STAT120C Homework 2

Due Monay April 15, 2019 by 5pm in the Dropbox in DBH

1. Consider the one-way layout. We use Y_{ij} to denote the measurment of the jth observation from the ith treatment, where $i = 1, \cdots, I$ and $j = 1, \cdots, J$. Define the following summary statistics

   $\bar{Y}_{i\cdot} = \frac{1}{J}\sum_{j=1}^{J} Y_{ij}$, $i = 1, \cdots, I$ and $\bar{Y}_{\cdot\cdot} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij}$

   (a) Show that $J\bar{Y}_{i\cdot} = \sum_{j=1}^{J} Y_{ij}$ and then conclude that $\sum_{j=1}^{J}(Y_{ij} - \bar{Y}_{i\cdot}) = 0$.

   Solution:

$$J\bar{Y}_{i\cdot} = J\left(\frac{1}{J}\sum_{j=1}^{J} Y_{ij} = \sum_{j} Y_{ij}\right)$$
$$\Rightarrow \sum_{j=1}^{J}(Y_{ij} - \bar{Y}_{i\cdot}) = \sum_{j} Y_{ij} - J\bar{Y}_{i\cdot} = 0$$

   (b) Use you result in (a) to prove that $\sum_{i=1}^{I}\sum_{j=1}^{J}[(Y_{ij} - \bar{Y}_{i\cdot})(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})] = 0$.

   Solution:

$$\begin{aligned}\sum_{i=1}^{I}\sum_{j=1}^{J}[(Y_{ij} - \bar{Y}_{i\cdot})(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})] &= \sum_{i}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})\sum_{j}(Y_{ij} - \bar{Y}_{i\cdot}) \\ &= 0, \quad \text{by (a)}\end{aligned}$$

   (c) Prove that both $\sum_{i=1}^{I}\bar{Y}_{i\cdot}$ and $\sum_{i=1}^{I}\bar{Y}_{\cdot\cdot}$ equal $\frac{1}{J}\sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij}$.

   Solution:

   Plug in definition of $\bar{Y}_{i\cdot}$.

   (d) Use (c) to conclude that $\sum_{i=1}^{I}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}) = 0$.

2. Assume that we have I independent random samples. For $i = 1, \cdots, I$, we assume that the ith random sample $(Y_{i1}, Y_{i2}, \cdots, Y_{iJ})$ came from the normal distribution with with mean μ_i and variance σ^2. These assumptions can be summarized using the following statistical model:

$$Y_{ij} = \mu_i + \epsilon_{ij}\ ,\ i = 1, \cdots, I\ ;\ j = 1, \cdots, J$$

   where $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$. Show that the MLE of μ_i is $\hat{\mu}_i = \bar{Y}_{i\cdot} = \frac{1}{J}\sum_{j=1}^{J} Y_{ij}$.

   Solution:

$$
\begin{aligned}
\mathcal{L}(\mu|Y) &= \prod_i \prod_j \frac{1}{2\pi\sigma^2} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - \mu_i)^2 \right\} \\
&= (2\pi\sigma^2)^{-IJ/2} \exp\left\{ \frac{-1}{2\sigma^2} \sum_i \sum_j (Y_{ij} - \mu_i)^2 \right\} \\
\ell(\mu) &= \frac{-IJ}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_i \sum_j (Y_{ij} - \mu_i)^2 \\
\frac{\partial \ell}{\partial \mu_i} &= \frac{1}{\sigma^2} \sum_j (Y_{ij} - \mu_i) \\
0 &= \sum_j (Y_{ij} - \hat{\mu}_i) \Rightarrow \hat{\mu}_i = \bar{Y}_{i\cdot}
\end{aligned}
$$

3. The statistical model of Problem 2 can also be written to

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \ , \, i = 1, \cdots, I \ ; \, j = 1, \cdots, J$$

where $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$ and $\sum_{i=1}^{I} \alpha_i = 0$. Derive the MLEs for μ, and α_i

Solution: Here are 2 methods to solve this problem.

The log-likelihood for this model is

$$\ell(\mu, \alpha_1, \ldots, \alpha_I, \sigma^2) = \frac{-IJ}{2} \log(2\pi) - \frac{IJ}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - \mu - \alpha_i)^2.$$

Method 1

$$
\begin{aligned}
\frac{\partial \ell}{\partial \mu} &= \frac{1}{\sigma^2} \sum_i \sum_j (Y_{ij} - \mu - \alpha_i) \\
0 &= \sum_i \sum_j Y_{ij} - \sum_i \sum_j \hat{m}u - \sum_i \sum_j \hat{\alpha}_i \\
&= \sum_i \sum_j Y_{ij} - IJ\hat{\mu}, \qquad \text{since} \sum_i \alpha_i = 0 \\
\hat{\mu} &= \bar{Y}_{\cdot\cdot}
\end{aligned}
$$

Using the fact that $\sum_{i=1}^{I} \alpha_i = 0$ impiles that $\alpha_I = -\sum_{i=1}^{I-1}$, we can reparameterize the log-likelihood in terms of $\mu, \alpha_1, \ldots, \alpha_{I-1}, \sigma^2$.

For $i = 1, \ldots, I-1$,

$$
\begin{aligned}
\frac{\partial \ell}{\partial \alpha_i} &= \frac{-1}{2\sigma^2}\frac{\partial}{\partial \alpha_i}\left[\sum_i\sum_j (Y_{ij}-\mu-\alpha_i)^2 + \sum_j (Y_{Ij}-\mu-\alpha_I)^2\right] \\
&= \frac{-1}{2\sigma^2}\left[\sum_j (Y_{ij}-\mu-\alpha_i)\cdot 2\cdot(-1) + \sum_j \frac{\partial}{\partial \alpha_I}(Y_{Ij}-\mu-\alpha_I)^2 \cdot \frac{\partial \alpha_I}{\partial \alpha_i}\right], \quad \text{using the chain rule.} \\
&= \frac{-1}{2\sigma^2}\left[-2\sum_j (Y_{ij}-\mu-\alpha_i) + 2\sum_j (Y_{Ij}-\mu-\alpha_I)\right] \\
&= \frac{1}{\sigma^2}\sum_j \left[-Y_{ij}+\mu+\alpha_i+Y_{Ij}-\mu-\alpha_I\right] \\
&= \frac{J}{\sigma^2}\left[\bar{Y}_{I\cdot}-\alpha_I-(\bar{Y}_{i\cdot}-\alpha_i)\right].
\end{aligned}
$$

Setting this to 0 for all $i = 1, \ldots, I-1$, we get the system of equations

$$
\begin{aligned}
\bar{Y}_{1\cdot}-\hat{\alpha}_1 &= \bar{Y}_{I\cdot}-\hat{\alpha}_I \\
\bar{Y}_{2\cdot}-\hat{\alpha}_2 &= \bar{Y}_{I\cdot}-\hat{\alpha}_I \\
&\vdots \\
\bar{Y}_{I-1\cdot}-\hat{\alpha}_{I-1} &= \bar{Y}_{I\cdot}-\hat{\alpha}_I.
\end{aligned}
$$

Adding up both sides gives

$$
\sum_{i=1}^{I-1}\bar{Y}_{i\cdot} - \sum_{i=1}^{I-1}\hat{\alpha}_i = (I-1)(\bar{Y}_{I\cdot}-\hat{\alpha}_I).
$$

Applying the constraint on α, we get

$$
\begin{aligned}
\sum_{i=1}^{I-1}\bar{Y}_{i\cdot}+\hat{\alpha}_I &= (I-1)\bar{Y}_{I\cdot}-(I-1)\hat{\alpha}_I \\
I\hat{\alpha}_I &= (I-1)\bar{Y}_{I\cdot}-\sum_{i=1}^{I-1}\bar{Y}_{i\cdot} \\
&= I\bar{Y}_{I\cdot}-\sum_{i=1}^{I}\bar{Y}_{i\cdot} = I\bar{Y}_{I\cdot}-\frac{1}{J}\sum_{i=1}^{I}\sum_{j=1}^{J}Y_{ij} \\
\hat{\alpha}_I &= \bar{Y}_{I\cdot}-\bar{Y}_{\cdot\cdot}.
\end{aligned}
$$

Plugging this solution back into the sysmte of equations for each $i = 1, \ldots, I-1$ gives

$$
\hat{\alpha}_i = \bar{Y}_{i\cdot}-\bar{Y}_{\cdot\cdot}.
$$

Method 2 We can instead incorporate the constraint using the method of Lagrange multipliers.

Let

$$Q(\mu, \alpha_1, \ldots, \alpha_I, \sigma^2, \lambda) = \ell(\mu, \alpha_1, \ldots, \alpha_I, \sigma^2) + \lambda \sum_{i=1}^{I} \alpha_i.$$

Finding the value for μ at the maximum, we have

$$\begin{aligned}
\frac{\partial Q}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - \mu - \alpha_i) \\
0 &= \sum \sum Y_{ij} - IJ\hat{\mu} - 0 \\
\hat{\mu} &= \bar{Y}_{..}
\end{aligned}$$

For the value of λ at the maximum, we get the constraint $0 = \sum_{i=1}^{I} \hat{\alpha}_i$.
For α_i, we have

$$0 = \frac{\partial Q}{\partial \alpha_i} = \frac{1}{\sigma^2} \sum_{j=1}^{J} (Y_{ij} - \mu - \alpha_i) + \lambda.$$

Summing over i and solving for λ,

$$\begin{aligned}
\hat{\lambda} &= \frac{-1}{I\sigma^2} \left[\sum_i \sum_j (Y_{ij} - \hat{\mu} - \hat{\alpha}_i) \right] \\
&= \frac{-1}{I\sigma^2} \left[\sum_i \sum_j (Y_{ij} - \hat{\mu}) \right], \quad \text{since } \sum \alpha i = 0 \\
&= \frac{-1}{I\sigma^2} \left[IJ\bar{Y}_{..} - IJ\hat{\mu} \right] = 0.
\end{aligned}$$

Setting $\lambda = 0$ in the equations for α_i and solving yields

$$\hat{\alpha}_i = \bar{Y}_{i.} - \bar{Y}_{..}.$$

4. Consider the balanced one-way ANOVA model with I treatment groups, and J observations for each group.

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij},$$

where the idiosyncratic errors are $\varepsilon \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$.

(a) Show that $SSW/\sigma^2 \sim \chi^2_{I(J-1)}$.
Solution: Shown in Rice.

(b) Show that $SSB/\sigma^2 \overset{H_0}{\sim} \chi^2_{I-1}$.

Solution: Under H_0, we have $SSTOT/\sigma^2 \overset{H_0}{\sim} \chi^2_{IJ-1}$, which follows from the one-sample normal case. Recall that for $U = V + W$ with $U \sim \chi^2_n, V \sim \chi^2_p$ and V and W independent, it follows that $W \sim \chi^2_{n-p}$. Combining (a) and (c), we conclude that $SSB/\sigma^2 \overset{H_0}{\sim} \chi^2_{I-1}$.

(c) Show that SSW and SSB are independent.

Solution: Write $SSW = (J-1)s_i^2$, where s_i^2 is the sample variance for group i, and write $SSB = J\sum_{i=1}^{I}\left(\bar{Y}_{i\cdot} - \frac{1}{I}\sum_{i'=1}^{I}\bar{Y}_{i'\cdot}\right)$. Then, since s_i and $\bar{Y}_{i\cdot}$ are independent for all i (Lemma, Ch. 6), and since functions of independent random variables are independent, we conclude that SSW and SSB are independent.

(d) What is the null distribution of $\frac{SSB/(I-1)}{SSW/(I(J-1))}$?

Solution: Combining the above results, we have

$$\frac{SSB/(I-1)}{SSW/[I(J-1)]} \overset{H_0}{\sim} F_{I-1,I(J-1)}.$$

Hint: See Theorem B given in Rice (Sec. 12-2, p482).

5. Consider two independent random samples. The first one $Y_{1,1},\cdots,Y_{1,9}$ is a random sample from $N(\mu_1,\sigma^2)$ and the second one $Y_{2,1},\cdots,Y_{2,9}$ is a random sample from $N(\mu_2,\sigma^2)$. The parameters μ_1,μ_2,σ^2 are unknown. We want to conduct hypothesis tesing

$$H_0: \mu_1 = \mu_2 \text{ v.s. } H_1: \mu_1 \neq \mu_2$$

If we use the two-sample t-test, we would calculate the following test statistic

$$T = \frac{\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot}}{\sqrt{s_p^2(\frac{1}{9}+\frac{1}{9})}}$$

where $\bar{Y}_{i\cdot} = \sum_{j=1}^{J} Y_{ij}, i = 1,2$ and $s_p^2 = \frac{\sum_{i=1}^{2}\sum_{j=1}^{9}(Y_{ij}-\bar{Y}_{i\cdot})^2}{9+9-2}$. If we use the F-test from one-way ANOVA, we would calculate the following test statistic

$$F = \frac{SSB/(2-1)}{SSW/(2\times(9-1))}$$

where $SSB = 9\sum_{i=1}^{2}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2$ and $SSW = \sum_{i=1}^{2}\sum_{j=1}^{9}(Y_{ij} - \bar{Y}_{i\cdot})^2$.

Show that $F = T^2$. (Hint: show that $\bar{Y}_{1\cdot} - \bar{Y}_{\cdot\cdot} = \frac{1}{2}(\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot})$ and $\bar{Y}_{2\cdot} - \bar{Y}_{\cdot\cdot} = -\frac{1}{2}(\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot})$)

Solution:

Plug in the result of the hint (which is straightforward to show from the definitions) to the F statistic:

$$\begin{aligned}
F &= \frac{\frac{1}{I-1}J\sum_i(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2}{\frac{1}{I(J-1)}\sum_i\sum_j(Y_{ij} - \bar{Y}_{\cdot\cdot})^2} \\
&= \frac{9\left[\frac{1}{4}(\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot})^2 + \frac{1}{4}(\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot})^2\right]}{\frac{1}{2(9-1)}\sum_i\sum_j(Y_{ij} - \bar{Y}_{\cdot\cdot})^2} \\
&= \frac{\frac{9}{2}(\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot})^2}{s_p^2} \\
&= \frac{(\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot})^2}{\frac{2}{9}s_p^2} = t^2.
\end{aligned}$$