① $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

a) $\mathbb{E}[\hat{\beta}_0] = \mathbb{E}[\bar{Y} - \hat{\beta}_1 \bar{X}] = \mathbb{E}[\bar{Y}] - \bar{X}\,\mathbb{E}[\hat{\beta}_1]$

$= \beta_0 + \beta_1 \bar{X} - \beta_1 \bar{X}$

$= \beta_0$

b) $\mathcal{L}(\beta_0, \beta_1, \sigma^2 \mid (X_i, Y_i)_{i=1}^n) = \prod \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-1}{2\sigma^2} (Y_i - X_i\beta_1 - \beta_0)^2 \right\}$

$\ell(\sigma^2) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 X_i)^2$

$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum (Y_i - \beta_0 - \beta_1 X_i)^2$

$\vdots$

$\hat{\sigma}^2 = \frac{\sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2}{n} = \frac{SSE}{n}$

c.) $\mathbb{E}[SSTot] = \mathbb{E}[\sum_i (Y_i - \bar{Y})^2]$

$= \sum_i \mathbb{E}(Y_i - \bar{Y})^2$

$= \sum_i [Var(Y_i - \bar{Y}) + (\mathbb{E}(Y_i - \bar{Y}))^2]$

$= \sum_i [\sigma^2 (\frac{n-1}{n}) + \beta_1^2 (X_i - \bar{X})^2]$

$\mathbb{E}[SSTot] = (n-1)\sigma^2 + \beta_1 \sum_i (X_i - \bar{X})^2$

$\mathbb{E}[SSR] = \sum_i \mathbb{E}[(\hat{\beta}_0 + \hat{\beta}_1 X_i - \bar{Y})^2] = \sum_i \mathbb{E}[\hat{\beta}_1^2 (X_i - \bar{X})^2]$

$= \mathbb{E}[\hat{\beta}_1^2] \sum (X_i - \bar{X})^2 = (Var(\hat{\beta}_1) + (\mathbb{E}(\hat{\beta}_1))^2) \sum (X_i - \bar{X})^2$

$= \left(\frac{\sigma^2}{\sum (X_i - \bar{X})^2} + \beta_1^2 \right) \sum (X_i - \bar{X})^2 = \sigma^2 + \beta_1^2 \sum (X_i - \bar{X})^2$

$\Rightarrow \mathbb{E}[SSE] = \mathbb{E}[SSTot] - \mathbb{E}[SSR] = (n-2)\sigma^2.$

(1). d.) An unbiased est. for σ^2 is $s^2 = \frac{SSE}{n-2}$

e.) Shown in (c).

f.) $\hat{\mu}(\bar{X}) = \hat{\beta}_0 + \hat{\beta}_1 \bar{X} = \bar{Y} - \hat{\beta}_1 \bar{X} + \hat{\beta}_1 \bar{X}$
$= \bar{Y}$

g.) $\hat{\beta}_0$ and $\hat{\beta}_1$ are linear transformations of the Y_i and are therefore normally distributed.

(2.) a.) $\mathcal{L}(\beta_1 \mid (X_i, Y_i)) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{\frac{-1}{2\sigma^2}(Y_i - \beta_1 X_i)^2\}$

$$\ell(\beta_1) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_i (Y_i - \beta_1 X_i)^2$$

$$\frac{\partial \ell}{\partial \beta_1} = \frac{-1}{2\sigma^2} \cdot \left(-2\sum_i X_i(Y_i - \beta_1 X_i)\right) = 0$$

$$\hat{\beta}_1 = \frac{\sum X_i Y_i}{\sum X_i^2}$$

b.) The model without β_0 forces the fitted line to have y-intercept $= 0$.