STAT 120 C

Introduction to Probability and Statistics III

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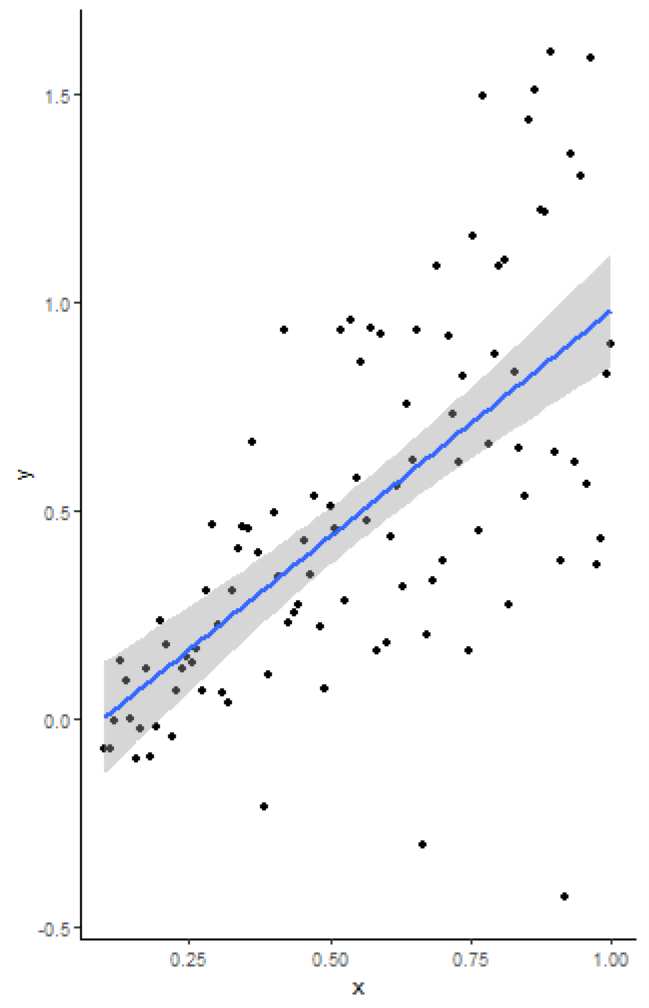
2019/04/01

Intro to STATS 120 C

STATS 120C is the last of a three-quarter series on introduction to probability and statistics. The goal of this course is to introduce basic principles of probability and statistical inference, and learn how these methods are applied to real world problems.

Topics that will be covered include **statistical hypothesis testing, linear regression, analysis of variance, and model checking.**



STAT 120C Course Info

Logistics

- Email: dpluta@uci.edu
- Website: https://github.com/dspluta/STAT120C
- Class Times: MWF 10 - 10:50am
- Room: ICS 174
- Office Hours: M 11:30am - 12:30pm; Th 2 - 3pm in DBH 2032
- TA Office Hours: W & Th 11am - 12pm in DBH 2013
- Discussion: MSTB 124
- Discussion Hours: W 5 - 5:50pm; 6 - 6:50pm

Materials

- **Text**: **Mathematical Statistics and Data Analysis**, 3rd Edition. John Rice. ISBN: 9788131519547.

- **Computing**: Many examples and problems will be given in `R`, which is available at http://www.r-project.org. I also recommend using the development environment RStudio: https://www.rstudio.com/. Please download and install `R` and RStudio before the next class meeting.

STAT 120C Course Info

Grading

- 30%: Eight (8) homework assignments
- 5%: Two (2) in-class quizzes
- 30%: Midterm Exam (Week 5)
- 35%: Final Exam (June 10th, 10:30am - 12:30pm)
- **Homework** will be assigned on Monday and **due the following Monday by 5pm** in the *STAT 120C Pluta* dropbox located near DBH 2013.

Policies

- Late homework will not be accepted!
- Exam make-ups will not be given except in case of emergency.
- One page of notes will be allowed for the midterm exam.
- Two pages of notes will be allowed for the final exam.
- Calculators will not be needed nor allowed.

Week 1

Review of 120 B

- Review Distributions: Normal, t, chi-squared, F
- NHST, reference distributions, rejection regions
- Equivalence of t-test and Likelihood Ratio Test

Review of 120B

Distributions

- We will mainly focus on continuous distributions in this course.
- The **cumulative density function (cdf)** of a random variable X is denoted $F(x)$, and is defined as the probability that $X < x$.

$$F(x) = P(X < x)$$

- The **probability density function (pdf)** is denoted $f(x)$, and is defined as the rate of change of the cumulative probability at x,

$$f(x) = F'(x).$$

- The **support** of a random variable X is the set of all values for which $f(x) \neq 0$

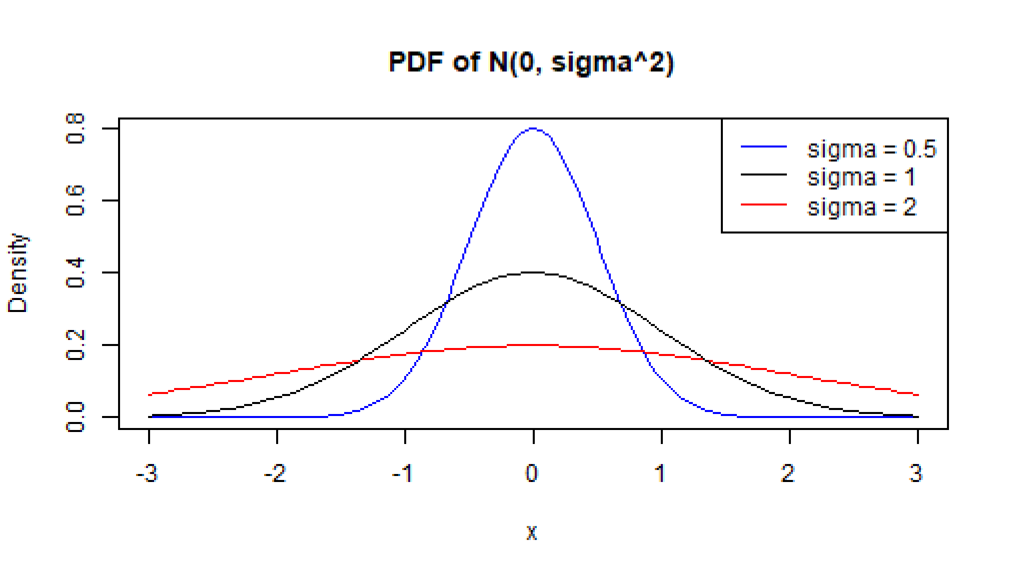
$$\text{Supp}(X) = \{x : f(x) \neq 0\}.$$

Review of 120B

Normal Distribution: PDF

The probability density function of $X \sim \mathcal{N}(\mu, \sigma^2)$ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$



Review of 120B

Sums of Normally Distributed Variables

Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and let a, b be real constants.

1. $aX_1 + bX_2 \sim \mathcal{N}(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$.
2. In particular, for $X_i \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2), i = 1, \ldots, n$, we have

$$\overline{X} := \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right).$$

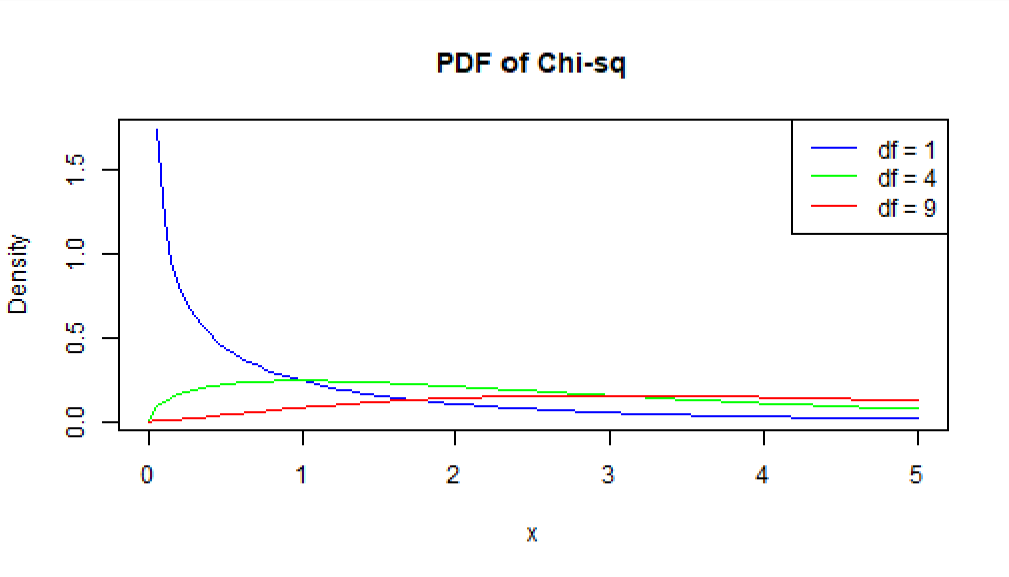
Review of 120B

χ^2 Distribution

$X \sim \chi^2_n$ has pdf

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}.$$

The parameter n is the *degrees of freedom* of the distribution.



Review of 120B

χ^2 Distribution

The following is a key property of the χ^2 distribution that we will use repeatedly throughout the course:

For $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$,

$$\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$$

.

Review of 120B

t Distribution

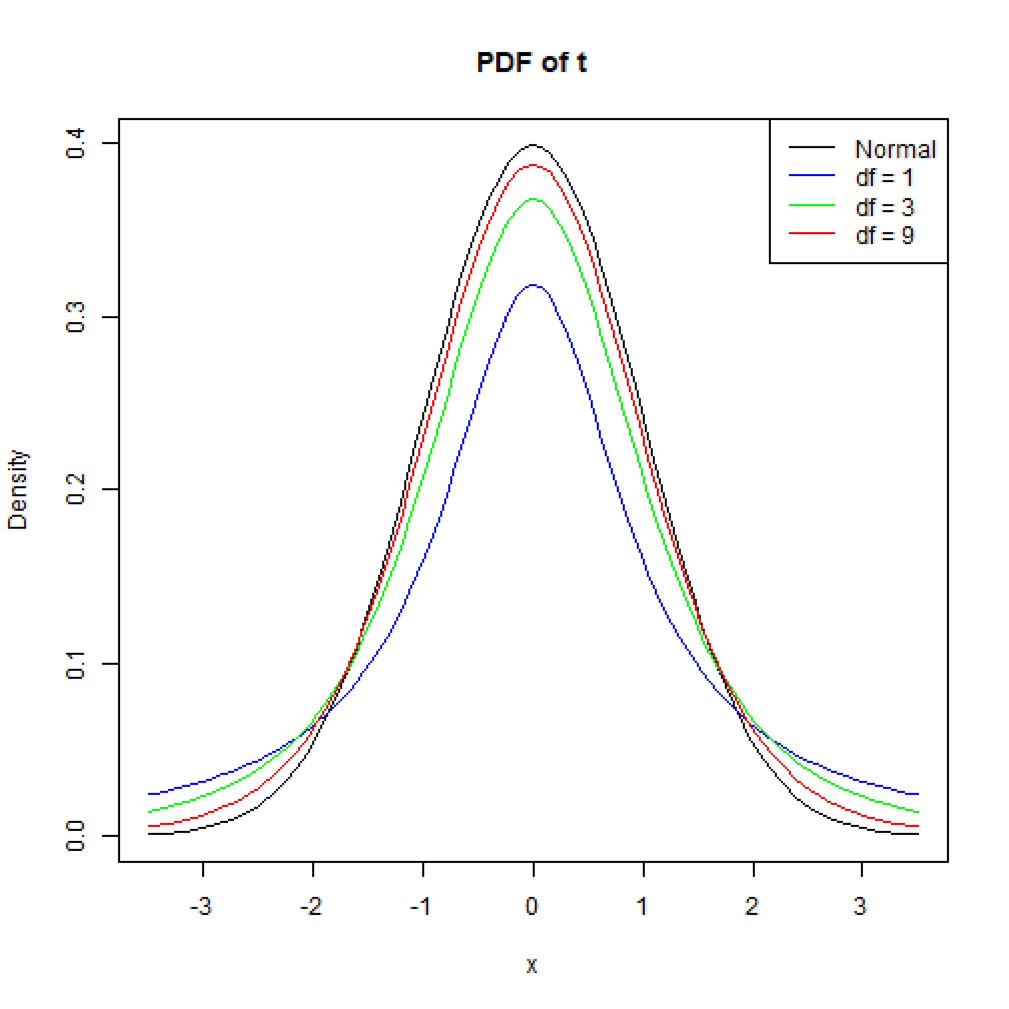
We will define the t distribution as a combination of a standard normal $Z \sim N(0, 1)$, and $V \sim \chi^2_v$:

$$T = \frac{Z}{\sqrt{V/v}} \sim t(v),$$

where v is the degrees of freedom of the distribution.

Review of 120B

t Distribution



Review of 120B

F Distribution

We will encounter the F distribution frequently throughout the course as well.

Let $U \sim \chi^2_u$ and $V \sim \chi^2_v$, with U and V independent. Then

$$X = \frac{U/u}{V/v} \sim F_{u,v},$$

where u and v are the degrees of freedom of the distribution.

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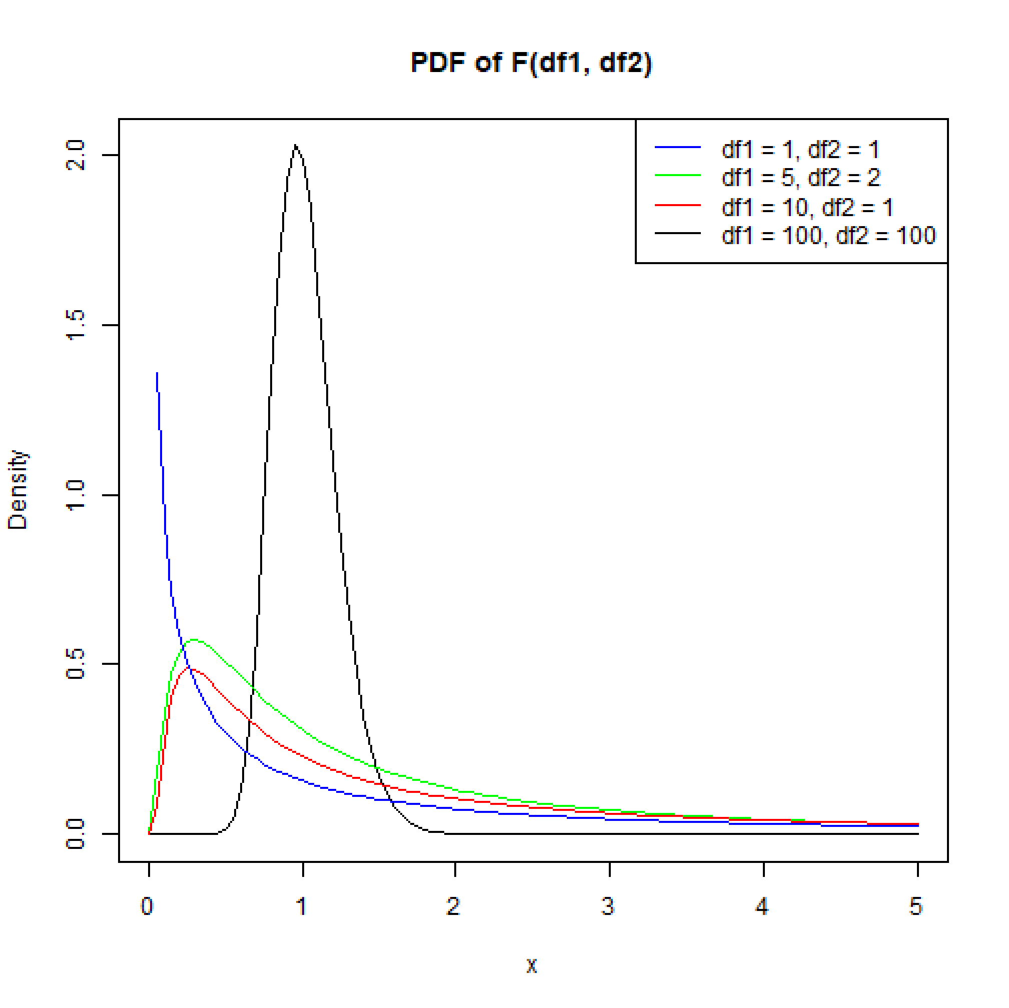
F Distribution



Review of 120 B

Types of Problems in Statistics

- **Hypothesis Testing**: Make a binary (Yes/No) decision regarding some unknown quantity.
- **Estimation**: Estimate the value of some unknown quantity, and characterize the uncertainty in the estimate.
- **Prediction**: Predict the values of new observations from existing observations.

We will primarily focus on a review of hypothesis testing this week.

Review of 120 B

Hypothesis Testing

In general, a Null Hypothesis Significance Test (NHST) has the form

$$\begin{aligned} H_0 &: \theta \in \Omega_0, \quad \text{(null hypothesis)} \\ H_1 &: \theta \in \Omega_1, \quad \text{(alternative hypothesis)} \end{aligned}$$

where $\Omega_0 \subset \mathbb{R}$ is the set of parameter values satisfying the null hypothesis, and similarly for Ω_1.

- When $\Omega_0 = \{\theta_0\}$ (contains a single value), then H_0 is $H_0 : \theta = \theta_0$, and is called a *simple hypothesis*.
- If Ω_0 contains more than one value, H_0 is called a *composite hypothesis*.

Review of 120B

Null Hypothesis Significance Testing

$$\begin{aligned} H_0 &: \theta \in \Omega_0, \quad \text{(null hypothesis)} \\ H_1 &: \theta \in \Omega_1, \quad \text{(alternative hypothesis)} \end{aligned}$$

at level of significance α, given a sample $X_1, \ldots, X_n$.

1. State the null and alternative hypotheses, the assumed sampling distribution of the data.

2. Choose an appropriate test statistic $T(X)$ for the null hypothesis.

3. Check model assumptions. (e.g. QQ-plot, histogram, scatterplot)

4. Compute the reference distribution and corresponding P-value for the test statistic.

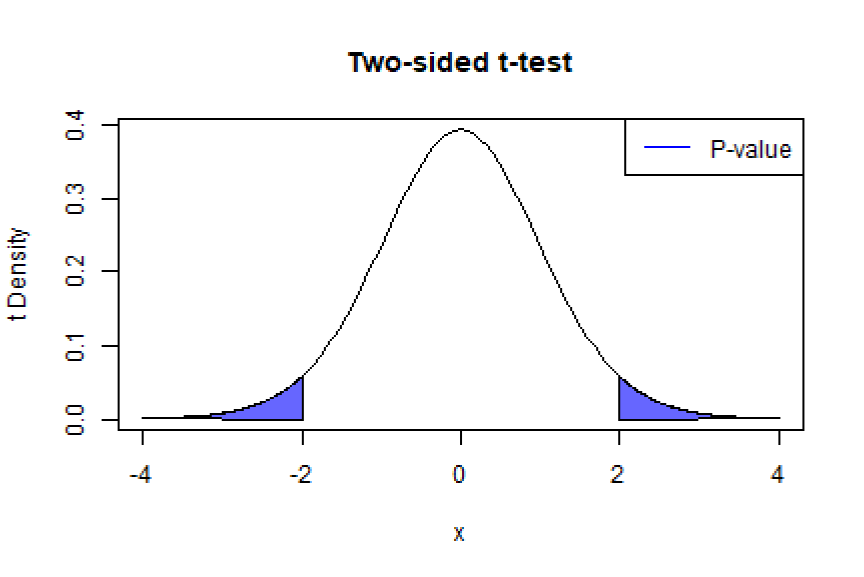
5. Conclude one of:

    - $P \geq \alpha \to$ **Fail to reject** H_0: There is insufficient evidence to reject the null hypothesis at the α level of significance.
    - $P < \alpha \to$ **Reject** H_0: There is sufficient evidence to reject the null hypothesis (and accept the alternative hypothesis) at the α level of significance.

Review of 120B

Definition: P-value

The **P-value** of a NHST is the probability of seeing a test statistic as extreme or more extremem than the observed test statistic, assuming the null hypothesis is true.



Review of 120B

One Sample z-test

Step 1

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, with σ^2 known.

We wish to test the hypothesis

$$\begin{aligned} H_0 &: \mu = \mu_0 \\ H_1 &: \mu > \mu_0. \end{aligned}$$

Review of 120B

One Sample z-test

Step 2

We will test H_0 with test statistic $T(X) = \frac{\overline{X} - \mu_0}{\sigma}$.

"True" Distribution: $\overline{X} \sim \mathcal{N}(\mu, \sigma^2/n)$

Null Distribution: $\overline{X} \overset{H_0}{\sim} \mathcal{N}(\mu_0, \sigma^2/n)$

Review of 120B

One Sample z-test

Step 2

We will test H_0 with test statistic $T(X) = \frac{\overline{X} - \mu_0}{\sigma}$.

"True" Distribution: $\overline{X} \sim \mathcal{N}(\mu, \sigma^2 / n)$

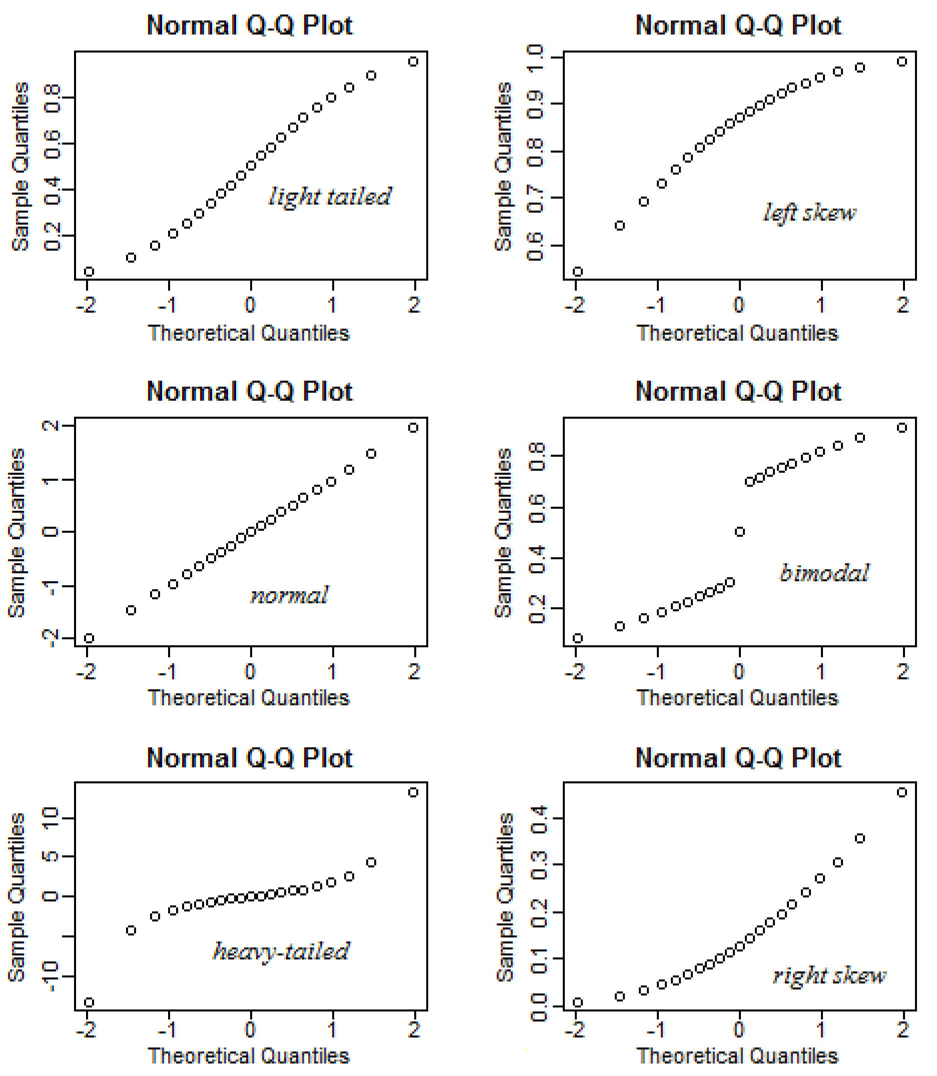
Null Distribution: $\overline{X} \overset{H_0}{\sim} \mathcal{N}(\mu_0, \sigma^2 / n)$

Note: A statistic based on $\overline{X}$ is a natural choice, and is also theoretically motivated, since it is

- The *minimum variance unbiased linear estimator* for μ
- The *Maximum Likelihood Estimator* for μ
- More on this later...

Review of 120B

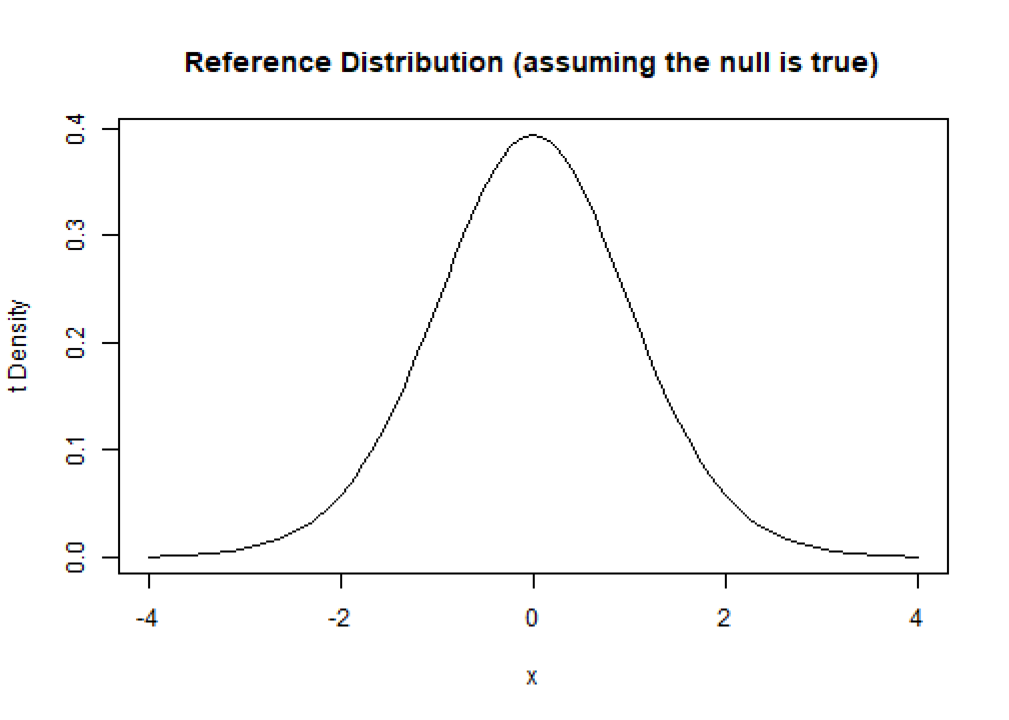
Step 3 Check model assumptions.



Review of 120B

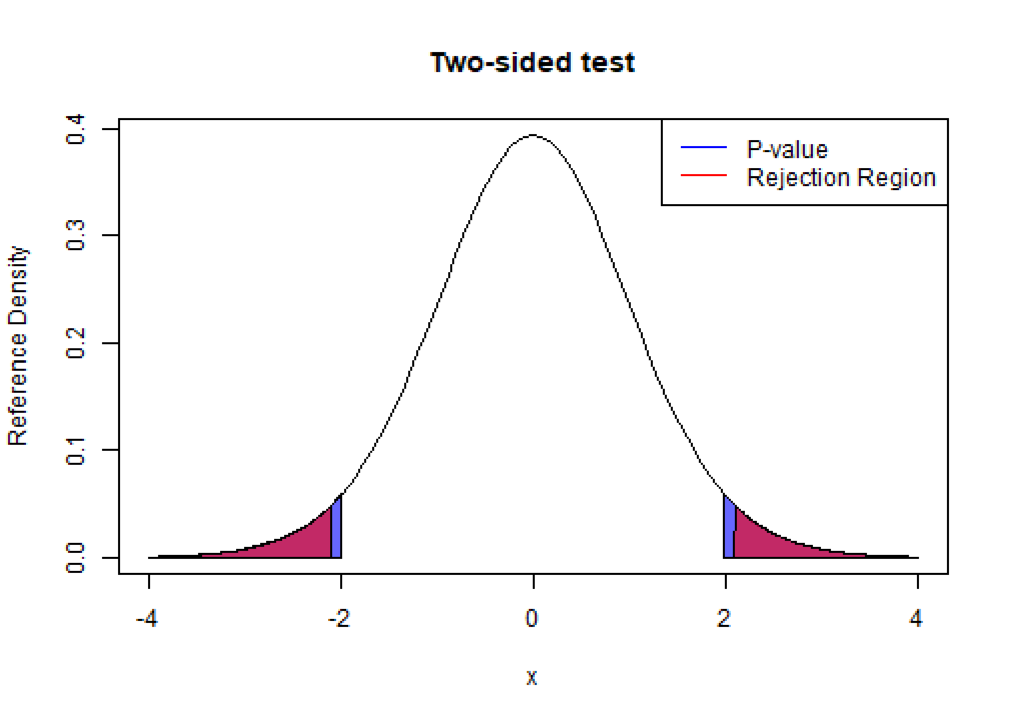
Step 4 Compute reference distribution.

- Reference Distribution: $T(X) \overset{H_0}{\sim} \mathcal{N}(0,1)$



Review of 120B

Step 5 Make conclusion.



Review of 120B

Hypothesis Testing Terminology

- **Type I Error**: $\alpha = P(\text{Reject } H_0 | H_0 \text{ is True})$
- **Type II Error**: $\beta = P(\text{Fail to reject } H_0 | H_0 \text{ is False})$
- **Power**: $1 - \beta = P(\text{Reject } H_0 | H_0 \text{ is False})$

Remarks

- In the NHST framework, α is selected by the researcher.
- Power is determined by the choice of α, as well as the sample size and the size of the effect being tested.

Review of 120B

One-sample z-test

- Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$. We wish to test $H_0 : \mu = \mu_0$. Assume σ^2 is known.

- Since $\bar{X}$ is an unbiased sufficient statistic for μ, we can use this estimator to construct our test statistic.

- We want to standardize the statistic to make it easy to compute the P-value.

- $\bar{X} \sim \mathcal{N}(\mu, \sigma^2)$, so

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \overset{H_0}{\sim} \mathcal{N}(0, 1).$$

Review of 120B

One-sample z-test

- Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$. We wish to test $H_0 : \mu = \mu_0$. Assume σ^2 is known.

- We can again use $\overline{X}$ to construct our test statistic, but we must now also estimate σ^2.

- Use the sample variance estimator, which is unbiased for σ^2:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2.$$

- Now consider test statistic $T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$. What is the reference distribution?

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One-sample z-test (cont'd)

- What is the reference distribution of $T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$?

- Recall that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$.

- We can rewrite T as

$$T = \frac{Z}{\sqrt{V/v}},$$

where $Z = \frac{(\overline{X} - \mu_0)}{\sigma}$ and $V = \frac{(n-1)s^2}{\sigma^2}$.

- Thus, $T \overset{H_0}{\sim} \chi^2_{n-1}$.

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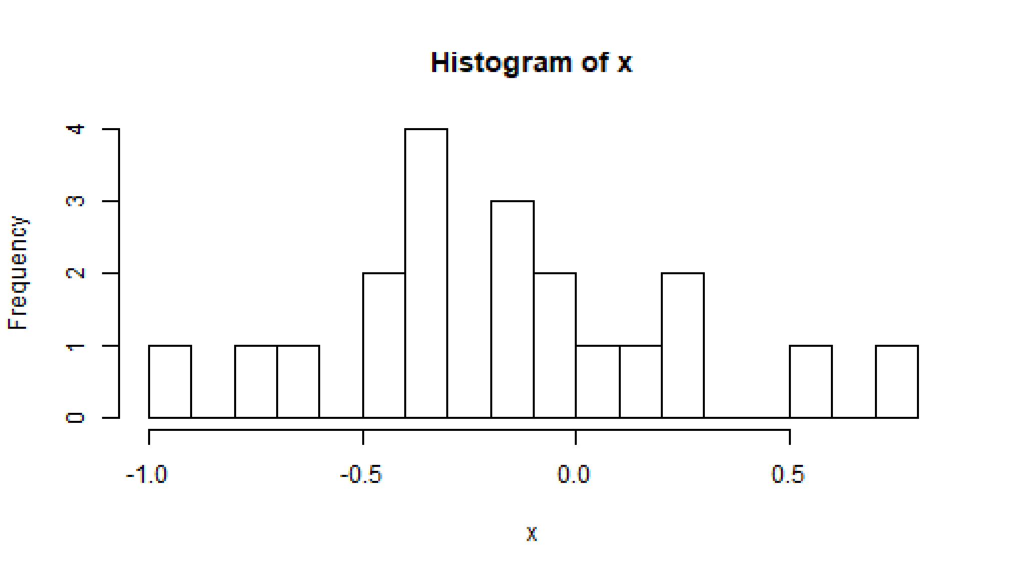
Example: One-sample t-test

```
set.seed(12)
n ← 20
mu ← 0
sigma ← 0.5
x ← rnorm(n, mu, sigma)
hist(x, breaks = 20)
```



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Example: One-sample t-test

```
x_bar ← sum(x) / n
s ← sqrt(sum((x - x_bar)^2) / (n - 1))
print(x_bar)
```

```
## [1] -0.1656045
```

```
print(s)
```

```
## [1] 0.4334393
```

```
test_stat ← (x_bar - 0) / (s / sqrt(n))
print(test_stat)
```

```
## [1] -1.708673
```

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Example: One-sample t-test

```
pnorm(test_stat)
```

```
## [1] 0.04375576
```

```
pt(test_stat, df = n - 1)
```

```
## [1] 0.05189839
```

- We see that the t-test gives a larger P-value than what one would get from the normal distribution.

- If one incorrectly applies a z-test instead of a t-test, the Type I error will be inflated, especially for small sample sizes.

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Likelihood Ratio Test

Review of 120B

Likelihood Ratio Test

```
set.seed(1234)
n ← 20
mu ← 0.9
sigma ← 0.5
x ← rnorm(n, mu, sigma)

lik ← function(mu, sigma = 1) {
  (2 * pi * sigma^2)^(-n / 2) * exp(- 1 / (2 * sigma^2) * sum((x - mu)^2))
}
mu_seq ← seq(0, 2, 0.01)
lik_vals ← sapply(X = mu_seq, FUN = lik)
```

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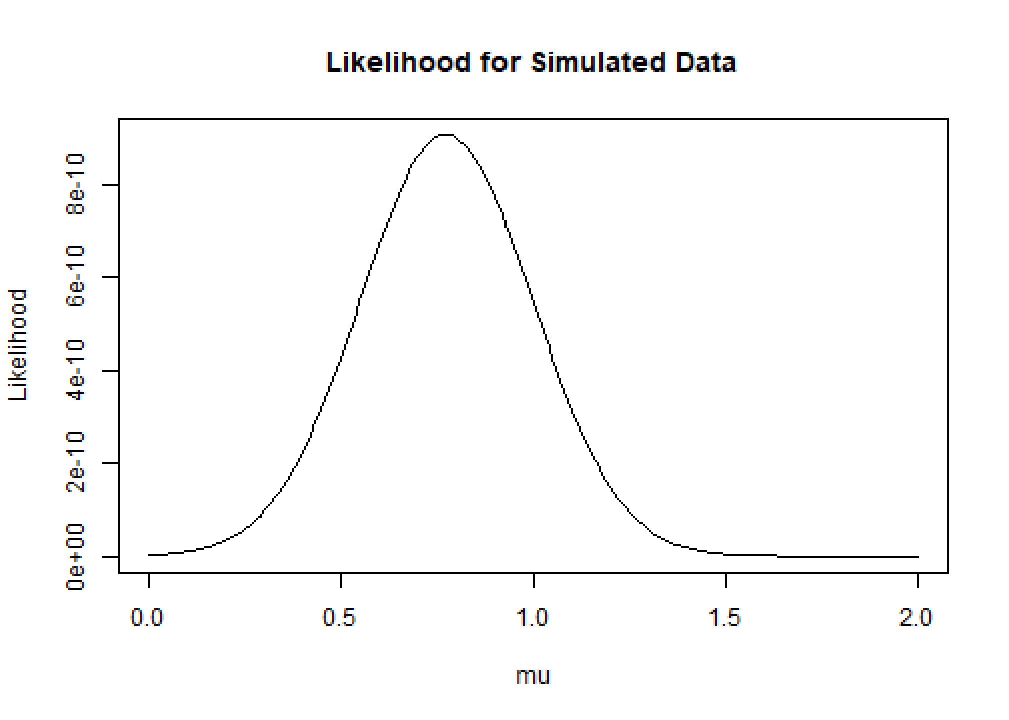
Likelihood Ratio Test

```
plot(mu_seq, lik_vals, ty = "l", ylab = "Likelihood", xlab = "mu",
     main = "Likelihood for Simulated Data")
```

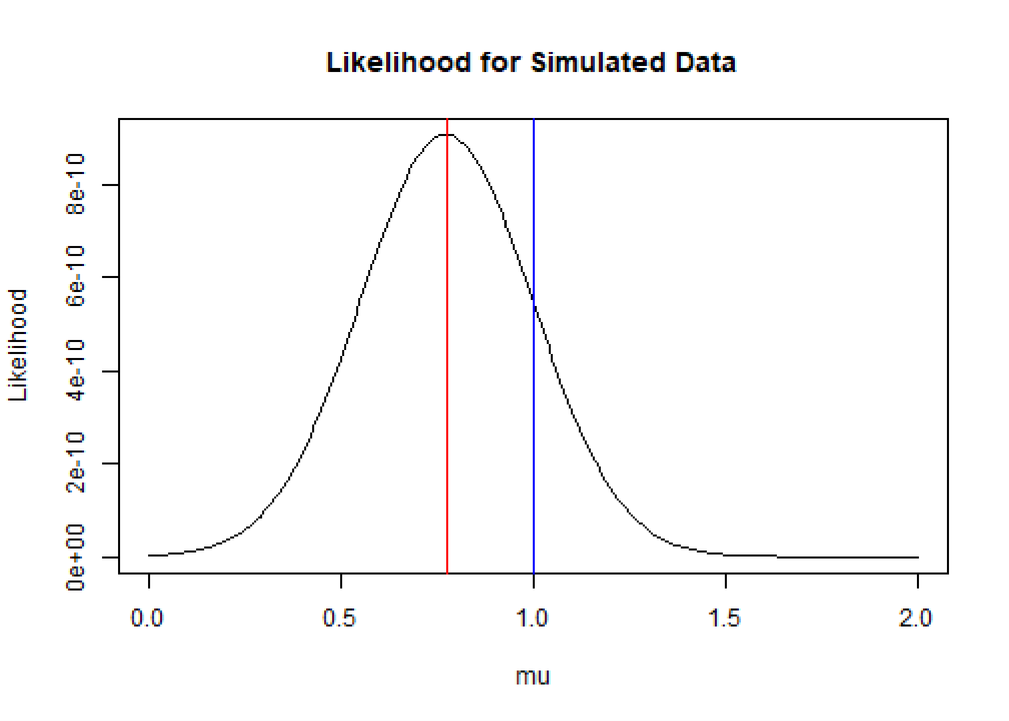


Review of 120B

Likelihood Ratio Test

Suppose we want to test $H_0 : \mu = 1$ with the LRT.

```
plot(mu_seq, lik_vals, ty = "l", ylab = "Likelihood", xlab = "mu",
     main = "Likelihood for Simulated Data")
abline(v = 1, col = "blue")
abline(v = mean(x), col = "red")
```



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Likelihood Ratio Test

Suppose we want to test $H_0 : \mu = 1$ using the LRT.

```
set.seed(1234)
n ← 20
mu ← 0.9
sigma ← 0.5
x ← rnorm(n, mu, sigma)
```

```
mu_0 ← 1
s_0 ← sqrt(1 / n * sum((x - mu_0)^2))
mu_hat ← mean(x)
s ← sqrt(1 / (n - 1) * sum((x - mu_hat)^2))
```

```
F_stat ← n * (mu_hat - mu_0)^2 / s^2
P_val ← pf(F_stat, df1 = 1, df2 = n - 1, lower.tail = FALSE)
P_val
```

```
## [1] 0.06141741
```

Review of 120B

Likelihood Ratio Test

```
t.test(x = x, mu = 1)
```

```
## 
##      One Sample t-test
## 
## data:  x
## t = -1.988, df = 19, p-value = 0.06142
## alternative hypothesis: true mean is not equal to 1
## 95 percent confidence interval:
##  0.5374297 1.0119063
## sample estimates:
## mean of x 
##  0.774668
```

```
T_stat <- (mu_hat - mu_0) / sqrt(s^2 / n)
2 * pt(T_stat, df = n - 1, lower.tail = T)
```

```
## [1] 0.06141741
```

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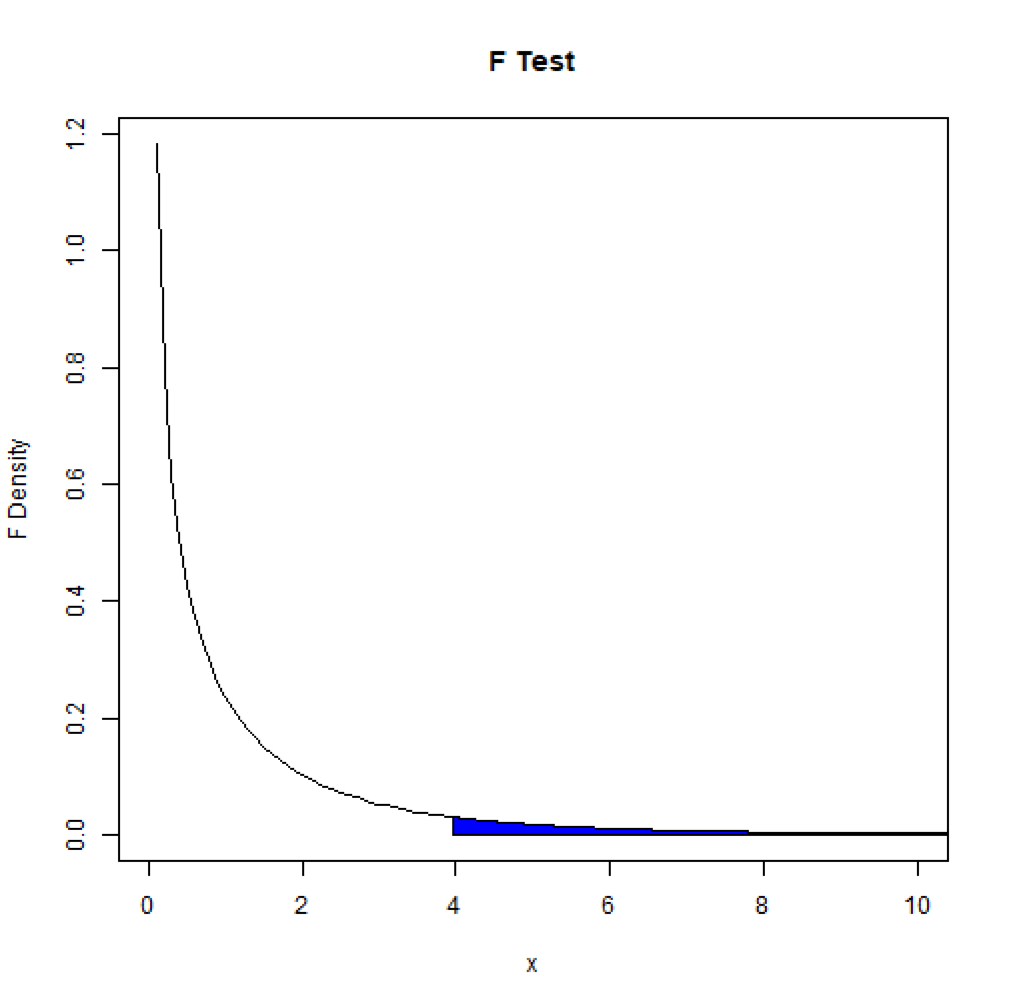
Likelihood Ratio Test



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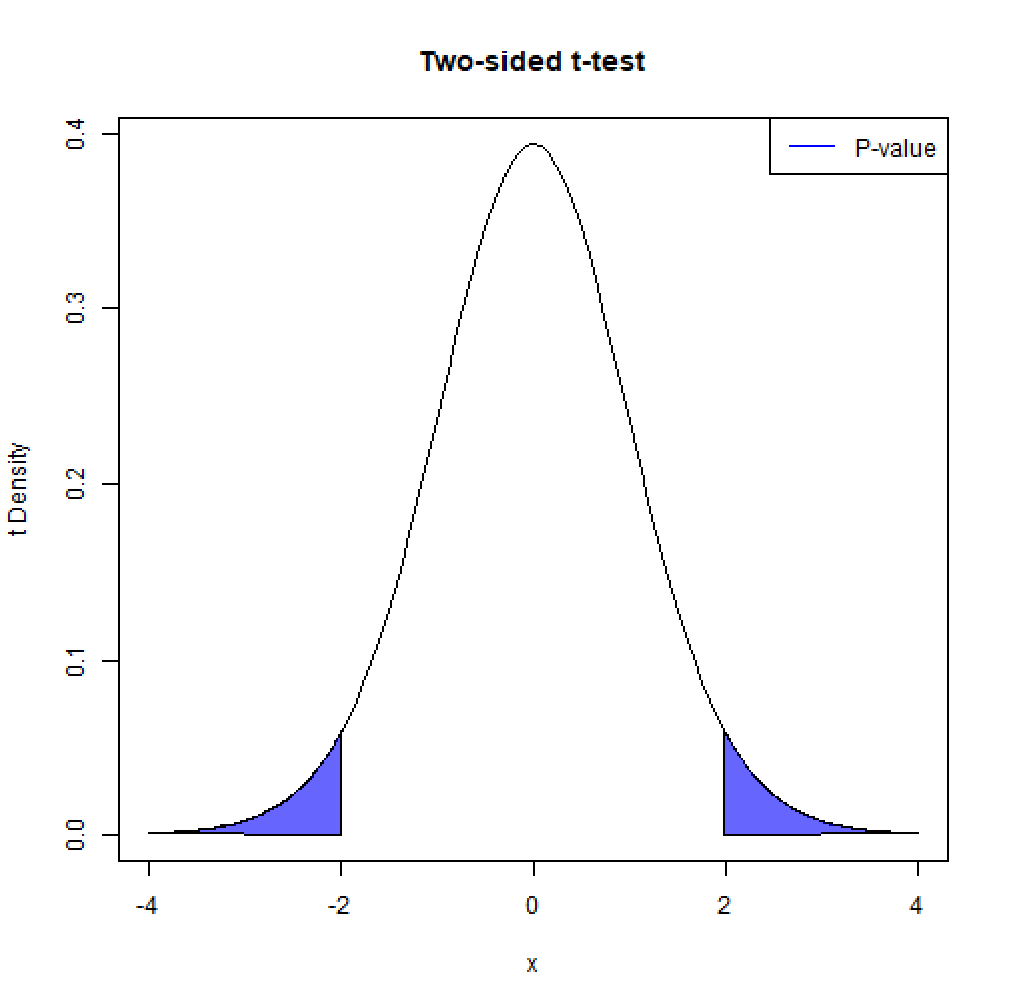
Likelihood Ratio Test



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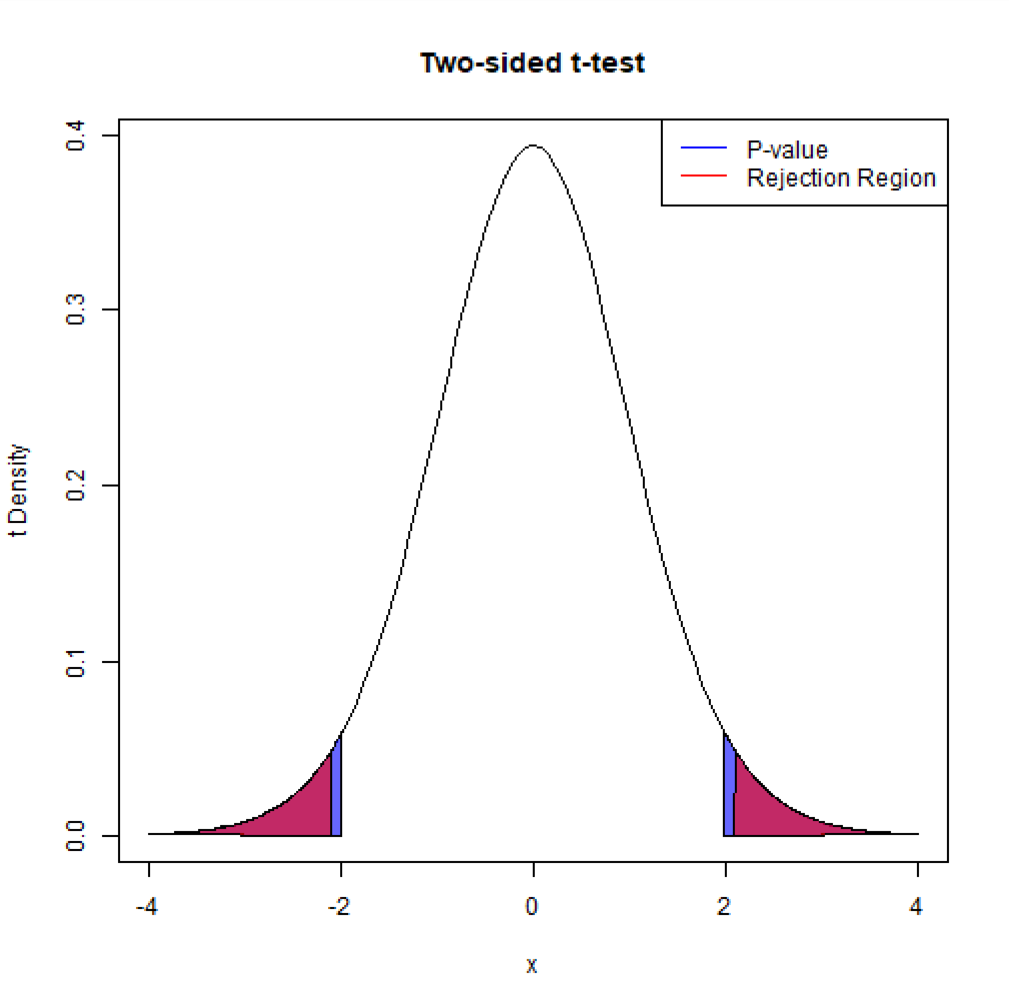
Likelihood Ratio Test



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