1. a.) $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \quad \varepsilon_i \sim N(0, \rho_i^2\sigma^2)$

where ρ_i are known constants

Then $z_i = \beta_0 u_i + \beta_1 v_i + \rho_i^{-1}\varepsilon_i \sim N(\beta_0 u_i + \beta_1 v_i, \sigma^2)$,

which satisfies the assumptions of independence of errors, normality, and constant variance.

b.) Let $\rho^{-1} = \begin{pmatrix} \rho_1^{-1} & & 0 \\ & \rho_2^{-1} & \\ & & \ddots \\ 0 & & & \rho_n^{-1} \end{pmatrix}$. Then the model in (a) can

be written as:

$$Z = \rho^{-1}X\beta + \rho^{-1}\varepsilon, \text{ where } Z = \rho^{-1}Y$$

Since the linear regression assumptions are satisfied, we can use the usual formula for $\hat{\beta}$, with the transformed response vector and design matrix:

$$\hat{\beta} = ((\rho^{-1}X)^T(\rho^{-1}X))^{-1}(\rho^{-1}X)^T(\rho^{-1}Y)$$

$$= (X^T\rho^{-2}X)^{-1}(X^T\rho^{-2}Y)$$

c.) $Var(\hat{\beta}) = Var((X^T\rho^{-2}X)^{-1}X^T\rho^{-2}Y)$

$$= (X^T\rho^{-2}X)^{-1}X^T\rho^{-2}\,Var(Y)\,\rho^{-2}X(X^T\rho^{-2}X)^{-1}$$

$$= \sigma^2(X^T\rho^{-2}X)^{-1}X^T\rho^{-2}X(X^T\rho^{-2}X)^{-1}$$

$Var(\hat{\beta}) = \sigma^2(X^T\rho^{-2}X)^{-1}$

(2) $P(Y=(y_1, y_2, y_3)) = \frac{n!}{y_1! y_2! y_3!} p_1^{y_1} p_2^{y_2} p_3^{y_3}$

where $y_1+y_2+y_3=n$.

a.) Show Y_1 follows Binomial(n, p_1) by showing that

$$P(Y_1=y_1) = \frac{n!}{y_1!(n-y_1)!} p_1^{y_1}(1-p_1)^{n-y_1}$$

$$\Rightarrow P(Y_1=y_1) = \sum_{y_2,y_3} P(Y_1=y_1, Y_2=y_2, Y_3=y_3)$$

$$= \sum_{y_2,y_3} \frac{n!}{y_1! y_2! y_3!} p_1^{y_1} p_2^{y_2} p_3^{y_3}$$

$$= \sum_{y_3} \frac{n!}{y_1! y_3!(n-y_1-y_3)!} p_1^{y_1} p_3^{y_3}(1-p_1-p_3)^{n-y_1-y_3}, \text{ using } \begin{array}{l} y_2=n-y_1-y_3 \\ p_2=1-p_1-p_3 \end{array}$$

$$= \frac{n!\, p_1^{y_1}}{y_1!} \sum_{y_3} \frac{1}{y_1!(n-y_1-y_3)!} p_3^{y_3}(1-p_1-p_3)^{n-y_1-y_3}$$

$$= \frac{n!\, p_1^{y_1}}{y_1!(n-y_1)!} \sum_{y_3} \frac{(n-y_1)!}{y_1!(n-y_1-y_3)!} p_3^{y_3}(1-p_1-p_3)^{n-y_1-y_3}$$

$$= \frac{n!\, p_1^{y_1}}{y_1!(n-y_1)!} \cdot (p_3+1-p_1-p_3)^{y_3+n-y_1-y_3}, \text{ using Binomial Thm.}$$

$$= \frac{n!}{y_1!(n-y_1)!} \cdot p_1^{y_1}(1-p_1)^{n-y_1}$$

② b.) $Cov(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2]$

$$E[Y_1 Y_2] = \sum_{y_1, y_2} y_1 y_2 \, P(Y_1 = y_1, Y_2 = y_2, Y_3 = n - y_1 - y_2)$$

$$= \sum_{y_1, y_2} y_1 y_2 \cdot \frac{n!}{y_1! \, y_2! \, (n-y_1-y_2)!} p_1^{y_1} p_2^{y_2} (1-p_1-p_2)^{n-y_1-y_2}$$

$$= \sum_{y_1, y_2} \frac{n!}{(y_1-1)! \, (y_2-1)! \, (n-y_1-y_2)!} p_1^{y_1} p_2^{y_2} (\cancel{1-}p_1-p_2)^{n-y_1-y_2}$$

$$= n(n-1) \sum_{y_1, y_2} \frac{(n-2)!}{(y_1-1)! \, (y_2-1)! \, (n-y_1-y_2)!} p_1^{y_1} p_2^{y_2} (1-p_1-p_2)^{n-y_1-y_2}$$

$= n(n-1) \cdot p_1 p_2$, by trinomial thm.

$E[Y_1] = np_1$, $E[Y_2] = np_2$

$$Cov(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2] = n(n-1)p_1p_2 - (np_1 \cdot np_2)$$

$$= -np_1p_2[-(n-1-n)]$$

$$= -np_1p_2$$

Similarly, can show $Cov(Y_1, Y_3) = -np_1p_3$, $Cov(Y_2, Y_3) = -np_2p_3$

3. a.) See uploaded R code.

b.) Pearson's χ^2 is not recommended here because the sample size is small, and Pearson's χ^2 test is asymptotic (based on CLT).

4. (a) + (b) see R code.

c.)
$$L_0 = \frac{n!}{n_{11}! \cdots n_{IJ}!} \prod_{i,j} \left(\hat{\pi}_{ij}^{(0)}\right)^{n_{ij}} = \frac{n!}{n_{11}! \cdots n_{IJ}!} \prod_{i,j} \left(\frac{n_{i\cdot} n_{\cdot j}}{n^2}\right)^{n_{ij}}$$

$$L_1 = \frac{n!}{n_{11}! \cdots n_{IJ}!} \prod_{i,j} \left(\hat{\pi}_{ij}\right)^{n_{ij}} = \frac{n!}{n_{11}! \cdots n_{IJ}!} \prod_{i,j} \left(\frac{n_{ij}}{n}\right)^{n_{ij}}$$

$$\Lambda = \frac{L_0}{L_1} = \prod_{i,j} \frac{\left(\frac{n_{i\cdot} n_{\cdot j}}{n^2}\right)^{n_{ij}}}{\left(n_{ij}/n\right)^{n_{ij}}} = \prod_{i,j} \left(\frac{n_{i\cdot} n_{\cdot j}}{n_{ij} \cdot n}\right)^{n_{ij}}$$

d.) Pearson's χ^2 Test stat $= 8.6 \overset{H_0}{\sim} \chi^2_3$

$\Rightarrow P = 0.035$

$\Rightarrow$ Reject H_0

$$-2\log(\Lambda) = 2\left(\log(L_1) - \log(L_0)\right) = 8.447 \overset{H_0}{\sim} \chi^2_3$$

$\Rightarrow P = 0.037$

$\Rightarrow$ Reject H_0.