Statistics

Two concepts and Two Theorems

Sample & Population

Theorem 1: Law of Large Numbers

Theorem 2: Central Limit Theorem

Important Distributions

Normal (mu, sigma2)

Chi-square (df)

T(df)

F(df1, df2)

Hypothesis testing and Two types of errors

Null hypothesis (H0) & Alternative hypothesis (H1)

False Positive & False Negative Errors

Statistical tests

One-sample t test

Two-sample t-test

Paired t-test

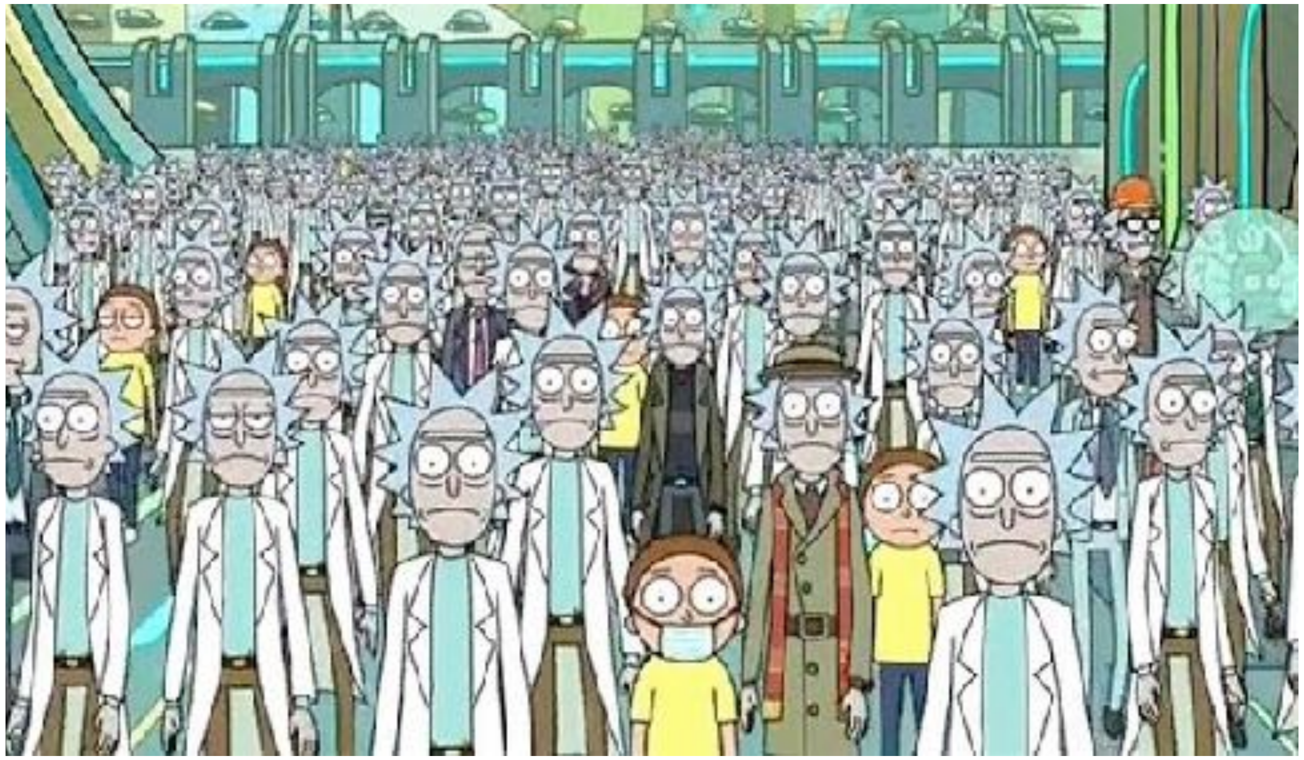
ANOVA

Repeated measure ANOVA

Regression

Sample: A fraction

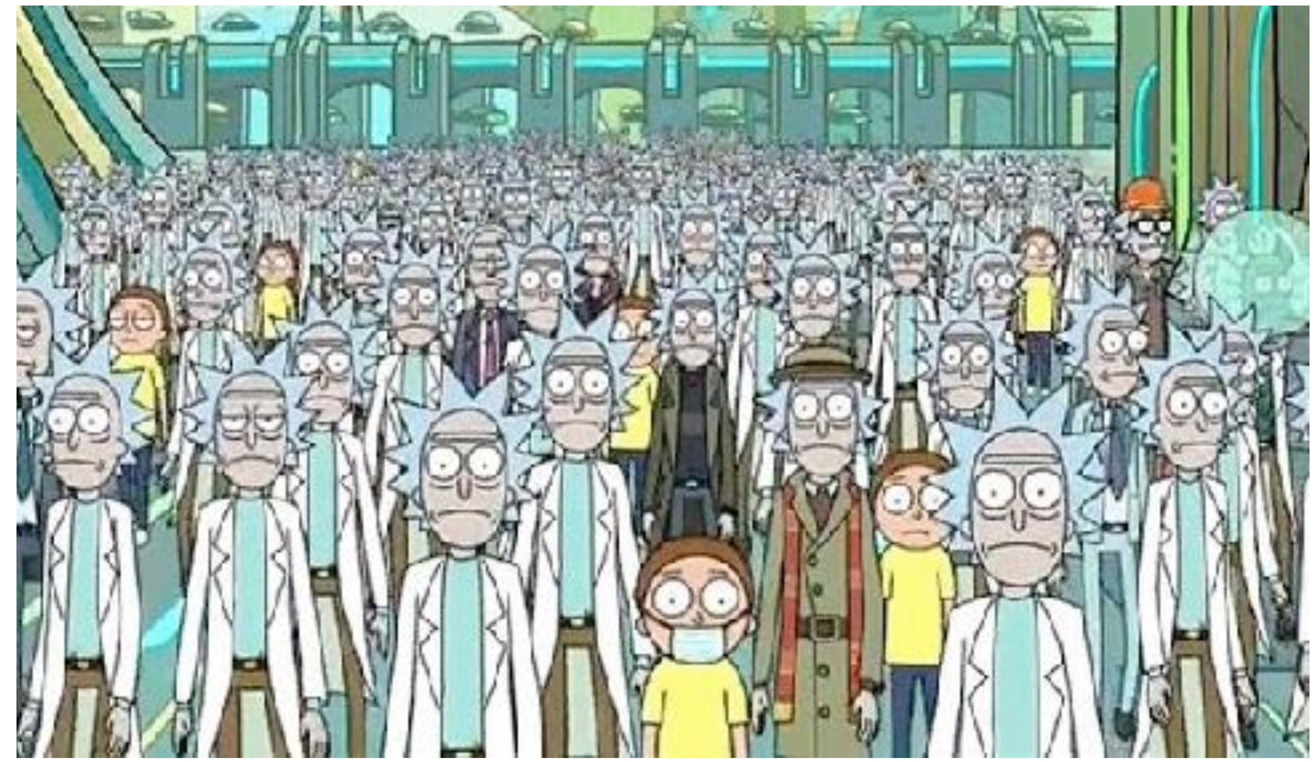
Population: A whole

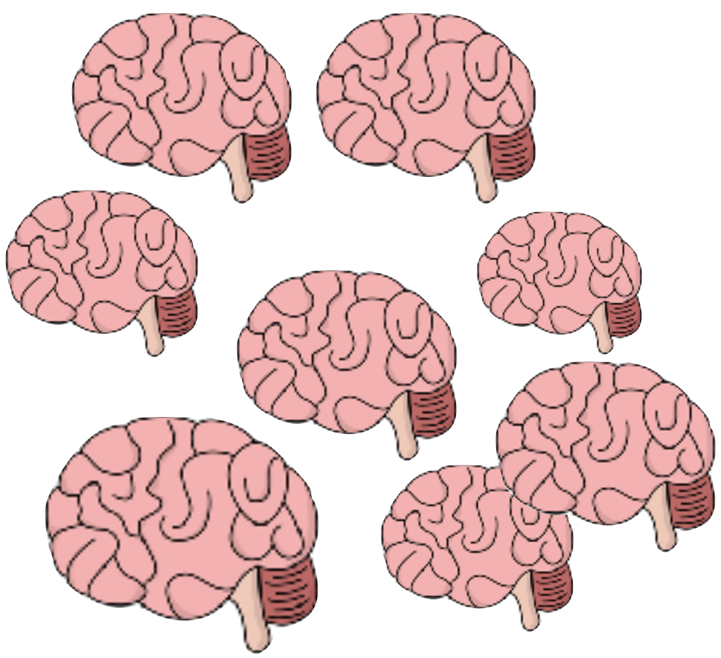


Theorem 1: Law of Large Numbers: the sample averages converge almost surely (converge in probability) to the expected value. In other words, as a sample size grows, its average gets closer to the average of the whole population.

Sample: A fraction

Population: A whole

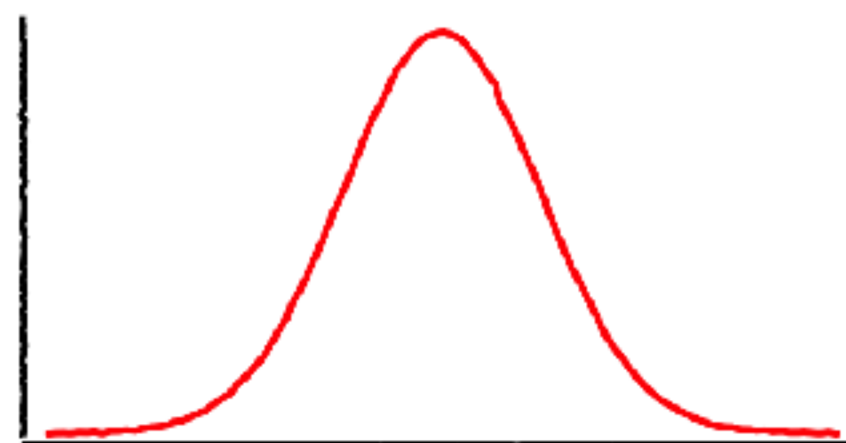




Sample (1) -> mean
Sample (2) -> mean

...

Sample (n) -> mean



NORMAL DISTRIBUTION

Theorem 1: Law of Large Numbers: the sample averages converge almost surely (converge in probability) to the expected value. In other words, as a sample size grows, its average gets closer to the average of the whole population.

Theorem 2: Central Limit Theorem: when independent random variables are summed up, their properly normalized sum tends toward a normal distribution, even if the original variables themselves are not normally distributed. In other words, as the sample size gets larger, the distribution of sample means approximates a normal distribution, regardless of the population's distribution.

What is the Hypothesis?

An assertion (conjecture) concerning one or more **populations**

A medication has an effect
Male's brain is larger than female

H0 (Null Hypothesis)
H1 (Alternative Hypothesis)

H0: Medication effect=0
<u>H1: Medication effect>0</u>

H0: Brain size: Male=Female
<u>H1: Brain size: Male>Female</u>

Alpha: the probability of rejecting the null hypothesis when it is true

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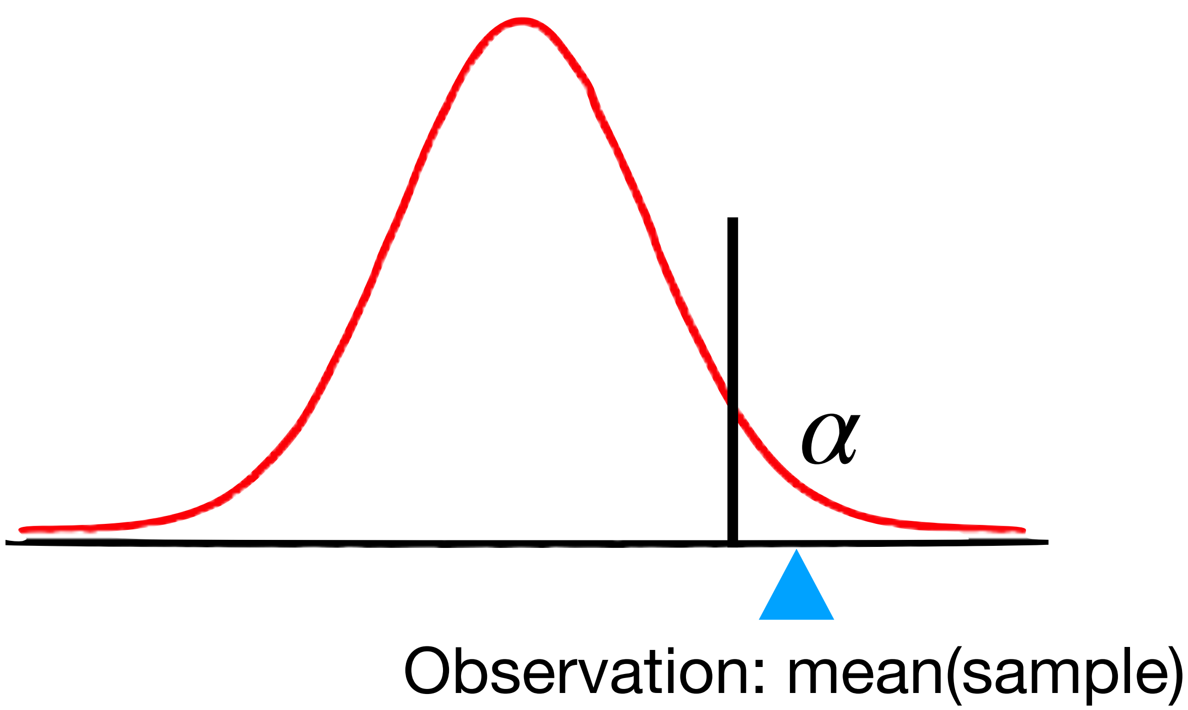
<u>H1: Medication effect>0</u>

H0: Brain size: Male=Female
<u>H1: Brain size: Male>Female</u>

H0: μ(population)=0
<u>H1: μ(population)>0</u>



Type I and Type II Error

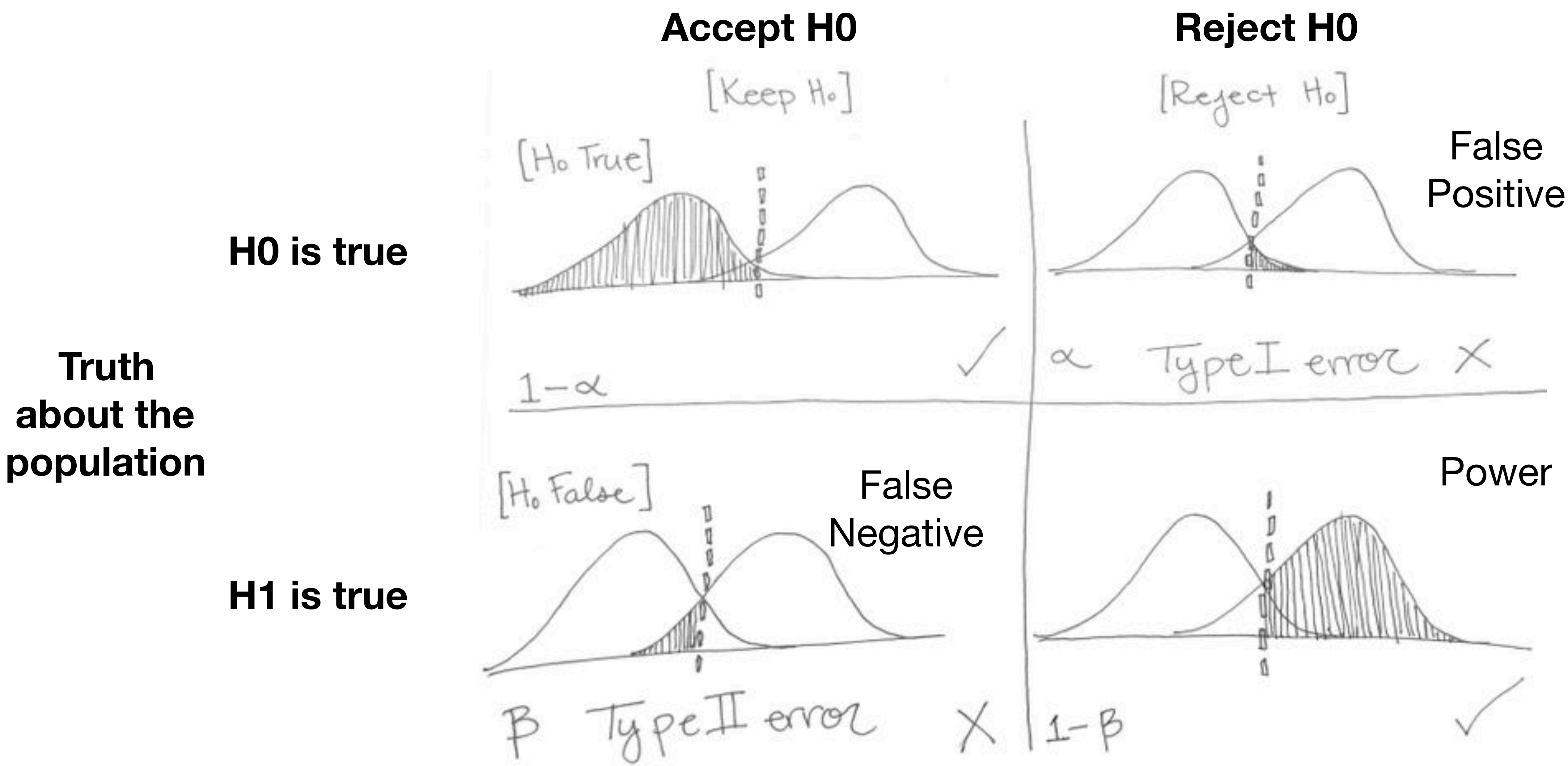
Type I Error (false-positive)

α ***(p-value)***

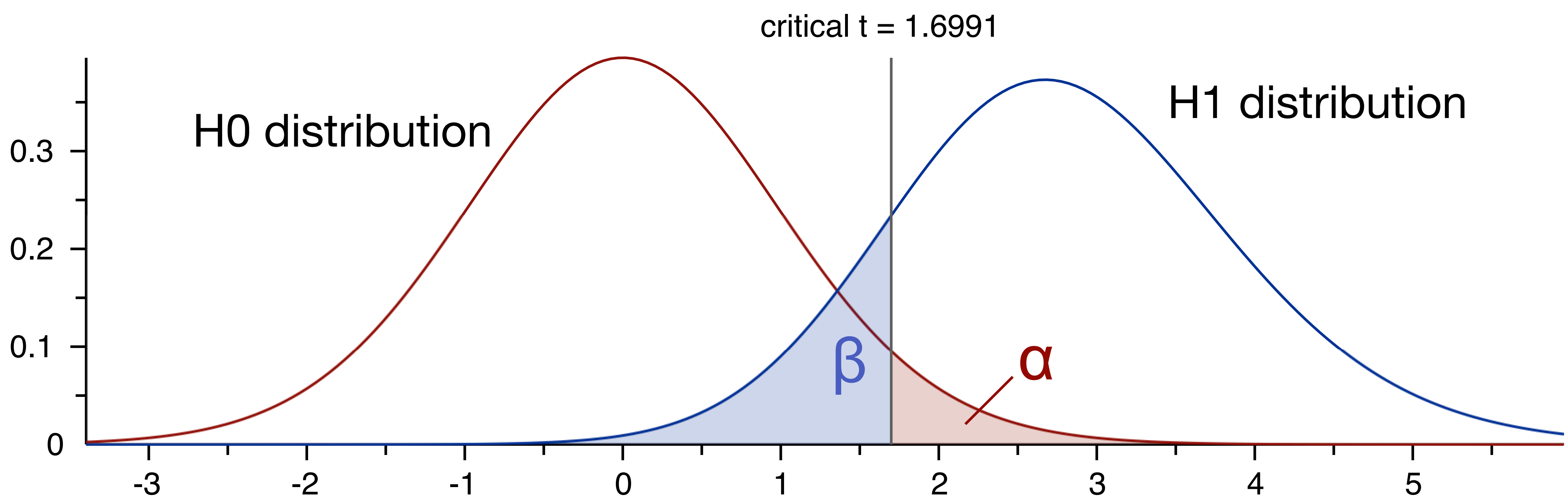
Type II Error (false-negative)

β

Decision based on sample



Alpha: the probability of rejecting the null hypothesis when it is true (risk of type-I error)
Power (1-beta): the probability of rejecting the null correctly



t tests - Means: Difference from constant (one sample t-test)

Input:	One Tail(s)	
	Effect size d	=0.5
	α err prob	=0.05
	Total sample size	=30
Output:		
	Critical t	=1.6991270
	Df	=29
	Power (1-β err prob)	=0.8482542

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Repeated measure ANOVA

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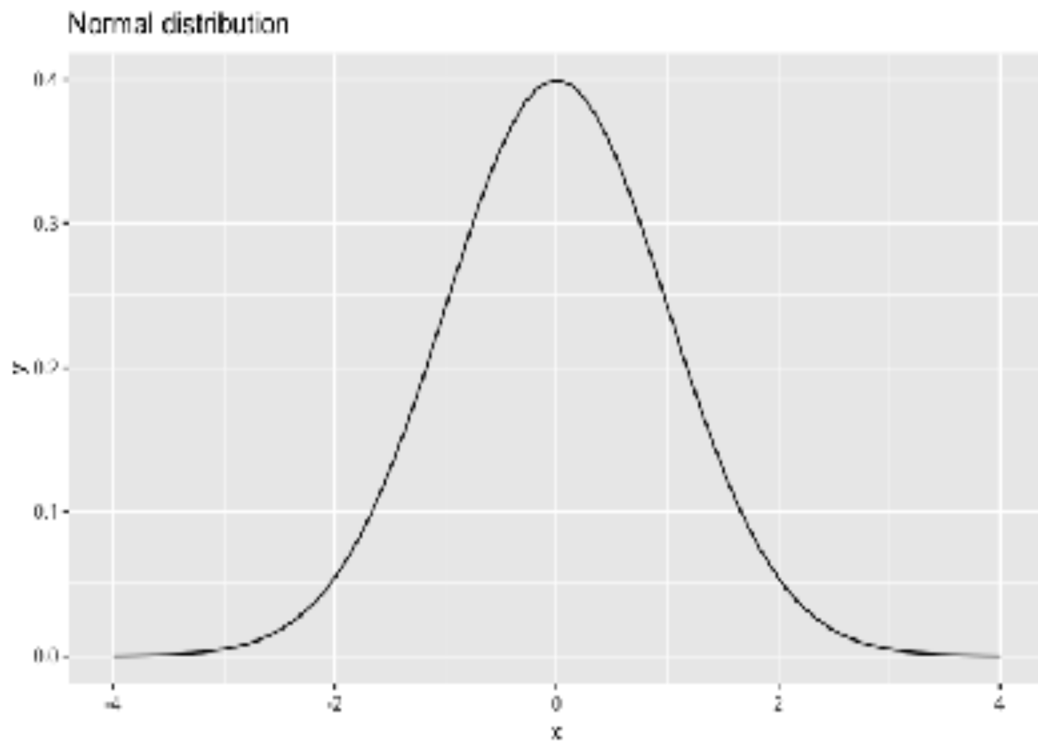
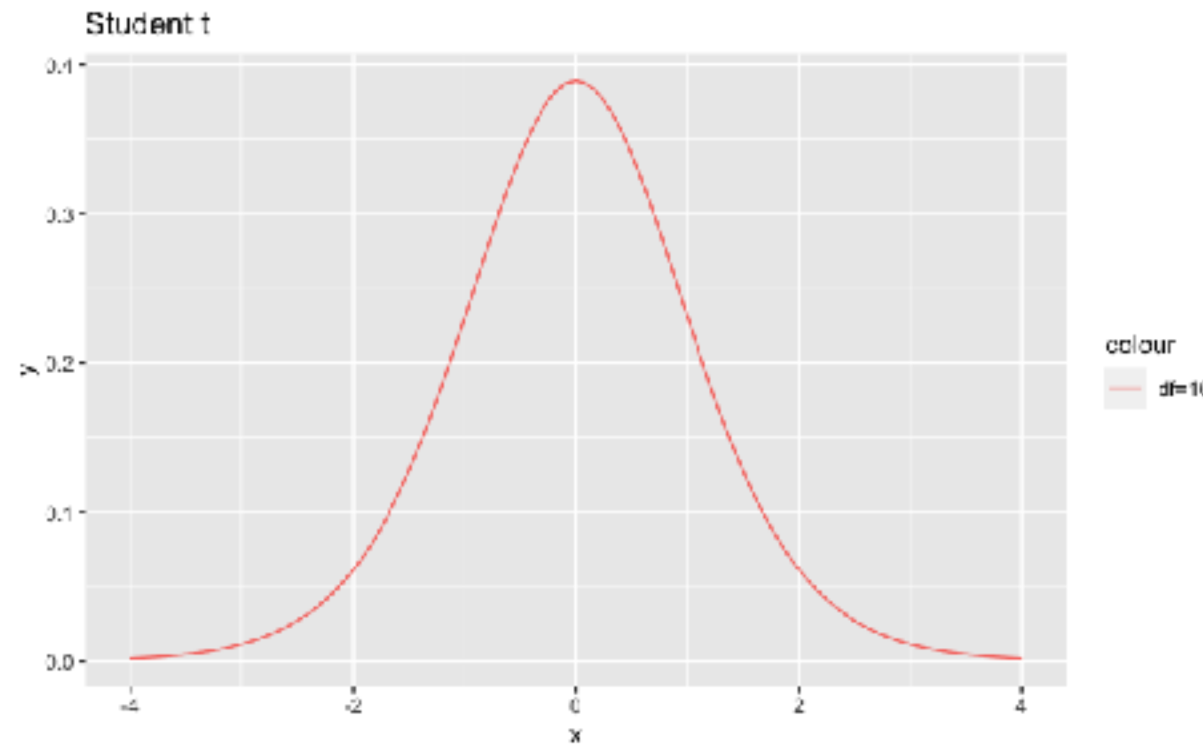
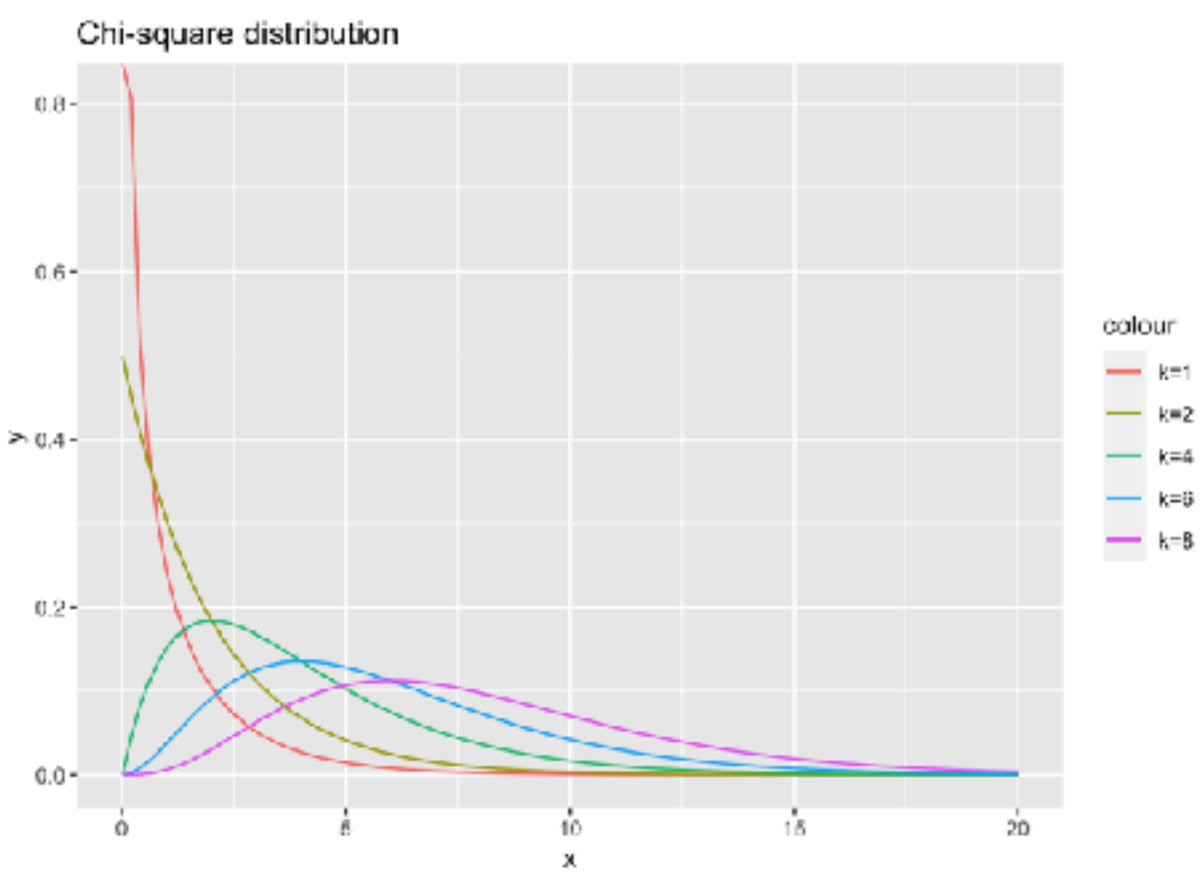
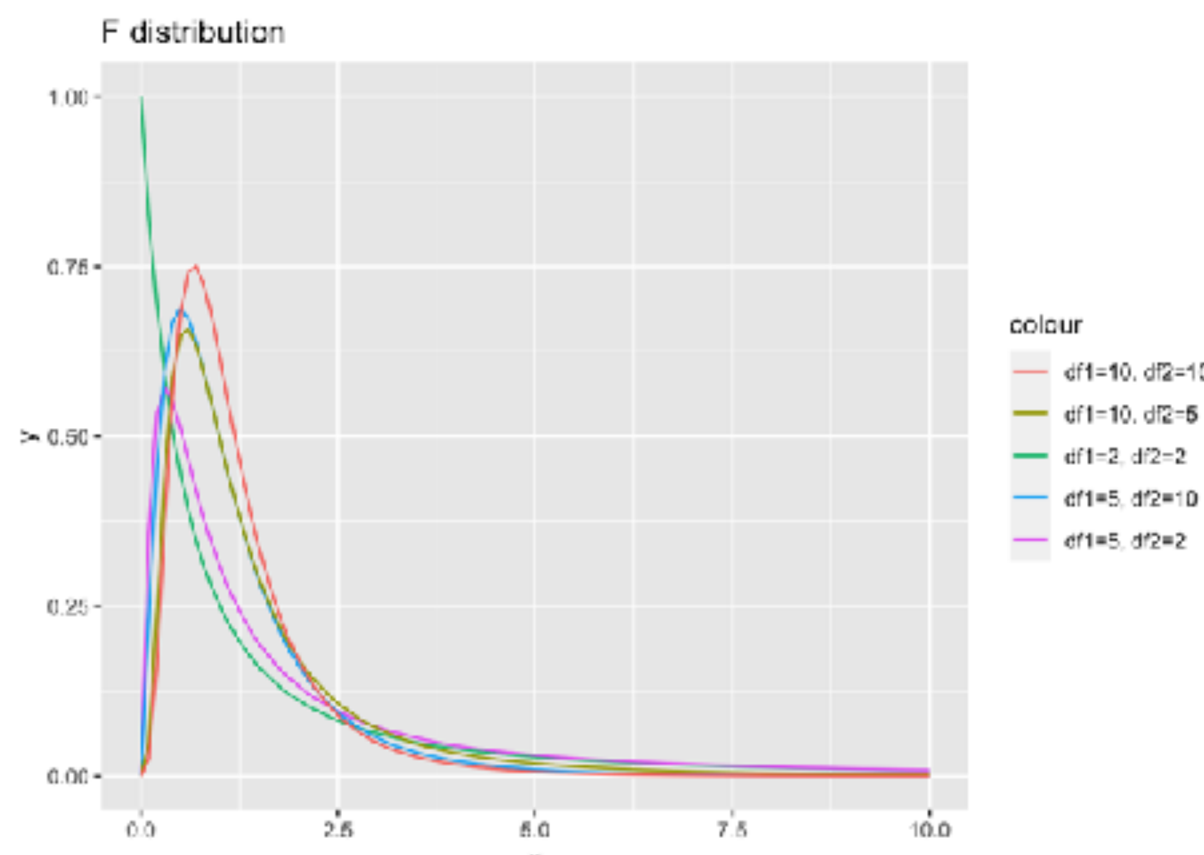
Distributions

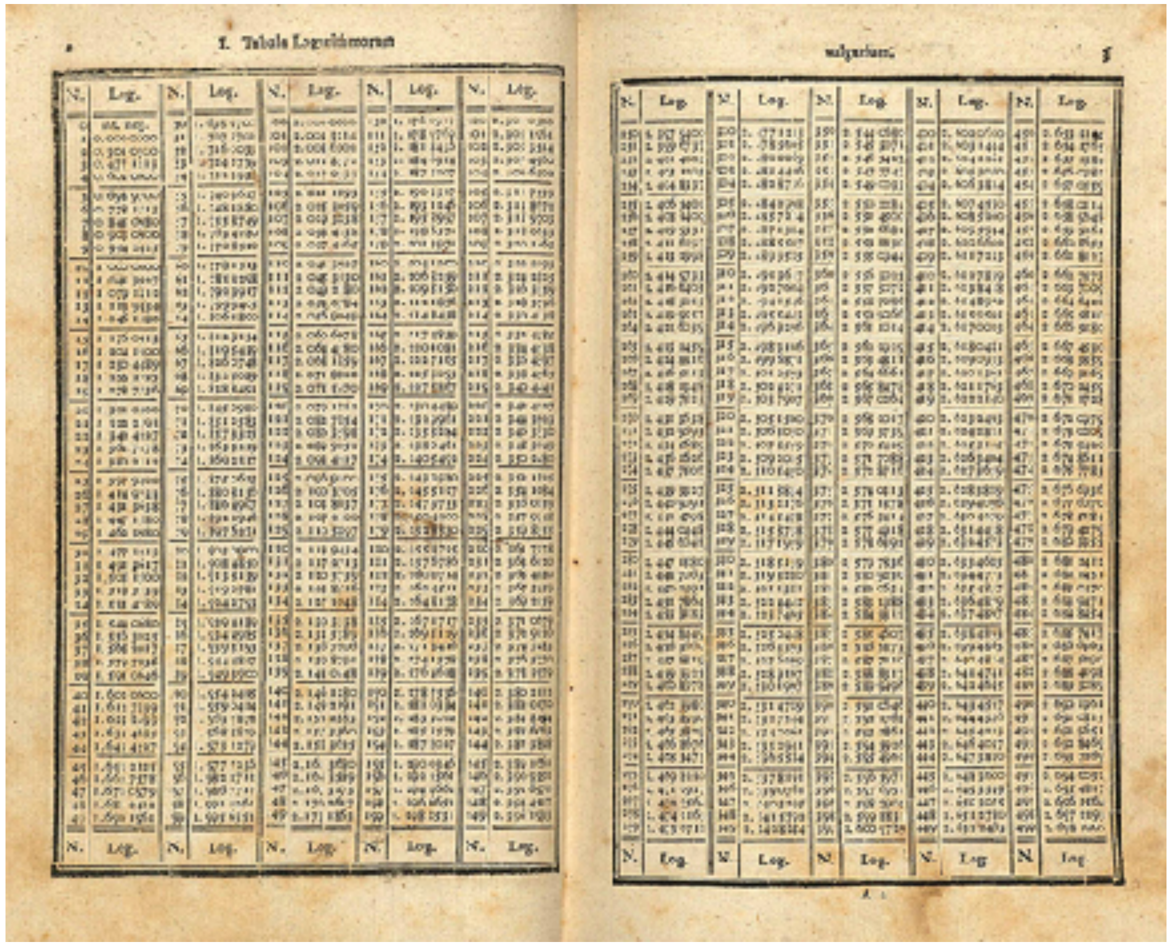


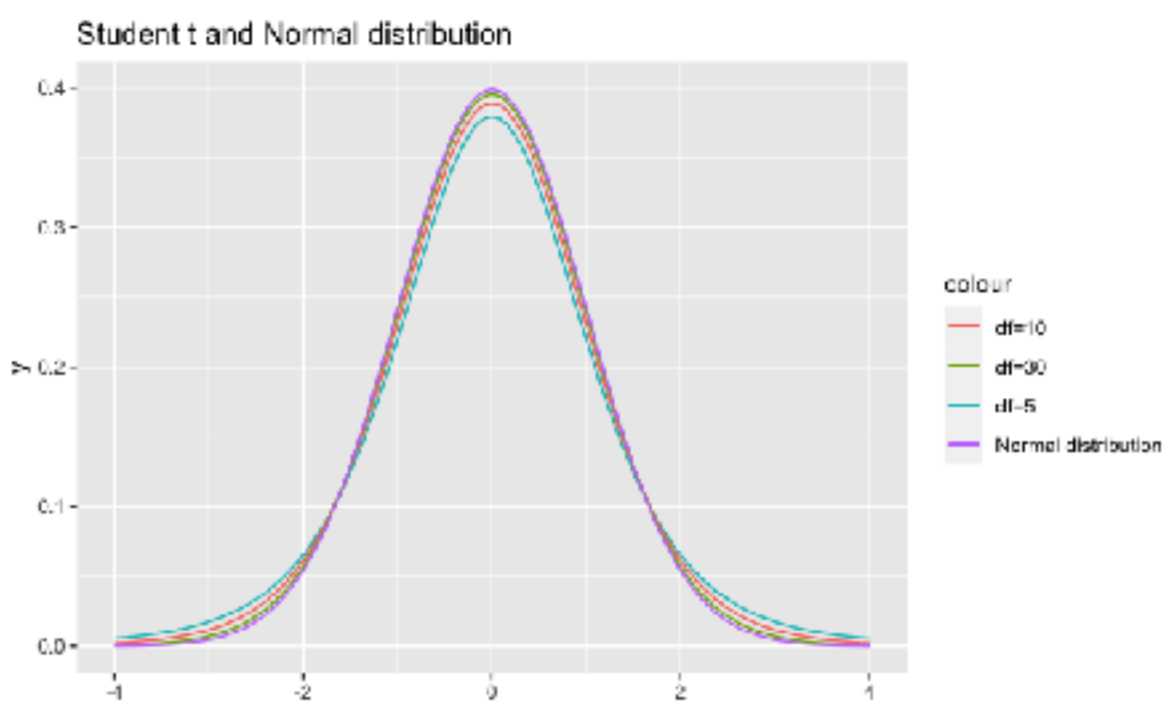






Mathematical table



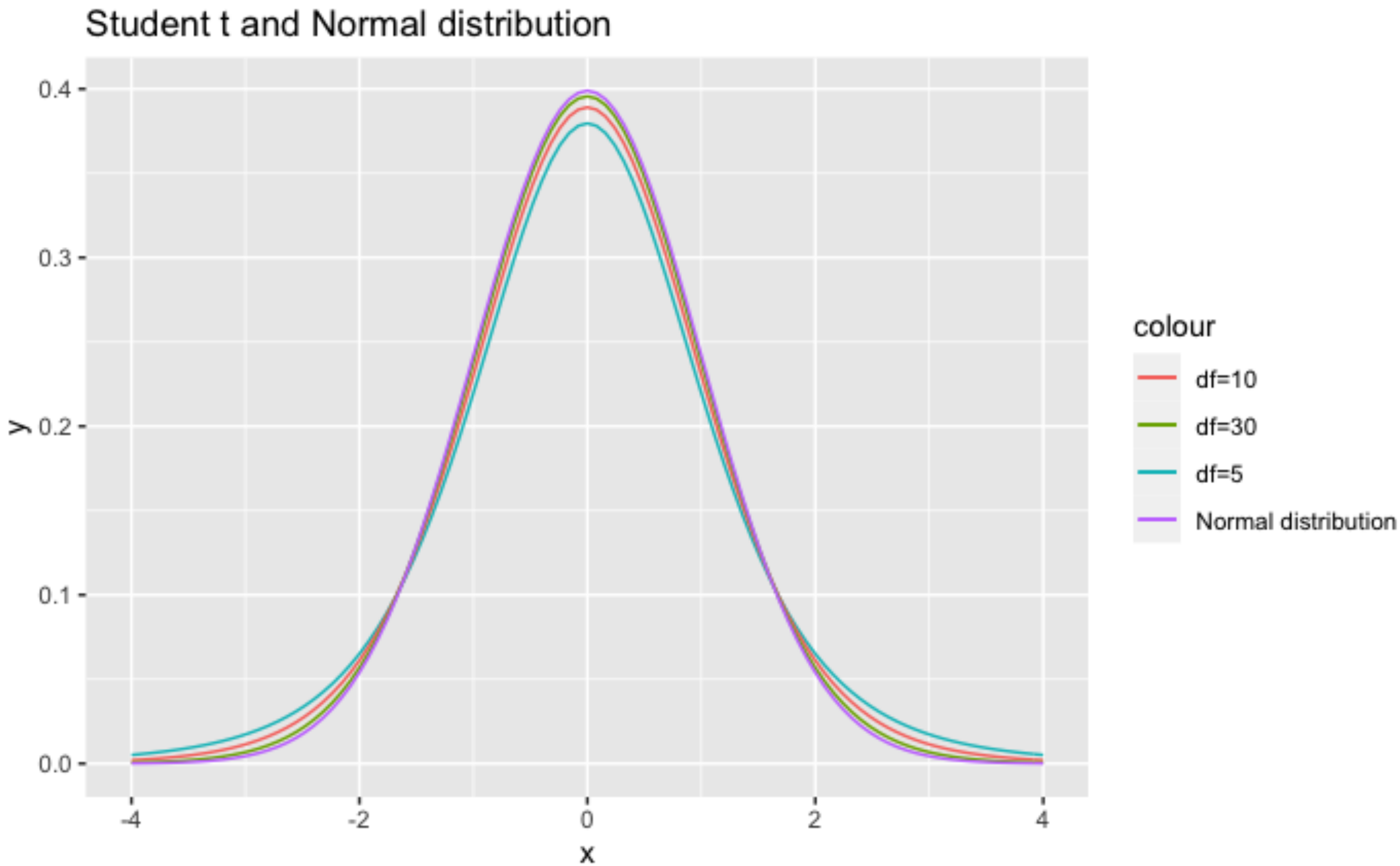


Z-value

z	.00
0.0	.5000
0.1	.5398
0.2	.5793
0.3	.6179
0.4	.6554
0.5	.6915
0.6	.7257
0.7	.7580
0.8	.7881
0.9	.8159
1.0	.8413
1.1	.8643
1.2	.8849
1.3	.9032
1.4	.9192
1.5	.9332
1.6	.9452
1.7	.9554
1.8	.9641
1.9	.9713
2.0	.9772
2.1	.9821
2.2	.9861
2.3	.9893
2.4	.9918
2.5	.9938
2.6	.9953
2.7	.9965
2.8	.9974
2.9	.9981
3.0	.9987
3.1	.9990
3.2	.9993
3.3	.9995
3.4	.9997

T-value

One sided	75%	80%	85%	90%	95%	97.5%	99%	99.5%	99.75%	99.9%	99.95%
Two-sided	50%	60%	70%	80%	90%	95%	98%	99%	99.5%	99.8%	99.9%
1	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.080	1.386	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390
120	0.677	0.845	1.041	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291


Student t and Normal distribution
0.4
0.3
0.2
0.1
0.0
y
-4
-2
0
2
4
x
colour
df=10
df=30
df=5
Normal distribution

Normal (Gaussian) distribution

$$Z \sim N(\mu, \sigma^2)$$

Chi-square (df)

$$If\ Z \sim N(0,1), and\ X = \sum_i^k (Z_i^2), then\ X \sim \chi^2(k)$$

Student T(df)

$$If\ Z \sim N(0,1),\ U \sim \chi^2(n),\ and\ X = \frac{Z}{\sqrt{U/n}},\ then\ X \sim t(n)$$

F(df1, df2)

$$If\ U1 \sim \chi^2(n1),\ U2 \sim \chi^2(n2),\ and\ X = \frac{U1/n1}{U2/n2},\ then\ X \sim F(n1,n2)$$

$$t(n)^2 = (\frac{Z}{\sqrt{U/n}})^2 = \frac{Z^2}{U/n} = F(1,n)$$

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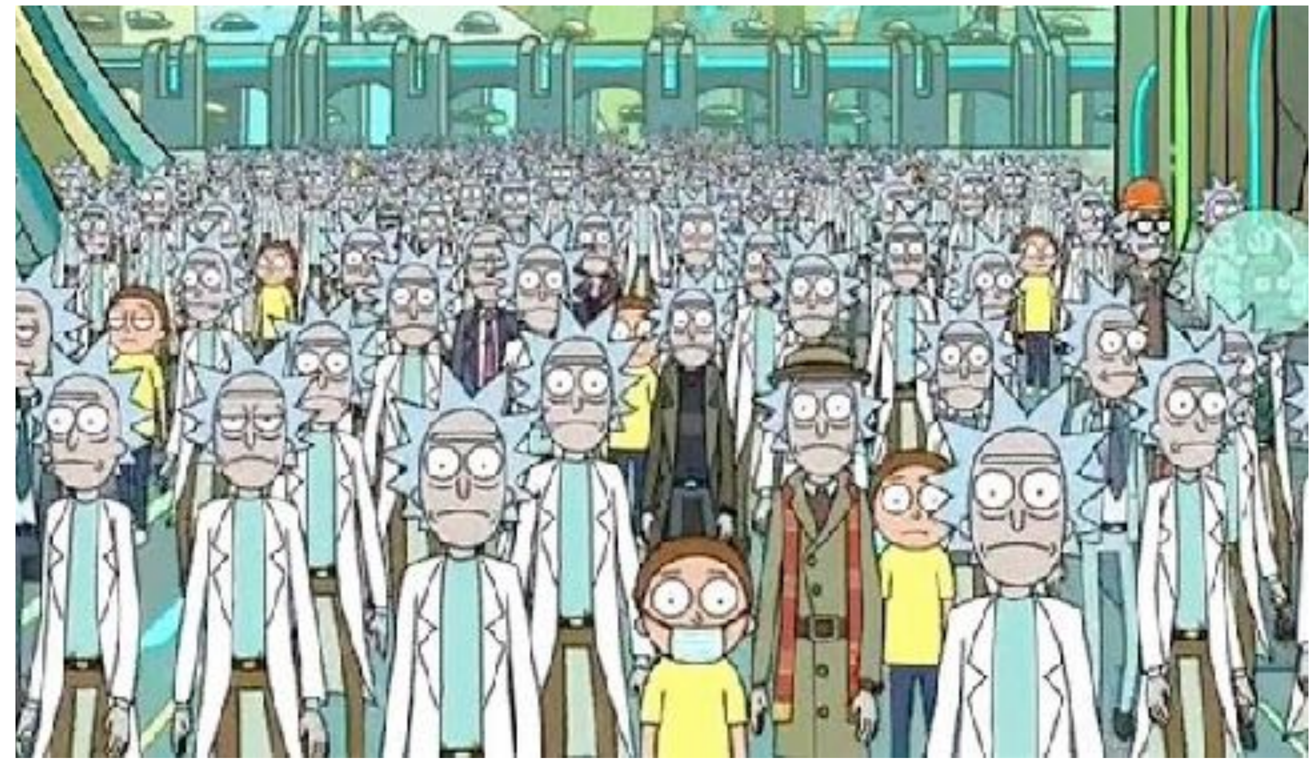
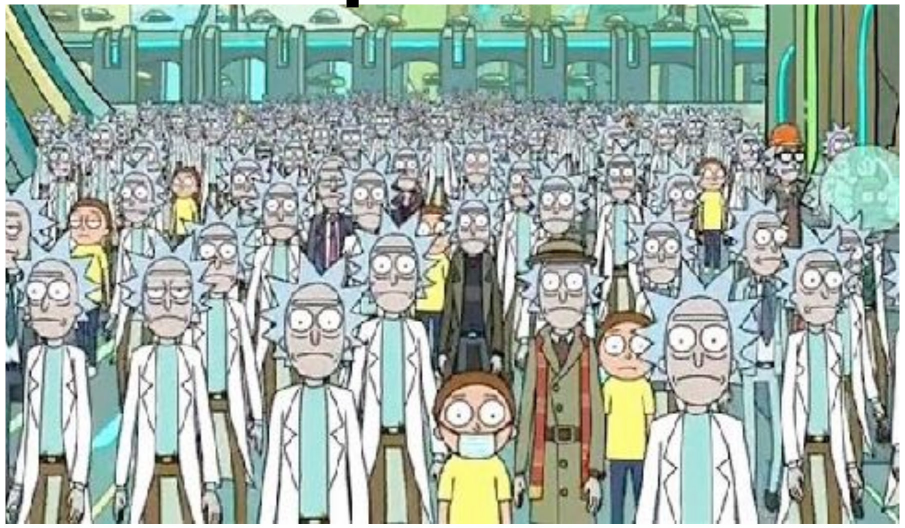


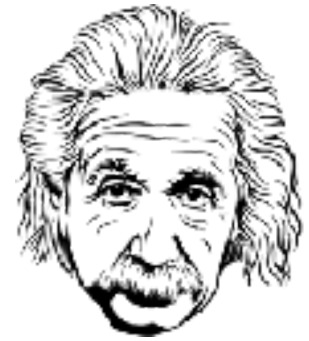
One-sample T test

Population

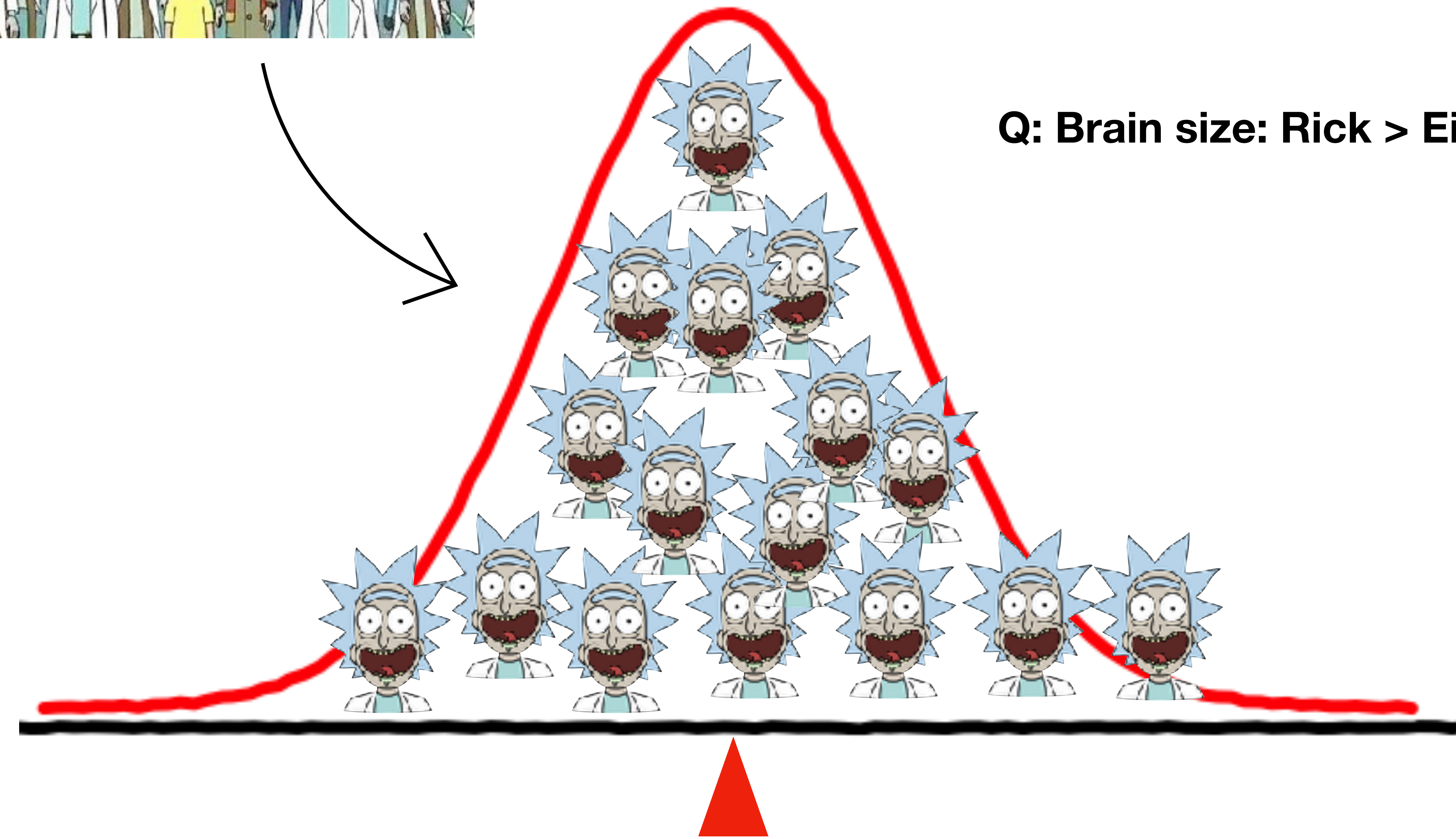
One Sample T-test



?



Q: Brain size: Rick > Einstein?

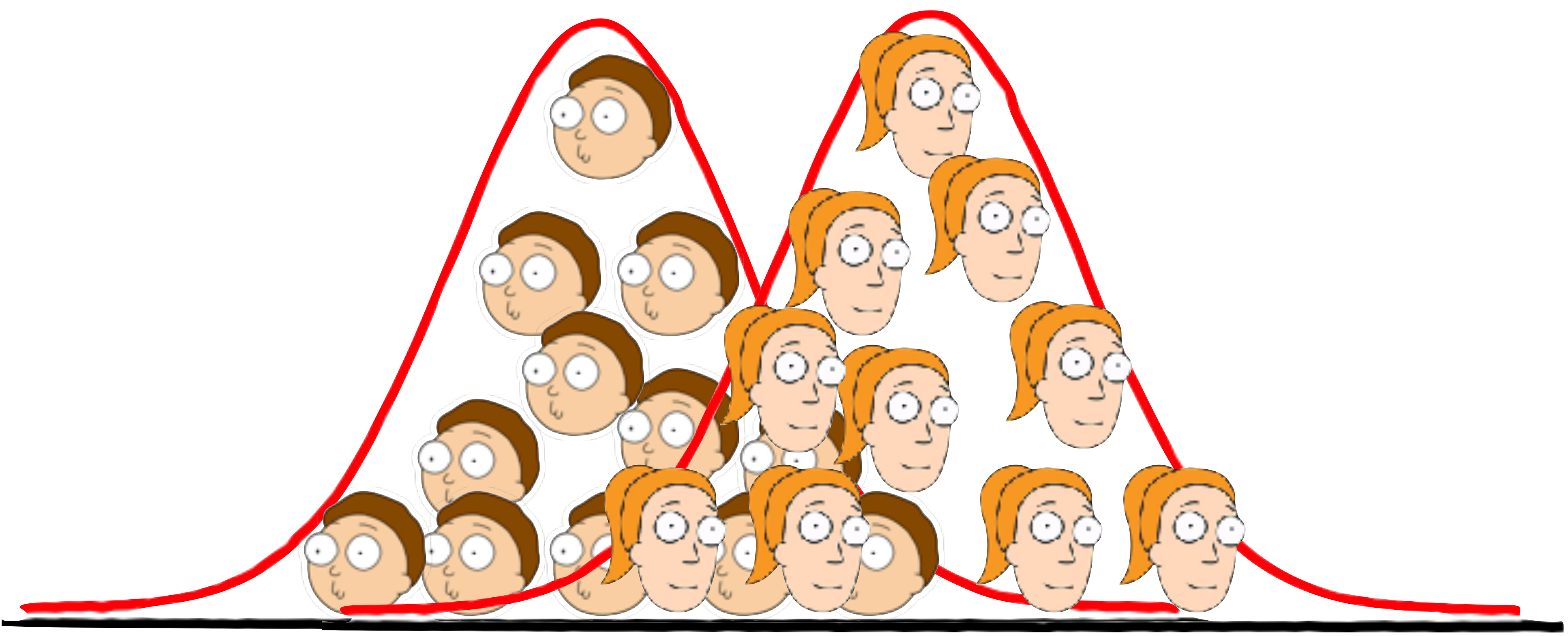


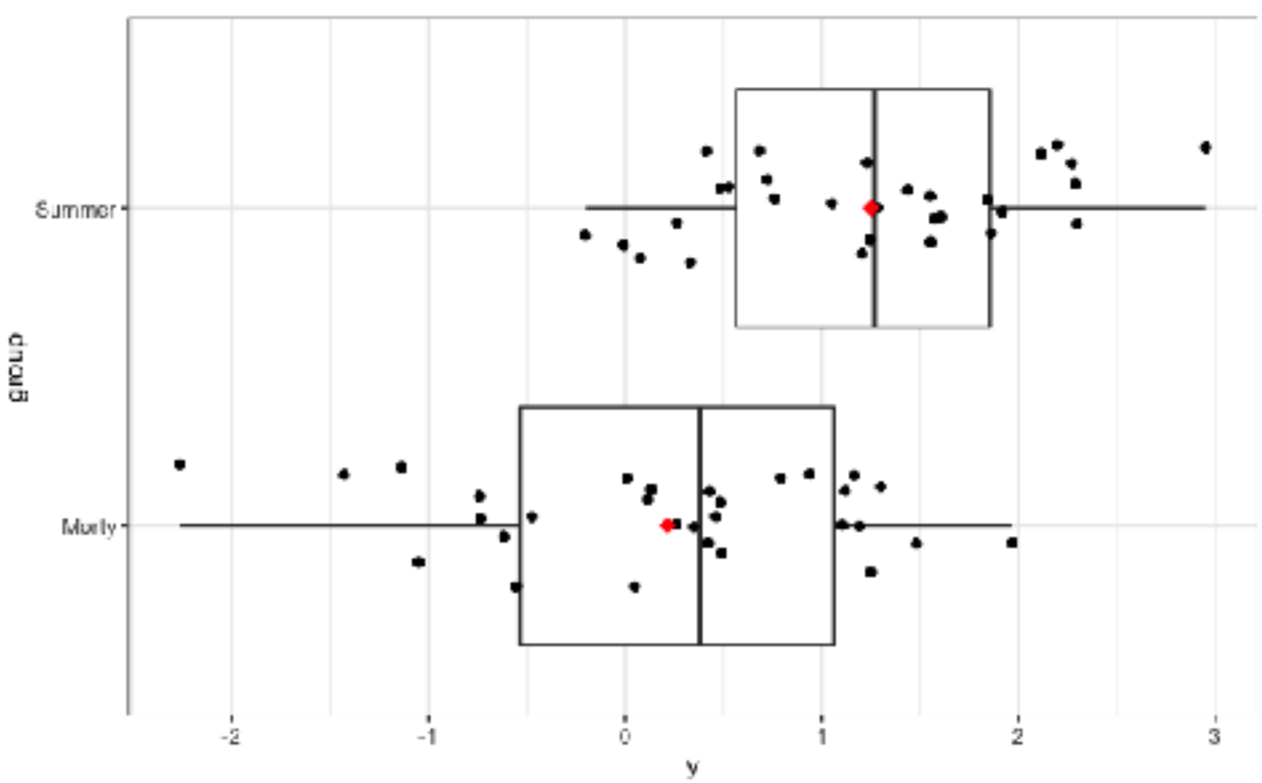
H0 (Null Hypothesis): Rick=Einstein
H1 (Alternative Hypothesis): Rick!=Einstein

Two-sample T test
One way ANOVA
Linear regression

Q: Difference: Summer > Morty?

Two Sample T-test

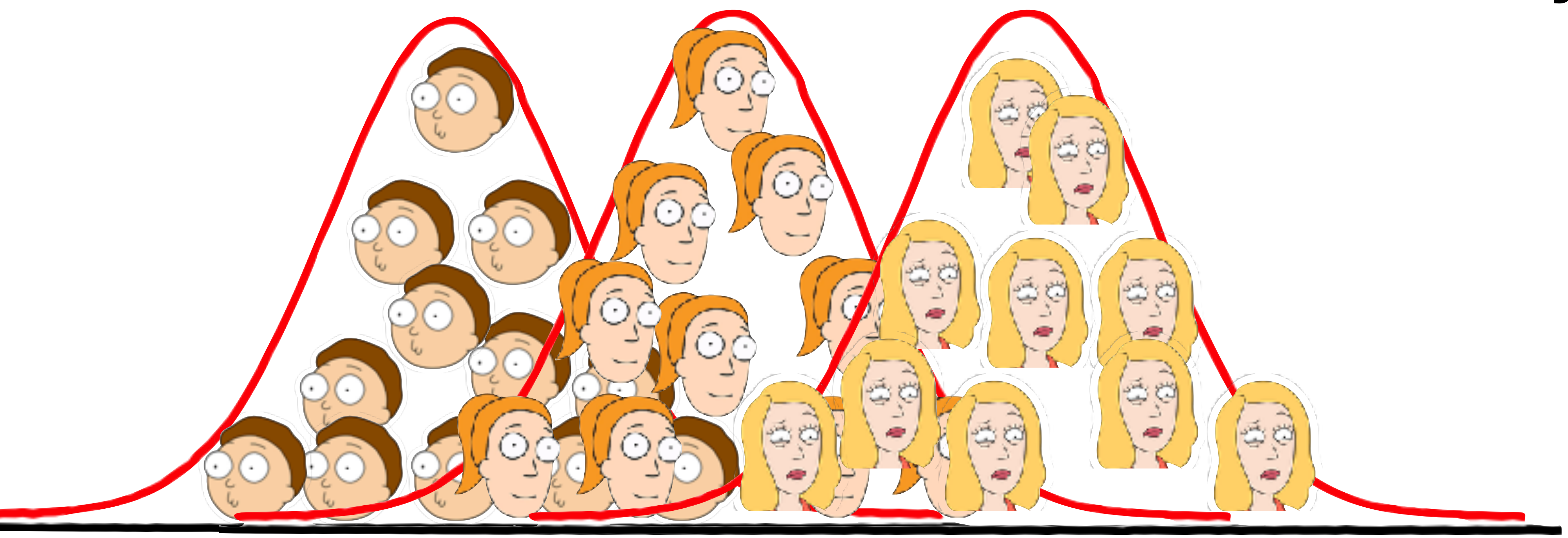


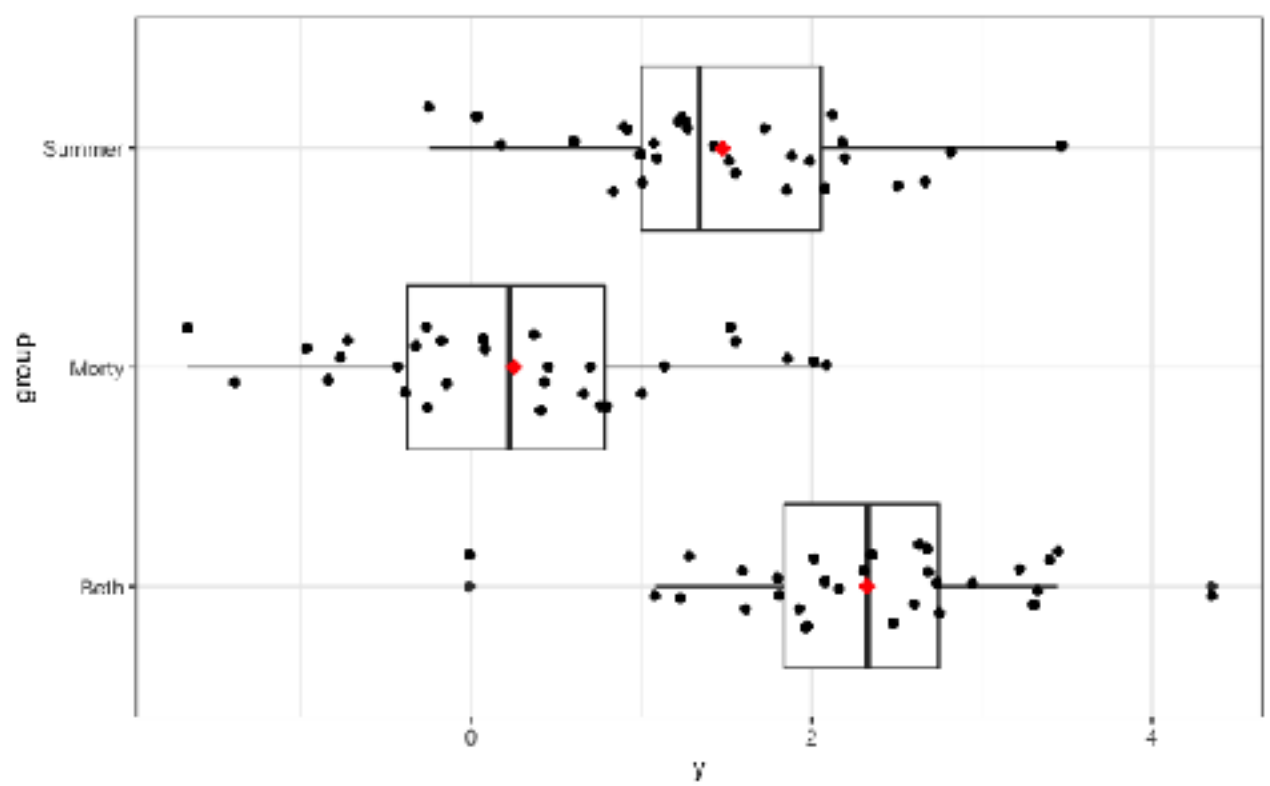



```
        Two Sample t-test

data:  y1 and y2
t = -5.2098, df = 58, p-value = 2.616e-06
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -1.6962870 -0.7545972
sample estimates:
mean of x mean of y
0.2528962 1.4783383
```

One-way ANOVA

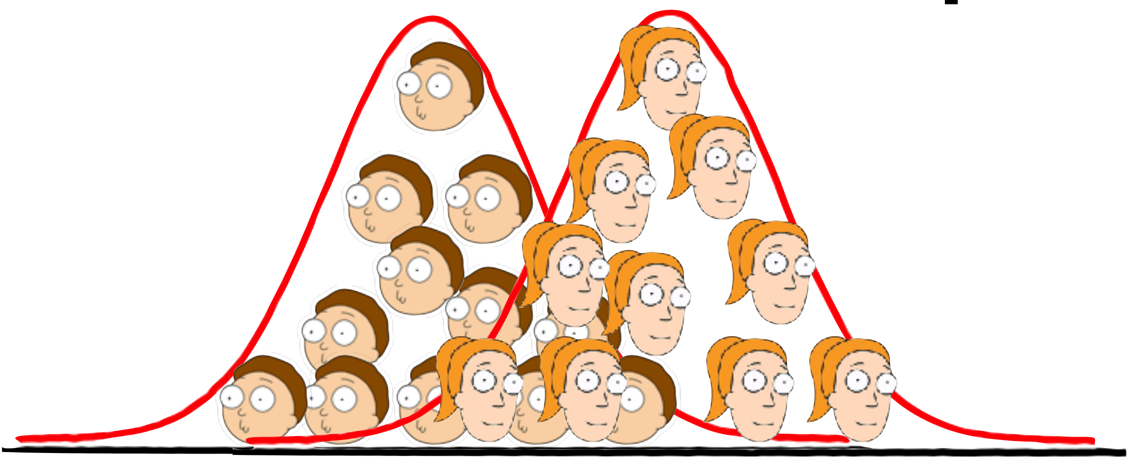





```
Anova Table (Type II tests)

Response: y
          Sum Sq Df F value    Pr(>F)
group     65.088  2  40.404 3.889e-13 ***
Residuals 70.076 87
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Two Sample T-test as One-way (2 levels) ANOVA



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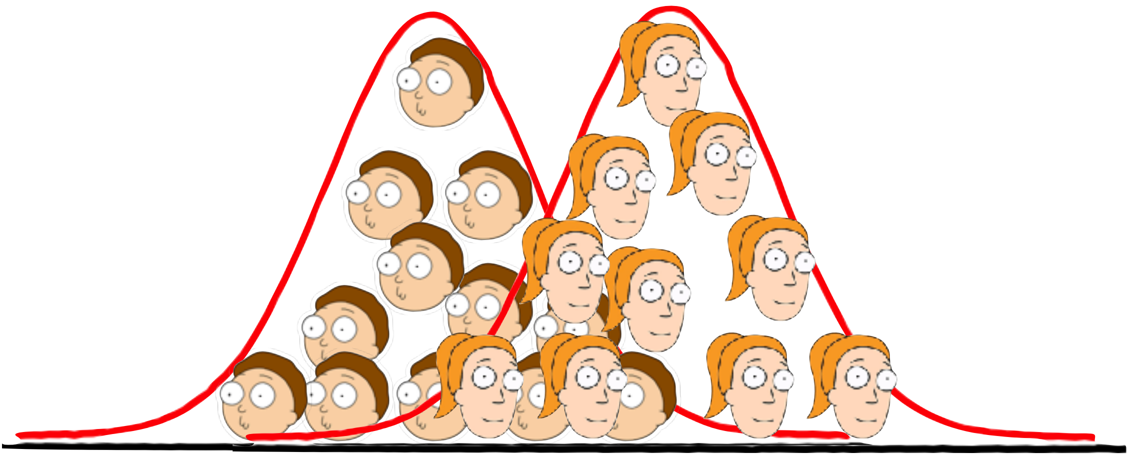
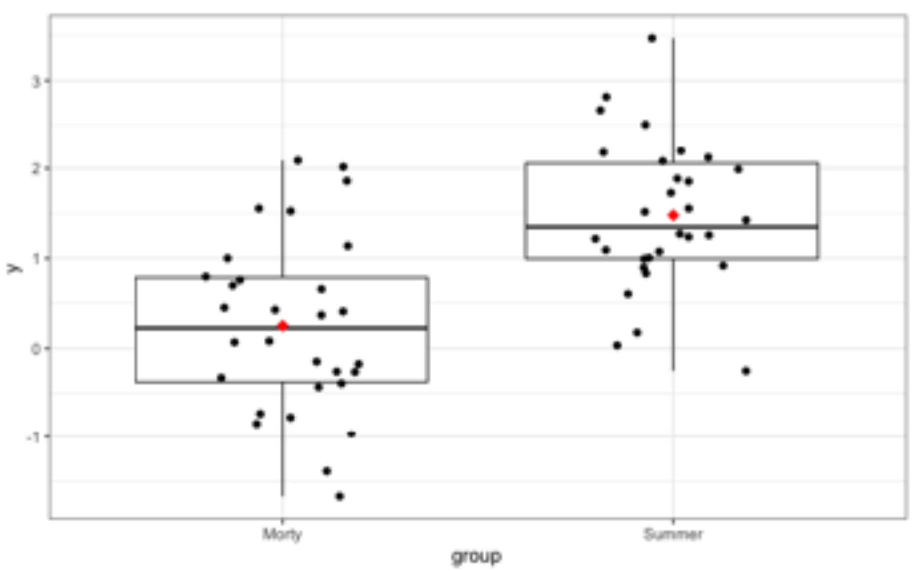
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -1.6962870 -0.7545972
sample estimates:
mean of x mean of y
0.2528962 1.4783383

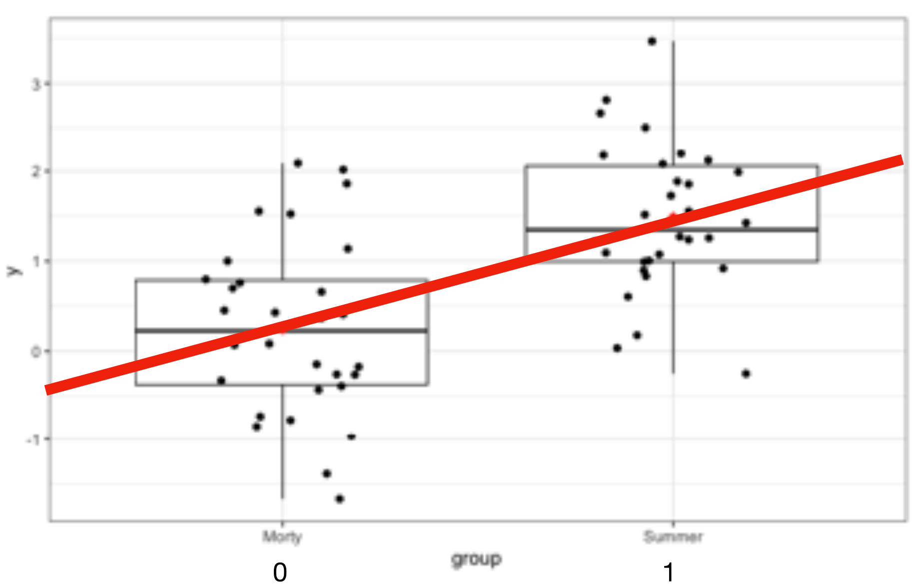
Anova Table (Type II tests)

Response: y
          Sum Sq Df F value    Pr(>F)
group     22.526  1  27.142 2.616e-06 ***
Residuals 48.136 58
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Two Sample T-test as linear regression







Two Sample t-test

```
data:  y1 and y2
t = -5.2098, df = 58, p-value = 2.616e-06
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -1.6962870 -0.7545972
sample estimates:
mean of x mean of y
0.2528962 1.4783383
```

Anova Table (Type II tests)

```
Response: y
          Sum Sq Df F value    Pr(>F)
group     22.526  1  27.142 2.616e-06 ***
Residuals 48.136 58
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Call:
lm(formula = y ~ 1 + group, data = df)

Residuals:
    Min      1Q  Median      3Q     Max
-1.9195 -0.5636 -0.1127  0.5591  1.9906

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.2529     0.1663    1.52    0.134
groupSummer   1.2254     0.2352    5.21 2.62e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

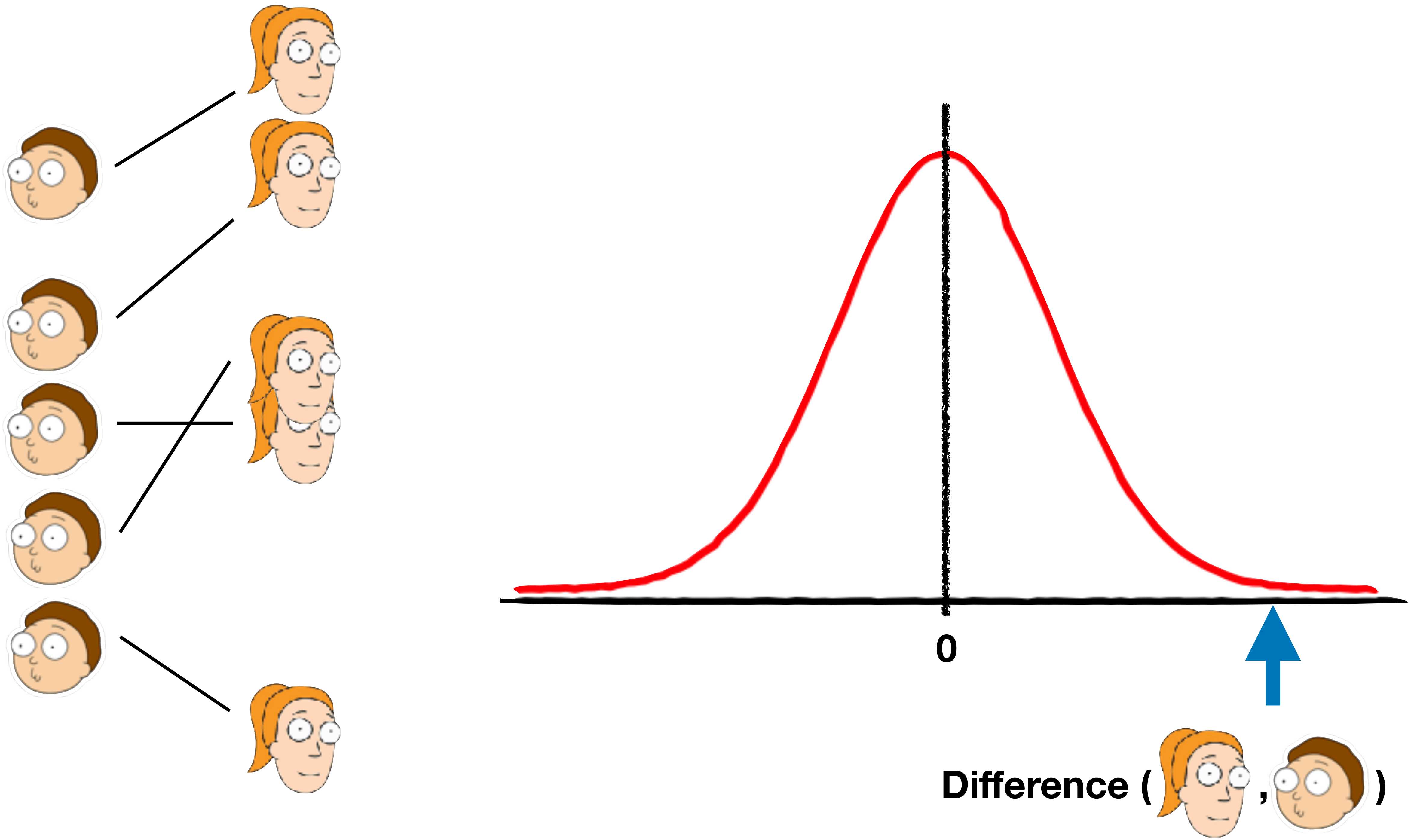
Residual standard error: 0.911 on 58 degrees of freedom
Multiple R-squared:  0.3188,      Adjusted R-squared:  0.307
F-statistic: 27.14 on 1 and 58 DF,  p-value: 2.616e-06
```

Paired t

Repeated Measure ANOVA

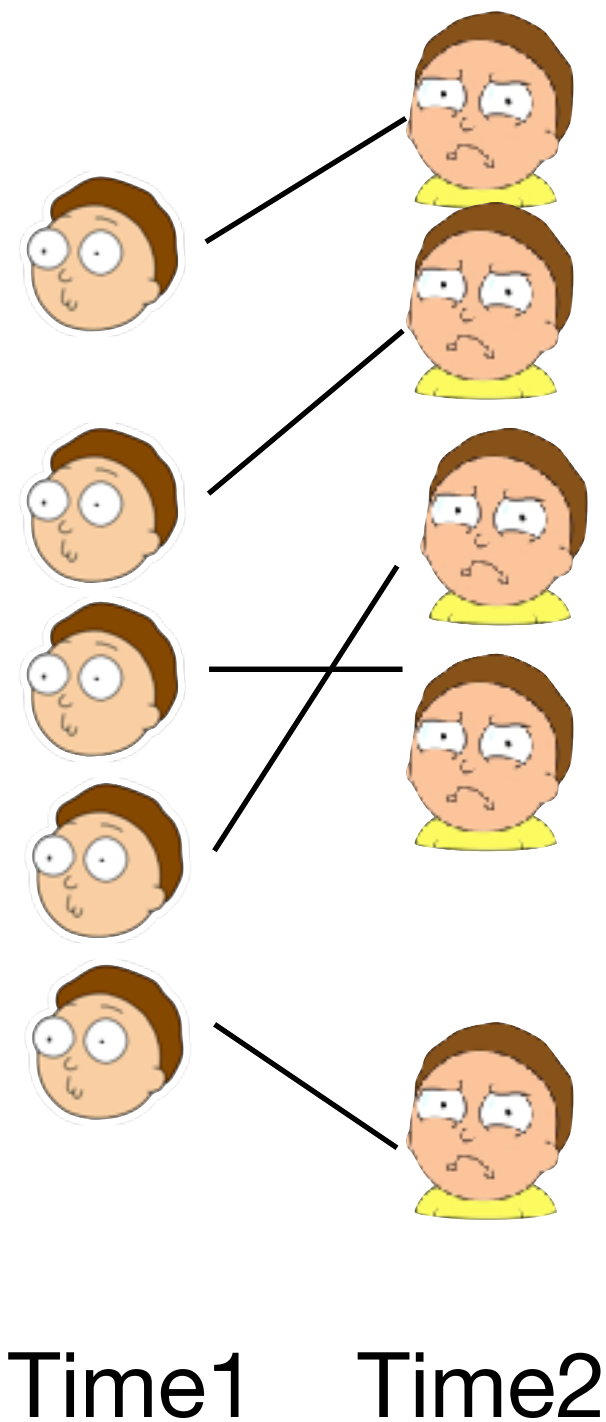
Linear Regression

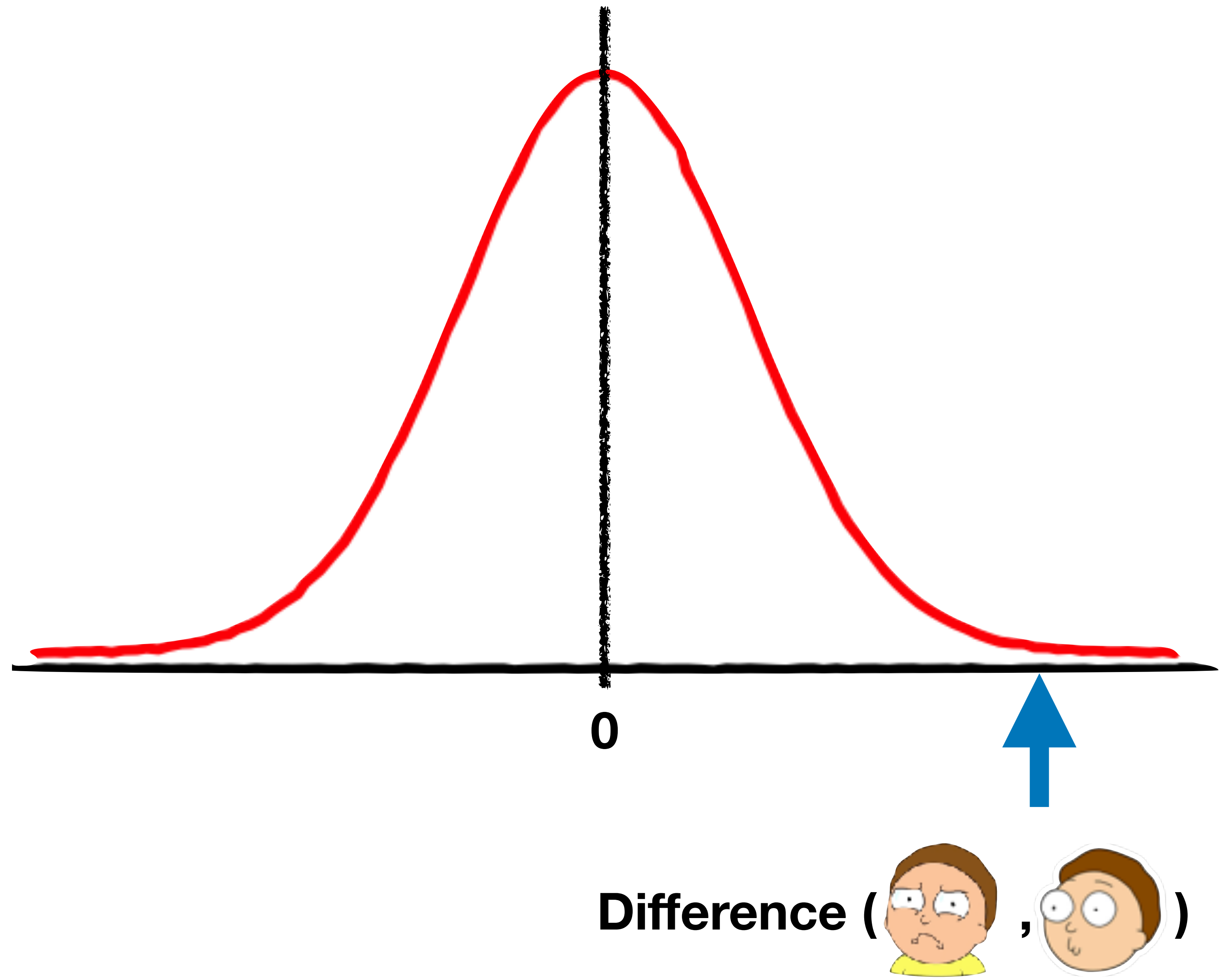
Paired Sample T-test



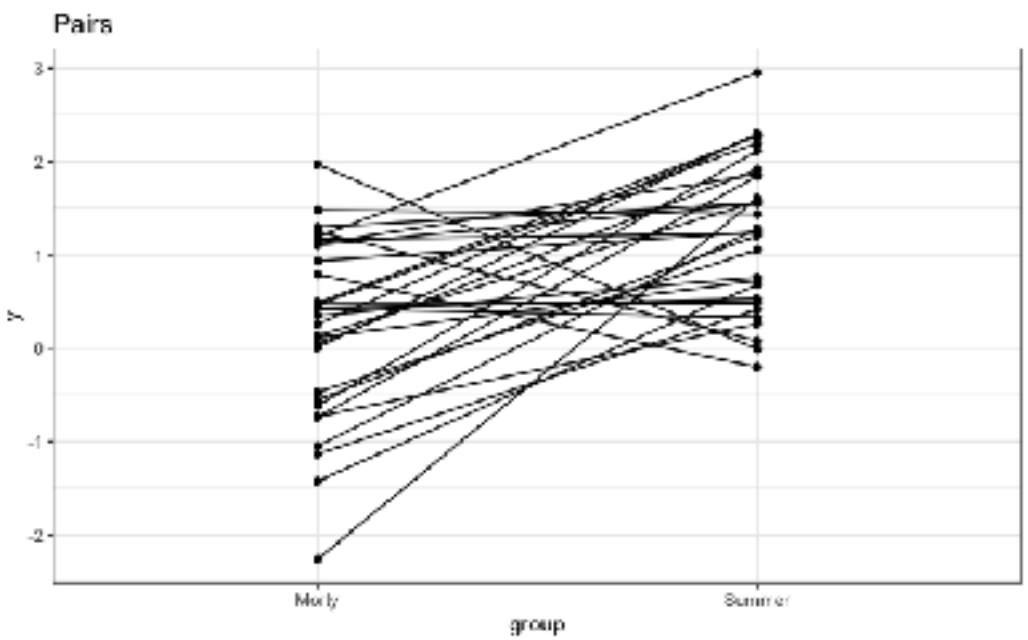
Paired T-test

Repeated Measure





Paired Sample T-test as One-sample T-test



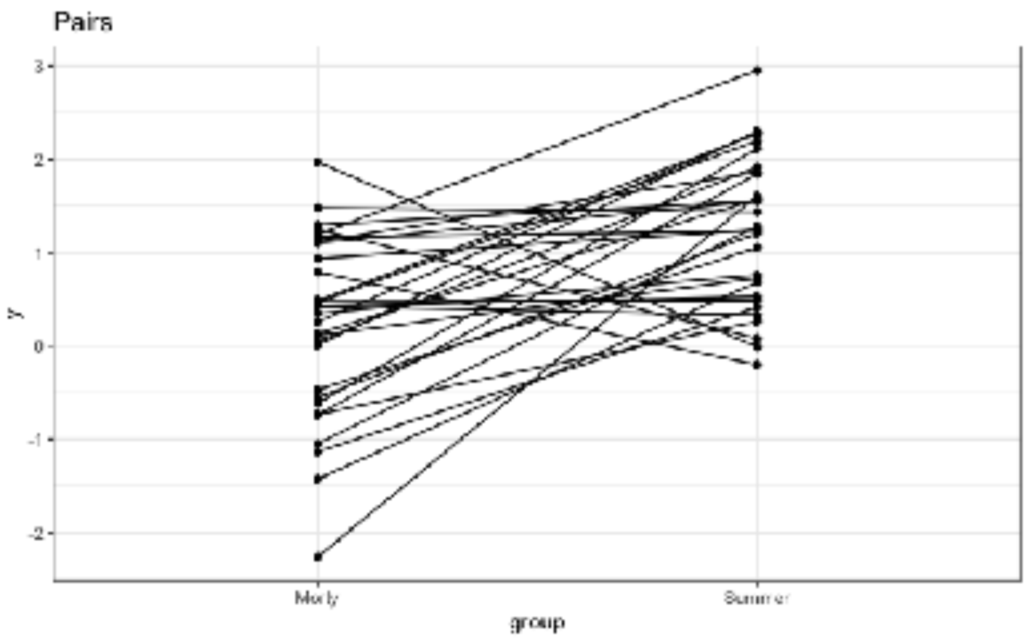

```
        Paired t-test

data:  y1 and y2
t = -4.8467, df = 29, p-value = 3.884e-05
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -1.7425597 -0.7083245
sample estimates:
mean of the differences
             -1.225442


        One Sample t-test

data:  y1 - y2
t = -4.8467, df = 29, p-value = 3.884e-05
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 -1.7425597 -0.7083245
sample estimates:
mean of x
-1.225442
```

Paired Sample T-test as repeated measure ANOVA



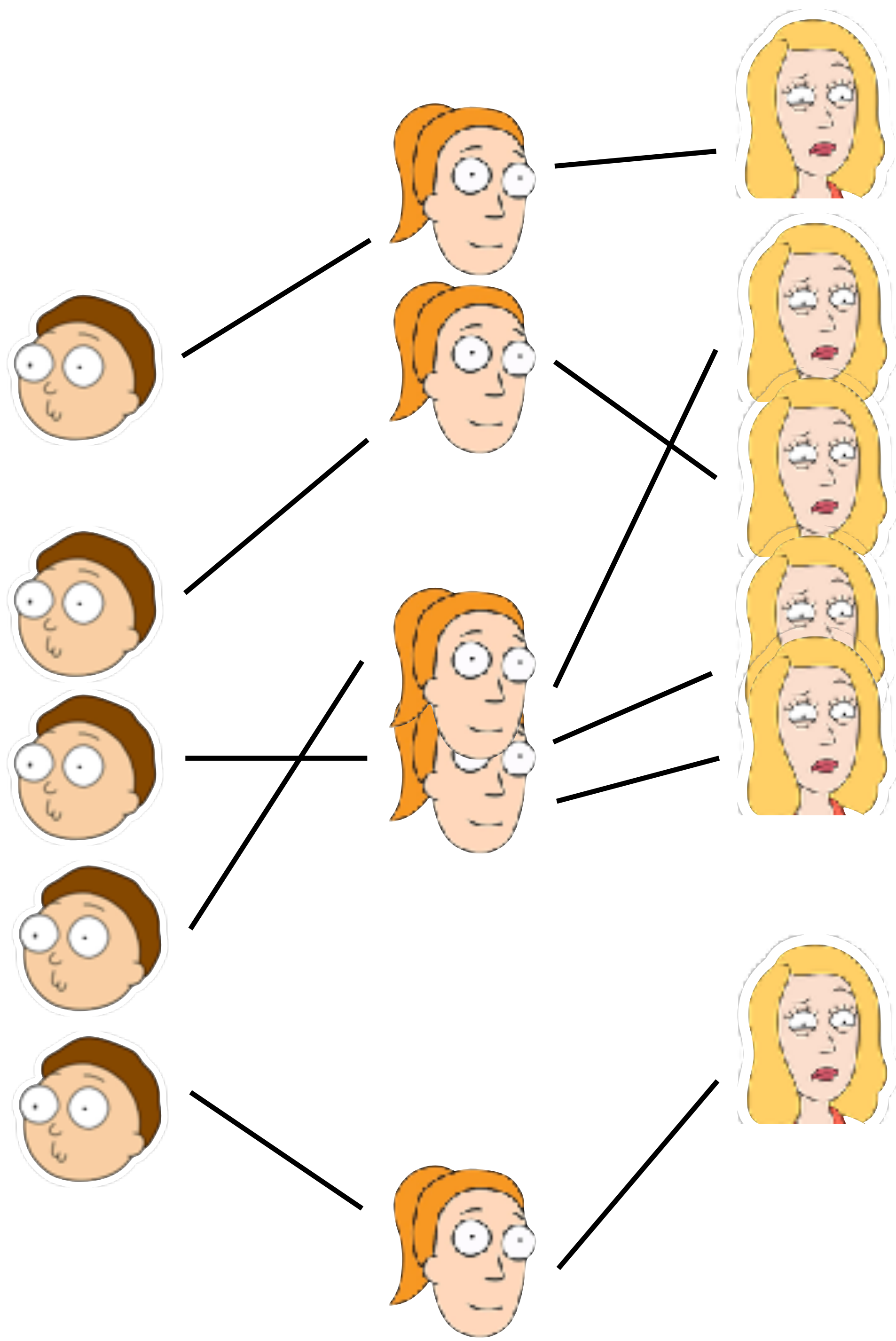

```
Paired t-test

data:  y1 and y2
t = -4.8467, df = 29, p-value = 3.884e-05
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -1.7425597 -0.7083245
sample estimates:
mean of the differences
              -1.225442

ANOVA Table (type III tests)

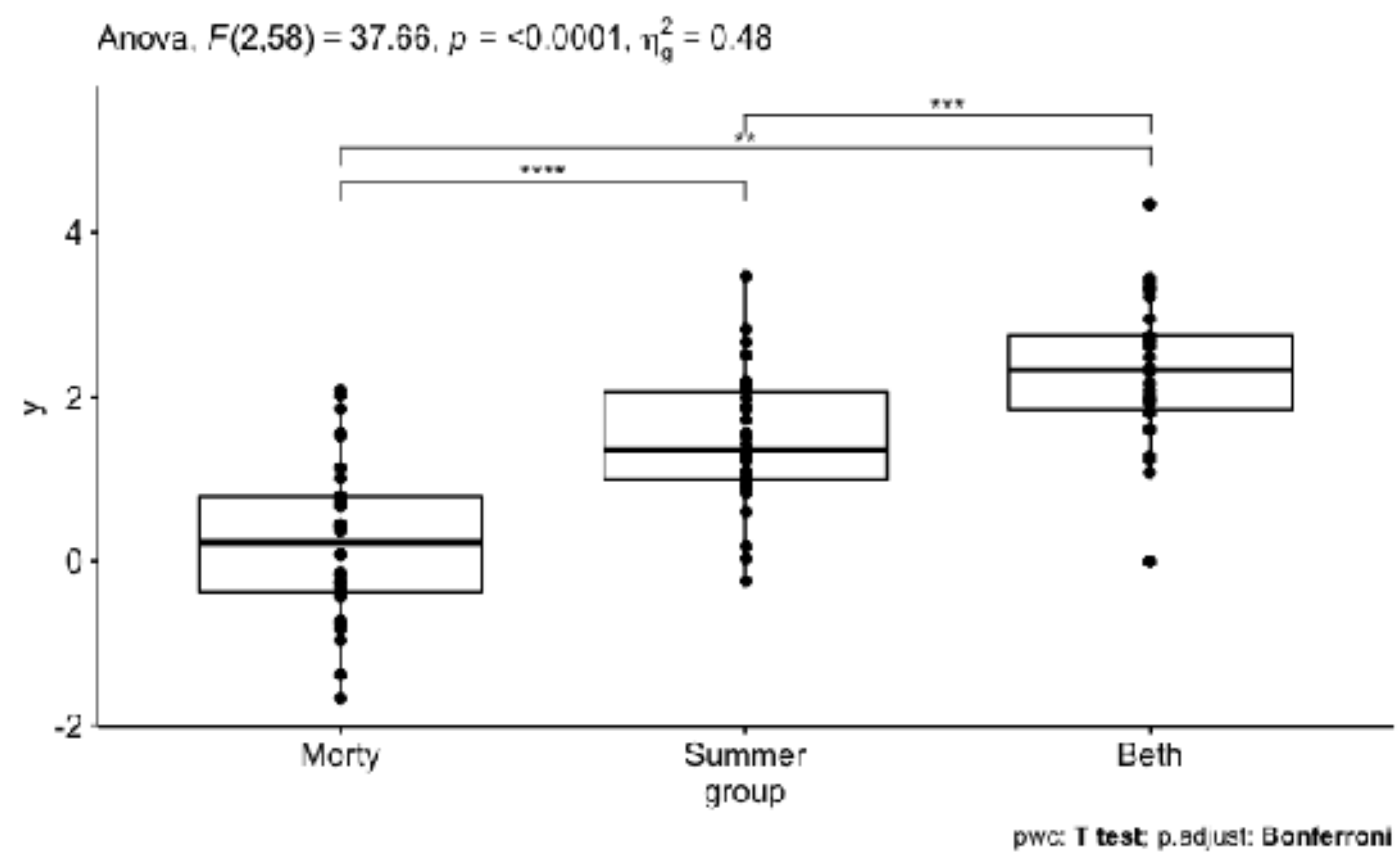
  Effect DFn DFd     F        p p<.05   ges
1  group   1  29 23.49 3.88e-05     * 0.319
```

One-way repeated measure ANOVA



```
ANOVA Table (type III tests)

  Effect DFn DFd      F        p p<.05   ges
1  group   2  58 37.664 3.29e-11     * 0.482
```

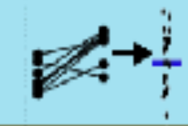
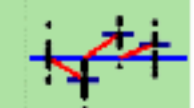
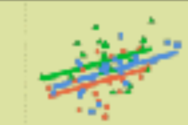


https://lindeloev.github.io/tests-as-linear/

Common statistical tests are linear models

Last updated: 02 April, 2019

See worked examples and more details at the accompanying notebook: https://lindeloev.github.io/tests-as-linear

	Common name	Built-in function in R	Equivalent linear model in R	Exact?	The linear model in words	Icon
Simple regression: lm(y ~ 1 + x)	**y is independent of x** P: One-sample t-test N: Wilcoxon signed-rank	t.test(y) wilcox.test(y)	lm(y ~ 1) lm(signed_rank(y) ~ 1)	✓ for N >14	One number (intercept, i.e., the mean) predicts **y**. - (Same, but it predicts the *signed rank* of **y**.)	
	P: Paired-sample t-test N: Wilcoxon matched pairs	t.test(y_1, y_2, paired=TRUE) wilcox.test(y_1, y_2, paired=TRUE)	lm(y_2 - y_1 ~ 1) lm(signed_rank(y_2 - y_1) ~ 1)	✓ for N >14	One intercept predicts the pairwise **y_2-y_1** differences. - (Same, but it predicts the *signed rank* of **y_2-y_1**.)	
	y ~ continuous x P: Pearson correlation N: Spearman correlation	cor.test(x, y, method='Pearson') cor.test(x, y, method='Spearman')	lm(y ~ 1 + x) lm(rank(y) ~ 1 + rank(x))	✓ for N >10	One intercept plus **x** multiplied by a number (slope) predicts **y**. - (Same, but with *ranked* **x** and **y**)	
	y ~ discrete x P: Two-sample t-test P: Welch's t-test N: Mann-Whitney U	t.test(y_1, y_2, var.equal=TRUE) t.test(y_1, y_2, var.equal=FALSE) wilcox.test(y_1, y_2)	lm(y ~ 1 + G_2)[A] gls(y ~ 1 + G_2, weights=...[B])[A] lm(signed_rank(y) ~ 1 + G_2)[A]	✓ ✓ for N >11	An intercept for **group 1** (plus a difference if **group 2**) predicts **y**. - (Same, but with one variance *per group* instead of one common.) - (Same, but it predicts the *signed rank* of **y**.)	
Multiple regression: lm(y ~ 1 + x_1 + x_2 +...)	P: One-way ANOVA N: Kruskal-Wallis	aov(y ~ group) kruskal.test(y ~ group)	lm(y ~ 1 + G_2 + G_3 +...+ G_N)[A] lm(rank(y) ~ 1 + G_2 + G_3 +...+ G_N)[A]	✓ for N >11	An intercept for **group 1** (plus a difference if group ≠ 1) predicts **y**. - (Same, but it predicts the *rank* of **y**.)	
	P: One-way ANCOVA	aov(y ~ group + x)	lm(y ~ 1 + G_2 + G_3 +...+ G_N + x)[A]	✓	- (Same, but plus a slope on **x**.) *Note: this is discrete AND continuous. ANCOVAs are ANOVAs with a continuous x.*	
	P: Two-way ANOVA	aov(y ~ group * sex)	lm(y ~ 1 + G_2 + G_3 + ... + G_N + S_2 + S_3+ ... + S_K + G_2*S_2+G_3*S_3+...+G_N*S_K)	✓	Interaction term: changing **sex** changes the **y ~ group** parameters. *Note: $G_{2 \text{ to } N}$ is an indicator (0 or 1) for each non-intercept levels of the **group** variable. Similarly for $S_{2 \text{ to } K}$ for sex. The first line (with G_i) is main effect of group, the second (with S_i) for sex and the third is the **group × sex** interaction. For two levels (e.g. male/female), line 2 would just be "S_2" and line 3 would be S_2 multiplied with each G_i.*	*[Coming]*
	Counts ~ discrete x N: Chi-square test	chisq.test(groupXsex_table)	**Equivalent log-linear model** glm(y ~ 1 + G_2 + G_3 + ... + G_N + S_2 + S_3+ ... + S_K + G_2*S_2+G_3*S_3+...+G_N*S_K, family=...)[A]	✓	Interaction term: (Same as Two-way ANOVA.) *Note: Run glm using the following arguments: `glm(model, family=poisson())` As linear-model, the Chi-square test is $\log(y_{ij}) = \log(N) + \log(\alpha_i) + \log(\beta_j) + \log(\alpha_i\beta_j)$ where α_i and β_j are proportions. See more info in the accompanying notebook.*	*Same as Two-way ANOVA*
	N: Goodness of fit	chisq.test(y)	glm(y ~ 1 + G_2 + G_3 +...+ G_N, family=...)[A]	✓	(Same as One-way ANOVA and see Chi-Square note.)	*1W-ANOVA*

List of common parametric (P) non-parametric (N) tests and equivalent linear models. The notation y ~ 1 + x is R shorthand for y = 1·b + a·x which most of us learned in school. Models in similar colors are highly similar, but really, notice how similar they *all* are across colors! For non-parametric models, the linear models are reasonable approximations for non-small sample sizes (see "Exact" column and click links to see simulations). Other less accurate approximations exist, e.g., Wilcoxon for the sign test and Goodness-of-fit for the binomial test. The signed rank function is `signed_rank = function(x) sign(x) * rank(abs(x))`. The variables G_i and S_i are "dummy coded" indicator variables (either 0 or 1) exploiting the fact that when Δx = 1 between categories the difference equals the slope. Subscripts (e.g., G_2 or y_1) indicate different columns in data. lm requires long-format data for all non-continuous models. All of this is exposed in greater detail and worked examples at https://lindeloev.github.io/tests-as-linear.

[A] See the note to the two-way ANOVA for explanation of the notation.
[B] Same model, but with one variance per group: `gls(value ~ 1 + G2, weights = varIdent(form = ~1|group), method="ML")`.

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