Chapter 4
Sets

Discrete Structures for Computing on January 4, 2023

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Contents

Course outcomes

	Course learning outcomes
L.O.1	Understanding of logic and discrete structures
	L.O.1.1 – Describe definition of propositional and predicate logic L.O.1.2 – Define basic discrete structures: set, mapping, graphs
L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing L.O.2.2 – Use proving methods: direct, contrapositive, induction L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
	L.O.3.1 – Define basic probability theory L.O.3.2 – Explain discrete random variables
L.O.4	Compute quantities of discrete structures and probabilities
	L.O.4.1 – Operate (compute/ optimize) on discrete structures L.O.4.2 – Compute probabilities of various events, conditional ones, Bayes theorem

Set Definition

- Set is a **fundamental** discrete structure on which all discrete structures are built
- Sets are used to group objects, which often have the **same properties**

Example

- Set of all the students who are currently taking Discrete Mathematics 1 course.
- Set of all the subjects that K2011 students have to take in the first semester.
- Set of natural numbers $\mathbb{N}$

Definition

A set is an unordered collection of objects.
The objects in a set are called the elements (*phần tử*) of the set.
A set is said to contain (*chứa*) its elements.

Notations

Definition

- $a \in A$: a is an element of the set A
- $a \notin A$: a is not an element of the set A

Definition (Set Description)

- The set V of all vowels in English alphabet, $V = \{a, e, i, o, u\}$
- Set of all real numbers greater than 1???
  $\{x \mid x \in \mathbb{R}, x > 1\}$
  $\{x \mid x > 1\}$
  $\{x \ : \ x > 1\}$

Equal Sets

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Definition

Two sets are equal iff they have the same elements.

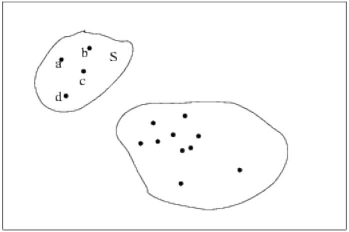
- $(A = B) \leftrightarrow \forall x (x \in A \leftrightarrow x \in B)$

Example

- $\{1, 3, 5\} = \{3, 5, 1\}$
- $\{1, 3, 5\} = \{1, 3, 3, 3, 5, 5, 5, 5\}$

Venn Diagram

- John Venn in 1881
- Universal set (*tập vũ trụ*) is represented by a rectangle
- Circles and other geometrical figures are used to represent sets
- Points are used to represent particular elements in set



Special Sets

- Empty set (*tập rỗng*) has no elements, denoted by $\emptyset$, or $\{\}$
- A set with one element is called a singleton set
- What is $\{\emptyset\}$?
- Answer: singleton

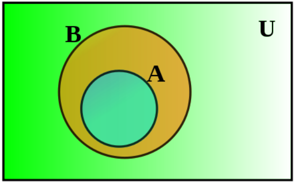
Subset

Definition

The set A is called a subset (*tập con*) of B iff every element of A is also an element of B, denoted by $A \subseteq B$.

If $A \neq B$, we write $A \subset B$ and say A is a proper subset (*tập con thực sự*) of B.

- $\forall x(x \in A \rightarrow x \in B)$
- For every set S,
  (i) $\emptyset \subseteq S$, (ii) $S \subseteq S$.



Cardinality

Definition

If S has exactly n distinct elements where n is non-negative integers, S is finite set (*tập hữu hạn*), and n is **cardinality** (*bản số*) of S, denoted by $|S|$.

Example

- A is the set of odd positive integers less than 10. $|A| = 5$.
- S is the letters in Vietnamese alphabet, $|S| = 29$.
- Null set $|\emptyset| = 0$.

Definition

A set that is infinite if it is not finite.

Example

- Set of positive integers is infinite

Power Set

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Definition

Given a set S, the power set (*tập lũy thừa*) of S is the set of all subsets of the set S, denoted by $P(S)$.

Example

What is the power set of $\{0, 1, 2\}$?
$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Example

- What is the power set of the empty set?
- What is the power set of the set $\{\emptyset\}$

Power Set

Theorem

If a set has n elements, then its power set has 2^n elements.

Prove using induction!

Ordered n-tuples

Definition

The ordered n-tuple (*dãy sắp thứ tự*) $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

Definition

Two ordered n-tuples $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ iff $a_i = b_i$, for $i = 1, 2, \ldots, n$.

Example

2-tuples, or **ordered pairs** (*cặp*), (a, b) and (c, d) are equal iff $a = c$ and $b = d$

Cartesian Product

- René Descartes (1596–1650)

Definition

Let A and B be sets. The Cartesian product (*tích Đề-các*) of A and B, denoted by $A \times B$, is the set of ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Example

Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

Show that $A \times B \neq B \times A$

Cartesian Product

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Definition

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Example

$A = \{0, 1\}, B = \{1, 2\}, C = \{0, 1, 2\}$. What is $A \times B \times C$?

$$\begin{aligned} A \times B \times C \quad = \quad & \{(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),\\ & (0,2,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),\\ & (1,2,1),(1,2,2)\} \end{aligned}$$

Union

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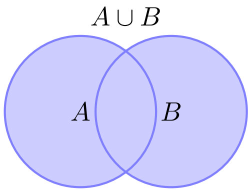
Definition

The union (*hợp*) of A and B

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$



- Example:
  - $\{1,2,3\} \cup \{2,4\} = \{1,2,3,4\}$
  - $\{1,2,3\} \cup \emptyset = \{1,2,3\}$

Intersection

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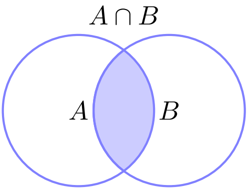
Definition

The intersection (*giao*) of A and B

$$A \cap B = \{x \mid x \in A \land x \in B\}$$



Example:

- $\{1,2,3\} \cap \{2,4\} = \{2\}$
- $\{1,2,3\} \cap \mathbb{N} = \{1,2,3\}$

Union/Intersection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = \{x \mid x \in A_1 \vee x \in A_2 \vee \ldots \vee x \in A_n\}$$

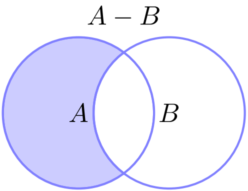
$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n = \{x \mid x \in A_1 \wedge x \in A_2 \wedge \ldots \wedge x \in A_n\}$$

Difference

Definition

The difference (*hiệu*) of A and B

$$A - B = \{x \mid x \in A \land x \notin B\}$$



Example:

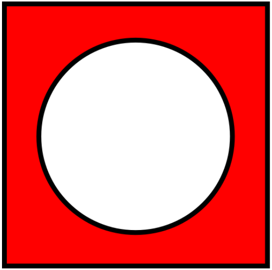
- {1,2,3} - {2,4} = {1,3}
- {1,2,3} - $\mathbb{N} = \emptyset$

Complement

Definition

The complement (*phần bù*) of A

$$\overline{A} = \{x \mid x \notin A\}$$



Example:

- A = {1,2,3} then $\overline{A}$ = ???
- **Note that A - B = A $\cap$ $\overline{B}$**

Set Identities

$A \cup \emptyset$	$=$	A	Identity laws
$A \cap U$	$=$	A	Luật đồng nhất
$A \cup U$	$=$	U	Domination laws
$A \cap \emptyset$	$=$	$\emptyset$	Luật nuốt
$A \cup A$	$=$	A	Idempotent laws
$A \cap A$	$=$	A	Luật lũy đẳng
$\overline{(\bar{A})}$	$=$	A	Complementation law
			Luật bù

Set Identities

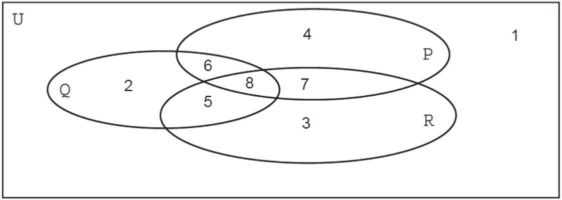
$A \cup B$	$=$	$B \cup A$	Commutative laws
$A \cap B$	$=$	$B \cap A$	Luật giao hoán
$A \cup (B \cup C)$	$=$	$(A \cup B) \cup C$	Associative laws
$A \cap (B \cap C)$	$=$	$(A \cap B) \cap C$	Luật kết hợp
$A \cup (B \cap C)$	$=$	$(A \cup B) \cap (A \cup C)$	Distributive laws
$A \cap (B \cup C)$	$=$	$(A \cap B) \cup (A \cap C)$	Luật phân phối
$\overline{A \cup B}$	$=$	$\overline{A} \cap \overline{B}$	De Morgan's laws
$\overline{A \cap B}$	$=$	$\overline{A} \cup \overline{B}$	Luật De Morgan

Method of Proofs of Set Equations

To prove $A = B$, we could use

- Venn diagrams
- Prove that $A \subseteq B$ and $B \subseteq A$
- Use membership table
- Use set builder notation and logical equivalences

Example (1)



Example

Verify the distributive rule $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$

Example (2)

Example

Prove: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(1) Show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

Suppose that $x \in \overline{A \cap B}$
By the definition of complement, $x \notin A \cap B$
So, $x \notin A$ or $x \notin B$
Hence, $x \in \bar{A}$ or $x \in \bar{B}$
We conclude, $x \in \overline{A} \cup \overline{B}$
Or, $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

(2) Show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Example (3)

Prove: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

A	B	$A \cap B$	$\overline{A \cap B}$	$\bar{A} \cup \bar{B}$
1	1	1	0	0
1	0	0	1	1
0	1	0	1	1
0	0	0	1	1

Example (4)

Prove: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$$\begin{aligned}
\overline{A \cap B} &= \{x | x \notin A \cap B\} \\
&= \{x | \neg(x \in A \cap B)\} \\
&= \{x | \neg(x \in A \wedge x \in B)\} \\
&= \{x | \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x | x \notin A \vee x \notin B\} \\
&= \{x | x \in \overline{A} \vee x \in \overline{B}\} \\
&= \{x | x \in \overline{A} \cup \overline{B}\}
\end{aligned}$$