STAT120C Homework 2

Due Monay April 15, 2019 by 5pm in the Dropbox in DBH

1. Consider the one-way layout. We use Y_{ij} to denote the measurment of the jth observation from the ith treatment, where $i = 1, \cdots, I$ and $j = 1, \cdots, J$. Define the following summary statistics $\bar{Y}_{i\cdot} = \frac{1}{J}\sum_{j=1}^{J} Y_{ij}$, $i = 1, \cdots, I$ and $\bar{Y}_{\cdot\cdot} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij}$
   (a) Show that $J\bar{Y}_{i\cdot} = \sum_{j=1}^{J} Y_{ij}$ and then conclude that $\sum_{j=1}^{J}(Y_{ij} - \bar{Y}_{i\cdot}) = 0$.
   (b) Use you result in (a) to prove that $\sum_{i=1}^{I}\sum_{j=1}^{J}[(Y_{ij} - \bar{Y}_{i\cdot})(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})] = 0$.
   (c) Prove that both $\sum_{i=1}^{I}\bar{Y}_{i\cdot}$ and $\sum_{i=1}^{I}\bar{Y}_{\cdot\cdot}$ equal $\frac{1}{J}\sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij}$.
   (d) Use (c) to conclude that $\sum_{i=1}^{I}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}) = 0$.

2. Assume that we have I independent random samples. For $i = 1, \cdots, I$, we assume that the ith random sample $(Y_{i1}, Y_{i2}, \cdots, Y_{iJ})$ came from the normal distribution with with mean μ_i and variance σ^2. These assumptions can be summarized using the following statistical model:

$$Y_{ij} = \mu_i + \epsilon_{ij} \ , \ i = 1, \cdots, I \ ; \ j = 1, \cdots, J$$

   where $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$. Show that the MLE of μ_i is $\hat{\mu}_i = \bar{Y}_{i\cdot} = \frac{1}{J}\sum_{j=1}^{J} Y_{ij}$.

3. The statistical model of Problem 2 can also be written to

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \ , \ i = 1, \cdots, I \ ; \ j = 1, \cdots, J$$

   where $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$ and $\sum_{i=1}^{I}\alpha_i = 0$. Derive the MLEs for μ, and α_i

4. Consider the balanced one-way ANOVA model with I treatment groups, and J observations for each group.

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij},$$

   where the idiosyncratic errors are $\varepsilon \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$.
   (a) Show that $SSW/\sigma^2 \sim \chi^2_{I(J-1)}$.
   (b) Show that $SSB/\sigma^2 \overset{H_0}{\sim} \chi^2_{I-1}$.
   (c) Show that SSW and SSB are independent.
   (d) What is the null distribution of $\frac{SSB/(I-1)}{SSW/(I(J-1))}$?

   Hint: See Theorem B given in Rice (Sec. 12-2, p482).

5. Consider two independent random samples. The first one $Y_{1,1}, \cdots, Y_{1,9}$ is a random sample from $N(\mu_1, \sigma^2)$ and the second one $Y_{2,1}, \cdots, Y_{2,9}$ is a random sample from $N(\mu_2, \sigma^2)$. The parameters μ_1, μ_2, σ^2 are unknown. We want to conduct hypothesis tesing

$$H_0 : \mu_1 = \mu_2 \text{ v.s. } H_1 : \mu_1 \neq \mu_2$$

   If we use the two-sample t-test, we would calculate the following test statistic

$$T = \frac{\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot}}{\sqrt{s_p^2(\frac{1}{9} + \frac{1}{9})}}$$

where $\bar{Y}_{i\cdot} = \sum_{j=1}^{J} Y_{ij}, i = 1, 2$ and $s_p^2 = \frac{\sum_{i=1}^{2}\sum_{j=1}^{9}(Y_{ij}-\bar{Y}_{i\cdot})^2}{9+9-2}$. If we use the F-test from one-way ANOVA, we would calculate the following test statistic

$$F = \frac{SSB/(2-1)}{SSW/(2 \times (9-1))}$$

where $SSB = 9\sum_{i=1}^{2}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2$ and $SSW = \sum_{i=1}^{2}\sum_{j=1}^{9}(Y_{ij} - \bar{Y}_{i\cdot})^2$.

Show that $F = T^2$. (Hint: show that $\bar{Y}_{1\cdot} - \bar{Y}_{\cdot\cdot} = \frac{1}{2}(\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot})$ and $\bar{Y}_{2\cdot} - \bar{Y}_{\cdot\cdot} = -\frac{1}{2}(\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot})$)