Chapter 3
Proving methods

Discrete Structures for Computing on January 4, 2023

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Course outcomes

	Course learning outcomes
L.O.1	Understanding of logic and discrete structures
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L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing L.O.2.2 – Use proving methods: direct, contrapositive, induction L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
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L.O.4	Compute quantities of discrete structures and probabilities
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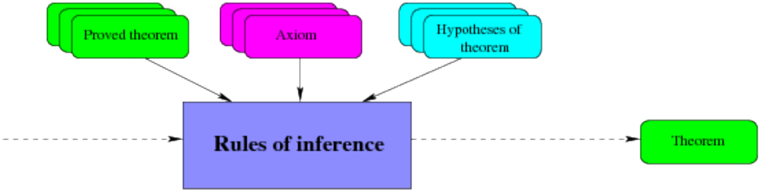
Introduction

Definition

A proof is a sequence of logical deductions from
- axioms, and
- previously proved theorems

that concludes with a new theorem.

Terminology



- **Theorem** (*định lý*) = a statement that can be shown to be true
- **Axiom** (*tiên đề*) = a statement we assume to be true
- **Hypothesis** (*giả thiết*) = the premises of the theorem

- **Lemma** (*bổ đề*) = less important theorem that is helpful in the proofs of other results
- **Corollary** (*hệ quả*) = a theorem that can be established directly from a proved theorem
- **Conjecture** (*phỏng đoán*) = statement being proposed to be true, when it is proved, it becomes theorem

Proving a Theorem

Many theorem has the form $\forall x P(x) \rightarrow Q(x)$

Goal:

- Show that $P(c) \rightarrow Q(c)$ is true with arbitrary c of the domain
- Apply universal generalization

$\Rightarrow$ How to show that conditional statement $p \rightarrow q$ is true.

Methods of Proof

- Direct proofs (*chứng minh trực tiếp*)
- Proof by contraposition (*chứng minh phản đảo*)
- Proof by contradiction (*chứng minh phản chứng*)
- Mathematical induction (*quy nạp toán học*)

Direct Proofs

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Definition

A direct proof shows that $p \to q$ is true by showing that if p is true, then q must also be true.

Example

Ex.: If n is an odd integer, then n^2 is odd.

Pr.: Assume that n is odd. By the definition, $n = 2k + 1$, $k \in \mathbb{Z}$. $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is an odd number.

Proof by Contraposition

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Definition

$p \to q$ can be proved by showing (directly) that its contrapositive, $\neg q \to \neg p$, is true.

Example

Ex.: Given an integer n, show that if $3n+2$ is odd, then n is odd.

Pr.: Assume that "n is even", so $n = 2k$, $k \in \mathbb{Z}$. Substituting $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$ is even. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, *Q.E.D.*

Proofs by Contradiction

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Definition

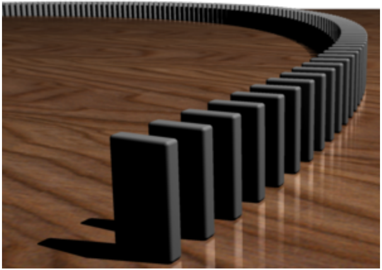
p is true if if can show that $\neg p \to (r \wedge \neg r)$ is true for some proposition r.

Example

Ex.: Prove that $\sqrt{2}$ is irrational.

Pr.: Let p is the proposition "$\sqrt{2}$ is irrational". Suppose $\neg p$ is true, which means $\sqrt{2}$ is rational. If so, $\exists a, b \in \mathbb{Z}, \sqrt{2} = a/b$, a, b have no common factors. Squared, $2 = a^2/b^2$, $2b^2 = a^2$, so a^2 is even, and a is even, too. Because of that $a = 2c, c \in \mathbb{Z}$. Thus, $2b^2 = 4c^2$, or $b^2 = 2c^2$, which means b^2 is even and so is b. That means 2 divides both a and b, contradict with the assumption.

Problem



Assume that we have an infinite domino string, we want to know whether every dominoes will fall, if we only know two things:

1. We can push the first domino to fall
2. If a domino falls, the next one will be fall

We can! Mathematical induction.

Mathematical Induction

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Definition (Induction)

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

- **Basis Step**: Verify that $P(1)$ is true.
- **Inductive Step**: Show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k

Logic form:

$[P(1) \land \forall k P(k) \to P(k+1))] \to \forall n P(n)$

What is $P(n)$ in domino string case?

Example on Induction

Example

Show that if n is a positive integer, then

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Solution

Let $P(n)$ be the proposition that sum of first n is $n(n+1)/2$

- Basis Step: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$
- Inductive Step:
  Assume that $1 + 2 + \ldots + k = \frac{k(k+1)}{2}$.
  Then:

$$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Example on Induction

Example

Prove that $n < 2^n$ for all positive integers n.

Solution

Let $P(n)$ be the proposition that $n < 2^n$.

- Basis Step: $P(1)$ is true, because $1 < 2^1 = 2$
- Inductive Step:
  Assume that $P(k)$ is true for the positive k, that is, $k < 2^k$.
  Add 1 to both side of $k < 2^k$, note that $1 \leq 2^k$.

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

  shows that $P(k + 1)$ is true, namely, that $k + 1 < 2^{k+1}$,
  based on the assumption that $P(k)$ is true.

Exercise

Prove that, if n is a non-negative integer and $7n + 9$ is an even number, then n is an odd number by three ways:

1. Directed proof.
2. Contraposition proof(phản đảo).
3. Contradiction.

Exercise

Directed proof

Assume: $7n + 9$ is the even number. Then, $7n + 9 = 2k, (k \in \mathbb{Z})$
So: $n = 2k - 6n - 9 = 2k - 6n - 10 + 1 = 2(k - 3n - 5) + 1$
That means n is an odd number.

Contrapositive proof (phản đảo)

To proof the above statement, firstly, we convert it into the logic expression: $p \rightarrow q$ with $p = 7n + 9$ is the even number and $q = n$ is the odd number.
Its contrapositive: "If n is not an odd number, then $7n + 9$ is not an even number". We can prove this statement follows this way:
If n is not an odd number, that means n can divisible 2. So that, $n = 2k, (k \in \mathbb{Z})$
We imply: $7n + 9 = 7(2k) + 9 = 14k + 9 = 2(7k + 4) + 1$
That means: $7n + 9$ is not the even number. Totally, we have proved the logic expression: $\neg q \rightarrow \neg p$. Therefore $p \rightarrow q$ is also truth.

Contradiction proof

Suppose $7n + 9$ is an even number and n is not an odd number or n is an even number.
Because n is an even number, then $n = 2k, (k \in \mathbb{Z})$
We infer: $7n + 9 = 7(2k) + 9 = 14k + 9 = 2(7k + 4) + 1$
Its means: $7n + 9$ is the odd number. We can show that, if n is an even number, then $7n + 9$ is an odd number. This contradicts with the hypothesis $7n + 9$ is an even number.

Exercise

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Which of the method of proof is used for proving the statement below:

To prove *"If m and n are integers, $m \times n$ is an even number, then either m is even or n is even"*, we follow these inferences:
Assume m and n are odd numbers. Then we can express: $m = 2k + 1$ and $n = 2l + 1$. So $mn = (2k + 1)(2l + 1) = 2(2kl + k + l) + 1$ is the odd number. Contradiction. That means either m is even or n even.

A) Directed proof.
B) Contradiction proof or contra-positive proof
C) inductive proof
D) All the above answers are incorrect.

Exercise

Which of the method of proof is used for proving the statement below:

To prove *"If m and n are integers, $m \times n$ is an even number, then either m is even or n is even"*, we follow these inferences:
Assume m and n are odd numbers. Then we can express: $m = 2k + 1$ and $n = 2l + 1$. So $mn = (2k + 1)(2l + 1) = 2(2kl + k + l) + 1$ is the odd number. Contradiction. That means either m is even or n even.

A) Directed proof.

B) Contradiction proof or contra-positive proof correct answer!

C) inductive proof

D) All the above answers are incorrect.

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What is wrong in the following induction to prove that all flowers are the same color?

1. Let $P(n)$ be all flowers in a set of n flowers have the same color.
2. We can easily infer that, $P(1)$ is a truth.
3. Assume that $P(n)$ is correct. Which means that all flowers in the set of n flowers have the same color .
4. Consider a set of $n+1$ flowers; numbering as $1, 2, 3, \ldots, n, (n+1)$.
5. Based on the assumption, Sequence of first n flowers has the same color, and sequence of n later flowers also has the same color.
6. Because the 2 set has same $n-1$ flowers, all $n+1$ flowers should have a same color.
7. Meaning $P(n+1)$ and the statement is proved by induction.

Consider a subset D ($\mathcal{N} \times N$) which defined in a recursive way:

(i) $(n, 0) \in D$.

(ii) If $(n, m) \in D$, $(n, n+m) \in D$.

Do

1) Calculate some elements from D.
2) prove by induction on k that 'if $m = k.n$, '$(n, m) \in D$'.
3) prove that if $(n, m) \in D$, we will get $m = kn$ $k \in \mathcal{N}$.

a) Prove that $\forall n \in \mathcal{N}^+$
$(n+1)^2 - (n+2)^2 - (n+3)^2 + (n+4)^2 = 4$.

b) The infer that for every natural number m, there exists natural number n that can represent m as a sum of squares 1^2, 2^2, ..., n^2, which mean: $\forall m \in \mathcal{N}^+$, $\exists n \in \mathcal{N}^+$, $\exists \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, $m = \varepsilon_1 1^2 + \varepsilon_2 2^2 + \ldots + \varepsilon_n n^2$.
(**Hint**: Try to display the values of $m \in \{1, 2, 3, 4, 5, 6\}$)

a) Prove that $\forall n \in \mathcal{N}^+$
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(**Hint**: Try to display the values of $m \in \{1, 2, 3, 4, 5, 6\}$)

- $m = 0$: $0 = 1^2 + 2^2 - 3^2 + 4^2 - 5^2 - 6^2 + 7^2$
- $m = 1$: $1 = 1^2$
- $m = 2$: $2 = -1^2 - 2^2 - 3^2 + 4^2$
- $m = 3$: $3 = -1^2 + 2^2$
- $m = 4$: $4 = 1^2 - 2^2 - 3^2 + 4^2$

a) Prove that $\forall n \in \mathcal{N}^+$
$(n+1)^2 - (n+2)^2 - (n+3)^2 + (n+4)^2 = 4.$

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(**Hint**: Try to display the values of $m \in \{1, 2, 3, 4, 5, 6\}$)

- $m = 0$: $0 = 1^2 + 2^2 - 3^2 + 4^2 - 5^2 - 6^2 + 7^2$
- $m = 1$: $1 = 1^2$
- $m = 2$: $2 = -1^2 - 2^2 - 3^2 + 4^2$
- $m = 3$: $3 = -1^2 + 2^2$
- $m = 4$: $4 = 1^2 - 2^2 - 3^2 + 4^2$
- $m = 5$: $5 = 1^2 + (2^2 - 3^2 - 4^2 + 5^2)$

a) Prove that $\forall n \in \mathcal{N}^+$
$(n+1)^2 - (n+2)^2 - (n+3)^2 + (n+4)^2 = 4.$

b) The infer that for every natural number m, there exists natural number n that can represent m as a sum of squares 1^2, 2^2, ..., n^2, which mean: $\forall m \in \mathcal{N}^+$, $\exists n \in \mathcal{N}^+$, $\exists \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, $m = \varepsilon_1 1^2 + \varepsilon_2 2^2 + \ldots + \varepsilon_n n^2$.
(**Hint**: Try to display the values of $m \in \{1, 2, 3, 4, 5, 6\}$)

- $m = 0$: $0 = 1^2 + 2^2 - 3^2 + 4^2 - 5^2 - 6^2 + 7^2$
- $m = 1$: $1 = 1^2$
- $m = 2$: $2 = -1^2 - 2^2 - 3^2 + 4^2$
- $m = 3$: $3 = -1^2 + 2^2$
- $m = 4$: $4 = 1^2 - 2^2 - 3^2 + 4^2$
- $m = 5$: $5 = 1^2 + (2^2 - 3^2 - 4^2 + 5^2)$
- $m = 6$: $6 = -1^2 - 2^2 - 3^2 + 4^2 + (5^2 - 6^2 - 7^2 + 8^2)$

Prove the following:

1. $1.2 + 2.5 + 3.8 + \ldots + n.(3n-1) = n^2(n+1), n \geq 1$
2. $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}, n \geq 1$
3. $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$
4. $\log_5(2)$ is an irrational number.

Solution

(1) $1.2 + 2.5 + 3.8 + \ldots + n.(3n-1) = n^2(n+1), n \geq 1$ (1)
- With $n = 1$ we have $1.2 = 1^2(1+1) \rightarrow$ (1) correct.
- Assume (1) is correct with $n = k$, then we have $1.2 + 2.5 + 3.8 + \ldots + k.(3k-1) = k^2(k+1)$
- We will prove (1) is correct with $n = k+1$, meaning $1.2+2.5+3.8+\ldots+k.(3k-1)+(k+1)(3k+2) = (k+1)^2(k+2)$.

In fact, $n = k+1$, meaning
$1.2 + 2.5 + 3.8 + \ldots + k.(3k-1) + (k+1)(3k+2) =$
$[1.2 + 2.5 + 3.8 + \ldots + k.(3k-1)] + (k+1)(3k+2) =$
$k^2(k+1) + (k+1)(3k+2) = (k+1)(k^2+3k+2) =$
$(k+1)(k+1)(k+2) = (k+1)^2(k+2)$.
The expression is correct with $n = k+1$.Thus, QED.

Solution

(2) $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=\frac{2n}{n+1}, n \geq 1$ (2)
- With $n=1$ we have P(1): $1=\frac{2}{1+1} \rightarrow$ (2) correct.
- Assume (2) is correct with $n=k$, Then we have P(k):
$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}=\frac{2k}{k+1}$
- We will prove (2) is correct with $n=k+1$, meaning P(k+1):
$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}+\frac{1}{1+2+\ldots+k+(k+1)}=$
$\frac{2k}{k+1}+\frac{1}{1+2+\ldots+k+(k+1)}=\frac{2k}{k+1}+\frac{2}{(k+1)(k+2)}=\frac{2k(k+2)+2}{(k+1)(k+2)}=$
$\frac{2(k^2+2k+1)}{(k+1)(k+2)}=\frac{2(k+1)^2}{(k+1)(k+2)}=\frac{2(k+1)}{k+2}$ [Noted that:
$1+2+\ldots+k+(k+1)=\frac{(k+1)(k+2)}{2}$]
The expression is correct with $n=k+1$.Thus, QED.

Solution

(3) $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$ (3)
- With $i = 1$ we have S(1): $\sum_{i=1}^{1} 2^{i-1} = 2^{1-1} = 2^1 - 1 \rightarrow$ (3) correct.
- Assume (3) is correct with $n = k$, Then we have S(k): $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$
- We will prove (3) is correct with $n = k + 1$, meaning S(k+1): $\sum_{i=1}^{k+1} 2^{i-1} = \sum_{i=1}^{k} 2^{i-1} + 2^k = 2^k - 1 + 2^k = 2^{k+1} - 1$
The expression is correct with $n = k + 1$.

(4) Prove that $\log_5(2)$ is an irrational number.
Assume the opposite $\log_5(2)$ is a rational number. Therefore, $\log_5(2) = \frac{a}{b}$, where $a, b \in \mathbb{Z}$, $b \neq 0$, GCD(a,b) = 1
Then, $5^{\frac{a}{b}} = \sqrt[b]{5^a} = 2 \Leftrightarrow 5^a = 2^b$. Because 5^a is always odd and 2^b is always even. So, this is a contradiction. Thus, QED.

Exercise

Prove the following statements by induction:

1. For every integer $n \geq 1$, $3^{2n-1} + 1$ is divisible by 4.
2. For every integer $n \geq 1$, $6^n - 1$ is divisible by 5.
3. For every integer $n \geq 1$, $5^{2n-1} + 1$ is divisible by 6.
4. For every integer $n \geq 1$, $8^n - 1$ is divisible by 7.
5. For every integer $n \geq 1$, $4^n + 15n - 1$ is divisible by 9.

Solution

(1) Prove that for every integer $n \geq 1$, $3^{2n-1} + 1$ is divisible by 4.

With $n = 1$, we have $4 \vdots 4$ - Truth.
On the other hand, There is an integer $k \geq 1$ so that $3^{2k-1} + 1$ is divisible by 4.
On the other hand, There is an integer m so that $3^{2k-1} + 1 = 4m$.
We need to prove $3^{2(k+1)-1} + 1$ is divisible by 4.
This is the equivalent to prove $3^{2k+1} + 1$ is divisible by 4.
On the other hand, we have:
$3^{2k+1} + 1 = 3^2(3^{2k-1} + 9 - 8 = 9(3^{2k-1} + 1) - 2.4 = 4(9m + 2)$.
Therefore, $3^{2k+1} + 1$ is a multiple of 4. The above statement is proved by induction

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Solution

(2) Prove that for every integer $n \geq 1$, $6^n - 1$ is divisible by 5.

With $n = 1$, we have $6^1 - 1 = 5\vdots 5$ - Truth.
On the other hand, There is an integer $n \geq 1$ so that $6^n - 1$ is divisible by 5.
On the other hand, There is an integer m so that $6^n - 1 = 5m$.
We need to prove $6^{n+1} - 1$ is divisible by 5.
On the other hand, we have:
$6^{n+1} - 1 = 6.6^n - 6 + 5 = 6(6^n - 1) + 5 = 30m + 5 = 5(6m + 1)$.
Therefore, $6^{n+1} - 1$ is a multiple of 5. The above statement is proved by induction.

Solution

(3) Prove that for every integer $n \geq 1$, $5^{2n-1} + 1$ is divisible by 6.

With $n = 1$, we have $5^{2-1} + 1 = 4 \vdots 4$ - Truth.
Assume there is an integer $n \geq 1$ so that $5^{2n-1} + 1$ is divisible by 6.
On the other hand, There is an integer m so that $5^{2n-1} + 1 = 6m$.
We have to prove $5^{2(n+1)-1} + 1$ is divisible by 6.
This is the equivalent to prove $5^{2n+1} + 1$ is divisible by 6.
On the other hand, we have:
$5^{2n+1} - 1 = 5^2.5^{2n-1} + 25 - 24 = 5^2(5^{2n-1} + 1) - 24 = 150m - 24 = 6(25m - 4)$. Therefore, $5^{2n-1} + 1$ is a multiple of 6. The above statement is proved by induction.

Solution

(4) Prove that for every integer $n \geq 1$, $8^n - 1$ is divisible by 7.

With $n = 1$, we have $8^1 - 1 = 7 \vdots 7$ - Truth.
Assume there is an integer $n \geq 1$ so that $8^n - 1$ is divisible by 7.
In other word, there exists an integer m so that $8^n - 1 = 7m$.
We need to prove that $8^{n+1} - 1$ is also divisible by 7.
On the other hand, We have:
$8^{n+1} - 1 = 8.8^n - 8 + 7 = 8(8^n - 1) + 7 = 56m + 7 = 7(8m + 1)$.
Thus, $8^{n+1} - 1$ is also a multiple of 7. The above statement is proved by induction.

Solution

(5) Prove that for all integers $n \geq 1$, $4^n + 15n - 1$ are divisible by 9.

With $n = 1$, we has $4^1 + 15 - 1 = 18 \vdots 9$ - Always true.
Assume there is an integer $n \geq 1$ so that $4^n + 15n - 1$ is divisible by 9
In other word, There is an integer m so that
$4^n + 15n - 1 = 9m$
We need to prove that $4^{n+1} + 15(n+1) - 1$ is also divisible by 9.

Solution

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On the other hand, we have:
$4^{n+1} + 15(n+1) - 1 = 4^n.4^1 + 15n + 15 - 1 =$
$3.4^n + 4^n + 15n + 15 - 1 = (4^n + 15n - 1) + 3.4^n + 15$

In which: $(4^n + 15n - 1) \vdots 9$ (based on the assumption)
And: For $3.4^n + 15 = 3(4^n + 5)$ divisible by 9, we need to prove

$(4^n + 5) \vdots 3$

Let B(k) = $4^k + 5$. With $k = 1$, $4^1 + 5 = 9 \vdots 3 \rightarrow$ correct with $k = 1$

Assume it is also true with $k = i \rightarrow$ B(i) = $(4^1 + 5) \vdots 3$. We prove that it is also correct with $k = i + 1 \rightarrow$ B(i+1) =
$4^{i+1} + 5 = 4^i.4 + 5 = 3.4^i + 4^i + 5 = (4^i + 5) + 3.4^i$. We have:

$(4^i + 5) \vdots 3$ based on the assumption

And: $3.4^i \vdots 3$. Therefore, $(4^n + 5) \vdots 3$. thus, QED.

Prove the following inequalities by induction:

1. For every integer $n \geq 1$, $3^n > n^2$.
2. For every integer $n \geq 4$, $n! > 2^n$.

Solution

(1) Prove that for every integer $n \geq 1$, $3^n > n^2$.
with $n = 1$, we have $3^1 = 3$ và $1^2 = 1$. Therefore, the statement is correct for $n = 1$.
With $n = 2$, we have $3^2 = 9$ and $2^2 = 4$. Therefore, the statement is also correct with $n = 2$.
Assume there is an integer $n \geq 2$ that $3^n > n^2$.
We need to prove 3^{n+1} is greater than $(n+1)^2$.
On the other hand, Expanding the right side of the inequality, we get: $(n+1)^2 = n^2 + 2n + 1$.
Because we only consider $n \geq 2$, $n^2 = n \times n \geq 2n$ and $3^n > 1$.
Consider the left side of the inequality:
$3^{n+1} = 3 \times 3^n = 3^n + 3^n + 3^n > n^2 + n^2 + 1 > n^2 + 2n + 1 = (n+1)^2$ (QED)
. Thus, The statement is proved by induction.

Solution

(2) Prove that for every integer $n \geq 4$, $n! > 2^n$.
with $n = 4$, we have $4! = 24 > 2^4 = 16$.
Assume there is an integer $n \geq 1$ that $n! > 2^n$.
We need to prove that $(n+1)!$ is greater than 2^{n+1}.
On the other hand, we have:
$(n+1)! = (n+1) \times n!$. based on the assumption, we have:
$(n+1)! > (n+1) \times 2^n$.
Moreover, the expression on the right side is greater than $2 \times 2^n = 2^{n+1}$ (vì $n \geq 4 > 2$) because of that, the above statement is proved by induction.

Exercise

Find the following values $f(1)$, $f(2)$ and $f(3)$ knowing that $f(n)$ is defined in the a recursive way with $f(0) = 1$ and $f(n+1) = f(n)^2 + f(n) + 1,\ n = 0, 1, 2, \ldots$

- A) 1, 3, 13
- B) 1, 3, 10
- C) 3, 10, 111
- D) 3, 13, 173
- E) None of the above

Exercise

Find the following values $f(1)$, $f(2)$ and $f(3)$ knowing that $f(n)$ is defined in the a recursive way with $f(0) = 1$ and $f(n+1) = f(n)^2 + f(n) + 1$, $n = 0, 1, 2, \ldots$

A) 1, 3, 13
B) 1, 3, 10
C) 3, 10, 111
D) 3, 13, 173
E) None of the above correct answer!

Exercise

A sequence $\{t(n)\}_n$ is defined by a recursive formula:

$$t(n) + t(n-1) - 6\,t(n-2) = 0 \quad (t \geq 3),$$

Knowing that $t(1) = 1$, $t(2) = 3$. The Explicit formula for the sequence is:

A) $t(n) = \frac{3}{5} \cdot 2^n + \frac{1}{15} \cdot (-3)^n$
B) $t(n) = \frac{3}{5} \cdot 2^n + \frac{7}{15} \cdot (-3)^n$
C) $t(n) = 3 \cdot (-2)^n + \frac{7}{3} \cdot 3^n$
D) $t(n) = -3 \cdot 2^n + \frac{5}{3} \cdot (-3)^n$
E) None of the above

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Exercise

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Find the first six numbers of the sequence defined by the following recursive formula.

$$a_0 = -1, a_n = -2a_{n-1}$$

A) -1, 2, -3, 4, -5, 6
B) 1, 2, 4, 8, 16, 32
C) 2, -4, 8, -16, 32, -64
D) -1, 2, -4, 8, -16, 32
E) None of the above

Exercise

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E) None of the above

Exercise

Recognize the pattern of the following integer sequence then find the next 4 numbers of the sequence.

$$1, 0, 2, 2, 0, 0, 4, 4, 4, 0, 0, 0, \ldots$$

A. Not exist
B. 8,8,8,8
C. 6,6,6,6
D. Either 8,8,8,8 or 0,8,8,8
E. None of the above.

Exercise

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$$1, 0, 2, 2, 0, 0, 4, 4, 4, 0, 0, 0, \ldots$$

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Exercise

Which of the following statement is correct?

A) A number is rational if and only if its square is a rational number.
B) integer n is odd if and only if $n^2 + 2n$ is odd.
C) number is irrational if and only if its square is an irrational number.
D) A number n is odd if and only if $n(n + 1)$ is even.
E) None of the above.

Exercise

Which of the following statement is correct?

A) A number is rational if and only if its square is a rational number.
B) integer n is odd if and only if $n^2 + 2n$ is odd. correct answer!
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E) None of the above.

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Exercise

When the sun rises, two models X and Y start walk along the seashore. X walk from A to B, and the other from B to A. At 12PM, the two meet the other, and continue walking with the same velocities. The first person arrives at B at 4M, and the other arrives at A at 9PM.
When did the sun rise?

A) 5:30 AM
B) 6:00 AM
C) 6:30 AM
D) 7:00 AM
E) None of the above

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Exercise

The Fibonacci numbers form the sequence of 1, 2, 3, 5, 8, 13, 21, 34, ..., with the following definition: $a_1 = 1$, $a_2 = 2$, và $a_{n+2} = a_{n+1} + a_n, \forall n \geq 1$. The greatest common divisor of a_{100} and a_{99} is:

A) 1
B) 2
C) 218922995834555169026
D) None of the above

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A) 1 correct answer!
B) 2
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D) None of the above

Exercise

In the Far Far Away kingdom, money is rarely used. People are often trading with their goods or with coins of value 5g or 7g. Is every good that has the price of 29g and above tradable using the above coins?
Choose the most correct answer for the above question.

A) No
B) The question should be “goods with the price above 25g”
C) Should be “goods with the price above 35g”
D) Should change the question to “goods with any price”
E) Yes
F) None of the above

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D) Should change the question to "goods with any price"
E) Yes correct answer!
F) None of the above