Calculus

HW3:Understanding Limit/Continuity and Crystal Growth Limit Obsrvation

組員：

411485018 蘇星丞

411485002 楊昕展

411485003 胡庭睿

411485042 黃柏崴

Basic Concepts for Limits

Q1

Evaluate the limit and justify each step by indicating the appropriate Limit Law(s)

1. $\lim_{x\to -3}(2x^3+6x^2-9)$
2. $\lim_{x\to 3}\sqrt[3]{x+5}(2x^2-3x)$

Ans1

1.

$$\begin{aligned}
&\lim_{x\to -3}(2x^3+6x^2-9) \\
&= \lim_{x\to -3}(2x^3) + \lim_{x\to -3}(6x^2) - \lim_{x\to -3}(9) && \text{(Sum/Difference Law)} \\
&= 2\lim_{x\to -3}(x^3) + 6\lim_{x\to -3}(x^2) - \lim_{x\to -3}(9) && \text{(Constant Multiple Law)} \\
&= 2(\lim_{x\to -3} x)^3 + 6(\lim_{x\to -3} x)^2 - (\lim_{x\to -3} 9) && \text{(Power Law)} \\
&= 2(-3)^3 + 6(-3)^2 - 9 && \text{(Appendix A 8,10)} \\
&= 2(-27) + 6(9) - 9 \\
&= -54 + 54 - 9 \\
&= -9
\end{aligned}$$

2.

$$\begin{aligned}
&\lim_{x\to 3}\sqrt[3]{x+5}(2x^2-3x) \\
&= \sqrt[3]{\lim_{x\to 3}(x+5)} \times \lim_{x\to 3}(2x^2-3x) && \text{(Root / Product Law)} \\
&= \sqrt[3]{\lim_{x\to 3}(x+5)} \times (2\lim_{x\to 3} x^2 - 3\lim_{x\to 3} x) && \text{(Sum / Difference / Constant Multiple Law)} \\
&= \sqrt[3]{8} \times (2\times 3^2 - 3\times 3) && \text{(Power Law)} \\
&= 2\times 9 \\
&= 18
\end{aligned}$$

Q2

Evaluate the limit, if it exists.

1. $\lim_{x\to -2}(3x-7)$
2. $\lim_{t^2-2t-8}(t-4)$
3. $\lim_{x\to 2}\frac{2-x}{\sqrt{x+2}-2}$
4. $\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}$

Ans2

1. Because $3x-7$ is a continuous function, which means $\lim_{x\to 2} 3x-7 = 3\times 2 - 7(\text{continuous definition:} \lim_{x\to a} f(x) = f(a))$,so we can find out that the limit is -1

2. The limit expression is mathematically ill-defined. The notation below $\lim$,which is $t^2 - 2t - 8$,is not a valid convergence condition. It fails to specify the value that the variable t is approaching (ex:$t \to 4$). Without this "target point," the limit cannot be calculated, so this problem has no solution.
3. We cannot apply Quotient Law because the limit of the denominator is 0, so we need to perform algebraic simplification on the expression first.

$$\begin{aligned}
&\lim_{x\to 2} \frac{2-x}{\sqrt{x+2}-2} \times \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} \\
&= \lim_{x\to 2} \frac{(2-x)(\sqrt{x+2}+2)}{(\sqrt{x+2})^2-4} \\
&= \lim_{x\to 2} \frac{(2-x)(\sqrt{x+2}+2)}{x+2-4} \\
&= \lim_{x\to 2} \frac{(2-x)(\sqrt{x+2}+2)}{x-2} \\
&= \lim_{x\to 2} \frac{-(x-2)(\sqrt{x+2}+2)}{x-2} \qquad \text{(Cancel x-2)} \\
&= \lim_{x\to 2} -(\sqrt{x+2}+2) \\
&= -(\sqrt{2+2}+2) \\
&= -(2+2) \\
&= -4
\end{aligned}$$

4. Same as the last question, because the limit of the denominator is 0, so we need to perform algebraic simplification on the expression first.

$$\begin{aligned}
&\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} \times \frac{x+3}{x+3} \\
&= \lim_{x\to 3} \frac{\frac{3}{x}-\frac{x}{3}}{x^2-9} \\
&= \lim_{x\to 3} \frac{\frac{1}{3x}(9-x^2)}{x^2-9} \qquad \text{(Cancel } x^2-9\text{)} \\
&= \lim_{x\to 3} \frac{1}{3x} \times -1 \\
&= \frac{1}{9} \times -1 \\
&= \frac{-1}{9}
\end{aligned}$$

Q3

Use Squeeze Theeorom to show that

$$\lim_{x\to 0} \sqrt{x^3+x^2} \sin\frac{\pi}{x}$$

Ans3

$$-1 \le \sin\frac{\pi}{x} \le 1$$

$$-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le \sqrt{x^3+x^2}$$

$$\lim_{x\to 0} -\sqrt{x^3+x^2} \le \lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le \lim_{x\to 0} \sqrt{x^3+x^2}$$

$$0 \le \lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le 0$$

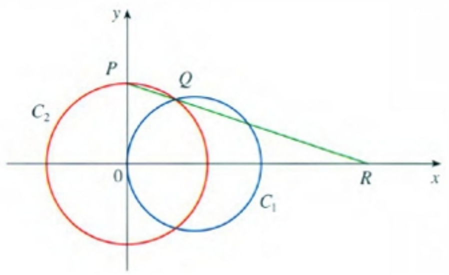
$$\lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x} = 0$$

Q4

The figure shows a fixed circle C_1 with equation $(x-1)^2+y^2=1$ and a shrinking circle C_2 with radius r and centering the origin

- Point $P(0,r)$
- Point Q is the upper intersection point of the two circles
- Point R is where line PQ intersects the x-axis

What happens to point R as circle C_2 shrinks (as $r \to 0^+$)?



Ans4

$$\begin{aligned}
&x^2+y^2=r^2 \qquad (1)\\
&(x-1)^2+y^2=1 \qquad (2)\\
&(1)\text{ - }(2)\text{ cancel } y^2\\
&=2x-1=r^2-1\\
&x=\frac{r^2}{2} \qquad y=\frac{r^2}{2}\\
&Q(x,y)=(\frac{r^2}{2},\frac{r^2}{2})\\
&P(0,r)
\end{aligned}$$

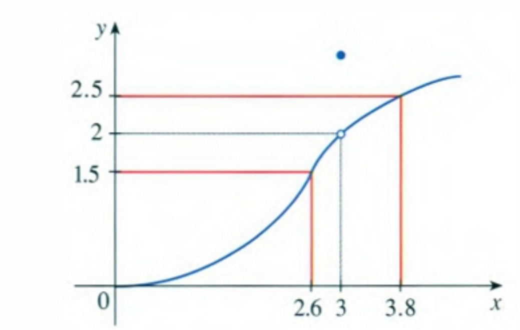
Find the slope $m_{\overline{PQ}} = 1-\frac{2}{r}$
Use P to find the equation of the line $L_{\overline{PQ}} = (1-\frac{2}{r})x = (y-r)$
so R is fix on x axias and to find R we assume $R=R(k,0)$
plug in line L which gives us the equation equation$(1-\frac{2}{r})k=-r$ then simplify the equation and we will get the new equation $k=-\frac{r^2}{r-2}$

$$\lim_{r \to 0^+} k = \lim_{r \to 0^+} -\frac{r^2}{r-2} = 0$$

so as $r \to 0^+$ $\quad R(k,0) \to (0,0)$

Q5

Use the given graph of f to find a number δ such that: $0 < |x-3| < \delta \implies |f(x)-2| < 0.5$



Ans5

$|f(x)-2| < 0.5$
$1.5 < f(x) < 2.5$
$f(x) = 1.5 \text{ as } x = 2.6$
$f(x) = 2.5 \text{ as } x = 3.8$
so when $x \in (2.6, 3.8)$ then the inequality $|f(x)-2| < 0.5$ will be satisfied
Given the inequality $0 < |x-3| < \delta$, the value of δ must ensure that the condition $|f(x)-2| < 0.5$ is s

Q6

Ans6

Q7

Ans7

Q8

Ans8

Basic Concepts for Continuity

Q1

Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a+h) = f(a)$$

Ans1

Prove: f is continuous at $a \iff \lim_{h \to 0} f(a+h) = f(a)$

From class's slides, we know that the definition of continuous is $\lim_{x \to a} f(x) = f(a)$

Part 1 : Foward direction's proof

Assume f is continuous at a, we know by the definition of continuous $\lim_{x \to a} f(x) = f(a)$.

Define a variable h , and set $x = a + h$, when $x \to a$, $h \to 0$

Now we can replace x with $a + h$ and the limit condition $x \to a$ with $h \to 0$

$$\lim_{h \to 0} f(a+h) = f(a)$$

From above, it have proven that if f is continuous at a, then the given limit expression holds true.

Part 2 : Reverse direction's proof

Assume $\lim_{h \to 0} f(a+h) = f(a)$, and we want to prove f is continuous at a

According to above, we define $x = a + h$, which means $h = x - a$

As $h \to 0$, the value of $x(\mathrm{a+h})$ must approach $a + 0$, so we can know as $h \to 0$,then $x \to a$

Now we can replace h with $x - a$ and the limit condition $h \to 0$ with $x \to a$

$$\lim_{x \to a} f(a + (x - a)) = f(a)$$

After simplify:

$$\lim_{x \to a} f(x) = f(a)$$

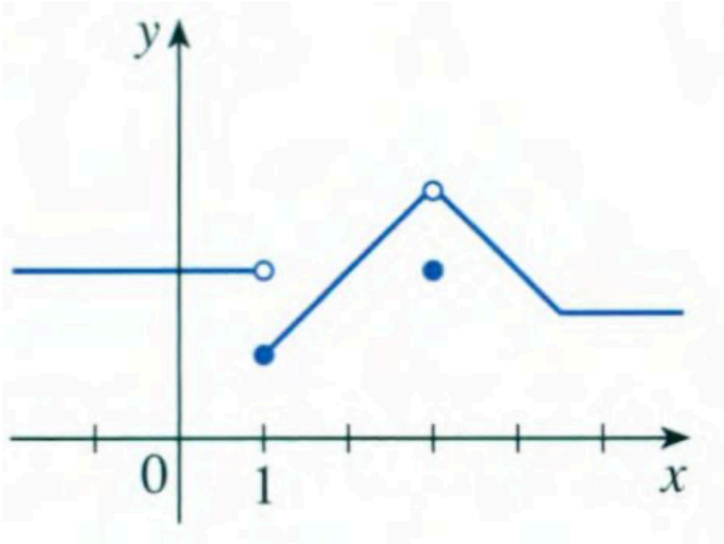
From above, it have proven that if $\lim_{h \to 0} f(a+h) = f(a)$ is true, f is continuous at a.

Conclusion

Since we have proven the statement in both directions , we have successfully shown that a function f is continuous at a if and only if $\lim_{h \to 0} f(a+h) = f(a)$.

Q2

The graph of a function f is given.



1. At what numbers a does $\lim_{x \to a} f(x)$ not exist?
2. At what numbers a is f not continuous?
3. At what numbers a does $\lim_{x \to a} f(x)$ exist but f is not continuous at a?

Ans2

- The criteria for limits, continuity, and differentiability can be found in Appendix B

1. $x = 1$, because $lim_{x \to 1-} f(x) \neq lim_{x \to 1+} f(x)$
2. $x = 3$, because $\lim_{x \to 3} f(x) \neq f(3)$
3. $x = 3$, according to Appendix B, although left-hand limit equals to right-hand limit and both exist(conditions for the existence of a limit), $\lim_{x \to 3} f(x) \neq f(3)$, from above we can know that f is not continuous at a.

Q3

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

Ans3

Q4

Ans4

Crystal Growth Inspection

?.py

?2.py

Work Division

學號/姓名	分配項目（寫）	分配項目（檢查）
411485002 楊昕展		
411485003 胡庭睿	part 1:1,2 part 2:1,2 Appendix A,B	
411485018 蘇星丞		
411485042 黃柏崴		

Challenges and Difficulties

此處由每位成員報告在擔任實作者或審核者時遇到的挑戰或未解決的問題，包含未來工作

姓名： 胡庭睿

遇到的困難與挑戰：

1. markdown轉pdf
   在上次作業中使用的extension(markdown pdf)，在轉換多行latex公式時(ex: \begin{aligned})會無法辨別，於是我在網上尋找許多不同方法，但是轉換的效果都不令我滿意，pandoc雖然能夠很好的渲染出latex公式，但是圖片的位置卻無法置中，換了很多個latex engine依然無解(pdflatex, mactex, xetex,basictex)再加上標題頁是用html寫的，也意味著標題頁需要重寫，但是我卻一直無法將它修改到我想要的樣式，最後我找到了一個叫做markdown preview enhanced的extension,對latex公式有很好的支援度，字體以及版面等也都可以透過.less控制，有不錯的自訂度，只是文檔寫的不太詳細，花了我一點時間才解決，不過我也學到了許多。
2. latex撰寫作業
   經過作業一的批改後，我更了解老師的評分標準，應該更詳細的列出思考以及計算過程，也代表用latex撰寫的佔比會更高，我學到了要打多行運算式時，可以用\begin{aligned} 和 \end{aligned}，用\begin{align}會讓每一行運算式後有編號，並且用&可以將算式對齊，
3. 全英文撰寫作業

姓名：

遇到的困難與挑戰：

姓名：

遇到的困難與挑戰：

姓名：

遇到的困難與挑戰：

Meeting Records

會議日期	會議方式 (線上/實體)	討論事項
[日期]	[方式]	1. 作業要求釐清 2. 進度規劃 3. 問題整合 4. 角色分配討論
...	...	...
...	...	...

Working Hours Records

組員姓名	工作時數	工作項目	工時高/低原因分析 (Bonus)
胡庭睿	15hr	1. part 1:1,2 2. part 2:1,2 3. Appendix A,B 4. the homework template 5. git and github tutorial 6. find ways of converting markdown to pdf	
[姓名2]	[時數]	1. [工作項目1] 2. [工作項目2]	[自行分析原因]
...	...	...	...

Reflection

學號：

姓名：

心得：

（應包含在作業中學習到的知識點，個人和小組整題的學習，工作風格的反思，總結在此次作業中遇到的挑戰和問題 (work hours, division)）

學號：

姓名：

心得：

學號：

姓名：

心得：

學號：

姓名：

心得：

Appendix

Appendix A

Limit Laws in class's slides:

1. Sum Law $(\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x))$
2. Difference Law $(\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x))$
3. Constant Multiple Law $(\lim_{x\to a} cf(x) = c \times \lim_{x\to a} f(x))$
4. Product Law $(\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x))$
5. Quotient Law $(\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{if } \lim_{x\to a} g(x) \neq 0)$
6. Power Law $(\lim_{x\to a}(f(x)^n)) = (\lim_{x\to a} f(x))^n)$
7. Root Law $(\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)})$
8. $\lim_{x\to a} c = c$ (c為常數)
9. $\lim_{x\to a} x = a$
10. $\lim_{x\to a} x^n = a^n$
11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

Appendix B

極限存在條件

1. $\lim_{x\to a-} f(x)$ 存在
2. $\lim_{x\to a+} f(x)$ 存在
3. $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$

在點**a**連續條件

1. $f(a)$ 存在

2. $\lim_{x \to a} f(x)$ 存在
3. $\lim_{x \to a} f(x) = f(a)$

在點**a**可微分條件

1. $f(a)$ 存在
2. $f(x)$ is continuous at a
3. $\lim_{x \to a-} \frac{f(x)-f(a)}{x-a}$ 存在
4. $\lim_{x \to a+} \frac{f(x)-f(a)}{x-a}$ 存在
5. $\lim_{x \to a-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a+} \frac{f(x)-f(a)}{x-a}$

三者關係

可微分一定連續，若連續，極限一定存在

Appendix C