STAT 120C - LRT

Dustin Pluta

April 2019

1 Likelihood Ratio Test

1.1 One Sample Normal, Mean and Variance Unknown

Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$, with unknown mean and variance. We wish to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ with the likelihood ratio test, which has test statistic

$$\Lambda(\mu, \sigma^2 | X) = \frac{\max_{\Omega_0} \mathcal{L}(\mu, \sigma^2 | X)}{\max_{\Omega} \mathcal{L}(\mu, \sigma^2 | X)},$$

where $\Omega_0 = \{(\mu, \sigma^2) \mid \mu = \mu_0, \sigma^2 > 0\}$, and $\Omega = \{(\mu, \sigma^2) \mid \mu \in \mathbb{R}, \sigma^2 > 0\}$, and $X = (X_1, X_2, \ldots, X_n)$ (the vector of the sample).

The likelihood is

$$\mathcal{L}(\mu, \sigma^2 | X) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2\right\}.$$

The log likelihood is

$$\ell(\mu, \sigma^2) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2.$$

Assuming H_0, we have $\mu = \mu_0$, so the maximum likelihood estimate $\hat{\sigma}_0^2$ of σ^2 over the null space is found by maximizing ℓ wrt σ^2 with $\mu = \mu_0$.

$$\begin{aligned}
\frac{d\ell(\mu_0, \sigma^2)}{d\sigma^2} &= -\frac{1}{2}\left[\frac{n}{\sigma^2} - \frac{1}{(\sigma^2)^2} \sum (X_i - \mu_0)^2\right] \\
0 &= n\hat{\sigma}_0^2 - \sum (X_i - \mu_0)^2 \\
\hat{\sigma}_0^2 &= \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2.
\end{aligned}$$

The MLEs of the parameters over the full parameter space are found by maximizing $\ell(\mu, \sigma^2)$, where both parameters are allowed to vary.
The MLE of μ is

$$\frac{\partial \ell(\mu,\sigma^2)}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu) \tag{1}$$

$$0 = \sum_{i=1}^{n}(X_i - \hat{\mu}) \tag{2}$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i. \tag{3}$$

The MLE of σ^2 follows from the derivation used for the null estimate, replacing μ by $\hat{\mu}$:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2.$$

The likelihood ratio test statistic is then

$$\Lambda(X) = \frac{(2\pi\hat{\sigma}_0^2)^{-n/2}\exp\left\{\frac{-1}{2\hat{\sigma}_0^2}\sum(X_i - \mu_0)^2\right\}}{(2\pi\hat{\sigma}^2)^{-n/2}\exp\left\{\frac{-1}{2\hat{\sigma}^2}\sum(X_i - \hat{\mu})^2\right\}}.$$

Recognizing that $\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \hat{\mu})^2$ and similar for $\hat{\sigma}_0^2$, the terms inside the exponential cancel to $-n/2$, and the exponential terms in the numerator and denominator cancel, giving

$$\Lambda(X) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{-n/2}.$$

We reject H_0 when $\Lambda(X)$ is small; equivalently, we reject H_0 when

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} = \frac{\sum(X_i - \mu_0)^2}{\sum(X_i - \hat{\mu})^2}$$

is large.

Recalling the identity $\sum(X_i - \mu_0)^2 = \sum(X_i - \bar{X})^2 - \sum(\bar{X} - \mu_0)^2$, we can write

$$
\begin{aligned}
\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} &= \frac{\sum (X_i - \mu_0)^2}{\sum (X_i - \hat{\mu})^2} \\
&= \frac{\sum (X_i - \bar{X})^2 - \sum (\bar{X} - \mu_0)^2}{\sum (X_i - \bar{X})^2} \\
&= 1 + \frac{n(\bar{X} - \mu_0)^2}{\sum (X_i - \bar{X})^2}.
\end{aligned}
$$

Thus, the LRT will reject H_0 when $\frac{n(\bar{X}-\mu_0)^2}{\sum (X_i - \bar{X})^2}$ is large.

Similar to our derivation of the distribution of the sample variance, we know $n(\bar{X} - \mu_0)^2/\sigma^2 \overset{H_0}{\sim} \chi_1^2$, and $\sum (X_i - \bar{X})^2/\sigma^2 \overset{H_0}{\sim} \chi_{n-1}^2$.

Using the fact that the numerator and denominator are independent (verify this is true for yourself), we can conclude with the following test statistic and reference distribution:

$$
T(X) = \frac{n(\bar{X} - \mu_0)^2}{\sum (X_i - \bar{X})^2/(n-1)} \overset{H_0}{\sim} F_{1,n-1}.
$$

The null hypothesis will be rejected when $T(X)$ is large. For a test with significance level α, H_0 will be rejected when $T(X) > F_{1,n-1}(1-\alpha)$. That is, the rejection region is defined by the $1-\alpha$ percentile of the $F_{1,n-1}$ distribution.