

Chapter 5

Functions

Discrete Structures for Computing on January 4, 2023

Functions

Nguyen An Khuong,
Tran Tuan Anh, Nguyen
Tien Thinh, Mai Xuan
Toan, Tran Hong Tai



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Recursion

Nguyen An Khuong, Tran Tuan Anh, Nguyen Tien Thinh, Mai
Xuan Toan, Tran Hong Tai
Faculty of Computer Science and Engineering
University of Technology - VNUHCM
trtanh@hcmut.edu.vn

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Course outcomes



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Course learning outcomes

L.O.1	Understanding of logic and discrete structures
	L.O.1.1 – Describe definition of propositional and predicate logic
	L.O.1.2 – Define basic discrete structures: set, mapping, graphs
L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing
	L.O.2.2 – Use proving methods: direct, contrapositive, induction
	L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
	L.O.3.1 – Define basic probability theory
	L.O.3.2 – Explain discrete random variables
L.O.4	Compute quantities of discrete structures and probabilities
	L.O.4.1 – Operate (compute/ optimize) on discrete structures
	L.O.4.2 – Compute probabilities of various events, conditional ones, Bayes theorem

Introduction

- Each student is assigned a grade from set $\{0, 0.1, 0.2, 0.3, \dots, 9.9, 10.0\}$ at the end of semester
- Function is extremely important in mathematics and computer science
 - linear, polynomial, exponential, logarithmic,...
- Don't worry! For discrete mathematics, we need to understand functions at a basic set theoretic level

Functions

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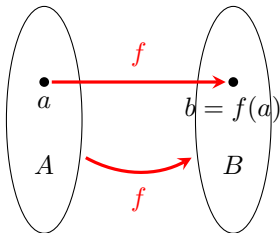
Recursion

Function

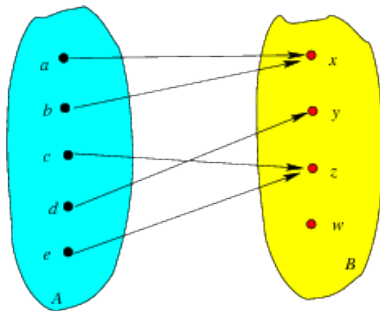
Definition

Let A and B be nonempty sets. A **function** f from A to B is an assignment of **exactly one** element of B to each element of A .

- $f : A \rightarrow B$
- A : **domain** (*miền xác định*) of f
- B : **codomain** (*miền giá trị*) of f
- For each $a \in A$, if $f(a) = b$
 - b is an **image** (*ảnh*) of a
 - a is **pre-image** (*ngược ảnh*) of $f(a)$
- **Range** of f is the set of all images of elements of A
- f **maps** (*ánh xạ*) A to B



Example



- Example:
 - y is an image of d
 - c is a pre-image of z



Example

Example

What are domain, codomain, and range of the function that assigns grades to students includes: student A: 5, B: 3.5, C: 9, D: 5.2, E: 4.9?

Example

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ assign the the square of an integer to this integer. What is $f(x)$? Domain, codomain, range of f ?

- $f(x) = x^2$
- **Domain**: set of all integers
- **Codomain**: Set of all integers
- **Range of f** : $\{0, 1, 4, 9, \dots\}$



Add and multiply real-valued functions

Definition

Let f_1 and f_2 be functions from A to \mathbb{R} . Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to \mathbb{R} defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

Example

Let $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + x - x^2 = x$$

$$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2(x - x^2) = x^3 - x^4$$

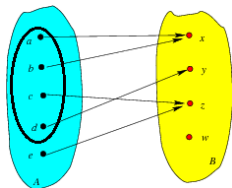


Image of a subset

Definition

Let $f : A \rightarrow B$ and $S \subseteq A$. The **image** of S :

$$f(S) = \{f(s) \mid s \in S\}$$



$$f(\{a, b, c, d\}) = \{x, y, z\}$$



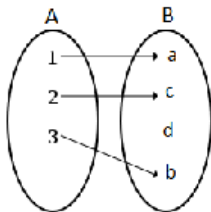
One-to-one



Definition

A function f is **one-to-one** or **injective** (*đơn ánh*) if and only if

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$$



- Is $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x + 1$ one-to-one?
- Is $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x^2$ one-to-one?

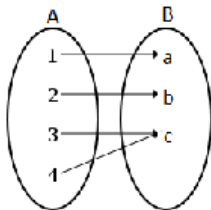
Onto



Definition

$f : A \rightarrow B$ is **onto** or **surjective** (*toàn ánh*) if and only if

$$\forall b \in B, \exists a \in A : f(a) = b$$

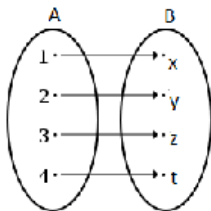


- Is $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x + 1$ onto?
- Is $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x^2$ onto?

One-to-one and onto (bijection)

Definition

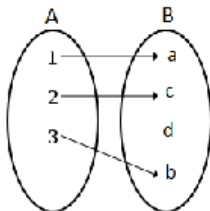
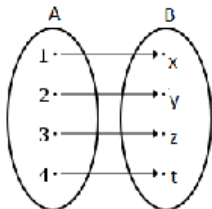
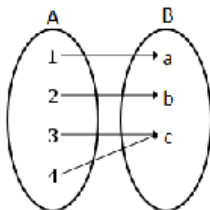
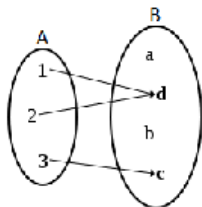
$f : A \rightarrow B$ is **bijective (one-to-one correspondence)** (*song ánh*) if and only if f is **injective** and **surjective**



- Let f be the function from $\{a, bc, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$. Is f a bijection?



Example



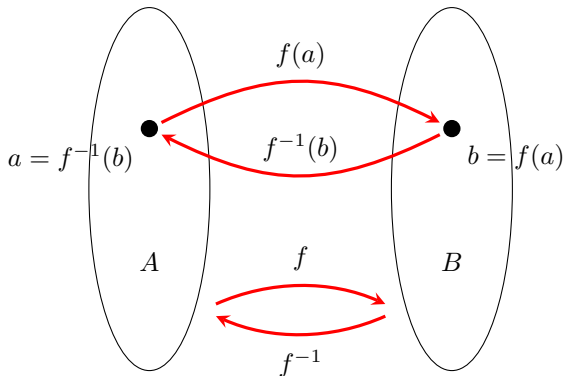
Inverse function (Hàm ngược)

Definition

Let $f : A \rightarrow B$ be a **bijection** then the **inverse of f** is the function $f^{-1} : B \rightarrow A$ defined by

$$\text{if } f(a) = b \text{ then } f^{-1}(b) = a$$

A one-to-one correspondence is call **invertible** (*khả nghịch*) because we can define the inverse of this function.



Example

Example

$A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ with

$$f(a) = 2 \quad f(b) = 3 \quad f(c) = 1$$

f is invertible and its inverse is

$$f^{-1}(1) = c \quad f^{-1}(2) = a \quad f^{-1}(3) = b$$

Example

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = x^2$. If f invertible?



Example

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = 2x + 1$$

$$f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$$

$$f^{-1}(x) = \frac{x - 1}{2}$$



Function Composition



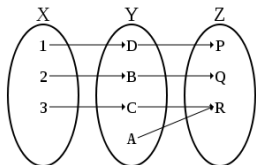
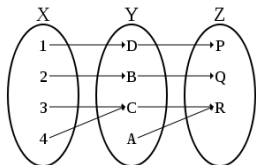
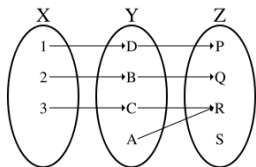
Definition

Given a pair of functions $g : A \rightarrow B$ and $f : B \rightarrow C$. Then the **composition** (*hợp thành*) of f and g , denoted $f \circ g$ is defined by

$$f \circ g : A \rightarrow C$$

$$f \circ g(a) = f(g(a))$$

Example



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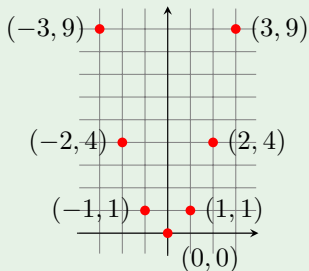
Sequences and Summation

Recursion

Graphs of Functions

Example

The graph of $f(x) = x^2$ from \mathbb{Z} to \mathbb{Z} .



Definition

Let f be a function from the set A to the set B . The **graph** of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.



Important Functions

Definition

Floor function (*hàm sàn*) of x ($\lfloor x \rfloor$): the largest integer $\leq x$

$$\lfloor \frac{1}{2} \rfloor = 0, \lfloor 3.1 \rfloor = 3, \lfloor 7 \rfloor = 7$$

Ceiling function (*hàm trần*) of x ($\lceil x \rceil$): the smallest integer $\geq x$

$$\lceil \frac{1}{2} \rceil = 1, \lceil 3.1 \rceil = 4, \lceil 7 \rceil = 7$$

Bảng: Properties (n is an integer, x is a real number)

$$(1a) \quad \lfloor x \rfloor = n \text{ iff } n \leq x < n + 1$$

$$(1b) \quad \lceil x \rceil = n \text{ iff } n - 1 < x \leq n$$

$$(1c) \quad \lfloor x \rfloor = n \text{ iff } x - 1 < n \leq x$$

$$(1d) \quad \lceil x \rceil = n \text{ iff } x \leq n < x + 1$$

$$(2) \quad x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1$$

$$(3a) \quad \lfloor -x \rfloor = -\lceil x \rceil$$

$$(3b) \quad \lceil -x \rceil = -\lfloor x \rfloor$$

$$(4a) \quad \lfloor x + n \rfloor = \lfloor x \rfloor + n$$

$$(4b) \quad \lceil x + n \rceil = \lceil x \rceil + n$$



Sequences

What are the rule of these sequences (dãy)?

Example

1, 3, 5, 7, 9, ... $a_n = 2n - 1$
Arithmetic sequence (cấp số cộng)

Example

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ $a_n = \frac{1}{2^{n-1}}$
Geometric sequence (cấp số nhân)

Example

$\{a_n\}$	5, 11, 17, 23, 29, 35, 41, 47, ...	$a_n = 6n - 1$
$\{b_n\}$	1, 7, 25, 79, 241, 727, 2185, ...	$b_n = 3^n - 2$



Recurrence Relations

Example

$\{a_n\}$ 5, 11, 17, 23, 29, 35, 41, 47, ...

$a_n = a_{n-1} + 6$ for $n = 2, 3, 4, \dots$ and $a_1 = 5$

Recurrence relations: công thức truy hồi

Definition (Fibonacci Sequence)

Initial condition: $f_0 = 0$ and $f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \dots$

Example

Find the Fibonacci numbers f_2, f_3, f_4, f_5 and f_6

$$f_2 = f_1 + f_0 = 1 + 0 = 1$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3$$

$$f_5 = f_4 + f_3 = 3 + 2 = 5$$

$$f_6 = f_5 + f_4 = 5 + 3 = 8$$



Exercise (1)

Initial deposit: \$10,000

Interest: 11%/year, **compounded** annually (*lãi suất kép*)

After 30 years, how much do you have in your account?



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Solution:

Let P_n be the amount in the account after n years. The sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10,000$





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The initial condition is $P_0 = 10,000$

Step 1. Solve the recurrence relation (iteration technique)

$$P_1 = (1.11)P_0$$

$$P_2 = (1.11)P_1 = (1.11)^2 P_0$$

$$P_3 = (1.11)P_2 = (1.11)^3 P_0$$

$$\vdots$$

$$P_n = (1.11)P_{n-1} = (1.11)^n P_0.$$



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\vdots

$$P_n = (1.11)P_{n-1} = (1.11)^nP_0.$$

Step 2. Calculate

$$P_{30} = (1.11)^{30}10,000 = \$228,922.97.$$

Exercise (2)

What is the 2012th number in the sequence $\{x_n\}$: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6,...





Exercise (2)

What is the **2012th number** in the sequence $\{x_n\}$: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6,...

Solution:

In this sequence, integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and so on. Therefore integer n appears n times in the sequence.

We can prove that (**try it!**)

$$\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

and can easily calculate that

$$\sum_{i=1}^{62} i = 1953$$

so the next 63 numbers (until 2016) is 63.

Therefore, 2012th number in the sequence is 63.



Theorem

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^n ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Chứng minh.

Let $S_n = \sum_{j=0}^n ar^j$.

$$\begin{aligned} rS_n &= r \sum_{j=0}^n ar^j \\ &= \sum_{j=0}^n ar^{j+1} \\ &= \sum_{k=1}^{n+1} ar^k \\ &= \left(\sum_{k=0}^n ar^k \right) + (ar^{n+1} - a) \\ &= S_n + (ar^{n+1} - a) \end{aligned}$$

Solving for S_n shows that if $r \neq 1$, then $S_n = \frac{ar^{n+1}-a}{r-1}$

If $r = 1$, then $S_n = \sum_{j=0}^n a = (n+1)a$



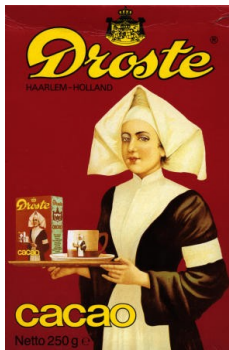
Recursion

Definition (Recurrence Relation)

An equation that **recursively defines** a sequence.

Definition (Recursion (đệ quy))

The act of defining an object (usually a function) in terms of that object itself.



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Recursive Algorithms

Definition

An algorithm is called **recursive** if it solves a problem by reducing it to an instance of the same problem with smaller input.

Example

Give a recursive algorithm for computing $n!$, where n is a nonnegative integer.

Solution. We base on the recursive definition of $n!$:
 $n! = n \cdot (n - 1)!$ and $0! = 1$.

```
procedure factorial( $n$ : nonnegative integer)
if  $n = 0$  then return 1
else return  $n \cdot \text{factorial}(n - 1)$ 
{output is  $n!$ }
```



Algorithms for Fibonacci Numbers

Recursive Algorithm

```
procedure fibonacci( $n$ : nonnegative integer)
if  $n = 0$  then return 0
else if  $n = 1$  then return 1
else return fibonacci( $n-1$ ) + fibonacci( $n-2$ )
{output is fibonacci( $n$ )}
```

Iterative Algorithm

```
procedure iterative fibonacci( $n$ : nonnegative integer)
if  $n = 0$  then return 0
else
     $x := 0$ 
     $y := 1$ 
    for  $i := 1$  to  $n - 1$ 
         $z := x + y$ 
         $x := y$ 
         $y := z$ 
    return  $y$ 
{output is the  $n$ th Fibonacci number}
```



Tower of Hanoi

There is a tower in Hanoi that has three pegs mounted on a board together with **64** gold disks of different sizes.

Initially, these disks are placed on the first peg in order of size, with the largest on the bottom.

The rules:

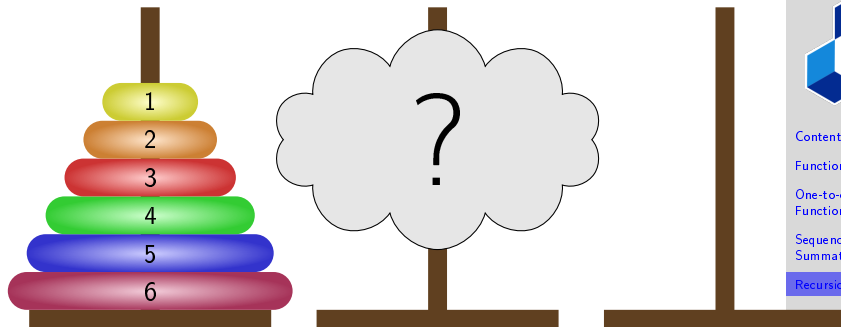
- ① Move one at a time from one peg to another
- ② A disk is never placed on top of a smaller disk

Goals: all the disks on the third peg in order of size.

The myth says that **the world will end** when they finish the puzzle.



Tower of Hanoi – 64 Discs



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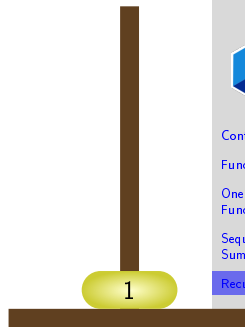
Tower of Hanoi – 1 Disc



Moved disc from peg 1 to peg 3.



Tower of Hanoi – 1 Disc



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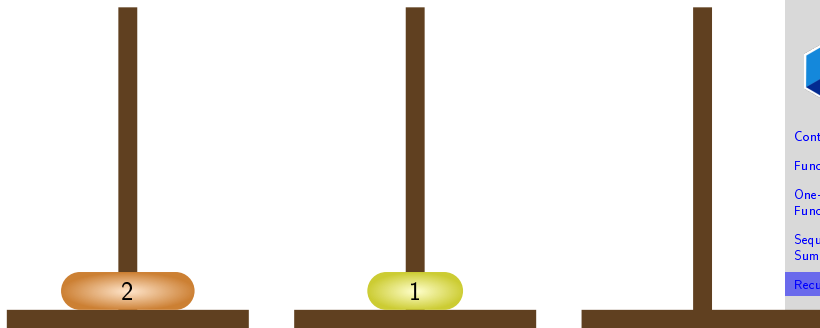
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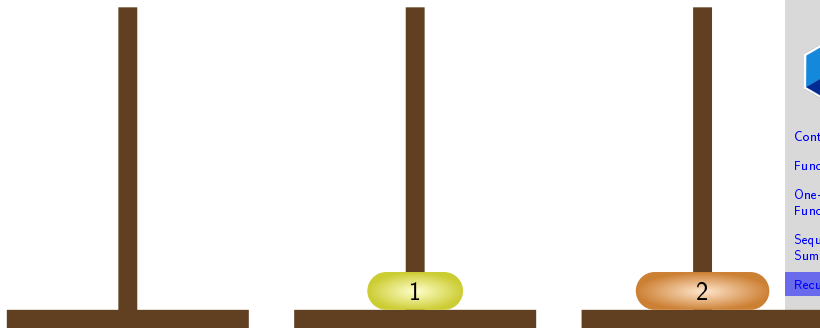
Tower of Hanoi – 2 Discs



Moved disc from peg 1 to peg 2.



Tower of Hanoi – 2 Discs



Moved disc from peg 1 to peg 3.



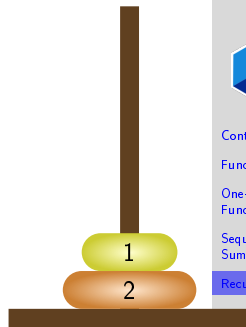
Tower of Hanoi – 2 Discs



Moved disc from peg 2 to peg 3.



Tower of Hanoi – 2 Discs



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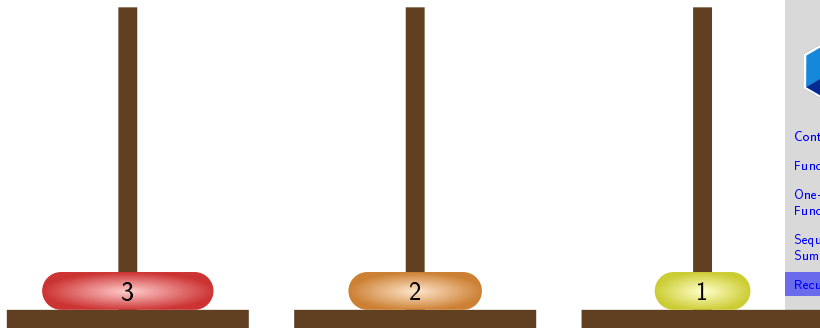
Tower of Hanoi – 3 Discs



Moved disc from peg 1 to peg 3.



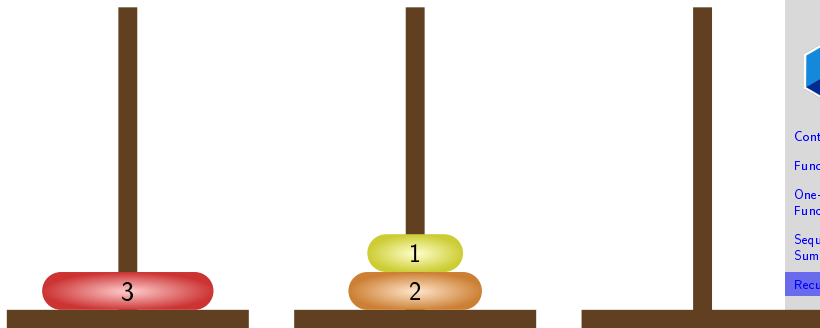
Tower of Hanoi – 3 Discs



Moved disc from peg 1 to peg 2.



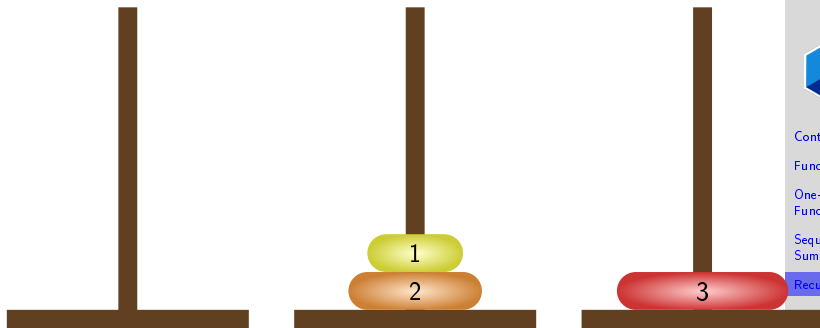
Tower of Hanoi – 3 Discs



Moved disc from peg 3 to peg 2.



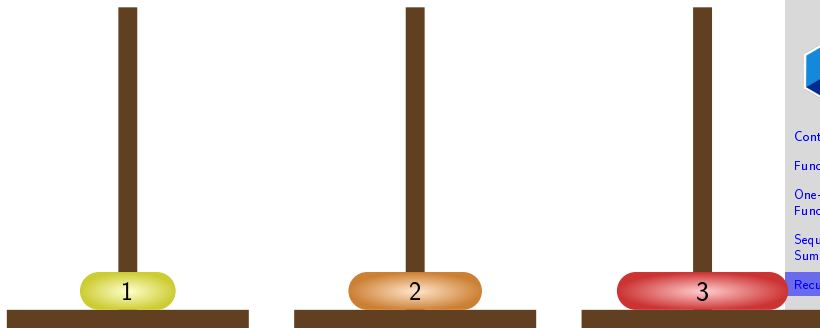
Tower of Hanoi – 3 Discs



Moved disc from peg 1 to peg 3.



Tower of Hanoi – 3 Discs



Moved disc from peg 2 to peg 1.



Tower of Hanoi – 3 Discs



Moved disc from peg 2 to peg 3.



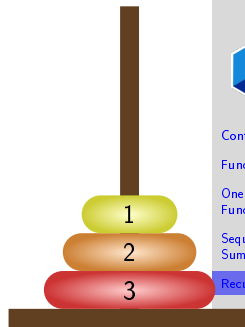
Tower of Hanoi – 3 Discs



Moved disc from peg 1 to peg 3.



Tower of Hanoi – 3 Discs



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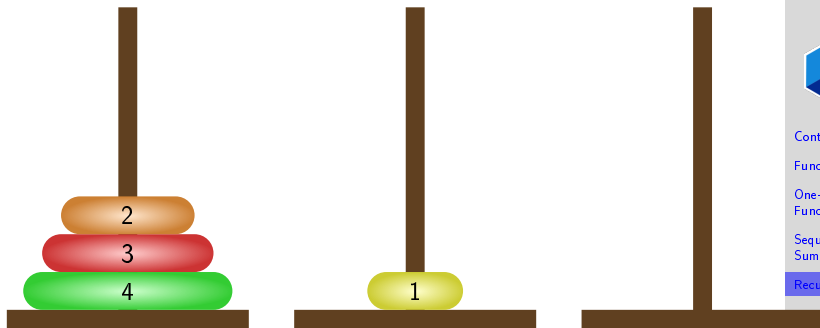
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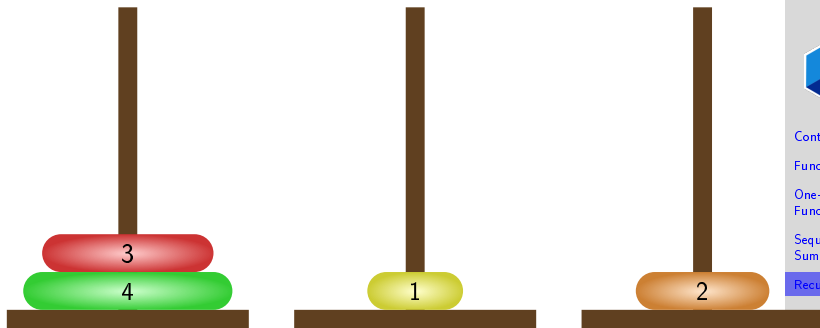
Tower of Hanoi – 4 Discs



Moved disc from peg 1 to peg 2.



Tower of Hanoi – 4 Discs



Moved disc from peg 1 to peg 3.



Tower of Hanoi – 4 Discs



Moved disc from peg 2 to peg 3.

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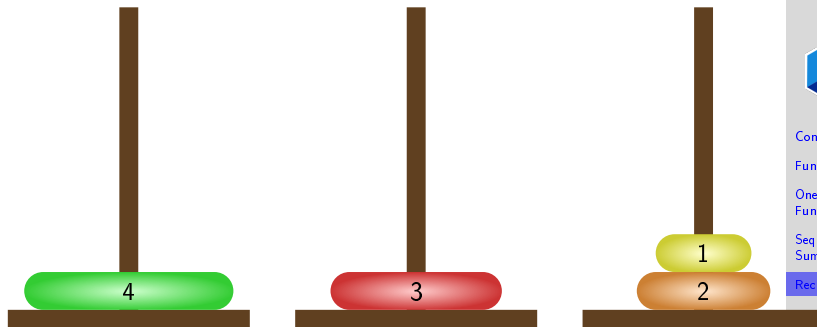
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Moved disc from peg 1 to peg 2.

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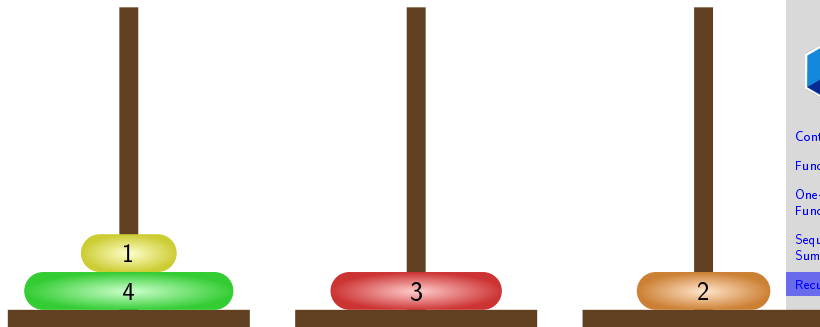
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Moved disc from peg 3 to peg 1.

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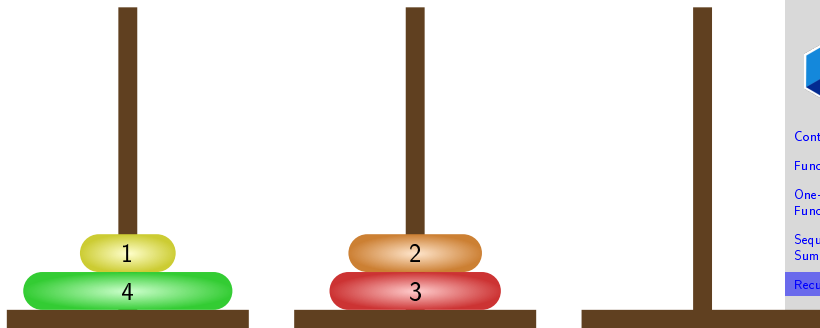
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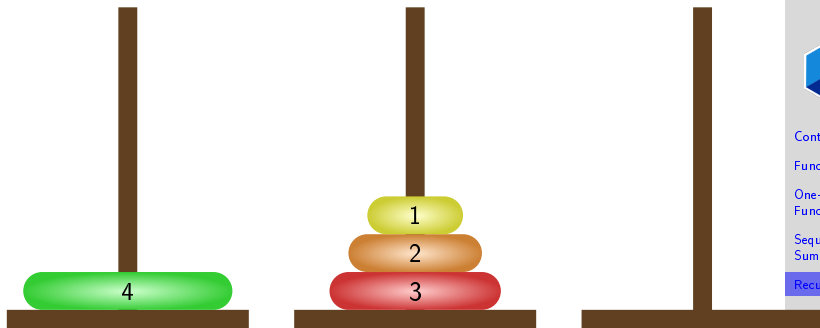
Tower of Hanoi – 4 Discs



Moved disc from peg 3 to peg 2.



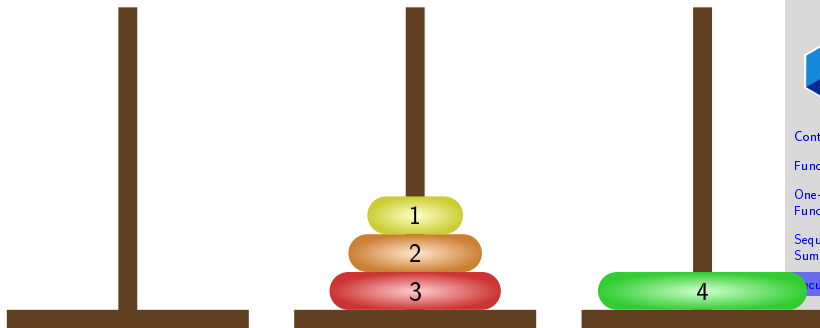
Tower of Hanoi – 4 Discs



Moved disc from peg 1 to peg 2.



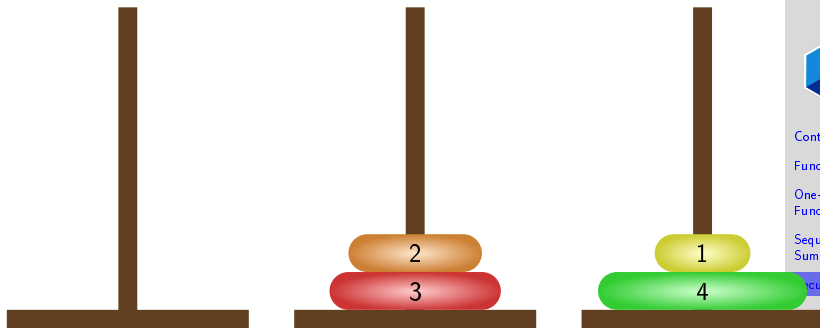
Tower of Hanoi – 4 Discs



Moved disc from peg 1 to peg 3.



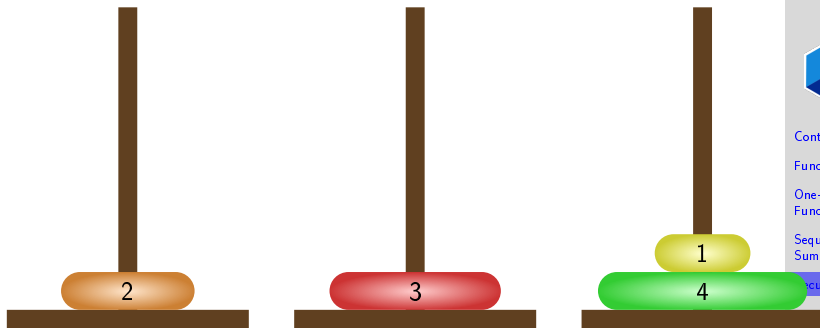
Tower of Hanoi – 4 Discs



Moved disc from peg 2 to peg 3.



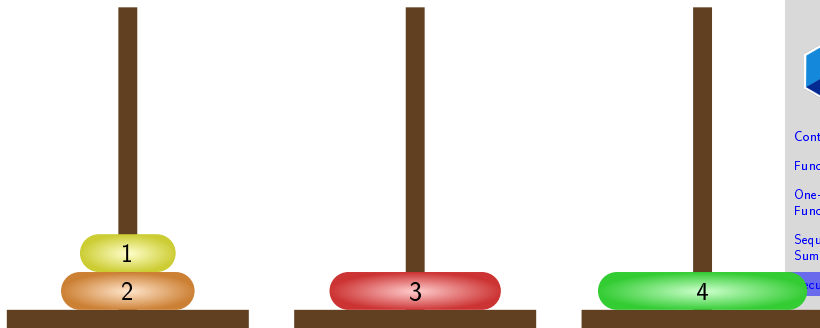
Tower of Hanoi – 4 Discs



Moved disc from peg 2 to peg 1.



Tower of Hanoi – 4 Discs



Moved disc from peg 3 to peg 1.



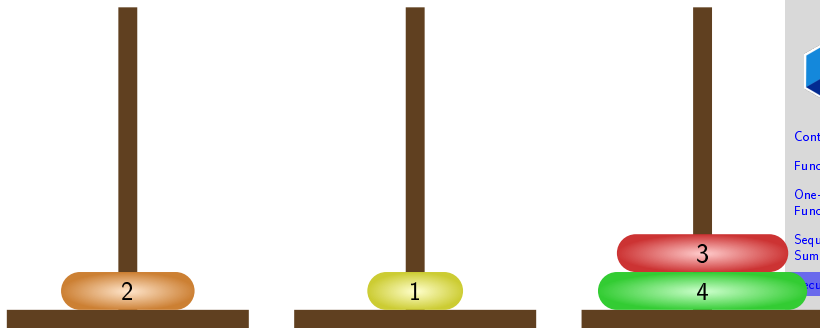
Tower of Hanoi – 4 Discs



Moved disc from peg 2 to peg 3.



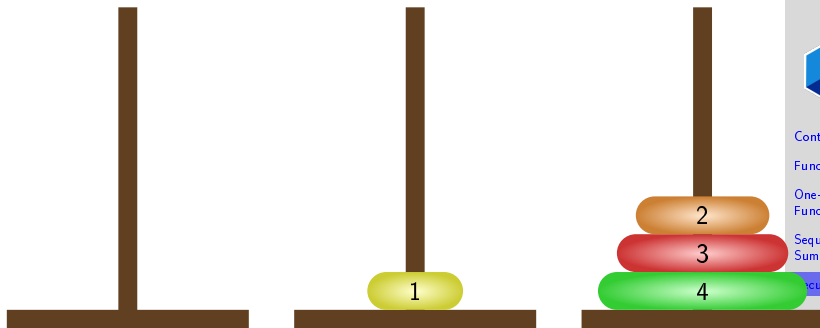
Tower of Hanoi – 4 Discs



Moved disc from peg 1 to peg 2.



Tower of Hanoi – 4 Discs



Moved disc from peg 1 to peg 3.

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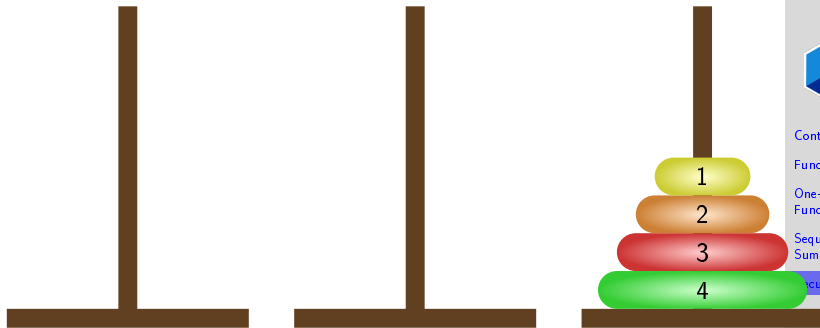
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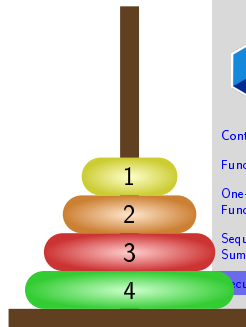
Tower of Hanoi – 4 Discs



Moved disc from peg 2 to peg 3.



Tower of Hanoi – 4 Discs



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Algorithm

```
procedure hanoi( $n$ , A, B, C)
if  $n = 1$  then move the disk from A to C
else
    call hanoi( $n - 1$ , A, C, B)
    move disk  $n$  from A to C
    call hanoi( $n - 1$ , B, A, C)
```

Recurrence Relation

$$H(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2H(n-1) + 1 & \text{if } n > 1. \end{cases}$$

Recurrence Solving

$$H(n) = 2^n - 1$$

If one move takes 1 second, for $n = 64$

$$\begin{aligned} 2^{64} - 1 &\approx 2 \times 10^{19} \text{ sec} \\ &\approx 500 \text{ billion years!} \end{aligned}$$

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