Computer Graphics
(Graphische Datenverarbeitung)

- Transformations -

Hendrik Lensch

WS 2021/2022

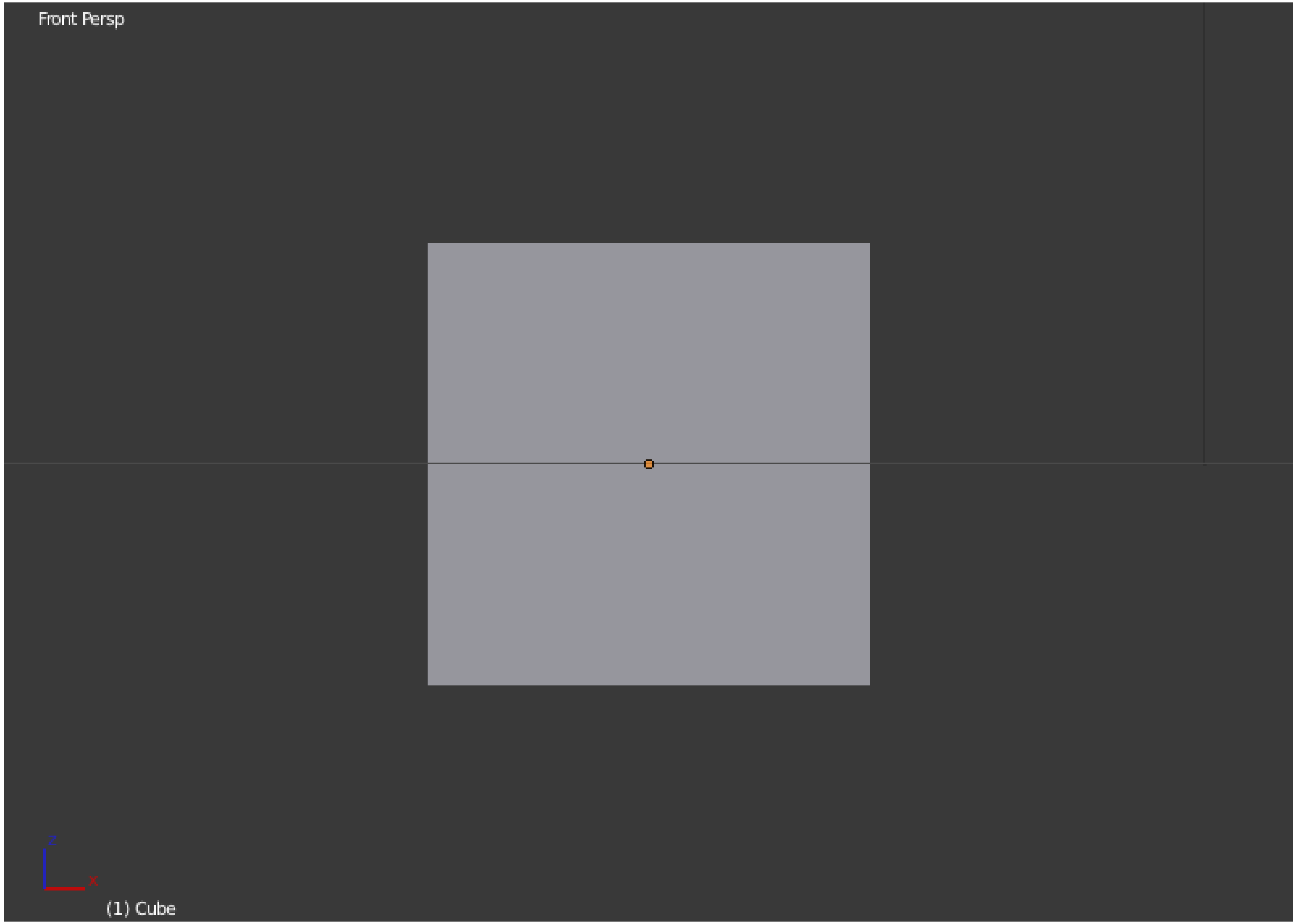
Corona

- Regular random lookup of the 3G certificates

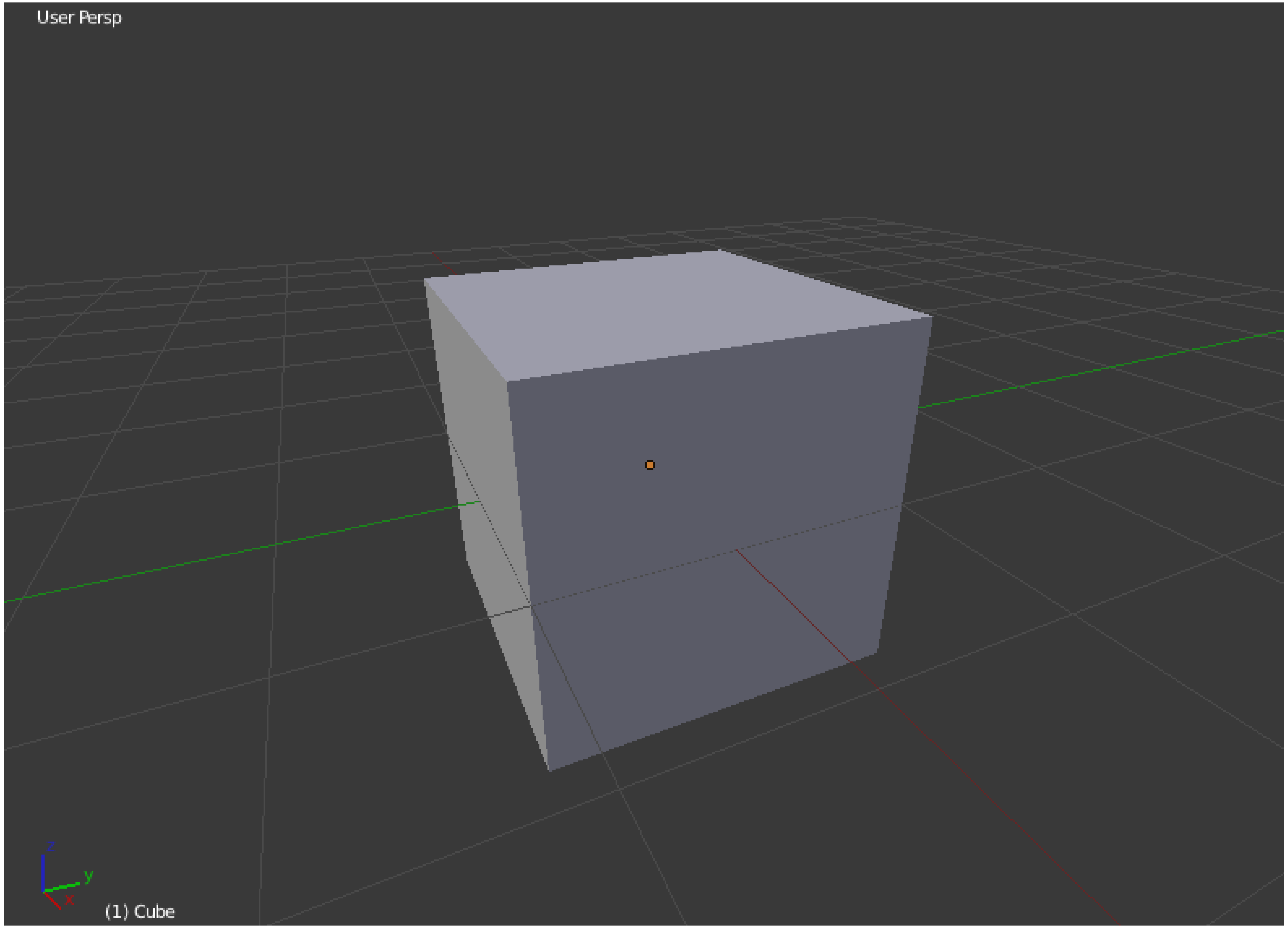
- Contact tracing: We need to know who is in the class room
  - New ILIAS group for every lecture slot
  - Register via ILIAS or this QR code (only if you are present in this room)

[Andrea Pozzo, St. Ignazio]

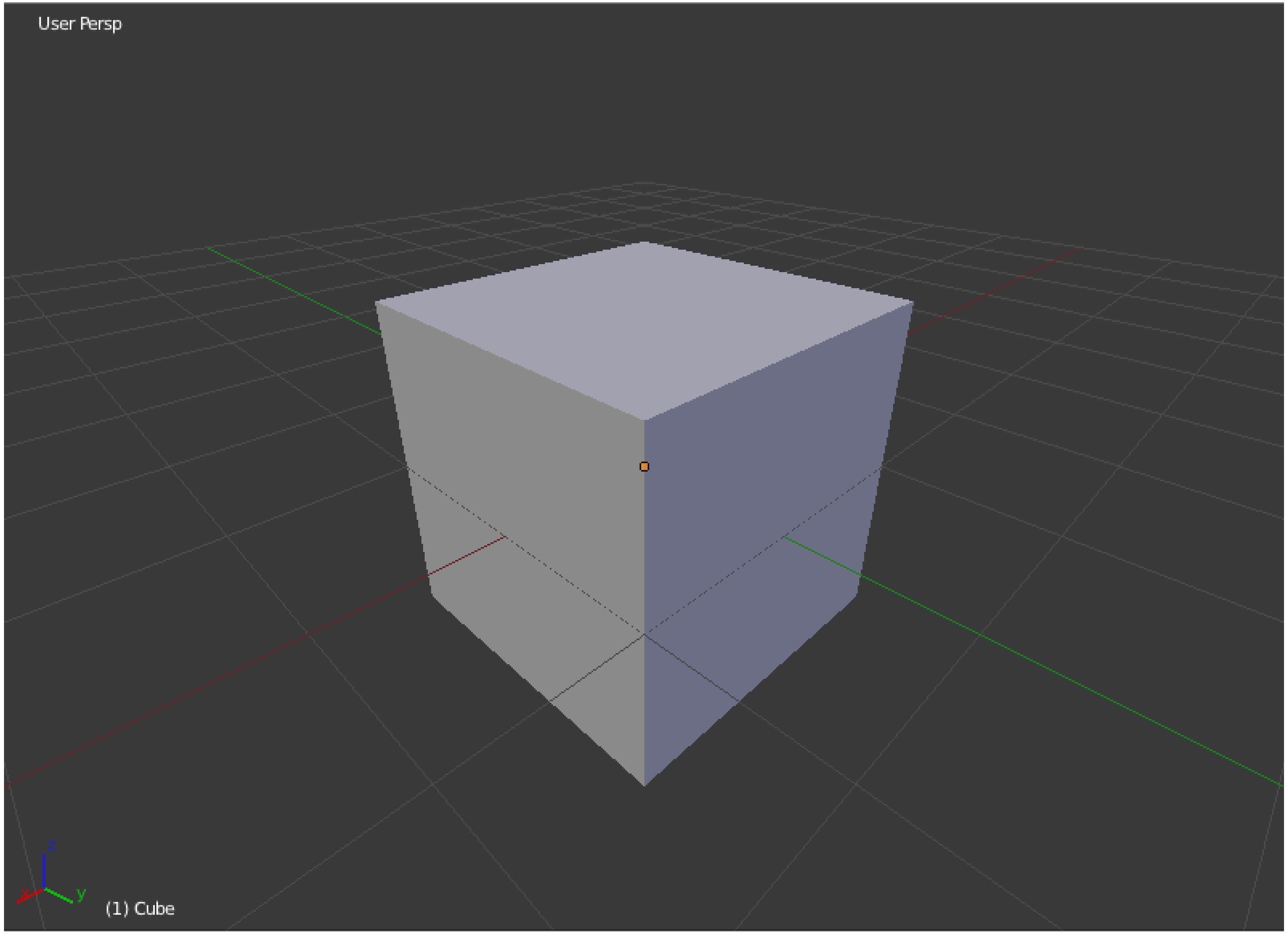
It’s a cube



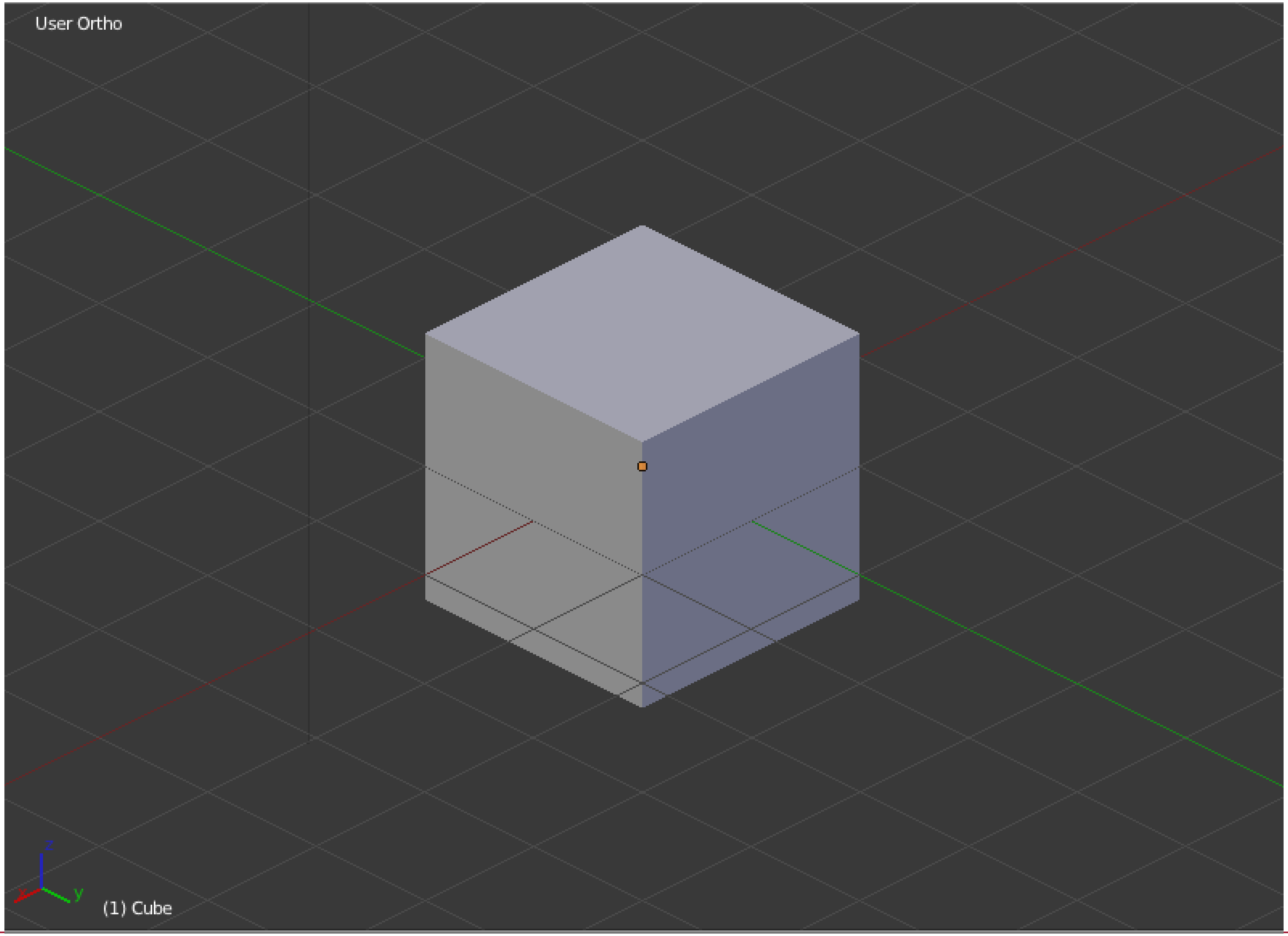
It’s a cube



It’s a cube



It’s a cube



Overview

- Last Time
  - Tone Mapping

- Today
  - Homogeneous Coordinates
  - Basic transformations in homogeneous coordinates
  - Concatenation of transformations
  - Projective transformations

- Next Lectures
  - Camera transformations
  - Rasterization

What you should learn

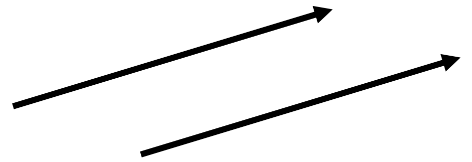
- Basic transformations in 3D
  - How to describe them

- Perspective transformations in 3D

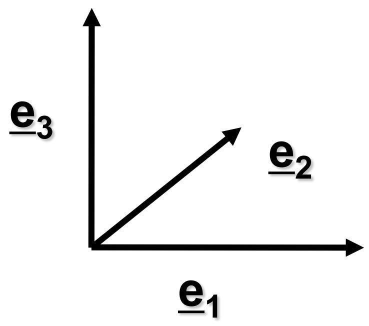
- Homogeneous Coordinates
  - How to express transformations conveniently

Euclidean Vector Space

- Known from Mathematics:
  - Elements of a 3D vector space
    - $\underline{v} = (v_1, v_2, v_3)^T \in V^3 = R^3$
  - Formally: Vectors written as column vectors (n x 1 matrix)!
  - Vectors describe directions – not positions!
  - 3 linear independent vectors create a basis:
    - $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$
  - Any vector can now uniquely be represented with coordinates
    - $\underline{v} = v_1\, \underline{e}_1 + v_2\, \underline{e}_2 + v_3\, \underline{e}_3 = (v_1, v_2, v_3)^T$
  - Operations
    - Addition, Subtraction, Scaling, …



- Metric
  - Dot/inner product:
    - Used for measurements of length ($|\underline{v}|^2 = \underline{v} \cdot \underline{v}$) and angles ($\cos(v_1, v_2) = v_1 \cdot v_2 \,/\, |v_1||v_2|$)
  - Orthonormal basis
    - $|\underline{e}_i| = 1$ $\qquad \underline{e}_i \cdot \underline{e}_j = \delta_{ij}$
    - right-/left handed: $\underline{e}_1 \times \underline{e}_2 = +/- \underline{e}_3$



Affine Space

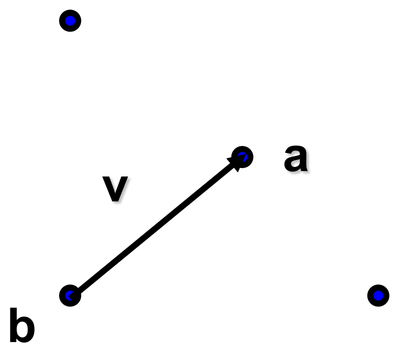
- Known from Mathematics

- Affine Space: A^3
  - Elements are positions – no directions!

- Defined via its associated vector space V^3
  - $a, b \in A^3 \Leftrightarrow v \in V^3$ with $v = b - a$

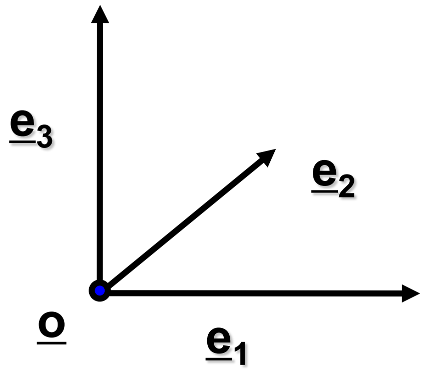
    $\rightarrow$: unique, $\leftarrow$: ambiguous
    - Addition of points and vectors ($p + v \in A^3$)
  - distance(a, b) = length(a - b)

- Operations on A^3
  - Subtraction yields a vector
  - No addition of affine elements



Affine Basis

- Affine Basis:
  - {o, e1, e2, e3}
    - Origin: $o \in A^3$ and
    - Basis of vector space
  - Position vector of point p
    - $(p - o) \in V3$

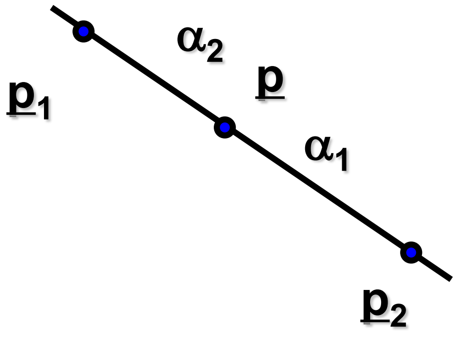


Affine Coordinates

- Affine Combination
  - Linear combination of (n+1) points
  - Weights form a **partition of unity**
  - $b_0, \dots, b_n \in A^n$

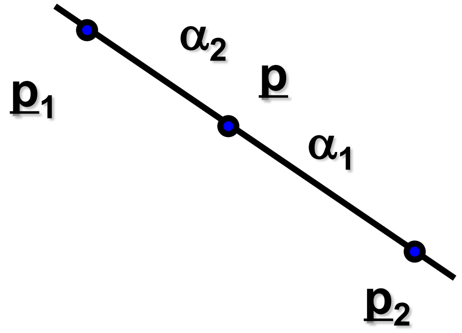
$$\underline{b} = \sum_{i=0}^{n} \alpha_i \underline{b}_i = \underline{b}_0 + \sum_{i=1}^{n} \alpha_i (\underline{b}_i - \underline{b}_0) = \underline{o} + \sum_{i=1}^{n} \alpha_i \underline{e}_i, \quad \text{with} \sum \alpha_i = 1$$

- Affine Coordinates
  - Barycentric coordinates
  - Center of mass ($R = \Sigma\, m_i\, r_i / \Sigma\, m_i$)
  - Affine weighted sum
    - Weights given by the splitting ratio
      - $\underline{p} = \alpha_1 \underline{p}_1 + \alpha_2 \underline{p}_2$
        $\alpha_1 + \alpha_2 = 1$



Affine Mappings

- Properties
  - Affine mapping (continuous, bijective, invertible)
    - T: $A^3 \to A^3$
  - Defined by two non-degenerated simplicies
    - 2D: Triangle, 3D: Tetrahedron, ...
  - Affine/Barycentric coordinates are invariant under affine transformations
  - Other invariants
    - Straight lines, parallelism, splitting ratios, surface/volume ratios
  - Characterization via fixed points and lines
    - Eigenvalues and eigenvectors of the mapping
- Representation
  - Linear mapping A plus a translation t
    - $\mathbf{T}\underline{p} = \mathbf{A}\,\underline{p} + \underline{t}$ with (n x n) matrix **A**
  - Invariance of affine coordinates
    - $Tp = T(\alpha_1\underline{p}_1 + \alpha_2\underline{p}_2) = A(\alpha_1\underline{p}_1 + \alpha_2\underline{p}_2) + \underline{t} =$
      $\alpha_1 A(\underline{p}_1) + \alpha_2 A(\underline{p}_2) + \alpha_1\underline{t} + \alpha_2\underline{t} = \alpha_1 T\underline{p}_1 + \alpha_2 T\underline{p}_2$



Homogeneous Coordinates

Homogeneous Coordinates for 3D

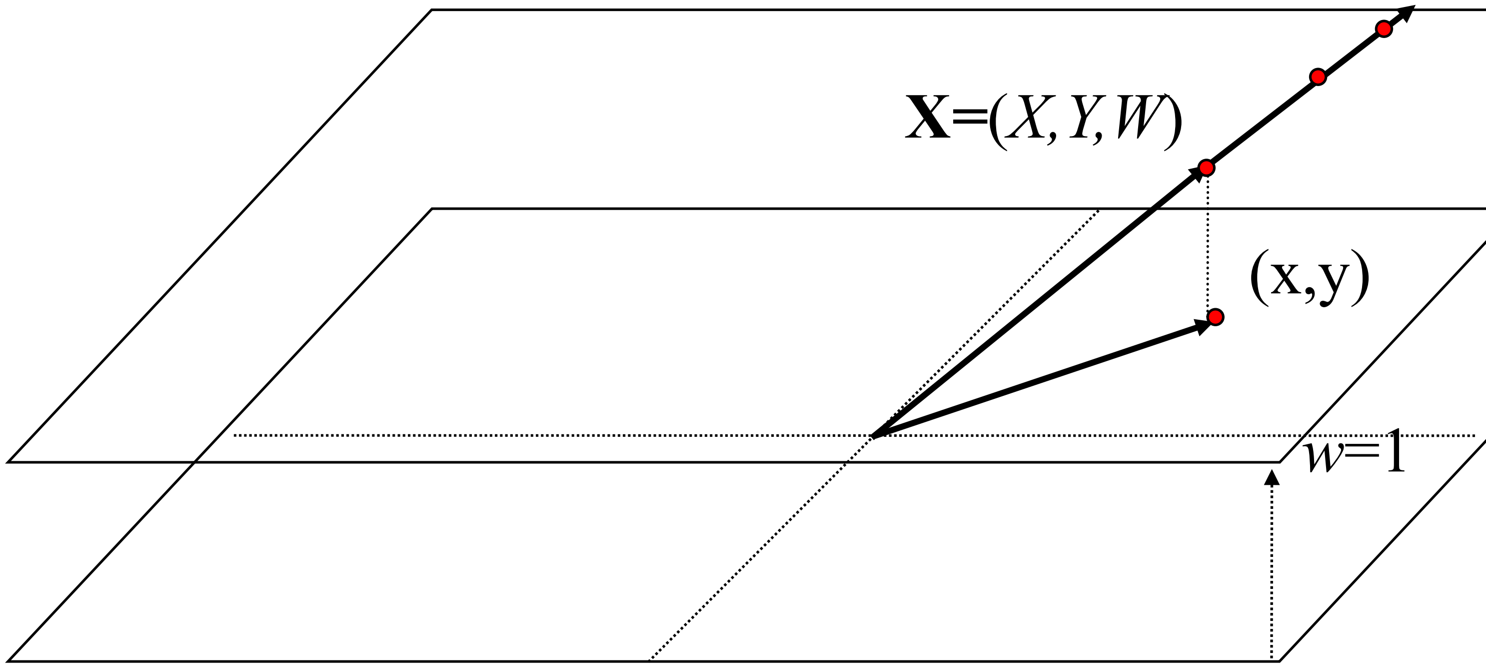
- Embedding of R^3 into $P(R^4)$
  - For the time being

$$R^3 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \in \mathrm{P}(\mathrm{R}^4), \quad \text{and} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} \to \begin{pmatrix} X/W \\ Y/W \\ Z/W \end{pmatrix}$$

  - Representation of transformations by 4x4 matrices
  - Mathematical trick
    - Convenient representation to express rotations and translations as matrix multiplications
    - Easy to find line through points, point-line/line-line intersections
  - Also important for projections (later)

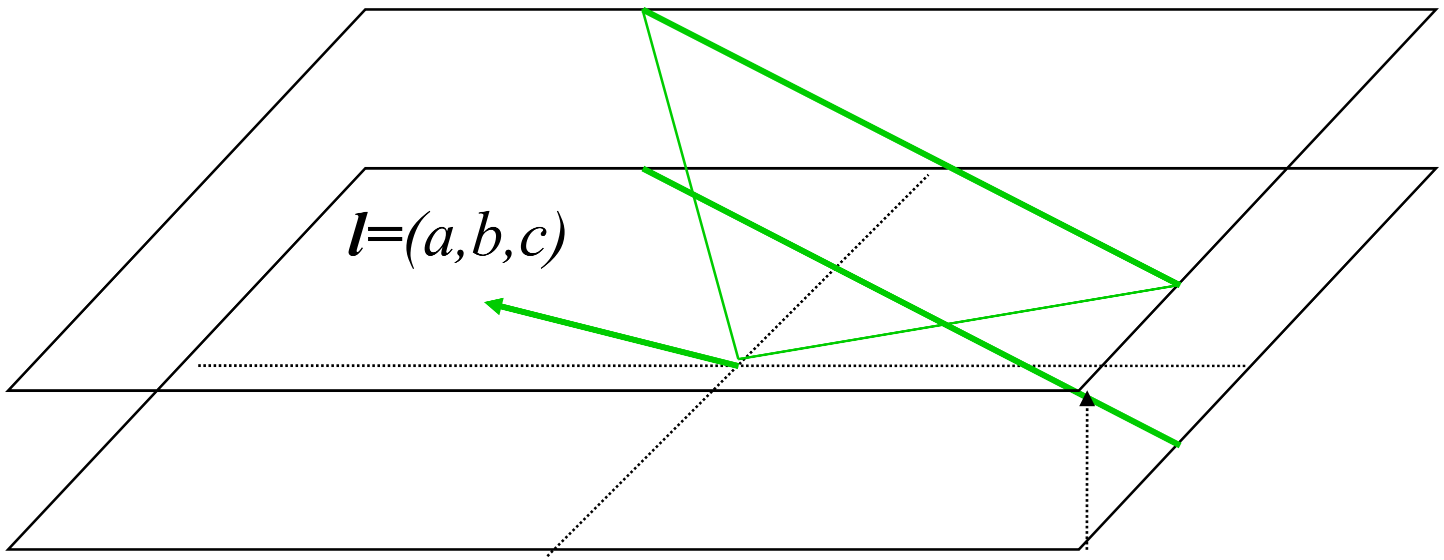
Points and Lines in Homogeneous Coordinates

Point Representation



$$x = \frac{X}{W} \qquad y = \frac{Y}{W}$$

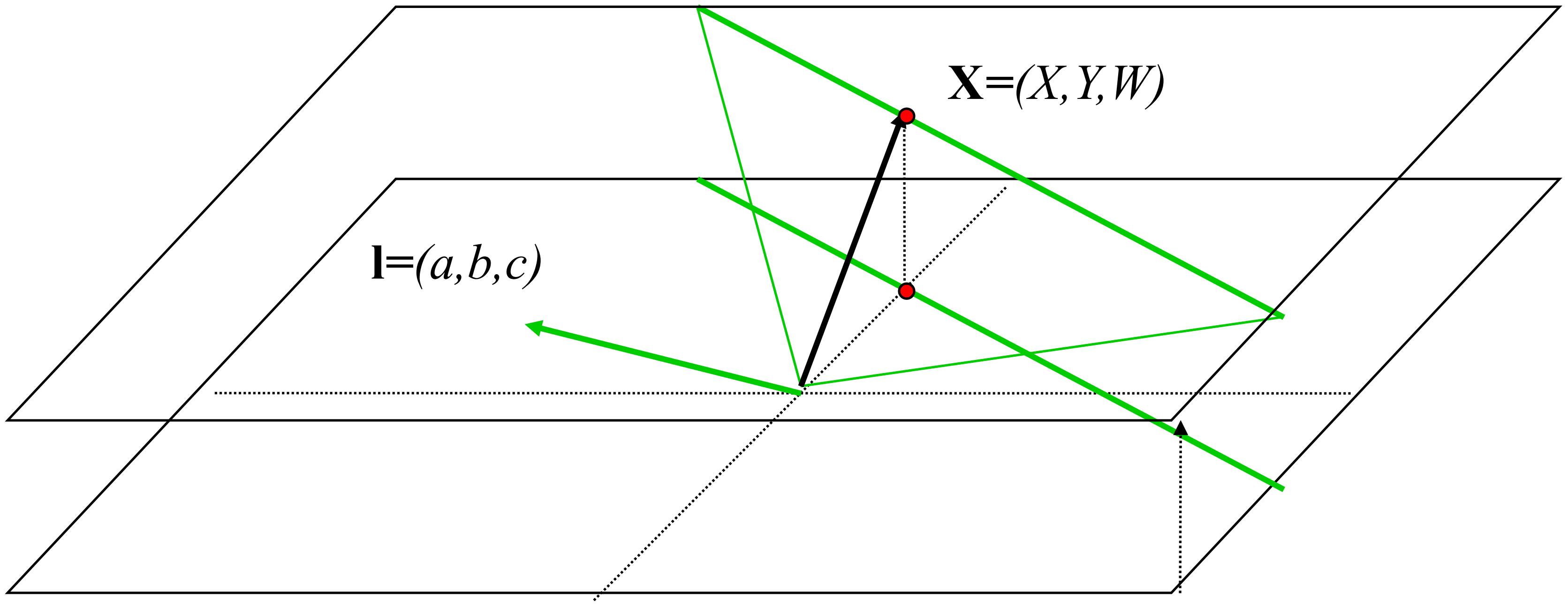
Line Representation



$$\boxed{ax+by+c=0}$$

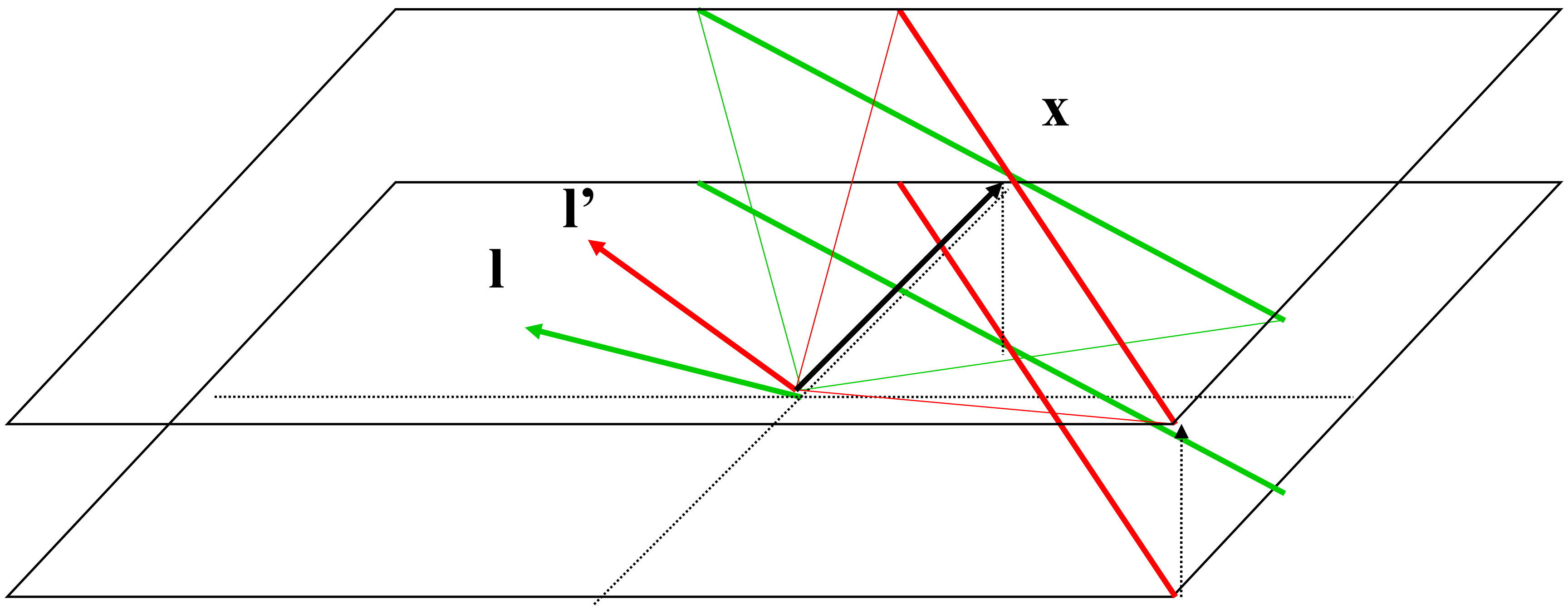
$$ax+by+c\cdot 1=0$$

Point on Line



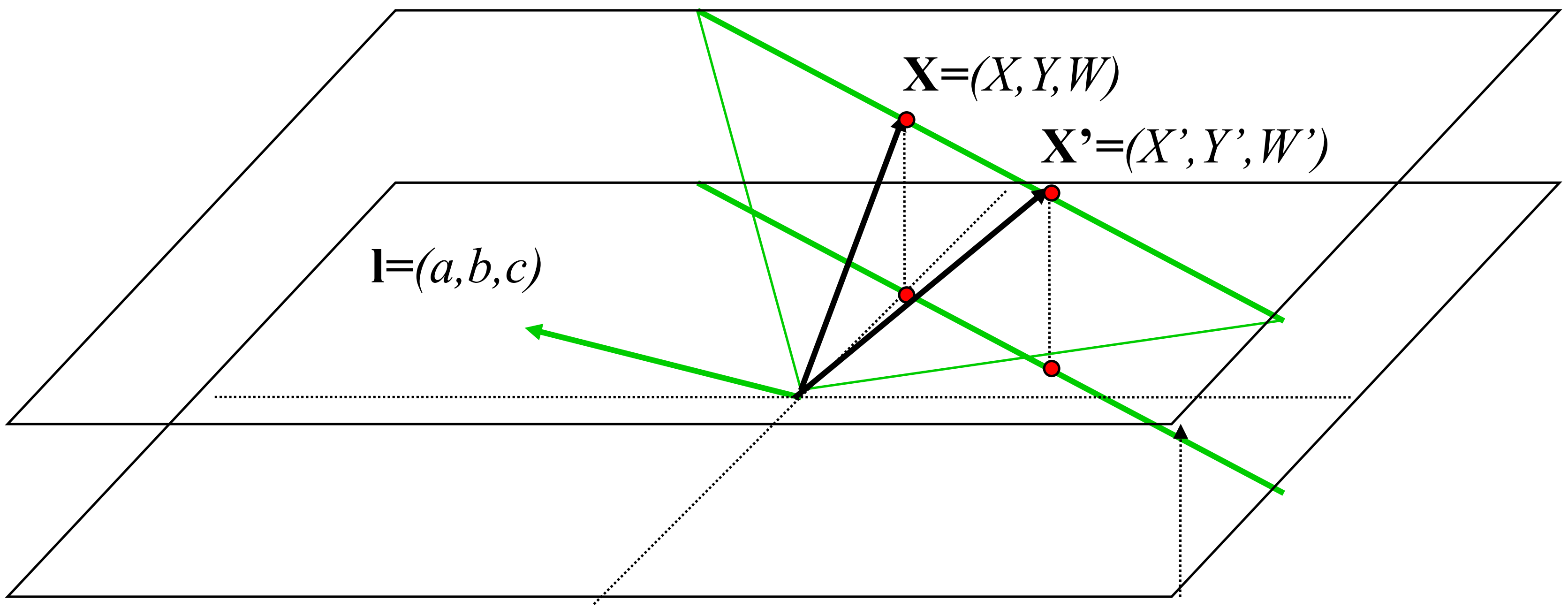
$$\mathbf{x} \cdot \mathbf{l} = 0$$

Intersection of Lines



$$\mathbf{l}' \times \mathbf{l} = \mathbf{x}$$

Line through 2 Points



$$\mathbf{x}' \times \mathbf{x} = \mathbf{l}$$

Basic Transformations

Linear Map = Matrix

- Vector Matrix Product
  - Action of a linear map on a vector
    - Multiplication of matrix with column vector

$$\underline{p}' = \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = T\,\underline{p} = T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} & t_{xw} \\ t_{yx} & t_{yy} & t_{yz} & t_{yw} \\ t_{zx} & t_{zy} & t_{zz} & t_{zw} \\ t_{wx} & t_{wy} & t_{wz} & t_{ww} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

  - Composition (first T_1, then T_2)
    - Matrix multiplication
    - $T_2 T_1 \underline{p} = T_2(T_1 \underline{p}) = (T_2 T_1) \underline{p} = T \underline{p}$
    - Warning: In general, matrix multiplications do not commute !!!

Basic Transformations

- Translation

$$T(d_x, d_y, d_z) = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{T}\underline{p} = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + d_x \\ p_y + d_y \\ p_z + d_z \\ 1 \end{pmatrix}$$

Translation of Vectors

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

- So far we looked at points (affine entities)
- Vectors are defined as the difference of two points
- Consequently, for vectors W is always equal to zero

$$\underline{v} = \underline{p} - \underline{q} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} - \begin{pmatrix} q_x \\ q_y \\ q_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x - q_x \\ p_y - q_y \\ p_z - q_z \\ 0 \end{pmatrix}$$

- This means that translations DO NOT act on vectors
  - Which is exactly what we expect to happen

$$\mathbf{T}\underline{v} = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} = \underline{v}$$

Translations

- Properties
  - $T(0,0,0) = \mathbf{1}$ (Identity Matrix)
  - $T(t_x, t_y, t_z)\, T(t_x', t_y', t_z') = T(t_x + t_x', t_y + t_y', t_z + t_z')$
  - $T(t_x, t_y, t_z)\, T(t_x', t_y', t_z') = T(t_x', t_y', t_z')\, T(t_x, t_y, t_z)$
  - $T^{-1}(t_x, t_y, t_z) = T(-t_x, -t_y, -t_z)$

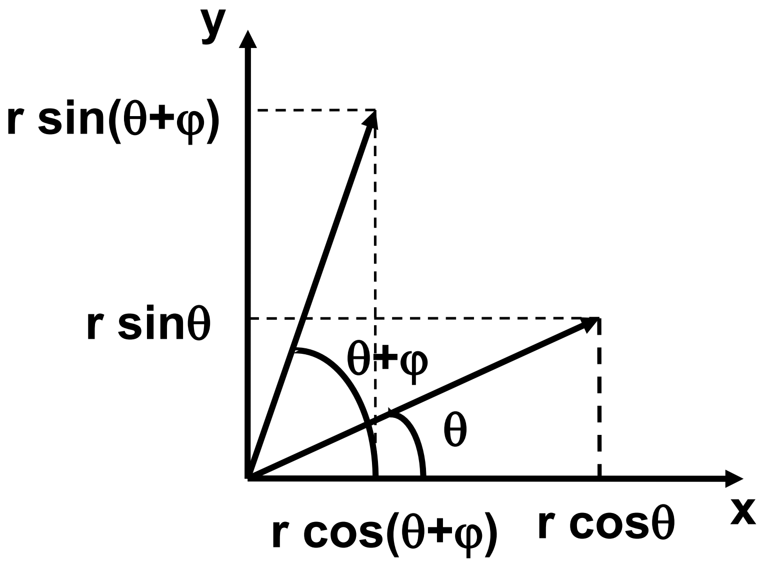
Rotation

- Rotation in 2D

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x', y'?$$



Rotation

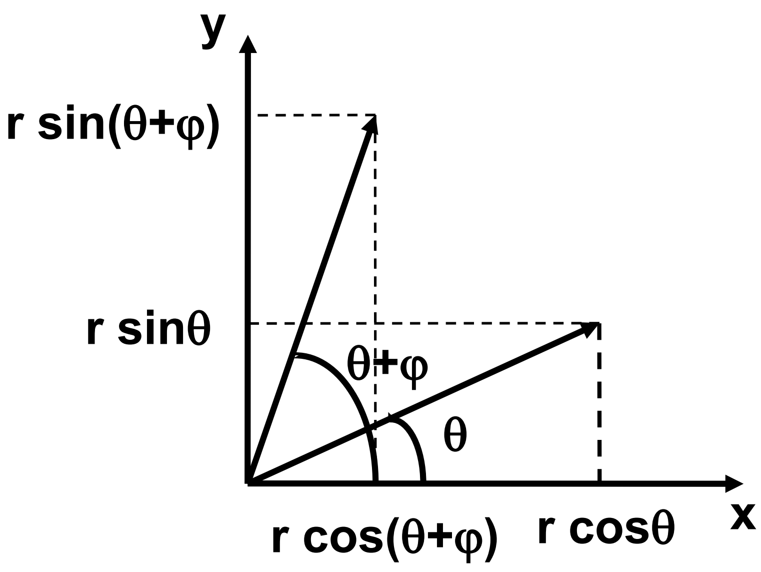
- Rotation in 2D

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$x' = r\cos(\theta+\varphi)$$
$$y' = r\sin(\theta+\varphi)$$

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$
$$\sin(\theta+\varphi) = \cos\theta\sin\varphi + \sin\theta\cos\varphi$$

$$x' = (r\cos\theta)\cos\varphi - (r\sin\theta)\sin\varphi = x\cos\varphi - y\sin\varphi$$
$$y' = (r\cos\theta)\sin\varphi + (r\sin\theta)\cos\varphi = x\sin\varphi + y\cos\varphi$$



Basic Transformations

- Rotation around major axis

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  - Assumes a right handed coordinate system

Rotation

- Properties
  - $R_a(0) = \mathbf{1}$
  - $R^{-1}{}_a(\theta) = R_a(-\theta)$
  - $R_a(\theta)\, R_a(\varphi) = R_a(\theta + \varphi)$
  - $R_a(\theta)\, R_a(\varphi) = R_a(\varphi)\, R_a(\theta)$
  - $R^{-1}{}_a(\theta) = R_a(-\theta) = R^T{}_a(\theta)$
  - BUT in general: $R_a(\theta)\, R_b(\varphi) \neq R_b(\varphi)\, R_a(\theta)$
    - For rotations around different axes, the order matters

Basic Transformations

- Scaling

$$S(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

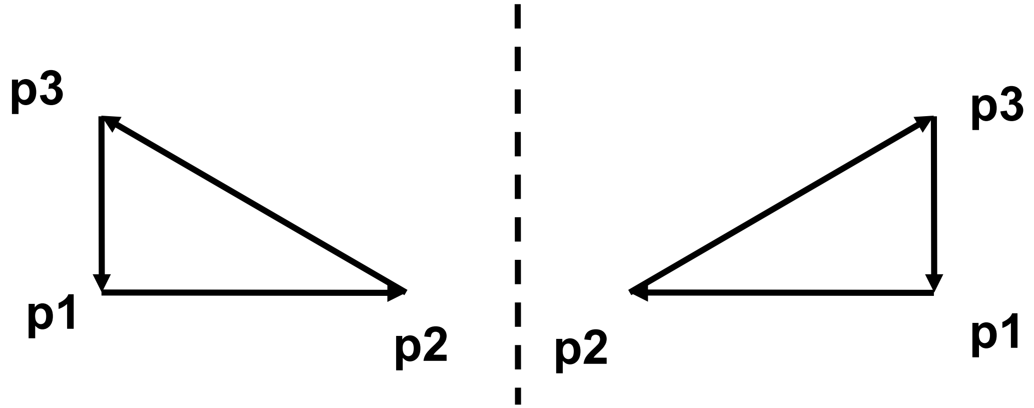
  - Uniform Scaling
    - sx= sy = sz

Basic Transformations

- Reflection at Z

$$M_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

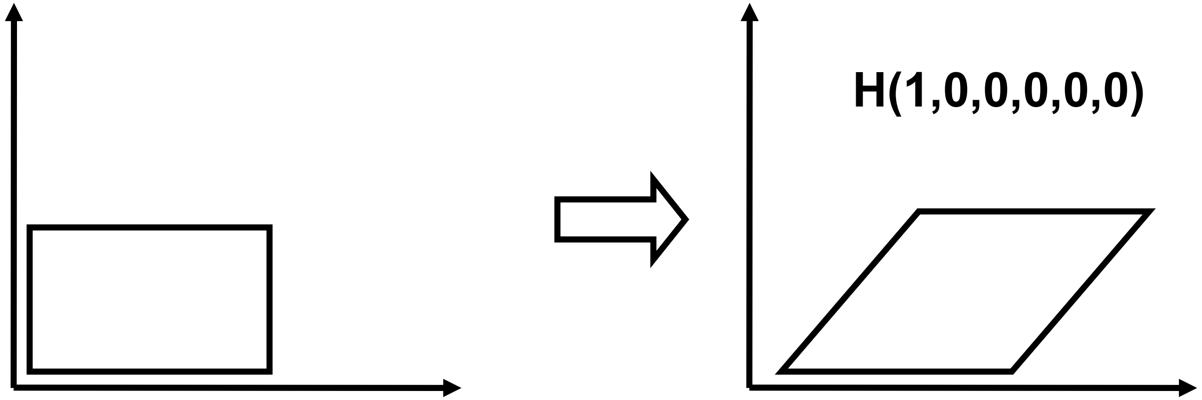
  - Warning: Change of orientation !



Basic Transformations

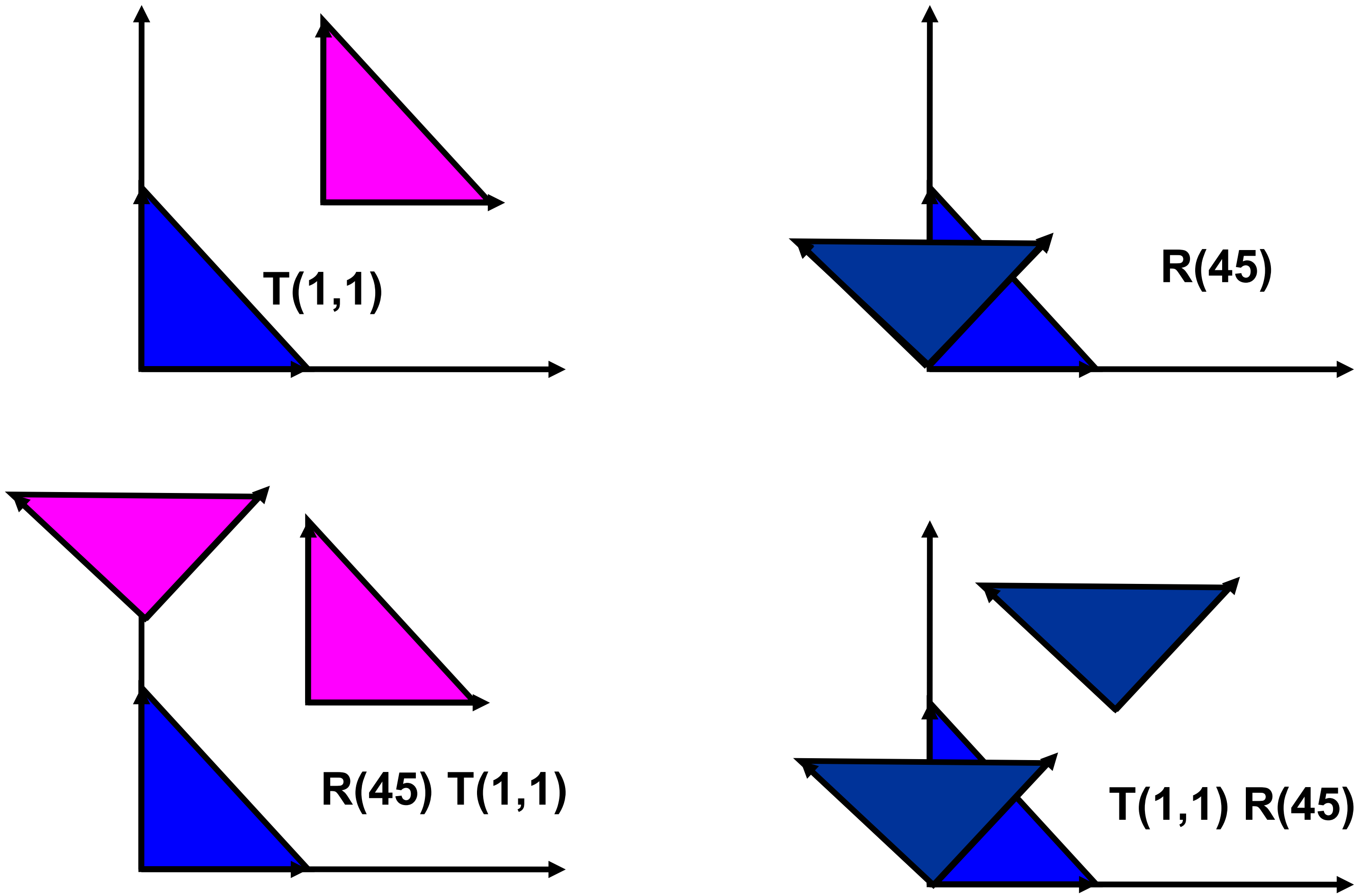
- Shear (deutsch: Scherung)

$$H(h_{xy}, h_{xz}, h_{yz}, h_{yx}, h_{zx}, h_{zy}) = \begin{pmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Concatenation of Transformations

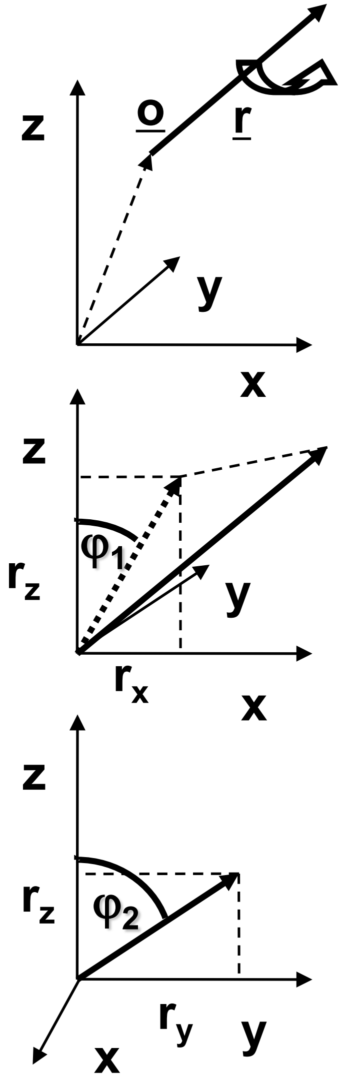
- In general, transformations do not commute



Rotation about Arbitrary Axis

- Move base point to origin
  - T(-o)
- Rotation around Y-axis, so that r is in YZ-plane
  - Use projection into XZ-plane
  - $R_y(-\varphi_1)$ $\tan(\varphi_1)= r_x/r_z$ $r' = R_y(-\varphi_1)\, r$
- Rotation around X-axis, so that r' is along Z-axis
  - $R_x(\varphi_2)$ $\tan(\varphi_2)= r'_y / r'_z$
- Rotation around Z-axis with angle φ
- Rotate back around X-axis
- Rotate back around Y-axis
- Translate back

• Together
- $R(\varphi, \underline{o}, \underline{r}) = T(\underline{o})R_y(\varphi_1)R_x(-\varphi_2)R_z(\varphi)R_x(\varphi_2)R_y(-\varphi_1)T(-\underline{o})$

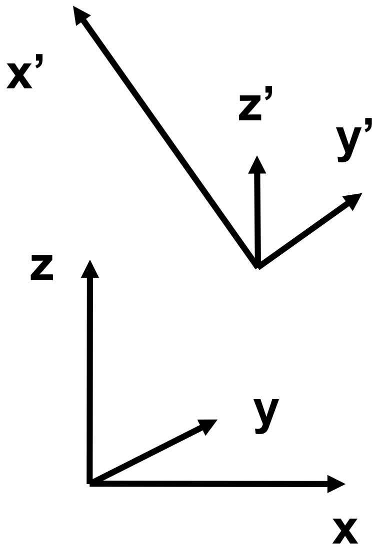


Matrices as Basis Transform

- Columns are transformed basis

$$p' = \begin{pmatrix} t_{xx} & t_{xy} & t_{xz} & t_{xw} \\ t_{yx} & t_{yy} & t_{yz} & t_{yw} \\ t_{zx} & t_{zy} & t_{zz} & t_{zw} \\ t_{wx} & t_{wy} & t_{wz} & t_{ww} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } , \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

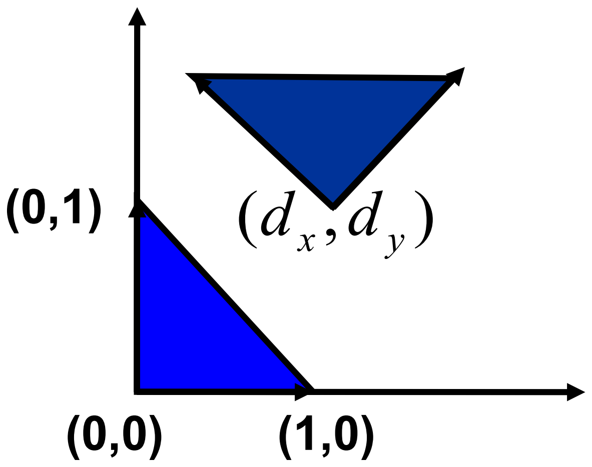
$$\begin{pmatrix} \underline{e}_x' & \underline{e}_y' & \underline{e}_z' & \underline{o}' \end{pmatrix} = M$$



- Transformation into new basis
  - Simple: Write new basis vectors into columns of matrix

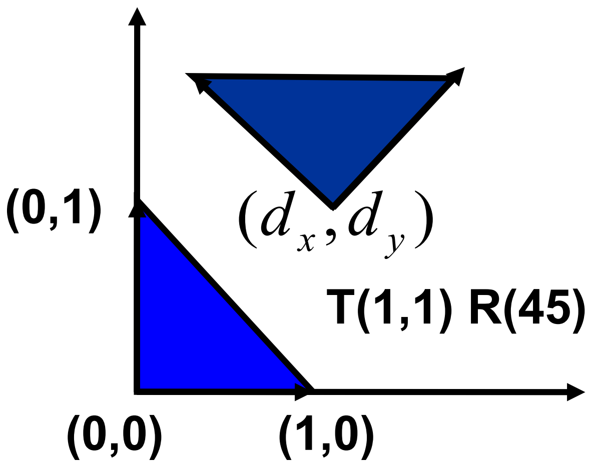
Complex Transformations

- Either by concatenation
- Or by coordinate transform
  - Translation plus transformation of basis vectors



Complex Transformations

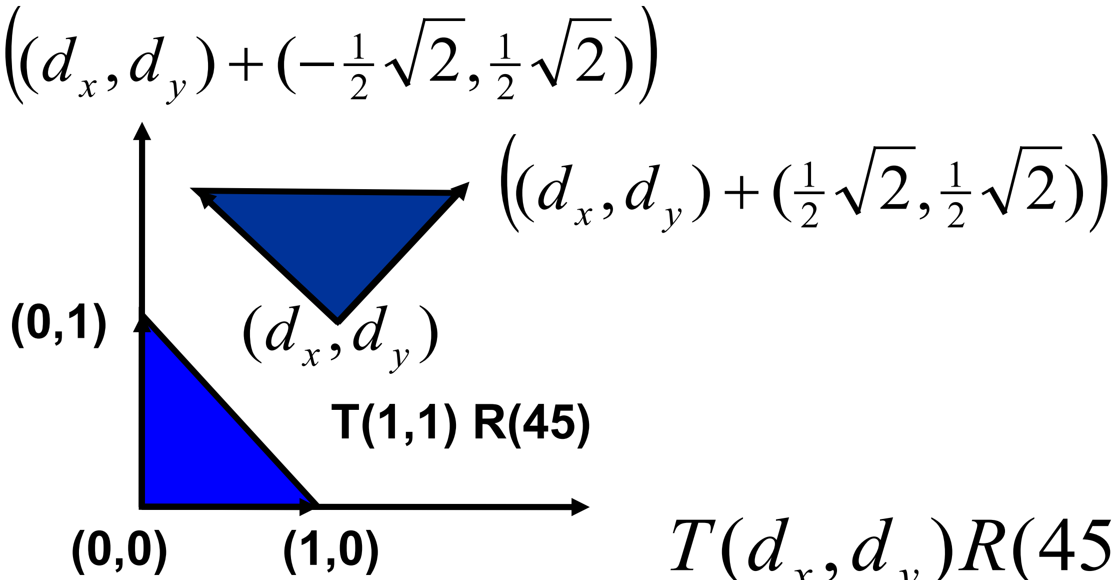
- Either by concatenation
- Or by coordinate transform
  - Translation plus transformation of basis vectors



$$T(d_x, d_y)R(45)$$

Complex Transformations

- Either by concatenation
- Or by coordinate transform
  - Translation plus transformation of basis vectors



$$T(d_x, d_y)R(45) = \begin{pmatrix} \tfrac{1}{2}\sqrt{2} & -\tfrac{1}{2}\sqrt{2} & d_x \\ \tfrac{1}{2}\sqrt{2} & \tfrac{1}{2}\sqrt{2} & d_y \\ 0 & 0 & 1 \end{pmatrix}$$

Orthonormal Matrices

- Orthonormal transformations
  - Images of basis vectors are again orthonormal
    - $e_i' \cdot e_j' = \delta_{ij}$

$$\mathbf{M}^T\mathbf{M} = \left(\underline{e}_x' \quad \underline{e}_y' \quad \underline{e}_z'\right)^T \left(\underline{e}_x' \quad \underline{e}_y' \quad \underline{e}_z'\right) =$$

$$= \begin{pmatrix} \underline{e}_x'\underline{e}_x' & \underline{e}_x'\underline{e}_y' & \underline{e}_x'\underline{e}_z' \\ \underline{e}_y'\underline{e}_x' & \underline{e}_y'\underline{e}_y' & \underline{e}_y'\underline{e}_z' \\ \underline{e}_z'\underline{e}_x' & \underline{e}_z'\underline{e}_y' & \underline{e}_z'\underline{e}_z' \end{pmatrix} = \mathbf{1}$$

Which means that

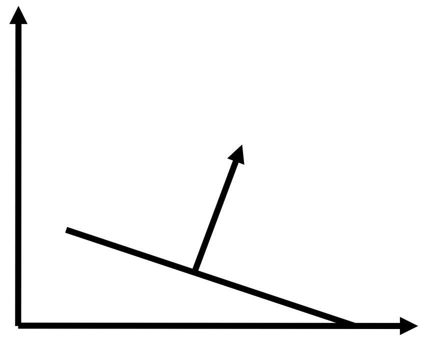
$$\mathbf{M}^T = \mathbf{M}^{-1}$$

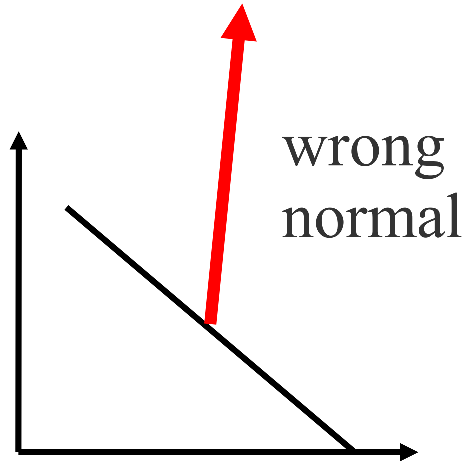
Transformation of Normals

Transformations

- Line
  - Transform end points
- Plane
  - Transform three points
- Vector
  - $\underline{v} = \underline{p} - \underline{q} = (x, z, y, 0)^T$
  - Translations to not act on vectors
- Normal vectors
  - Problem: e.g. with non-uniform scaling

input





scale y by 2

Transforming Normals

- Dot product as matrix multiplication

$$\underline{v} \cdot \underline{w} = v^T * w = (v_x, v_y, v_z) * \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

Matrix multiplication

Dot product

- Normal N on a plane
  - For any vector T in the plane: $\underline{N}^T * \underline{T} = 0$
  - Given M, find transformation M' for normal vector, such that

$$(\mathbf{M}' * \underline{N})^T * (\mathbf{M} * \underline{T}) = 0 = \underline{N}^T * (\mathbf{M}'^T * \mathbf{M}) * \underline{T}$$

$$\mathbf{M}'^T * \mathbf{M} = 1$$

$$\mathbf{M}' = (\mathbf{M}^{-1})^T$$

Transforming Normals

- Remember:

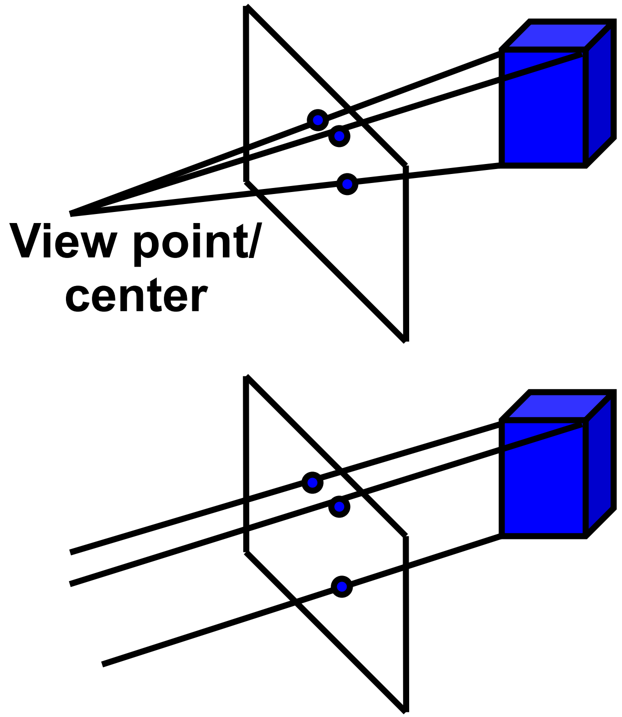
  Normals are transformed by the transpose of the inverse of the 4x4 transformation matrix of points and vectors

- No problem with orthogonal transformations
  - E.g. rotation, uniform scaling
    - $M^{-1} = M^T$
    - $M^{-1^T} = M^{T^T} = M$

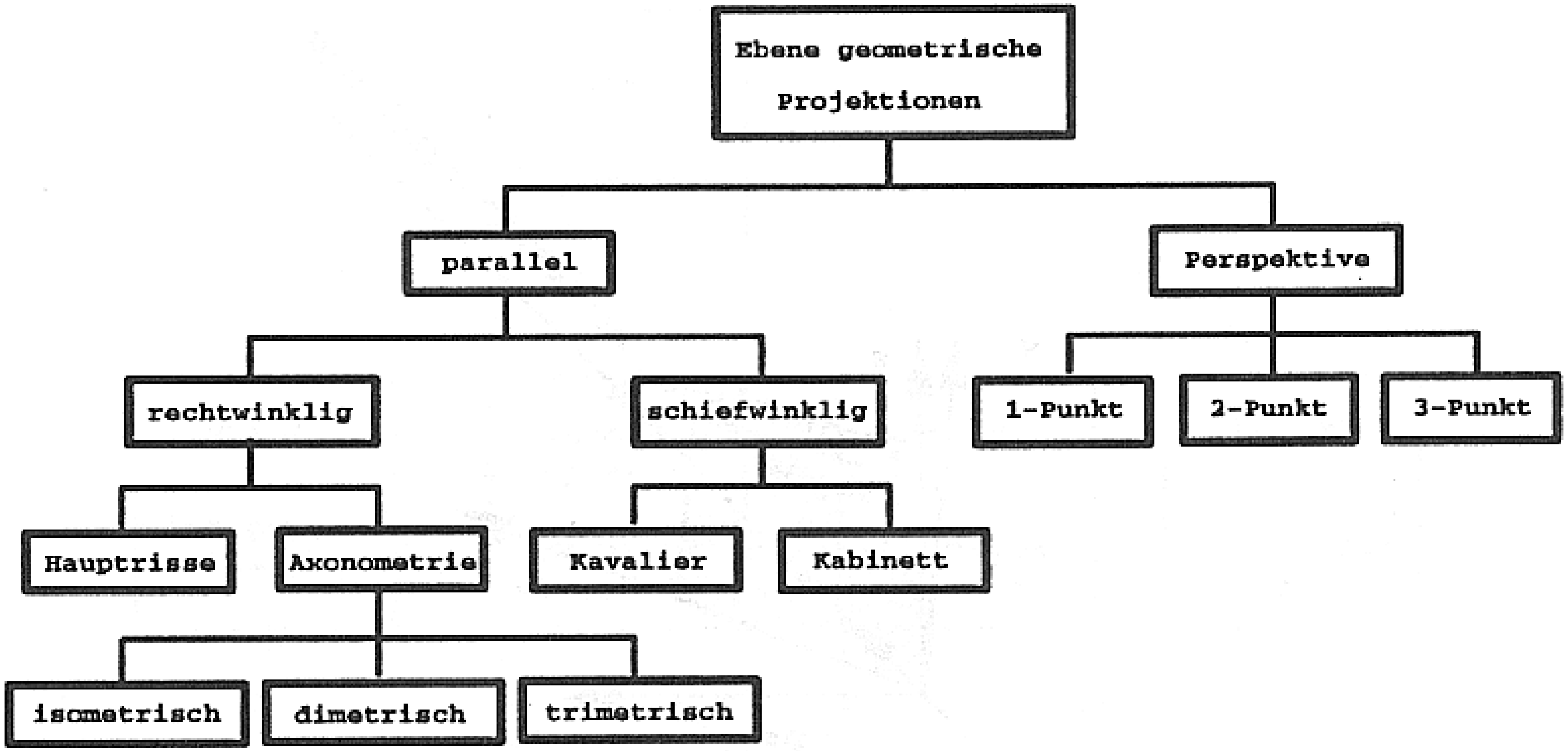
Projections

Projections

- Definition: Projection
  - Mapping from 3D to 2D
  - Results in loss of information
  - Non invertible
- Planar perspective projections
  - Projection along lines onto a projection plane
  - Perspective projection (central projection)
    - Lines intersect in a single point
  - Special case: Orthographic projection
    - Parallel lines
    - View point at infinity

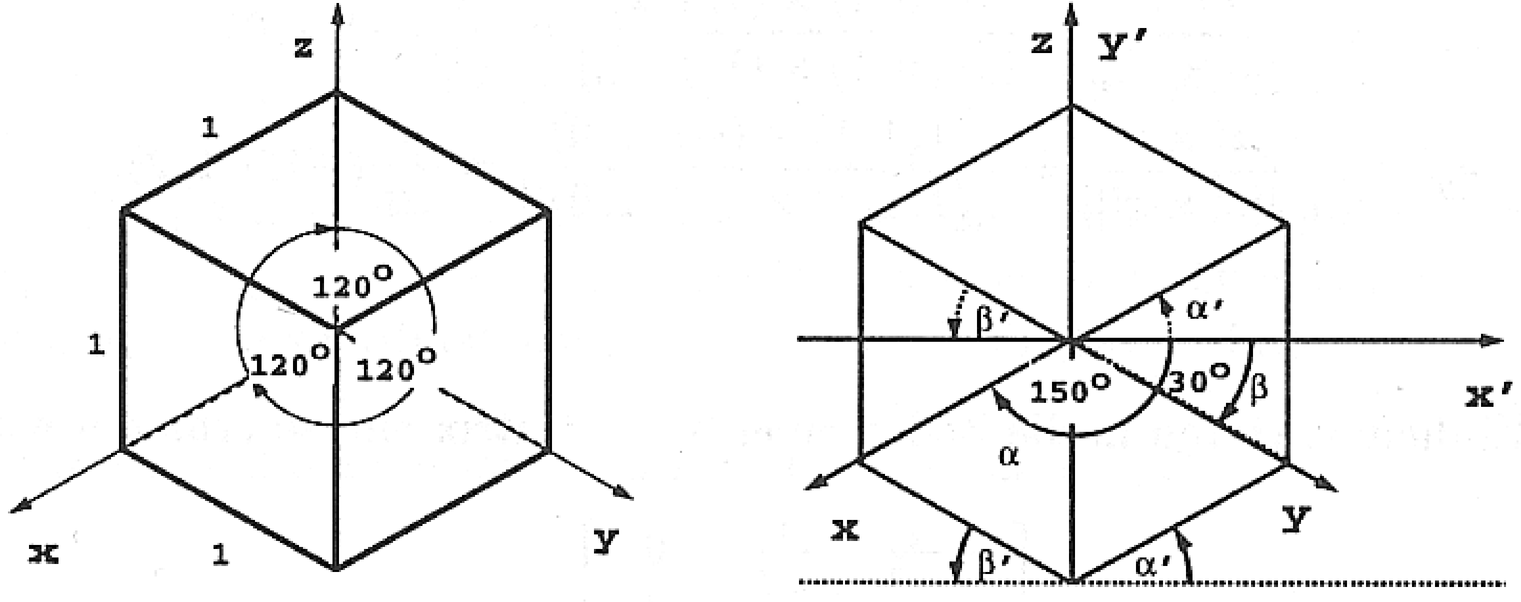


Classification of Projections



Axonometric Projection

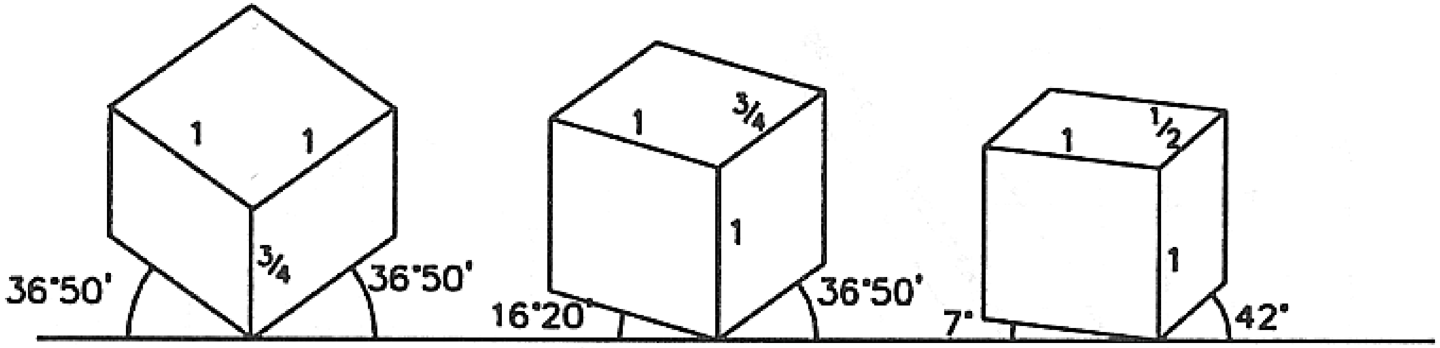
- Properties
  - Parallel/orthographic projection
  - Projection plane orthogonal to projection direction
- Isometric Projection
  - Projektion direction has same angle with every coordinate axis
    - Lengths are maintained



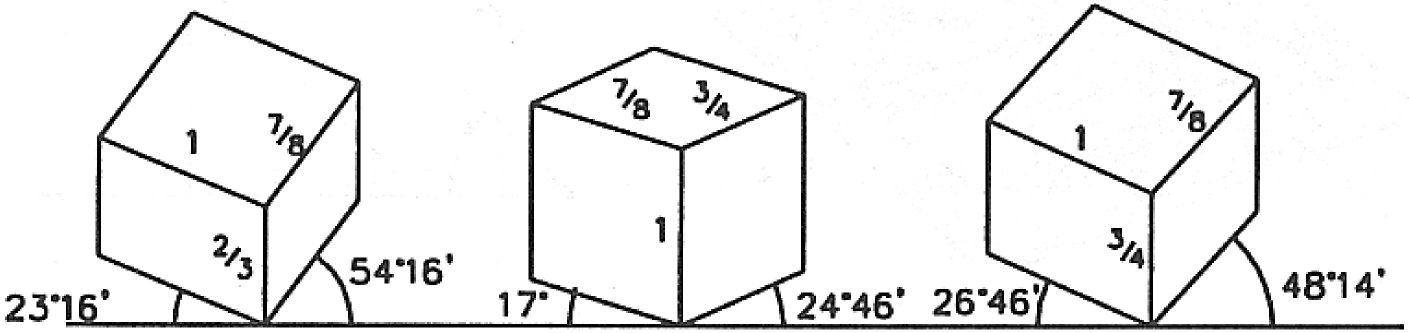
a) b)

Axonometric Projection

- Dimetric and Trimetric Projection
  - Same angle with 2 axes
    - Two lengths are maintained, one is scaled



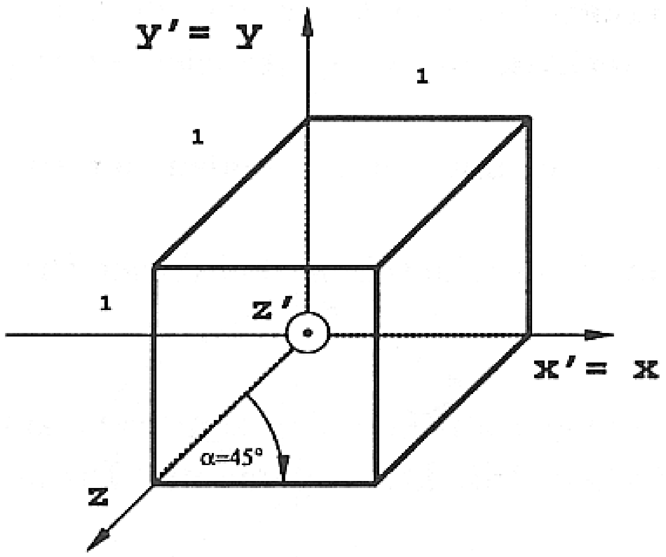
  - Same angle with one axis
    - One length is maintained, two are scaled

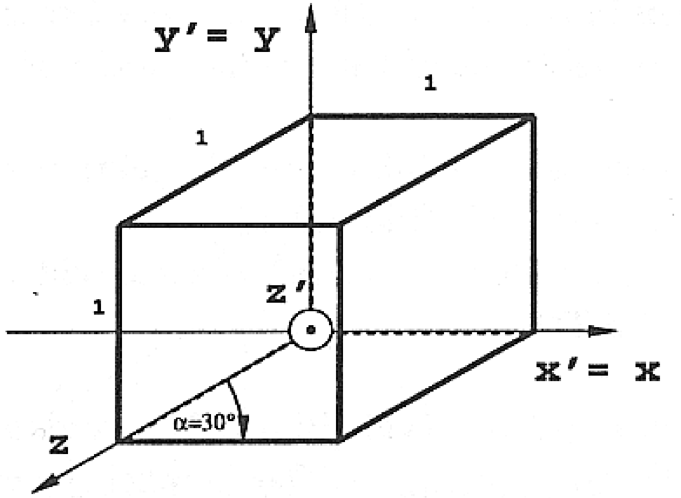


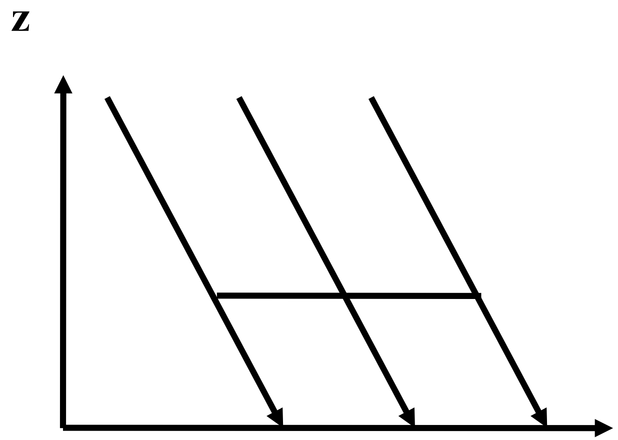
Sheared Perspective

Oblique Projection

- Properties
  - Parallel projection
  - Projektion plane parallel to two coordinate axes (e.g. x, y)
  - Projection direction ***not*** orthogonal to plane
- Cavalier Projection
  - Same length on all axes

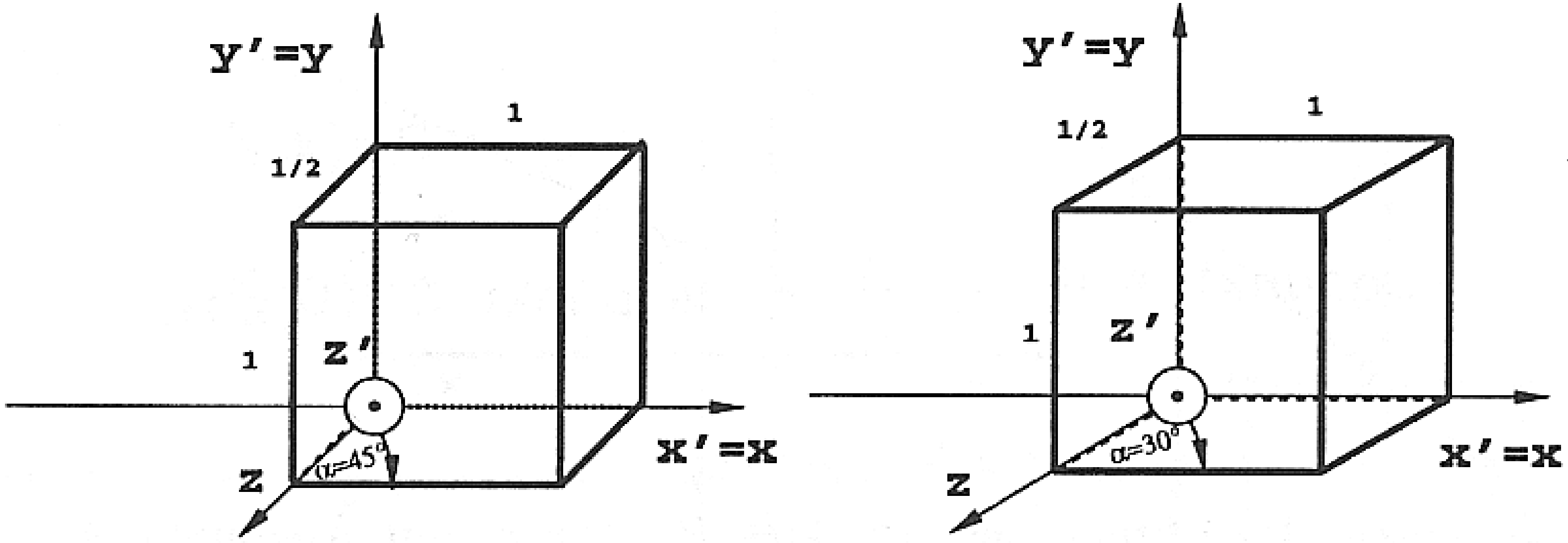




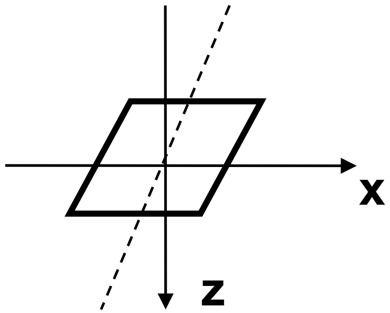


Oblique Projection

- Cabinet Projection
  - Foreshortening of ½ orthogonal to projection plane

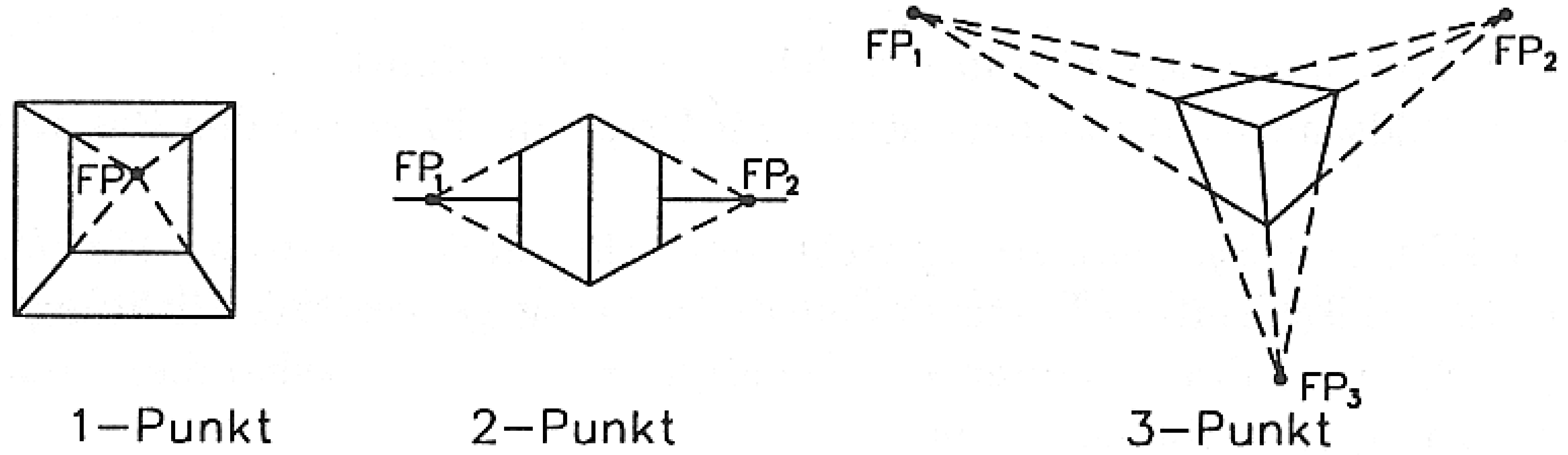


- Implementation of Oblique Projections
  - Shearing plus parallel projection

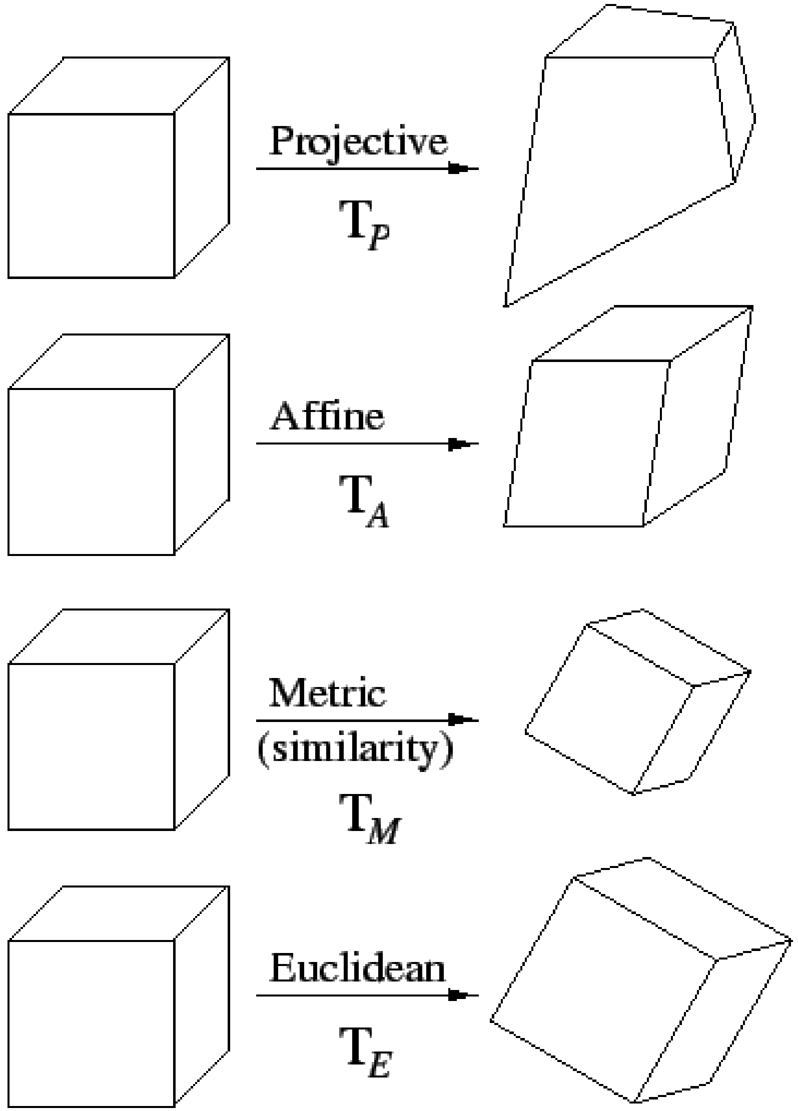


Planar Perspective Projection

- Properties
  - Projection onto plane along lines through a projection point
  - Parallel lines do ***NOT*** stay parallel
    - Not an affine transformation
- Vanishing point
  - Projections of intersection points axis-parallel lines at infinity
  - N-point perspective
- Details later

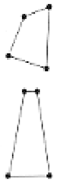
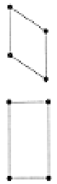
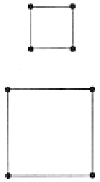
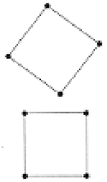


Projective Transformations in 3D



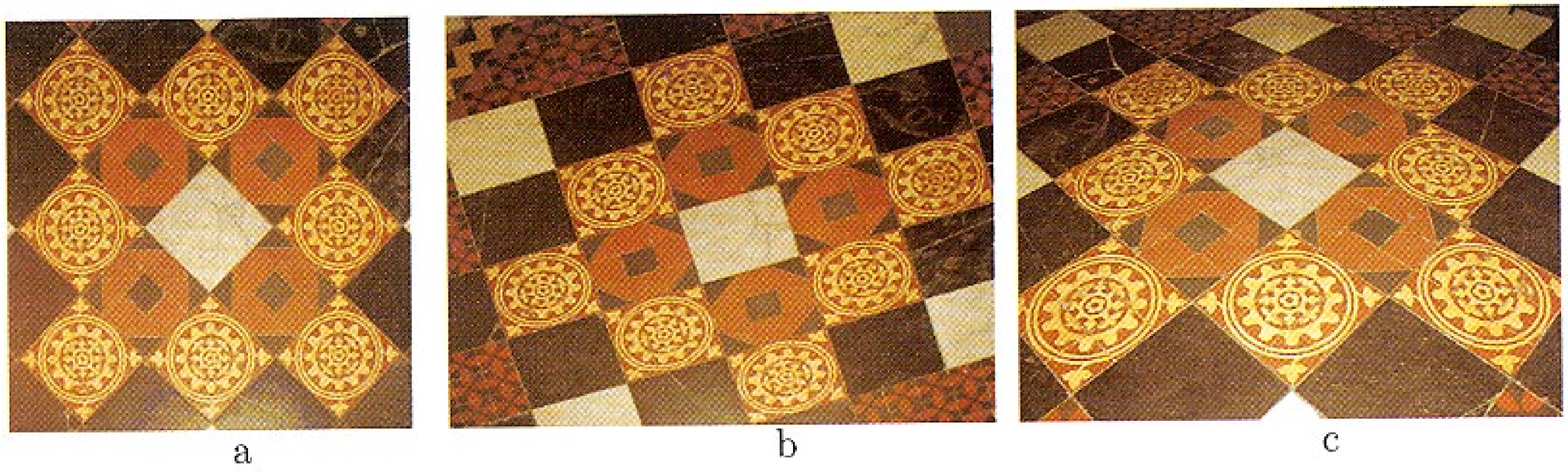
Taxonomy of Projective Transformations

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, $\mathbf{I}, \mathbf{J}$ (see section 1.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Distortions under Central Projection

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN



- **Similarity**: circle remains circle, square remains square

⇒ line orientation is preserved

- **Affine**: circle becomes ellipse, square becomes rhombus

⇒ parallel lines remain parallel

- **Projective**: imaged object size depends on distance from camera

⇒ parallel lines converge

Summary

- Vector space and Affine Space
- Affine transformations
- Homogeneous coordinates
- Basic transformations
- Concatenation vs. basistransform
- Treatment of normals
- Projections

Overview

- Last Time
  - Tone Mapping
- Today
  - Homogeneous Coordinates
  - Basic transformations in homogeneous coordinates
  - Concatenation of transformations
  - Projective transformations
- Next Lectures
  - Camera transformations
  - Rasterization