EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

FACULTY OF
SCIENCE
Computer Graphics

Computer Graphics (Graphische Datenverarbeitung)

- Aliasing & Fourier Transform -

WS 2021/2022

Corona

- Regular random lookup of the 3G certificates

- Contact tracing: We need to know who is in the class room
  - New ILIAS group for every lecture slot
  - Register via ILIAS or this QR code (only if you are present in this room)

Overview

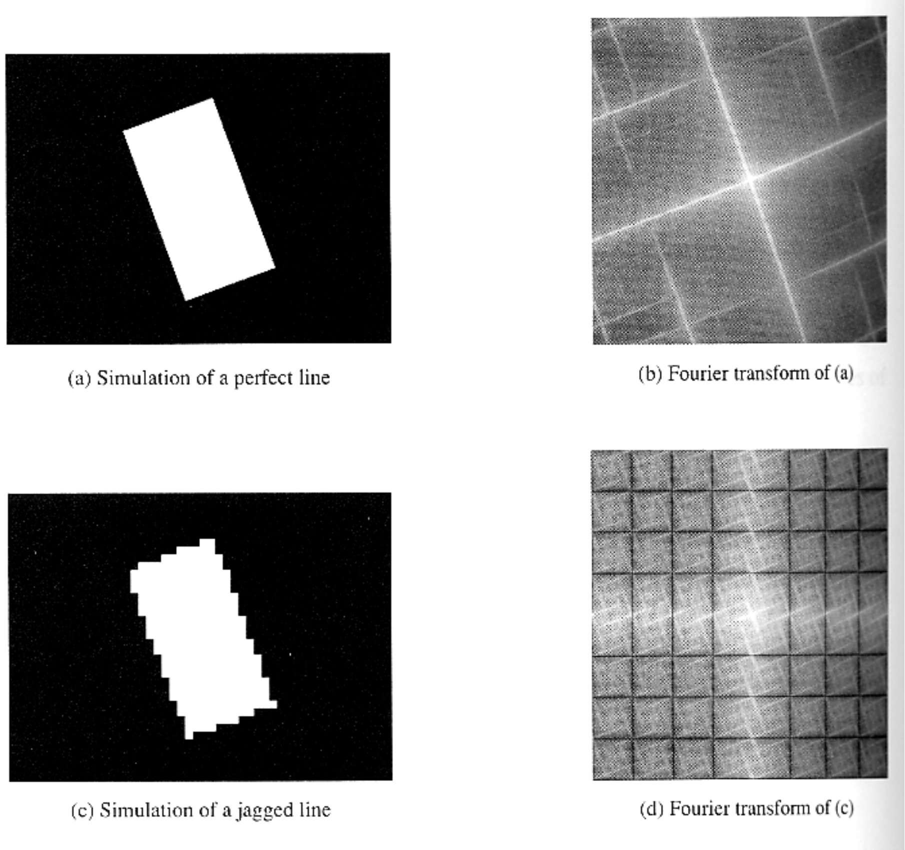
- Last lectures
  - Texture Parameterization
  - Procedural Shading
  - Various texture maps
  - Texture Filtering

- Today
  - Aliasing
  - Fourier Transform

- Next
  - Sampling Theory
  - Anti-aliasing

Aliasing

Aliasing

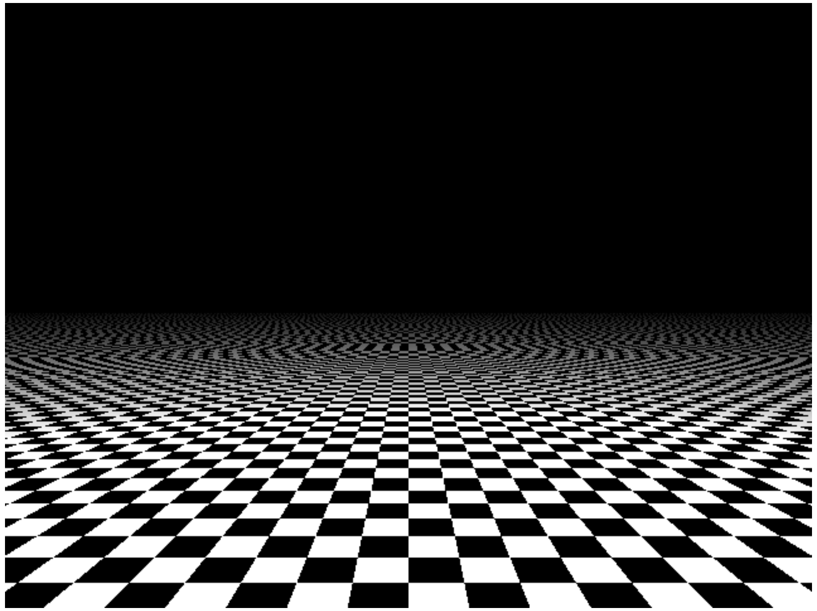


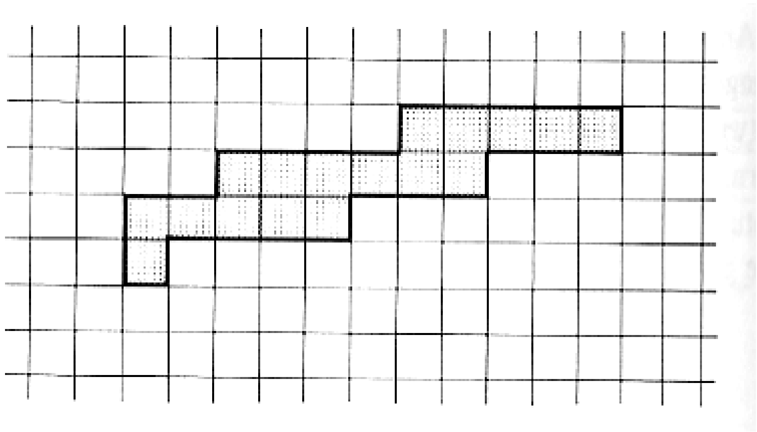
(a) Simulation of a perfect line

(b) Fourier transform of (a)

(c) Simulation of a jagged line

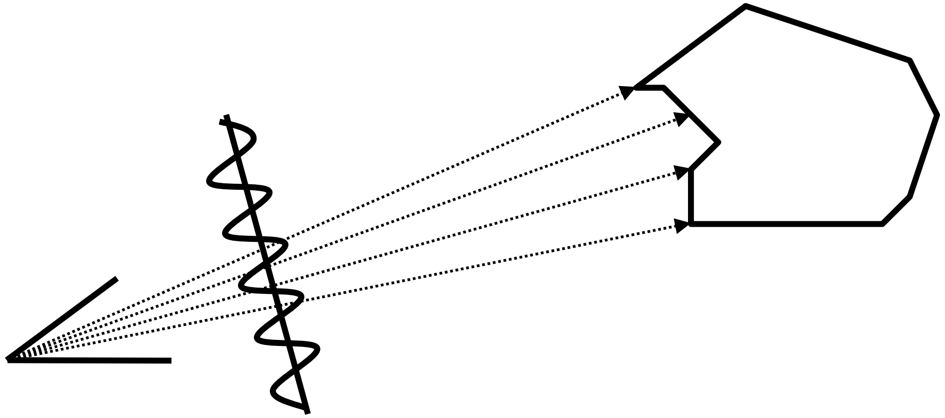
(d) Fourier transform of (c)

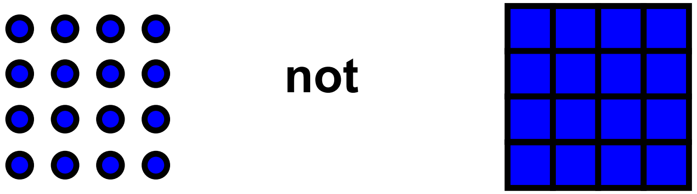




The Digital Dilemma

- Nature: continuous signal (2D/3D/4D with time)
  - Defined at every point
- **Acquisition: sampling**
  - Rays, pixel/texel, spectral values, frames, ...
- Representation: discrete data
  - Discrete points, discretized values
- **Reconstruction: filtering**
  - Mimic continuous signal
- Display and perception: faithful
  - Hopefully similar to the original signal, no artifacts





Sensors

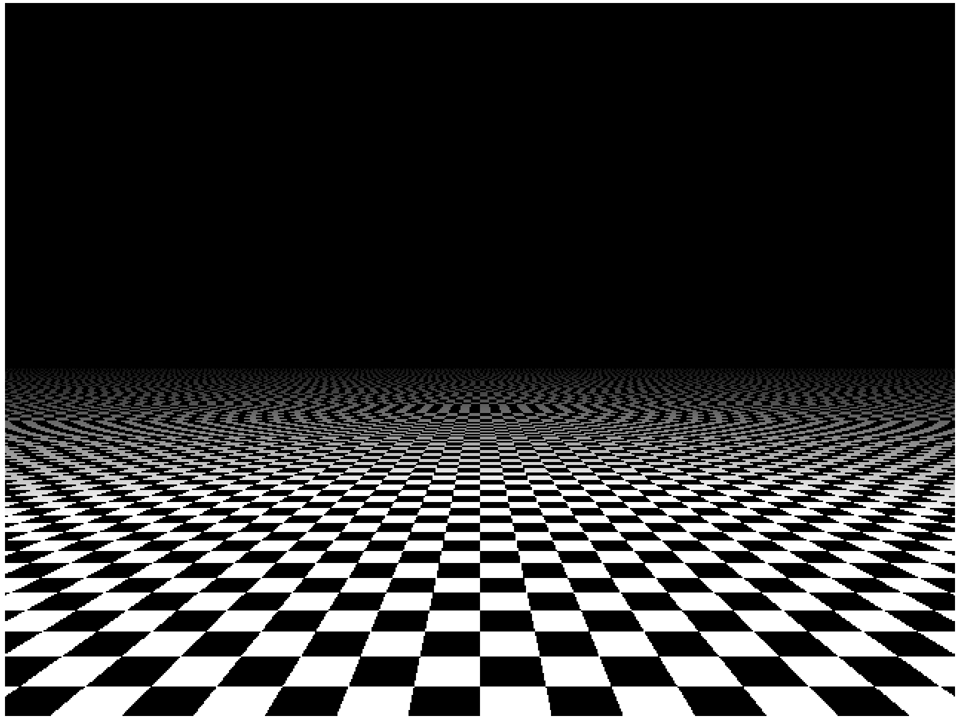
- Sampling of signals
  - Conversion of a continuous signal to discrete samples by integrating over the sensor field
  - Required by physical processes

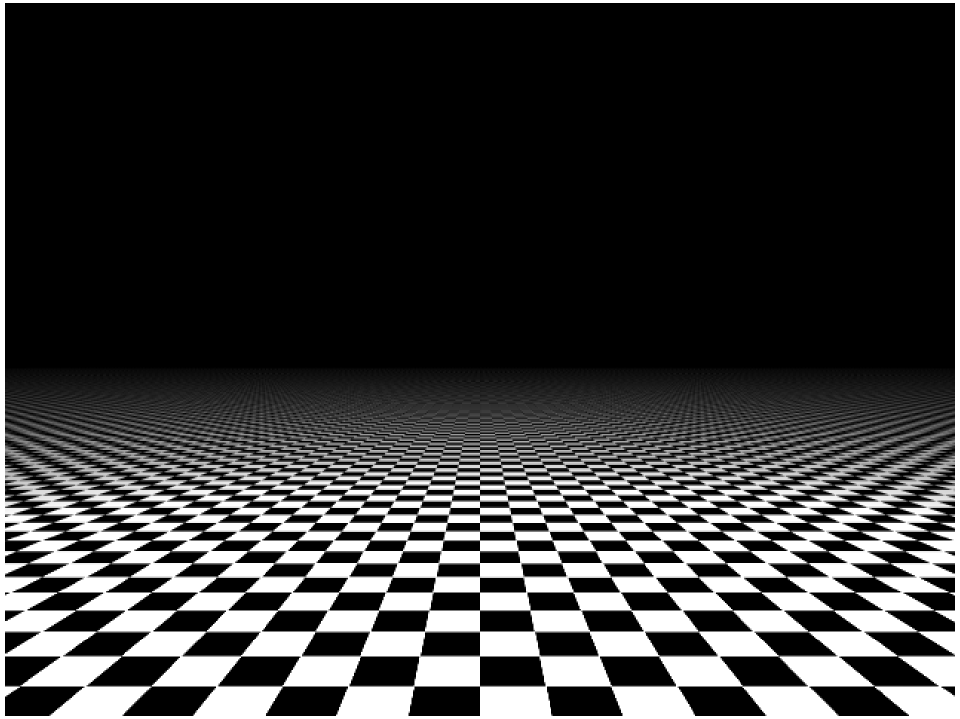
$$R(i,j)=\int_{A_{i\ \ j}} E(x,y)P_{i\ \ j}(x,y)\,d\ \ d\ x$$

- Examples
  - Photo receptors in the retina
  - CCD or CMOS cells in a digital camera
- Virtual cameras in computer graphics
  - Integration is too expensive and usually avoided
  - Ray tracing: mathematically ideal point samples
    - Origin of aliasing artifacts !

Aliasing

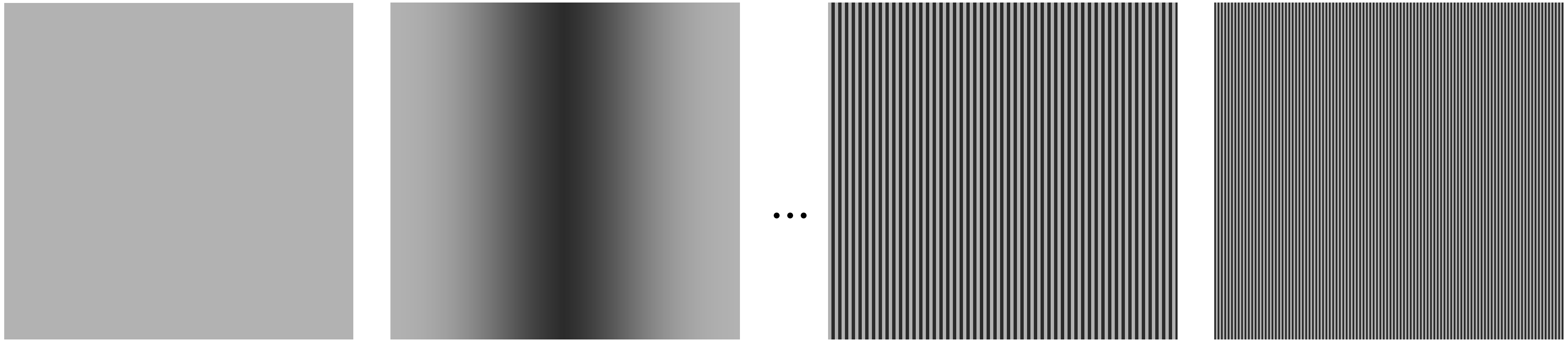
- Ray tracing
  - Textured plane with one ray for each pixel (say, at pixel center)
    - No texture filtering: equivalent to modeling with b/w tiles
  - Checkerboard period becomes smaller than two pixels
    - At the Nyquist limit
  - Hits textured plane at only one point, black or white by chance





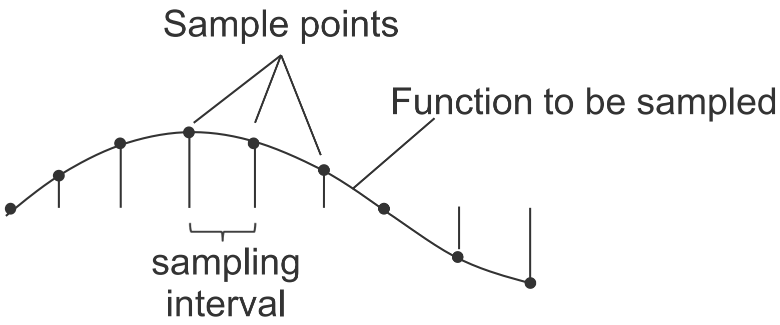
Spatial Frequency

- Frequency: period length of some structure in an image
  - Unit [1/pixel]
  - Range: -0.5…0.5 (-π…π)
- Lowest frequency
  - Image average
- Highest frequency: Nyquist limit
  - In nature: defined by wavelength of light
  - In graphics: defined by image resolution

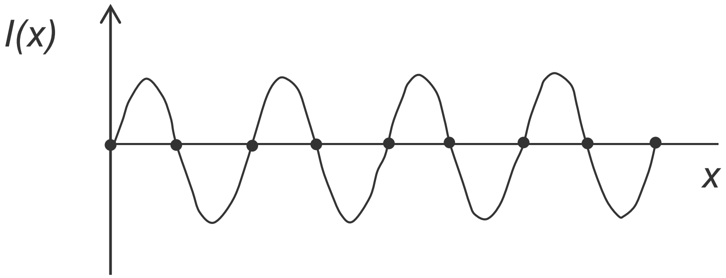


Nyquist Frequency

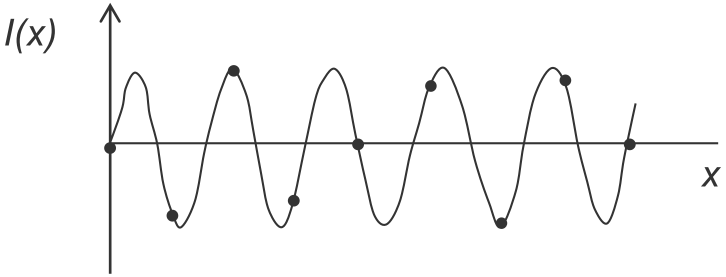
- Highest (spatial) frequency that can be represented
- Determined by image resolution (pixel size)



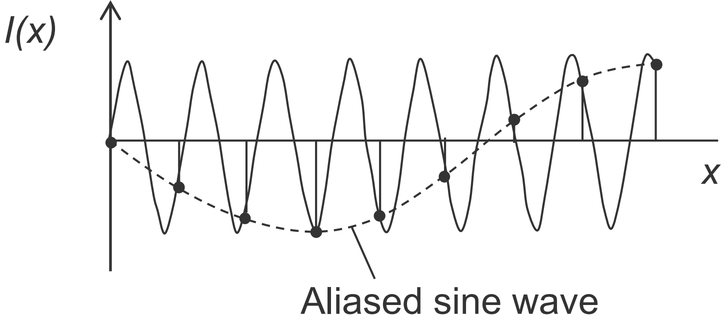
Spatial frequency < Nyquist



Spatial frequency = Nyquist
2 samples / period



Spatial frequency > Nyquist



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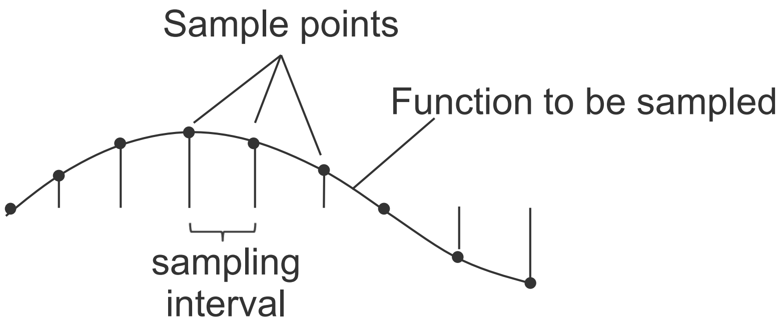

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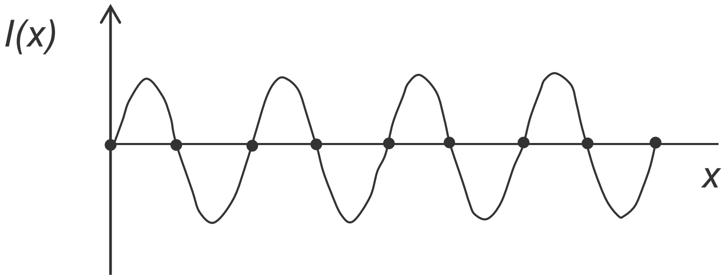

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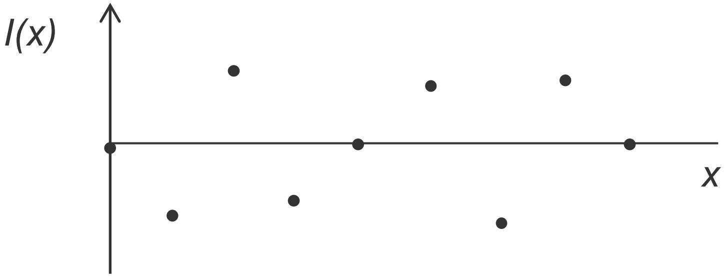

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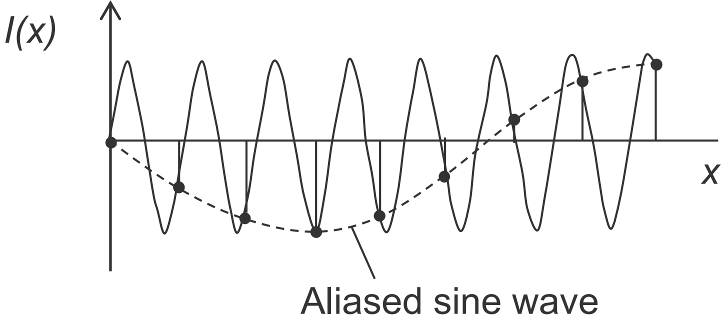

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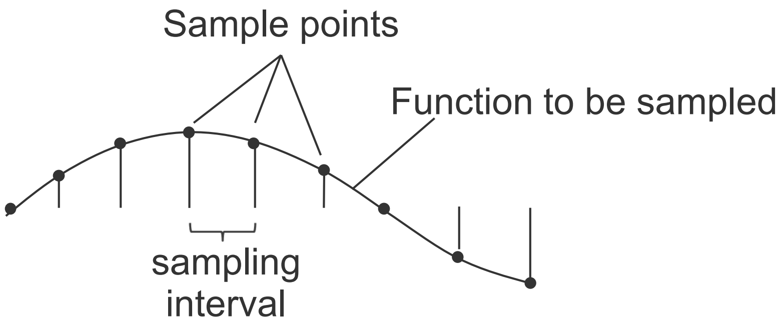
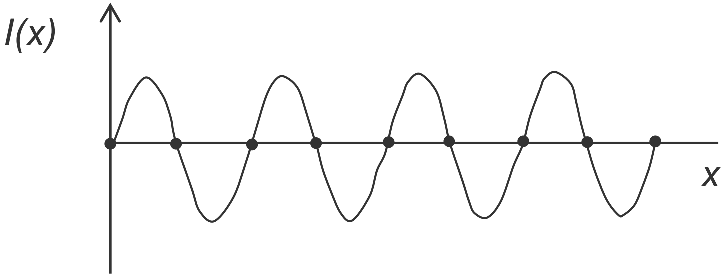


Spatial frequency < Nyquist



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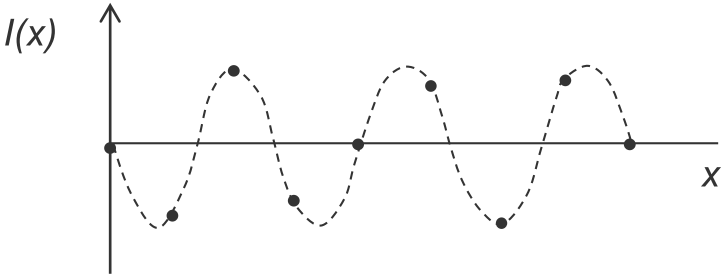

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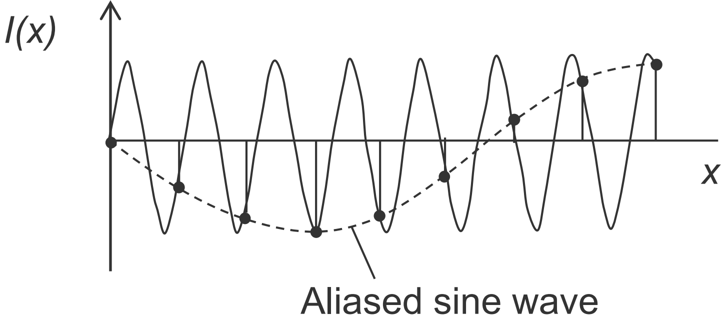

Spatial frequency >> Nyquist

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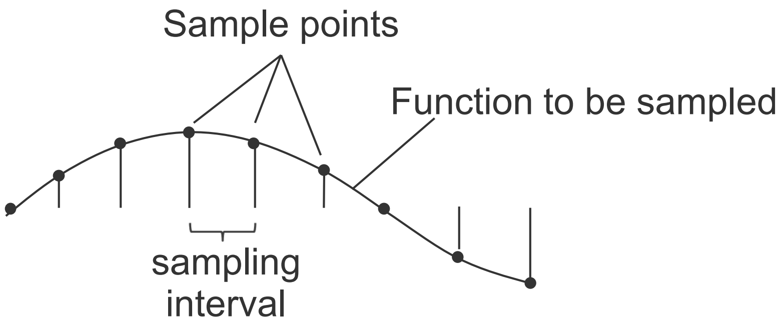

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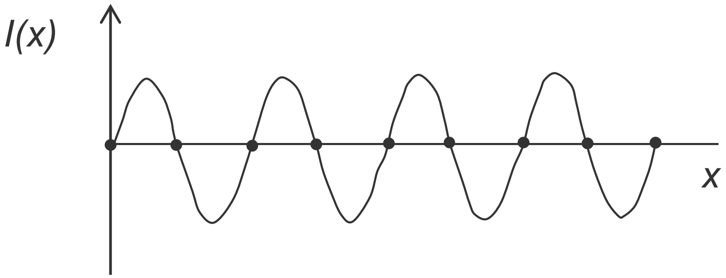

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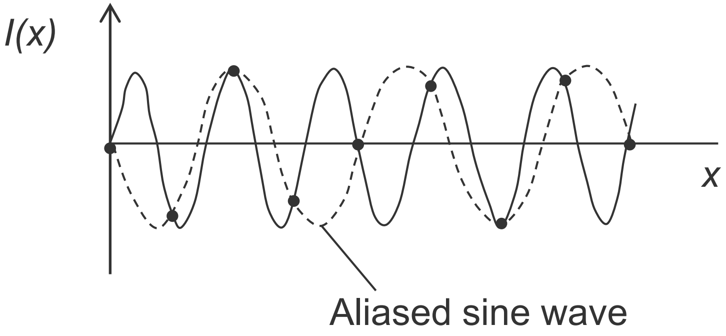
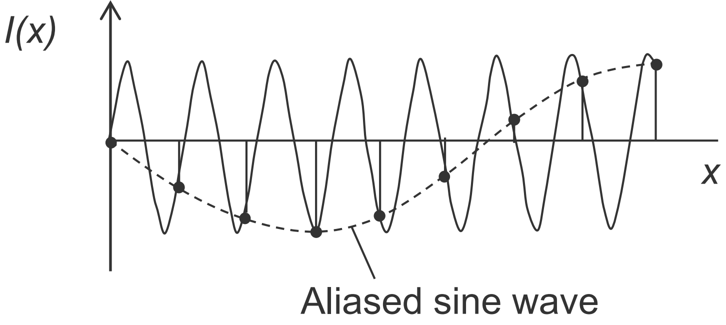


Spatial frequency > Nyquist



Spatial frequency >> Nyquist

Beispiel

http://www.falstad.com/fourier

Fourier Transform

Fourier Series

- Any periodic, continuous function can be expressed as the sum of an (infinite) number of sine or cosine waves:

$$f(x)=\Sigma_k \, a_k \sin(2\pi*k*x) + b_k \cos(2\pi*k*x)$$

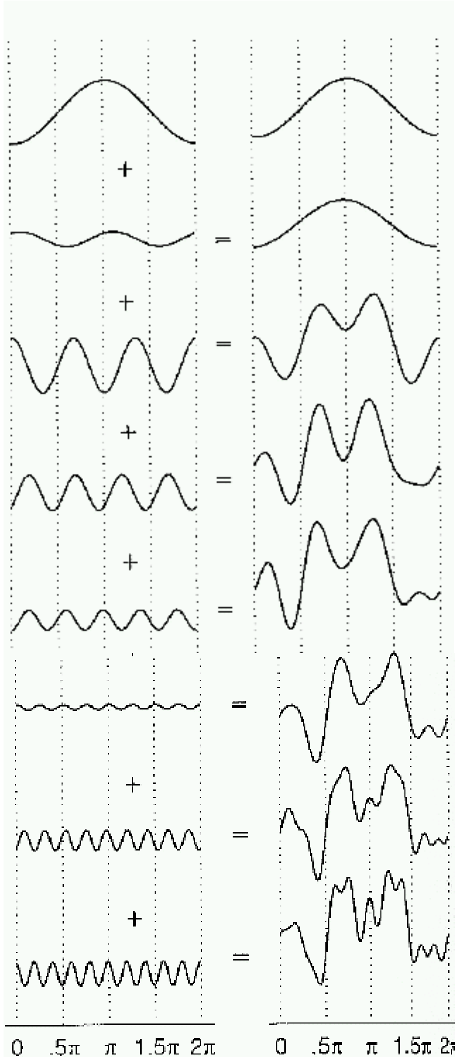
  - Decomposition of signal into different frequency bands
    - Spectral analysis
  - k: frequency band
    - k=0 mean value
    - k=1 function period, lowest possible frequency
    - k_{max}? band limit, no higher frequency present in signal
  - a_k,b_k : (real-valued) Fourier coefficients
    - Even function $f(x)= f(-x)$: $a_k = 0$
    - Odd function $f(x)= -f(-x)$: $b_k = 0$

Fourier Transformation

- Any continuous function f(x) can be expressed as an integral over sine and cosine waves:

$$F(k) = F_x\left[f(x)\right](k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x}\, dx \quad \text{Analysis}$$

$$f(x) = F_x^{-1}\left[F(k)\right](x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i k x}\, dk \quad \text{Synthesis}$$



$$e^{2\pi i \omega x} = \cos(2\pi\omega x) + i \sin(2\pi\omega x)$$

Fourier Transformation

- Any continuous function f(x) can be expressed as an integral over sine and cosine waves:

$$F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} dx \quad \text{Analysis}$$

$$f(x) = F_x^{-1}[F(k)](x) = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx} dk \quad \text{Synthesis}$$

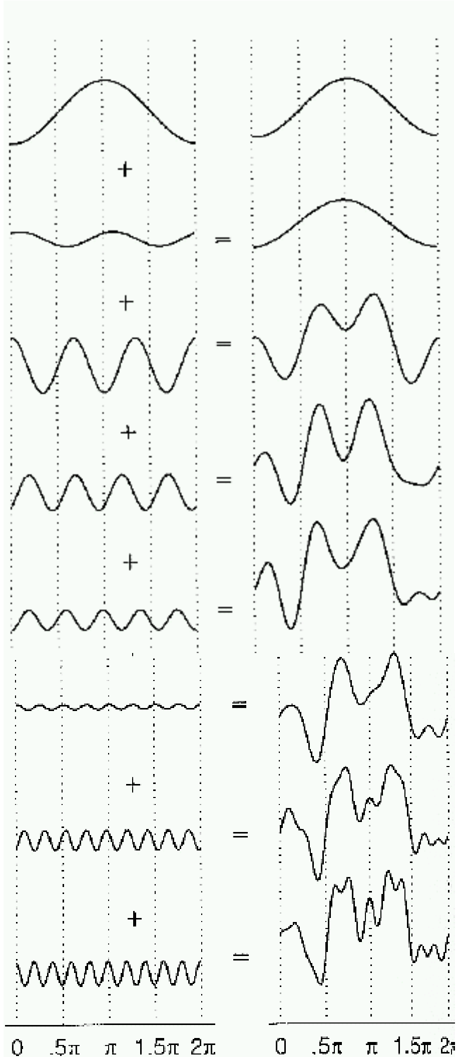
- Division into even and odd parts

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = E(x) + O(x)$$

- Transform of each part

$$F[f(x)](k) = \int_{-\infty}^{\infty} E(x)\cos(2\pi kx)dx - i\int_{-\infty}^{\infty} O(x)\sin(2\pi kx)dx$$

$$e^{2\pi i\omega x} = \cos(2\pi\omega x) + i\sin(2\pi\omega x)$$



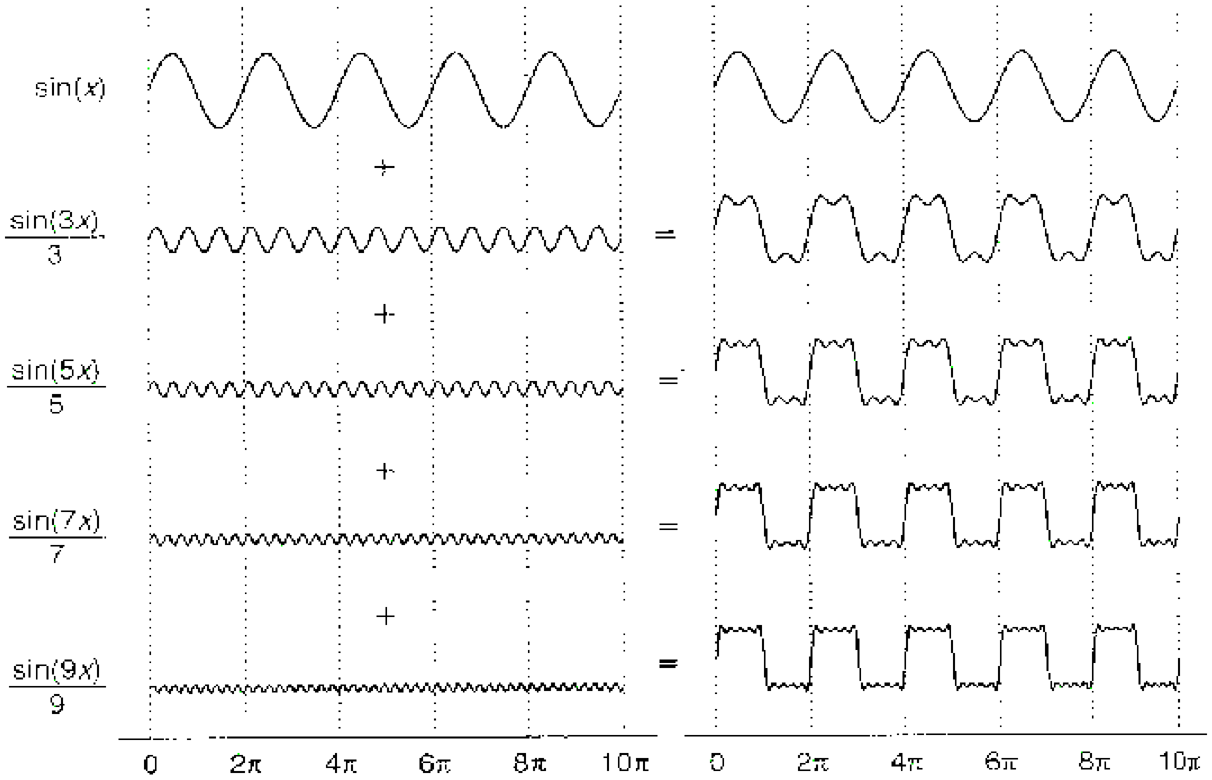
Fourier Synthesis Example

- Periodic, odd function: square wave

$$f(x) = 0.5 \;\forall\; 0 < (x \bmod 2\pi) < \pi$$
$$= -0.5 \;\forall\; \pi < (x \bmod 2\pi) < 2\pi$$

$$a_k = \int \sin(k*x)* f(x)\, dx \qquad f(x)=\Sigma_k\, a_k \sin(k*x)$$

- $a_0 = 0$
- $a_1 = 1$
- $a_2 = 0$
- $a_3 = 1/3$
- $a_4 = 0$
- $a_5 = 1/5$
- $a_6 = 0$
- $a_7 = 1/7$
- $a_8 = 0$
- $a_9 = 1/9$
- …



Discrete Fourier Transform

- N Equally-spaced function samples f_i
  - Function values known only at discrete points
    - Physical measurements
    - Pixel positions in an image !
- Fourier Analysis

$$\mathbf{a_k = 1/\ N\ \ \sum_i \cos(2\pi\ k\ i\ /\ N)\ \ f_i\ ,\qquad b_k = 1/\ N\ \ \sum_i \sin(2\pi\ k\ i\ /\ N)\ \ f_i}$$

  - Sum over all measurement points N
  - k=0,1,2, …, ? Highest possible frequency ?

$\Rightarrow$**Nyquist frequency**

  - Sampling rate N_i
  - 2 samples / period $\Leftrightarrow$ 0.5 cycles per pixel

$\Rightarrow k \leq N / 2$

Fourier Spectrum

$$F(f(x)) = F(\omega) = \int f(x) e^{-2\pi i \omega x} dx$$

$$e^{2\pi i \omega x} = \cos(2\pi\omega x) + i\sin(2\pi\omega x)$$

$$F(\omega) = a(\omega) + ib(\omega)$$

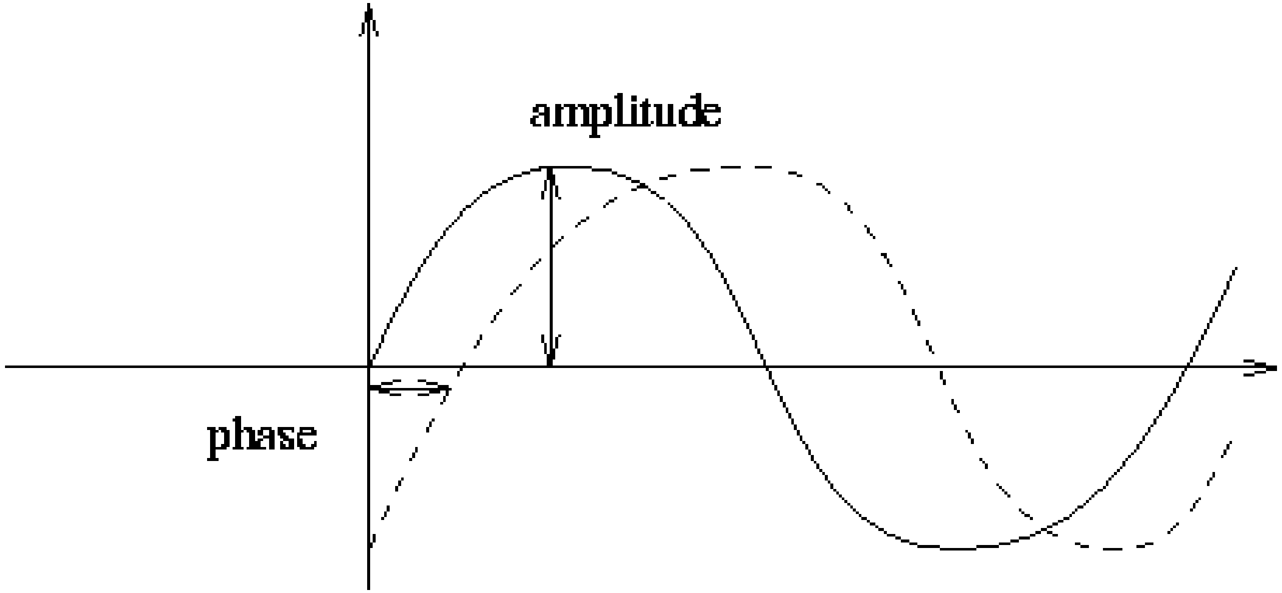
$$= \left|F(\omega)\right| e^{i\Phi(\omega)}$$

- Amplitude

$$\left|F(\omega)\right| = \sqrt{a^2 + b^2}$$

- Phase

$$\Phi(\omega) = \tan^{-1}\left(\frac{b}{a}\right)$$



2D – Fourier Transform

$$F(f(x,y)) = F(\omega_k, \omega_l) = \int\int f(x,y) e^{-2\pi i(\omega_k x + \omega_l y)} dxdy$$

- Exponential function is separable:

$$g(x,y) = e^{-2\pi i(\omega_k x + \omega_l y)} = e^{-2\pi i(\omega_k x)} \cdot e^{-2\pi i(\omega_l y)} = u(x)v(y)$$

- Transform first in *x* and then in *y*.

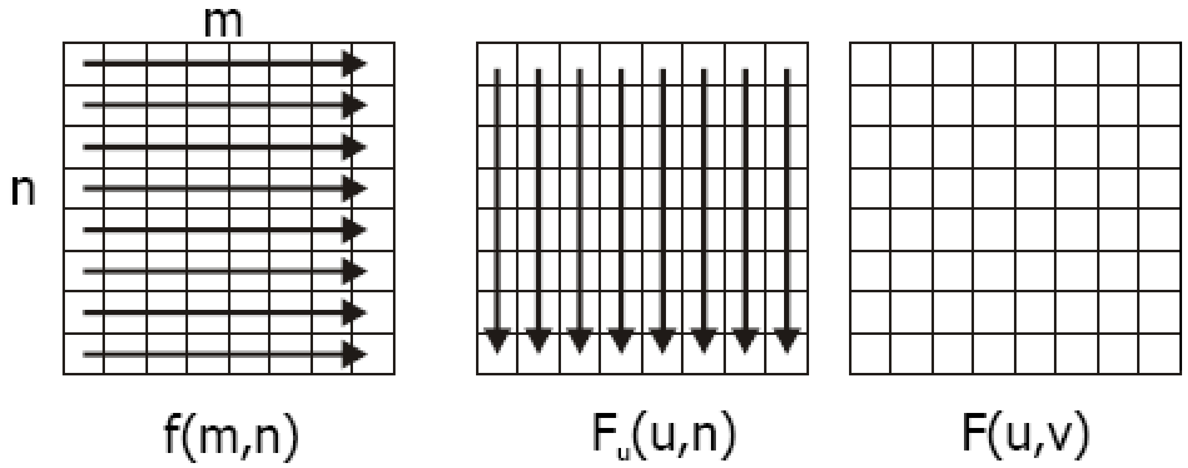
$$F(\omega_k, \omega_l) = \int \left(\int f(x,y) e^{-2\pi i \omega_k x} dx \right) e^{-2\pi i \omega_l y} dy$$

2D – Fourier Transform

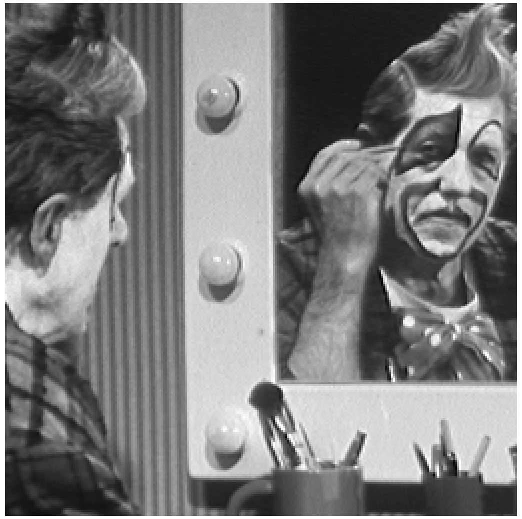
- As the Fourier transform is separable we can apply it in one dimension for all rows and then transform the intermediate result along the second dimension.

$$\begin{aligned} F(u,v) &= 1/N^2 \sum_m \sum_n f(m,n)\cdot \exp(-i2\pi(um+vn)/N) \\ &= 1/N^2 \sum_m \sum_n f(m,n)\cdot \exp(-i2\pi um/N)\cdot \exp(-i2\pi vn/N) \\ &= 1/N^2 \sum_n \left[\sum_m f(m,n)\cdot \exp(-i2\pi um/N)\right]\cdot \exp(-i2\pi vn/N) \\ &= 1/N^2 \sum_n F_u(u,n)\cdot \exp(-i2\pi vn/N) \end{aligned}$$

- reduction O(N4) -> O(N3)

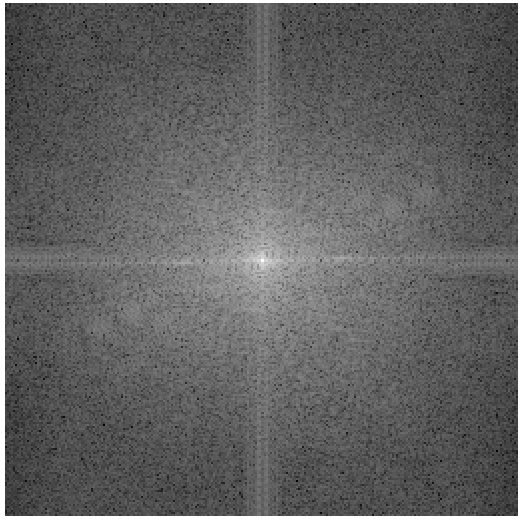


An Example

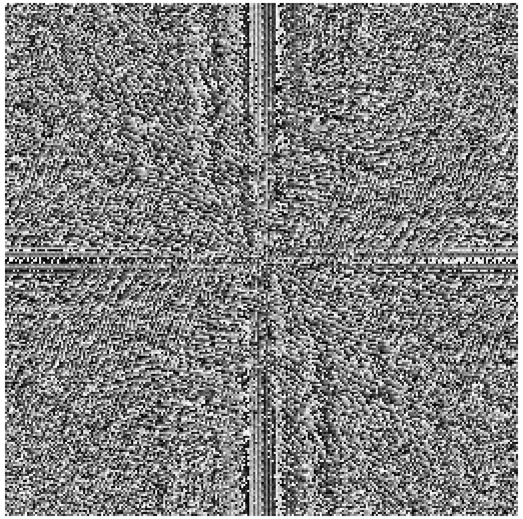


f(x)

Fourier transformed

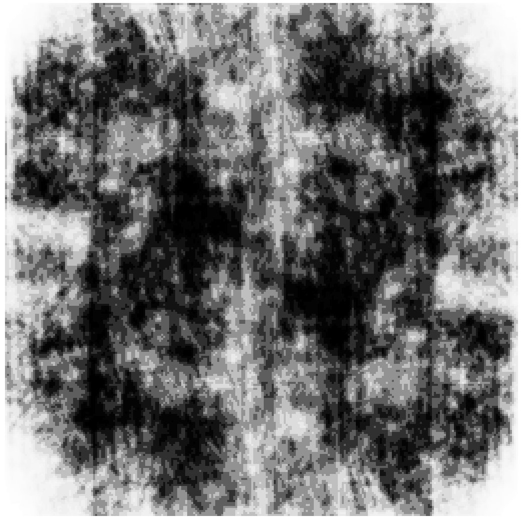


Amplitude

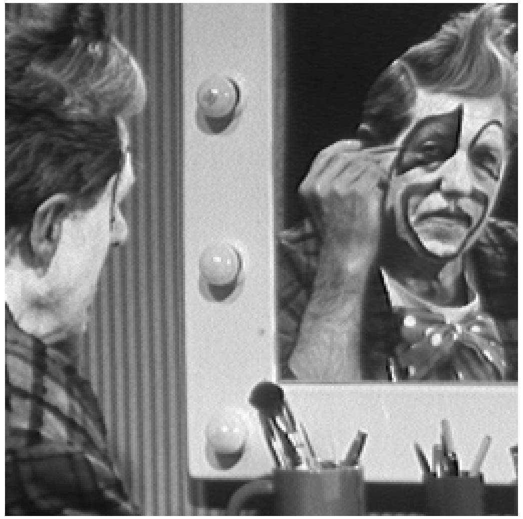


Phase

reconstructed



ignoring Phase



using Phase+Amplitude

Fast Fourier Transform - FFT

- discrete Fourier Transform for 2n samples

$$f_m = \sum_{k=0}^{2n-1} x_k e^{-\frac{2\pi i}{2n}mk} \qquad m = 0,1,\ldots,2n-1$$

m - discrete frequency

- split series of samples into even and odd numbered ones

$$\vec{x} = [x_0, x_1, x_2, x_3, \ldots., x_{2n-2}, x_{2n-1}]$$

$$\begin{aligned} f_m &= \sum_{k=0}^{n-1} x_{2k} e^{-\frac{2\pi i}{2n}m2k} + \sum_{k=0}^{n-1} x_{2k+1} e^{-\frac{2\pi i}{2n}m(2k+1)} \\ &= \sum_{k=0}^{n-1} x'_k e^{-\frac{2\pi i}{n}mk} + e^{-\frac{\pi i}{n}m} \sum_{k=0}^{n-1} x''_k e^{-\frac{2\pi i}{n}mk} \\ &= \begin{cases} f_m' + e^{-\frac{\pi i}{n}m} f_m'' & when \;\; m < n \\ f'_{m-n} - e^{-\frac{\pi i}{n}(m-n)} f''_{m-n} & when \;\; m \geq n \end{cases} \end{aligned}$$

Fast Fourier Transform - FFT

```
Function:   $fft(n, \vec{f})$
  if (n=1)
    return  $\vec{f}$
  else
          $\vec{g} = fft(\frac{n}{2}, (f_o, f_2, \ldots, f_{n-2}))$
          $\vec{u} = fft(\frac{n}{2}, (f_1, f_3, \ldots, f_{n-1}))$
    for  $k = 0$  to  $\frac{n}{2} - 1$
             $c_k = g_k + u_k e^{-2\pi i k/n}$
             $c_{k+n/2} = g_k - u_k e^{-2\pi i k/n}$
    return  $\vec{c}$
```

Properties of the Fourier Transform

Symmetries:

For $f(x) \in \Re$ the Fourier transform is symmetric, i.e., $\frown{f}(\omega) = \frown{f}^*(-\omega)$

For $f(x) = f(-x)$ the transform is real-valued, i.e., $\frown{f}(\omega) \in \Re$

For $f(x) = -f(-x)$ the transform is imaginary, i.e., $i\frown{f}(\omega) \in \Re$

Properties of the Fourier Transform

- Shift Property:
  - the amplitude spectrum is invariant to translation. The phase spectrum is not.

$$F[f(x-x_0)] = e^{-i\omega x_0}\widehat{f}(\omega)$$

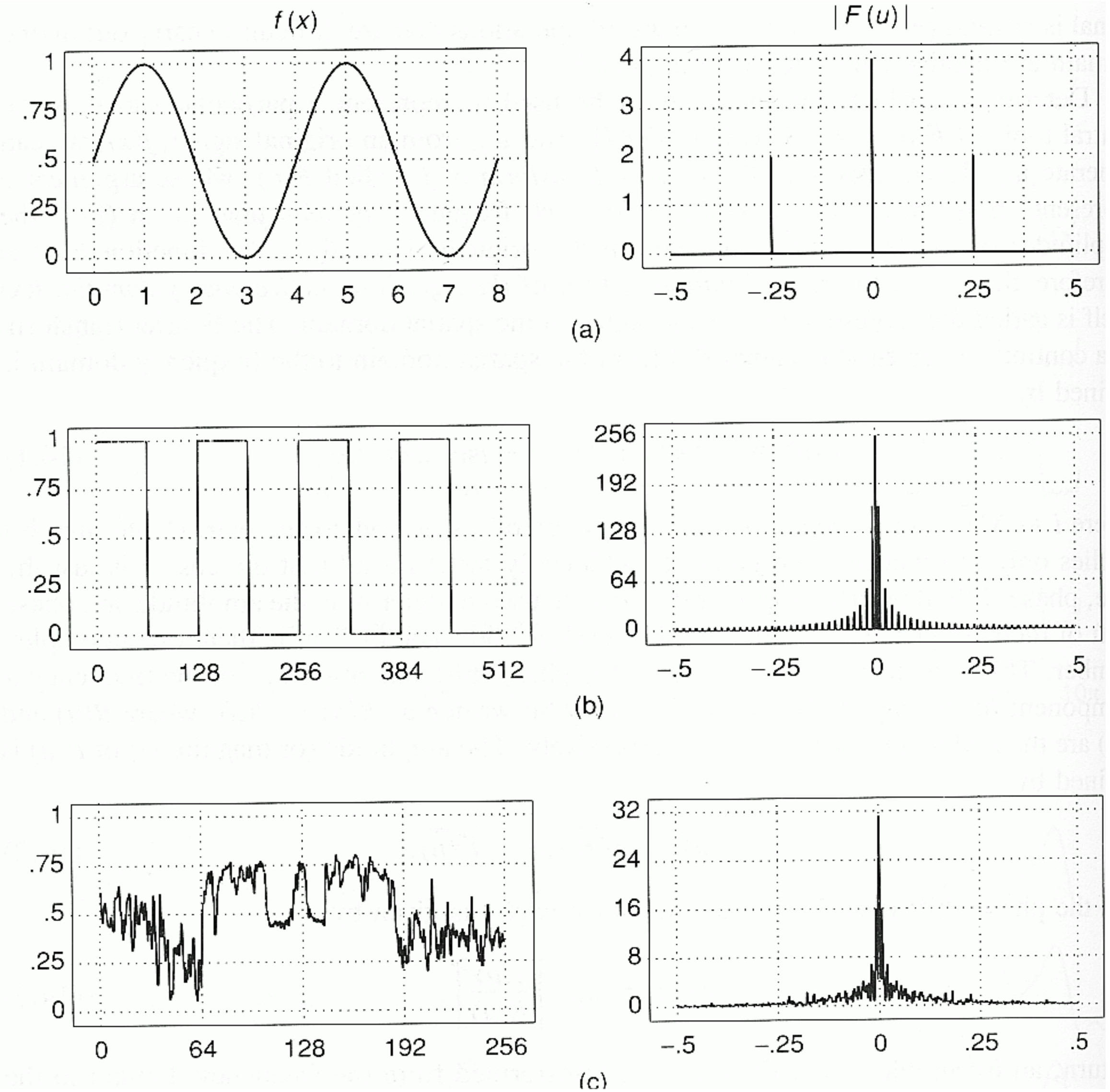
- Linear Scaling:
  - Scaling the signal domain causes inverse scaling of the Fourier domain, i.e. given

$$F[f(ax)] = \frac{1}{a}\widehat{f}(\omega / a) \qquad a \in \Re$$

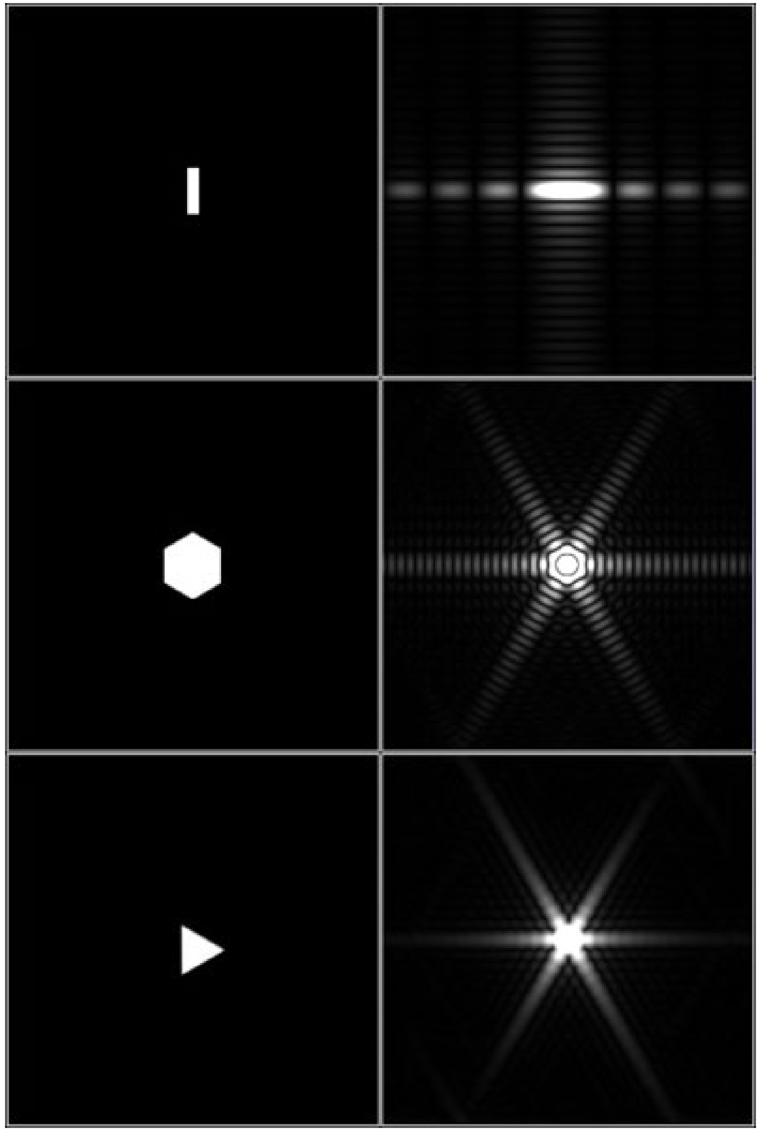
- Parseval’s Theorem:
  - Sum of squared Fourier coefficients is a constant multiple of the sum of squared signal values.

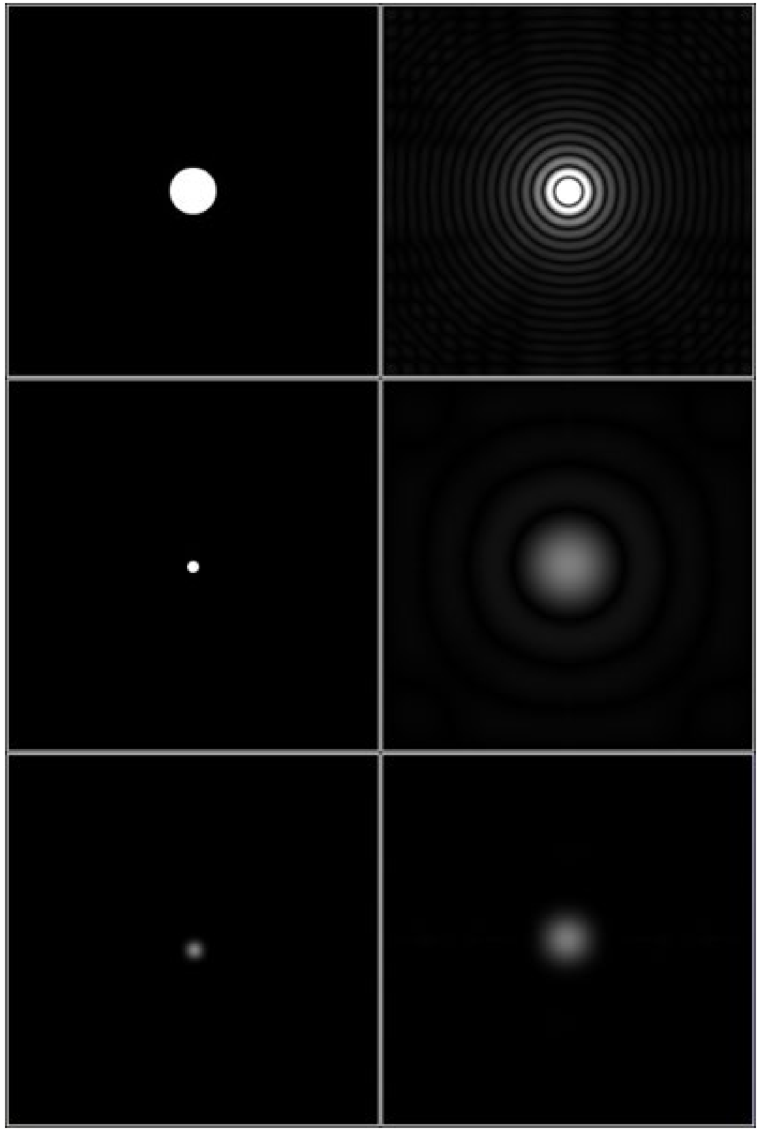
Spatial vs. Frequency Domain

- Examples (pixel vs cycles per pixel)
  - Sine wave with positive offset
  - Square wave
  - Scanline of an image



2D Fourier Transforms

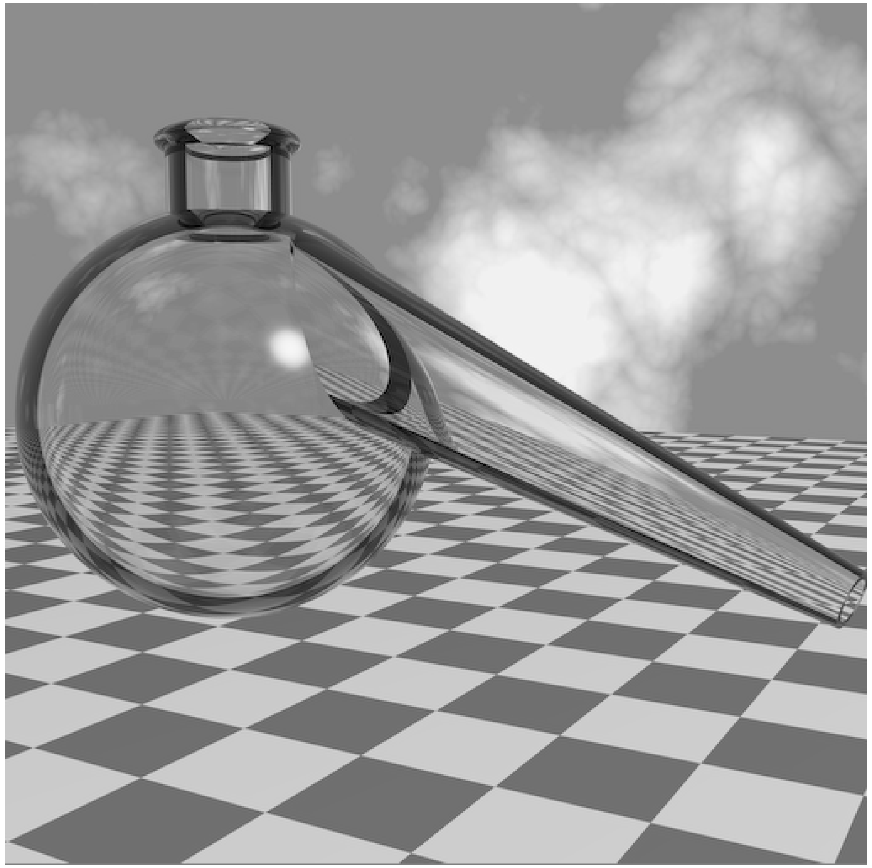


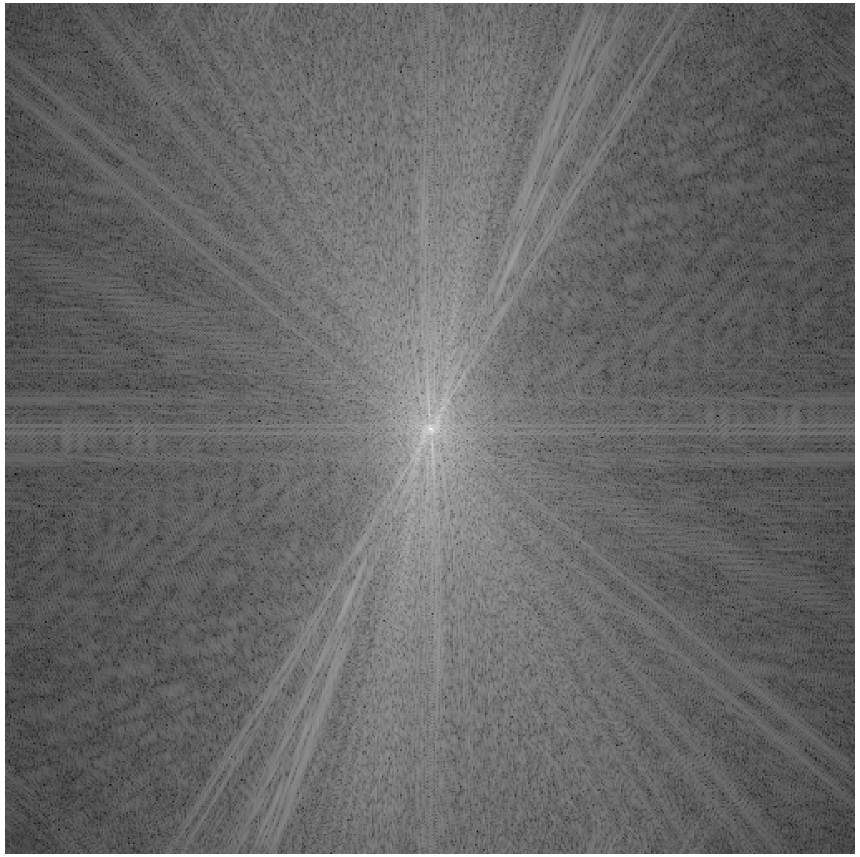


[Wikipedia]

2D Fourier Transform

- 2 separate 1D Fourier transformations along x- and y-direction
- Discontinuities: orthogonal direction in Fourier domain !





Spatial vs. Frequency Domain

- Important basis functions
  - Box ↔ sinc

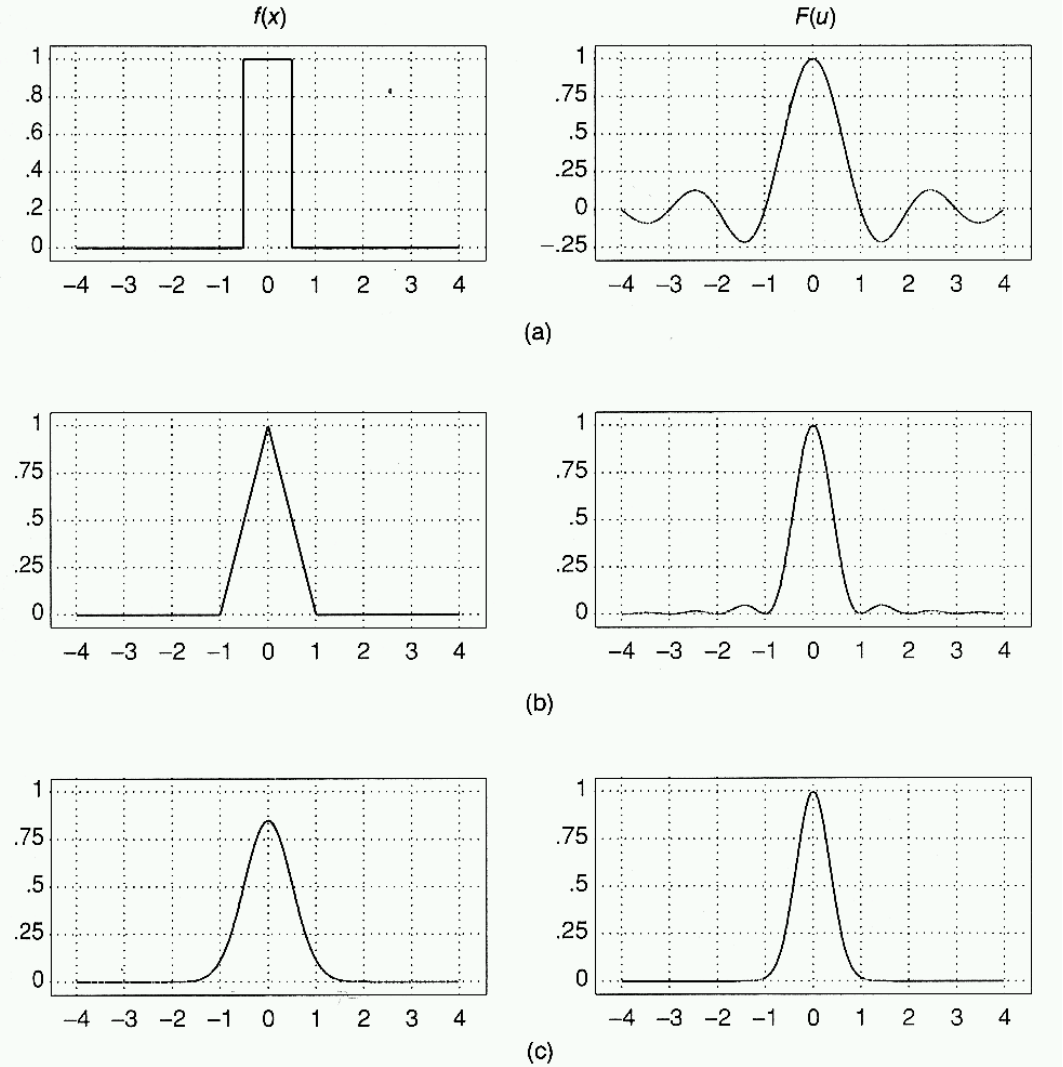
$$\mathrm{sinc}(x) = \frac{\sin(x\pi)}{x\pi}$$

$$\mathrm{sinc}(0) = 1$$

$$\int \mathrm{sinc}(x)\,dx = 1$$

    - Wide box ↔ small sinc
    - Negative values
    - Infinite support

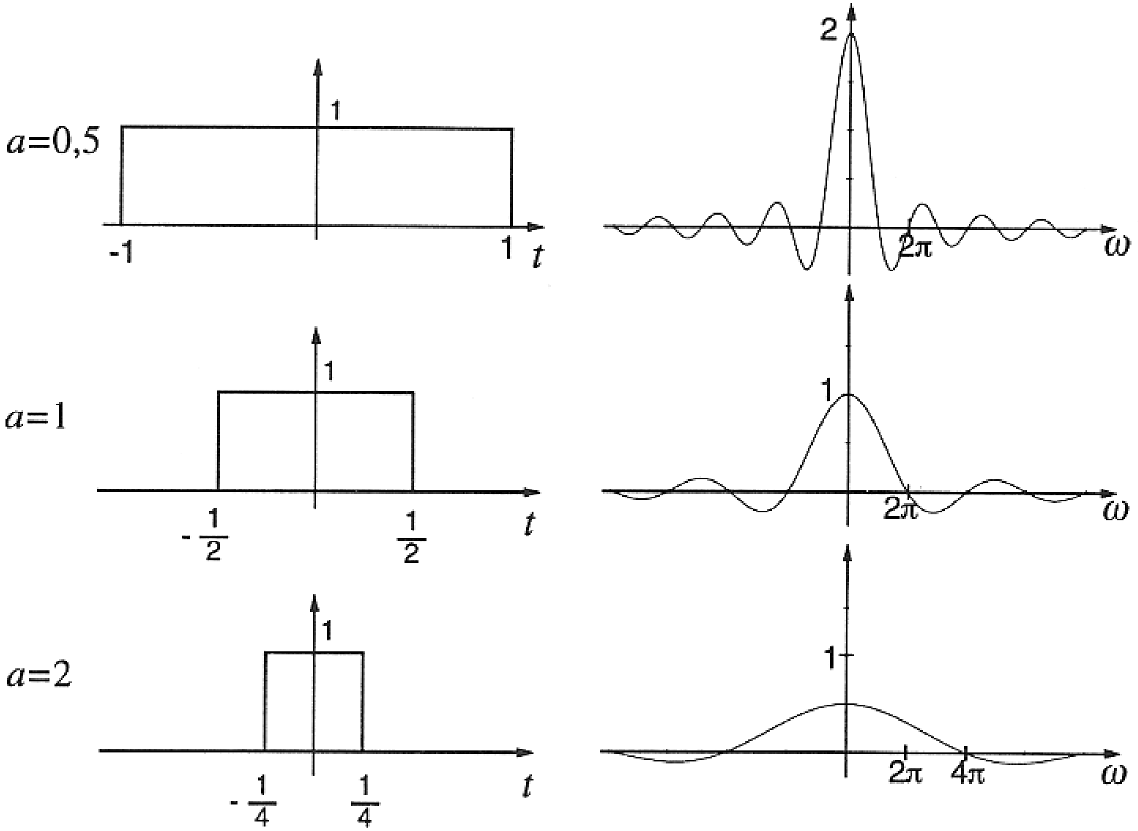
  - Triangle ↔ sinc2
  - Gauss ↔ Gauss



Spatial vs. Frequency Domain

- Transform behavior
- Example: **box function**
  - Fourier transform: sinc
  - Wide box: narrow sinc
  - Narrow box: wide sinc

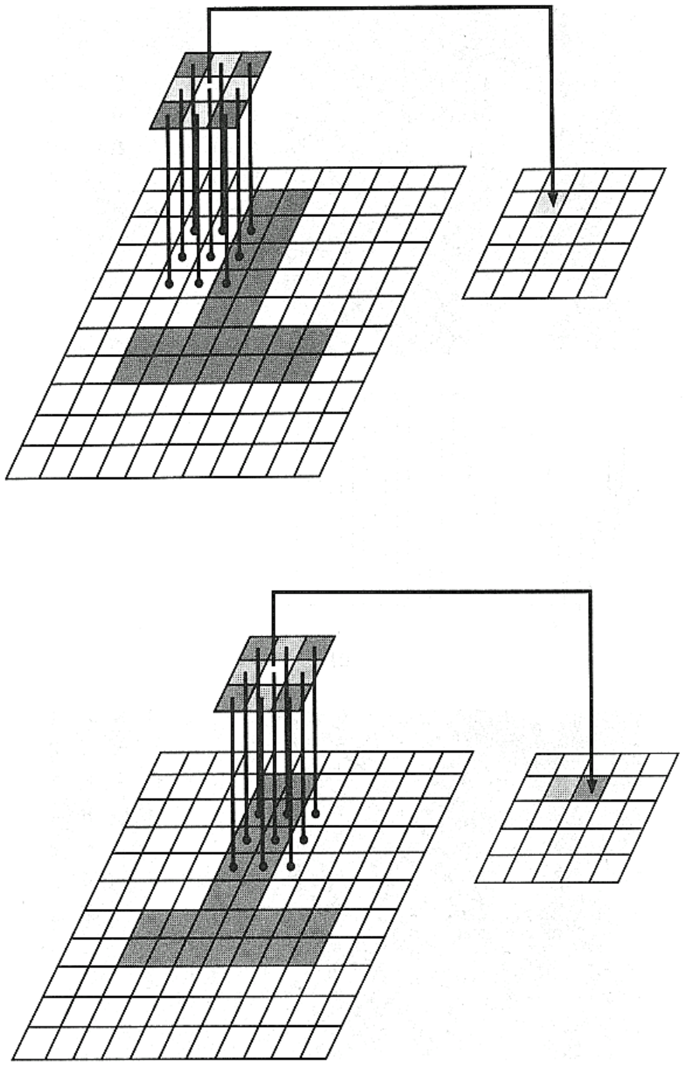
$$\boxed{\operatorname{rect}(at) \quad \circ\!\!-\!\!\bullet \frac{1}{|a|}\,\mathrm{si}\left(\frac{\omega}{2a}\right) .}$$



Filtering and Convolution

Convolution and Filtering

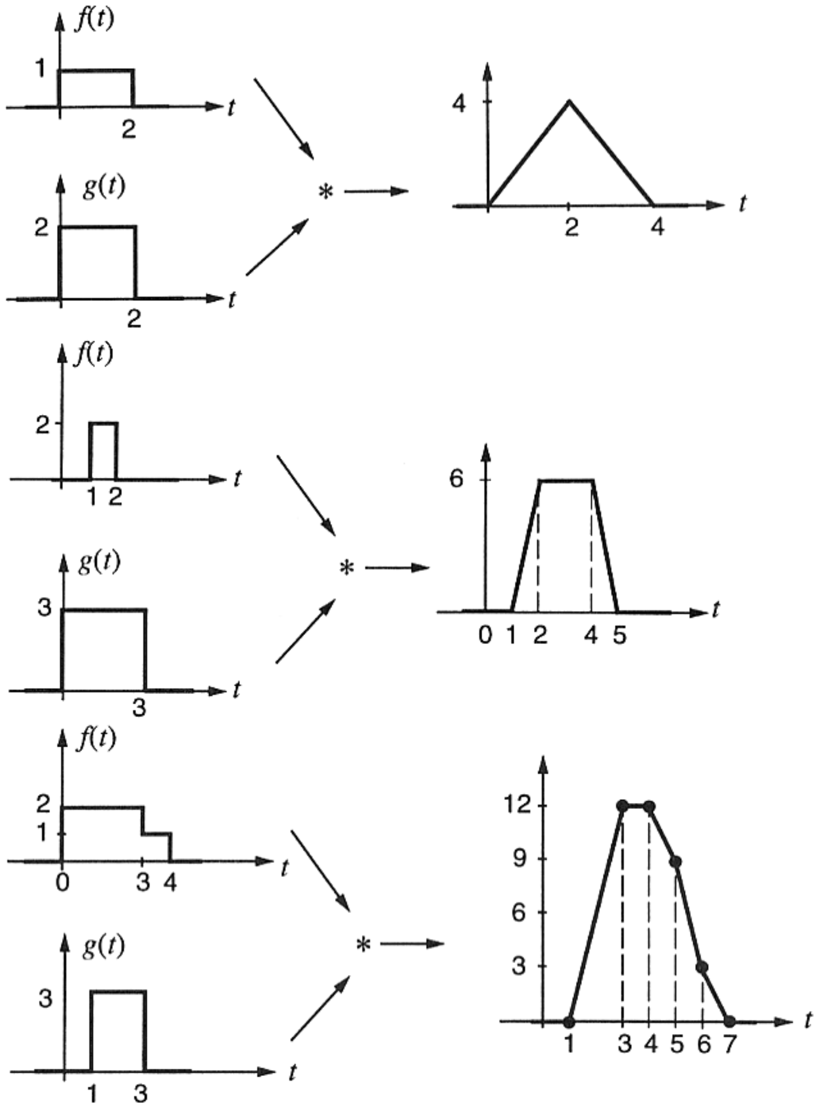
- Technical Realization
  - In image domain
  - Pixel mask with weights
  - OpenGL: Convolution extension
- Problems (e.g. sinc)
  - Large filter support
    - Large mask
    - A lot of computation
  - Negative weights
    - Negative light?



Convolution

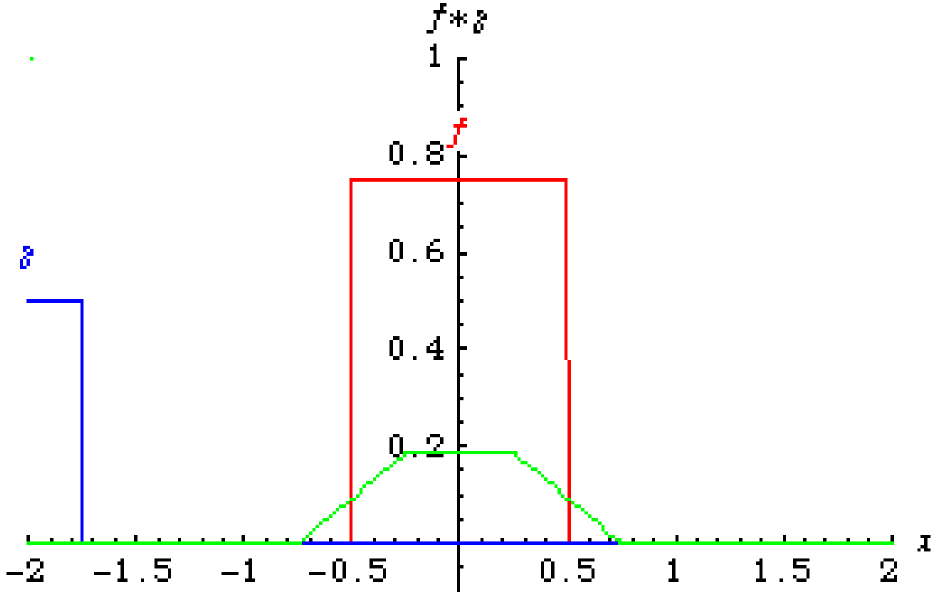
$$(f \otimes g)(x) = \int_{-\infty}^{\infty} f(\tau)\, g(x-\tau)\, d\tau$$

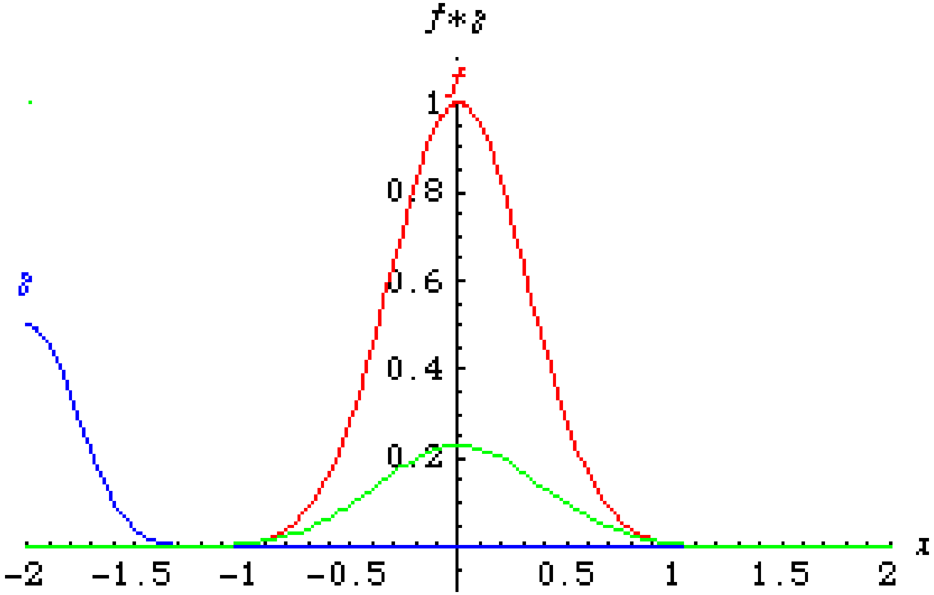
- Two functions *f, g*
- Shift one function against the other by *x*
- Multiply function values
- Integrate overlapping region
- Numerical convolution:
  Expensive operation
  - For each *x*:
    integrate over non-zero domain



Convolution

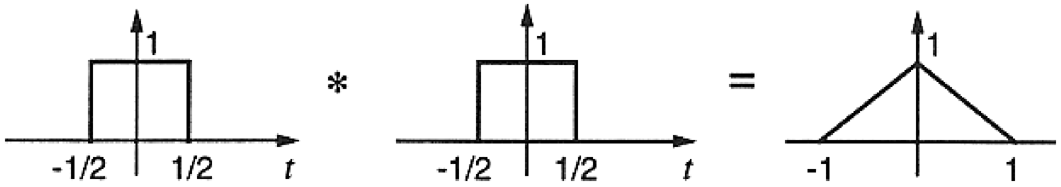
- Examples
  - Box functions
  - Gauss functions





Convolution Theorem

- **Convolution in image domain ↔ multiplication in Fourier domain**
- **Convolution in Fourier domain ↔ multiplication in image domain**
  - Multiplication much cheaper than convolution !

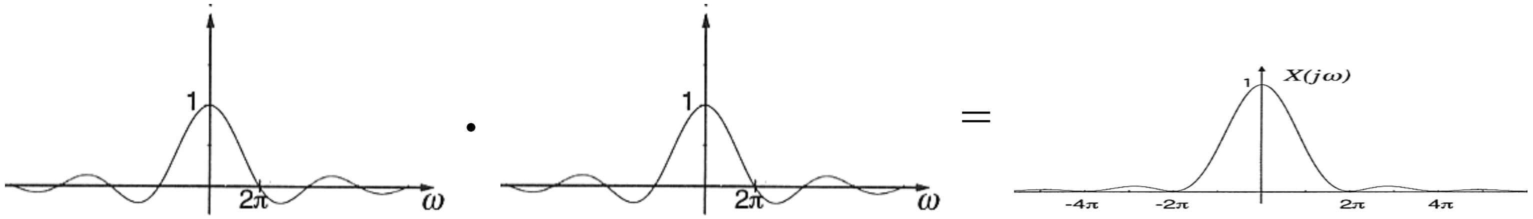


$$x(t) = \begin{cases} t+1 & \text{für } -1 \leq t < 0 \\ -t+1 & \text{für } 0 \leq t < 1 \\ 0 & \text{sonst} \end{cases}$$

$$\mathrm{rect}(t) \ast \mathrm{rect}(t) = x(t)$$

$$\circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet$$

$$\mathrm{si}\left(\frac{\omega}{2}\right) \cdot \mathrm{si}\left(\frac{\omega}{2}\right) = X(j\omega)\,.$$



$$X(j\omega) = \mathrm{si}^2\left(\frac{\omega}{2}\right).$$

Convolution Theorem

$$F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)] = \widehat{f}(\omega) \cdot \widehat{g}(\omega)$$

$$F[f * g] = \sum_{n} \left(f * g \right) e^{-i\omega n} = \sum_{n} \sum_{m} f(m) g(n-m) e^{-i\omega n}$$

$$= \sum_{m} f(m) \sum_{n} g(n-m) e^{-i\omega n}$$

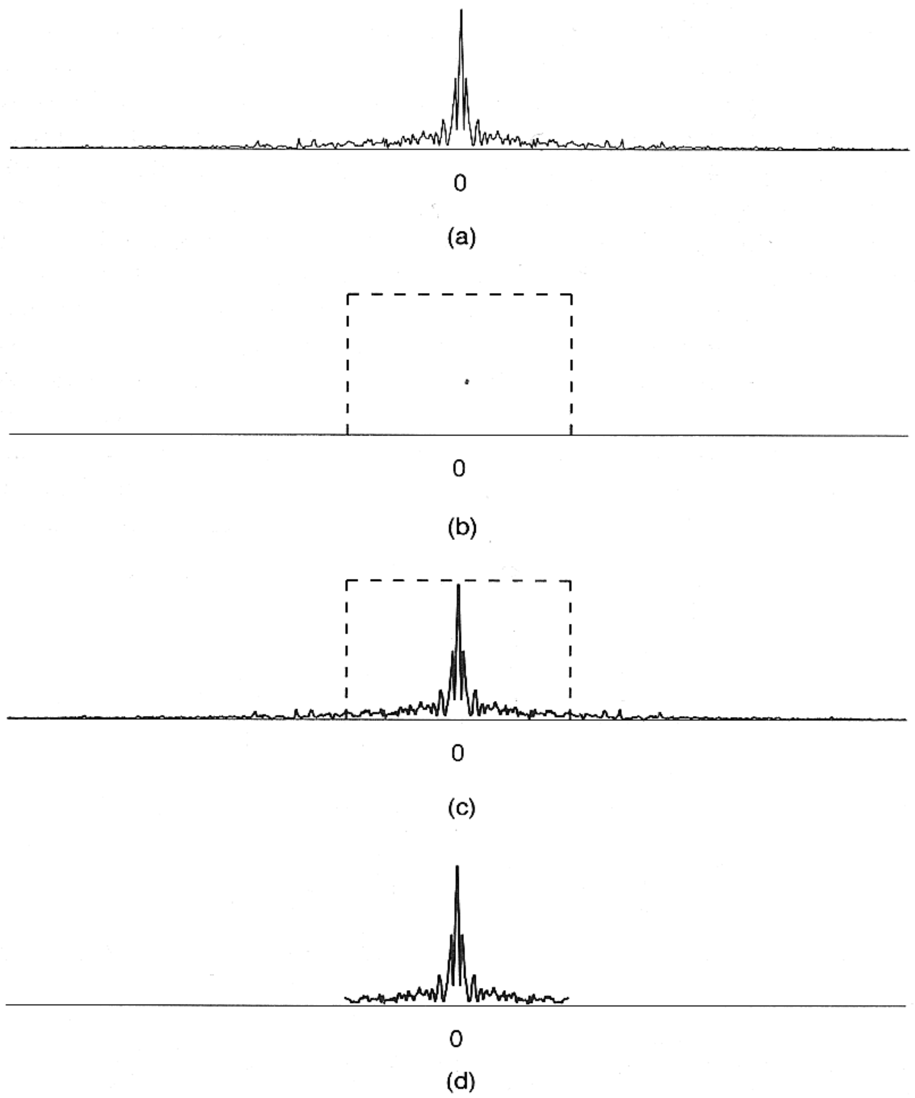
(shift property)

$$= \sum_{m} f(m) \widehat{g}(\omega) e^{-i\omega m}$$

$$= \widehat{g}(\omega) \widehat{f}(\omega)$$

Filtering

- **Low-pass filtering**
  - Convolution with sinc in spatial domain, or
  - Multiplication with box in frequency domain
- High-pass filtering
  - Only high frequencies
- Band-pass filtering
  - Only intermediate



Low-pass filtering in frequency domain: multiplication with box

Filtering

- Low-pass filtering
  - Convolution with sinc in spatial domain, or
  - Multiplication with box in frequency domain
- **High-pass filtering**
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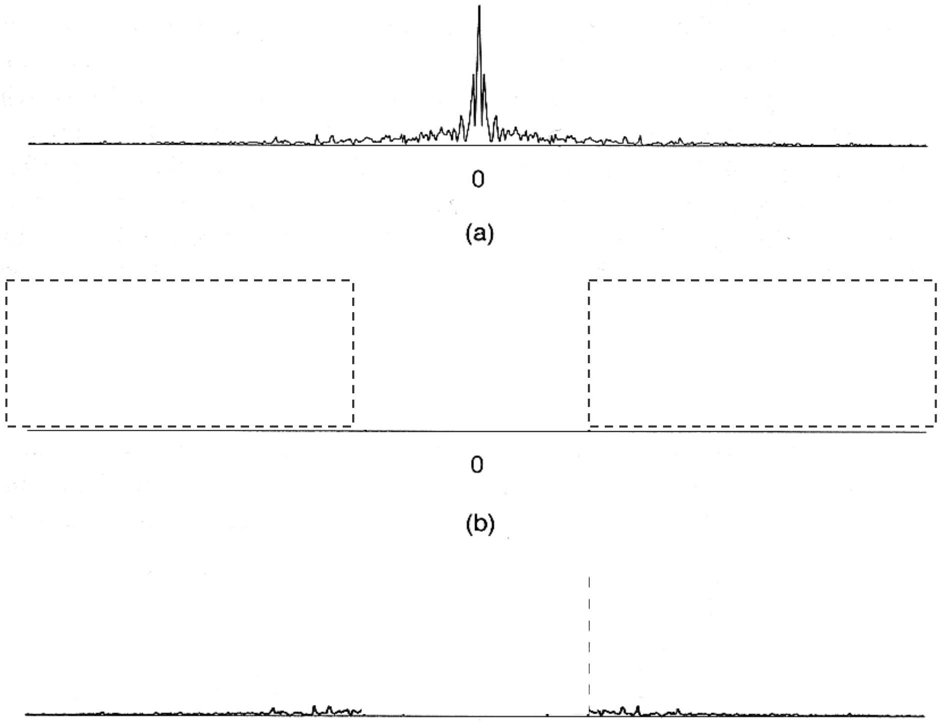

Filtering

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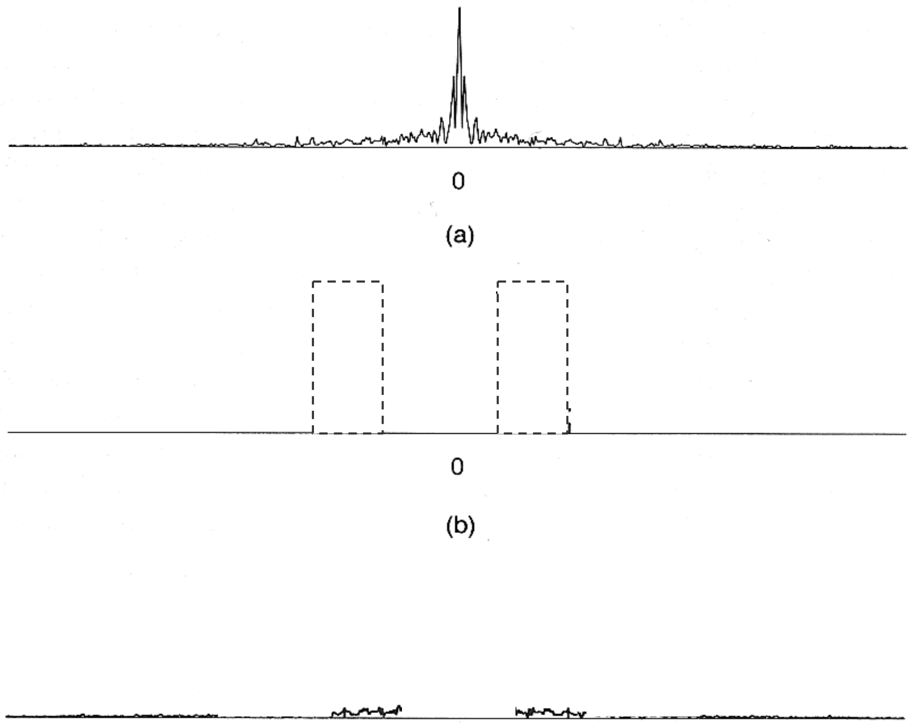
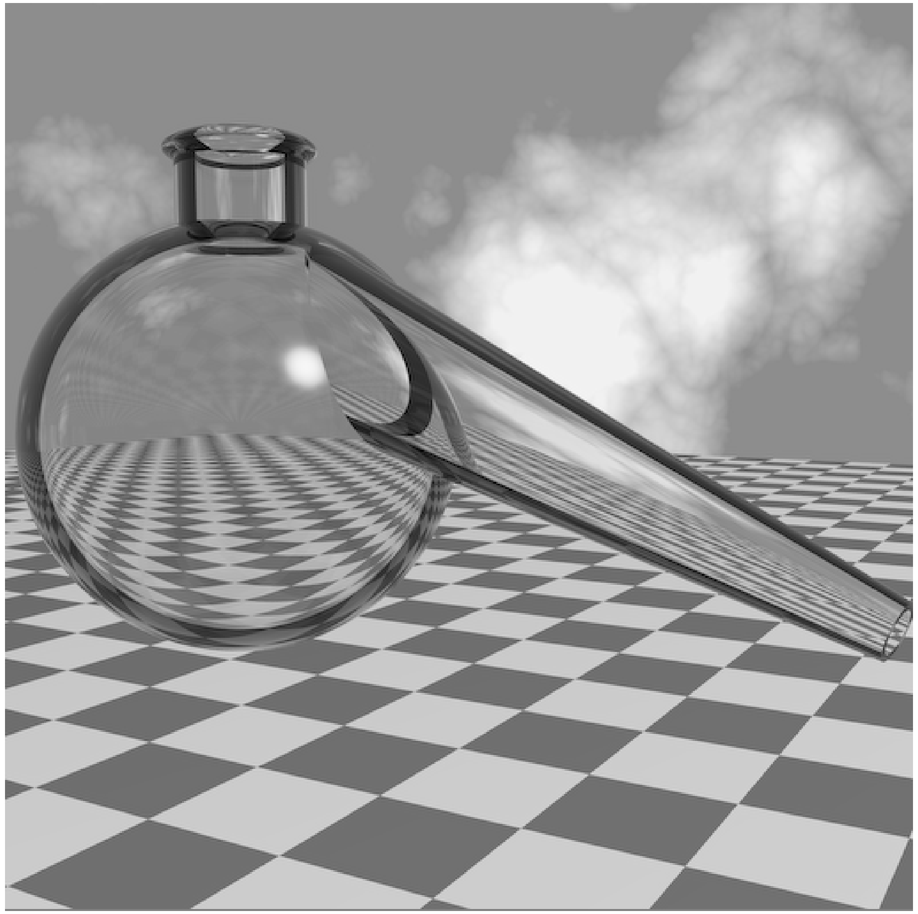
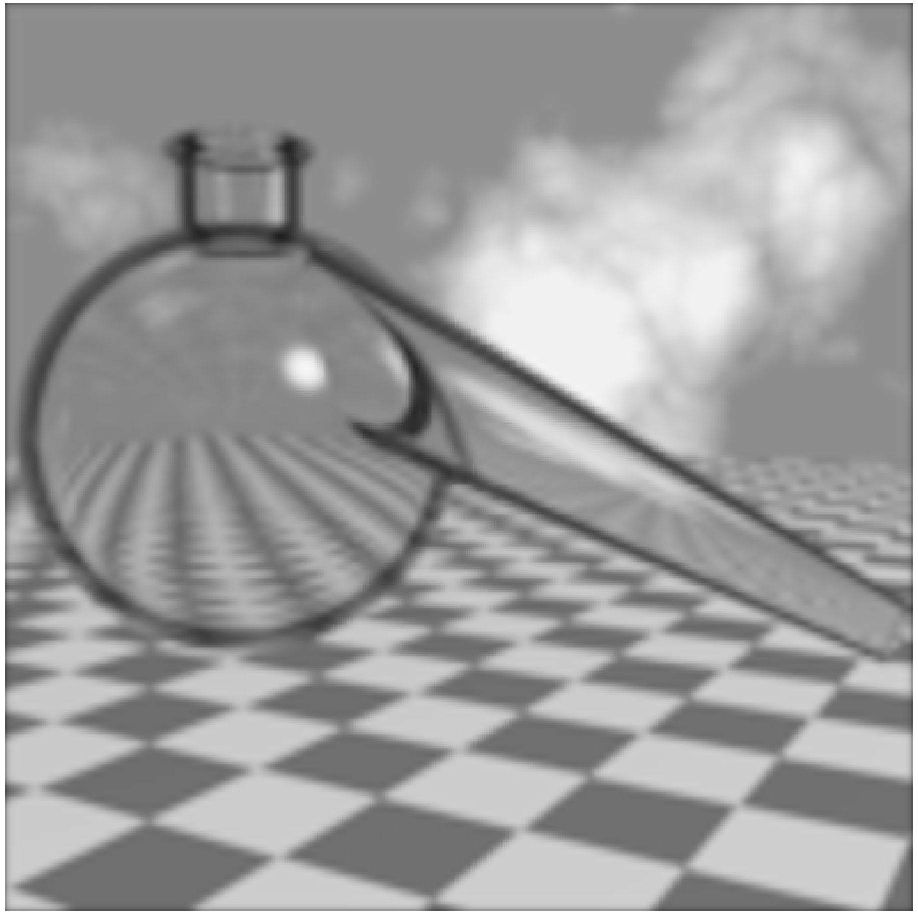


Low-Pass Filtering

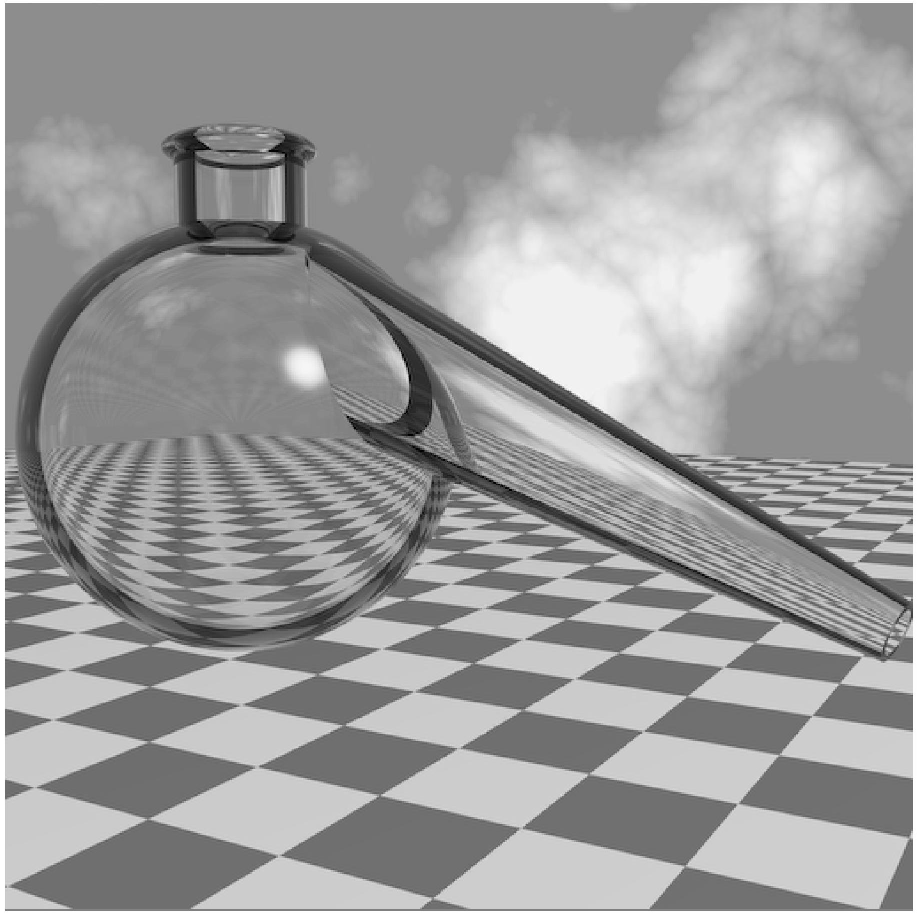
- „Blurring“

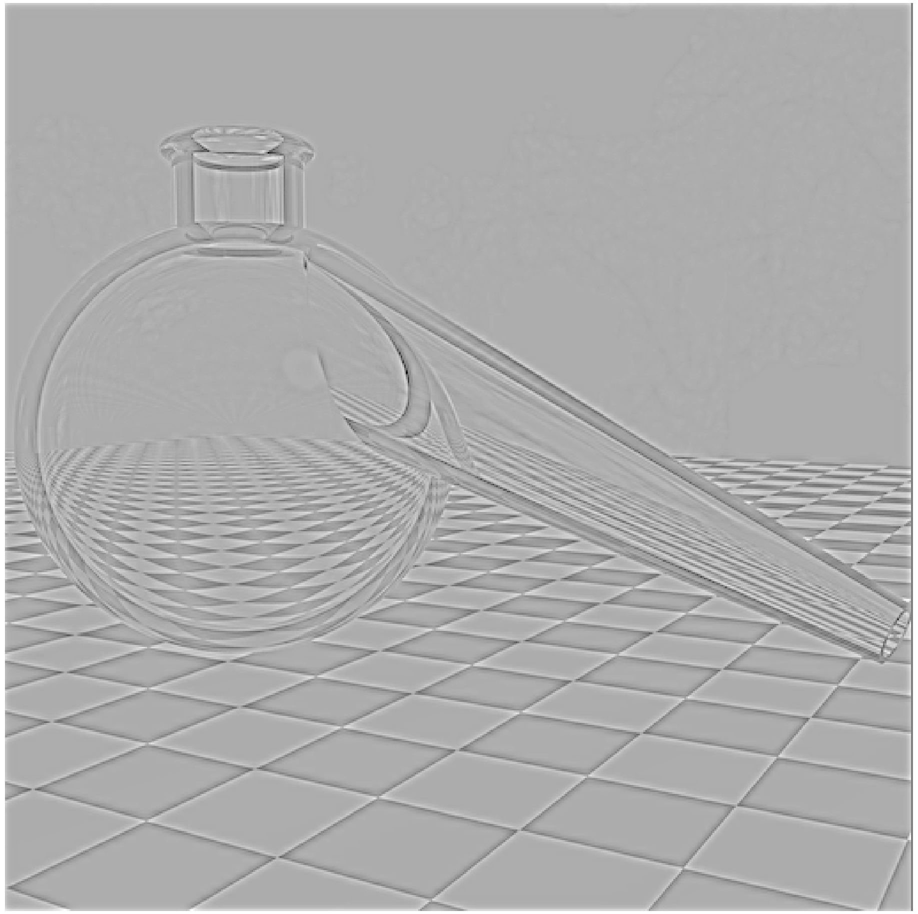




High-Pass Filtering

- Enhances discontinuities in image
  - Useful for edge detection





Questions

- What does the Fourier transform do?
- How does the power spectrum looks like for a horizontal/ vertical bar, set of dots, circles of varying sizes?
- How to perform a convolution?
- What is a low-pass, high-pass, band-pass filter?

Summary

- Fourier Transform
- Importance of Amplitude and Phase
- Spatial Extent vs. Frequency
- Low-Pass Filtering

Next Lecture

- Filtering and Reconstruction
- Anti-Aliasing