Exercises for Chapter 3
Set & Function

1 Introduction

In the exercise below, we will familiarize ourselves with knowledge related to set theory and function.
Students should review the theory of chapter 3 before doing the exercises.

2 Sample exercises

Exercise 1.

a) $0 \in \emptyset$

b) $\{\emptyset \in \{\{\emptyset\}\}$

c) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

Solution.

a) Incorrect

b) Incorrect

c) Correct

$\square$

Exercise 2.
Sketch Venn diagram for the following problems

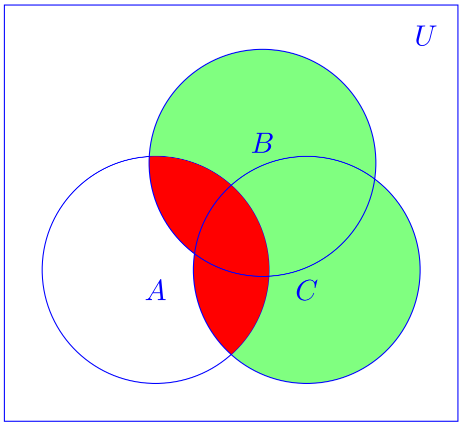
a) $A \cap (B \cup C)$

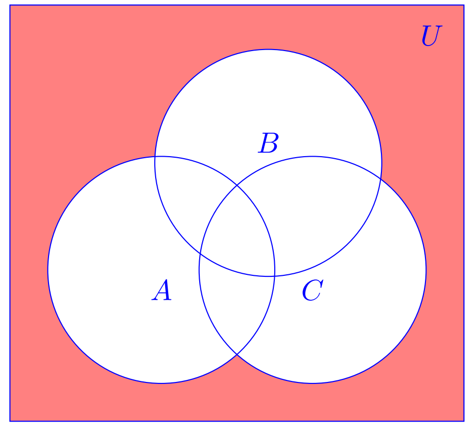
b) $\overline{A} \cap \overline{B} \cap \overline{C}$

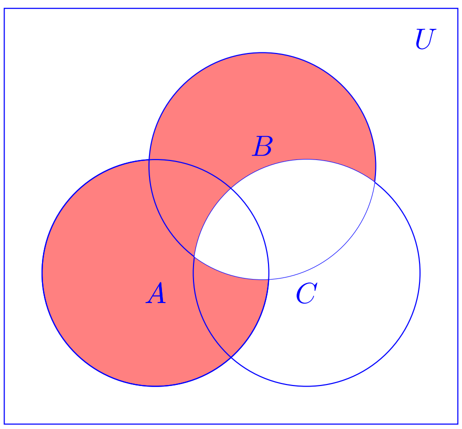
c) $(A - B) \cup (A - C) \cup (B - C)$

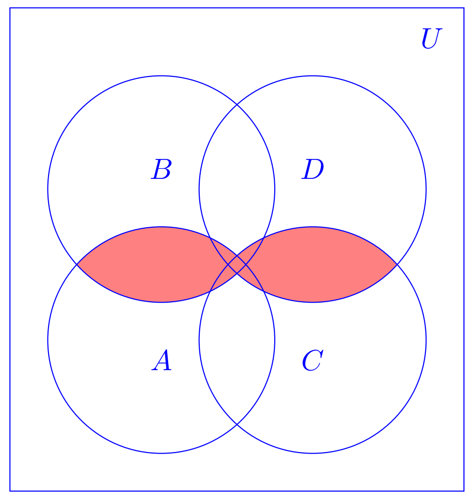
d) $(A \cap B) \cup (C \cap D)$

Solution.

a) 

b) 

c) 

d) 

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3 Practice exercises

Exercise 3.

List the members of these sets.

a) $\{x \quad — \quad x$ is a real number such that $x^2 = 1\}$

b) $\{x \quad — \quad x$ is a positive integer less than $12\}$

c) $\{x \quad — \quad x$ is the square of an integer and $x < 100\}$

d) $\{x \quad — \quad x$ is an integer such that $x^2 = 2\}$

Exercise 4.
For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbb{R} \mid x \text{ is an integer greater than } 1\}$

b) $\{x \in \mathbb{R} \mid x \text{ is the square of an integer}\}$

c) $\{2, \{2\}\}$

d) $\{\{2\}, \{\{2\}\}\}$

e) $\{\{2\}, \{2, \{2\}\}\}$

f) $\{\{\{2\}\}\}$

Exercise 5.
Determine whether each of these statements is true or false.

a) $0 \in \emptyset$

b) $\emptyset \in \{0\}$

c) $\{0\} \subset \emptyset$

d) $\emptyset \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\emptyset\} \subseteq \{\emptyset\}$

Exercise 6.
Determine whether each of these statements is true or false.

a) $x \in \{x\}$

b) $\{x\} \subseteq \{x\}$

c) $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$

e) $\emptyset \subseteq \{x\}$

f) $\emptyset \in \{x\}$

Exercise 7.
Use Venn diagram to illustrate each of statements is WRONG.

a) $(A \cup B) \cap C = A \cup (B \cap C)$

b) $(A - B) \cap (C - B) = A - (B \cup C)$

Exercise 8.
Suppose that A, B, C are sets such that $A \subseteq B$ và $B \subseteq C$. Show that $A \subseteq C$.

Exercise 9.
What is the cardinality of each of these sets?

a) $\{a\}$ 1

b) $\{\{a\}\}$ 1

c) $\{a, \{a\}\}$ 2

d) $\{a, \{a\}, \{a, \{a\}\}\}$ 3

e) $\{\emptyset\}$ 1

Exercise 10.
Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$

b) $\{a, b\}$

c) $\{\emptyset, \{\emptyset\}\}$

Exercise 11.
Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) $A \times B$

b) $B \times A$

Exercise 12.
Let $A = \{a, b, c\}, B = \{x, y\}$, và $C = \{0, 1\}$. Find

a) $A \times B$

b) $B \times A$

c) $A \times B \times C$

d) $C \times B \times A$

e) $C \times A \times B$

f) $B \times B \times B$

Exercise 13.
Find A^2 if

a) $A = \{0, 1, 3\}$

b) $A = \{1, 2, a, b\}$

Exercise 14.
Use a Venn diagram to illustrate the relationship

a) $A \subseteq B, C \subseteq B, A \cap C = \emptyset$

b) $A \supseteq C, B \cap C = \emptyset$

Exercise 15.
A chessboard with 64 black and white squares, 8 rows (labelled from 1 to 8), and 8 columns (labelled from a to h), each of these squares is expressed in a pair of numbers (row,column)

a) The Knight is at (d,3). List locations of regular movement (only 1 move for each movement).

b) If R={1,2,3,4,5,6,7,8} is the set of rows on the chessboard, C={a,b,c,d,e,f,g,h} is the set of columns on the chessboard, do define set P that expresses as all the locations on the chessboard.

c) The rook is at (g,2), if T={2} and G={g}, do define the set of regular movement (only 1 move for each movement).

Exercise 16.

a) How many different elements does $A \times B$ have if A has m elements and B has n elements?

b) How many different dlements does A^n have when A has m elements and n is a positive integer?

Exercise 17.
Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cap B$

b) $A \cup B$

c) $A - B$

d) $B - A$

Exercise 18.
Show that if A and B are sets, then

a) $A - B = A \cap \overline{B}$

b) $(A \cap B) \cup (A \cap \overline{B}) = A$

Exercise 19.
Prove that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

a) by showing each side is a subset of the order side.

b) using a membership table.

Exercise 20.
Determine whether f is a function if

a) $f(x) = 1/x$, from $\mathbb{R}$ to $\mathbb{R}$

b) $f(n) = \pm n$, from $\mathbb{Z}$ to $\mathbb{R}$

c) $f(n) = \sqrt{n^2 + 1}$, from $\mathbb{Z}$ to $\mathbb{N}$

d) $f(n) = 1/(n^2 - 4)$, from $\mathbb{Z}$ to $\mathbb{R}$

e) $f(S)$ is the position of a 0 bit in S, from the set of all bit strings to the set of integers.

f) $f(S)$ is the number of 1 bits in S, from the set of all bit strings to the set of integers .

Exercise 21.
Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each bit string the number of ones in the string minus the number of zeros in the string.

b) the function that assigns to each bit string twice the number of zeros in that string.

c) the function that assigns to each positive integer the largest perfect square not exceeding this integer.

d) the function that assigns to each nonnegative integer its last digit.

e) the function that assigns to a bit string the longest string of ones in the string.

f) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer.

Exercise 22.
Give an explicit formula for a function from the set of integers to the set of positive integers that is

a) one-to-one, but not onto.

b) onto, but not one-to-one.

c) one-to-one and onto.

d) neither one-to-one nor onto.

Exercise 23.
Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

a) $f(x) = 2x + 5$

b) $f(x) = x^2 + 5$

c) $f(x) = (x + 1)/(x + 5)$

d) $f(x) = x^5 + 1$

e) $f(x) = (x^2 + 1)/(x^2 + 2)$

f) $f(x) = \lfloor x + \frac{1}{2} \rfloor$

Exercise 24.
Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.

Exercise 25.
Show that the function $f(x) = |x|$ from the set of real numbers to the set of nonnegative real numbers is not invertible, but if the domain is restricted to the set of nonnegative real numbers, the resulting function is invertible.

Exercise 26.
Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Exercise 27.
Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbb{R}$ to $\mathbb{R}$.

Exercise 28.
If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

Exercise 29.
If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

Exercise 30.
Let f be a function from A to B. Let S and T be subsets of B. Show that

a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

Exercise 31.
What are the values of these sums?

a) $\displaystyle\sum_{k=1}^{5}(k+1)$

b) $\displaystyle\sum_{j=0}^{4}(-2)^j$

c) $\displaystyle\sum_{i=3}^{10}3$

d) $\displaystyle\sum_{j=0}^{8}(2^{j+1}-2^j)$