

Lab. 13 (보충설명) Dijkstra's Shortest Path Algorithm



정보통신공학과
교수 김 영 탁

(Tel : +82-53-810-2497; E-mail : ytkim@yu.ac.kr)

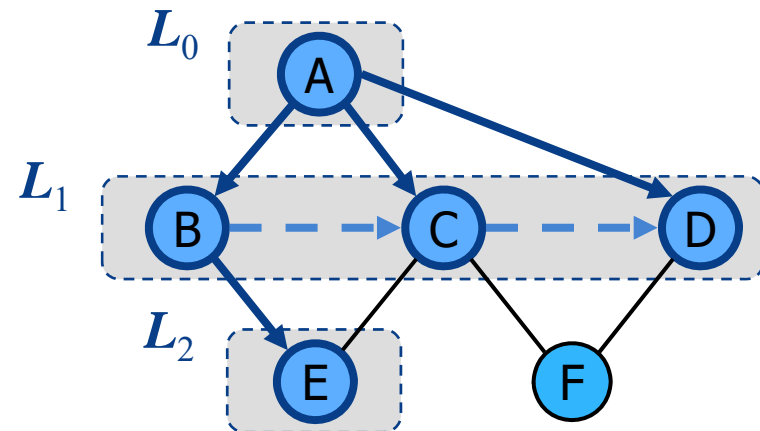
Breadth-First Search

◆ Breadth-first search

- Breadth-first search (BFS) is a general technique for traversing a graph

◆ A BFS traversal of a graph G

- Visits all the vertices and edges of G
- Determines whether G is connected
- Computes the connected components of G
- Computes a spanning forest of G



Breadth-First Search

- ◆ **BFS on a graph with n vertices and m edges takes $O(n + m)$ time**
- ◆ **BFS can be further extended to solve other graph problems**
 - Find and report a path with the minimum number of edges between two given vertices
 - Find a simple cycle, if there is one



BFS Algorithm

- ◆ The algorithm uses a mechanism for setting and getting “labels” of vertices and edges

Algorithm *BFS*(*G*)

Input graph *G*

Output labeling of the edges
and partition of the
vertices of *G*

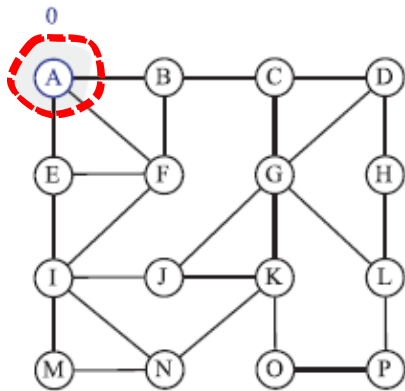
```
for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
        BFS(G, v)
```

Algorithm *BFS*(*G*, *s*)

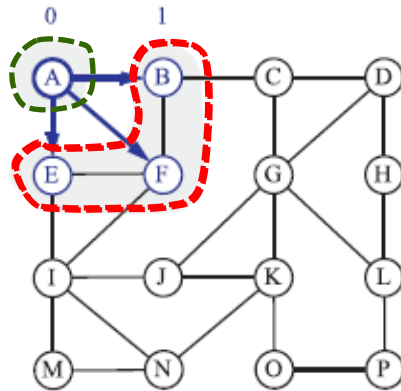
```
L0 ← new empty sequence
L0.insertLast(s)
setLabel(s, VISITED)
i ← 0
while (!Li.isEmpty())
    Li+1 ← new empty sequence
    for all v ∈ Li.elements()
        for all e ∈ G.incidentEdges(v)
            if getLabel(e) = UNEXPLORED
                w ← opposite(v, e)
                if getLabel(w) = UNEXPLORED
                    setLabel(e, DISCOVERY)
                    setLabel(w, VISITED)
                    Li+1.insertLast(w)
                else // w is in same or next level
                    setLabel(e, CROSS)
    i ← i + 1
```



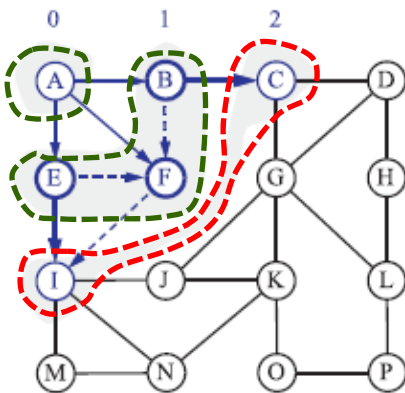
Example of BFS



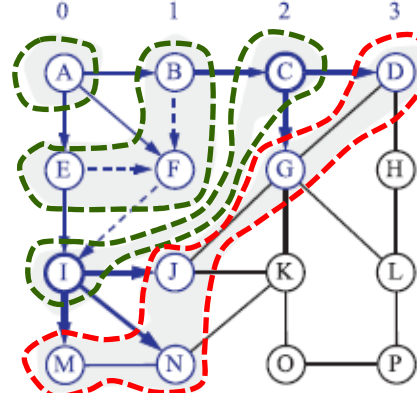
(a)



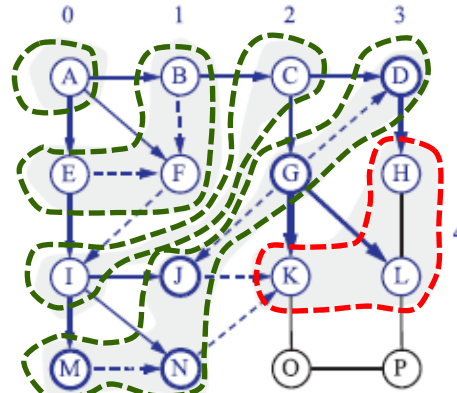
(b)



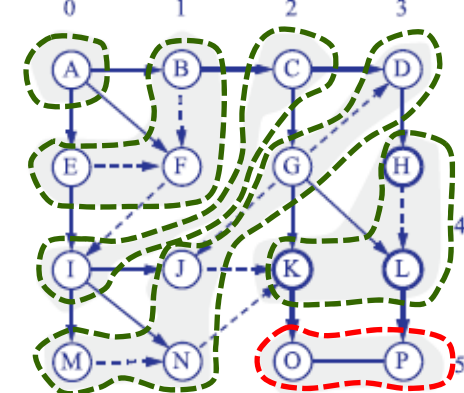
(c)



(d)



(e)



(f)



Properties

Notation

G_s : connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

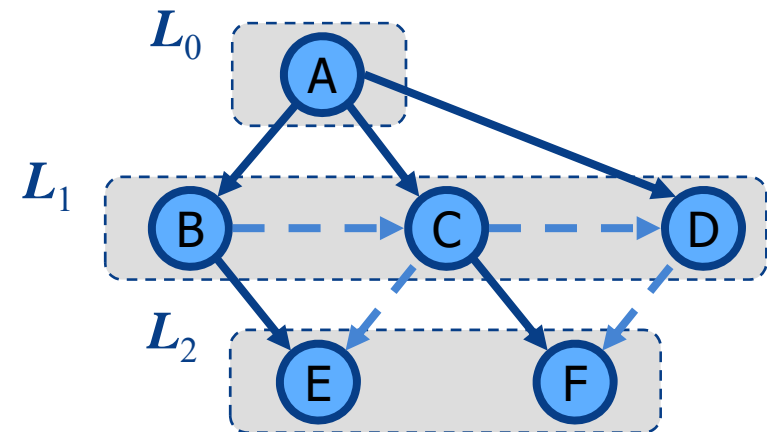
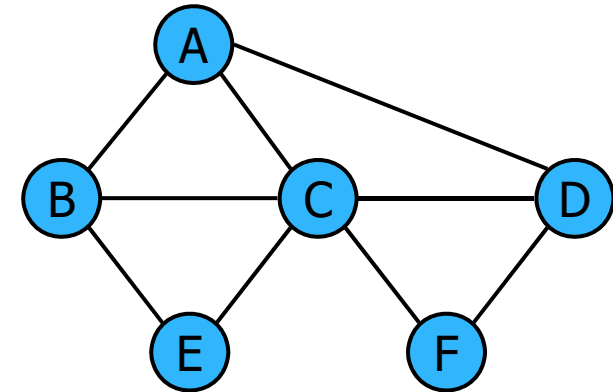
Property 2

The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Property 3

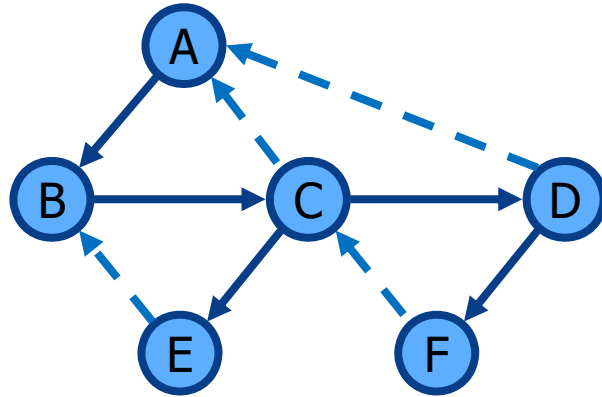
For each vertex v in L_i

- The path of T_s from s to v has i edges
- Every path from s to v in G_s has at least i edges

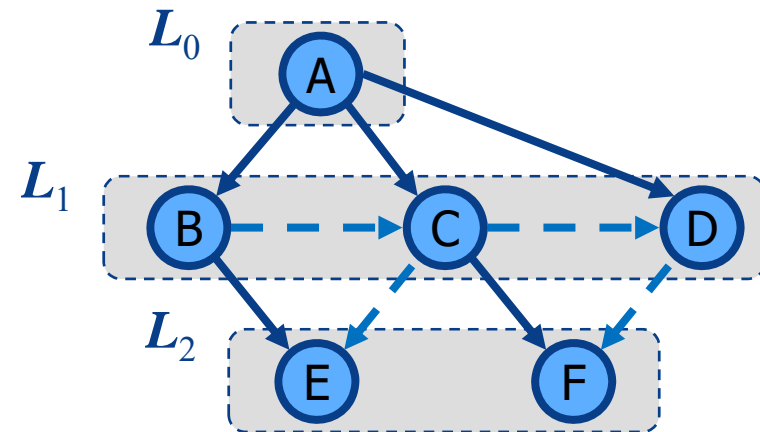


DFS vs. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



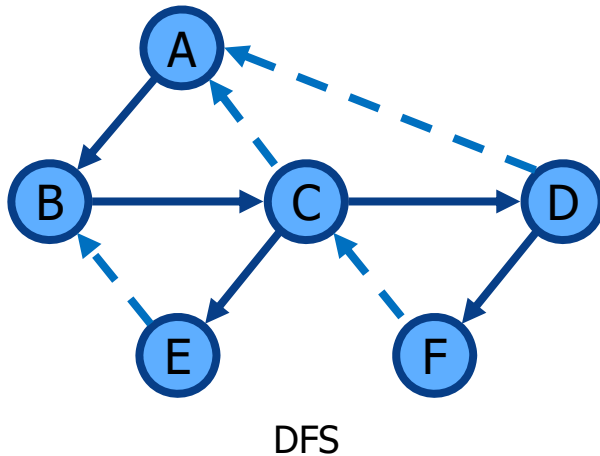
BFS



DFS vs. BFS (cont.)

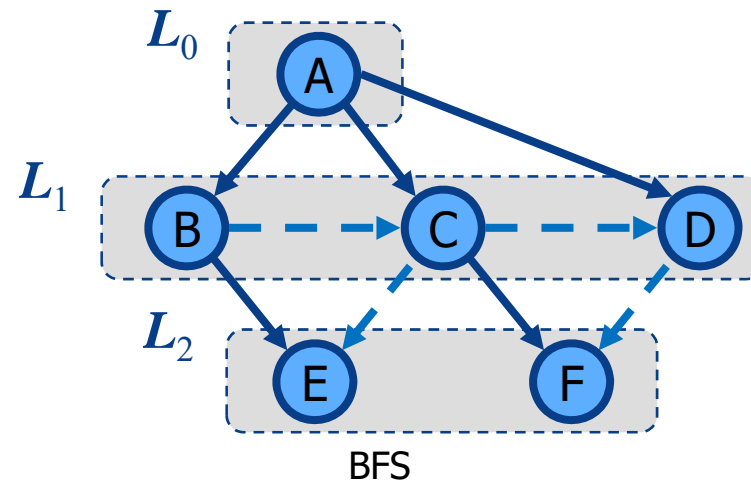
Back edge (v, w)

- w is an ancestor of v in the tree of discovery edges



Cross edge (v, w)

- w is in the same level as v or in the next level in the tree of discovery edges



Shortest Paths

◆ Weighted graphs

- Shortest path problem
- Shortest path properties

◆ Dijkstra's algorithm

- Algorithm
- Edge relaxation

◆ The Bellman-Ford algorithm

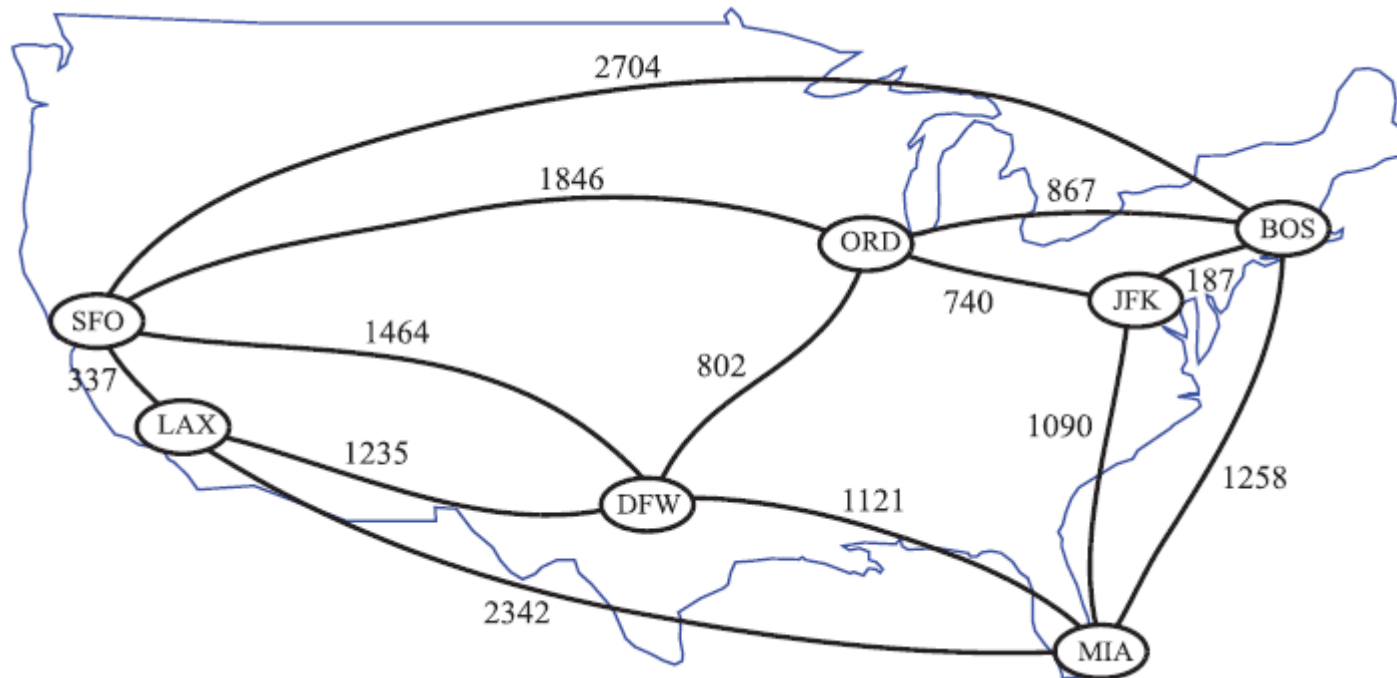
◆ Shortest paths in DAGs (Directed Acyclic Graphs)

◆ All-pairs shortest paths



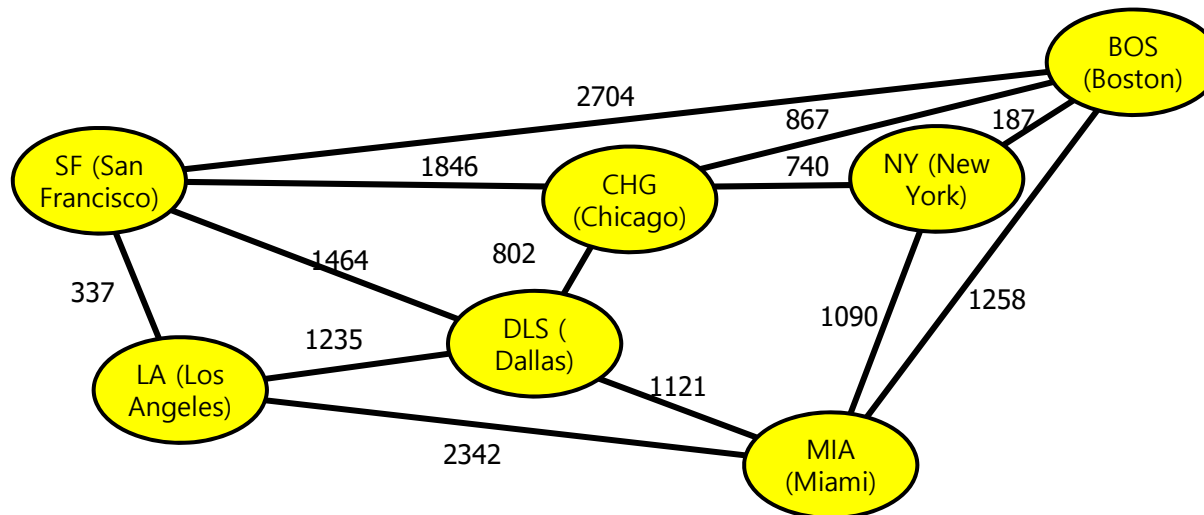
Weighted Graphs

- ◆ In a weighted graph, each edge has an associated numerical value, called the **weight** of the edge
- ◆ Edge weights may represent distances, costs, etc.
- ◆ **Example:**
 - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



Shortest Path Problem

- ◆ Given a weighted graph and two vertices u and v , we want to find a path of minimum total weight between u and v .
 - Length of a path is the sum of the weights of its edges.
- ◆ **Example:**
 - Shortest path between Providence (Rhode Island, USA) and Honolulu (Hawaii, USA)
- ◆ **Applications**
 - Internet packet routing
 - Flight reservations
 - Driving directions



Dijkstra's Algorithm

- ◆ The distance of a vertex v from a vertex s is the length of a shortest path between s and v
- ◆ Dijkstra's algorithm computes the distances of all the vertices from a given start vertex s
- ◆ Assumptions:
 - the graph is connected
 - the edges are undirected
 - the edge weights are nonnegative
- ◆ We grow a "cloud" of vertices, beginning with s and eventually covering all the vertices
- ◆ We store with each vertex v a label $d(v)$ representing the distance of v from s in the subgraph consisting of the cloud and its adjacent vertices
- ◆ At each step
 - We add to the cloud the vertex u outside the cloud with the smallest distance label, $d(u)$
 - We update the labels of the vertices adjacent to u



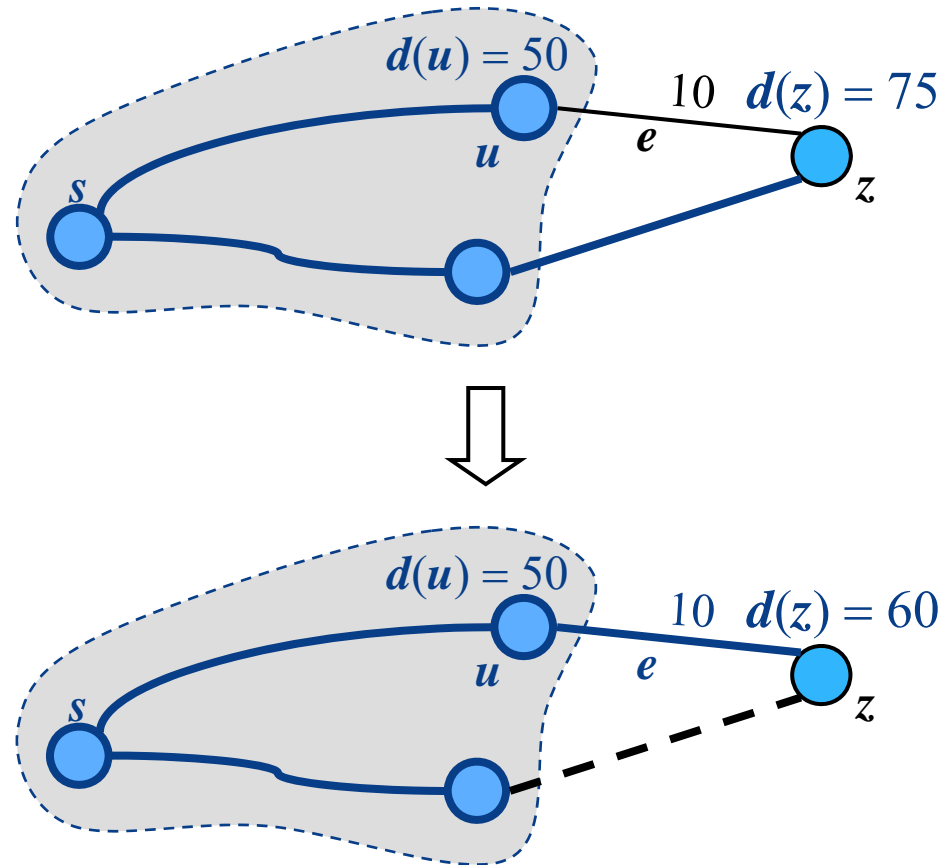
Edge Relaxation

◆ Consider an edge $e = (u, z)$ such that

- u is the vertex most recently added to the cloud
- z is not in the cloud

◆ The *relaxation* of edge e updates distance $d(z)$ as follows:

$$d(z) \leftarrow \min\{d(z), d(u) + \text{weight}(e)\}$$



class BreadthFirstSearch

```
/** BFS_Dijkstra.h (1)*/

#ifndef BFS_DIJKSTRA_H
#define BFS_DIJKSTRA_H

#include "Graph.h"
#include <fstream>
using namespace std;

typedef Graph::Vertex Vertex;
typedef Graph::Edge Edge;
typedef std::list<Graph::Vertex> VrtxList;
typedef std::list<Graph::Edge> EdgeList;
typedef std::list<Graph::Vertex>::iterator VrtxItr;
typedef std::list<Graph::Edge>::iterator EdgeItr;

class BreadthFirstSearch
{
protected:
    Graph& graph;
    bool done;// flag of search done
    int **ppDistMtrx; // distance matrix

    void initialize();
    bool isValidID(int vid) { return graph.isValidID(vid); }
    int getNumVertices() { return graph.getNumVertices(); }
```



```

/** BFS_Dijkstra.h (2)*/

public:
    BreadthFirstSearch(Graph& g) : graph(g) {
        int num_nodes;

        num_nodes = g.getNumVertices();
        // initialize DistMtrx
        for (int i = 0; i<num_nodes; i++)
            ppDistMtrx = new int*[num_nodes];
        for (int i = 0; i<num_nodes; i++)
            ppDistMtrx[i] = new int[num_nodes];

        for (int i = 0; i<num_nodes; i++) {
            for (int j = 0; j<num_nodes; j++)
            {
                ppDistMtrx[i][j] = PLUS_INF;
            }
        }
    }
    void initDistMtrx();
    void fprintDistMtrx(ofstream& fout);
    void DijkstraShortestPathTree(ofstream& fout, Vertex& s, int* pPrev);
    void DijkstraShortestPath(ofstream& fout, Vertex& s, Vertex& t, VrtxList& path);
    Graph& getGraph() { return graph; }
    int** getppDistMtrx() { return ppDistMtrx; }
};
#endif

```



```

/** DijkstraShortestPath() (1) */
void BreadthFirstSearch::DijkstraShortestPath(ofstream& fout, Vertex& start, Vertex &target,
VrtxList& path)
{
    int** ppDistMtrx;
    int* pLeastCost;
    int num_nodes, num_selected;
    int minID, minCost;
    BFS_PROCESS_STATUS* pBFS_Process_Stat;
    int *pPrev;

    Vertex* pVrtxArray;
    Vertex vrtx, *pPrevVrtx, v;
    Edge e;
    int start_vID, target_vID, curVID, vID;
    EdgeList* pAdjLstArray;

    pVrtxArray = graph.getpVrtxArray();
    pAdjLstArray = graph.getpAdjLstArray();
    start_vID = start.getID();
    target_vID = target.getID();

    num_nodes = getNumVertices();
    ppDistMtrx = getppDistMtrx();

    pLeastCost = new int[num_nodes];
    pPrev = new int[num_nodes];
    pBFS_Process_Stat = new BFS_PROCESS_STATUS[num_nodes];

```




```

/** DijkstraShortestPath() (2) */

for (int i = 0; i < num_nodes; i++)
{
    pLeastCost[i] = ppDistMtrx[start_vID][i];
    pPrev[i] = start_vID;
    pBFS_Process_Stat[i] = NOT_SELECTED;
}
pBFS_Process_Stat[start_vID] = SELECTED;
num_selected = 1;
path.clear();

int round = 0;
int cost;
string vName;

fout << "Dijkstra::Least Cost from Vertex (" << start.getName() << ") at each round : " << endl;
fout << "      |";
for (int i = 0; i < num_nodes; i++)
{
    vName = pVrtxArray[i].getName();
    fout << setw(5) << vName;
}
fout << endl;
fout << "-----+";
for (int i = 0; i < num_nodes; i++)
{
    fout << setw(5) << "-----";
}
fout << endl;

```



```

/** DijkstraShortestPath() (3) */

while (num_selected < num_nodes)
{
    round++;
    fout << "round [" << setw(2) << round << "]" |";
    // find current node with LeastCost
    minID = -1;    minCost = PLUS_INF;
    for (int i = 0; i<num_nodes; i++)
    {
        if ((pLeastCost[i] < minCost) && (pBFS_Process_Stat[i] != SELECTED)) {
            minID = i;
            minCost = pLeastCost[i];
        }
    }
    if (minID == -1) {
        fout << "Error in Dijkstra() -- found not connected vertex !!" << endl;
        break;
    } else {
        pBFS_Process_Stat[minID] = SELECTED;
        num_selected++;

        if (minID == target_vID)
        {
            fout << endl << "reached to the target node (" << pVrtxArray[minID].getName()
                << ") at Least Cost = " << minCost << endl;
            vID = minID;
            do {
                vrtx = pVrtxArray[vID];
                path.push_front(vrtx);
                vID = pPrev[vID];
            } while (vID != start_vID);
            vrtx = pVrtxArray[vID];
            path.push_front(vrtx); // start node
            break;
        }
    }
}

```



```

/** DijkstraShortestPath() (4) */

/* Edge relaxation */
int pLS, ppDistMtrx_i;
for (int i = 0; i < num_nodes; i++)
{
    pLS = pLeastCost[i];
    ppDistMtrx_i = ppDistMtrx[minID][i];

    if ((pBFS_Process_Stat[i] != SELECTED) && (pLeastCost[i] >
        (pLeastCost[minID] + ppDistMtrx[minID][i])))
    {
        pPrev[i] = minID;
        pLeastCost[i] = pLeastCost[minID] + ppDistMtrx[minID][i];
    }
}

// print out the pLeastCost[] for debugging
for (int i = 0; i < num_nodes; i++)
{
    cost = pLeastCost[i];
    if (cost == PLUS_INF)
        fout << " +oo";
    else
        fout << setw(5) << pLeastCost[i];
}
fout << " ==> selected vertex : " << pVrtxArray[minID];
fout << endl;

} // end while()

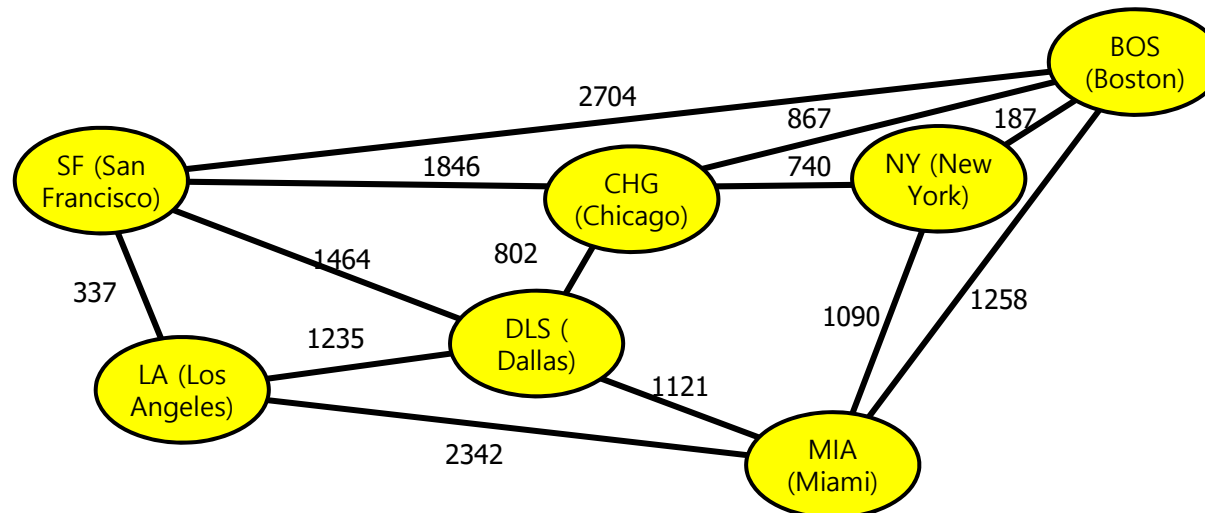
}

```



Sample Graph

◆ Simplified USA Topology



```

/** GRAPH_USA_7_NODES (1) */
#define NUM_NODES 7
#define NUM_EDGES 26

Vertex v[NUM_NODES] = // 7 nodes
{
    Vertex("SF", 0),
    Vertex("LA", 1),
    Vertex("DLS", 2),
    Vertex("CHG", 3),
    Vertex("MIA", 4),
    Vertex("NY", 5),
    Vertex("BOS", 6)
};

```

```

Graph::Edge edges[NUM_EDGES] = // 70 edges
{
    Edge(v[0], v[1], 337), Edge(v[1], v[0], 337),
    Edge(v[0], v[2], 1464), Edge(v[2], v[0], 1464),
    Edge(v[0], v[3], 1846), Edge(v[3], v[0], 1846),
    Edge(v[0], v[6], 2704), Edge(v[6], v[0], 2704),
    Edge(v[1], v[2], 1235), Edge(v[2], v[1], 1235),
    Edge(v[1], v[4], 2342), Edge(v[4], v[1], 2342),
    Edge(v[2], v[3], 802), Edge(v[3], v[2], 802),
    Edge(v[2], v[4], 1121), Edge(v[4], v[2], 1121),
    Edge(v[3], v[5], 740), Edge(v[5], v[3], 740),
    Edge(v[3], v[6], 867), Edge(v[6], v[3], 867),
    Edge(v[5], v[4], 1090), Edge(v[4], v[5], 1090),
    Edge(v[5], v[6], 187), Edge(v[6], v[5], 187),
    Edge(v[4], v[6], 1258), Edge(v[6], v[4], 1258),
};

int test_start = 1;
int test_end = 6;

```



```
/** GRAPH_USA_20_NODES (2) */
```

```
Edge edges[NUM_EDGES] =  
{  
    Edge(v[0], v[1], 820), Edge(v[1], v[0], 820),  
    Edge(v[0], v[3], 828), Edge(v[3], v[0], 828),  
    Edge(v[0], v[4], 1144), Edge(v[4], v[0], 1144),  
    Edge(v[1], v[2], 380), Edge(v[2], v[1], 380),  
    Edge(v[1], v[3], 745), Edge(v[3], v[1], 745),  
    Edge(v[2], v[3], 688), Edge(v[3], v[2], 688),  
    Edge(v[2], v[6], 381), Edge(v[6], v[2], 381),  
    Edge(v[3], v[4], 657), Edge(v[4], v[3], 657),  
    Edge(v[3], v[5], 521), Edge(v[5], v[3], 521),  
    Edge(v[4], v[5], 389), Edge(v[5], v[4], 389),  
    Edge(v[4], v[7], 611), Edge(v[7], v[4], 611),  
    Edge(v[5], v[6], 816), Edge(v[6], v[5], 816),  
    Edge(v[5], v[7], 920), Edge(v[7], v[5], 920),  
    Edge(v[5], v[8], 780), Edge(v[8], v[5], 780),  
    Edge(v[5], v[12], 861), Edge(v[12], v[5], 861),  
    Edge(v[6], v[8], 1067), Edge(v[8], v[6], 1067),  
    Edge(v[7], v[13], 409), Edge(v[13], v[7], 409),  
    Edge(v[8], v[9], 246), Edge(v[9], v[8], 246),  
    Edge(v[8], v[11], 454), Edge(v[11], v[8], 454),  
    Edge(v[9], v[10], 352), Edge(v[10], v[9], 352),  
    Edge(v[10], v[11], 393), Edge(v[11], v[10], 393),  
    Edge(v[10], v[15], 861), Edge(v[15], v[10], 861),  
    Edge(v[10], v[16], 473), Edge(v[16], v[10], 473),
```

```
/** GRAPH_USA_20_NODES (3) */
```

```
    Edge(v[11], v[12], 285), Edge(v[12], v[11], 285),  
    Edge(v[11], v[16], 394), Edge(v[16], v[11], 394),  
    Edge(v[12], v[13], 297), Edge(v[13], v[12], 297),  
    Edge(v[12], v[17], 845), Edge(v[17], v[12], 845),  
    Edge(v[13], v[14], 286), Edge(v[14], v[13], 286),  
    Edge(v[14], v[17], 534), Edge(v[17], v[14], 534),  
    Edge(v[14], v[18], 640), Edge(v[18], v[14], 640),  
    Edge(v[14], v[19], 834), Edge(v[19], v[14], 834),  
    Edge(v[15], v[16], 661), Edge(v[16], v[15], 661),  
    Edge(v[16], v[17], 632), Edge(v[17], v[16], 632),  
    Edge(v[17], v[18], 237), Edge(v[18], v[17], 237),  
    Edge(v[18], v[19], 211), Edge(v[19], v[18], 211)  
};  
  
int test_start = 0;  
int test_end = 15;
```



```

/** main.cpp (1) */
. . . . // include necessary header files and definitions

void main()
{
    . . . . . // include necessary elements
    Graph simpleGraph("GRAPH_SIMPLE_USA_7_NODES", NUM_NODES);

    fout << "Inserting vertices .." << endl;
    . . . . // insert vertices

    VrtxList vtxLst;
    simpleGraph.vertices(vtxLst);

    fout << "Inserted vertices: ";
    . . . // printout inserted vertices

    fout << "Inserting edges .." << endl;
    . . . . // insert edges

    fout << "Inserted edges: " << endl;
    . . . . // printout inserted edges

    fout << "Print out Graph based on Adjacency List .." << endl;
    simpleGraph.fprintGraph(fout);
}

```



```

/** main.cpp (2) */

/* ===== */
VrtxList path;
BreadthFirstSearch bfsGraph(simpleGraph);

fout << "\nTesting Breadth First Search with Dijkstra Algorithm" << endl;

bfsGraph.initDistMtrx();

//fout << "Distance matrix of BFS for Graph:" << endl;
bfsGraph.fprintDistMtrx(fout);

path.clear();
fout << "\nDijkstra Shortest Path Finding from " << v[test_start].getName() << " to "
    << v[test_end].getName() << " .... " << endl;
bfsGraph.DijkstraShortestPath(fout, v[test_start], v[test_end], path);
fout << "Path found by DijkstraShortestPath from " << v[test_start] << " to " << v[test_end] << " : ";
for (VrtxItr vltor = path.begin(); vltor != path.end(); ++vltor)
{
    fout << *vltor;
    if (*vltor != v[test_end])
        fout << " -> ";
}
fout << endl;

fout.close();
}

```



◆ Execution results (1)

```
Inserting vertices ..
Inserted vertices: SF, LA, DLS, CHG, MIA, NY, BOS,
Inserting edges ..
Inserted edges:
Edge(SF, LA, 337), Edge(SF, DLS, 1464), Edge(SF, CHG, 1846), Edge(SF, BOS, 2704), Edge(LA, SF, 337),
Edge(LA, DLS, 1235), Edge(LA, MIA, 2342), Edge(DLS, SF, 1464), Edge(DLS, LA, 1235), Edge(DLS, CHG, 802),
Edge(DLS, MIA, 1121), Edge(CHG, SF, 1846), Edge(CHG, DLS, 802), Edge(CHG, NY, 740), Edge(CHG, BOS, 867),
Edge(MIA, LA, 2342), Edge(MIA, DLS, 1121), Edge(MIA, NY, 1090), Edge(MIA, BOS, 1258), Edge(NY, CHG, 740),
Edge(NY, MIA, 1090), Edge(NY, BOS, 187), Edge(BOS, SF, 2704), Edge(BOS, CHG, 867), Edge(BOS, NY, 187),
Edge(BOS, MIA, 1258),
Print out Graph based on Adjacency List ..
GRAPH_SIMPLE_USA_7_NODES with 7 vertices has following adjacency lists :
vertex ( SF) : Edge(SF, LA, 337) Edge(SF, DLS, 1464) Edge(SF, CHG, 1846) Edge(SF, BOS, 2704)
vertex ( LA) : Edge(LA, SF, 337) Edge(LA, DLS, 1235) Edge(LA, MIA, 2342)
vertex (DLS) : Edge(DLS, SF, 1464) Edge(DLS, LA, 1235) Edge(DLS, CHG, 802) Edge(DLS, MIA, 1121)
vertex (CHG) : Edge(CHG, SF, 1846) Edge(CHG, DLS, 802) Edge(CHG, NY, 740) Edge(CHG, BOS, 867)
vertex (MIA) : Edge(MIA, LA, 2342) Edge(MIA, DLS, 1121) Edge(MIA, NY, 1090) Edge(MIA, BOS, 1258)
vertex ( NY) : Edge(NY, CHG, 740) Edge(NY, MIA, 1090) Edge(NY, BOS, 187)
vertex (BOS) : Edge(BOS, SF, 2704) Edge(BOS, CHG, 867) Edge(BOS, NY, 187) Edge(BOS, MIA, 1258)
```

Testing Breadth First Search with Dijkstra Algorithm

Distance Matrix of Graph (GRAPH_SIMPLE_USA_7_NODES) :

	SF	LA	DLS	CHG	MIA	NY	BOS
SF	0	337	1464	1846	+oo	+oo	2704
LA	337	0	1235	+oo	2342	+oo	+oo
DLS	1464	1235	0	802	1121	+oo	+oo
CHG	1846	+oo	802	0	+oo	740	867
MIA	+oo	2342	1121	+oo	0	1090	1258
NY	+oo	+oo	+oo	740	1090	0	187
BOS	2704	+oo	+oo	867	1258	187	0

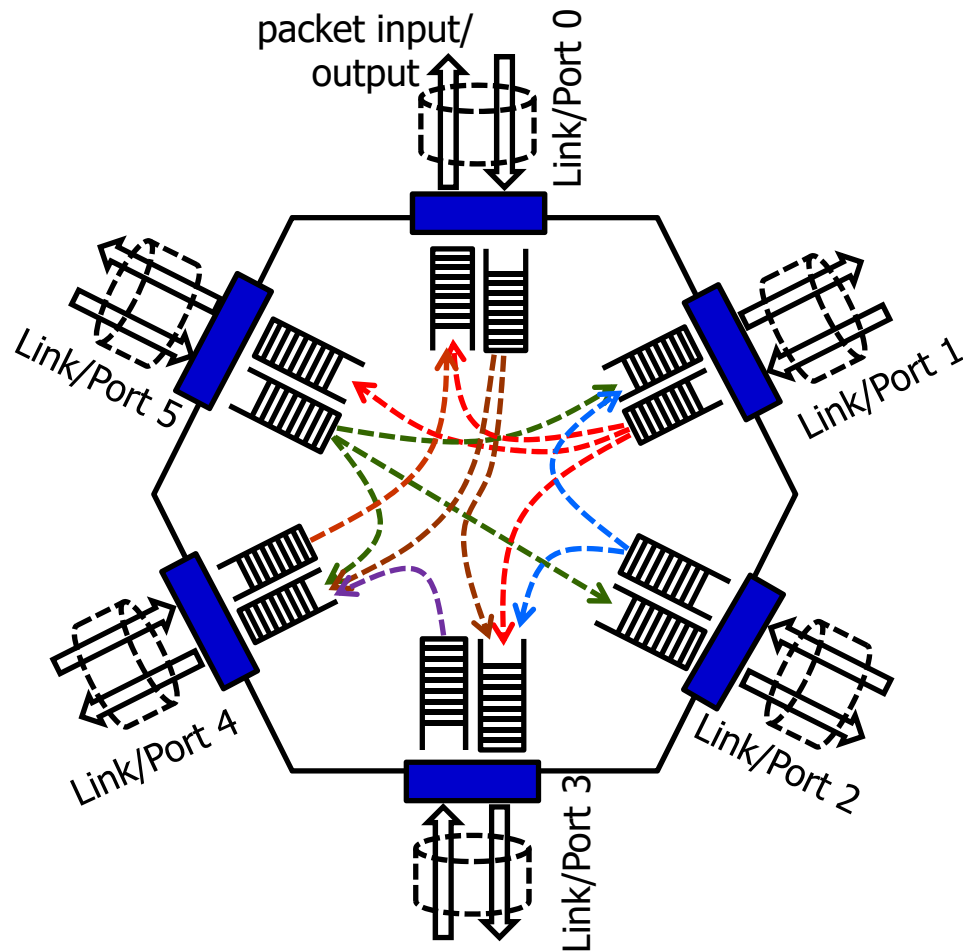


◆ Execution results (2)

```
Dijkstra Shortest Path Finding from LA to BOS ....
Dijkstra::Least Cost from Vertex (LA) at each round :
      |  SF  LA  DLS  CHG  MIA  NY  BOS
-----+-----
round [ 1] | 337   0 1235 2183 2342 +oo 3041 ==> selected vertex : SF
round [ 2] | 337   0 1235 2037 2342 +oo 3041 ==> selected vertex : DLS
round [ 3] | 337   0 1235 2037 2342 2777 2904 ==> selected vertex : CHG
round [ 4] | 337   0 1235 2037 2342 2777 2904 ==> selected vertex : MIA
round [ 5] | 337   0 1235 2037 2342 2777 2904 ==> selected vertex : NY
round [ 6] |
reached to the target node (BOS) at Least Cost = 2904
Path found by DijkstraShortestPath from LA to BOS : LA -> DLS -> CHG -> BOS
```



Internet Protocol (IP) Router Model



Packet Forwarding Table

Dest IP address	Next hop	Tx Link/Port No



Implementation of Router

◆ Basic Operation of Router

- obtain the network topology data: nodes, links
- calculate **shortest paths tree** to all other nodes from this node
- configure **forwarding table** (next hop for a target address)
- (optional for simulator) generate packets to all nodes (including itself)
- check the input queue for packet arrivals
- obtain the target address of the arrived packet
- if a packet to this node is arrived:
 print the route, and delete
- if a transit packet (to other node) is arrived:
 push the router ID into the packet to record route
 forward the packet to "next_hop" router



Packet Forwarding Table

◆ Initialization of Packet Forwarding Table

- for each destination address, the next hop (from this node) is found from the shortest paths tree (calculated by Dijkstra's algorithm)

◆ Packet Forwarding at each Router

- for each arrived packet, obtain the target address
- if a packet to this node is arrived:
print the route, and delete
- if a transit packet (to other node) is arrived:
push the router ID into the packet to record route
forward the packet to "next_hop" router



Oral Test 13 (1)

13.1 그래프를 표현하기 위하여 사용되는 자료구조들을 그림으로 표현하고, 그 복잡도 (complexity)를 정점의 개수와 간선의 개수로 표현하라.

<Key Points>

- (1) Vertex List, Edge List
- (2) Adjacency List
- (3) Adjacency Matrix
- (4) 복잡도 분석표

13.2 2 x 3 격자형 그래프에 대한 깊이 우선 탐색 (Depth First Search) 알고리즘을 실행하기 위하여 구현되는 dfsTraversal() 멤버 함수를 pseudocode로 표현하고, 상세 동작 절차를 주어진 2 x 3 격자형 그래프의 노드를 사용하여 설명하라.

<Key Points>

- (1) 2 x 3 격자형 그래프 (vertex 0, 1, 2, 3, 4, 5)
- (2) vertex 0으로 부터의 dfsTraversal() 실행에서 정점의 방문 순서와 간선의 구분 (discovery, back)
- (3) DFS로 탐색된 vertex 0 -> vertex 5의 경로
- (4) vertex 5로 부터의 dfsTraversal() 실행하는 경우 vertex 5 -> vertex 0의 경로
- (5) 위 (3)과 (4) 경로에 대한 비교 분석



Oral Test 13

13.3 그래프에 대한 깊이 우선 탐색 (Depth First Search) 알고리즘, 넓이 우선 탐색 (Breadth First Search) 알고리즘, Dijkstra 알고리즘의 차이점에 대하여 설명하라.

<Key Points>

- (1) Breadth First Search (BFS) 알고리즘의 동작 절차
- (2) Breadth First Search (BFS)의 한 종류인 Dijkstra 알고리즘을 사용한 최단거리 경로 탐색 (shortest path search) 기능의 동작 절차
- (3) Depth First Search (DFS)와 Breadth First Search (BFS) 알고리즘의 기능적 차이점과 주요 응용 분야에 대한 비교 설명

13.4 그래프의 자료구조의 주요 응용 분야에 대하여 상세하게 설명하라.

<Key Points>

- (1) 자동차/스마트폰의 네비게이션
- (2) 인터넷의 패킷 라우팅
- (3) 전자회로 부품 배치
- (4) 데이터 베이스의 연관 정보 검색을 통한 상관관계 분석

- 주요 응용 분야에서 정점과 간선은 어떤 정보를 의미하는가?
- 이 응용 분야에서 그래프 탐색의 목적은 무엇인가?

