

Calculus

HW3: Understanding Limit/Continuity and Crystal Growth
Limit Observation

組員：

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Basic Concepts for Limits

Q1

Evaluate the limit and justify each step by indicating the appropriate Limit Law(s)

1. $\lim_{x \rightarrow -3} (2x^3 + 6x^2 - 9)$
2. $\lim_{x \rightarrow 3} \sqrt[3]{x+5}(2x^2 - 3x)$

Ans1

1.

$$\begin{aligned} & \lim_{x \rightarrow -3} (2x^3 + 6x^2 - 9) \\ &= \lim_{x \rightarrow -3} (2x^3) + \lim_{x \rightarrow -3} (6x^2) - \lim_{x \rightarrow -3} (9) \quad (\text{Sum/Difference Law}) \\ &= 2 \lim_{x \rightarrow -3} (x^3) + 6 \lim_{x \rightarrow -3} (x^2) - \lim_{x \rightarrow -3} (9) \quad (\text{Constant Multiple Law}) \\ &= 2 \left(\lim_{x \rightarrow -3} x \right)^3 + 6 \left(\lim_{x \rightarrow -3} x \right)^2 - \left(\lim_{x \rightarrow -3} 9 \right) \quad (\text{Power Law}) \\ &= 2(-3)^3 + 6(-3)^2 - 9 \quad (\text{Appendix A 8,10}) \\ &= 2(-27) + 6(9) - 9 \\ &= -54 + 54 - 9 \\ &= -9 \end{aligned}$$

2.

$$\begin{aligned} & \lim_{x \rightarrow 3} \sqrt[3]{x+5}(2x^2 - 3x) \\ &= \sqrt[3]{\lim_{x \rightarrow 3} (x+5)} \times \lim_{x \rightarrow 3} (2x^2 - 3x) \quad (\text{Root / Product Law}) \\ &= \sqrt[3]{\lim_{x \rightarrow 3} (x+5)} \times (2 \lim_{x \rightarrow 3} x^2 - 3 \lim_{x \rightarrow 3} x) \quad (\text{Sum / Difference / Constant Multiple Law}) \\ &= \sqrt[3]{8} \times (2 \times 3^2 - 3 \times 3) \quad (\text{Power Law}) \\ &= 2 \times 9 \\ &= 18 \end{aligned}$$

Q2

Evaluate the limit, if it exists.

1. $\lim_{x \rightarrow -2} (3x - 7)$
2. $\lim_{t^2 - 2t - 8} (t - 4)$
3. $\lim_{x \rightarrow 2} \frac{2-x}{\sqrt{x+2}-2}$
4. $\lim_{x \rightarrow 3} \frac{x-3}{x-3}$

Ans2

1. Because $3x - 7$ is a continuous function, which means $\lim_{x \rightarrow 2} 3x - 7 = 3 \times 2 - 7$ (continuous definition: $\lim_{x \rightarrow a} f(x) = f(a)$), so we can find out that the limit is -1

2. The limit expression is mathematically ill-defined. The notation below \lim , which is $t^2 - 2t - 8$, is not a valid convergence condition. It fails to specify the value that the variable t is approaching (ex: $t \rightarrow 4$). Without this "target point," the limit cannot be calculated, so this problem has no solution.
3. We cannot apply Quotient Law because the limit of the denominator is 0, so we need to perform algebraic simplification on the expression first.

$$\begin{aligned}
 & \lim_{x \rightarrow 2} \frac{2-x}{\sqrt{x+2}-2} \times \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} \\
 &= \lim_{x \rightarrow 2} \frac{(2-x)(\sqrt{x+2}+2)}{(\sqrt{x+2})^2 - 4} \\
 &= \lim_{x \rightarrow 2} \frac{(2-x)(\sqrt{x+2}+2)}{x+2-4} \\
 &= \lim_{x \rightarrow 2} \frac{(2-x)(\sqrt{x+2}+2)}{x-2} \\
 &= \lim_{x \rightarrow 2} \frac{-(x-2)(\sqrt{x+2}+2)}{x-2} \quad (\text{Cancel } x-2) \\
 &= \lim_{x \rightarrow 2} -(\sqrt{x+2}+2) \\
 &= -(\sqrt{2+2}+2) \\
 &= -(2+2) \\
 &= -4
 \end{aligned}$$

4. Same as the last question, because the limit of the denominator is 0, so we need to perform algebraic simplification on the expression first.

$$\begin{aligned}
 & \lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3} \times \frac{x+3}{x+3} \\
 &= \lim_{x \rightarrow 3} \frac{\frac{\frac{3}{x} - \frac{x}{3}}{x^2 - 9}}{x^2 - 9} \\
 &= \lim_{x \rightarrow 3} \frac{\frac{1}{3x}(9-x^2)}{x^2 - 9} \quad (\text{Cancel } x^2 - 9) \\
 &= \lim_{x \rightarrow 3} \frac{1}{3x} \times -1 \\
 &= \frac{1}{9} \times -1 \\
 &= \frac{-1}{9}
 \end{aligned}$$

Q3

Use Squeeze Theorem to show that

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x}$$

Ans3

$$-1 \leq \sin \frac{\pi}{x} \leq 1$$

$$-\sqrt{x^3 + x^2} \leq \sqrt{x^3 + x^2} \sin \frac{\pi}{x} \leq \sqrt{x^3 + x^2}$$

$$\lim_{x \rightarrow 0} -\sqrt{x^3 + x^2} \leq \lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} \leq \lim_{x \rightarrow 0} \sqrt{x^3 + x^2}$$

$$0 \leq \lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} \leq 0$$

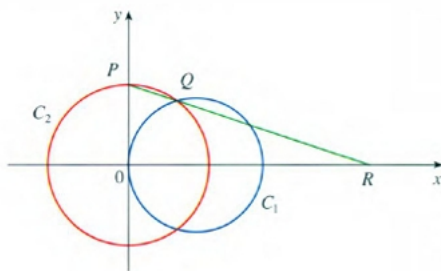
$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Q4

The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and centering the origin

- Point $P(0, r)$
- Point Q is the upper intersection point of the two circles
- Point R is where line PQ intersects the x-axis

What happens to point R as circle C_2 shrinks (as $r \rightarrow 0^+$)?



Ans4

$$x^2 + y^2 = r^2 \quad (1)$$

$$(x - 1)^2 + y^2 = 1 \quad (2)$$

$$(1) - (2) \text{ cancel } y^2$$

$$= 2x - 1 = r^2 - 1$$

$$x = \frac{r^2}{2} \quad y = \frac{r^2}{2}$$

$$Q(x, y) = \left(\frac{r^2}{2}, \frac{r^2}{2}\right)$$

$$P(0, r)$$

Find the slope $m_{\overline{PQ}} = 1 - \frac{2}{r}$

Use P to find the equation of the line $L_{\overline{PQ}} = (1 - \frac{2}{r})x = (y - r)$

so R is fix on x axis and to find R we assume $R = R(k, 0)$

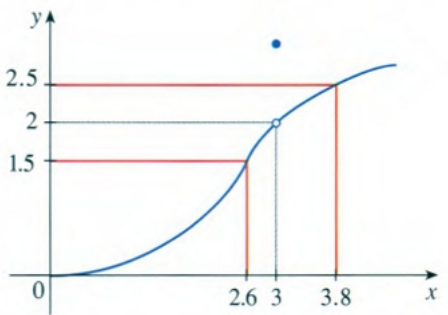
plug in line L which gives us the equation $(1 - \frac{2}{r})k = -r$ then simplify the equation and we will get the new equation $k = -\frac{r^2}{r-2}$

$$\lim_{r \rightarrow 0^+} k = \lim_{r \rightarrow 0^+} -\frac{r^2}{r-2} = 0$$

so as $r \rightarrow 0^+$ $R(k, 0) \rightarrow (0, 0)$

Q5

Use the given graph of f to find a number δ such that: $0 < |x - 3| < \delta \implies |f(x) - 2| < 0.5$



Ans5

$$|f(x) - 2| < 0.5$$

$$1.5 < f(x) < 2.5$$

$$f(x) = 1.5 \text{ as } x = 2.6$$

$$f(x) = 2.5 \text{ as } x = 3.8$$

so when $x \in (2.6, 3.8)$ then the inequality $|f(x) - 2| < 0.5$ will be satisfied

Given the inequality $0 < |x - 3| < \delta$, the value of δ must ensure that the condition $|f(x) - 2| < 0.5$ is :

Q6

Ans6

Q7

Ans7

Q8

Ans8

Basic Concepts for Continuity

Q1

Prove that f is continuous at a if and only if

$$\lim_{h \rightarrow 0} f(a + h) = f(a)$$

Ans1

Prove: f is continuous at $a \iff \lim_{h \rightarrow 0} f(a + h) = f(a)$

From class's slides, we know that the definition of continuous is $\lim_{x \rightarrow a} f(x) = f(a)$

Part 1 : Forward direction's proof

Assume f is continuous at a , we know by the definition of continuous $\lim_{x \rightarrow a} f(x) = f(a)$.

Define a variable h , and set $x = a + h$, when $x \rightarrow a$, $h \rightarrow 0$

Now we can replace x with $a + h$ and the limit condition $x \rightarrow a$ with $h \rightarrow 0$

$$\lim_{h \rightarrow 0} f(a + h) = f(a)$$

From above, it have proven that if f is continuous at a , then the given limit expression holds true.

Part 2 : Reverse direction's proof

Assume $\lim_{h \rightarrow 0} f(a + h) = f(a)$, and we want to prove f is continuous at a

According to above, we define $x = a + h$, which means $h = x - a$

As $h \rightarrow 0$, the value of $x(a+h)$ must approach $a + 0$, so we can know as $h \rightarrow 0$, then $x \rightarrow a$

Now we can replace h with $x - a$ and the limit condition $h \rightarrow 0$ with $x \rightarrow a$

$$\lim_{x \rightarrow a} f(a + (x - a)) = f(a)$$

After simplify:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

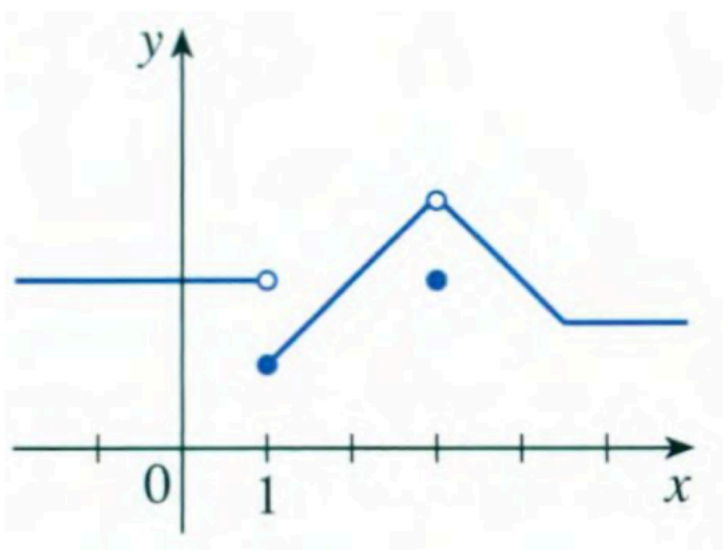
From above, it have proven that if $\lim_{h \rightarrow 0} f(a + h) = f(a)$ is true, f is continuous at a .

Conclusion

Since we have proven the statement in both directions, we have successfully shown that a function f is continuous at a if and only if $\lim_{h \rightarrow 0} f(a + h) = f(a)$.

Q2

The graph of a function f is given.



1. At what numbers a does $\lim_{x \rightarrow a} f(x)$ not exist?
2. At what numbers a is f not continuous?
3. At what numbers a does $\lim_{x \rightarrow a} f(x)$ exist but f is not continuous at a ?

Ans2

- The criteria for limits, continuity, and differentiability can be found in Appendix B
1. $x = 1$, because $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$
 2. $x = 3$, because $\lim_{x \rightarrow 3} f(x) \neq f(3)$
 3. $x = 3$, according to Appendix B, although left-hand limit equals to right-hand limit and both exist (conditions for the existence of a limit), $\lim_{x \rightarrow 3} f(x) \neq f(3)$, from above we can know that f is not continuous at a .

Q3

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a .

Ans3

Q4

Ans4

Crystal Growth Inspection

?.py

?2.py

Work Division

學號/姓名	分配項目（寫）	分配項目（檢查）
411485002 楊昕展		
411485003 胡庭睿	part 1:1,2 part 2:1,2 Appendix A,B	
411485018 蘇星丞		
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Challenges and Difficulties

此處由每位成員報告在擔任實作者或審核者時遇到的挑戰或未解決的問題，包含未來工作

姓名：胡庭睿

遇到的困難與挑戰：

1. markdown轉pdf

在上次作業中使用的extension(markdown pdf)，在轉換多行latex公式時(ex: $\begin{aligned}$)會無法辨別，於是我在網上尋找許多不同方法，但是轉換的效果都不令我滿意，pandoc雖然能夠很好的渲染出latex公式，但是圖片的位置卻無法置中，換了很多個latex engine依然無解(pdflatex, mactex, xetex, basictex)再加上標題頁是用html寫的，也意味著標題頁需要重寫，但是我卻一直無法將它修改到我想要的樣式，最後我找到了一個叫做markdown preview enhanced的extension,對latex公式有很好的支援度，字體以及版面等也都可以透過.less控制，有不錯的自訂度，只是文檔寫的不太詳細，花了我一點時間才解決，不過我也學到了許多。

2. latex撰寫作業

經過作業一的批改後，我更了解老師的評分標準，應該更詳細的列出思考以及計算過程，也代表用latex撰寫的佔比會更高，我學到了要打多行運算式時，可以用 $\begin{aligned}$ 和 $\end{aligned}$ ，用 \begin{align} 會讓每一行運算式後有編號，並且用&可以將算式對齊，在這次作業中也花了不少時間學習latex的語法。不過像part 1 ans 1中的公式，雖然可以使它們與第一行算式對齊，但是我並沒有辦法控制所有算式的位置是在中間還是左邊，寫的方式完全一樣，但是卻依然無法改善此問題，推測是markdown轉換為html時出錯。

3. 全英文撰寫作業

全英文撰寫作業是這次作業我給自己的挑戰，我發現在上課的簡報中，太容易依賴翻譯去查詢那些數學專有名詞，寫上中文解釋後下次上課依然還是會忘記，真正用英文開始寫後，我發現有好多在數學算式中常用到的詞彙，像是假

設(assume)，代數化簡(algebraic simplification)，function的定義錯誤(ill-definition)等，很多都是生活中常用到的單字但是會讓人想不到可以這樣運用，透過這次作業也讓我更熟悉這些數學的英文專有名詞，也期許自己能夠越來越熟悉，到最後可以能夠像母語一樣順利的運用。

姓名：

遇到的困難與挑戰：

姓名：

遇到的困難與挑戰：

姓名：

遇到的困難與挑戰：

Meeting Records

會議日期	會議方式 (線上/實體)	討論事項
[日期]	[方式]	1. 作業要求釐清 2. 進度規劃 3. 問題整合 4. 角色分配討論
...
...

Working Hours Records

組員姓名	工作時數	工作項目	工時高/低原因分析 (Bonus)
胡庭睿	20hr	1. part 1:1,2 (1hr) 2. part 2:1,2 (2hr) 3. Appendix A,B (4hr) 4. the report template (3hr) 5. git and github tutorial(2hr) 6. find ways of converting markdown to pdf (8hr) 7. improve whole team workflow on report(2hr)	
[姓名2]	[時數]	1. [工作項目1] 2. [工作項目2]	[自行分析原因]
...
...

Reflection

(應包含在作業中學習到的知識點，個人和小組整題的學習，團隊工作風格的反思，總結在此次作業中遇到的挑戰和問題(work hours, division))

學號：411485003

姓名：胡庭睿

心得：

這次作業的時間點剛好在作業一的成績發佈之後，老師對作業的layouts, writing styles, reflection and team's distribution to the work也變得更清楚，爲了更好的統一writing styles，首先我根據作業二用markdown混合latex的方式撰寫了修正過後的模板，不僅能夠統一組員的writing styles也可以更貼近老師的要求，再來用markdown + latex的好處是不用碰觸到latex排版，同時也能學習如何用latex的語法寫數學公式，最後則是處理上次作業中提及的共編問題，我寫了教學git 和 github的教學文檔，教了組員們必要的git知識，即使還是遇到了常見的版本衝突問題，花了一段時間才解決，但是我也同樣學到了許多，

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Appendix

Appendix A

Limit Laws in class's slides:

1. Sum Law ($\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$)
2. Difference Law ($\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$)
3. Constant Multiple Law ($\lim_{x \rightarrow a} cf(x) = c \times \lim_{x \rightarrow a} f(x)$)
4. Product Law ($\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x)$)

5. Quotient Law ($\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$, if $\lim_{x \rightarrow a} g(x) \neq 0$)
6. Power Law ($\lim_{x \rightarrow a} (f(x)^n) = (\lim_{x \rightarrow a} f(x))^n$)
7. Root Law ($\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$)
8. $\lim_{x \rightarrow a} c = c$ (c 為常數)
9. $\lim_{x \rightarrow a} x = a$
10. $\lim_{x \rightarrow a} x^n = a^n$
11. $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$

Appendix B

極限存在條件

1. $\lim_{x \rightarrow a-} f(x)$ 存在
2. $\lim_{x \rightarrow a+} f(x)$ 存在
3. $\lim_{x \rightarrow a-} f(x) = \lim_{x \rightarrow a+} f(x)$

在點 a 連續條件

1. $f(a)$ 存在
2. $\lim_{x \rightarrow a} f(x)$ 存在
3. $\lim_{x \rightarrow a} f(x) = f(a)$

在點 a 可微分條件

1. $f(a)$ 存在
2. $f(x)$ is continuous at a
3. $\lim_{x \rightarrow a-} \frac{f(x)-f(a)}{x-a}$ 存在
4. $\lim_{x \rightarrow a+} \frac{f(x)-f(a)}{x-a}$ 存在
5. $\lim_{x \rightarrow a-} \frac{f(x)-f(a)}{x-a} = \lim_{x \rightarrow a+} \frac{f(x)-f(a)}{x-a}$

三者關係

可微分一定連續，若連續，極限一定存在

Appendix C