GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

CRC 1456 MATHEMATICS OF EXPERIMENT

Learned infinite elements for the vector-valued time-harmonic Galbrun's equation

Janosch Preuss[1], Paul Stocker[2], Tim van Beeck[3]

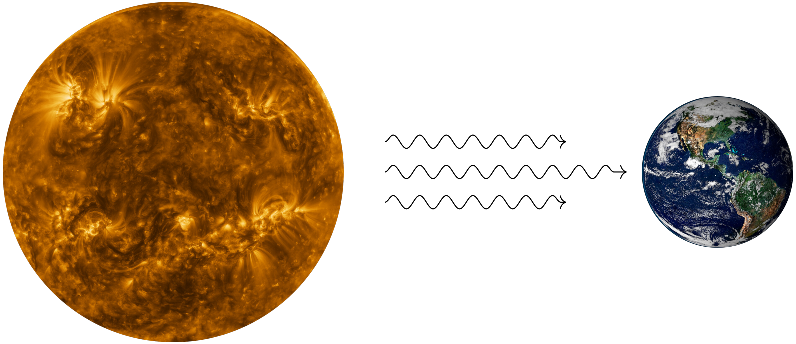
Based on previous work by: Damien Fournier, Laurent Gizon, Martin Halla, Thorsten Hohage & Christoph Lehrenfeld

[1]Inria Makutu; [2]University of Vienna; [3]University of Göttingen

Conference on Mathematics of Wave Phenomena 2025, Karlsruhe
Feb. 26th, 2025.

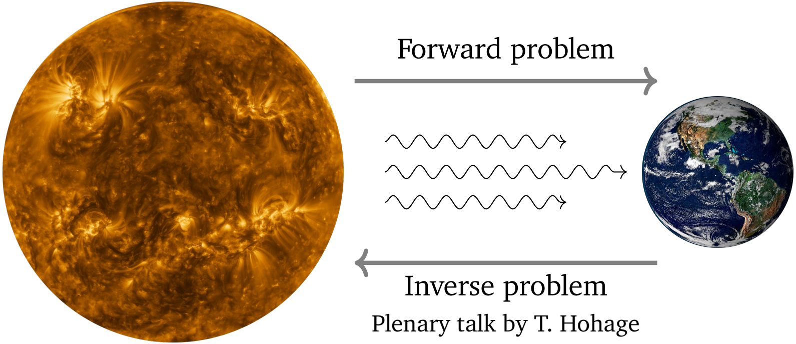
Motivation: Helioseismology

- Helioseismology $\widehat{=}$ study of the Sun through its *oscillations*



Motivation: Helioseismology

- Helioseismology $\widehat{=}$ study of the Sun through its *oscillations*



Solar and stellar oscillations

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

➔ Solar oscillations modeled with *Galbrun's equation*:
Find $\boldsymbol{u} : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{C}^3$, $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\mathcal{O}$, s.t.

$$\begin{aligned}\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2\boldsymbol{u} - \nabla\left(c_s^2\rho \operatorname{div}\boldsymbol{u}\right) + (\operatorname{div}\boldsymbol{u})\nabla p - \nabla(\nabla p \cdot \boldsymbol{u})&\\ + (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\boldsymbol{u} - i\omega\gamma\rho\boldsymbol{u} = \boldsymbol{f} \quad &\text{in } \mathcal{O},\end{aligned}$$

ρ: density, c_s: sound-speed, p: pressure, ϕ: gravitational potential, $\boldsymbol{b}$: background flow, Ω: rotation of the frame, ω: frequency, γ: damping coefficient

[1] M. Halla, C. Lehrenfeld, P. Stocker, *A new T-compatibility cond. & its application to the discr. of the damped time-harmonic Galbrun's eq.*, arXiv, 2022.
[2] M. Halla *Convergence analysis of nonconform H(div)-finite elements for the damped time-harmonic Galbrun's eq.*, arXiv, 2023.
[3] TvB, *On stable discontinuous Galerkin discretizations for Galbrun's equation*, Master's thesis, 2023.
[4] M. Halla, C. Lehrenfeld, TvB, *HDG discretizations for the damped time-harmonic Galbrun's equation*, in preparation.

Solar and stellar oscillations

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

- Solar oscillations modeled with *Galbrun's equation*:
  Find $\boldsymbol{u} : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{C}^3$, $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\mathcal{O}$, (assume $p = \text{const.}$, $\phi = \text{const.}$) s.t.

$$\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u} - \nabla\left(c_s^2 \rho \operatorname{div} \boldsymbol{u}\right) - i\omega\gamma\rho\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathcal{O},$$

ρ: density, c_s: sound-speed, p: pressure, ϕ: gravitational potential, $\boldsymbol{b}$: background flow, Ω: rotation of the frame, ω: frequency, γ: damping coefficient

[1] M. Halla, C. Lehrenfeld, P. Stocker, *A new T-compatibility cond. & its application to the discr. of the damped time-harmonic Galbrun's eq.*, arXiv, 2022.
[2] M. Halla *Convergence analysis of nonconform H(div)-finite elements for the damped time-harmonic Galbrun's eq.*, arXiv, 2023.
[3] TvB, *On stable discontinuous Galerkin discretizations for Galbrun's equation*, Master's thesis, 2023.
[4] M. Halla, C. Lehrenfeld, TvB, *HDG discretizations for the damped time-harmonic Galbrun's equation*, in preparation.

Solar and stellar oscillations

- Solar oscillations modeled with *Galbrun's equation*:
  Find $\boldsymbol{u} : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{C}^3$, $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\mathcal{O}$, (assume $p = \text{const.}$, $\phi = \text{const.}$) s.t.

$$\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u} - \nabla\left(c_s^2 \rho \operatorname{div} \boldsymbol{u}\right) - i\omega\gamma\rho\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathcal{O},$$

  ρ: density, c_s: sound-speed, p: pressure, ϕ: gravitational potential, $\boldsymbol{b}$: background flow, Ω: rotation of the frame, ω: frequency, γ: damping coefficient
- so far: numerical analysis[1,2,3,4] mostly focussed on $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ on $\partial\mathcal{O}$

[1] M. Halla, C. Lehrenfeld, P. Stocker, *A new T-compatibility cond. & its application to the discr. of the damped time-harmonic Galbrun's eq.*, arXiv, 2022.
[2] M. Halla *Convergence analysis of nonconform H(div)-finite elements for the damped time-harmonic Galbrun's eq.*, arXiv, 2023.
[3] TvB, *On stable discontinuous Galerkin discretizations for Galbrun's equation*, Master's thesis, 2023.
[4] M. Halla, C. Lehrenfeld, TvB, *HDG discretizations for the damped time-harmonic Galbrun's equation*, in preparation.

Solar and stellar oscillations

- Solar oscillations modeled with *Galbrun's equation*:
  Find $\boldsymbol{u} : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{C}^3$, $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\mathcal{O}$, (assume $p = \text{const.}$, $\phi = \text{const.}$) s.t.

$$\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u} - \nabla\left(c_s^2 \rho \operatorname{div} \boldsymbol{u}\right) - i\omega\gamma\rho\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathcal{O},$$

  ρ: density, c_s: sound-speed, p: pressure, ϕ: gravitational potential, $\boldsymbol{b}$: background flow, Ω: rotation of the frame, ω: frequency, γ: damping coefficient
- so far: numerical analysis[1,2,3,4] mostly focussed on $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ on $\partial\mathcal{O}$
- waves do not stop propagating, what is ∂Sun?

[1] M. Halla, C. Lehrenfeld, P. Stocker, *A new T-compatibility cond. & its application to the discr. of the damped time-harmonic Galbrun's eq.*, arXiv, 2022.
[2] M. Halla *Convergence analysis of nonconform H(div)-finite elements for the damped time-harmonic Galbrun's eq.*, arXiv, 2023.
[3] TvB, *On stable discontinuous Galerkin discretizations for Galbrun's equation*, Master's thesis, 2023.
[4] M. Halla, C. Lehrenfeld, TvB, *HDG discretizations for the damped time-harmonic Galbrun's equation*, in preparation.

Solar and stellar oscillations

- Solar oscillations modeled with *Galbrun's equation*:
  Find $\boldsymbol{u} : \mathcal{O} \subset \mathbb{R}^3 \to \mathbb{C}^3$, $\boldsymbol{\nu} \cdot \boldsymbol{u} = 0$ on $\partial\mathcal{O}$, (assume $p = \text{const.}$, $\phi = \text{const.}$) s.t.

$$\rho(\omega + i\partial_{\boldsymbol{b}} + i\Omega\times)^2 \boldsymbol{u} - \nabla\left(c_s^2 \rho \operatorname{div} \boldsymbol{u}\right) - i\omega\gamma\rho\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathcal{O},$$

  ρ: density, c_s: sound-speed, p: pressure, ϕ: gravitational potential, $\boldsymbol{b}$: background flow, Ω: rotation of the frame, ω: frequency, γ: damping coefficient
- so far: numerical analysis[1,2,3,4] mostly focussed on $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ on $\partial\mathcal{O}$
- waves do not stop propagating, what is ∂Sun?
- want to impose transparent boundary conditions!

---

[1] M. Halla, C. Lehrenfeld, P. Stocker, *A new T-compatibility cond. & its application to the discr. of the damped time-harmonic Galbrun's eq.*, arXiv, 2022.
[2] M. Halla *Convergence analysis of nonconform H(div)-finite elements for the damped time-harmonic Galbrun's eq.*, arXiv, 2023.
[3] TvB, *On stable discontinuous Galerkin discretizations for Galbrun's equation*, Master's thesis, 2023.
[4] M. Halla, C. Lehrenfeld, TvB, *HDG discretizations for the damped time-harmonic Galbrun's equation*, in preparation.

Genesis of the project

➔ Focus on the Forward problem

Well-posedness

Halla & Hohage 2021

Ext. Domain

Halla 2022

?

Learned Infinite elements

Hohage, Lehrenfeld & Preuss 2021

Preuss 2021

Discretizations

Halla, Lehrenfeld & Stocker 2022

Halla 2023

vB 2023

Other works by: T. Hohage, B. Müller, L. Gizon, D. Fournier, H. Barucq, F. Faucher, H. Pham, ...

Genesis of the project

➔ Focus on the Forward problem

Well-posedness

Halla & Hohage 2021

Ext. Domain

Halla 2022

?

Learned Infinite elements

Hohage, Lehrenfeld & Preuss 2021

Preuss 2021

Discretizations

Halla, Lehrenfeld & Stocker 2022

Halla 2023

vB 2023

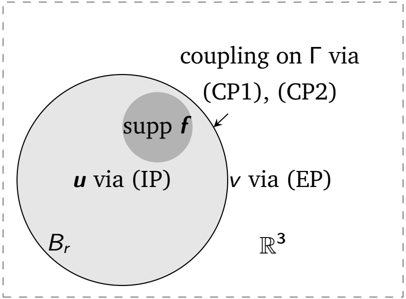
Other works by: T. Hohage, B. Müller, L. Gizon, D. Fournier, H. Barucq, F. Faucher, H. Pham, ...

The exterior problem (simplified)[5]

Further simplifications: no rotation $\Omega = 0$, no background flow $\boldsymbol{b} = 0$.

Set $\sigma^2 := \omega^2 + i\omega\gamma$, $B_r \subset \mathbb{R}^3$ s.t. $\operatorname{supp}(\boldsymbol{f}) \subset B_r$

$$
\begin{aligned}
-\nabla\left(c_s^2 \rho \operatorname{div} \boldsymbol{u}\right) - \rho\sigma^2 \boldsymbol{u} &= \rho \boldsymbol{f} && \text{in } B_r, && \text{(IP)}\\
-\operatorname{div}\left(\frac{1}{\rho}\nabla v\right) - \frac{\sigma^2}{c_s^2\rho} v &= 0 && \text{in } B_r^c, && \text{(EP)}\\
\boldsymbol{\nu} \cdot \frac{1}{\rho}\nabla v &= \sigma^2 \boldsymbol{\nu} \cdot \boldsymbol{u} && \text{on } \Gamma, && \text{(CP1)}\\
\operatorname{div}\left(\frac{1}{\rho}\nabla v\right) &= \sigma^2 \operatorname{div} \boldsymbol{u} && \text{on } \Gamma. && \text{(CP2)}
\end{aligned}
$$



[5] Derived & Analyzed (for the full problem) in:
M. Halla, *On the Treatment of Ext. Domains for the Time-Harmonic Eqs. of Stellar Oscillations*. SIAM J. Math. Anal., 2022.

From the coupled problem (1), we obtain the sesquilinear form

$$
\begin{aligned}
a[(\boldsymbol{u}, v), (\boldsymbol{u}', v')] := & \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle_{\boldsymbol{L}^2(B_r)} - \langle \rho\sigma^2 \boldsymbol{u}, \boldsymbol{u}' \rangle_{\boldsymbol{L}^2(B_r)} && \text{(from (IP))} \\
& + \langle \frac{1}{\rho} \nabla v, \nabla v' \rangle_{L^2(B_r^c)} - \langle \frac{\sigma^2}{c_s^2 \rho} v, v' \rangle_{L^2(B_r^c)} && \text{(from (EP))} \\
& + \langle v, \boldsymbol{\nu} \cdot \boldsymbol{u}' \rangle_{\partial B_r} - \langle \sigma^2 \boldsymbol{\nu} \cdot \boldsymbol{u}, v' \rangle_{\partial B_r} && \text{(from (CP1) \& (CP2))}
\end{aligned}
$$

$\mathcal{DtN}$-operator

From the coupled problem (1), we obtain the sesquilinear form

$$\begin{aligned} a[(\boldsymbol{u}, v), (\boldsymbol{u}', v')] :=& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle_{\boldsymbol{L}^2(B_r)} - \langle \rho \sigma^2 \boldsymbol{u}, \boldsymbol{u}' \rangle_{\boldsymbol{L}^2(B_r)} && \text{(from (IP))} \\ &+ \langle \frac{1}{\rho} \nabla v, \nabla v' \rangle_{L^2(B_r^c)} - \langle \frac{\sigma^2}{c_s^2 \rho} v, v' \rangle_{L^2(B_r^c)} && \text{(from (EP))} \\ &+ \langle v, \boldsymbol{\nu} \cdot \boldsymbol{u}' \rangle_{\partial B_r} - \langle \sigma^2 \boldsymbol{\nu} \cdot \boldsymbol{u}, v' \rangle_{\partial B_r} && \text{(from (CP1) \& (CP2))} \end{aligned}$$

Let $\mathcal{DtN} \boldsymbol{u}^{\text{int}} := -\nabla v^{\text{ext}} \cdot \boldsymbol{\nu}$, where v^{ext} solves (EP). Then, we can write

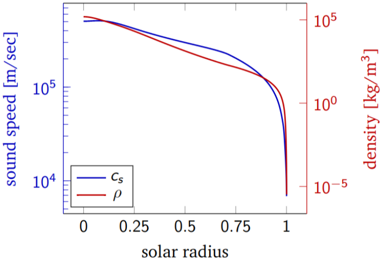
$$\begin{aligned} a[(\boldsymbol{u}, v), (\boldsymbol{u}', v')] =& \langle c_s^2 \rho \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{u}' \rangle_{\boldsymbol{L}^2(B_r)} - \langle \rho \sigma^2 \boldsymbol{u}, \boldsymbol{u}' \rangle_{\boldsymbol{L}^2(B_r)} \\ &+ \langle v, \boldsymbol{\nu} \cdot \boldsymbol{u}' \rangle_{\partial B_r} - \langle \sigma^2 \boldsymbol{\nu} \cdot \boldsymbol{u}, v' \rangle_{\partial B_r} + \langle \frac{1}{\rho} \mathcal{DtN} v, v' \rangle_{\partial B_r} \end{aligned}$$

➔ we need an (efficient) implementation of $\mathcal{DtN}$

[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Implementing $\mathcal{DtN}$

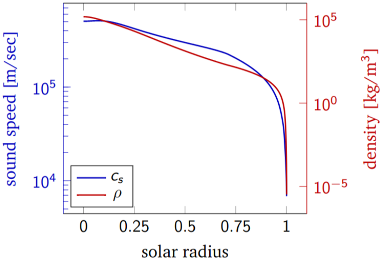
- we need an (efficient) implementation of $\mathcal{DtN}$
- classical methods (PML, infinite elements, ...) are not applicable
  - strongly varying non-analytic coefficients



[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Implementing $\mathcal{DtN}$

- we need an (efficient) implementation of $\mathcal{DtN}$
- classical methods (PML, infinite elements, ...) are not applicable
  - strongly varying non-analytic coefficients
- Learned Infinite Elements[6,7] have been developed to address this problem



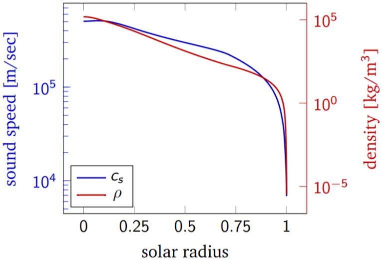
[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.

[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Implementing $\mathcal{DtN}$

- we need an (efficient) implementation of $\mathcal{DtN}$
- classical methods (PML, infinite elements, ...) are not applicable
  - strongly varying non-analytic coefficients
- Learned Infinite Elements[6,7] have been developed to address this problem

Disclaimer: "Learned" $\nRightarrow$ blackbox neural network



[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

Idea: take the *algebraic structure* of classical infinite element methods and *optimize* over infinite element matrices such that their *Schur complement* approximates $\mathcal{DtN}$

[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.

[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

Idea: take the *algebraic structure* of classical infinite element methods and *optimize* over infinite element matrices such that their *Schur complement* approximates $\mathcal{DtN}$

To be precise: The goal is to replace

$$\begin{bmatrix} L_{II} & L_{I\Gamma} \\ L_{\Gamma I} & L_{\Gamma\Gamma}^{\text{int}} + M\text{DtN}^{\text{ext}} \end{bmatrix} \begin{bmatrix} u_I \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} f_I \\ f_\Gamma^{\text{int}} \end{bmatrix}$$

by

$$\begin{bmatrix} L_{II} & L_{I\Gamma} & 0 \\ L_{\Gamma I} & L_{\Gamma\Gamma}^{\text{int}} + L_{\Gamma\Gamma} & L_{\Gamma E} \\ 0 & L_{\Gamma E} & L_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_I \\ u_\Gamma \\ u_E \end{bmatrix} = \begin{bmatrix} f_I \\ f_\Gamma \\ 0 \end{bmatrix}$$

such that $\text{DtN}^{\text{ext}} \approx \text{DtN} := M^{-1}(L_{\Gamma\Gamma} - L_{\Gamma E} L_{EE}^{-1} L_{E\Gamma})$.

[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.

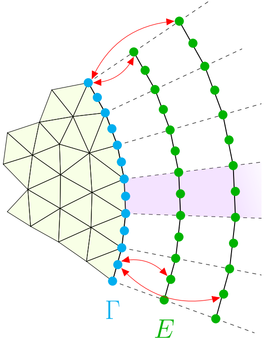
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

Tensor product ansatz $A, B \in \mathbb{C}^{(N+1)\times(N+1)}$ (M = mass matrix, K = stiffness matrix):

$$\begin{bmatrix} L_{\Gamma\Gamma} & L_{\Gamma E} \\ L_{E\Gamma} & L_{EE} \end{bmatrix} = \left[\begin{pmatrix} A_{\Gamma\Gamma} & A_{\Gamma E} \\ A_{E\Gamma} & A_{EE} \end{pmatrix} \otimes M + \begin{pmatrix} B_{\Gamma\Gamma} & B_{\Gamma E} \\ B_{E\Gamma} & B_{EE} \end{pmatrix} \otimes K \right]$$

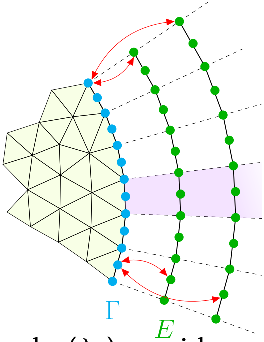
➔ independent of interior discretization



[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.

[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]



Tensor product ansatz $A, B \in \mathbb{C}^{(N+1)\times(N+1)}$ (M = mass matrix, K = stiffness matrix):

$$\begin{bmatrix} L_{\Gamma\Gamma} & L_{\Gamma E} \\ L_{E\Gamma} & L_{EE} \end{bmatrix} = \left[\begin{pmatrix} A_{\Gamma\Gamma} & A_{\Gamma E} \\ A_{E\Gamma} & A_{EE} \end{pmatrix} \otimes M + \begin{pmatrix} B_{\Gamma\Gamma} & B_{\Gamma E} \\ B_{E\Gamma} & B_{EE} \end{pmatrix} \otimes K \right]$$

➔ independent of interior discretization

➔ Let (λ_ℓ, v_ℓ) be eigenpairs of $-\Delta_{\Gamma,h} = M^{-1}K$. Then $\mathrm{DtN} v_\ell = \mathrm{dtn}(\lambda_\ell) v_\ell$, with

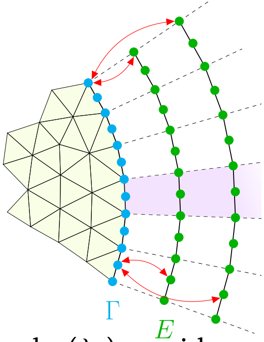
$$\mathrm{dtn}(\lambda) := A_{\Gamma\Gamma} + \lambda B_{\Gamma\Gamma} - (A_{\Gamma E} + \lambda B_{\Gamma E})(A_{EE} + \lambda B_{EE})^{-1}(A_{E\Gamma} + \lambda B_{E\Gamma})$$

---

6 J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.

7 T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]



Tensor product ansatz $A, B \in \mathbb{C}^{(N+1)\times(N+1)}$ (M = mass matrix, K = stiffness matrix):

$$\begin{bmatrix} L_{\Gamma\Gamma} & L_{\Gamma E} \\ L_{E\Gamma} & L_{EE} \end{bmatrix} = \left[\begin{pmatrix} A_{\Gamma\Gamma} & A_{\Gamma E} \\ A_{E\Gamma} & A_{EE} \end{pmatrix} \otimes M + \begin{pmatrix} B_{\Gamma\Gamma} & B_{\Gamma E} \\ B_{E\Gamma} & B_{EE} \end{pmatrix} \otimes K \right]$$

→ independent of interior discretization

→ Let (λ_ℓ, v_ℓ) be eigenpairs of $-\Delta_{\Gamma,h} = M^{-1}K$. Then $\mathrm{DtN}\, v_\ell = \mathrm{dtn}(\lambda_\ell) v_\ell$, with

$$\mathrm{dtn}(\lambda) := A_{\Gamma\Gamma} + \lambda B_{\Gamma\Gamma} - (A_{\Gamma E} + \lambda B_{\Gamma E})(A_{EE} + \lambda B_{EE})^{-1}(A_{E\Gamma} + \lambda B_{E\Gamma})$$

→ $\mathrm{DtN} = \sum_\ell \mathrm{dtn}(\lambda_\ell)\langle v_\ell, v_\ell\rangle_M =: \mathrm{dtn}(-\Delta_{\Gamma,h})$

[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

➔ Intuition: this works well, if $\mathrm{DtN}^{\mathrm{ext}} = \mathrm{dtn}^{\mathrm{ext}}(-\Delta_{\Gamma,h})$ & $\mathrm{dtn}^{\mathrm{ext}}$ can be extended to a *well-behaved* smooth function

[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

- **Intuition:** this works well, if $\mathrm{DtN}^{\mathrm{ext}} = \mathrm{dtn}^{\mathrm{ext}}(-\Delta_{\Gamma,h})$ & $\mathrm{dtn}^{\mathrm{ext}}$ can be extended to a *well-behaved* smooth function
- **Assumption:** the exterior problem is separable, i.e. $\exists$ coordinate transf. Ψ s.t.

$$(\mathcal{L}u) \circ \Psi = (\mathcal{A} \otimes \mathsf{Id}_\Gamma + \mathcal{B} \otimes -\Delta_\Gamma)(u \circ \Psi)$$

[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

- **Intuition:** this works well, if $\mathrm{DtN}^{\mathrm{ext}} = \mathrm{dtn}^{\mathrm{ext}}(-\Delta_{\Gamma,h})$ & $\mathrm{dtn}^{\mathrm{ext}}$ can be extended to a *well-behaved* smooth function
- **Assumption:** the exterior problem is separable, i.e. $\exists$ coordinate transf. Ψ s.t.

$$(\mathcal{L}u) \circ \Psi = (\mathcal{A} \otimes \mathrm{Id}_\Gamma + \mathcal{B} \otimes -\Delta_\Gamma)(u \circ \Psi)$$

$\Longrightarrow$ $\mathcal{DtN} = \mathcal{dtn}(-\Delta_\Gamma)$ with $\mathcal{dtn}(\lambda) := -\partial_r \Lambda_r(\Gamma)|_{r=a}$ where for $\lambda \in \{\lambda_\ell\}$

$$[\mathcal{A} + \lambda\mathcal{B}]\,\Lambda_r(\lambda) = 0 \text{ for } r \in [a, \infty),$$
$$\Lambda_a(\lambda) = 1, \quad \Lambda_r \text{ satisfies the radiation condition.}$$

6 J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.

7 T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

- **Intuition:** this works well, if $\mathrm{DtN}^{\mathrm{ext}} = \mathrm{dtn}^{\mathrm{ext}}(-\Delta_{\Gamma,h})$ & $\mathrm{dtn}^{\mathrm{ext}}$ can be extended to a *well-behaved* smooth function
- **Assumption:** the exterior problem is separable, i.e. $\exists$ coordinate transf. Ψ s.t.

$$(\mathcal{L}u) \circ \Psi = (\mathcal{A} \otimes \mathrm{Id}_\Gamma + \mathcal{B} \otimes -\Delta_\Gamma)(u \circ \Psi)$$

$\Longrightarrow$ $\mathcal{DtN} = dtn(-\Delta_\Gamma)$ with $dtn(\lambda) := -\partial_r \Lambda_r(\Gamma)|_{r=a}$ where for $\lambda \in \{\lambda_\ell\}$

$$[\mathcal{A} + \lambda \mathcal{B}]\, \Lambda_r(\lambda) = 0 \text{ for } r \in [a, \infty),$$
$$\Lambda_a(\lambda) = 1, \quad \Lambda_r \text{ satisfies the radiation condition.}$$

- Analytic solutions often available, otherwise solution of ODE required (cheap)

[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

➔ Recall our *discrete* DtN function:

$$\mathrm{dtn}(\lambda) := A_{\Gamma\Gamma} + \lambda B_{\Gamma\Gamma} - (A_{\Gamma E} + \lambda B_{\Gamma E})(A_{EE} + \lambda B_{EE})^{-1}(A_{E\Gamma} + \lambda B_{E\Gamma})$$

[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.
[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Learned Infinite Elements (LIE)[6,7]

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

- Recall our *discrete* DtN function:

$$\mathrm{dtn}(\lambda) := A_{\Gamma\Gamma} + \lambda B_{\Gamma\Gamma} - (A_{\Gamma E} + \lambda B_{\Gamma E})(A_{EE} + \lambda B_{EE})^{-1}(A_{E\Gamma} + \lambda B_{E\Gamma})$$

- Choose $A, B \in \mathrm{argmin}_{A,B\in\mathbb{C}^{(N+1)\times(N+1)}} J(A, B)$ with

$$J(A, B) := \frac{1}{2}\sum_{\ell} |w_\ell(dtn(\lambda_\ell) - \mathrm{dtn}(\lambda_\ell))|.$$

w_ℓ : appropriate weights, λ_ℓ : eigenvalues of (continuous) $-\Delta_\Gamma$

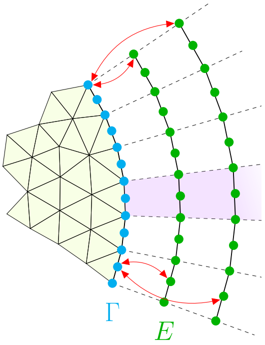
[6] J. Preuss, *Learned infinite elements for Helioseismology*. PhD thesis, 2021.

[7] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

In summary

Assumptions: tensor-product structure, exterior PDE separable

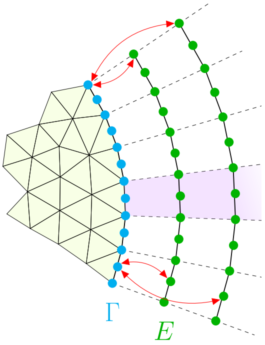
➔ guarantees that $\mathcal{DtN} = dtn(-\Delta_\Gamma)$



In summary

Assumptions: tensor-product structure, exterior PDE separable

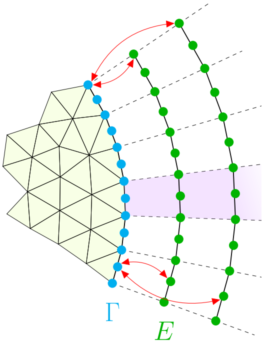
- ➔ guarantees that $\mathcal{DtN} = dtn(-\Delta_\Gamma)$
- ➔ compute dtn either analytically or solve the ODE numerically



In summary

Assumptions: tensor-product structure, exterior PDE separable

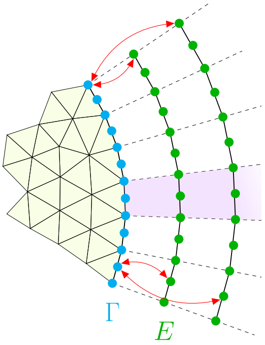
- ➔ guarantees that $\mathcal{DtN} = \mathit{dtn}(-\Delta_\Gamma)$
- ➔ compute dtn either analytically or solve the ODE numerically
- ➔ compute $A, B \in \mathbb{C}^{(N+1)\times(N+1)}$ to optimize $J(A, B)$



In summary

Assumptions: tensor-product structure, exterior PDE separable

- ➔ guarantees that $\mathcal{DtN} = dtn(-\Delta_\Gamma)$
- ➔ compute dtn either analytically or solve the ODE numerically
- ➔ compute $A, B \in \mathbb{C}^{(N+1)\times(N+1)}$ to optimize $J(A, B)$
- ➔ use $A \otimes M + B \otimes K$ to implement $\mathcal{DtN}$



Example: (Scalar) Helmholtz equation[6]

- Consider the scalar Helmholtz eq.:

$$-\Delta u - k^2 u = \delta_y$$

in a homogeneous medium

[6] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Example: (Scalar) Helmholtz equation[6]

- Consider the scalar Helmholtz eq.:

$$-\Delta u - k^2 u = \delta_y$$

  in a homogeneous medium
- δ_y point source, $y = (0.95, 0)$,
  $\mathcal{O} = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$

[6] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

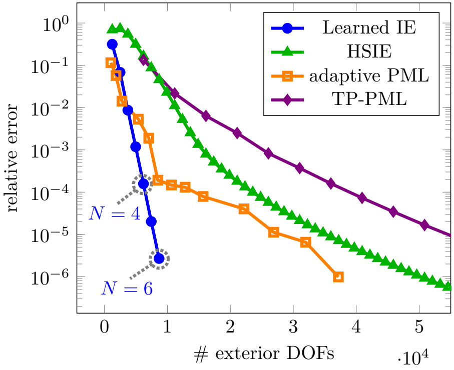
Example: (Scalar) Helmholtz equation[6]

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

- Consider the scalar Helmholtz eq.:

$$-\Delta u - k^2 u = \delta_y$$

  in a homogeneous medium
- δ_y point source, $y = (0.95, 0)$, $\mathcal{O} = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$
- relative error of LIE compared with:
  - Hardy-space infinite elements
  - tensor product PML
  - adaptive PML

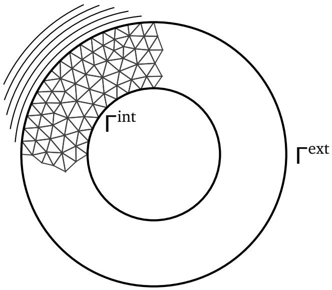


[6] T. Hohage, C. Lehrenfeld, J. Preuss, *Learned infinite elements.*. SIAM J. Sci. Comput., 2021.

Vector-Scalar coupling: A first example

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

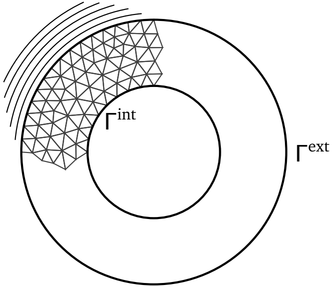
➔ constant coefficients $c_s = \rho = 1$, $\gamma = 0$



On Γ^{int}: $\operatorname{div} \boldsymbol{u}_h = \operatorname{div} \phi$

Vector-Scalar coupling: A first example

- constant coefficients $c_s = \rho = 1$, $\gamma = 0$
- (irrotational) vector-valued Helmholtz equation

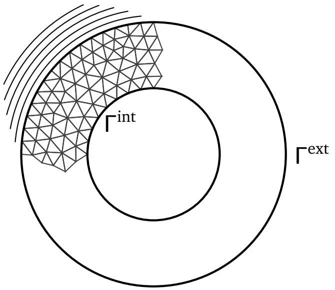


On Γ^{int}: $\operatorname{div} \boldsymbol{u}_h = \operatorname{div} \phi$

Vector-Scalar coupling: A first example

- constant coefficients $c_s = \rho = 1$, $\gamma = 0$
- (irrotational) vector-valued Helmholtz equation
- coupled problem reads as:

$$
\begin{aligned}
-\nabla\big(\operatorname{div} \boldsymbol{u}\big) - \omega^2 \boldsymbol{u} &= \boldsymbol{f} && \text{in } B_r, \\
\Delta v - \omega^2 v &= 0 && \text{in } B_r^c, \\
\nabla v \cdot \boldsymbol{\nu} &= \omega^2 \boldsymbol{u} \cdot \boldsymbol{\nu} && \text{on } \Gamma = \Gamma^{\text{ext}}, \\
\operatorname{div} \nabla v &= \omega^2 \operatorname{div} \boldsymbol{u} && \text{on } \Gamma = \Gamma^{\text{ext}}.
\end{aligned}
$$



On Γ^{int}: $\operatorname{div} \boldsymbol{u}_h = \operatorname{div} \phi$

Vector-Scalar coupling: A first example

- constant coefficients $c_s = \rho = 1$, $\gamma = 0$
- (irrotational) vector-valued Helmholtz equation
- coupled problem reads as:

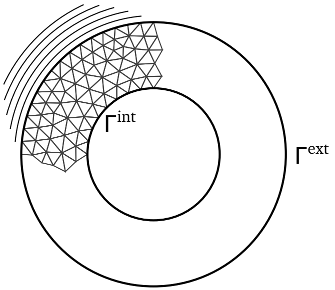
$$
\begin{aligned}
-\nabla\big(\operatorname{div} \boldsymbol{u}\big) - \omega^2 \boldsymbol{u} &= \boldsymbol{f} && \text{in } B_r,\\
\Delta v - \omega^2 v &= 0 && \text{in } B_r^c,\\
\nabla v \cdot \boldsymbol{\nu} &= \omega^2 \boldsymbol{u} \cdot \boldsymbol{\nu} && \text{on } \Gamma = \Gamma^{\text{ext}},\\
\operatorname{div} \nabla v &= \omega^2 \operatorname{div} \boldsymbol{u} && \text{on } \Gamma = \Gamma^{\text{ext}}.
\end{aligned}
$$

- $\phi(x, x_0) := \frac{i}{4} H_0^{(1)}(\omega|x - x_0|)$ solves:

$$-\Delta_x \phi(x, x_0) - \omega^2 \phi(x, x_0) = \delta(x - x_0)$$

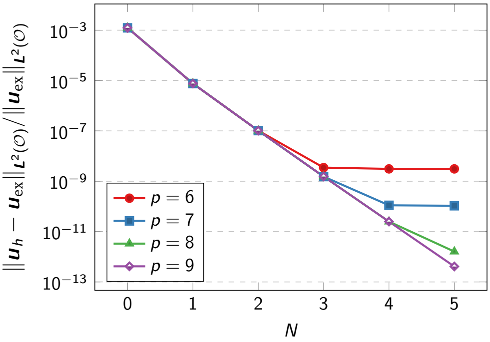
$\Rightarrow \boldsymbol{\phi} := \nabla \phi$ solves

$$-\nabla(\operatorname{div} \boldsymbol{\phi}) - \omega^2 \boldsymbol{\phi} = \nabla \delta(x - x_0).$$

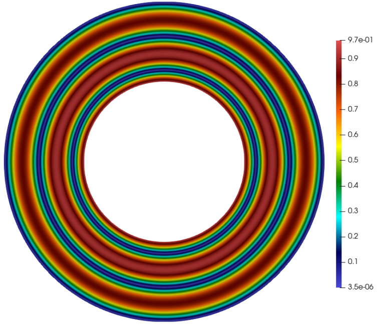


On Γ^{int}: $\operatorname{div} \boldsymbol{u}_h = \operatorname{div} \boldsymbol{\phi}$

Vector-Scalar coupling: A first example



(a) Relative error



(b) Solution

Towards computational Helioseismology

→ how can we validate our results?

[8] D. Fournier, T. Hohage, J. Preuss, L. Gizon, *LIE for Helioseismology – Learning transparent BCs for the solar atmosphere*. A&A, 2024.

Towards computational Helioseismology

- how can we validate our results?
- comparison with scalar model problem: with $\psi := c_s^2 \rho \operatorname{div} \boldsymbol{u}$

$$-\nabla(c_s \rho \operatorname{div} \boldsymbol{u}) - \rho\sigma\boldsymbol{u} = \rho\boldsymbol{f} \quad \rightarrow \quad -\operatorname{div}\left(\frac{1}{\rho}\nabla\psi\right) - \frac{\sigma^2}{c_s^2 \rho}\psi = \operatorname{div} \boldsymbol{f}$$

[8] D. Fournier, T. Hohage, J. Preuss, L. Gizon, *LIE for Helioseismology – Learning transparent BCs for the solar atmosphere*. A&A, 2024.

- how can we validate our results?
- comparison with scalar model problem: with $\psi := c_s^2 \rho \operatorname{div} \boldsymbol{u}$

$$-\nabla(c_s \rho \operatorname{div} \boldsymbol{u}) - \rho\sigma \boldsymbol{u} = \rho \boldsymbol{f} \quad \rightarrow \quad -\operatorname{div}\left(\frac{1}{\rho}\nabla\psi\right) - \frac{\sigma^2}{c_s^2 \rho}\psi = \operatorname{div} \boldsymbol{f}$$

- considered in combination with LIE[8]

[8] D. Fournier, T. Hohage, J. Preuss, L. Gizon, *LIE for Helioseismology – Learning transparent BCs for the solar atmosphere*. A&A, 2024.

Towards computational Helioseismology

- how can we validate our results?
- comparison with scalar model problem: with $\psi := c_s^2 \rho \operatorname{div} \boldsymbol{u}$

$$-\nabla(c_s \rho \operatorname{div} \boldsymbol{u}) - \rho\sigma \boldsymbol{u} = \rho \boldsymbol{f} \quad \rightarrow \quad -\operatorname{div}(\frac{1}{\rho}\nabla\psi) - \frac{\sigma^2}{c_s^2 \rho}\psi = \operatorname{div} \boldsymbol{f}$$

- considered in combination with LIE[8]
- we set $c_s = \rho = 1$ and choose rhs $\boldsymbol{f}$ s.t. $\operatorname{div} \boldsymbol{f} = \delta$

[8] D. Fournier, T. Hohage, J. Preuss, L. Gizon, *LIE for Helioseismology – Learning transparent BCs for the solar atmosphere*. A&A, 2024.

Towards computational Helioseismology

- how can we validate our results?
- comparison with scalar model problem: with $\psi := c_s^2 \rho \operatorname{div} \boldsymbol{u}$

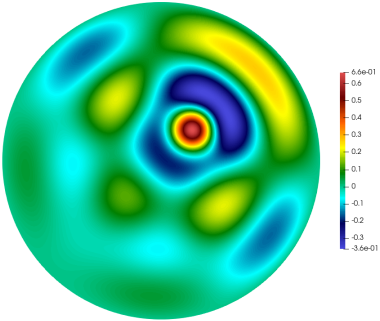
$$-\nabla(c_s \rho \operatorname{div} \boldsymbol{u}) - \rho\sigma \boldsymbol{u} = \rho \boldsymbol{f} \quad \rightarrow \quad -\operatorname{div}\left(\frac{1}{\rho}\nabla\psi\right) - \frac{\sigma^2}{c_s^2 \rho}\psi = \operatorname{div} \boldsymbol{f}$$

- considered in combination with LIE[8]
- we set $c_s = \rho = 1$ and choose rhs $\boldsymbol{f}$ s.t. $\operatorname{div} \boldsymbol{f} = \delta$
- we compare ψ_h and $\operatorname{div} \boldsymbol{u}_h$ for:
  (a) natural BCs ($\psi_h = \operatorname{div} \boldsymbol{u}_h = 0$), (b) transparent BCs with LIE

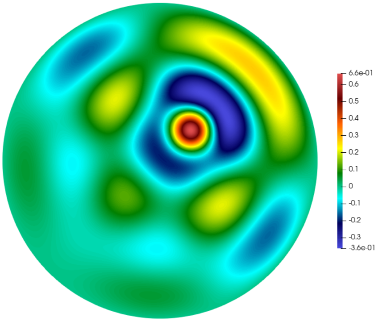
[8] D. Fournier, T. Hohage, J. Preuss, L. Gizon, *LIE for Helioseismology – Learning transparent BCs for the solar atmosphere*. A&A, 2024.

Visual comparison in 2D - natural BCs

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

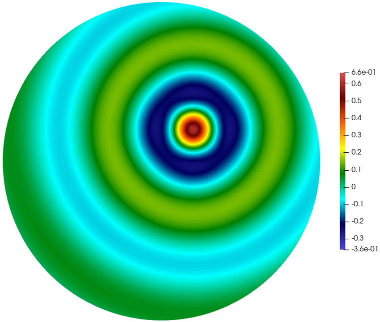


scalar

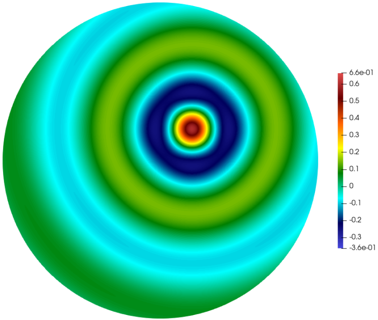


vector

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737



scalar



vector

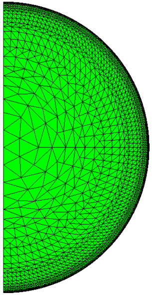
Physicall interesting computations

- sun coefficients from `modelS`[9] + `VAL-C`[10]

[9] J. Christensen-Dalsgaard et al., *The current state of solar modeling.* Science 272, 1996.
[10] J. E. Vernazza, E. H. Avrett, R. Loeser, *Structure of the solar chro- mosphere. III. Models of the EUV brightness components of the quiet sun.* ApJ, 1981.

Physicall interesting computations

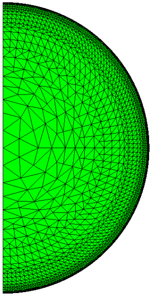
- sun coefficients from `modelS`[9]+ `VAL-C`[10]
- layered mesh adapted to frequency & coefficients

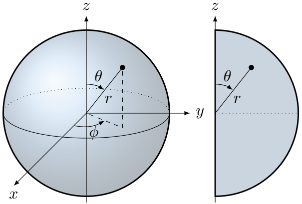


9 J. Christensen-Dalsgaard et al., *The current state of solar modeling.* Science 272, 1996.
10 J. E. Vernazza, E. H. Avrett, R. Loeser, *Structure of the solar chro- mosphere. III. Models of the EUV brightness components of the quiet sun.* ApJ, 1981.

Physicall interesting computations

- sun coefficients from `modelS`[9] + `VAL-C`[10]
- layered mesh adapted to frequency & coefficients
- use axisymmetry to reduce computational costs





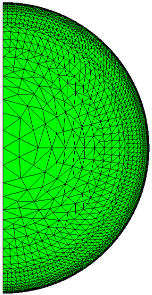
[9] J. Christensen-Dalsgaard et al., *The current state of solar modeling.* Science 272, 1996.

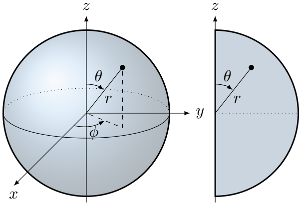
[10] J. E. Vernazza, E. H. Avrett, R. Loeser, *Structure of the solar chro- mosphere. III. Models of the EUV brightness components of the quiet sun.* ApJ, 1981.

Physicall interesting computations

- sun coefficients from `modelS`[9] + `VAL-C`[10]
- layered mesh adapted to frequency & coefficients
- use axisymmetry to reduce computational costs
- Power Spectrum: project Green's function into spherical harmonics

$$P^{\ell}(\omega) = \frac{\Pi(\omega)}{\omega} \int_0^{\pi} \Im\left(G(\theta,\omega)\right) P_{\ell}(\cos(\theta)) \sin(\theta) d\theta$$





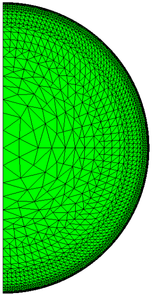
[9] J. Christensen-Dalsgaard et al., *The current state of solar modeling.* Science 272, 1996.
[10] J. E. Vernazza, E. H. Avrett, R. Loeser, *Structure of the solar chro- mosphere. III. Models of the EUV brightness components of the quiet sun.* ApJ, 1981.

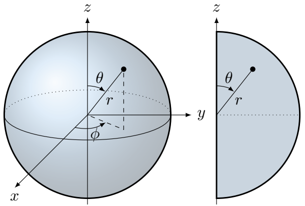
Physicall interesting computations

- sun coefficients from `modelS`[9] + `VAL-C`[10]
- layered mesh adapted to frequency & coefficients
- use axisymmetry to reduce computational costs
- Power Spectrum: project Green's function into spherical harmonics

$$P^{\ell}(\omega) = \frac{\Pi(\omega)}{\omega} \int_0^{\pi} \Im\left(G(\theta,\omega)\right) P_{\ell}(\cos(\theta)) \sin(\theta) d\theta$$

- Future goal: compare computed power spectrum to observational data



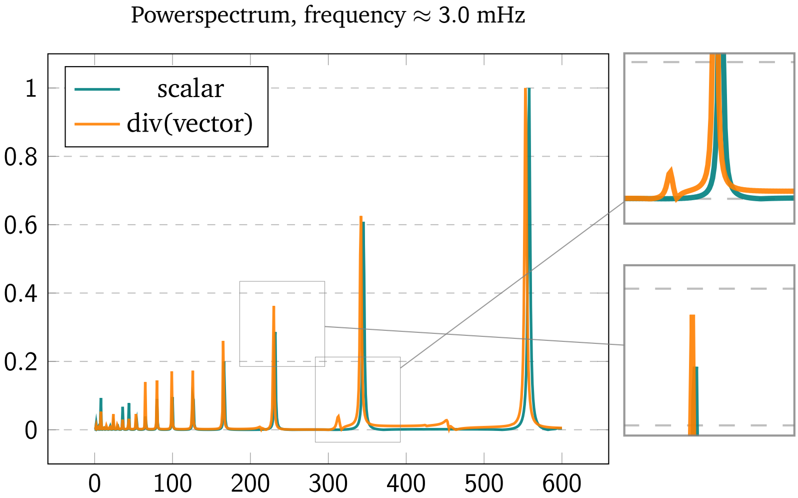


[9] J. Christensen-Dalsgaard et al., *The current state of solar modeling.* Science 272, 1996.

[10] J. E. Vernazza, E. H. Avrett, R. Loeser, *Structure of the solar chro- mosphere. III. Models of the EUV brightness components of the quiet sun.* ApJ, 1981.

Scalar-Vector comparison

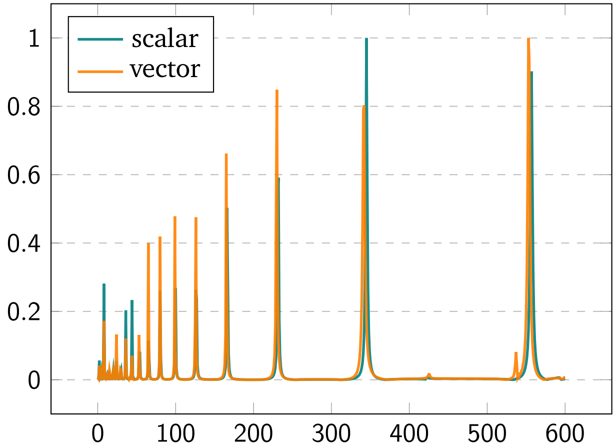
→ choose $\boldsymbol{f}$ s.t. $\operatorname{div} \boldsymbol{f} = \delta$, computed with $H^1/\boldsymbol{H}^1$-conforming FE method ($p = 6$)



Vector-valued power spectrum

➔ $\boldsymbol{f} = \delta(x,y)^T$, $f = \delta$, computed with $H^1/\boldsymbol{H}^1$-conforming FE method ($p = 6$)

Powerspectrum, frequency $\approx$ 3.0 mHz



Extensions

- incorporate background flow $\boldsymbol{b} \neq 0$
  - ➔ relatively straightforward, since $\operatorname{supp} \boldsymbol{b} \subset B_r$
  - ➔ only affects interior discretization

Extensions

- incorporate background flow $\boldsymbol{b} \neq 0$
  - ➔ relatively straightforward, since supp $\boldsymbol{b} \subset B_r$
  - ➔ only affects interior discretization
- incorporate pressure and gravitational potential
  - ➔ exterior problem has to be adjusted

$$-\operatorname{div}\left(\frac{e^{2\eta}}{\rho}(\underline{\underline{m}} + i\omega\gamma)^{-1}\nabla v\right) - \frac{e^{2\eta}}{c_s^2\rho} v = 0 \text{ in } B_r^c$$

with $\underline{\underline{m}} := -\rho\left(\mathrm{Hess}(p) - \rho\,\mathrm{Hess}(\phi) - \frac{1}{c_s^2\rho}\nabla p \otimes \nabla p\right)$

  - ➔ calculating $\underline{\underline{m}}^{-1}$ computationally involved, but doable
  - ➔ separability assumption?

Extensions

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

- incorporate background flow $\boldsymbol{b} \neq 0$
  - → relatively straightforward, since supp $\boldsymbol{b} \subset B_r$
  - → only affects interior discretization
- incorporate pressure and gravitational potential
  - → exterior problem has to be adjusted

$$-\operatorname{div}\left(\frac{e^{2\eta}}{\rho}(\underline{\underline{m}} + i\omega\gamma)^{-1}\nabla v\right) - \frac{e^{2\eta}}{c_s^2\rho} v = 0 \text{ in } B_r^c$$

with $\underline{\underline{m}} := -\rho\left(\mathrm{Hess}(p) - \rho\,\mathrm{Hess}(\phi) - \frac{1}{c_s^2\rho}\nabla p \otimes \nabla p\right)$

  - → calculating $\underline{\underline{m}}^{-1}$ computationally involved, but doable
  - → separability assumption?
- incorporate rotation $\Omega \neq 0$
  - → separation assumption not satisfied
  - → LIE-framework would need to be extended, not realistic in the near future

Conclusions

- LIE offer a flexible & efficient way to implement $\mathcal{DtN}$

Conclusions

- LIE offer a flexible & efficient way to implement $\mathcal{D}t\mathcal{N}$
- through the *scalar* exterior problem, we can apply LIE to the *vector*-valued Galbrun's equation

Conclusions

- LIE offer a flexible & efficient way to implement $\mathcal{DtN}$
- through the *scalar* exterior problem, we can apply LIE to the *vector*-valued Galbrun's equation
- in principle, the combination seems to work well

Conclusions

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

- LIE offer a flexible & efficient way to implement $\mathcal{DtN}$
- through the *scalar* exterior problem, we can apply LIE to the *vector*-valued Galbrun's equation
- in principle, the combination seems to work well
- problem is computational expensive, in particular for physically relevant scenarios

Conclusions

- LIE offer a flexible & efficient way to implement $\mathcal{DtN}$
- through the *scalar* exterior problem, we can apply LIE to the *vector*-valued Galbrun's equation
- in principle, the combination seems to work well
- problem is computational expensive, in particular for physically relevant scenarios
- incorporating rotation ($\Omega \neq 0$) is an open problem

Conclusions

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

- LIE offer a flexible & efficient way to implement $\mathcal{DtN}$
- through the *scalar* exterior problem, we can apply LIE to the *vector*-valued Galbrun's equation
- in principle, the combination seems to work well
- problem is computational expensive, in particular for physically relevant scenarios
- incorporating rotation ($\Omega \neq 0$) is an open problem

Thank you for your attention!