Chapter 6
Relations

Discrete Structures for Computing on January 4, 2023

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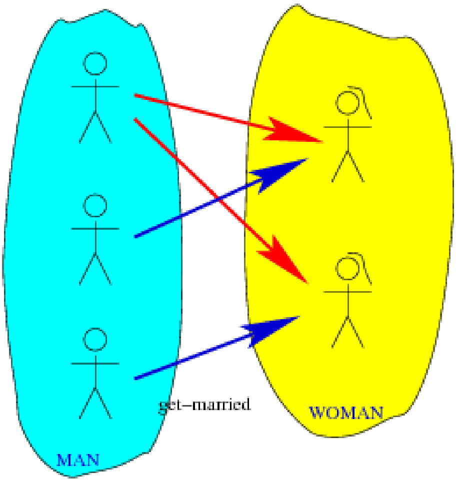
Contents

Course outcomes

	Course learning outcomes
L.O.1	Understanding of logic and discrete structures
	L.O.1.1 – Describe definition of propositional and predicate logic L.O.1.2 – Define basic discrete structures: set, mapping, graphs
L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing L.O.2.2 – Use proving methods: direct, contrapositive, induction L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
	L.O.3.1 – Define basic probability theory L.O.3.2 – Explain discrete random variables
L.O.4	Compute quantities of discrete structures and probabilities
	L.O.4.1 – Operate (compute/ optimize) on discrete structures L.O.4.2 – Compute probabilities of various events, conditional ones, Bayes theorem

Introduction



Function?

Relation

Definition

Let A and B be sets. A **binary relation** (*quan hệ hai ngôi*) from a set A to a set B is a set

$$R \subseteq A \times B$$

- Notations:

  $(a, b) \in R \longleftrightarrow aRb$

- **n-ary relations?**

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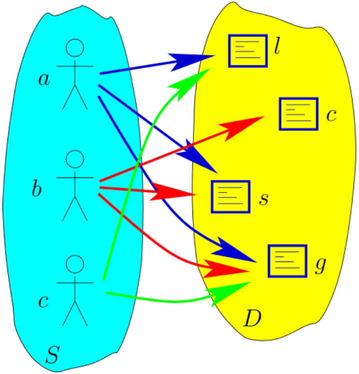
Example

Example

Let $A = \{a, b, c\}$ be the set of students, $B = \{l, c, s, d\}$ be the set of the available optional courses. We can have relation R that consists of pairs (a, b), where a is a student enrolled in course b.



$$R = \{(a,l),(a,s),(a,g),(b,c),(b,s),(b,g),(c,l),(c,g)\}$$

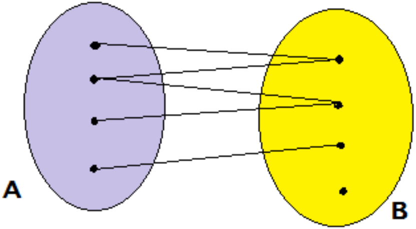
R	l	c	s	g
a	x		x	x
b		x	x	x
c	x			x

Functions as Relations

- **Is a function a relation?**
- **Yes!**
- $f: A \to B$
  $R = \{(a, b) \mid b = f(a)\}$

Functions as Relations

- **Is a relation a function?**
- **No**



- Relations are a generalization of functions

Relations on a Set

Definition

A relation on the set A is a relation from A to A.

Example

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ (a *là ước số của* b)?

Solution:

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$$

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Properties of Relations

Reflexive (*phản xạ*)	$xRx, \forall x \in A$
Symmetric (*đối xứng*)	$xRy \to yRx, \forall x, y \in A$
Antisymmetric (*phản đối xứng*)	$(xRy \land yRx) \to x = y, \forall x, y \in A$
Transitive (*bắc cầu*)	$(xRy \land yRz) \to xRz, \forall x, y, z \in A$

Example

Example

Consider the following relations on $\{1, 2, 3, 4\}$:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$,
$R_2 = \{(1,1),(1,2),(2,1)\}$,
$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,
$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,
$R_5 = \{(3,4)\}$

Solution:

- Reflexive: R_3
- Symmetric: R_2, R_3
- Antisymmetric: R_4, R_5
- Transitive: R_4, R_5

Example

Example

What is the properties of the **divides** (*ước số*) relation on the set of positive integers?

Solution:

- $\forall a \in \mathbb{Z}^+, a \mid a$: reflexive
- $1 \mid 2$, but $2 \nmid 1$: not symmetric
- $\exists a, b \in \mathbb{Z}^+, (a \mid b) \wedge (b \mid a) \rightarrow a = b$: antisymmetric
- $a \mid b \Rightarrow \exists k \in \mathbb{Z}^+, b = ak; b \mid c \Rightarrow \exists l \in \mathbb{Z}^+, c = bl$. Hence, $c = a(kl) \Rightarrow a \mid c$: transitive

Example

Example

What are the properties of these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

Combining Relations

Because relations from A to B are **subsets** of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Example

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. List the combinations of relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$.

Solution: $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$ and $R_2 - R_1$.

Example

Let A and B be the set of all students and the set of all courses at school, respectively. Suppose $R_1 = \{(a, b) \mid a$ has taken the course $b\}$ and $R_2 = \{(a, b) \mid a$ requires course b to graduate$\}$. What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, $R_2 - R_1$?

Composition of Relations

Definition

Let R be **relations** from A to B and S be from B to C. Then the **composite** (*hợp thành*) of S and R is

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B\ (aRb \wedge bSc)\}$$

Example

$R = \{(0, 0), (0, 3), (1, 2), (0, 1)\}$
$S = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$
$S \circ R = \{(0, 0), (0, 1), (1, 1)\}$

Power of Relations

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Definition

Let R be a relation on the set A. The **powers** (*lũy thừa*) $R^n, n = 1, 2, 3, \ldots$ are defined recursively by

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R.$$

Example

Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers $R^n, n = 2, 3, 4, \ldots$.

Solution:
$R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$
$R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$
$R^4 = \{(1,1), (2,1), (3,1), (4,1)\}$
$\ldots$

Representing Relations Using Matrices

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Definition

Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, R can be represented by the **matrix** $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if} (a_i, b_j) \in R \\ 0 & \text{if} (a_i, b_j) \notin R \end{cases}$$

Example

R is relation from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$. Let $R = \{(2,1), (3,1), (3,2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Determine whether the relation has certain properties (reflexive, symmetric, antisymmetric,...)

Representing Relations Using Digraphs

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Definition

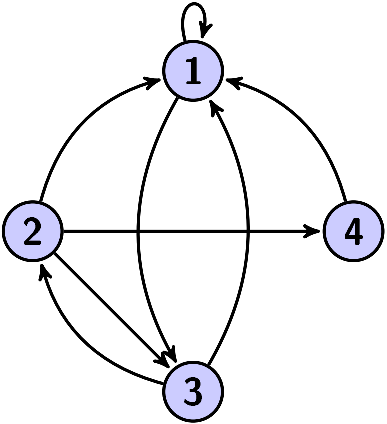
Suppose R is a relation in $A = \{a_1, a_2, \ldots, a_m\}$, R can be represented by the **digraph** (*đồ thị có hướng*) $G = (V, E)$, where

$$V = A$$
$$(a_i, a_j) \in E \text{ if } (a_i, a_j) \in R$$

Example

Given a relation on $A = \{1, 2, 3, 4\}$,
$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$
Draw corresponding digraph.

Resulting digraph



Closure

Definition

The **closure** (*bao đóng*) of relation R with respect to **property** P is the relation S that

i. **contains** R

ii. **has** property P

iii. is **contained in any** relation satisfying **(i) and (ii)**.

S is the "smallest" relation satisfying (i) & (ii)

Reflexive Closure

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Example

Let $R = \{(a,b),(a,c),(b,d),(d,c)\}$

The **reflexive closure** of R

$$\{(a,b),(a,c),(b,d),(d,c),(a,a),(b,b),(c,c),(d,d)\}$$

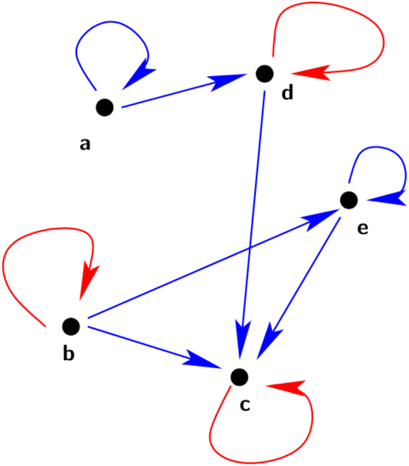
$$R \cup \Delta$$

where

$$\Delta = \{(a,a) \mid a \in A\}$$

diagonal relation (*quan hệ đường chéo*).

Reflexive Closure



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Symmetric Closure

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Example

Let $R = \{(a,b),(a,c),(b,d),(c,a),(d,e)\}$

The **symmetric closure** of R

$$\{(a,b),(a,c),(b,d),(c,a),(d,e),(b,a),(d,b),(e,d)\}$$

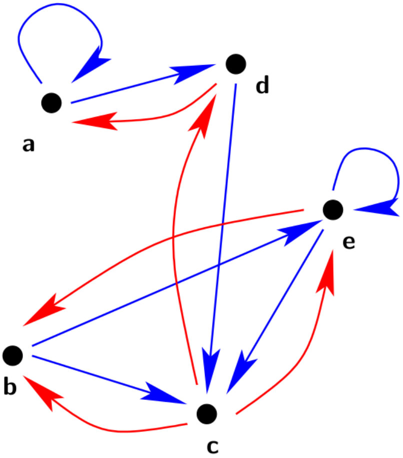
$$R \cup R^{-1}$$

where

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

inverse relation (*quan hệ ngược*).

Symmetric Closure



Transitive Closure

Example

Let R = $\{(a, b), (a, c), (b, d), (d, e)\}$

The **transitive closure** of R

$$\{(a, b), (a, c), (b, d), (d, e), (a, d), (b, e), (a, e)\}$$

$$\cup_{n=1}^{\infty} R^n$$

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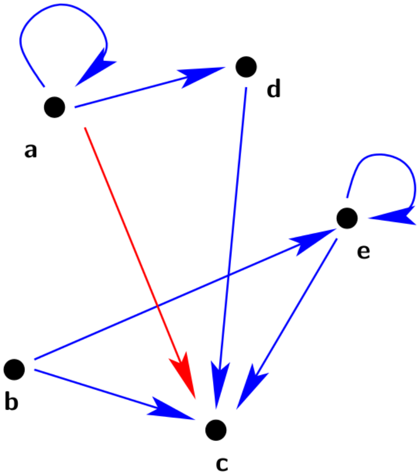
Transitive Closure



Equivalence Relations

Definition

A relation on a set A is called an **equivalence relation** (*quan hệ tương đương*) if it is reflexive, symmetric and transitive.

Example (1)

The relation $R = \{(a, b) | a \text{ and } b \text{ are in the same provinces}\}$ is an equivalence relation. a is equivalent to b and vice versa, denoted $a \sim b$.

Example (2)

$$R = \{(a, b) \mid a = b \vee a = -b\}$$

R is an equivalence relation.

Example (3)

$$R = \{(x, y) \mid |x - y| < 1\}$$

Is R an equivalence relation?

Example

Example (Congruence Modulo m - Đồng dư modulo m)

Let m be a positive integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Equivalence Classes

Definition

Let R be an **equivalence relation** on the set A. The set of all elements that are related to an element a of A is called the **equivalence class** (*lớp tương đương*) of a, denoted by

$$[a]_R = \{s \mid (a, s) \in R\}$$

Example

The equivalence class of "Thủ Đức" for the equivalence relation "in the same provinces" is { "Thủ Đức", "Gò Vấp", "Bình Thạnh", "Quận 10",...}

Example

Example

What are the equivalence classes of 0, 1, 2, 3 for congruence modulo 4?

Solution:

$$[0]_4 = \{..., -8, -4, 0, 4, 8, ...\}$$
$$[1]_4 = \{..., -7, -3, 1, 5, 9, ...\}$$
$$[2]_4 = \{..., -6, -2, 2, 6, 10, ...\}$$
$$[3]_4 = \{..., -5, -1, 3, 7, 11, ...\}$$

Equivalence Relations and Partitions

Theorem

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

i aRb

ii $[a] = [b]$

iii $[a] \cap [b] \neq \emptyset$

Example 1

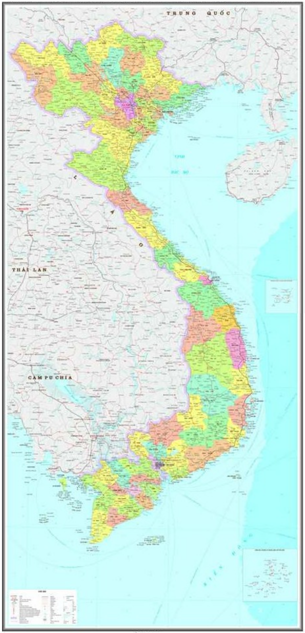
Example

Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S, because these sets are disjoint and their union is S

The equivalence classes of an equivalence relation R on a set S form a **partition** of S.

Every partition of a set can be used to form an equivalence relation.

Example 2



Example

Divides set of all cities and towns in Vietnam into set of 64 provinces. We know that:

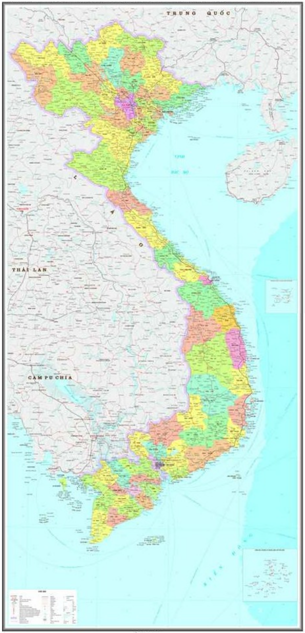
- there are no provinces with no cities or towns
- no city is in more than one province
- every city is accounted for

Definition

A **partition** of a Vietnam is a collection of non-overlapping non-empty subsets of Vietnam (provinces) that, together, make up all of Vietnam.

Contents

Relation in a Partition



- We divided based on relation

$$R = \{(a, b) | a \text{ and } b \text{ are in the same provinces}$$

- "Thủ Đức" is related (equivalent) to "Gò Vấp"
- "Đà Lạt" is not related (not equivalent) to "Long Xuyên"

Partial Order Relations

- Order words such that x comes before y in the dictionary
- Schedule projects such that x must be completed before y
- Order set of integers, where $x < y$

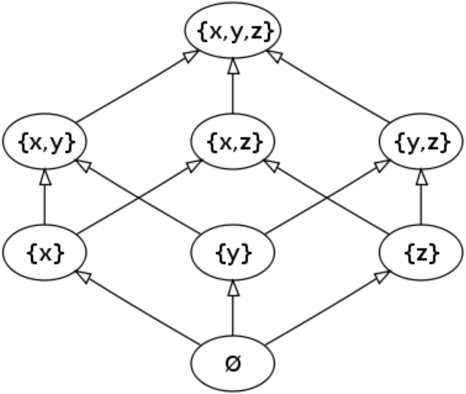
Definition

A relation R on a set S is called a **partial ordering** (*có thứ tự bộ phận*) if it is reflexive, antisymmetric and transitive. A set S together with a partial ordering R is called a partially ordered set, or **poset** (*tập có thứ tự bộ phận*), and is denoted by (S, R) or $(S, \preccurlyeq)$.

Example

- $(\mathbb{Z}, \geq)$ is a poset
- Let S a set, $(P(S), \subseteq)$ is a poset

Example



Totally Order Relations

Example

In the poset $(\mathbb{Z}^+, |)$, 3 and 9 are comparable (*so sánh được*), because $3 \mid 9$, but 5 and 7 are not, because $5 \nmid 7$ and $7 \nmid 5$.

$\rightarrow$ That's why we call it **partially** ordering.

Definition

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a **totally ordered** (*có thứ tự toàn phần*). A totally ordered set is also called a **chain** (*dây xích*).

Example

The poset $(\mathbb{Z}, \leq)$ is totally ordered.

Maximal & Minimal Elements

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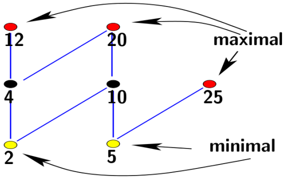
Definition

- a is **maximal** (*cực đại*) in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$.
- a is **minimal** (*cực tiểu*) in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \prec a$.

Example

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are minimal and maximal?



Greatest Element& Least Element

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Definition

- a is the **greatest element** (*lớn nhất*) of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$.
- a is the **least element** (*nhỏ nhất*) of the poset $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.

The greatest and least element are unique if it exists.

Example

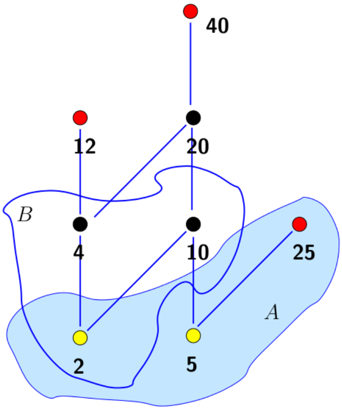
Let S be a set. In the poset $(P(S), \subseteq)$, the least element is $\emptyset$ and the greatest element is S.

Upper Bound & Lower Bound

Definition

Let $A \subseteq (S, \preccurlyeq)$.

- If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** (*cận trên*) of A.
- If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** (*cận dưới*) of A.



Example

- Subset A does not have upper bound and lower bound.
- The upper bound of B are $20, 40$ and the lower bound is 2.

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