EBERHARD KARLS UNIVERSITÄT TÜBINGEN

FACULTY OF SCIENCE
Computer Graphics

Computer Graphics (Graphische Datenverarbeitung)

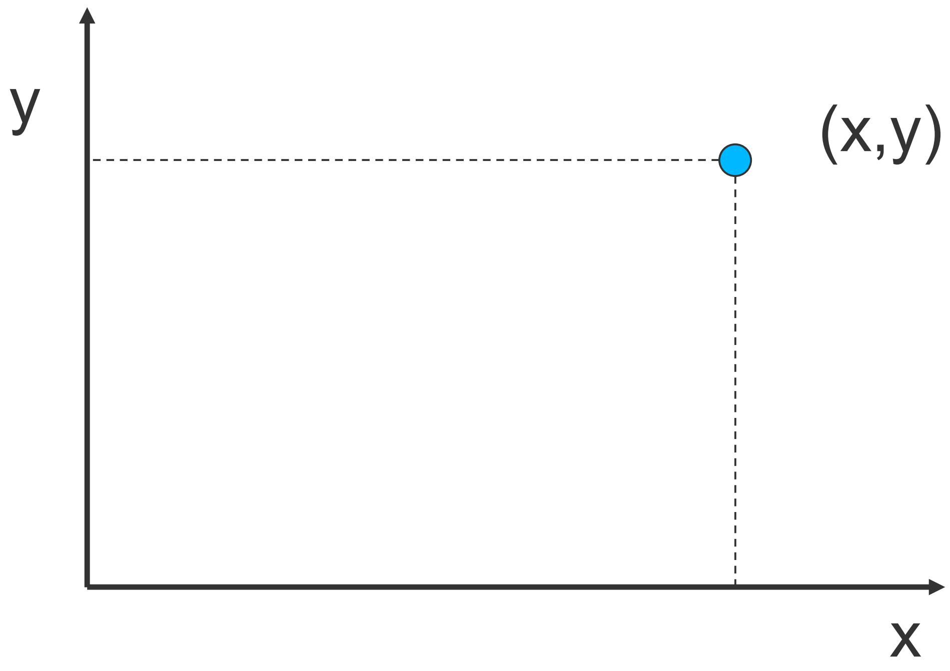
- Math Primer -

WS 2021/2022

Trigonometry

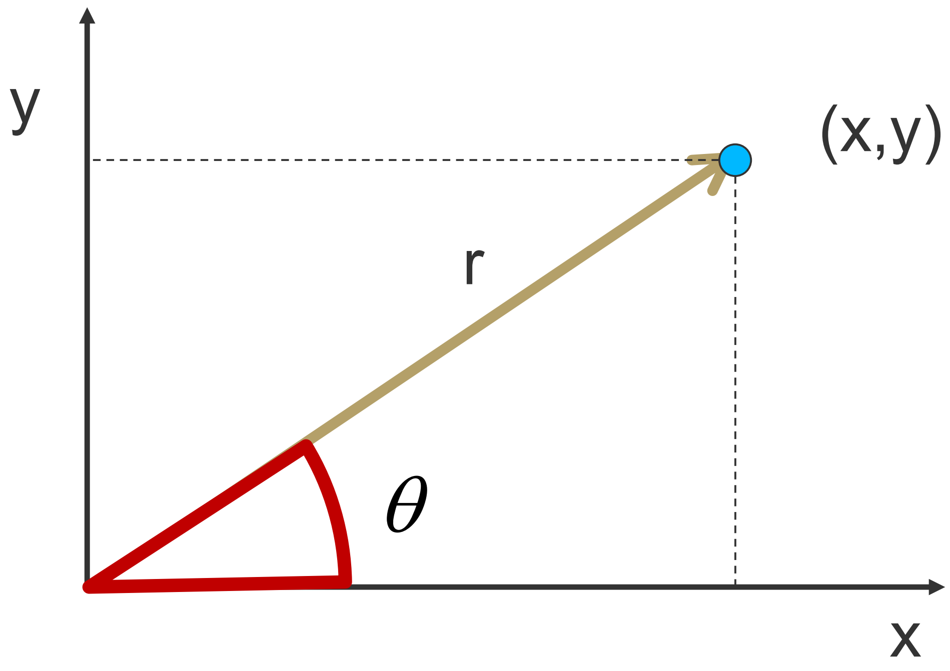
Trigonometry

- Basic rules on angles and triangles
- Point in plane given by x and y coordinates



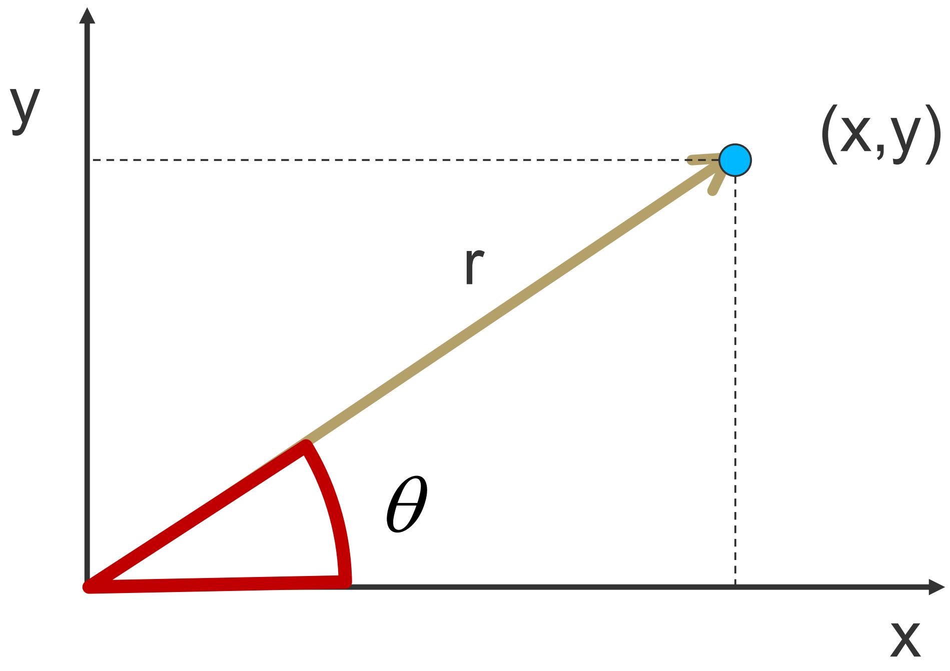
Trigonometry

- Point in plane given by distance r to origin and angle θ



Trigonometry

- Angles and lengths



$$\sin\theta = y / r$$

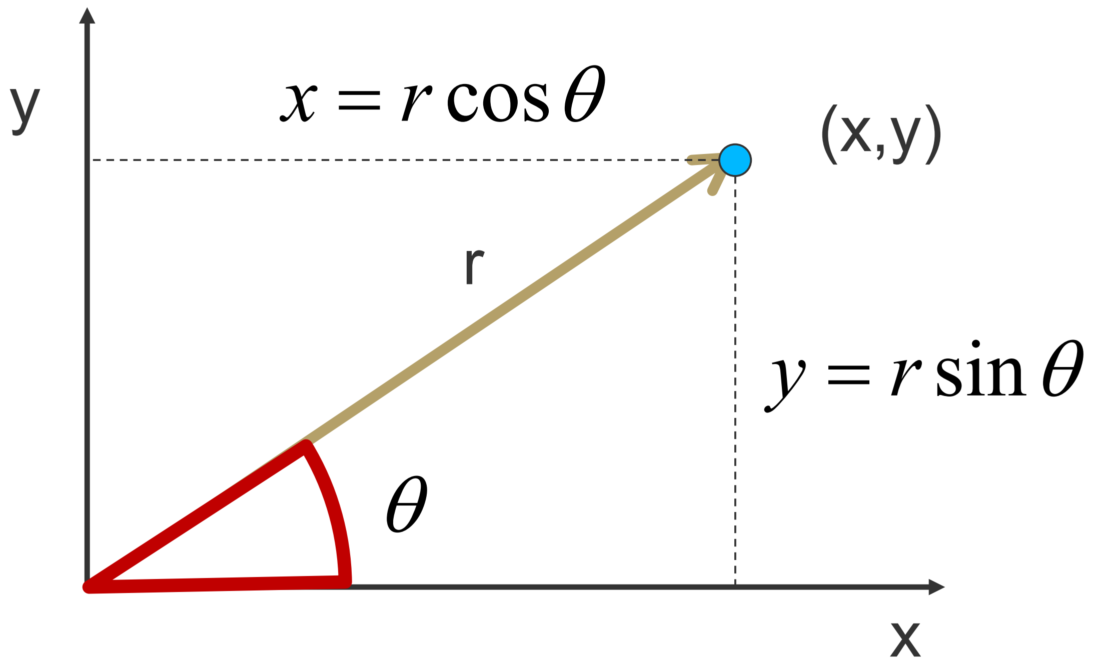
$$\cos\theta = x / r$$

$$\tan\theta = y / x$$

$$\cot\theta = x / y$$

Trigonometry

- Angles and lengths



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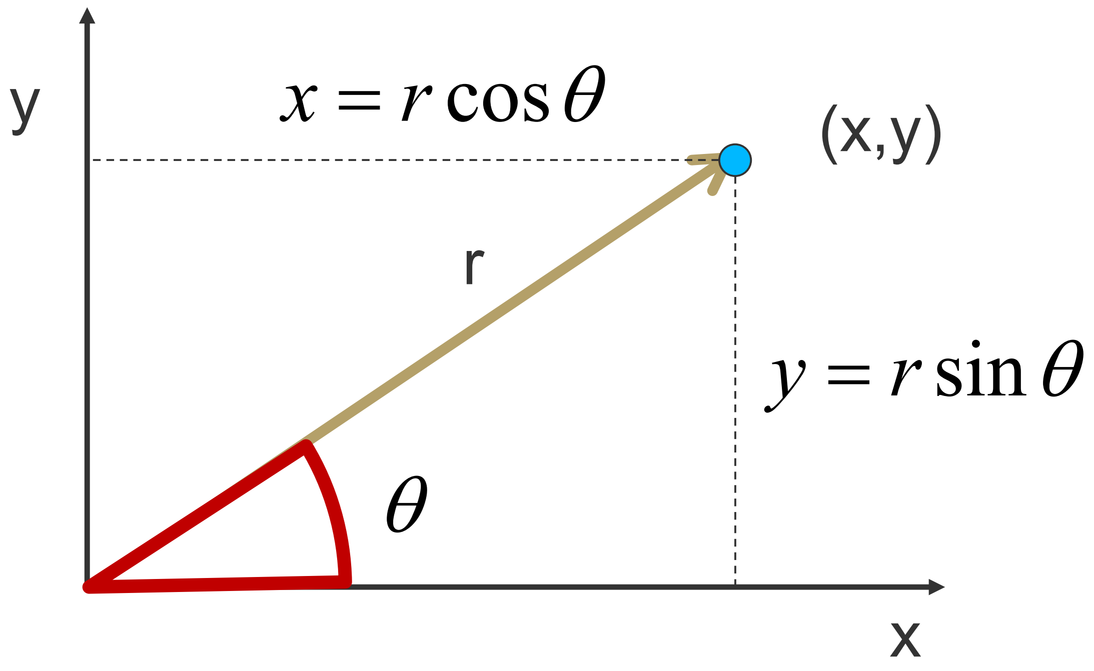
Trigonometry

- Angles and lengths



$$\sin\theta = y / r$$

$$\cos\theta = x / r$$

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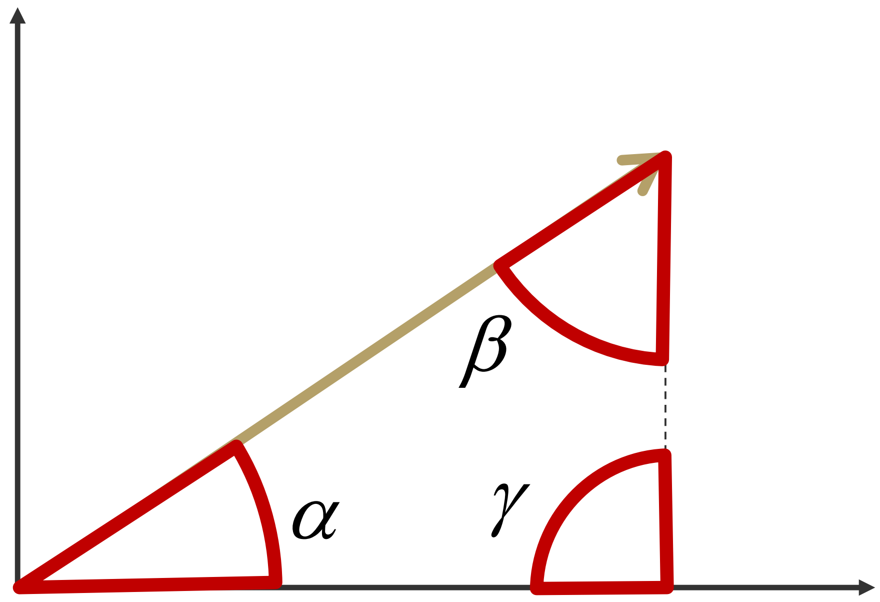
Trigonometry

- Angles measured in radians or degree

$$1 \; radian = 180 / \pi \qquad\qquad 1^{\circ} = \pi / 180$$

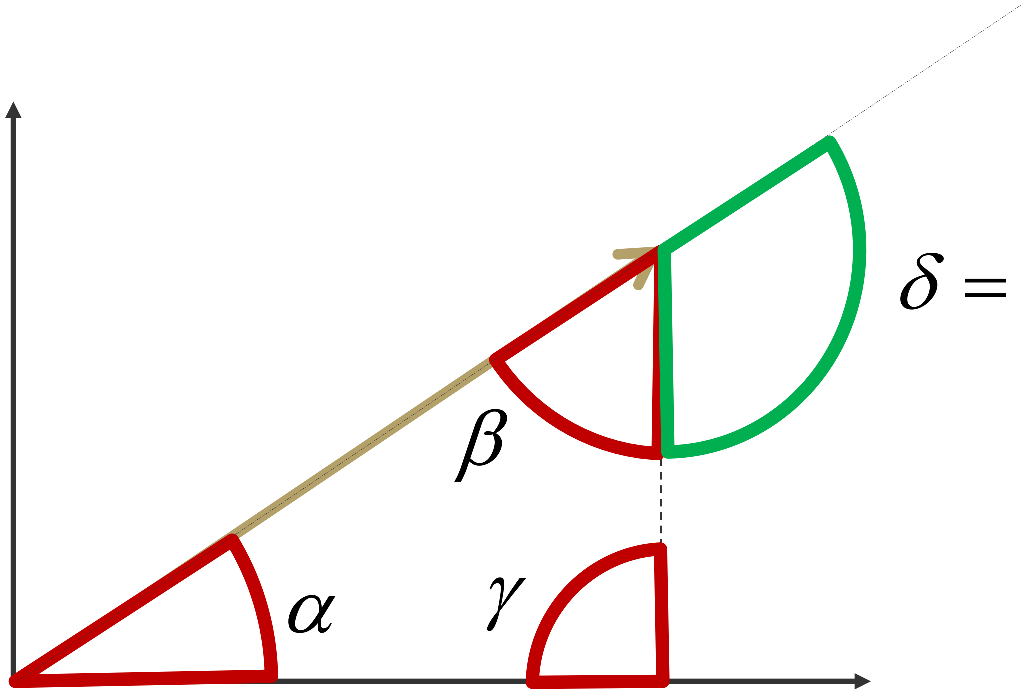


$$\alpha + \beta + \gamma = \pi$$

Trigonometry

- Angles measured in radians or degree

$$1\ radian = 180/\pi \qquad 1^\circ = \pi/180$$



$$\delta = \pi - \beta = \alpha + \gamma$$

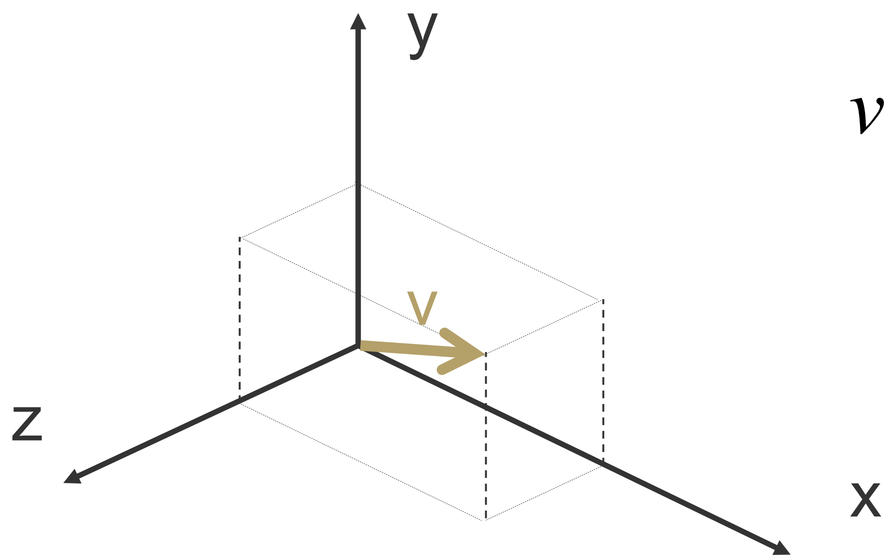
$$\alpha + \beta + \gamma = \pi$$

Vectors

Vectors

- Useful for describing
  - Positions
  - Directions
  - …

- 3D Cartesian System



$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Vectors

- Useful for describing
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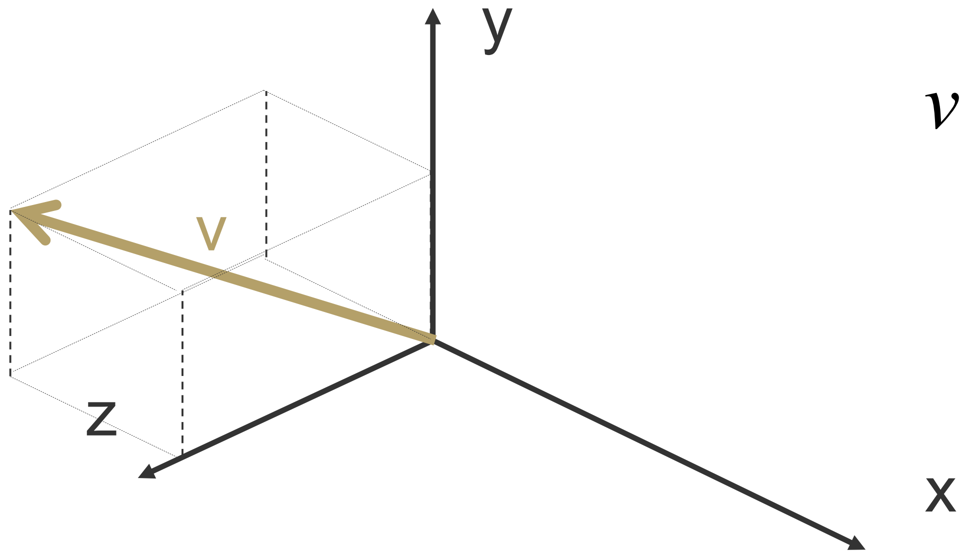
  - Directions
  - …

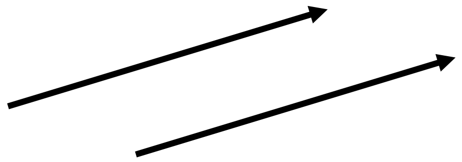
- 3D Cartesian System

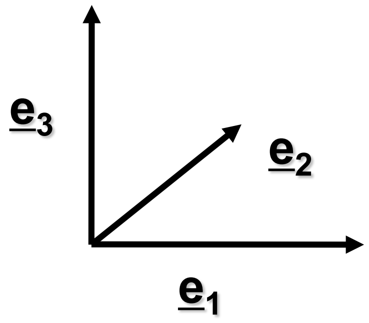


$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Euclidean Vector Space

- Known from Mathematics:
  - Elements of a 3D vector space
    - $\underline{v} = (v_1, v_2, v_3)^T \in V^3 = R^3$
  - Formally: Vectors written as column vectors (n x 1 matrix)!
  - Vectors describe directions – not positions!
  - 3 linear independent vectors create a basis:
    - $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$
  - Any vector can now uniquely be represented with coordinates
    - $\underline{v} = v_1\,\underline{e}_1 + v_2\,\underline{e}_2 + v_3\,\underline{e}_3 = (v_1, v_2, v_3)^T$
  - Operations
    - Addition, Subtraction, Scaling, …





Vectors

- Typical variable names (small letters)

$$v,\ \vec{v},\ \overline{v},\ldots.$$

- Representation

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad x = \begin{pmatrix} 4 & 1 & 2 \end{pmatrix}^T$$

Vectors - Operations

- Multiplication/Division by scalar value

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad 3v = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

- Addition/Subtraction of two vectors

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad v + w = \begin{pmatrix} 1+a \\ 2+b \\ 3+c \end{pmatrix}$$

Vectors - Operations

- Multiplication/Division by scalar value

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad 3v = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

- Addition/Subtraction of two vectors

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad v - w = \begin{pmatrix} 1-a \\ 2-b \\ 3-c \end{pmatrix}$$

Vectors – Scalar Product

- Also called dot product

- Sum of the products of components of two vectors

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad w = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$< v, w >= v \cdot w = v^T w = (1a + 2b + 3c)$$

- Induced Norm – Length of a vector

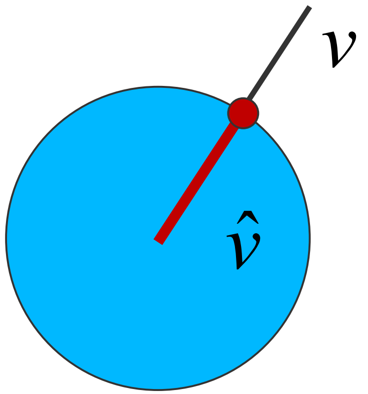
$$|v| = \sqrt{< v, v >} = \sqrt{(1^2 + 2^2 + 3^2)} = \sqrt{14}$$

Vectors – Scalar Product

- Normalizing vectors (resulting length = 1)

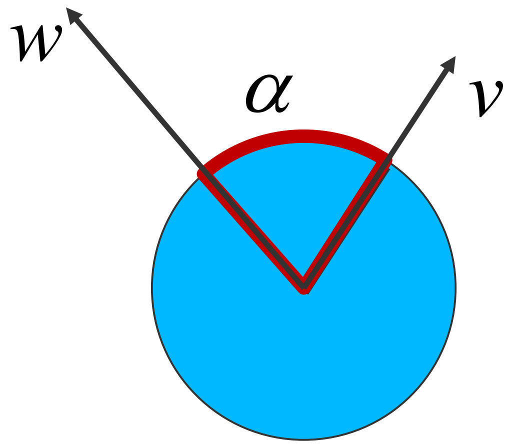
$$\hat{v} = \frac{v}{|v|} = \frac{1}{|v|}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

- All normalized vectors end on a ball of radius 1
- Normalized vectors indicate a direction
  (2 degrees of freedom only)



Vectors – Scalar Product

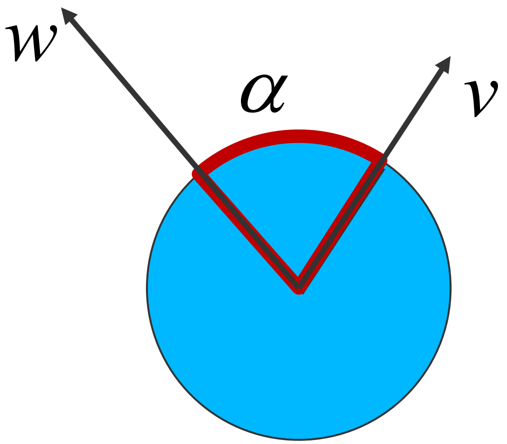
- The scalar product between two normalized vectors corresponds to the cosine of the angle between the two directions



$$<\hat{v}, \hat{w}> = \hat{v} \cdot \hat{w} = \cos\alpha$$

Vectors – Scalar Product

- The scalar product between two unnormalized vectors corresponds to the cosine of the angle between the two directions weighted by the length of the vectors



- Length of arc on unit circle corresponds to angle in radians

$$< v, w >= v \cdot w = |v| \cdot |w| \cdot \cos\alpha$$

Vectors – Scalar Product

Special configurations

- perpendicular $v \perp w$

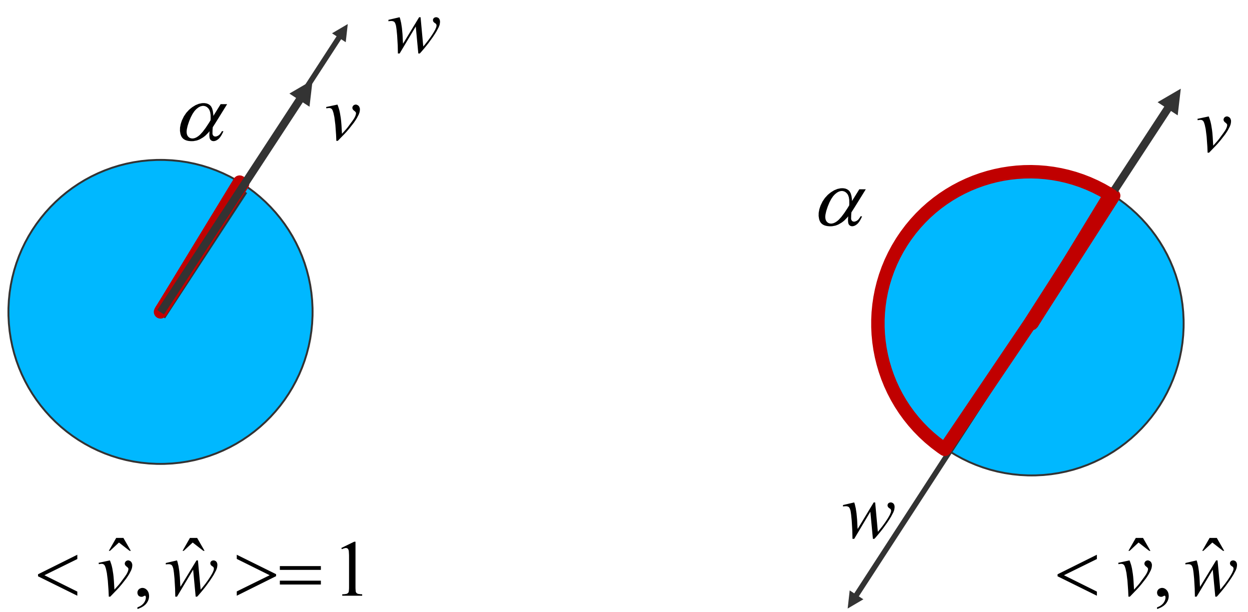
$$< \hat{v}, \hat{w} > = < v, w > = 0$$

$$\alpha = 90^{\circ} = \frac{\pi}{2}$$

Vectors – Scalar Product

Special configurations

- collinear $v \,||\, w$



$$< \hat{v}, \hat{w} >= 1$$

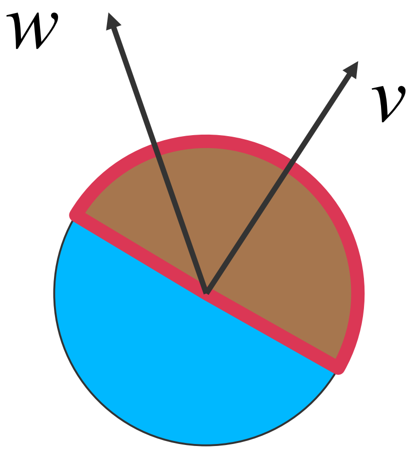
$$\alpha = 0$$

$$< \hat{v}, \hat{w} >= -1$$

$$\alpha = 180^\circ = \pi$$

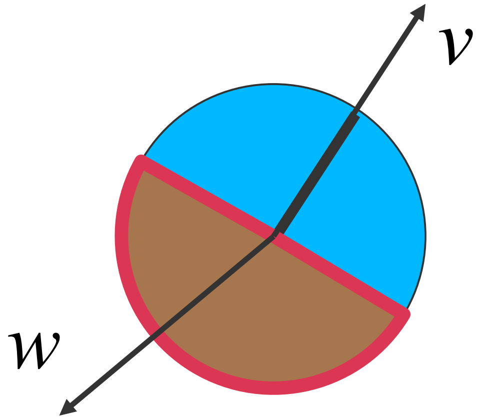
Special configurations

- Front-facing / back-facing



$$<\hat{v},\hat{w}> > 0$$

$$\alpha < 90^\circ$$



$$<\hat{v},\hat{w}> < 0$$

$$\alpha > 90^\circ$$

Linear Combination

- Basis

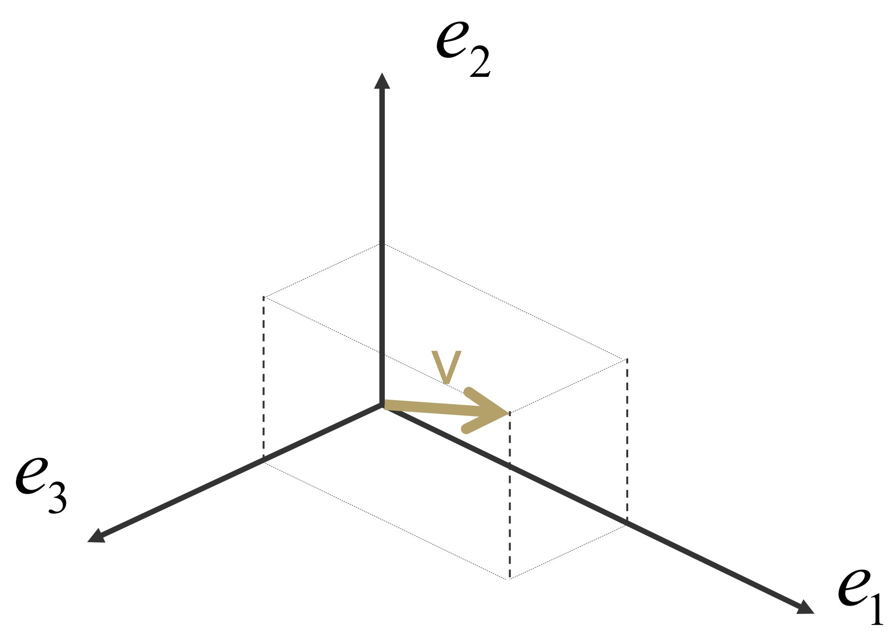
$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Vector in basis

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad v = 1e_1 + 2e_2 + 3e_3$$

Scalar Product as Projected Length

$$v = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \qquad v = 3e_1 + 1e_2 + 1e_3$$



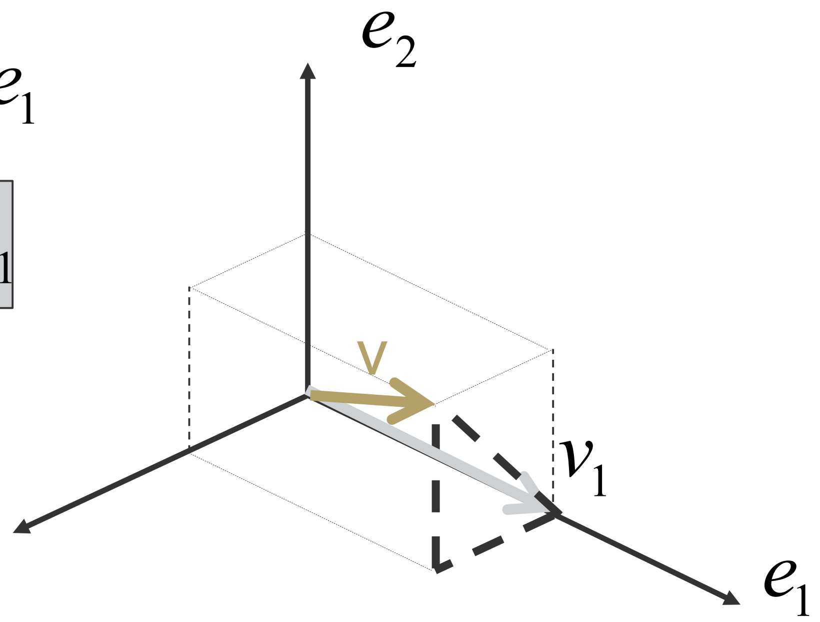
Scalar Product as Projected Length

$$v = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \qquad v = \boxed{3e_1} + 1e_2 + 1e_3$$

- projected on v

$$e_1$$

$$\boxed{v_1 = < v, e_1 > e_1}$$

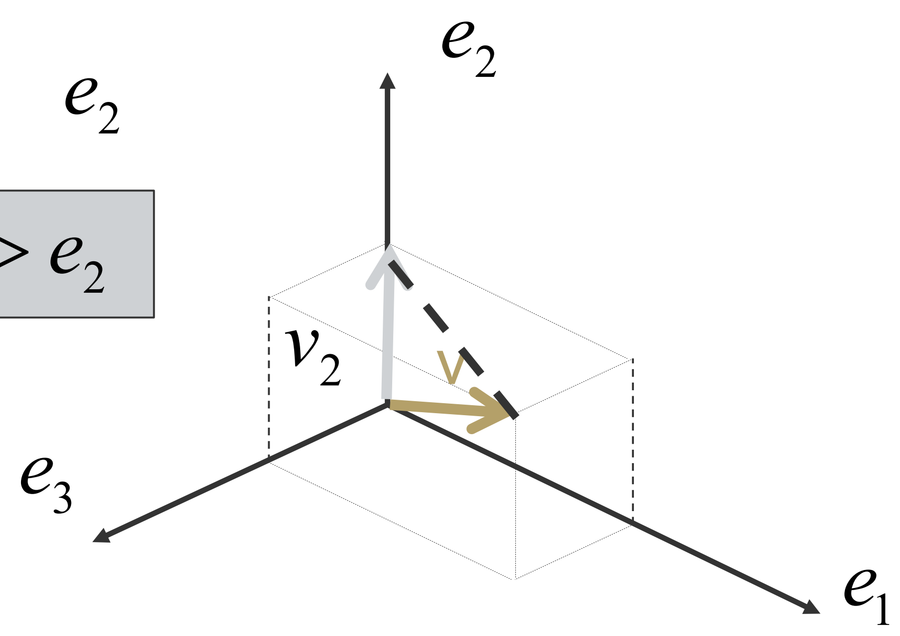


Scalar Product as Projected Length

$$v = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \qquad v = 3e_1 + \boxed{1e_2} + 1e_3$$

- projected on v

$$\boxed{v_2 = \langle v, e_2 \rangle e_2}$$

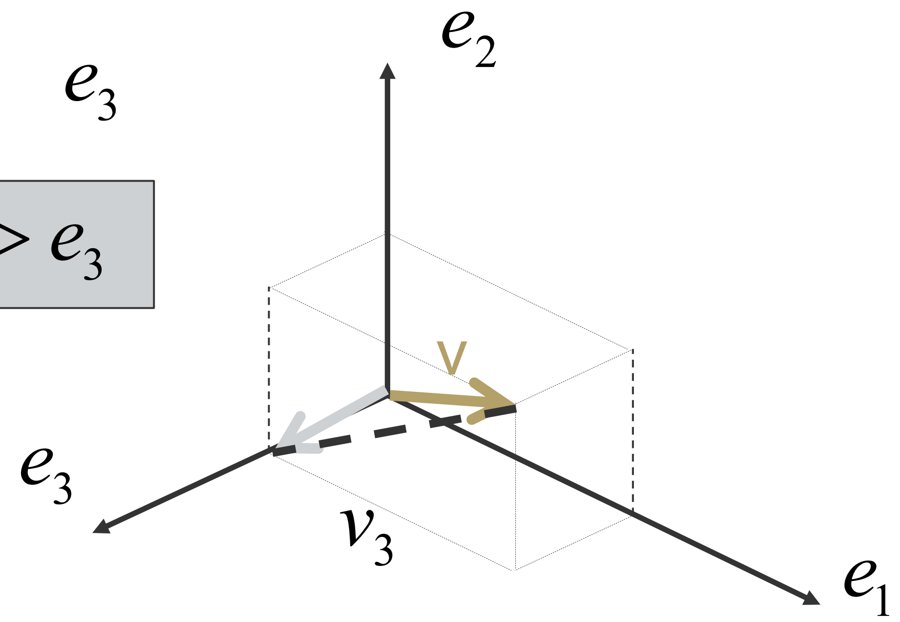


Scalar Product as Projected Length

$$v = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \qquad v = 3e_1 + 1e_2 + \boxed{1e_3}$$

- projected on v

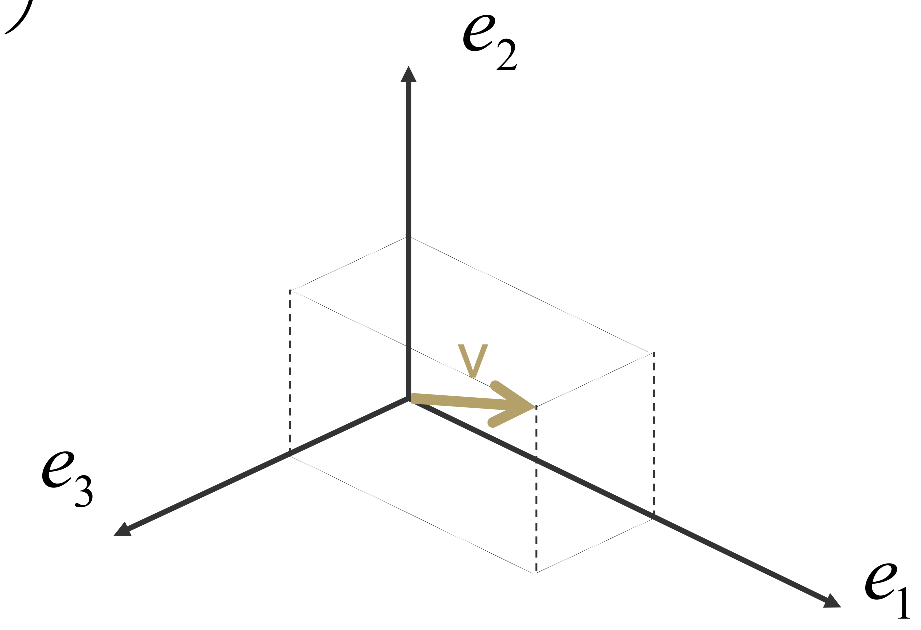
$$\boxed{v_3 = \langle v, e_3 \rangle e_3}$$



Projection onto Planes

- Onto coordinate planes:
  - set corresponding coordinate to 0

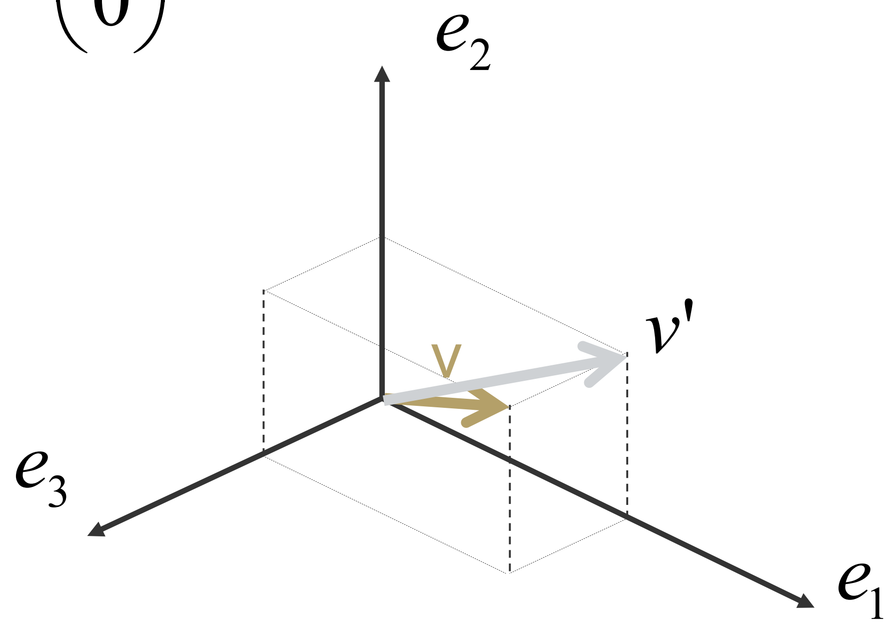
$$v = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \qquad v = 3e_1 + 1e_2 + 1e_3$$



Projection onto Planes

- Onto e_1, e_2 plane:
  - third component = 0

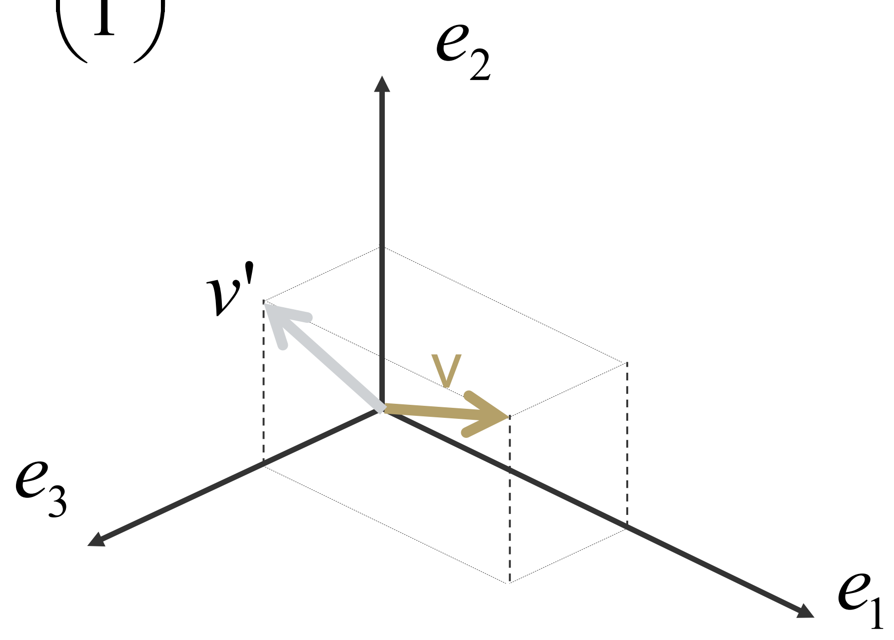
$$v = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \quad v' = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \qquad v' = 3e_1 + 1e_2 + 0e_3$$



Projection onto Planes

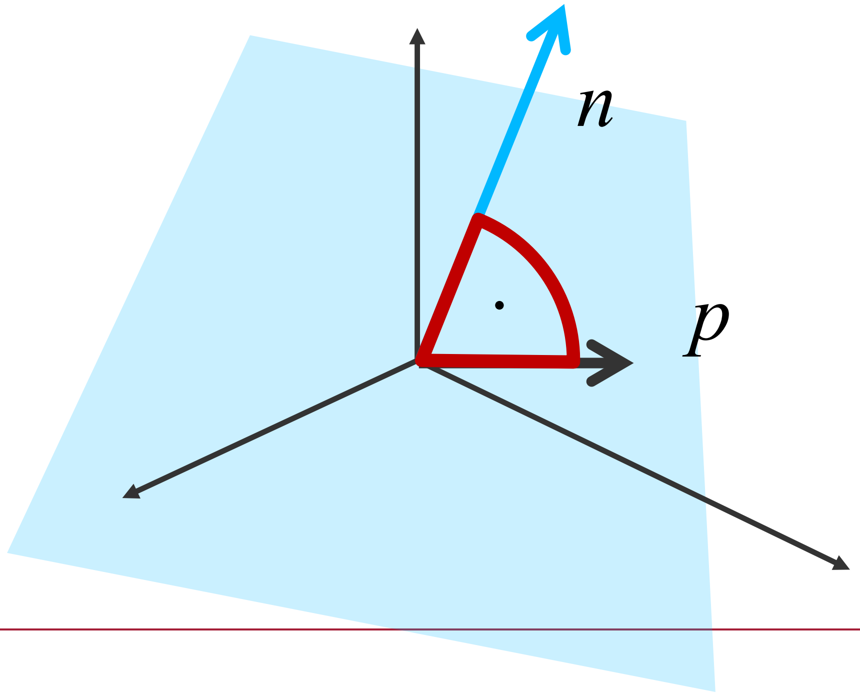
- Onto e_2, e_3 plane:
  - first component = 0

$$v = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \quad v' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad v' = 0e_1 + 1e_2 + 1e_3$$



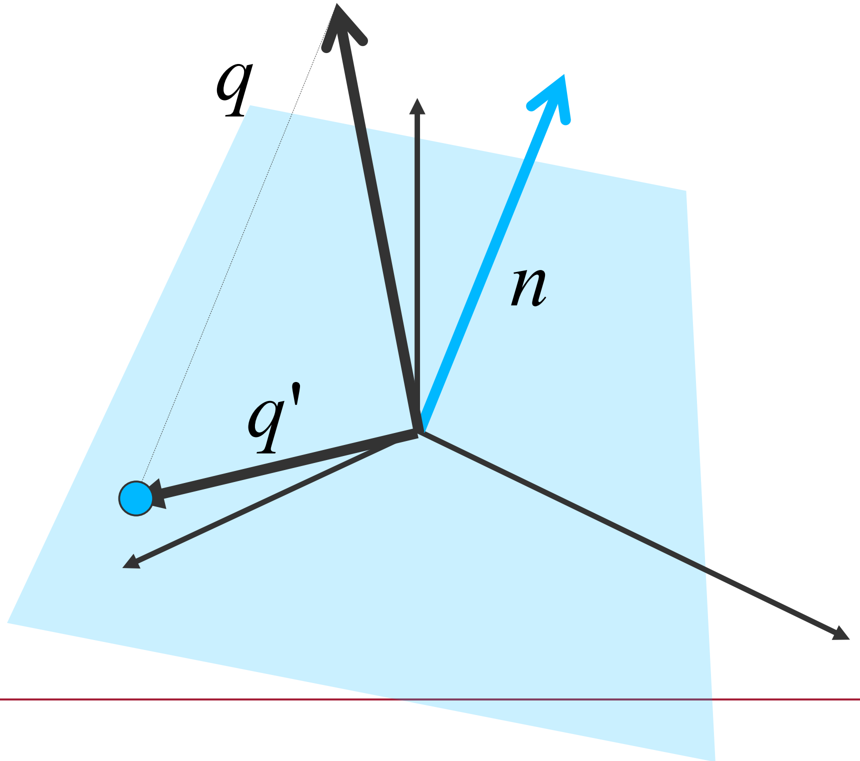
Projection onto Arbitrary Planes

- Assumption: plane through origin
- Plane specified by normal n
- For all points p on plane: $< p, n >= 0$ $\qquad p \perp n$



Projection onto Arbitrary Planes

- Assumption: plane through origin
- Plane specified by normal n
- For all points p on plane: $< p, n >= 0$ $\quad p \perp n$
- Projection: q projected on plane: q'



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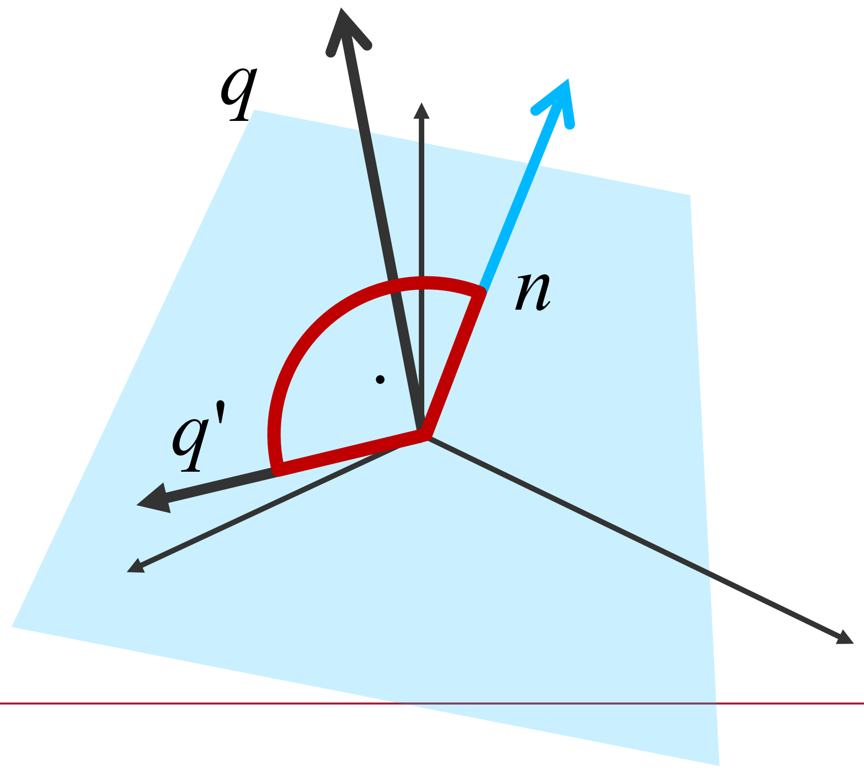

Projection onto Arbitrary Planes

- Assumption: plane through origin
- Plane specified by normal n
- For all points p on plane: $< p, n >= 0$ $\qquad p \perp n$
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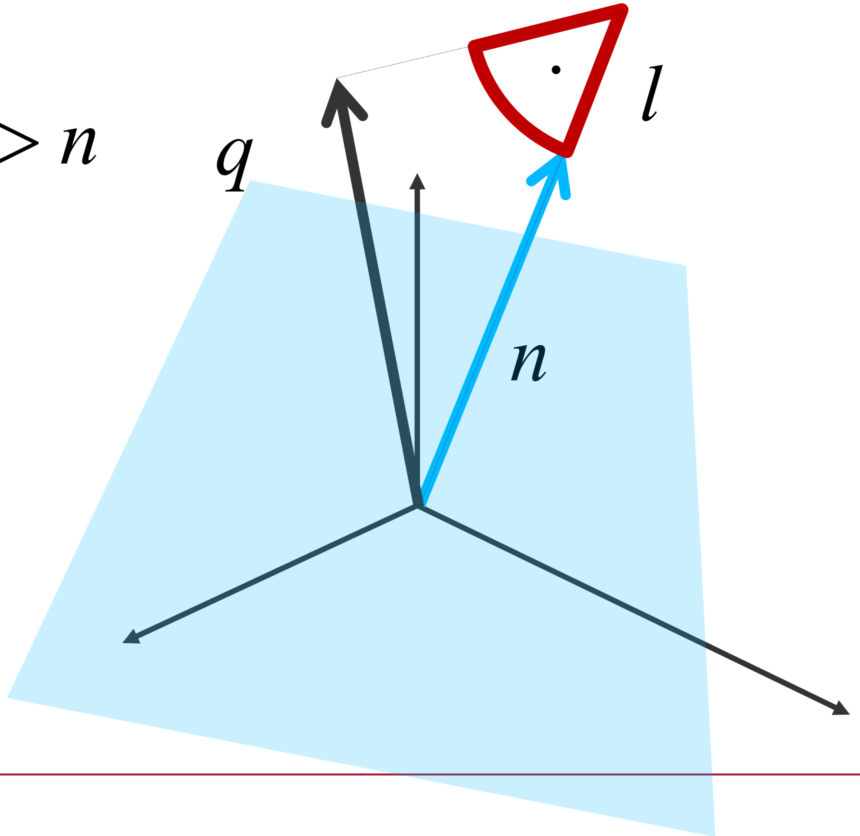
- q projected on n :

$$l =< q, n > n$$



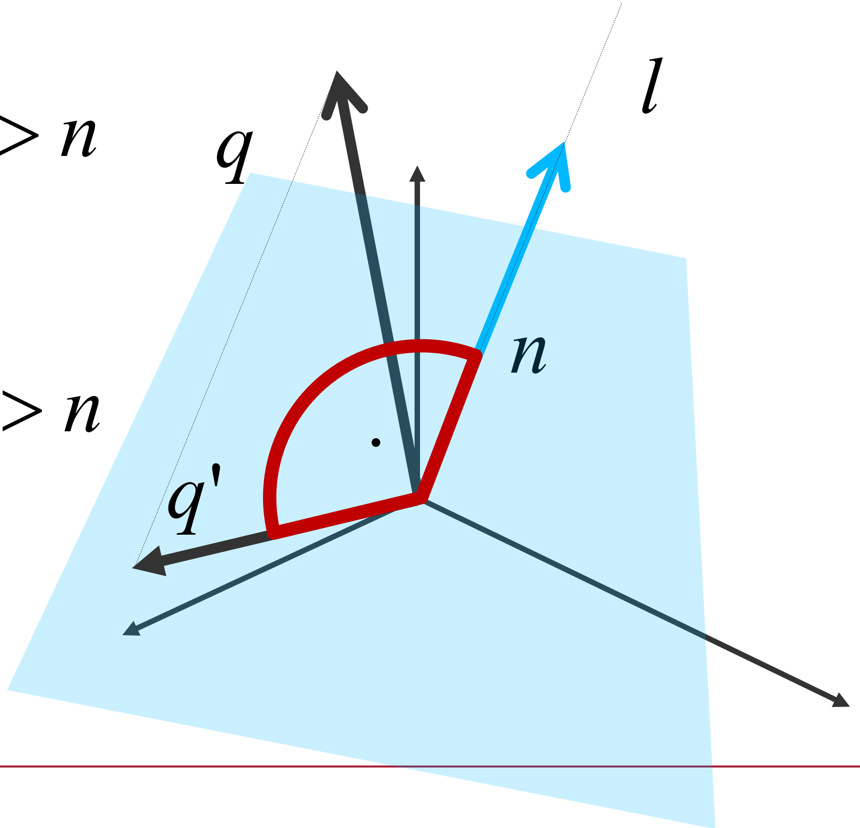
Projection onto Arbitrary Planes

- Assumption: plane through origin
- Plane specified by normal n
- For all points p on plane: $< p, n >= 0$ $\quad p \perp n$
- Projection: q projected on plane: q' $\quad < q', n >= 0$
- q projected on n :

$$l =< q, n > n$$

$$q' = q - l$$

$$= q - < q, n > n$$



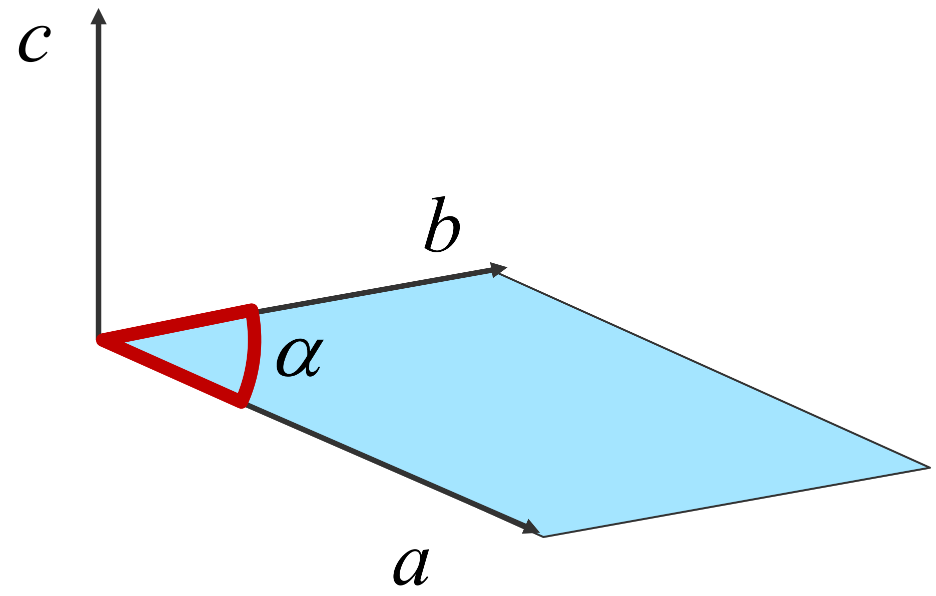
Cross Product (Vector Product)

- Constructing perpendicular vectors

$$c = a \times b$$

- Length of c corresponds to area of parallelogram

$$|c| = |a| \cdot |b| \sin \alpha$$

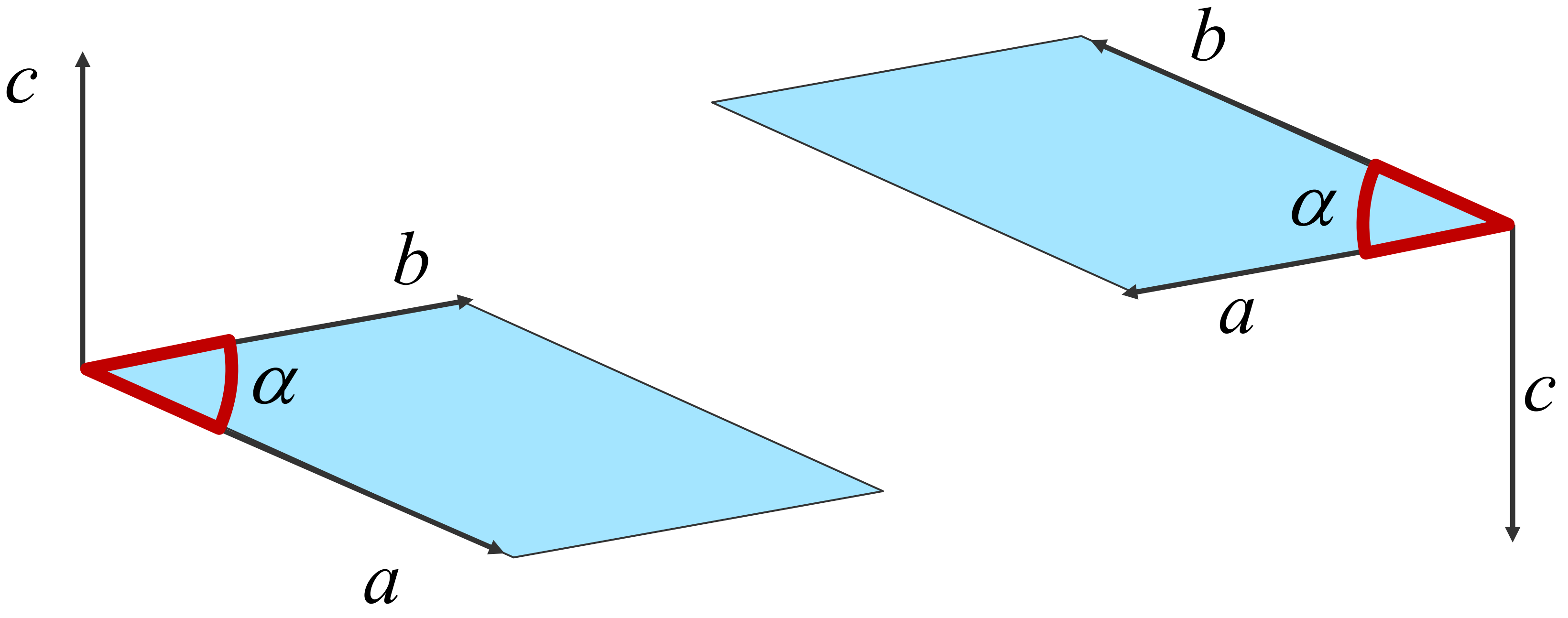


Cross Product (Vector Product)

- Constructing perpendicular vectors

$$c = a \times b$$

- Right Hand Rule



Cross Product (Vector Product)

- Calculation

$$c = a \times b = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

Cross Product (Vector Product)

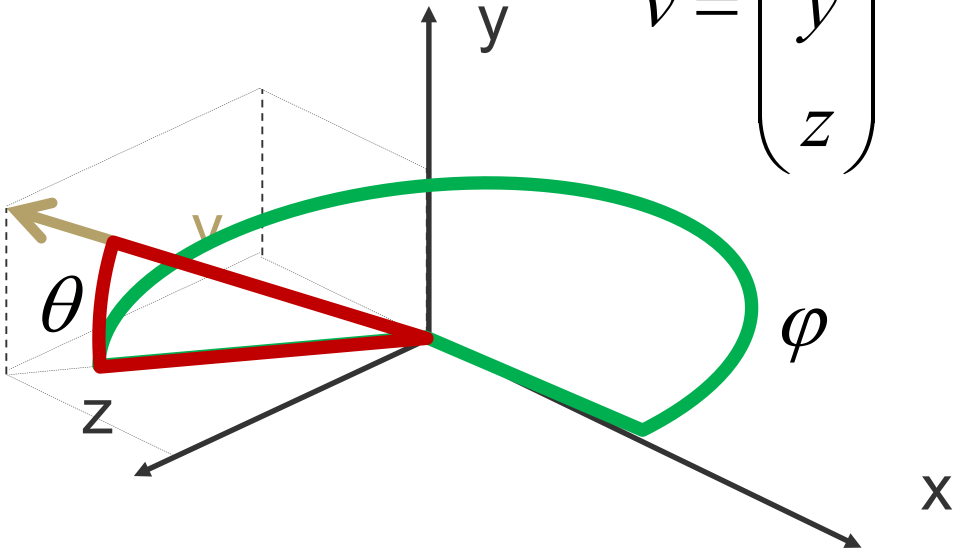
- Calculation (based on the determinant)

$$c = a \times b = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= e_1(a_y b_z - a_z b_y) - e_2(a_x b_z - a_z b_x) + e_3(a_x b_y - a_y b_x)$$

Polar Coordinates

- Expressing directions by two angles



$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad v = (\theta, \varphi)$$

Matrices

Matrices

- Typical variable names (capital letters)

- Representation
- 2D arrays of numbers
  - rows and columns

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

2x3 matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

3x3 matrix

Vector Matrix Product

$$y = Ax$$

- Dimension of Vector must match number of columns

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Vector Matrix Product

$$y = Ax$$

- Yields vector of dimension number of rows
- Each component in y is computed as row vector dot x

$$y = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1a & +2b & +3c \\ 4a & +5b & +6c \end{pmatrix}$$

Matrix Matrix Product

$$C = AB$$

- Each column of C is computed as the matrix vector product of A and the corresponding column of B

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad B = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \qquad C = AB$$

2x3 matrix 3x3 matrix 2x3 matrix

output columns

output rows

Matrix Matrix Product

$$C = AB$$

- Each column of C is computed as the matrix vector product of A and the corresponding column of B

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad B = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \qquad C = AB$$

2x3 matrix 3x3 matrix 2x3 matrix

dot product

Matrix Matrix Product

$$C = AB$$

- Each column of C is computed as the matrix vector product of A and the corresponding column of B

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Matrix Matrix Product

$$C = AB$$

- Each column of C is computed as the matrix vector product of A and the corresponding column of B

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$$= \begin{pmatrix} 1a + 2b + 3c & 1d + 2e + 3f & 1g + 2h + 3i \\ 4a + 5b + 6c & 4d + 5e + 6f & 4g + 5h + 6i \end{pmatrix}$$

Matrix Matrix Product

$$C = AB$$

- Each column of C is computed as the matrix vector product of A and the corresponding column of B

$$C = \begin{pmatrix} A^T{}_1 \\ A^T{}_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix}$$

$$= \begin{pmatrix} A_1^T B_1 & A_1^T B_2 & A_1^T B_3 \\ A_2^T B_1 & A_2^T B_2 & A_2^T B_3 \end{pmatrix}$$

Matrix Transpose

- Interchanging rows and columns

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Matrix Inverse

- The inverse of a matrix inverts the effect of a matrix vector multiplication

$$y = Ax$$
$$x = A^{-1}y$$

- The inverse does not exist for all matrices
- Methods for computing the inverse
  - Gauss-Jordan, SVD, (QR), ...

Matrix Inverse - Rules

- If the inverse exists:

$$A^{-1}A = Id$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

$$\left(A^{-1}\right)^{-1} = A$$

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

$$(kA)^{-1} = k^{-1}A^{-1} = \frac{1}{k}A^{-1}$$

Transformations

- Translation
- Rotation
- Scaling
- Shear
- …
- Projections (later)

- Most transformations on vectors can be expressed as matrix vector multiplications

Translation

- Move one object along one direction
- Simple addition of two vectors

$$p_{new} = p_{old} + t \cdot v = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} + t \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

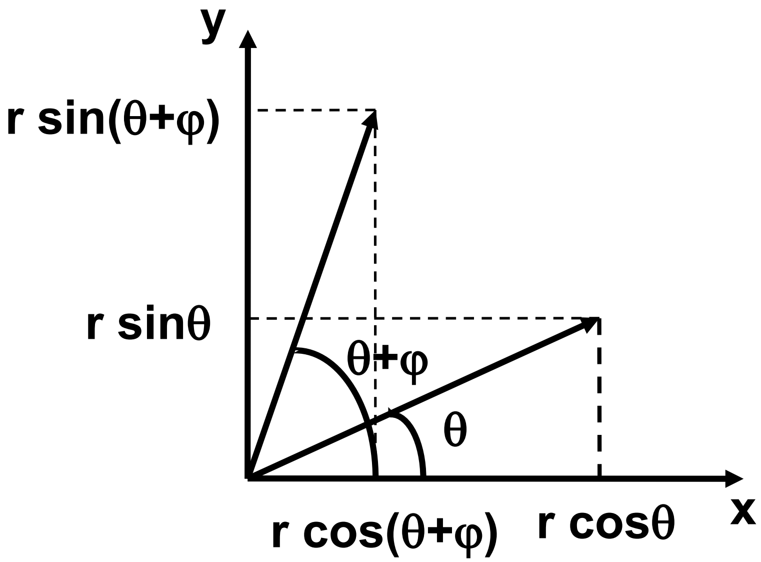
Rotation

- Rotation in 2D

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$x' = r\cos(\theta+\varphi)$$
$$y' = r\sin(\theta+\varphi)$$

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$
$$\sin(\theta+\varphi) = \cos\theta\sin\varphi + \sin\theta\cos\varphi$$



$$x' = (r\cos\theta)\cos\varphi - (r\sin\theta)\sin\varphi = x\cos\varphi - y\sin\varphi$$
$$y' = (r\cos\theta)\sin\varphi + (r\sin\theta)\cos\varphi = x\sin\varphi + y\cos\varphi$$

Basic Transformations

- Rotation around major axis

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  - Assumes a right handed coordinate system

Basic Transformations

- Scaling

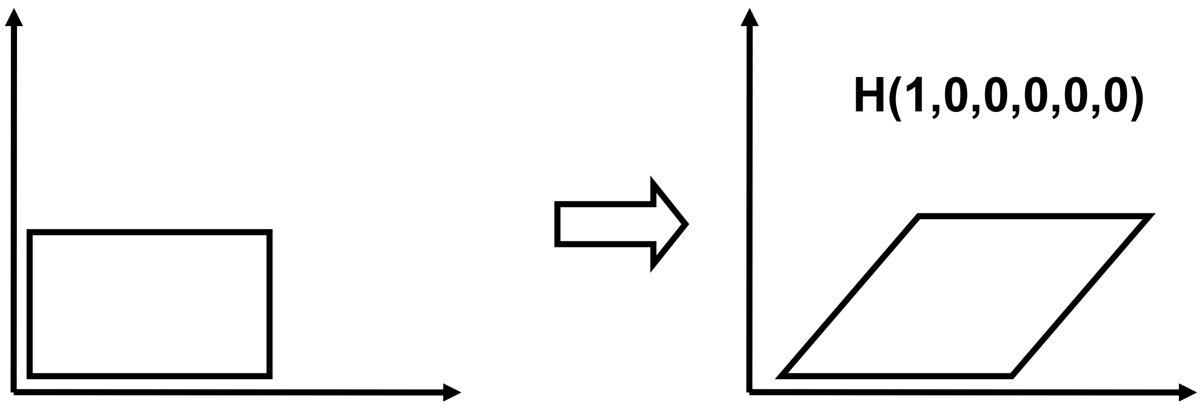
$$S(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix}$$

  - Uniform Scaling
    - sx= sy = sz

Basic Transformations

- Shear (deutsch: Scherung)

$$H(h_{xy},h_{xz},h_{yz},h_{yx},h_{zx},h_{zy}) = \begin{pmatrix} 1 & h_{xy} & h_{xz} \\ h_{yx} & 1 & h_{yz} \\ h_{zx} & h_{zy} & 1 \end{pmatrix}$$



Concatenation of Transformations

- Multiple transformations can be expressed by matrix matrix multiplications

$$y = T_1 x$$

$$w = T_2 y$$

$$w = T_2 y = T_2 T_1 x = (T_2 T_1) x$$

References

- Morris, Dan; Essential Mathematics for Computer Graphics, CS148, 2005

- Rudolph, Alexander, 3D-Spiele mit C++ und DirectX in 21 Tagen, Markt und Technik, 2003, Chap. 3+4

- (both you can find online)

Wrap-Up

- Vectors
  - Operations
  - Scalar Products / Length
  - Projections
  - Cross Product

- Matrices
  - Matrix Vector Product
  - Matrix Matrix Product
  - Transpose
  - Inverse
  - Transformations

- Next lecture
  - Ray Tracing