Rotations

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Useful relationships

Skew-symmetric matrices

- Property:

$$\mathbf{S}^{\mathrm{T}} = -\mathbf{S}$$

- Example in $\mathbb{R}^3$:

$$\mathbf{S} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \quad \mathbf{S}^{\mathrm{T}} = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} = -\mathbf{S}$$

Skew-symmetric matrices

- For a vector:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

- define the skew-symmetric matrix:

$$\mathbf{S}_{\mathbf{x}} := \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Skew-symmetric matrices and cross product

- then, it follows for the product with another vector $\mathbf{u}$

$$\mathbf{S}_{\mathbf{x}}\mathbf{u} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} yw - zv \\ zu - xw \\ xv - yu \end{bmatrix}$$

- This is exactly the cross product of $\mathbf{x}$ with $\mathbf{u}$:

$$\mathbf{x} \times \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} yw - zv \\ zu - xw \\ xv - yu \end{bmatrix}$$

To memorize: determinant

$$\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ x & y & z \\ u & v & w \end{vmatrix}$$

Skew-symmetric matrices and cross product

- The cross product is antisymmetric:

$$\mathbf{u} \times \mathbf{x} = -\mathbf{x} \times \mathbf{u}$$

- Therefore:

$$\mathbf{S}_{\mathbf{u}}\mathbf{x} = -\mathbf{x} \times \mathbf{u} \qquad \text{or} \qquad \mathbf{x} \times \mathbf{u} = -\mathbf{S}_{\mathbf{u}}\mathbf{x}$$

- Since S is skew-symmetric, $-\mathbf{S} = \mathbf{S}^{\mathrm{T}}$ and therefore, one can write:

$$\mathbf{x} \times \mathbf{u} = \mathbf{S}_{\mathbf{u}}^{\mathrm{T}}\mathbf{x}$$

- Therefore, one can write the vector product in two different ways: either $\mathbf{x}$ as matrix, or $\mathbf{u}$:
  - Either: $\mathbf{x} \times \mathbf{u} = \mathbf{S}_{\mathbf{x}}\mathbf{u}$
  - Or: $\mathbf{x} \times \mathbf{u} = \mathbf{S}_{\mathbf{u}}^{\mathrm{T}}\mathbf{x}$

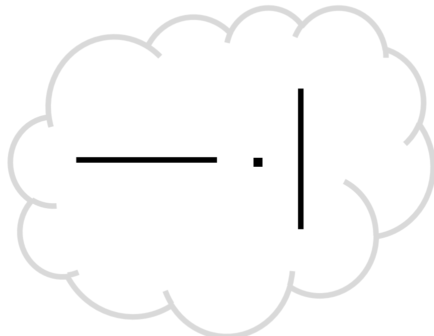
The dyadic product

- Two matrices may only be multiplied if the dimensions allow it, e.g.

$$\underset{2\times\mathbf{3}}{\mathbf{A}} \cdot \underset{\mathbf{3}\times 5}{\mathbf{B}} = \underset{2\times 5}{\mathbf{C}} \qquad\qquad \underset{3\times 1}{\mathbf{x}} \cdot \underset{3\times 1}{\mathbf{u}} = \text{⚡}$$

- To multiply two vectors:
  - If the **first** vector is transposed, one obtains the well-known scalar product, also written as: $\langle \mathbf{x}, \mathbf{u} \rangle$

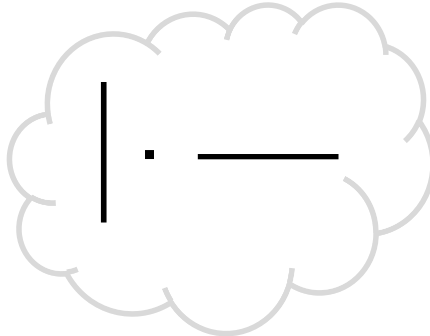
$$\underbrace{\mathbf{x}^{\mathrm{T}}}_{1\times 3} \cdot \underset{3\times 1}{\mathbf{u}} = \underset{1\times 1}{s} = xu + yv + zw \;\in\; \mathbb{R}$$



The dyadic product

- If instead the **second** vector is transposed, one obtains the dyadic product:

$$\underset{3\times 1}{\mathbf{x}} \cdot \underbrace{\mathbf{u}^{\mathrm{T}}}_{1\times 3} = \underset{3\times 3}{\mathbf{D}} = \begin{bmatrix} xu & xv & xw \\ yu & yv & yw \\ zu & zv & zw \end{bmatrix}$$



- Sometimes, one needs the dyadic product of a vector with itself:

$$\mathbf{D}_{\mathbf{x}} := \mathbf{x} \cdot \mathbf{x}^{\mathrm{T}} = \begin{bmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{bmatrix} = \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}$$

Properties of S_x and D_x

- The following holds:

$$(\mathbf{D}_\mathbf{x})^\mathrm{T} = \mathbf{D}_\mathbf{x}$$

$$(\mathbf{S}_\mathbf{x})^\mathrm{T} = -\mathbf{S}_\mathbf{x}$$

- For $\|\mathbf{x}\| = 1$ the following holds:

$$(\mathbf{D}_\mathbf{x})^2 = \mathbf{D}_\mathbf{x}$$

$$(\mathbf{S}_\mathbf{x})^2 = \mathbf{D}_\mathbf{x} - \mathbf{I}$$

Check e.g.: $(\mathbf{D}_\mathbf{x})^2 = \mathbf{x}\mathbf{x}^\mathrm{T}\mathbf{x}\mathbf{x}^\mathrm{T} = \mathbf{x}(\mathbf{x}^\mathrm{T}\mathbf{x})\mathbf{x}^\mathrm{T} = \mathbf{x}\mathbf{x}^\mathrm{T}$

Rotations

Rotation matrices

- Rotation group, special orthogonal group, *SO(n)*
  - Distance preserving transformation in Euclidean space, det = 1
- First, in $\mathbb{R}^2$: *SO(2)*
- Often, the following is used:
  - Parameterization using sin, cos
  - Turn by angle α:

$$\mathbf{R}_\alpha := \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

- Test: this has to be an **orthogonal matrix**:

$$\mathbf{R}_\alpha^{\mathrm{T}}\mathbf{R}_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Rotation matrices

- Test: must be a **special orthogonal** matrix:

$$\det(\mathbf{R}_\alpha) = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = +1$$

- Note: the following matrix is **orthogonal**, but **not a special orthogonal matrix** (it contains a reflection):

$$\mathbf{T}_\alpha := \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

$$\mathbf{T}_\alpha^{\mathrm{T}}\mathbf{T}_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \mathbf{I}$$

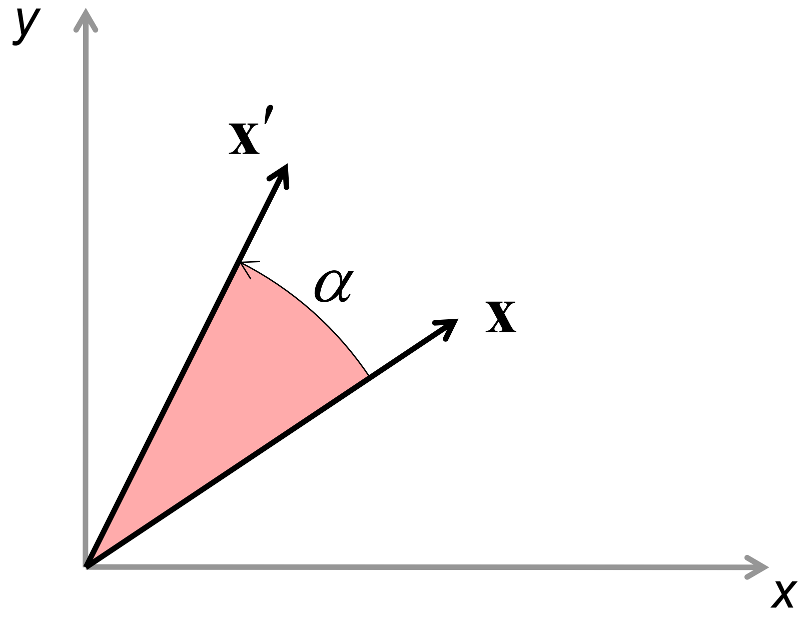
$$\det(\mathbf{T}_\alpha) = -\cos^2\alpha - \sin^2\alpha = -1$$

Rotation matrices

- Rotation matrix corresponds to a left rotation (math. positive, counter-clockwise) by α :

$$\mathbf{R}_\alpha := \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{R}_\alpha \mathbf{x}$$



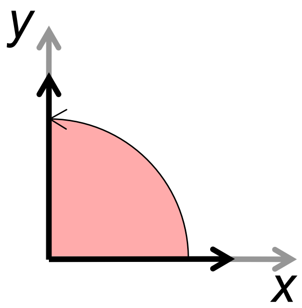
Rotation matrices

- The turn direction can be easily memorized, e.g. turn by 90 degrees:

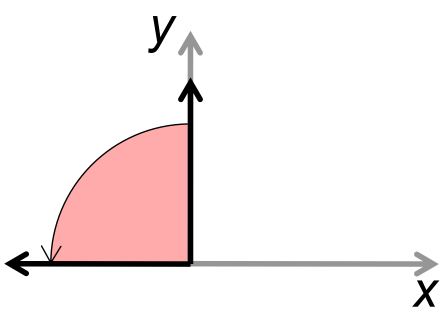
$$\mathbf{R}_{90^\circ} := \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- I.e.:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



Rotation matrices

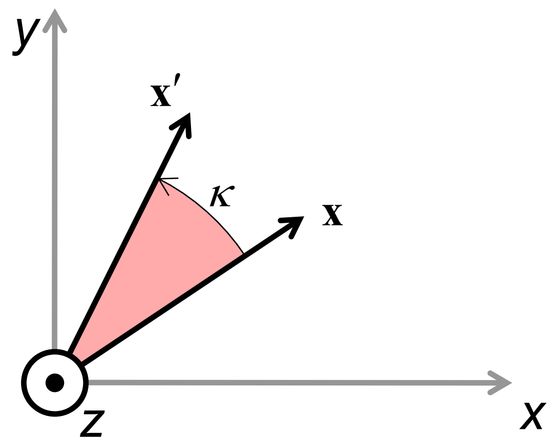
- Note:
  - One can either transform the objects...
  - ...or the coordinate system (reference system)
- A rotation of the objects by α corresponds to a rotation of the coordinate system by $-\alpha$
- Inverse = transposed = rotation by the negative angle:

$$(\mathbf{R}_\alpha)^{-1} = \mathbf{R}_\alpha^{\mathrm{T}} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} = \mathbf{R}_{-\alpha}$$

Rotation matrices in 3D

- Frequently, rotations in $\mathbb{R}^3$ are specified using a concatenation of elementary rotation matrices
- Example for an elementary rotation matrix: rotation around the z-axis (axis 3) by the angle κ :

$$\mathbf{R}_3(\kappa) := \left[\begin{array}{cc|c} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$



- Result:
  - The z-coordinate does not change
  - In the xy plane, there is a rotation in the mathematically positive sense, as discussed on the previous slides

Elementary rotations

Around z: $$\mathbf{R}_3(\kappa) = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\kappa & -s_\kappa & 0 \\ s_\kappa & c_\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Around x: $$\mathbf{R}_1(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\omega & -s_\omega \\ 0 & s_\omega & c_\omega \end{bmatrix}$$

Around y: $$\mathbf{R}_2(\varphi) = \begin{bmatrix} c_\varphi & 0 & s_\varphi \\ 0 & 1 & 0 \\ -s_\varphi & 0 & c_\varphi \end{bmatrix}$$

Note (for y):

$$\begin{array}{ccc|c} c_\varphi & 0 & s_\varphi & c_\varphi \\ 0 & 1 & 0 & 0 \\ -s_\varphi & 0 & c_\varphi & -s_\varphi \\ \hline c_\varphi & 0 & s_\varphi & c_\varphi \end{array}$$

Concatenation of elementary rotations

- Consecutive rotations around 1., 2., 3. axis:

$$\boxed{\mathbf{x}' = \mathbf{R}\mathbf{x} = \mathbf{R}_3(\kappa)\mathbf{R}_2(\varphi)\mathbf{R}_1(\omega)\mathbf{x}}$$

- If this is fully expanded:

$$\mathbf{R} = \begin{bmatrix} c_\varphi c_\kappa & s_\omega s_\varphi c_\kappa - c_\omega s_\kappa & c_\omega s_\varphi c_\kappa + s_\omega s_\kappa \\ c_\varphi s_\kappa & s_\omega s_\varphi s_\kappa + c_\omega c_\kappa & c_\omega s_\varphi s_\kappa - s_\omega c_\kappa \\ -s_\varphi & s_\omega c_\varphi & c_\omega c_\varphi \end{bmatrix}$$

A note on rotations by small angles

- For small rotation angles $\alpha, \beta << 1$ it holds (first order terms):

$$\sin\alpha \approx \alpha$$

$$\cos\alpha \approx 1$$

$$\sin\alpha \sin\beta \approx 0$$

- Therefore, it follows for the rotation matrix:

$$\mathbf{R} \approx \begin{bmatrix} 1 & -\kappa & \varphi \\ \kappa & 1 & -\omega \\ -\varphi & \omega & 1 \end{bmatrix} = \mathbf{I} + \mathbf{S}_{(\omega,\varphi,\kappa)}$$

cf. original: $\mathbf{R} = \begin{bmatrix} c_\varphi c_\kappa & s_\omega s_\varphi c_\kappa - c_\omega s_\kappa & c_\omega s_\varphi c_\kappa + s_\omega s_\kappa \\ c_\varphi s_\kappa & s_\omega s_\varphi s_\kappa + c_\omega c_\kappa & c_\omega s_\varphi s_\kappa - s_\omega c_\kappa \\ -s_\varphi & s_\omega c_\varphi & c_\omega c_\varphi \end{bmatrix}$

Notes on the concatenation of rotation matrices

- The rotations given are for "fixed axes":

$$\mathbf{R} = \mathbf{R}_3(\kappa)\mathbf{R}_2(\varphi)\mathbf{R}_1(\omega)$$

- The rotation using "rotated axes" corresponds to

$$\mathbf{R} = \mathbf{R}_1(\omega)\mathbf{R}_2(\varphi)\mathbf{R}_3(\kappa)$$

- If the rotation does not rotate the object, but rather the reference system, there are two more possibilities for the rotation matrix:

$$\mathbf{R} = \mathbf{R}_3^{\mathrm{T}}(\kappa)\mathbf{R}_2^{\mathrm{T}}(\varphi)\mathbf{R}_1^{\mathrm{T}}(\omega)$$

$$\mathbf{R} = \mathbf{R}_1^{\mathrm{T}}(\omega)\mathbf{R}_2^{\mathrm{T}}(\varphi)\mathbf{R}_3^{\mathrm{T}}(\kappa)$$

Notes on the concatenation of rotation matrices

- Note: transposed = inverse = rotation using negative angle:

$$\mathbf{R}_i^{\mathrm{T}}(\alpha) = \mathbf{R}_i(-\alpha)$$

- Therefore, the following holds:
  - For three given angles, there are, depending on convention (fixed vs. rotating axis, rotation of object vs. reference system) four different rotation matrices which are built from elementary rotations around the three axes:

$$\mathbf{R}(\omega,\varphi,\kappa) := \mathbf{R}_3(\kappa)\mathbf{R}_2(\varphi)\mathbf{R}_1(\omega)$$
$$\mathbf{R}(-\omega,-\varphi,-\kappa) = \mathbf{R}_3(-\kappa)\mathbf{R}_2(-\varphi)\mathbf{R}_1(-\omega)$$
$$\mathbf{R}^{\mathrm{T}}(\omega,\varphi,\kappa) = \mathbf{R}_1(-\omega)\mathbf{R}_2(-\varphi)\mathbf{R}_3(-\kappa)$$
$$\mathbf{R}^{\mathrm{T}}(-\omega,-\varphi,-\kappa) = \mathbf{R}_1(\omega)\mathbf{R}_2(\varphi)\mathbf{R}_3(\kappa)$$

Notes on the concatenation of rotation matrices

- Due to differences in the used convention and the complexity of the expanded matrix, the actual matrices shown in text books (or on slides…) may be wrong → check on your own
- In practice, it is often useful to simply check the four variants (listed on the previous slide)

- There are further conventions regarding the combination of elementary matrices
  - Euler-angles around 3. and 1. axis (with rotating axes)

$$\mathbf{R}(\alpha,\beta,\gamma) := \mathbf{R}_3(\gamma)\mathbf{R}_1(\beta)\mathbf{R}_3(\alpha)$$

  - Euler-angles around 3. and 2. axis (with rotating axes)

$$\mathbf{R}(\alpha,\beta,\gamma) := \mathbf{R}_3(\gamma)\mathbf{R}_2(\beta)\mathbf{R}_3(\alpha)$$

Disadvantages of this representation

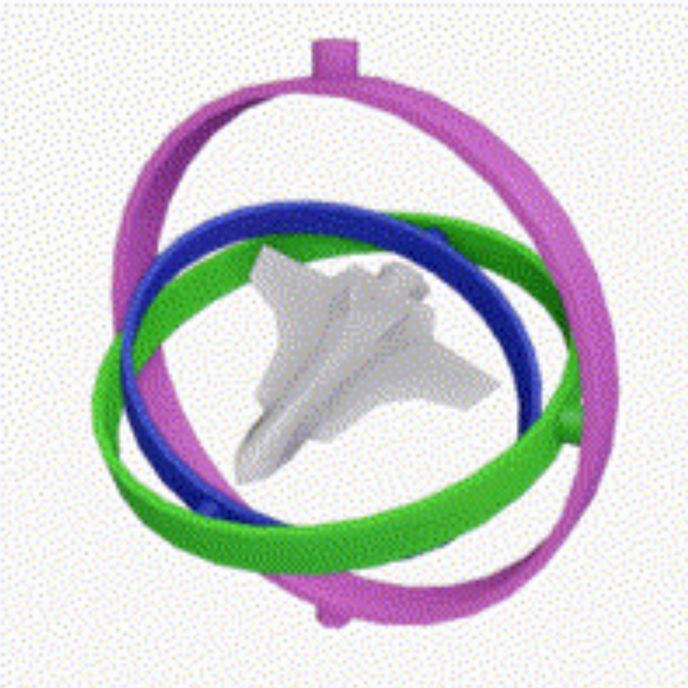
- No simple concatenation in terms of the three angles:

$$\mathbf{R}(\omega',\varphi',\kappa')\circ\mathbf{R}(\omega,\varphi,\kappa)\neq\mathbf{R}(\omega'+\omega,\varphi'+\varphi,\kappa'+\kappa)$$

- Rotation matrix:
  - Transcendent functions (sin, cos)
  - (Partial) derivatives yield complicated terms

Disadvantages of this representation

- Singularities in the mapping: **gimbal lock**
  - For certain angles, the matrix loses one degree of freedom
  - → It is *singular*
  - Mechanical gimbal: when two axes align
  - Then, there is one d.o.f. which cannot be mapped to a $(\omega, \varphi, \kappa)$ change → “lock”
  - For example, at $(0, \pm\frac{\pi}{2}, 0)$.



(Animation source: en.wikipedia.org)

Understanding gimbal lock

- Approximating the rotation matrix for small angles
  - …at $(\omega, \phi, \kappa) = (0,0,0)$
- This is the rotation matrix for small angles we got earlier

$$\mathbf{R}(0+\Delta\omega, 0+\Delta\varphi, 0+\Delta\kappa) = \begin{bmatrix} 1 & -\Delta\kappa & \Delta\varphi \\ \Delta\kappa & 1 & -\Delta\omega \\ -\Delta\varphi & \Delta\omega & 1 \end{bmatrix} + \text{h.o.t.}$$

- From this matrix, we see that
  - E.g. the vector $[1 \quad 0 \quad 0]^\mathrm{T}$ will map to $[1 \quad \Delta\kappa \quad -\Delta\varphi]^\mathrm{T}$
  - Any small rotation of this vector from $[1 \quad 0 \quad 0]^\mathrm{T}$ to $[1 \quad \Delta y \quad \Delta z]^\mathrm{T}$ can be achieved by $\Delta\kappa = \Delta y$ and $\Delta\varphi = -\Delta z$
- Similar arguments hold for $[0 \quad 1 \quad 0]^\mathrm{T}$ and $[0 \quad 0 \quad 1]^\mathrm{T}$.

Understanding gimbal lock

- Approximating the rotation matrix for small angles
  - **...now at** $(\omega, \phi, \kappa) = (0, \boldsymbol{\pi/2}, 0)$
  - (airplane nose pointing downwards)

$$\mathbf{R}\left(0+\Delta\omega, \frac{\pi}{2}+\Delta\varphi, 0+\Delta\kappa\right) = \begin{bmatrix} -\Delta\varphi & -(\Delta\kappa-\Delta\omega) & 1 \\ 0 & 1 & \Delta\kappa-\Delta\omega \\ -1 & 0 & -\Delta\varphi \end{bmatrix} + \text{h.o.t.}$$

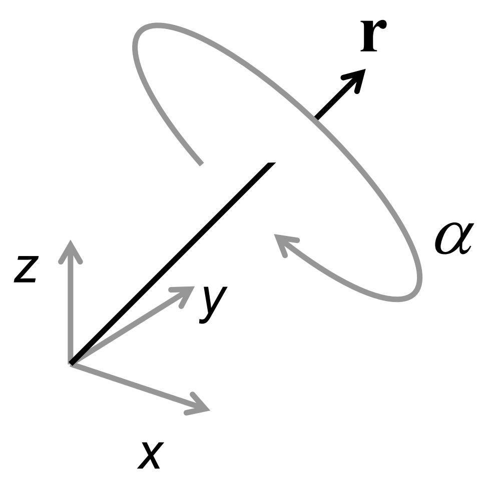
- From the first order matrix, we now see that
  - E.g. the vector $[1 \quad 0 \quad 0]^{\mathrm{T}}$ will map to $[-\Delta\varphi \quad 0 \quad -1]^{\mathrm{T}}$
  - A given small rotation from $[0 \quad 0 \quad -1]^{\mathrm{T}}$ to $[\Delta x \quad \Delta y \quad -1]^{\mathrm{T}}$ cannot be represented, since $\Delta y = 0$, independent of any choice of $(\Delta\omega, \Delta\varphi, \Delta\kappa)$
- The singularity can also be seen from this argument:
  - The matrix depends only on $\Delta\varphi$ (first degree of freedom) and the difference $(\Delta\kappa - \Delta\omega)$ (second degree of freedom).

Representation of rotation matrices by: rotation axis & rotation angle

Rotation: axis and angle

- Any rotation can be specified using a rotation axis and a rotation angle:
  - Rotation axis: unit vector

$$\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \qquad \|\mathbf{r}\| = 1$$



  - Rotation angle: α
  - Note: 3 degrees of freedom, direction (2), angle (1)

Rotation: axis and angle

- The resulting rotation matrix is given by the following formula:
  - $\mathbf{D_r}$ dyadic product, $\mathbf{S_r}$ skew-symmetric matrix

$$\boxed{\mathbf{R}(\mathbf{r},\alpha) \quad = \quad \cos\alpha\,\mathbf{I} + (1-\cos\alpha)\,\mathbf{D_r} + \sin\alpha\,\mathbf{S_r}}$$

  - Expanded:

$$\mathbf{R}(\mathbf{r},\alpha)$$

$$= \cos\alpha \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + (1-\cos\alpha) \begin{bmatrix} r_x^2 & r_x r_y & r_x r_z \\ r_x r_y & r_y^2 & r_y r_z \\ r_x r_z & r_y r_z & r_z^2 \end{bmatrix} + \sin\alpha \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

Rotation: axis and angle

- Is R a orthogonal matrix? To show: $\mathbf{R}^{\mathrm{T}}(\mathbf{r},\alpha)\mathbf{R}(\mathbf{r},\alpha)=\mathbf{I}$
  - Note:

$$\begin{aligned}\mathbf{R}(\mathbf{r},\alpha) &= \cos\alpha\,\mathbf{I}+(1-\cos\alpha)\,\mathbf{D_r}+\sin\alpha\,\mathbf{S_r}\\ \mathbf{R}^{\mathrm{T}}(\mathbf{r},\alpha) &= \cos\alpha\,\mathbf{I}+(1-\cos\alpha)\,\mathbf{D_r}-\sin\alpha\,\mathbf{S_r}\end{aligned}$$

  - Then,

$$\begin{aligned}\mathbf{R}^{\mathrm{T}}(\mathbf{r},\alpha)\mathbf{R}(\mathbf{r},\alpha) &= \left(\cos\alpha\,\mathbf{I}+(1-\cos\alpha)\mathbf{D_r}\right)^2-\left(\sin\alpha\,\mathbf{S_r}\right)^2\\ &= \cos^2\alpha\,\mathbf{I}+2\cos\alpha(1-\cos\alpha)\mathbf{D_r}\\ &\quad+(1-\cos\alpha)^2\mathbf{D_r^2}-\sin^2\alpha\,\mathbf{S_r^2}\\ &= \mathbf{I}\end{aligned}$$

  - obtained using the following identities:

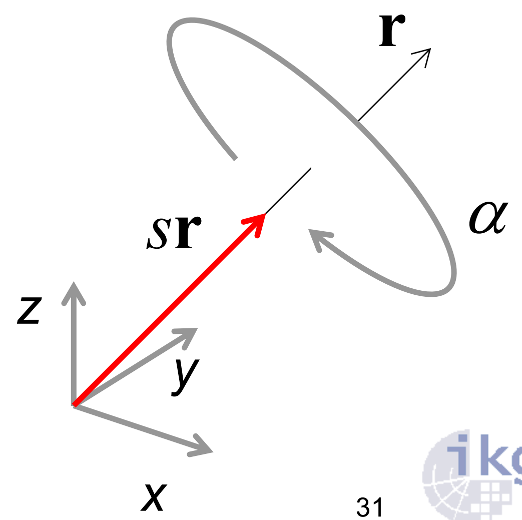
$$(a-b)(a+b)=a^2-b^2$$

$$\mathbf{D_r^2}=\mathbf{D_r}\qquad \mathbf{S_r^2}=\mathbf{D_r}-\mathbf{I}$$

Rotation: axis and angle

- Is R a rotation around the vector $\mathbf{r}$?
  - First step: R does not modify vectors parallel to $\mathbf{r}$

$$
\begin{aligned}
\mathbf{R}(\mathbf{r},\alpha)\cdot\mathbf{r} &= \left(\cos\alpha\,\mathbf{I} + (1-\cos\alpha)\,\mathbf{D}_\mathbf{r} + \sin\alpha\,\mathbf{S}_\mathbf{r}\right)\mathbf{r} \\
&= \left(\cos\alpha\,\mathbf{r} + (1-\cos\alpha)\mathbf{r}\underbrace{\mathbf{r}^\mathrm{T}\mathbf{r}}_{=1} + \sin\alpha\,\underbrace{\mathbf{S}_\mathbf{r}\mathbf{r}}_{=\mathbf{r}\times\mathbf{r}=\mathbf{0}}\right) \\
&= \mathbf{r}
\end{aligned}
$$



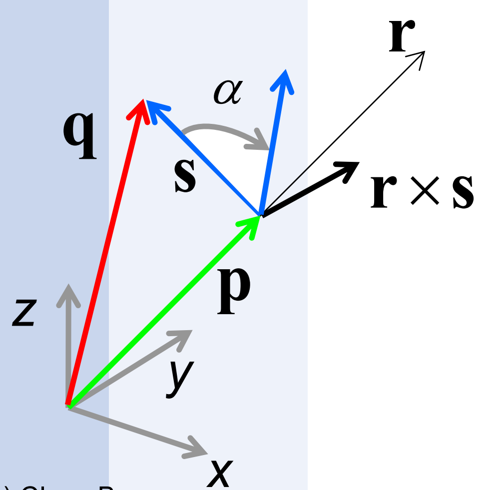
Rotation: axis and angle

- Is R a rotation around the vector $\mathbf{r}$?
  - Second step: now use an arbitrary position vector $\mathbf{q}$
  - Split into a component parallel and another perpendicular to $\mathbf{r}$:

$$\mathbf{q} = \mathbf{p} + \mathbf{s}$$

  - Then, it follows:

$$\begin{aligned}
\mathbf{R}(\mathbf{r},\alpha)\cdot\mathbf{q} &= \mathbf{R}(\mathbf{r},\alpha)\cdot(\mathbf{p}+\mathbf{s}) \\
&= \mathbf{p} + \mathbf{R}(\mathbf{r},\alpha)\mathbf{s} \\
&= \mathbf{p} + \left(\cos\alpha\,\mathbf{I} + (1-\cos\alpha)\,\mathbf{D}_{\mathbf{r}} + \sin\alpha\,\mathbf{S}_{\mathbf{r}}\right)\mathbf{s} \\
&= \mathbf{p} + \cos\alpha\,\mathbf{s} + (1-\cos\alpha)\mathbf{r}\underbrace{\mathbf{r}^{\mathrm{T}}\mathbf{s}}_{=0} + \sin\alpha\,\mathbf{S}_{\mathbf{r}}\mathbf{s} \\
&= \mathbf{p} + \cos\alpha\,\mathbf{s} + \sin\alpha\left(\mathbf{r}\times\mathbf{s}\right)
\end{aligned}$$



Rotation by α in the plane spanned by $\mathbf{s}$ and $\mathbf{r}\times\mathbf{s}$

Rotation: axis and angle

- Specifying rotations in the form “axis + angle” is common e.g. in computer graphics
- If the axis vector and the angle are both inverted, this leads to the same rotation:

$$\mathbf{R}(\mathbf{r},\alpha) = \mathbf{R}(-\mathbf{r},-\alpha)$$

Quaternions

Excursion: complex numbers

- A complex number z is given by its real and imaginary part:

$$z \in \mathrm{C} \qquad z = a + ib \qquad i = \sqrt{-1}$$

- Rules for complex arithmetic
- Especially multiplication: note that

$$i^2 = -1$$

- Thus:

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2 bd \\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

Excursion: complex numbers

- If complex numbers are identified with vectors in $\mathbb{R}^2$:

$$x = a + ib \quad \leftrightarrow \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad y = c + id \quad \leftrightarrow \quad \mathbf{y} = \begin{bmatrix} c \\ d \end{bmatrix}$$

- Then it follows for the multiplication:
  - "real part times identity plus skew-symmetric matrix of the imaginary part"

$$\begin{aligned} xy &= (ac - bd) + i(ad + bc) \\ &\leftrightarrow \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix} \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \\ &= \left(a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \right) \begin{bmatrix} c \\ d \end{bmatrix} \\ &= (a\mathbf{I} + \mathbf{S}_b) \begin{bmatrix} c \\ d \end{bmatrix} \end{aligned}$$

Quaternions

- Quaternions are 4-tuples
- Similar to complex numbers, there is a real part and a complex part
- The real part is a scalar, while the complex part is a 3-vector
- Instead of only i there are now i, j and k

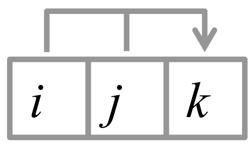
$$q = q_0 + q_1 i + q_2 j + q_3 k$$

- Scalar part: q_0
- Vector part: $(q_1, q_2, q_3)^{\mathrm{T}}$

Quaternions

- Arithmetic rules for i, j, k:

$$
\begin{array}{lll}
i \cdot i = -1 & i \cdot j = k & i \cdot k = -j \\
j \cdot i = -k & j \cdot j = -1 & j \cdot k = i \\
k \cdot i = j & k \cdot j = -i & k \cdot k = -1
\end{array}
$$



- Therefore, the multiplication of two quaternions leads to:

$$
\begin{aligned}
q \cdot r &= (q_0 + q_1 i + q_2 j + q_3 k) \cdot (r_0 + r_1 i + r_2 j + r_3 k) \\
&= (q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3) \\
&\quad + i(q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2) \\
&\quad + j(q_0 r_2 + q_2 r_0 + q_3 r_1 - q_1 r_3) \\
&\quad + k(q_0 r_3 + q_3 r_0 + q_1 r_2 - q_2 r_1)
\end{aligned}
$$

Quaternions

- Similar to complex numbers, quaternions can be identified with vectors (in this case, from $\mathbb{R}^4$)

$$q \quad \leftrightarrow \quad \begin{bmatrix} q_0 \\ \hline q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ \hline \mathbf{q} \end{bmatrix}$$

- Now, the multiplication can be expressed in terms of vector operations:

$$q \cdot r = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} \cdot \begin{bmatrix} r_0 \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} q_0 r_0 - \mathbf{q}^{\mathrm{T}} \mathbf{r} \\ r_0 \mathbf{q} + q_0 \mathbf{r} + \mathbf{q} \times \mathbf{r} \end{bmatrix}$$

Quaternions

- Fully expanded:

$$\begin{bmatrix} q_0 r_0 - \mathbf{q}^{\mathrm{T}}\mathbf{r} \\ \hline r_0\mathbf{q} + q_0\mathbf{r} + \mathbf{q}\times\mathbf{r} \end{bmatrix} = \begin{bmatrix} q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 \\ \hline r_0 q_1 + q_0 r_1 + q_2 r_3 - q_3 r_2 \\ r_0 q_2 + q_0 r_2 + q_3 r_1 - q_1 r_3 \\ r_0 q_3 + q_0 r_3 + q_1 r_2 - q_2 r_1 \end{bmatrix}$$

Quaternions

- Remember: cross-product can be written in terms of a product matrix x vector:

$$\mathbf{x} \times \mathbf{u} = \mathbf{S_x u}$$

- Similarly, the multiplication of quaternions can be written as a multiplication „matrix x vector“:

$$q \cdot r = \begin{bmatrix} q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 \\ r_0 q_1 + q_0 r_1 + q_2 r_3 - q_3 r_2 \\ r_0 q_2 + q_0 r_2 + q_3 r_1 - q_1 r_3 \\ r_0 q_3 + q_0 r_3 + q_1 r_2 - q_2 r_1 \end{bmatrix} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \cdot \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \mathbf{T}_q \begin{bmatrix} r_0 \\ \mathbf{r} \end{bmatrix}$$

Quaternions

- Alternatively, this result can be obtained as follows:

$$\begin{bmatrix} q_0 r_0 - \mathbf{q}^{\mathrm{T}}\mathbf{r} \\ r_0\mathbf{q} + q_0\mathbf{r} + \mathbf{q}\times\mathbf{r} \end{bmatrix} = \left[\begin{array}{c|c} q_0 & -\mathbf{q}^{\mathrm{T}} \\ \hline \mathbf{q} & q_0\mathbf{I} + \mathbf{S}_{\mathbf{q}} \end{array}\right] \cdot \begin{bmatrix} r_0 \\ \mathbf{r} \end{bmatrix}$$

$$\left[\begin{array}{c|ccc} q_0 & -q_1 & -q_2 & -q_3 \\ \hline q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{array}\right]$$

Quaternions

- Remember: we could either multiply $\mathbf{S}_x$ with **u,** or $\mathbf{S}_u$ with **x:**

$$\mathbf{x}\times\mathbf{u}=\mathbf{S}_{\mathbf{x}}\mathbf{u} \qquad \mathbf{x}\times\mathbf{u}=\mathbf{S}_{\mathbf{u}}^{\mathrm{T}}\mathbf{x}=-\mathbf{S}_{\mathbf{u}}\mathbf{x}$$

- This works here too, since:

$$\begin{bmatrix} q_0 r_0-\mathbf{q}^{\mathrm{T}}\mathbf{r} \\ r_0\mathbf{q}+q_0\mathbf{r}+\mathbf{q}\times\mathbf{r} \end{bmatrix}=\left[\begin{array}{c|c} r_0 & -\mathbf{r}^{\mathrm{T}} \\ \hline \mathbf{r} & r_0\mathbf{I}-\mathbf{S}_{\mathbf{r}} \end{array}\right]\cdot\begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix}=\hat{\mathbf{T}}_r\begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix}$$

- I.e., if q and r are interchanged, this is equivalent to the following matrix operation:

$$q\cdot r=\hat{\mathbf{T}}_r\begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix}$$

What do we know up to now?

- Quaternions
  - Notation $q = q_0 + q_1 i + q_2 j + q_3 k$
  - Or, identified with vectors in $\mathbb{R}^4$: $\begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^{\mathrm{T}}$
- The multiplication can be written as:

$$q \cdot r = \begin{bmatrix} q_0 r_0 - \mathbf{q}^{\mathrm{T}} \mathbf{r} \\ r_0 \mathbf{q} + q_0 \mathbf{r} + \mathbf{q} \times \mathbf{r} \end{bmatrix} = \mathbf{T}_q r = \hat{\mathbf{T}}_r q$$

- where

$$\mathbf{T}_q = \begin{bmatrix} q_0 & -\mathbf{q}^{\mathrm{T}} \\ \mathbf{q} & q_0 \mathbf{I} + \mathbf{S}_{\mathbf{q}} \end{bmatrix} \qquad \hat{\mathbf{T}}_r = \begin{bmatrix} r_0 & -\mathbf{r}^{\mathrm{T}} \\ \mathbf{r} & r_0 \mathbf{I} - \mathbf{S}_{\mathbf{r}} \end{bmatrix}$$

Further identities

- Norm: $|q|^2 := q_0^2 + q_1^2 + q_2^2 + q_3^2$
- Conjugated quaternion: $q^* = q_0 - q_1 i - q_2 j - q_3 k = [q_0 \quad -\mathbf{q}]^{\mathrm{T}}$
- Inverse: $q^{-1} = q^* / |q|^2$

- Neutral (identity) element: 1 or: $[1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$

- Conjugation of a product:

$$(rq)^* = q^* r^*$$

- From this it follows also that:

$$(rq)^{-1} = q^{-1} r^{-1}$$

E.g. proof for the inverse of a quaternion

$$q \cdot q^{-1} = q \cdot \frac{1}{|q|^2} \cdot q^* = \frac{1}{|q|^2} qq^* = \frac{1}{|q|^2} \begin{bmatrix} q_0 q_0 - \mathbf{q}^\mathrm{T}(-\mathbf{q}) \\ q_0\mathbf{q} + q_0(-\mathbf{q}) - \mathbf{q} \times \mathbf{q} \end{bmatrix}$$

$$= \frac{1}{|q|^2} \begin{bmatrix} |q|^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(this is the neutral element)

$$q \cdot r = \begin{bmatrix} q_0 r_0 - \mathbf{q}^\mathrm{T}\mathbf{r} \\ r_0\mathbf{q} + q_0\mathbf{r} + \mathbf{q} \times \mathbf{r} \end{bmatrix}$$

...what has all of this to do with rotations?

- Multiplying a quaternion p, from the left by q and from the right by the inverse of q...

$$\boxed{p' = qpq^{-1}}$$

- ...rotates the vector part of p
- ...and leaves the scalar part of p as is.
- Therefore, the quaternion q represents a rotation.

- Why? And which rotation?

Quaternions as rotations

- Proof: compute the transformation in terms of a matrix operation:

$$
\begin{aligned}
qpq^{-1} &= \left(\mathbf{T}_q p\right)q^{-1} = \frac{1}{|q|^2}\left(\mathbf{T}_q p\right)q^{*} \\
&= \frac{1}{|q|^2}\hat{\mathbf{T}}_{q^*}\left(\mathbf{T}_q p\right) = \left(\frac{1}{|q|^2}\hat{\mathbf{T}}_{q^*}\mathbf{T}_q\right)p
\end{aligned}
$$

- To proof now: the following 4x4 matrix leaves the scalar part constant and rotates the vector part:

$$
\frac{1}{|q|^2}\hat{\mathbf{T}}_{q^*}\mathbf{T}_q
$$

Quaternions as rotations

- First observe
  - The norm of q is not relevant, since it cancels out:

$$p' = qpq^{-1} = qp\frac{1}{|q|^2}q^*$$

  - Therefore, without loss of generality, one can assume a unit quaternion,

$$|q|^2 = 1$$

  - Therefore, what needs to be shown is the property from the previous slide for:

$$\hat{\mathbf{T}}_{q^*}\mathbf{T}_q$$

Quaternions as rotations

- Just compute this matrix:

$$
\begin{aligned}
\hat{\mathbf{T}}_{q^*}\mathbf{T}_q &= \begin{bmatrix} q_0 & \mathbf{q}^{\mathrm{T}} \\ -\mathbf{q} & q_0\mathbf{I}+\mathbf{S}_{\mathbf{q}} \end{bmatrix}\begin{bmatrix} q_0 & -\mathbf{q}^{\mathrm{T}} \\ \mathbf{q} & q_0\mathbf{I}+\mathbf{S}_{\mathbf{q}} \end{bmatrix} \\
&= \begin{bmatrix} q_0^2+\mathbf{q}^{\mathrm{T}}\mathbf{q} & -q_0\mathbf{q}^{\mathrm{T}}+q_0\mathbf{q}^{\mathrm{T}}+\mathbf{q}^{\mathrm{T}}\mathbf{S}_{\mathbf{q}} \\ -q_0\mathbf{q}^{\mathrm{T}}+q_0\mathbf{q}^{\mathrm{T}}+\mathbf{S}_{\mathbf{q}}\mathbf{q} & \mathbf{q}\mathbf{q}^{\mathrm{T}}+\left(q_0\mathbf{I}+\mathbf{S}_{\mathbf{q}}\right)^2 \end{bmatrix} \\
&= \left[\begin{array}{c|c} 1 & \mathbf{0}^{\mathrm{T}} \\ \hline \mathbf{0} & \mathbf{q}\mathbf{q}^{\mathrm{T}}+\left(q_0\mathbf{I}+\mathbf{S}_{\mathbf{q}}\right)^2 \end{array}\right]
\end{aligned}
$$

- First result: the scalar part remains unchanged – OK
- Still to prove: lower right 3x3 sub-matrix is a rotation matrix:

$$\mathbf{q}\mathbf{q}^{\mathrm{T}}+\left(q_0\mathbf{I}+\mathbf{S}_{\mathbf{q}}\right)^2$$

Quaternions as rotations

- Observe the following: any unit quaternion can be written as*:

$$q = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \cos\alpha/2 \\ \sin\alpha/2 \cdot \mathbf{r} \end{bmatrix}$$

- ...where r is a unit vector from $\mathbb{R}^3$.

* this is actually true for any 4-vector of length 1.

Quaternions as rotations

$$q = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \cos\alpha/2 \\ \sin\alpha/2 \cdot \mathbf{r} \end{bmatrix}$$

- ...now we substitute this unit quaternion into our matrix:

$$\mathbf{S}_{\mathbf{r}}^2 = \mathbf{r}\mathbf{r}^{\mathrm{T}} - \mathbf{I}$$

$$\begin{aligned}
&\mathbf{q}\mathbf{q}^{\mathrm{T}} + \left(q_0\mathbf{I} + \mathbf{S}_{\mathbf{q}}\right)^2 = \\
= \;& \sin^2(\alpha/2)\mathbf{r}\mathbf{r}^{\mathrm{T}} + \left(\cos(\alpha/2)\mathbf{I} + \sin(\alpha/2)\mathbf{S}_{\mathbf{r}}\right)^2 \\
= \;& \sin^2(\alpha/2)\mathbf{r}\mathbf{r}^{\mathrm{T}} + \cos^2(\alpha/2)\mathbf{I} + 2\sin(\alpha/2)\cos(\alpha/2)\mathbf{S}_{\mathbf{r}} + \sin^2(\alpha/2)\mathbf{S}_{\mathbf{r}}^2 \\
= \;& \underbrace{\left(\cos^2(\alpha/2) - \sin^2(\alpha/2)\right)}_{=\cos\alpha}\mathbf{I} + \underbrace{2\sin^2(\alpha/2)}_{1-\cos\alpha}\mathbf{r}\mathbf{r}^{\mathrm{T}} + \underbrace{2\sin(\alpha/2)\cos(\alpha/2)}_{\sin\alpha}\mathbf{S}_{\mathbf{r}} \\
= \;& \cos\alpha\,\mathbf{I} + \left(1 - \cos\alpha\right)\mathbf{D}_{\mathbf{r}} + \sin\alpha\,\mathbf{S}_{\mathbf{r}}
\end{aligned}$$

- ...as it turns out: this is exactly the formula for a rotation given by “axis & angle”!

$$\boxed{\mathbf{R}(\mathbf{r},\alpha) \quad = \quad \cos\alpha\,\mathbf{I} + (1-\cos\alpha)\,\mathbf{D}_{\mathbf{r}} + \sin\alpha\,\mathbf{S}_{\mathbf{r}}}$$

Conclusions

- Any quaternion *q* can be viewed as the representation of a rotation
- The rotation can be specified by:

$$p' = qpq^{-1}$$

  which leaves the scalar part of *p* unchanged and rotates the vector part
- Multiplying *q* by a scalar $s \neq 0$ does not change the represented rotation $q \cong sq$
- Therefore, without loss of generality a unit quaternion can be used for *q*
- *q* can be interpreted as a rotation around the axis **r** by the angle α :

$$q = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \cos \alpha/2 \\ \sin \alpha/2 \cdot \mathbf{r} \end{bmatrix}$$

Advantages

- The representation of rotations by quaternions is unique (except for the sign) and nonsingular
- The representation in the form of a matrix does not need any transcendental functions
- In expanded form, this is:

$$\frac{1}{|q|^2}\left\{\mathbf{q}\mathbf{q}^{\mathrm{T}} + \left(q_0\mathbf{I} + \mathbf{S}_{\mathbf{q}}\right)^2\right\} =$$

$$= \frac{1}{q_0^2 + q_1^2 + q_2^2 + q_3^2}\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_2 q_1 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_3 q_1 - q_0 q_2) & 2(q_3 q_2 + q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

(Orthogonal matrix)

Advantages

- Concatenation
  - The concatenation of two rotations can be computed directly:

$$p' = r\left(qpq^{-1}\right)r^{-1} = \left(rq\right)p\left(q^{-1}r^{-1}\right) = \left(rq\right)p\left(rq\right)^{-1}$$

  - That is, applying the rotation r after the rotation q is identical to applying the (quaternion) product of r and q,

$$rq$$

Numerical example

- Rotation using the angle α around the z-axis $\mathbf{r} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$
- Substitute:

$$q = \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2)\cdot \mathbf{r} \end{bmatrix} = \begin{bmatrix} \cos(\alpha/2) \\ 0 \\ 0 \\ \sin(\alpha/2) \end{bmatrix}$$

into:

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_2q_1+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_3q_1-q_0q_2) & 2(q_3q_2+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

- Using $c := \cos(\alpha/2)$, $s := \sin(\alpha/2)$ it follows:

$$\begin{bmatrix} c^2-s^2 & -2cs & 0 \\ 2cs & c^2-s^2 & 0 \\ 0 & 0 & c^2+s^2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Where the addition theorems were used:

$$\cos^2(\alpha/2) - \sin^2(\alpha/2) = \cos\alpha$$

$$2\sin(\alpha/2)\cos(\alpha/2) = \sin\alpha$$

OK, well-known rotation matrix!

Reference

- Manual of Photogrammetry, 5th ed., ASPRS