Chapter 1
Logics

Discrete Structures for Computing on January 4, 2023

Nguyen An Khuong, Tran Tuan Anh, Nguyen Tien Thinh, Mai Xuan Toan, Tran Hong Tai
Faculty of Computer Science and Engineering
University of Technology - VNUHCM
trtanh@hcmut.edu.vn

Contents

Course outcomes

	Course learning outcomes
L.O.1	Understanding of logic and discrete structures
	L.O.1.1 – Describe definition of propositional and predicate logic L.O.1.2 – Define basic discrete structures: set, mapping, graphs
L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing L.O.2.2 – Use proving methods: direct, contrapositive, induction L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
	L.O.3.1 – Define basic probability theory L.O.3.2 – Explain discrete random variables
L.O.4	Compute quantities of discrete structures and probabilities
	L.O.4.1 – Operate (compute/ optimize) on discrete structures L.O.4.2 – Compute probabilities of various events, conditional ones, Bayes theorem

Logic

Definition (Averroes)

The tool for distinguishing between the true and the false.

Definition (Penguin Encyclopedia)

The formal systematic study of the principles of valid inference and correct reasoning.

Definition (Discrete Mathematics - Rosen)

Rules of logic are used to distinguish between valid and invalid mathematical arguments.

Applications in Computer Science

- Design of computer circuits
- Construction of computer programs
- Verification of the correctness of programs
- Constructing proofs automatically
- Artificial intelligence
- Many more...

Propositional Logic

Definition

A proposition is a declarative sentence that is either true or false, but not both.

Examples

- Hanoi is the capital of Vietnam.
- New York City is the capital of USA.
- $1 + 1 = 2$
- $2 + 2 = 3$

Examples

Examples (Which of these are propositions?)

- How easy is logic!
- Read this carefully.
- H1 building is in Ho Chi Minh City.
- $4 > 2$
- $2^n \geq 100$
- The Sun circles the Earth.
- Today is Thursday.
  - Proposition only when the time is **specified**

Notations

- Propositions are denoted by $p, q, \ldots$
- The **truth value** (”chân trị”) is true (T) or false (F)

Operators

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Negation - ''Phủ định'': $\neg p$

Bảng: Truth Table for Negation

p	$\neg p$
T	F
F	T

Operators

Conjunction - ''Hội'': $p \wedge q$
"p and q"

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

I'm teaching DM1 and it is raining today.

Disjunction - ''Tuyển'': $p \vee q$
"p or q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

We need students who have experience in Java or C++.
Tomorrow, I will eat Pho or Bun bo.

Operators

Exclusive OR - *Tuyển loại*: $p \oplus q$
"p or q (but not both)"

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication - *Kéo theo*: $p \rightarrow q$
"if p, then q"

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If it rains, the pavement will be wet.

More Expressions for Implication $p \to q$

- if p, then q
- p implies q
- p is sufficient for q
- q if p
- p only if q
- q unless $\neg p$
- If you get 100% on the final, you will get 10 grade.
- If you feel asleep this afternoon, then $2 + 3 = 5$.

Conditional Statements From $p \to q$

- $q \to p$ (converse - *đảo*)
- $\neg q \to \neg p$ (contrapositive - *phản đảo*)
- Prove that only contrapositive have the same truth table with $p \to q$

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Exercise

What are the converse and contrapositive of the following conditional statement

"If he plays online games too much, his girlfriend leaves him."

- Converse: If his girlfriend leaves him, then he plays online games too much.
- Contrapositive: If his girlfriend does not leave him, then he does not play online games too much.

Biconditionals

$$p \leftrightarrow q$$

"p if and only if q"

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- "p is necessary and sufficient for q".
- "if p then q, and conversely".
- "p iff q".

The order of operators

- 1. in the bracket()
- 2. negation $\neg$
- 3. $\vee, \wedge, \oplus$
- 4. $\rightarrow$
- 5. $\leftrightarrow$

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Translating Natural Sentences

p $\quad p \to (q \vee \neg r)$

Exercise

I will buy a new phone **only if** I have enough money to buy iPhone 4 **or** my phone is not working.

r

- p: I will buy a new phone
- q: I have enough money to buy iPhone 4
- r: My phone is working

- $p \to (q \vee \neg r)$

$(p \to q) \vee \neg r$

Translating Natural Sentences

Exercise

He will not run the red light if he sees the police unless he is too risky.

Bi P: He

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Construct Truth Table

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Exercise

Construct the truth table of the compound proposition
$$(p \vee \neg q) \to (p \wedge q).$$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Exercise - Truth table

$\neg p \to (\neg q \vee r)$

p	q	r	$\neg p$	$\neg q$	$\neg q \vee r$	$\neg p \to (\neg q \vee r)$
T	T	T	F	F	T	T
T	T	F	F	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	T	T

a) $(p \wedge q) \to \neg q$
b) $(p \vee r) \to (r \vee \neg p)$
c) $(p \to q) \vee (q \to p)$
d) $(p \vee \neg q) \wedge (\neg p \vee q)$
e) $(p \to \neg q) \vee (q \to \neg p)$
f) $\neg(\neg p \wedge \neg q)$
g) $(p \vee q) \to (p \oplus q)$
h) $(p \wedge q) \vee (r \oplus q)$

Applications

- System specifications
  - "When a user clicked on *Help* button, a pop-up will be shown up"
- Boolean search
  - type "dai hoc bach khoa" in Google
  - means "dai AND hoc AND bach AND khoa"

Applications (cont.)

- **Logic puzzles**
  - There are two kinds of inhabitants on an island, knights, who always tell the truth, and their opposites, knaves, who may lie. You encounter two people A and B. What are A and B if A says **"B is a knight"** and B says **"The two of us are opposite types"**?
- **Bit operations**
  - 101010011 is a bit string of length nine.

Tautology and Contradiction

Definition

A compound proposition that is always true (false) is called a tautology - hằng đúng (contradiction - hằng sai).

- Tautology: *hằng đúng*
- Contradiction: *mâu thuẫn*

Example

- $p \vee \neg p$ (tautology)
- $p \wedge \neg p$ (contradiction)

Question

Which of the following is a tautology
Hint: Apply truth table.

a) $(p \vee q) \rightarrow (p \wedge q)$

b) $(p \wedge q) \rightarrow (p \vee q)$

c) $p \rightarrow (\neg q \rightarrow p)$

d) $p \rightarrow (p \rightarrow q)$

e) $p \rightarrow (p \rightarrow p)$

f) $(p \rightarrow q) \rightarrow [(p \rightarrow r) \rightarrow (q \rightarrow r)]$

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Proposition? Truth value?

a) "Fansipan is the highest mountain in Vietnam."
b) "Two coprime numbers have the only common divisor of 1."
c) "The product of 3 continuous integers is divisible by 3."
d) "Stand up!"
e) "x+1=0"
f) "Hexagons have 8 vertices."
g) "0 is a positive number."
h) "The equation: $x^2 + 5x + 6 = 0$ has no root."
i) "is 2 a prime number?"
j) "The equation $mx^2 + 2x - 1 = 0$ has a single root if and only if m=-1."
k) "There is a prime that is even."
l) "$x^2 + 1 > 0$."
m) "When will our class go camping?"
n) "Mercury is not a metal."
o) "$3^{20} > 2^{30}$."
p) "Airplanes are the fastest transport."
q) "2002 is a leap year."
r) "There are infinite prime numbers."
s) "$2^{10} - 1$ is divisible by 11."
t) "No smoking in public place."
u) "All even positive integer is a summation of 2 prime numbers."
v) "x is a prime number if it doesn't have any divisor other than 1 and x."

Logical Equivalences

Definition

The compound compositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology, denoted $p \equiv q$.

Example

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Logical Equivalences

$p \wedge \mathbf{T}$	$\equiv$	p	Identity laws
$p \vee \mathbf{F}$	$\equiv$	p	Luật đồng nhất
$p \vee \mathbf{T}$	$\equiv$	$\mathbf{T}$	Domination laws
$p \wedge \mathbf{F}$	$\equiv$	$\mathbf{F}$	Luật nuốt
$p \vee p$	$\equiv$	p	Idempotent laws
$p \wedge p$	$\equiv$	p	Luật lũy đẳng
$\neg(\neg p)$	$\equiv$	p	Double negation law
			Luật phủ định kép

Logical Equivalences

$p \vee q$	$\equiv$	$q \vee p$	Commutative laws
$p \wedge q$	$\equiv$	$q \wedge p$	Luật giao hoán
$(p \vee q) \vee r$	$\equiv$	$p \vee (q \vee r)$	Associative laws
$(p \wedge q) \wedge r$	$\equiv$	$p \wedge (q \wedge r)$	Luật kết hợp
$p \vee (q \wedge r)$	$\equiv$	$(p \vee q) \wedge (p \vee r)$	Distributive laws
$p \wedge (q \vee r)$	$\equiv$	$(p \wedge q) \vee (p \wedge r)$	Luật phân phối
$\neg(p \wedge q)$	$\equiv$	$\neg p \vee \neg q$	De Morgan's law
$\neg(p \vee q)$	$\equiv$	$\neg p \wedge \neg q$	Luật De Morgan
$p \vee (p \wedge q)$	$\equiv$	p	Absorption laws
$p \wedge (p \vee q)$	$\equiv$	p	Luật hút thu

Logical Equivalences

Equivalence		
$p \vee \neg p$	$\equiv$	$\mathbf{T}$
$p \wedge \neg p$	$\equiv$	$\mathbf{F}$
$p \rightarrow q$	$\equiv$	$\neg p \vee q$
$(p \rightarrow q) \wedge (p \rightarrow r)$	$\equiv$	$p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r)$	$\equiv$	$(p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r)$	$\equiv$	$p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r)$	$\equiv$	$(p \wedge q) \rightarrow r$
$p \leftrightarrow q$	$\equiv$	$(p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q$	$\equiv$	$(\neg p \vee q) \wedge (p \vee \neg q)$

Constructing New Logical Equivalences

Example

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently, $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

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Exercise

Negate the following proposition and try to simplify it.

Example

$p \to (\neg q \wedge r)$
By using the truth table, we can prove that $p \to q$ and $\neg p \vee q$ are logical equivalence.
Negate: $\neg(p \to (\neg q \wedge r))$
$\equiv \neg(\neg p \vee (\neg q \wedge r))$
$\equiv p \wedge \neg(\neg q \wedge r)$
$\equiv p \wedge (q \vee \neg r)$

a) $p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)$
b) $(p \wedge q) \to r$
c) $p \vee q \vee (\neg p \wedge \neg q \wedge r)$
d) $[[[(p \wedge q) \wedge r] \vee [(p \wedge r) \wedge \neg r]] \vee \neg q] \to s$

Exercise

Prove the following proposition are logical equivalence.
<u>Hint:</u> Apply truth table or the series of logical equivalences.

a) $\neg(p \leftrightarrow q)$ và $\neg p \leftrightarrow q$
b) $(p \to q) \wedge (p \to r)$ và $p \to (q \wedge r)$
c) $(p \to r) \wedge (q \to r)$ và $(p \vee q) \to r$
d) $(p \to q) \vee (p \to r)$ và $p \to (q \vee r)$
e) $\neg p \to (q \to r)$ và $q \to (p \vee r)$
f) $p \leftrightarrow q$ và $(p \to q) \wedge (q \to p)$

Exercise

The following proposition are logical equivalence? Prove it or give an example?

a) $p \wedge (p \to q)$ và $p \wedge q$
b) $p \to q$ và $\neg p \vee (p \wedge q)$
c) $p \to q$ và $\neg p \vee \neg q$
d) $\neg p$ và $\neg(p \vee q) \vee (\neg p \wedge q)$
e) $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$ và $[(p \to q) \wedge (q \to r) \wedge (r \to p)]$
f) $[(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)]$ và $[(p \vee q) \wedge (q \vee r) \wedge (r \vee p)]$

Exercise

Determine the truth value and find the contrapositions as well as the contradictions of the following propositions.

a) "If ABCD is a rectangle, AB and CD are perpendicular."
b) "If 14 is an odd number, 15 is divisible by 4."
c) "Two equal triangles have the same area."
d) "If the quadratic equation $ax^2 + bx + c = 0$ has $a.c < 0$, it has root."
e) "If two numbers x and y are both divisible by n, $(x + y)$ is also divisible by n."
f) "If 45 ended with 5, 45 is divisible by 5."
g) "If $\sqrt{2}$ is an irrational number then $\sqrt{2}.\sqrt{2}$ is an irrational number."
h) "If Pythagoras is French, Vietnam belongs to Asia."
i) "If $3n + 2$ is an odd integer, n is an odd integer."
j) "If 8 < 9, 5 is a prime number."
k) "A quadrilateral is a rhombus when it has 2 perpendicular diagonals."
l) "If 5 < 3, 7 is a prime number."

Exercise

Let p and q be:

- p: "Brandon likes reading"
- q: "Brandon is a good student"

The statement that formalize "If Brandon likes reading, Brandon is a good student, vice versa, If Brandon is a good student, Brandon like reading" is:

A) $(p \wedge q) \rightarrow r$
B) $p \rightarrow q$
C) $p \vee q$
D) $p \wedge q$
E) $p \leftrightarrow q$
F) $\neg p \rightarrow \neg q$
G) $\neg p \vee (p \wedge q)$
H) None of the others.

Exercise

Let P, Q, R be:

- P: "Potter is studying Math".
- Q: "Potter is studying Computer science".
- R: "Potter is studying English".

Formalize the following statement using the propositional connectives.

Example

Potter is studying Math and English but not Computer science: $P \wedge R \wedge \neg Q$

a) Potter is studying Math and Computer science but not Computer science and English at the same time.
b) It is not true that Potter is studying English and not Math.
c) It is not true that Potter is studying English or Computer science and not Math.
d) Potter is not studying both Computer science and English but is studying Math.

Exercise

Determine the wrong statement among the following.

a) $x \in \{x\}$

b) $\{x\} \subseteq \{x\}$

c) $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$

e) $\emptyset \subseteq \{x\}$

A) a

B) b

C) c

D) d

E) none of the others.

Exercise

Which of the following proposition is a truth.

A) $(p \vee \neg q) \rightarrow q$

B) $p \rightarrow (p \wedge q)$

C) $\neg p \rightarrow (p \rightarrow q)$

D) $\neg(p \rightarrow q) \rightarrow q$

E) none of the others.

Exercise

Let's consider a propositional language where:

- p: "ABC *is an isosceles triangle*".
- q: "ABC *is an equilateral triangle*".
- r: "ABC *has a* 60^o *angle*".

Which of the following compounds formalize the theorem: *"if ABC is an isosceles triangle and has a 60^o angle then it is an equilateral triangle"* ?

A) $(p \land q) \to r$
B) $(p \land r) \to q$
C) $(p \land r) \lor q$
D) $q \to (p \lor r)$
E) none of the others.

Exercise

There are 6 soccer teams A, B, C, D, E, F contested in a tournament. The following are statements on which two teams are in the grand final:

a. A and C
b. B and E
c. B and F
d. A and F
e. A and D

Knowing that there are 4 half true statements and 1 totally false statement. What teams are in the grand final?

Exercise

Find the truth values of the following statements (with brief explanations):

a) "$\forall x \in N, x^2 + 5x + 6$ is not a prime number."
b) "$\exists x \in R, x^2 + x + 1 \leq 0$"
c) "$\exists n \in N, (n^3 - n)$ is not a multiple of 3."
d) "$\forall n \in N*, n^2 - 1$ is a multiple of 3."
e) "$\forall x, \forall y \in R, x^2 + y^2 > 2xy$"
f) "$\exists r \in Q, 3 < r < \pi$"
g) "$\exists n \in N, n^2 + 1$ divisible by 8"
h) "$\forall x \in R, |x| < 3 \Leftrightarrow x^2 < 9$"
i) "$\exists a, b \in R, (a + b)^2 > 2(a^2 + b^2)$"
j) "All real numbers are positive."
k) "There is a liquid metal."
l) "All equilateral triangles are equal."
m) "All gases are non-conductive."
n) "There exist quadrilaterals which don't have circumcircles."
o) "There is a natural number n that, for all real numbers x, we have $f(x) = x^2 - 2x + n$ is not negative."
p) "For all positive integers x and y we have $x \leq y$."
q) "For all positive integers x, there is a positive integer y so that $x \leq y$."
r) "There is a positive integer x that, for all positive integers y, we have $x \leq y$."
s) "There exist positive integers x and y so that $x \leq y$."