_X \text{ for every } x \in X.$$

We denote the space of all bounded linear operators from X to Y by $L(X, Y)$. and define the operator norm of A as the smallest such constant, i.e.

$$\|A\|_{L(X,Y)} := \sup_{x \in X, \|x\|_X \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{x \in X, \|x\|_X = 1} \|Ax\|_Y.$$

In the following, we write $L(X) := L(X, X)$. Note further, that all linear operators on finite-dimensional spaces are bounded.

Definition A.3 (Compact operator)**.** A bounded linear operator $A \in L(X, Y)$ is called A *compact*, if it maps bounded sets in X to precompact sets in Y, i.e. for every bounded set $X' \subset X$, $\overline{A(X')} \subset Y$ is compact.

Lemma A.4. *Let $A \in L(X, Y)$. Then, the following are equivalent*

(i) *A is compact in the sense of definition A.3.*

(ii) *If $(x_n)_{n\in\mathbb{N}} \subset X$ is a bounded sequence, then $(Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y.*

(iii) *The set $\overline{\{Ax : x \in X, \|x\|_X \leq 1\}} \subset Y$ is compact.*

Proof. See [BS18, Lemma 4.2.1] □

We call two an embedding between two Banach spaces X and Y, $X \subset Y$, *compact*, if the embedding operator $\iota : X \to Y$ is a compact operator. The following well-known theorem states that the embedding between certain Sobolev spaces is compact.

Theorem A.5 (Rellich-Kondrachov)**.** *Let $D \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain. Further, let $k > 0$ and $1 \leq p \leq \infty$. Then the following embeddings are compact*

(i) *If $sp \leq d$, $W^{s,p}(D) \hookrightarrow L^q(D)$ for all $1 \leq q \leq \frac{pd}{d-sp}$.*

(ii) *If $sp > d$, $W^{s,p}(D) \hookrightarrow C^0(\overline{D})$.*

(iii) *$W^{s,p}(D) \hookrightarrow W^{s',p}$ for all $s > s'$.*

Proof. See [EG21a, Theorem 2.35] and the references therein. □

Definition A.6. A bounded linear operator $A \in L(X, Y)$ is said to be of *finite rank*, if $\operatorname{ran}(A) \subset Y$ is finite-dimensional.

Definition A.7 (Adjoint operator on Hilbert spaces)**.** Let V and W be Hilbert spaces and $A \in L(V, W)$. We call an operator $A^* \in L(W, V)$ such that

$$(Au, v) = (u, A^* v) \text{ for all } u \in V, v \in W$$

the *adjoint* operator of A. If $V = W$ and $A = A^*$ we call A *self-adjoint* or *symmetric*. We note that $A^{**} = A$.

Lemma A.8. *Let V and W be Hilbert spaces and $A \in L(V, W)$ be a bounded linear operator. Then, we have that*

$$\operatorname{ran}(A)^\perp = \ker(A^*),$$
$$\ker(A)^\perp = \overline{\operatorname{ran}(A^*)}.$$

Theorem A.9 (Riesz representation theorem)**.** *Let V be a Hilbert space and let $F : V \to \mathbb{K}$ be a (anti-)linear map. Then there exists a unique element $u \in V$ such that*

$$F(v) = (u, v)_V \text{ for all } v \in V.$$

Furthermore, there holds

$$\|F\|_{V'} = \|u\|_V$$

Proof. See [BS18, Thm. 1.4.4]. □

The Riesz representation theorem implies the following useful result that we will employ to speak simultaneously of sesquilinear forms and linear operators.

Corollary A.10. *Let V be a Hilbert space and $a : V \times V \to \mathbb{K}$ be a bounded sesquilinear (bilinear) form. Then there exists a unique operator $A \in L(V)$ such that*

$$A(u, v) = (Au, v)_V \text{ for all } u, v \in V.$$

Remark A.11. *Using the Riesz representation theorem A.9, we can show that for any linear operator $A \in L(V, W)$ the adjoint operator from definition A.7 exists and is unique. Thus, it makes sense to speak of "the" adjoint operator.*

Definition A.12 (Projection operator)**.** Let X be a Banach space. We call $P \in L(X)$ a *projection operator* if $P^2 = P$. If V is a Hilbert space, we call P an *orthogonal projection* if P is a projection operator with

$$(Pu, (\mathrm{Id} - P)v)_V = 0 \quad \text{for all } u, v \in V.$$

Recall that we call a vector space Y the direct algebraic sum of subspaces $Y_1, \dots, Y_N \subset Y$ and write

$$Y = \bigoplus_{i=1,\dots,N} Y_i,$$

if each element $y \in Y$ can be uniquely written as $y = \sum_{i=1}^N y_i$, $y_i \in Y_i$ for $i = 1, \dots, N$. In this case, there exists projection operators $P_{Y_i} : Y \to Y_i, y \mapsto y_i$. An algebraic decomposition of a Hilbert space Y is called *topological decomposition*, denoted by $\oplus$, if the associated projection operators P_{Y_i} are continuous, $i = 1, \dots, N$.
The following lemma shows that there is a one-to-one correspondence between direct sums and projection operators.

Lemma A.13. *Let X be a Banach space such that $X = V \oplus W$, i.e. for each $u \in X$ we can write $u = v + w$, $v \in V$, $w \in W$. Set $Pu = v$ for all $u \in X$. Then we have that $P \in L(X)$, $(\mathrm{Id} - P)u = w$ for all $u \in X$ and $P^2 = P$. Furthermore, it holds that*

$$v \in V \Leftrightarrow Pv = v \quad \textit{and} \quad w \in W \Leftrightarrow (\mathrm{Id} - P)w = w.$$

Conversely, if $P \in L(X)$ is a projection operator, then

$$X = \mathrm{ran}(P) \oplus \mathrm{ran}(\mathrm{Id} - P).$$

Proof. See [Zei90a, Lemma 21.37]. □

A.1.1 Spectral theory for compact self-adjoint operators

Let V be Hilbert and $A \in L(V)$ be a bounded linear operator. We define the *spectrum* of A as

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda \, \mathrm{Id} - A \text{ is not bijective}\}.$$

The following result shows in particular that the spectrum of a compact operator consists of its eigenvalues and possibly zero.

Lemma A.14. *Let $A \in L(V)$ be a compact operator.*

(i) If $\dim V = \infty$, then $0 \in \sigma(A)$.

(ii) If $\lambda \in \sigma(A)$, $\lambda \neq 0$, then λ is an eigenvalue of A, i.e. there exits an eigenvector $u \in V$ such that $Au = \lambda u$.

Proof. To show (i), assume that $0 \notin \sigma(A)$. Then, T is bijective and $\mathrm{Id} = TT^{-1}$ is compact. By Lemma A.4, the unit ball $B_1 := \{v \in V : \|v\|_V \leq 1\} \subset V$ is compact. This, however, implies that V is finite-dimensional which is a direct consequence of the Riesz lemma [BS18, Lemma 1.2.12] and contradicts the assumption that $\dim V = \infty$. For (ii), we note that by Corollary A.24, $\lambda \, \mathrm{Id} - A$ is Fredholm with index zero. Hence, due to Lemma A.26, $\lambda \, \mathrm{Id} - A$ is either bijective or neither injective nor surjective. Since $\lambda \in \sigma(A)$, $\lambda \, \mathrm{Id} - A$ cannot be bijective and thus, there exists $u \in V$ such that $(\lambda \, \mathrm{Id} - A)u = 0$. □

If the operator is not only compact but also self-adjoint, the spectrum can be classified further.

Lemma A.15. *If $A \in L(V)$ is compact and self-adjoint, then $\sigma(A) \subset \mathbb{R}$.*

Proof. See [BS18, Thm. 5.3.15, (i)]. □

Theorem A.16 (Spectral theorem for compact self-adjoint operators)**.** *Let H be a nonzero complex or real Hilbert space and $A \in L(H)$ be a self-adjoint and compact operator. Then there exists an orthonormal basis $(e_i)_{i\in N}$ of H consisting of eigenvectors of A and eigenvalues $(\lambda_i)_{i\in N}$, $N \subset \mathbb{N}$, such that*

$$Au = \sum_{i=1}^{N} \lambda_i (e_i, u)_H e_i. \tag{A.1}$$

If $N = \mathbb{N}$, then $\lim_{i\to\infty} \lambda_i = 0$.

Proof. See [BS18, Thm. 5.3.15, (v)]. □

Remark A.17 (Eigenbasis of L^2 w.r.t. the Laplacian)**.** *We can apply the previous theorem to the Laplace operator $\Delta : H_0^1(D) \to L^2(D)$ with Dirichlet boundary conditions. Assuming that $D \subset \mathbb{R}^d$ is an open and bounded Lipschitz domain, the Rellich-Kondrachov theorem A.5 yields that the embedding operator $\iota : H_0^1(D) \to L^2(D)$ is compact. Furthermore, we note that the inverse of the Laplacian $\Delta^{-1} : L^2(D) \to H_0^1(D)$ exists and is a bounded linear operator. Thence, the composition $\iota \circ \Delta^{-1} : L^2(D) \to L^2(D)$ is a compact operator and it is self-adjoint since the Δ is self-adjoint. Thus, we can apply the spectral theorem A.16 to the operator $\iota \circ \Delta^{-1}$ and conclude the existence of an eigenbasis of $L^2(D)$ consisting of eigenvectors of Δ.*

A.2 Fredholm operators

In this section, we will introduce Fredholm operators and present results to which we refer on numerous occasions in section 2. Fredholm operators are applied in a variety of mathematical fields, for instance, the study of linear integral equations, K-theory or the analysis of elliptic PDEs. This section is based on [BS18, Chapter 4] and [GGK90, Chapter XI].
Intuitively speaking, a Fredholm operator is "almost" invertible in the following sense. An operator $A : X \to Y$ between vector spaces X and Y is an isomorphism if and only if it is injective and surjective, that means if and only if $\ker(A) = \{0\}$ and $\operatorname{ran}(A) = Y$. Equivalently, A is an isomorphism if and only if $\dim\ker(A) = 0 = \dim\operatorname{coker}(A)$. Therefore, the invertibility of A is related to the dimensions of the operator's kernel and cokernel. Thus, with the following definition, we can understand a Fredholm operator as being "almost" invertible.

Definition A.18 (Fredholm Operator)**.** We call a bounded linear operator $A \in L(X, Y)$ *semifredholm*, if $\operatorname{ran}(A)$ is closed and either $\ker(A)$ or $\operatorname{coker}(A)$ is finite-dimensional. If $\operatorname{ran}(A)$ is closed and both, $\ker(A)$ and $\operatorname{coker}(A)$, are finite-dimensional, we call the operator *Fredholm*. We define the *index* of the operator as

$$\operatorname{ind} A = \dim\ker(A) - \dim\operatorname{coker}(A). \tag{A.2}$$

Remark A.19 (On different notations)**.** *In some literature, for instance, in [Vai76], the definition of Fredholm operators is formulated in terms of the codimension rather than the cokernel. For a subspace $U \subset V$ of a vector space V, we define its codimension as*

$$\operatorname{codim} U := \dim(V/U). \tag{A.3}$$

Note, that for finite dimensional vector spaces $U \subset V$, we have that $\operatorname{codim} U = \dim V - \dim U$. *Consequently, with the notation from above, we have that*

$$\dim \operatorname{coker}(A) = \dim(Y/\operatorname{ran}(A)) = \operatorname{codim} \operatorname{ran}(A). \tag{A.4}$$

In the interpretation of Fredholm operators above, we have only considered the dimensions of the nullspace and the cokernel, whereas the formal definition also requires the range of the operator to be closed. It turns out that the condition that $\operatorname{ran}(A)$ is closed is redundant, as the following lemma shows.

Lemma A.20. *Let $A \in L(X,Y)$ be a bounded linear operator with* $\dim \operatorname{coker}(A) < \infty$. *Then* $\operatorname{ran}(A) \subset Y$ *is a closed subspace.*

Proof. See Lemma 4.3.2. in [BS18]. □

When we are considering discrete approximation schemes, we often only consider finite-dimensional spaces X_n. In this case, we can apply the following lemma.

Lemma A.21. *Let X and Y be finite-dimensional and $A \in L(X,Y)$ be a bounded linear operator. Then A is Fredholm with index*

$$\operatorname{ind} A = \dim X - \dim Y. \tag{A.5}$$

Proof. By the rank theorem from linear algebra we have that

$$\dim \ker(A) + \dim \operatorname{ran}(A) = \dim X, \tag{A.6}$$

and furthermore $\dim \operatorname{coker}(A) = \dim Y - \dim \operatorname{ran}(A)$. Therefore, both $\ker(A)$ and $\operatorname{ran}(A)$ are finite dimensional. Hence, A is Fredholm and

$$\begin{aligned} \operatorname{ind} A &= \dim \ker(A) - \dim \operatorname{coker}(A) \\ &= \dim X - \dim \operatorname{ran}(A) - \dim Y + \dim \operatorname{ran}(A) \\ &= \dim X - \dim Y. \end{aligned}$$

□

The following theorems show that the composition of Fredholm operators is a Fredholm operator itself and that the perturbation by compact operator preserves Fredholmness.

Theorem A.22. *Let $A \in L(X,Y)$ and $B \in L(Y,Z)$ be Fredholm operators. Then $BA \in L(X,Z)$ is a Fredholm operator with* $\operatorname{ind}(BA) = \operatorname{ind}(A) + \operatorname{ind}(B)$.

Proof. See [GGK90, Section XI.3, Thm. 3.2]. □

Theorem A.23. *Let $A \in L(X,Y)$ be a Fredholm operator and $K \in L(X,Y)$ be compact. Then $A + K$ is a Fredholm operator with* $\operatorname{ind}(A+K) = \operatorname{ind}(A)$.

Proof. See [BS18, Thm. 4.4.2] or [GGK90, Section XI.3, Lemma 4.2]. □

Corollary A.24. *Let $K \in L(X,Y)$ be compact. Then $\operatorname{Id} - K$ is a Fredholm operator with* $\operatorname{ind}(\operatorname{Id} - K) = 0$.

Proof. Apply Thm. A.23 with $A = \operatorname{Id}$ and note that $\operatorname{ind}(\operatorname{Id}) = 0$. □

Theorem A.25. *An operator $A \in L(X,Y)$ is a Fredholm operator with* $\operatorname{ind} A = 0$ *if and only if there exists an operator $F \in L(X,Y)$ of finite rank such that $A + F$ is invertible.*

Proof. We follow the lines of [GGK90, Thm. 5.3, p. 191]. Suppose that A is Fredholm with $\operatorname{ind} A = 0$. We can decompose $X = X_0 \oplus \ker A$ and $Y = Y_0 \oplus \operatorname{ran} A$. Since $\operatorname{ind} A = 0$, we have that $\dim \ker A = \dim Y_0$, and thus there exists a bijective operator $F_0 : \ker A \to Y_0$. We define $F = F_0(\mathrm{Id} - P)$, where P is the projection onto $\ker A$. Then F is a finite rank operator and $A + F$ is bijective. The other direction follows from Thm. A.23. □

Fredholm operators with index zero are useful because it is easier to show that they are bijective. The following lemma shows that operators of this type are either injective and surjective or neither.

Lemma A.26. *Let $A \in L(X, Y)$ be a Fredholm operator with $\operatorname{ind}(A) = 0$. If A is injective, then A is also surjective.*

Proof. We have that $0 = \operatorname{ind}(A) = \dim \ker(A) - \dim \operatorname{coker}(A)$. If A is injective, then $\ker(A) = \{0\}$ and thus $\dim \operatorname{coker}(A) = 0$, which implies that A is surjective. □

To close this section, let us come back to the initial motivation of Fredholm operators being "almost" invertible. This motivation can be considered formally in two different ways. For the first interpretation, we consider the case $X = Y$ and denote by $\mathcal{K}(X) \subset L(X)$ the space of compact operators on X. Then, the quotient space $L(X)/\mathcal{K}(X)$ together with the operator $[C][D] = [CD]$ defines an algebra, called *Calkin Algebra*. The following theorem allows us to interpret Fredholm operators as being invertible modulo compact operators.

Theorem A.27 (Atkinson)**.** *Let $A \in L(X)$. The following are equivalent*

(i) *A is Fredholm.*

(ii) *There exists an operator $T \in L(X)$ such that $\mathrm{Id} - TA$ and $\mathrm{Id} - AT$ are compact*[1].

(iii) *$[A]$ has an inverse in the Calkin Algebra $L(X)/\mathcal{K}(X)$.*

Proof. See [GGK90, Thms. 5.1 and 5.2]. □

Remark A.28. *The first equivalence $(i) \Leftrightarrow (ii)$ also holds in the case where $X \neq Y$ with $A, T \in L(X, Y)$.*

For the second interpretation, we say that an operator $A \in L(X, Y)$ has a *generalized inverse* $Q \in L(Y, X)$ if it holds that

$$AQA = A \text{ and } QAQ = Q.$$

We can show that every Fredhlom operator has such a generalized inverse.

Lemma A.29. *Every Fredholm operator has a generalized inverse.*

Proof. [GGK90, Cor. 6.2, p. 192] □

A.3 Inequalities

This section of the Appendix contains some elementary inequalities that we apply numerous times in this thesis. Furthermore, we also recall some standard norm estimates from finite element theory and approximation results.

[1] can be replaced by the condition that $\mathrm{Id} - TA$ and $\mathrm{Id} - AT$ are of finite rank.

A.3.1 Elementary inequalities

In this section, let $(V, (\cdot,\cdot)_V)$ be a complex Hilbert space and $\|\cdot\|_V := \sqrt{(\cdot,\cdot)_V}$ be the norm induced by the scalar product.

Lemma A.30 (Cauchy-Schwarz inequality)**.** *For all $v, w \in V$ it holds that*

$$|(v,w)_V| \leq \|v\|_V \|w\|_V. \tag{A.7}$$

For two non-negative real numbers $a, b \in \mathbb{R}$, we have *Young's inequality* in the following form. If $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{A.8}$$

In particular, the case $p = q = 2$ is often applied in the following way: For every $\epsilon > 0$, $\epsilon \in \mathbb{R}$, we have that

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2. \tag{A.9}$$

Furthermore, Young's inequality can also be applied on Hilbert spaces.

Lemma A.31 (Young's inequality)**.** *For every positive real number $\gamma \in \mathbb{R}$, $\gamma > 0$, we have that*

$$|(v,w)_V| \leq \frac{\gamma}{2}\|v\|_V^2 + \frac{1}{2\gamma}\|w\|_V^2. \tag{A.10}$$

There are different version of the *Poincaré-inequality*, but most often we consider the following case: Let $D \subset \mathbb{R}^n$ be a Lipschitz domain. Then, there is a constant $C > 0$ such that

$$C\|v\|_{L^2(D)} \leq \|\nabla v\|_{L^2(D).} \tag{A.11}$$

The following lemma shows a very general version of the *Poincaré-inequality* that covers the most common applications.

Lemma A.32. *Let $D \subset \mathbb{R}^n$ be a Lipschitz domain and $p \in [1, \infty)$. Furthermore, let f be a bounded linear form on $W^{1,p}(D)$ such that the restriction of f to constant functions is not zero. Then, there exists a constant $C_{PS,p} > 0$ such that for all $v \in W^{1,p}(D)$, we have that*

$$C_{PS,p}\|v\|_{L^p(D)} \leq diam(D)\|\nabla v\|_{L^p(D)} + |f(v)|. \tag{A.12}$$

In particular, we have that

$$C_{PS,p}\|v\|_{L^p(D)} \leq diam(D)\|\nabla v\|_{L^p(D)} \qquad \forall v \in \ker(f). \tag{A.13}$$

Proof. See [EG21a, Lemma 3.30]. □

A.3.2 Classical FEM inequalities

In this section, we recall some standard norm estimates from finite element theory.

Lemma A.33 (Discrete trace inequality)**.** *For all $u \in \mathcal{P}^k(D)$ and $\tau \in \mathcal{T}_h$, there exists a constant $C_{dt} > 0$ such that*

$$\|u\|_{L^2(\partial\tau)} \leq C_{dt} h_\tau^{-1/2} \|u\|_{L^2(\tau)}. \tag{A.14}$$

Proof. We refer to [EG21a, Lem. 12.8]. □

Lemma A.34 (Discrete inverse inequality)**.** *For $u \in \mathcal{P}^k(D)$ and $\tau \in \mathcal{T}_h$, there exists a constant $C_{inv} > 0$ such that*

$$|u|_{H^1(\tau)} \leq C_{inv} h_\tau^{-1} \|u\|_{L^2(\tau)}.$$

Proof. We refer to [EG21a, Lem. 12.15]. □

A.3.3 Interpolation operators

The following theorem is a classical result about the local interpolation error in finite element spaces. For more details, we refer to [EG21a, Sec. 11 & 12].

Theorem A.35 (Local interpolation)**.** *Let $\{\hat{K}.\hat{P},\hat{\Sigma}\}$ be a finite element with associated normed vector space $V(\hat{K})$ and let $1 \leq p \leq \infty$. Furthermore, assume that there exists $k \in \mathbb{N}$ such that*

$$\mathbb{P}^k \subset \hat{P} \subset W^{k+1,p}(\hat{K}) \subset V(\hat{K}).$$

Let $\mathcal{I}_K^k$ be the canonical local interpolation operator and $0 \leq l \leq k$ be such that $W^{1+l,p}(\hat{K}) \subset V(\hat{K})$, where the embedding is continuous. Then, for all $m \in \{0, \ldots, l+1\}$ and all $v \in W^{1+l,p}(K)$, it holds that

$$|v - \mathcal{I}_K^k v|_{W^{m,p}(K)} \lesssim h_K^{l+1-m} |v|_{W^{1+l,p}(K)}.$$

Furthermore, for all $F \in \mathcal{F}_n$ such that $F \subset \partial K$, it holds that

$$\|v - \mathcal{I}_K^k(v)|_{W^{m,p}(F)} \lesssim h_K^{r-1/p} |v|^{W^{r,p}(K)}. \tag{A.15}$$

Proof. We refer to [EG21a, Thm. 11.13] and [EG21a, Rem. 12.17]. □

List of Figures

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